

APPROXIMATE ANALYSIS OF NONLINEAR
STOCHASTIC SYSTEMS

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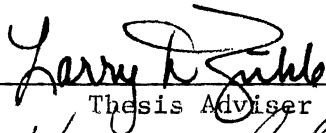
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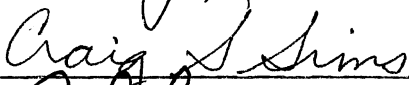
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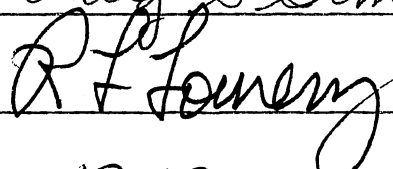
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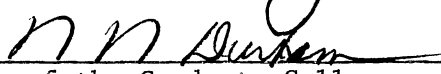


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PREFACE

This study is concerned with the development of a method of calculating the statistics of processes in nonlinear stochastic systems. The main objective is to evaluate the statistical characteristics of the processes with sufficient accuracy. The accurate values are used to check the accuracy of the method of statistical linearization (MSL). The MSL is a very general and powerful method of nonlinear systems analysis and synthesis. The only drawback of the method has been the lack of a means of checking its accuracy and, therefore, its applicability. The method developed in this study fulfills these needs, to a large extent.

I wish to express my appreciation to my major adviser, Dr. Larry Zirkle, for his guidance and assistance throughout this study. Appreciation is also expressed to the chairman of the thesis committee, Dr. Richard Lowery, for his encouragement and counsel; to Dr. Henry Sebesta and Dr. Craig Sims, members of the committee, who provided invaluable assistance and criticism throughout my program.

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LIST OF SYMBOLS

α_n	Initial moment of order n
β_n	Quasi-moment of order n
E	Mathematical expectation operator
$f(z)$	Characteristic function of a process
γ_1	Coefficient of skewness
γ_2	Coefficient of excess
$G(j\omega)$	Transfer function of a linear system
$g(z)$	Log characteristic function of a process
H_k	Chebyshev-Hermite polynomials of order k
k_0	Static gain corresponding to the mean value
k_1	Static gain corresponding to centered component
K	Matrix of statics gains for centered components
λ_n	Semi-invariant of order n
m_x	Mean value of x
μ_n	Central moment of order n
P	Covariance matrix
$p_0^*(x)$	Normal density function of independent variables
$p_0(x)$	Normal density function, general
$p(x)$	Arbitrary density function
$\phi(.)$	Nonlinear characteristic of arbitrary form
ϕ_0	Statistical characteristic of a nonlinearity
$R_x(.)$	Correlation function of x

$R_{xy}(\cdot)$	Crosscorrelation of x and y
ρ	Correlation coefficient
$S(\omega)$	Spectral density function
σ_x	Standard deviation of x
τ	Correlation time
$W(t,s)$	Weighting function of a linear system
ω	Frequency, radians per second
ω_c	Cut-off frequency of a linear system
\longleftrightarrow	Fourier transform pair indicator

CHAPTER I

INTRODUCTION

Background

In most engineering disciplines linear problems were the first to be raised and solved, owing to the relative simplicity of their solutions. Development, however, soon confronted the engineer with problems impossible to approach by linear methods. This tended to focus attention on a more profound study of nonlinear systems. In recent years, indeed, nonlinearities have increasingly been regarded as difficult to treat mathematically, but also as the source of advantages.

In control engineering, control systems meeting specific operational requirements can be realized. Examples include relay systems and the constrained control-variable systems of minimum time, minimum fuel or energy consumption. Such problems either cannot be solved by linear systems at all or only exceptionally in a very roundabout manner.

Every physical system is essentially nonlinear. A physical system may also have parameters varying more or less with time. In every physical system, certain limitations will sooner or later set in, either with decreasing amplitudes (insensitivity bands) or with their increasing amplitudes (saturation phenomena). Also, time brings about fatigue and aging phenomena.

In the investigation of engineering systems a central position is held by the statistical approach to the problems. The pressure fields

generated by jet propulsion devices fluctuate in a random manner. They contain a wide spectrum of frequencies that may result in severe vibrations in the aircraft. Earthquakes excite severe vibrations which may cause failure in structures. These excitations are also random. The motion of a ship in a confused sea and the motion of an aircraft through a turbulent atmosphere can only be described statistically. Dynamic systems may also have parameters that vary in a random manner. Servomechanisms with randomly varying transfer functions, random disturbances on instruments and rate gyros in an aircraft, and randomly pulsating loads on a structure all involve random parameters.

Confronted with the difficulties that nonlinear closed-loop stochastic systems present, engineers directed their attention to methods of approximate analysis. Many such methods have been developed and most of them are only applicable to certain types of problems. At the present time, three main methods of analysis exist.

1. The first of these methods is based on representing an arbitrary nonlinear operation with memory by means of a system of special operations (Pugachev, 1965) (Wiener, 1958). The analysis problem is reduced to seeking the parameters of this representation. This is done by computing the crosscorrelations of the outputs of the actual nonlinear operations and of the model when the signals acting on them are of the white noise type. This method, being experimental-analytic in principle, makes it possible to construct a practically exact model of the arbitrary nonlinear operation but is extremely laborious even at the modern level of development of computation techniques. Apparently, there has been little follow up or results of practical applications of this approach.

2. Another method is that of constructing a Markov model of the nonlinear system. In principle, it allows us to solve for the statistics of a large class of nonlinear operations. However, in practice we can obtain results with relative ease only if the system is described by differential equations of first or second order.

3. Finally, the third method is that of linearization. Linearization about an operating point, or direct linearization, is common in Western literature and consists of retaining the linear terms of the Taylor series expansion of the nonlinear relation. Evidently, this method is limited to small variations about the point of linearization. It also requires that the nonlinear characteristic be analytic. A more general method of linearization is the method of statistical linearization (MSL), also called quasi-linear method. The method applies to small and large nonlinearities, small and large perturbations and to continuous and discontinuous nonlinearities. The method is described in some detail in Chapter II. The investigation of the various aspects of this method is a main objective in this work.

Purpose and Method of Investigation

The purpose of the investigation is the development of a method of assessing the accuracy and applicability of the MSL. Means of improving the present method are also explored as a side issue.

The method of investigation depends heavily on an approximate analytical representation of the probability density function of a random process. Since the representation has been treated only partially (mainly in one dimension) in the literature, a detailed treatment of the problem is included as an appendix at the end of this work. In this appendix,

the results available in the literature have been organized and the results of the author's own investigations are added, especially those of multi-dimensional representations.

A brief summary of the methods of nonlinear analysis of stochastic systems has been mentioned above. Since the main concern here is the accuracy of the MSL, the method is surveyed in an organized manner in Chapter II. The material is taken from several sources in the literature. The main purpose of the survey is to expose the various aspects of method: its generality, simplicity and power in problem solving; and to indicate the areas which need further investigation.

A method of nonlinear analysis is developed in Chapter III. It is based on the theory of Markov processes and is the main contribution in this work. Coupled with the use of the density function representation, referred to above, the method enables the evaluation of system statistics of various orders. The method is quite general for the class of problems common in engineering systems and can be made as accurate as desired.

The method of nonlinear analysis is used to assess the accuracy of the MSL. This is done by comparing the values of the system statistics obtained by the latter with the accurate values obtained by the former. This is the subject of Chapter IV.

In Chapter V, the applicability of the MSL is discussed by investigating the various classes of problems to which the method may be applied with confidence. The author developed a modified version of the MSL which is more accurate than the presently used version, but which requires additional computations. The development is also based on the probability density representation. Finally, some areas of potential

applications of the MSL to the synthesis problem are briefly mentioned. These applications are included to show cases which could be evaluated with the technique presented in this dissertation.

Summary of Main Results

Two main results have been achieved in this work:

1. A method has been developed which allows the calculation of the statistical characteristics with arbitrary accuracy.
2. The accuracy of the MSL has been investigated for typical cases of some general nature.

The investigation of the accuracy of the MSL revealed several observations:

- a. It has been found that the assumption of a Gaussian signal at the input of the nonlinearity is the main source of error when the MSL yields inaccurate results. This occurs in systems with narrow band inputs. In this case, the assumption does not hold and the computed rms values involved large errors.
- b. Limited comparison of the linear models based on the minimum mean-square-error and on the equivalence of the correlation functions indicates that the former is a preferred criterion for choosing a linear model.
- c. The method of nonlinear analysis developed in this work is accurate and thus serves to assess the accuracy of the results obtained by the MSL.
- d. The MSL is found to give acceptable results for inputs with spectral densities peaking beyond the passband of the system. The MSL

is also accurate for systems with a systematic narrow-band signal corrupted with wideband noise.

e. As a general trend, the accuracy of the MSL increases with the increase in the order of the system.

f. The accuracy of the MSL for non-zero-mean systems is not as good as that for zero-mean systems. The method developed in this work, on the other hand, maintains the same accuracy in both types of systems.

CHAPTER II

THE METHOD OF STATISTICAL LINEARIZATION

Introduction

The Method of Statistical Linearization (MSL) also called Quasi-Linear Method, was introduced about the same time by Booton (1953) and Kazakov (1954). The method is based on the replacement of the nonlinear element by a linear model which is equivalent to it in some statistical sense. The method was examined later by several workers in the field (Barrett and Coales, 1955) (Douce and Roberts, 1963) (Leland, 1960) (Brown, 1964) (Pupkov, 1960).

The main criticism of the method has been on the assumption of a normal distribution to the signal at the input of the nonlinearity in the closed loop. The method has been found to give sufficiently accurate second-order statistics when such an assumption holds, while excessive errors have been encountered when it does not hold. This suggested the need for

1. investigating the accuracy of the method based on accurate general procedures which do not involve assumptions that are hard to justify,
2. determining on this basis the areas of applicability of the method and
3. if possible, rendering the method more general without losing its chief merit of simplicity.

Only a few efforts, with limited results, have been made in these areas (Smith, 1966) (Kolovskii, 1960) (Pyatnitskii, 1960). Several

suggestions have been made to improve the accuracy of the method by accounting for the distortion effects of the nonlinearity, on the signal spectrum, which the method neglects (Pupkov, 1960) (West, Douce and Leary, 1960) (Douce and Roberts, 1963). However, all these attempts retained the basic assumption of a normal signal at the input of the nonlinearity. Among the major contributions to the method has been Kazakov's (1965) extension of the method to multidimensional nonlinearities with nonadditive dependent inputs. This extension forms the basis of a general theory of treating nonlinear characteristics of the hysteresis type and the analysis of nonlinearities with random parameters. The theory was also applied to self-oscillatory systems in the steady state (Evlanov and Kazakov, 1969) and in the transient state (Evlanov and Kazakov, 1970).

In this chapter, the MSL is presented in a unified manner. The material is based on several sources in the literature. First, the general formulation of the method is outlined. The purpose of this formulation is to show the manner in which a linear model can be determined. The simplest form of the MSL is presented next. This is the form which has been used so far. It uses a simple static model for the nonlinearity. The extension of the MSL to multidimensional nonlinearities is described next. This is the basis of a general theory of the MSL which is of great generality. Finally, the application of the MSL to the analysis of nonlinear systems is outlined.

A General Formulation of the MSL

To determine a linear model which is statistically equivalent to the nonlinearity, a criterion for statistical equivalence is to be chosen. Two criteria have survived through the literature. The first is based on the equivalence of the mean values and correlation functions of the model and the nonlinear transformation. The second is based on the minimum mean-square error which results from the substitution of the linear model for the nonlinear element.

Let $X(t)$ be the input to the nonlinearity. It is written in the convenient form

$$X(t) = m_x(t) + X^0(t) \quad (2.1)$$

$m_x(t)$ is the mean value of $X(t)$ and $X^0(t)$ is the centered random component of the signal. In general, these two quantities are functions of time.

The nonlinear function is written as

$$Y(t) = \phi\{X(t), t\} \quad (2.2)$$

The linear equivalent model is written as

$$U(t) = m_u(t) + U^0(t) \quad (2.3)$$

$m_u(t)$ is the mean value of $U(t)$, given by

$$m_u(t) = k_0(t) \int_{t_0}^t W(t, \tau) m_x(\tau) d\tau \quad (2.4)$$

where $k_0(t)$ is a statistical gain, $W(.,.)$ is the weighting function of the linear model and t_0 is the initial time. $k_0(t)$ is used with odd symmetric nonlinearities. For other types of nonlinearities

(2.4) is replaced by

$$m_u(t) = \phi_o(t) \quad (2.5)$$

where $\phi_o(t)$ is called the statistical characteristic of the nonlinearity.

$U^o(t)$ is the centered random component of $U(t)$, given by

$$U^o(t) = k_1(t) \int_{t_o}^t W(t,\tau) X^o(\tau) d\tau \quad (2.6)$$

where $k_1(t)$ is a statistical gain. This is the most general formulation of the linear model, a dynamic one.

The problem then reduces to the evaluation of the parameters k_o or ϕ_o , k_1 and $W(.,.)$ which characterize the linear model. These are determined according to one of the two equivalence criteria.

1. Equivalence of the mean values and correlation functions of the model and the nonlinearity. This means that

$$m_y(t) = m_u(t)$$

or

$$\begin{aligned} E\{\phi(x,t)\} &= k_o(t) \int_{t_o}^t W(t,\tau) m_x(\tau) d\tau \text{ (for symmetric odd} \\ &\hspace{15em} \text{nonlinearities)} \\ &= \phi_o(t) \text{ (otherwise)} \end{aligned} \quad (2.7)$$

and

$$R_y(t,s) = R_u(t,s)$$

or

$$E \left\{ \left[\phi(x,t) - m_y(t) \right] \left[\phi(x,s) - m_y(s) \right] \right\} = k_1(t)k_1(s) \int_t^s \int_s^t W(t,\tau)W(s,\tau') R_x(\tau,\tau') d\tau d\tau' \quad (2.8)$$

where $R_x(\cdot)$ is the correlation function of the input signal $X(t)$ and

$$m_y(t) = E\{\phi(x,t)\}$$

We take

$$k_1(t) = \{R_y(t,t)/R_x(t,t)\}^{1/2}$$

and solve (2.8) for $W(\cdot)$ and (2.7) for $k_o(t)$ or $\phi_o(t)$. The solution of (2.8) for $W(\cdot)$ is difficult. For a stationary system, we take

$$k_1 = \{R_y(0)/R_x(0)\}^{1/2}$$

and $W(t,s) = W(t-s)$ can be determined by frequency analysis methods. In particular, we have

$$W(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(j\omega) e^{j\omega t} d\omega \quad (2.9)$$

where $G(j\omega)$ is the frequency (transfer) function corresponding to $W(t)$, and is given by

$$|G(j\omega)|^2 = \frac{1}{k_1^2} \frac{S_y(\omega)}{S_x(\omega)} \quad (2.10)$$

where $S(\omega)$ is the spectral density function of a signal.

To calculate ϕ_o , k_1 , and $G(j\omega)$ or $W(t)$, it is necessary to know the distribution of $X(t)$. Here the major assumption in the method occurs,

that $X(t)$ is normally distributed. With this assumption the parameters can be determined without difficulty. In particular, it is readily shown that, under this assumption

$$R_y(\tau) = \sum_{k=1}^{\infty} \frac{\rho_x^k(\tau)}{k!} a_k^2 \quad (2.11)$$

where $\rho_x(\tau) = R_x(\tau)/\sigma_x^2$, $\sigma_x^2 = R_x(0)$

and a_k are given by

$$a_k = \int_{-\infty}^{\infty} \phi(x) H_k(x^0/\sigma_x) p_0(x) dx \quad (2.12)$$

where $H_k(\cdot)$ are the Chebychev-Hermite polynomials and $p_0(x)$ is the density function corresponding to a normal $X(t)$.

Equation (2.11) yields

$$S_y(\omega) = \sum_{k=1}^{\infty} \frac{S_x^k(\omega)}{k!} a_k^2, \quad S_x^k(\omega) \leftrightarrow \rho_x^k(\tau) \quad (2.13)$$

from which it is easy to show that

$$|G(j\omega)|^2 = \frac{1}{k_1^2 \sigma_x^2} \sum_{k=1}^{\infty} b_k \frac{a_k^2}{k!} \quad (2.14)$$

where

$$b_k = S_x^k(\omega)/S_x(\omega)$$

But,

$$\sigma_y^2 = \sum_{k=1}^{\infty} a_k^2/k! \quad (2.15)$$

Therefore,

$$k_1 = 1/\sigma_x \left[\sum_{k=1}^{\infty} a_k^2/k! \right]^{1/2} \quad (2.16)$$

and

$$|G(j\omega)|^2 = \frac{\sum_{k=1}^{\infty} b_k a_k^2/k!}{\sum_{k=1}^{\infty} a_k^2/k!} \quad (2.17)$$

Thus, the equivalent linear model is truly a dynamic one. Usually, the first two or three terms in the infinite series are sufficient.

2. Minimum mean-square error criterion. This requires minimizing

$$E\{[Y(t) - U(t)]^2\}$$

By conventional variational methods, it is easy to show that this is satisfied when

$$m_y(t) = m_u(t) \quad (2.18)$$

or

$$\begin{aligned} E\{\phi(x,t)\} &= \int_{t_0}^t k_0(t)W(t,\tau)m_x(\tau)d\tau \quad (\text{for odd symmetric} \\ &\hspace{15em} \text{nonlinearities}) \\ &= \phi_0(t) \quad (\text{otherwise}) \end{aligned} \quad (2.19)$$

as in the first criterion, and

$$E\{X^0(t)Y(s)\} = k_1(t) \int_{t_0}^t R_x(t, \tau) W(\tau, s) d\tau \quad (2.20)$$

We take

$$k_1(t) = E\{X^0(t)Y(t)\}/R_x(t, t) \quad (2.21)$$

and solve (2.20) for $W(\cdot)$. In particular, (2.20) becomes

$$E\{X^0(t)Y(s)\} = k_1(t) \int_{-\infty}^t R_x(t, \tau) W(\tau, s) d\tau \quad (2.22)$$

This is a Wiener-Hopf equation. Methods of solving (2.22) are available in the literature. For the special case of a stationary system, we have

$$W(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(j\omega) e^{j\omega t} d\omega$$

$$G(j\omega) = \frac{1}{2\pi k_1 \Psi(j\omega)} \int_0^{\infty} e^{-j\omega t} dt \int_{-\infty}^{\infty} \frac{S_{xy}(\omega)}{\Psi^*(j\omega)} e^{j\omega t} d\omega \quad (2.23)$$

where

$$\Psi(j\omega) \Psi^*(j\omega) = S_x(\omega)$$

$S_{xy}(\omega)$ is the spectral density function corresponding to

$$E\{X^0(t)Y(t+\tau)\}$$

and

$$k_1(t) = E\{X^0(t)Y(t)\}/R_x(0)$$

We have thus obtained a dynamic model for this case, too. Again, with $X(t)$ normal, it is easy to show that

$$E\{X^0 Y\} = \sigma_x a_1 \rho_x(\tau) \quad (2.24)$$

from which

$$k_1 = a_1 / \sigma_x \quad (2.25)$$

and (2.23) is solved easily.

From the above relations one can verify the relations

$$R_u^{(1)}(\tau) = k_1^{(1)2} R_x(\tau) \geq R_y(\tau) \geq R_u^{(2)}(\tau) = k_1^{(2)2} R_x(\tau)$$

where the superscripts (1) and (2) refer to the first and second criteria, respectively.

The BMSL

The form of the MSL that has been used by most workers in systems analysis is a special case of the previous general form. It is designated, the "Basic Method of Statistical Linearization" (BMSL). In this form the linear model is without dynamics and is characterized by $k_0(t)$ or $\phi_0(t)$ and $k_1(t)$ only. The linear model for the nonstationary case is

$$U(t) = \phi_0(t) + k_1(t) X^0(t) \quad (2.26)$$

where, for an odd symmetric nonlinearity,

$$\phi_0(t) = k_0(t) m_x(t)$$

The parameters $\phi_o(t)$, $k_o(t)$ and $k_1(t)$ corresponding to the BMSL are easily derived from the general expressions just developed. They are given by

$$\phi_o(t) = E\{\phi(x,t)\} \quad (2.27)$$

$$k_1^{(1)}(t) = 1/\sigma_x E\{\phi^2(x,t) - \phi_o^2(t)\}^{1/2} \quad (2.28)$$

$$k_1^{(2)}(t) = E\{\phi(x,t)X^o(t)\}/\sigma_x^2 \quad (2.29)$$

For a stationary system ϕ_o , k_o and k_1 are constants, otherwise they may be functions of time.

Using the assumption that $X(t)$ is normal one can determine ϕ_o , $k_1^{(1)}$ and $k_1^{(2)}$. These can be evaluated beforehand for various nonlinear characteristics and tables would be prepared for them. These parameters will be function of m_x and σ_x which characterize the density function $p_o(x)$.

Important questions include:

How well does a static model represent the behavior of the original nonlinearity? Which criterion is to be preferred as the basis for statistical equivalence? The minimum mean-square-error criterion involves less computational effort than the first criterion. Limited evidence based on sample problems indicates that this criterion is also more representative of the original element. The minimum mean-square-error criterion is used in this work.

In order to gain an appraisal of what to expect from the BMSL in the way of representation, consider two nonlinear characteristics. The first is a cubic

$$\phi(x) = N x^3$$

and the second is an ideal relay

$$\phi(x) = L \operatorname{sgn} x$$

The correlation functions of the output of the actual nonlinearity and the linear models based on the two criteria are shown in Figures 1 and 2 for these two nonlinearities. The spectral densities are shown in Figure 3 for the relay. The calculations are based on a normal $X(t)$. The figures show that the correlation functions based on the first and second criteria form an upper and lower bounds, to the actual correlation function, respectively. This fact has been pointed out previously. Figure 3 indicates that the model based on the minimum mean-square-error criterion is more representative of the signal spectrum than that based on the first criterion.

Multidimensional Nonlinearities

The multidimensional nonlinearities are characterized by the single-valued transformation

$$Y = \phi(x_1, x_2, \dots, x_n) \quad (2.30)$$

where x_1, \dots, x_n are random dependent variables which, in the general case, enter nonadditively and $\phi(\cdot)$ is a nonlinear single-valued function or characteristic, of arbitrary form. As usual, we write

$$x_i = m_{x_i} + x_i^0 \quad i = 1, \dots, n$$

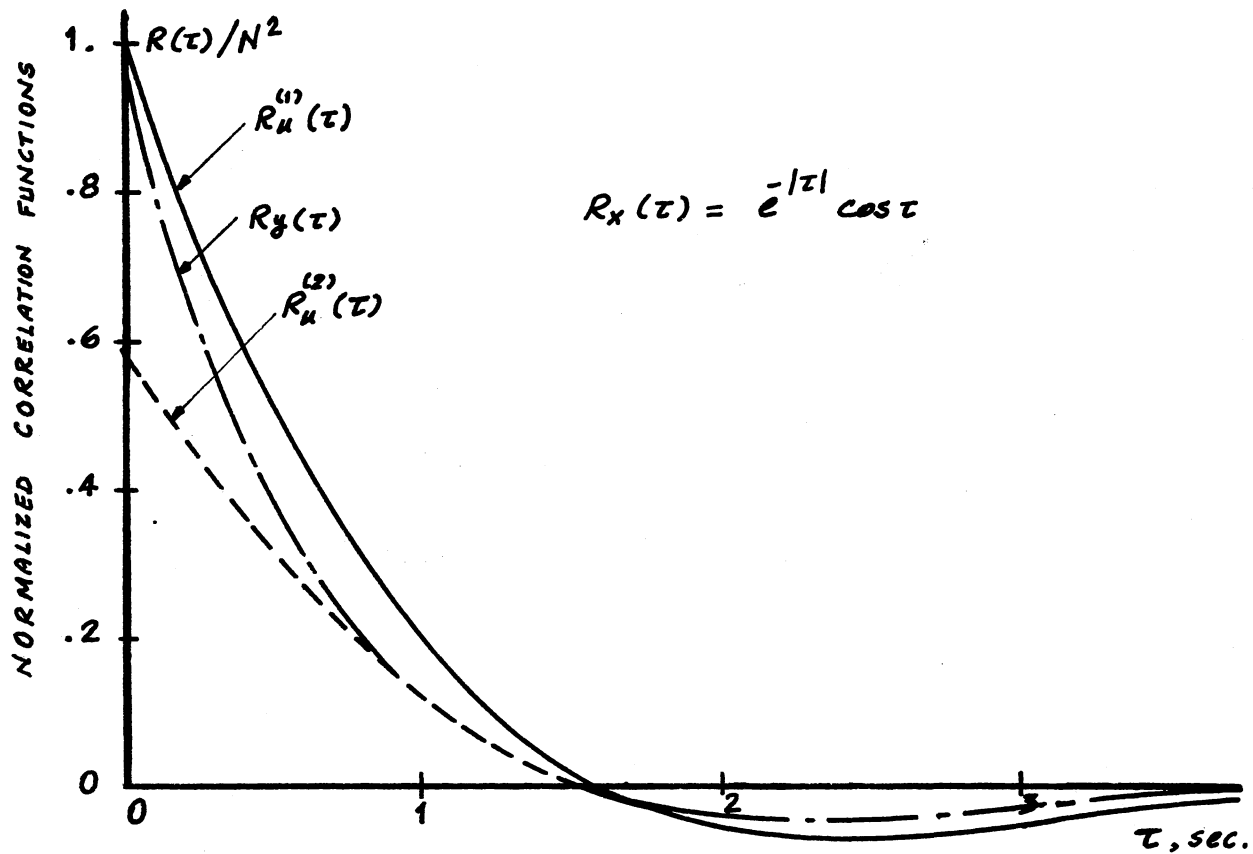


Figure 1. Correlation Functions of the Output of the Cubic Nonlinearity and its Equivalent Linear Models

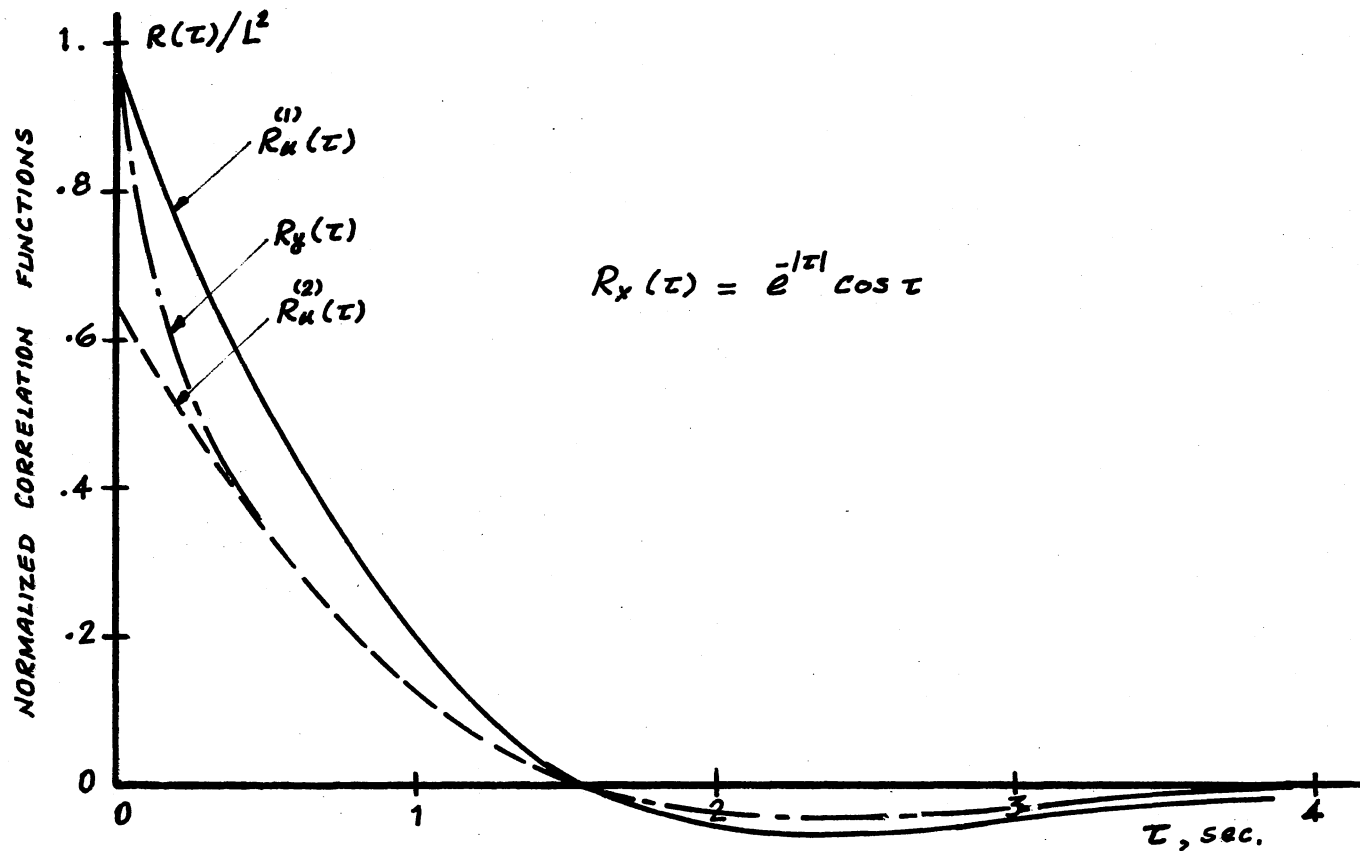


Figure 2. Correlation Functions of the Output of the Relay Nonlinearity and its Equivalent Linear Models

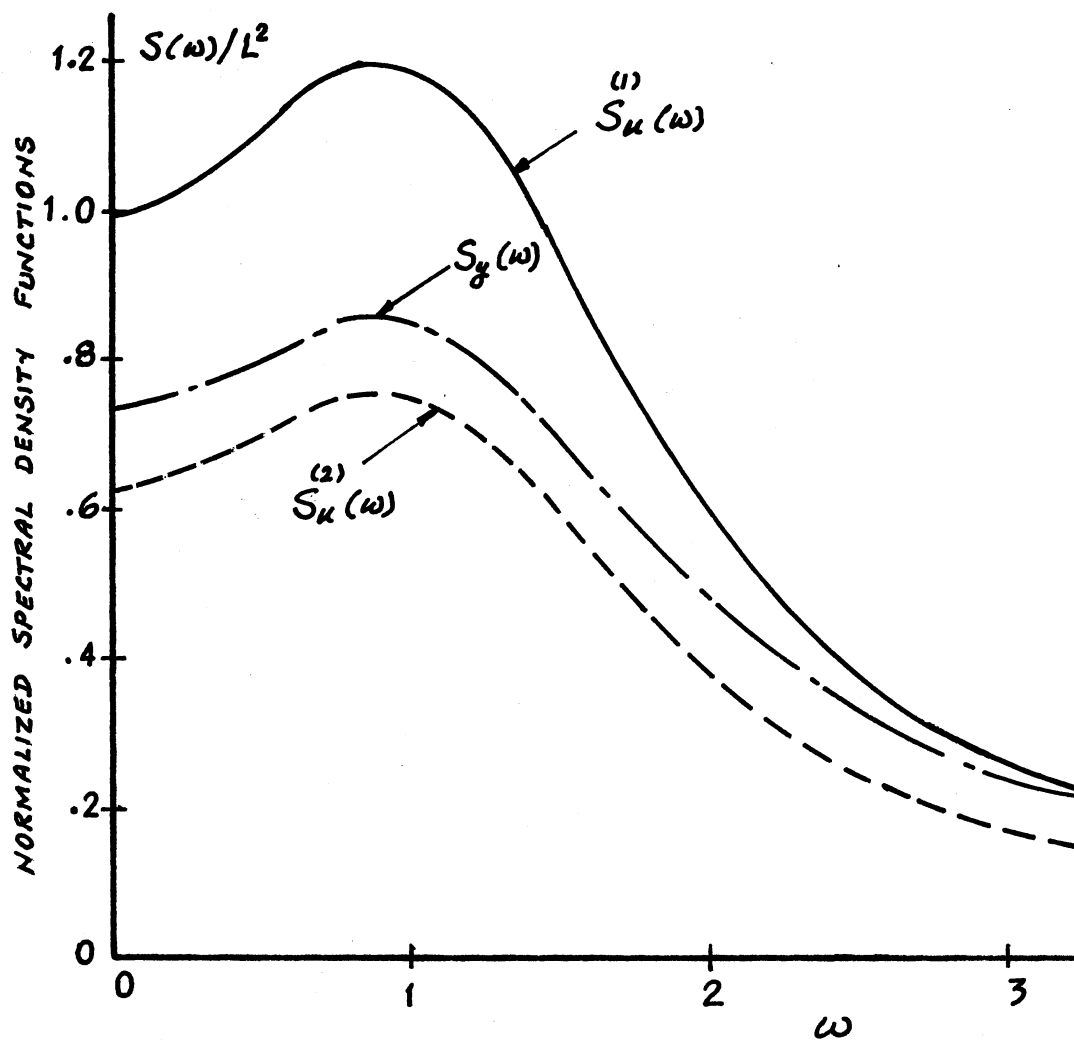


Figure 3. Spectral Density Functions of the Output of the Relay Nonlinearity and its Equivalent Linear Models

The linear model is then

$$U = \phi_o + \sum_{i=1}^n k_{1i} x_i^o \quad (2.31)$$

where ϕ_o is, as usual, a nonrandom function which is the statistical characteristic of the nonlinearities, and k_{1i} are the statistical gains.

ϕ_o and k_{1i} are chosen to minimize

$$E\left\{\left[\phi - \left(\phi_o + \sum_{i=1}^n k_{1i} x_i^o\right)\right]^2\right\} \quad (2.32)$$

Carrying out the minimization operation, we have

$$\phi_o = E\{\phi(x_1, \dots, x_n)\} \quad (2.33)$$

$$\sum_{i=1}^n k_{1i} R_{ji} = E\{\phi(x_1, \dots, x_n) x_j^o\}, j = 1, 2, \dots, n \quad (2.34)$$

where $R_{ij} = E\{x_i^o(t) x_j^o(t)\}$.

Solving (2.34), we get

$$k_{1i} = \sum_{j=1}^n (-1)^{i+j} \frac{\Delta_j^i}{\Delta} E\{x_j^o \phi\} \quad (2.35)$$

where

$$\Delta = \begin{vmatrix} R_{11} & \dots & R_{1n} \\ \cdot & \dots & \cdot \\ R_{n1} & \dots & R_{nn} \end{vmatrix} \quad (2.36)$$

and Δ_j^i is the cofactor of the element in the i th column and j th row of the determinant Δ .

To compute ϕ_0 and k_{1i} according to the BMSL, the variables x_i are assumed to have a multidimensional normal density function

$$p_0(x_1, \dots, x_n) = \frac{1}{\sqrt{2^n \Delta}} e^{\Delta^*/2\Delta} \quad (2.37)$$

where Δ^* is the bordered determinant

$$\Delta^* = \begin{vmatrix} R_{11} & \cdots & R_{1n} & x_1^{-m} x_1 \\ \cdot & \cdots & \cdot & \cdot \\ R_{n1} & \cdots & R_{nn} & x_n^{-m} x_1 \\ x_1^{-m} x_1 & \cdots & x_n^{-m} & 0 \end{vmatrix} \quad (2.38)$$

ϕ_0 is then given by

$$\phi_0 = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \phi(x_1, \dots, x_n) p_0(x_1, \dots, x_n) dx_1 \cdots dx_n \quad (2.39)$$

Differentiating (2.39) with respect to m_i and using (2.35) we get for k_{1i} the following convenient form

$$k_{1i} = \partial \phi_0 / \partial m_i, \quad m_i = m_{x_i} \quad (2.40)$$

Thus, the statistical linearization of the function ϕ reduces essentially to the computation of ϕ_0 . The obtaining of the formulas for ϕ_0 by computing the multiple integrals in (2.39) raises, in general, significant technical difficulties. Methods of obtaining formulas for the expectation operation of this kind which do not require the computation of multiple integrals are thus highly desired. The approximate analytic expansions for the density function in the appendix have been very helpful for this purpose.

The general theory just presented extends the application of the BMSL to a large class of nonlinear dependencies of arbitrary forms. In particular, a hysteresis-type nonlinearity of n input variables may be represented by a $2n$ -dimensional single-valued dependency of the form (2.30) and treated according to the above procedure. Moreover, the variables x_i may include random parameters which can also be interrelated, or sinusoidal signals with random phases.

Applications

In this section the application of the BMSL to the analysis of closed-loop systems is outlined. For stationary systems, frequency methods of analysis are used. For nonstationary systems, the state-variable representation is preferred.

Stationary Systems

A closed-loop system of a general type is shown in Figure 4. The system is described by the equation

$$Q(p) Y + R(p) X = S(p) Z \quad (2.41)$$

$$Y = \phi(X)$$

$Z(t)$ is a stationary random process with a rational spectral density function. Replacing Y by $\phi_0 + k_1 X^0$ and taking the mathematical expectations of both sides in (2.41) we have

$$Q(0) m_y + R(0) m_x = S(0) m_z \quad (2.42)$$

$$m_y = \phi_0$$

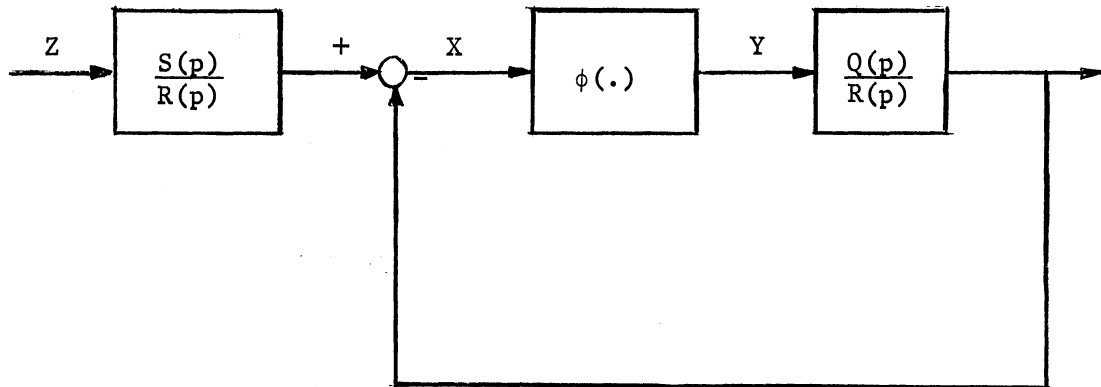


Figure 4. A Closed-Loop System of General Form- Stationary

Subtracting (2.42) from (2.41) we have for the centered components

$$Q(p) k_1 X^0 + R(p) X^0 = S(p) Z^0 \quad (2.43)$$

Equations (2.42) and (2.43) are related since both ϕ_0 and k_1 depend on m_x and σ_x . Therefore, they must be solved simultaneously.

The transfer function for X^0 is

$$G(p) = \frac{S(p)}{R(p) + k_1} \quad (2.44)$$

The variance of $X^0(t)$ is then

$$\sigma_x^2 = 1/2\pi \int_{-\infty}^{\infty} S_x(\omega) d\omega \quad (2.45)$$

where

$$S_x(\omega) = |G(j\omega)|^2 S_z(\omega)$$

From equation (2.42) we also have

$$\phi_0 = \frac{S(0)}{Q(0)} m_z - \frac{R(0)}{Q(0)} m_x \quad (2.46)$$

Equations (2.45) and (2.46) together with the relations defining ϕ_0 and k_1 in terms of m_x and σ_x are sufficient to evaluate m_x , σ_x , ϕ_0 , and k_1 . Once these are determined analysis of the system proceeds according to linear theory. For simple problems, graphical methods may be used to solve for the parameters. For high dimensions, the equations are solved by iterative procedures. A formulation will be described in connection

with nonstationary systems that can be used to evaluate the steady state values of the parameters.

Integrals of the form (2.45) are common in the study of optimal systems. Tables are available for their evaluation (Newton, Gould, and Kaisar, 1957). A fast algorithm is also available for the same purpose which is more useful than tables for high-order systems.

Nonstationary Systems

A nonstationary system is best described in the state-variable notation. A large class of systems may be described by differential equations of the form

$$\dot{\underline{x}}(t) = \underline{\phi}(\underline{x}, t) + B(t)\underline{v}(t), \underline{x}(0) = \underline{c} \quad (2.47)$$

where $\underline{x}(t)$ is the n -dimensional state vector, $\underline{\phi}(\underline{x}, t)$ is a vector of nonlinear characteristics; $B(t)$ is the matrix of coefficients and $\underline{v}(t)$ is a vector of white noise with intensity matrix equal to the identity matrix. \underline{c} is the n -dimensional vector of initial conditions, assumed normally distributed.

Applying the BMSL to (2.47), $\underline{\phi}(t, \underline{x})$ is replaced by

$$\underline{U} = \underline{\phi}_0(t) + K(t) \underline{x}^0(t)$$

where $\underline{\phi}_0$ is the statistical characteristics vector and K is the matrix of statistical gains. The linearized system equation is then

$$\underline{\dot{x}}(t) = \underline{\phi}_0(t) + K(t) \underline{x}^0(t) + B(t) \underline{v}(t) \quad (2.48)$$

$$\underline{x}(0) = \underline{c}$$

It is clear that $\underline{x}(t)$ described by (2.48) is a Markov-Gaussian process. So it is completely characterized by its mean vector, $\underline{m}_x(t)$, and the matrix of correlation functions, $R(t,s)$. A less complete, but sometimes sufficient, representation is obtained if only the covariance matrix is available in addition to the mean vector.

From (2.48), the mean vector satisfies the differential equation

$$\dot{\underline{m}}_x(t) = \underline{\phi}_0(t) + B(t) \underline{m}_v(t) \quad (2.49)$$

$$\underline{m}_x(0) = \underline{m}_c$$

The covariance matrix satisfies the differential equation

$$\dot{P}(t) = K(t) P(t) + P(t) K^T(t) + B(t) B^T(t) \quad (2.50)$$

$$P(0) = P_c$$

Equations (2.49) and (2.50) are easily solved on the digital computer. In general, there will be a total of $n(n+1)/2$ equations. In the steady state, the set of differential equations reduces to a set of algebraic equations.

The correlation functions satisfy the relations

$$\begin{aligned} \frac{\partial R(t,s)}{\partial t} &= K(t) R(t,s) & t > s \\ R(t,t) &= P(s) \end{aligned} \quad (2.51)$$

$$\begin{aligned} \frac{\partial R(t,s)}{\partial s} &= R(t,s) K^T(t) & t < s \\ R(s,s) &= P(t) \end{aligned} \quad (2.52)$$

Equations (2.51) are integrated for fixed values of s thus obtaining sections of $R(t,s)$ that are parallel to the t -axis. Similarly, (2.52) is integrated for fixed values of t giving sections of $R(t,s)$ parallel to the s -axis.

If the system is stationary, $R(t,s) = R(t-s) = R(\tau)$ and

$$\frac{dR(\tau)}{d\tau} = KR(\tau) \quad , \quad R(0) = P \quad (2.53)$$

where K is now a matrix of constant static gains.

Summary

In this Chapter the MSL has been presented in a unified manner. The BMSL has been emphasized as the simple and most practical version of the general formulation. The static model based on the minimum mean-square-error criterion requires less computations and limited evidence indicates that it may even be superior in representing the behavior of the original system. This model is adopted throughout this work.

The extension of the BMSL to multidimensional nonlinearities allows the application of the method to a wide variety of nonlinear dependencies and increases the generality of the method as a tool for systems analysis and design.

CHAPTER III

A METHOD OF NONLINEAR ANALYSIS

Introduction

In this chapter a method of nonlinear analysis which has great generality is developed. Of course, we should not expect to find a universal method of nonlinear analysis that will be devoid of any limitations. In this work, it is assumed that the processes involved are Markov.

One favorable circumstance is that a non-Markov process can be approximated to within any required precision by a multidimensional Markov process. However, it should be understood that the application of the theory of Markov processes is essentially connected with some approximations. In physical systems the processes possess a series of "good" properties like smoothness and analyticity which are incompatible with the Markov characteristics of the processes. The replacement of a physical process with a Markov process resembles, to a certain extent, the central limit theory, having to do with the convergence of an arbitrary process to the Gaussian. The analogy shows the wide scope of the problem.

Experience has shown that since the statistics of the random processes at the input of the system are known only approximately, taking such processes as Gaussian, or for that matter multidimensional Markov is a good engineering strategy in a great number of problems.

The theory of Markov processes is well covered in the basic literature (Doob, 1953) (Bharuch-Reid, 1960) (Stratonovich, 1968). The convergence of physical processes to Markov processes is also treated in Stratonovich (1968), among others.

A continuous stochastic system with state vector $\underline{x}(t) = \{x_1(t), \dots, x_n(t)\}$, which can be considered as a Markov process is described by the Ito stochastic differential equation

$$d \underline{x}(t) = \underline{A}(\underline{x}, t) dt + D(\underline{x}, t) d\underline{W}(t) \quad (3.1)$$

where,

$\underline{x}(t) = \{x_i(t)\}$, $i=1, \dots, n$ is the n -dimensional state vector.

The initial conditions $\underline{x}(t_0)$ can be a constant vector or a random vector with specified distribution or given statistics.

$\underline{A}(\underline{x}, t) = \{A_i(\underline{x}, t)\}$, $i=1, \dots, n$ is an n -dimensional vector function of \underline{x} and t .

$D(\underline{x}, t) = \{D_{ij}(\underline{x}, t)\}$, $i, j=1, \dots, n$ is the matrix of coefficients depending on \underline{x} and t .

$\underline{W}(t) = \{W_i(t)\}$, $i=1, \dots, n$ is an n -dimensional Wiener process with the incremental properties

$$E\{dW_i(t)\} = 0$$

$$E\{dW_i(t) dW_j(t)\} = Q_{ij}(t) dt, \quad i, j=1, \dots, n$$

The elements of $\underline{W}(t)$ are assumed independent of the initial conditions $\underline{x}(t_0)$.

The differentials are understood in the Ito sense.

Equation (3.1) may be written in the more convenient form

$$d \underline{x}(t) = \underline{A}(\underline{x},t) dt + B(\underline{x},t) d\underline{w}(t) \quad (3.2)$$

where,

$$B(\underline{x},t) = \{B_{ij}(\underline{x},t)\} , i,j=1,\dots,n$$

and
$$= D(\underline{x},t) Q^{\frac{1}{2}}(t)$$

$d\underline{w}(t) = Q^{-\frac{1}{2}}(t) d\underline{W}(t)$, $\underline{w}(t)$ is a normalized Wiener process with the incremental properties

$$E\{d\underline{w}(t)\} = \underline{0}$$

$$E\{d\underline{w}(t) d\underline{w}^T(t)\} = I dt, I \text{ being the identity matrix.}$$

A wide class of dynamic systems can be reduced to the form in (3.1) or (3.2) by the shaping filter methods. This class includes systems with random parameters where the randomness can be reduced to a Markov process.

The process $\underline{x}(t)$ described by the Ito differential equation (3.1) or (3.2) is also called an Ito process and the study of the properties of this process is termed "Ito calculus". It is known that the transient probability density function $p(\underline{x},t)$ of the process $\underline{x}(t)$ satisfies the famous Fokker-Plank-Kolmogorov differential equation

$$\partial p(\underline{x},t)/\partial t = - \sum_{i=1}^n \frac{\partial}{\partial x_i} \{A_i p\} + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} \{B_{ij} p\} \quad (3.3)$$

In the steady state, if there is one, (3.3) becomes

$$\sum_{i=1}^n \frac{\partial}{\partial x_i} \{A_i p\} = \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} \{B_{ij} p\} \quad (3.4)$$

The integration of equation (3.4), and, in particular, of (3.3) is very difficult; very often, it is not possible. The integration has been carried out in few special cases of limited practical value. Thus, only numerical solutions by means of digital computers are possible in a great majority of problems. But the calculations required are so tedious that it is possible to accomplish them only when the order of the system is not too high. The volume of calculations involved and the necessary storage capacity of computers increase rapidly with the increase in the order of the system. This makes resort to approximate calculations of the system statistics the only way, for the time being, to tackle complex problems. One such approximate method is developed in this chapter. In this method the probability density function $p(\underline{x}, t)$ is not evaluated directly. Instead, the statistics of the process, its moments, correlation functions, semi-invariants, are evaluated. It is known that these statistics characterize the random process satisfactorily. The characterization process is described in the appendix.

A Basic Result

An important result in the Ito calculus is Ito's differential rule. Several important results can be obtained from this rule. Consider a scalar-valued real function $G(\underline{x}, t)$, a function of \underline{x} and t . Let $G(\underline{x}, t)$ be continuously differentiable in t and twice continuously differentiable in $\underline{x}(t)$. Let $G_{\underline{x}}(t) = \partial G / \partial \underline{x}$, $G_{\underline{xx}}(t) = \partial^2 G / \partial \underline{x}^2$ and $G_t = \partial G / \partial t$. With $\underline{x}(t)$ defined by equation (3.2), Ito's differential rule states that

$$dG = \left\{ G_{\underline{x}}^T A + \frac{1}{2} \text{tr} (G_{\underline{xx}} B B^T) + G_t \right\} dt + G_{\underline{x}} B d\omega(t) \quad (3.5a)$$

or

$$dG = \left\{ \sum_{i=1}^n G_{x_i} A_i + \frac{1}{2} \sum_{i,j=1}^n G_{x_i x_j} (BB^T)_{ij} + G_t \right\} dt + \sum_{i=1}^n G_{x_i} (B dw(t))_i \quad (3.5b)$$

where

$$G = G(\underline{x}, t) \quad , \quad \underline{A} = \underline{A}(\underline{x}, t) \quad , \quad B = B(\underline{x}, t)$$

$$\underline{G}_x = \{G_{x_1} \quad G_{x_2} \quad \dots \quad G_{x_n}\}^T$$

$$G_{xx} = \begin{bmatrix} G_{x_1 x_1} & G_{x_1 x_2} & \dots & G_{x_1 x_n} \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ G_{x_n x_1} & G_{x_n x_2} & \dots & G_{x_n x_n} \end{bmatrix}$$

Taking the expectation of both sides in (3.5a) then dividing by dt, we get

$$\frac{d}{dt} E\{G\} = E\{\underline{G}_x^T \underline{A}\} + \frac{1}{2} E\{\text{tr} G_{xx} BB^T\} + E\{G_t\} \quad (3.6a)$$

where we have used the independence of $w_i(t)$ on the initial conditions $\underline{x}(t_0)$ and thus on $\underline{x}(t)$.

Similarly, from (3.5b) we have

$$\frac{d}{dt} E\{G\} = \sum_{i=1}^n E\{G_{x_i} A_i\} + \frac{1}{2} \sum_{i,j=1}^n E\{G_{x_i x_j} \beta_{ij}\} + E\{G_t\} \quad (3.6b)$$

where

$$\beta_{ij} = (BB^T)_{ij}$$

Equations (3.6) were derived by Cumming (1967) directly from the system equation (3.1). Equations (3.6) will be used to obtain the equations for the system statistics.

Equations for the System Statistics

The process $\underline{x}(t)$ is completely specified by the transient density function and the initial conditions. It is known that $p(\underline{x},t)$ for the process $\underline{x}(t)$ described by (3.1) satisfies equation (3.3). We have pointed out that this equation is difficult to solve in practical problems. Another way of characterizing the process $\underline{x}(t)$ is through its statistics. The differential equations for these statistics are easily derived from equations (3.6). In what follows some of these equations are derived.

The Moments Equations

Differential equations for the initial moments (moments about the origin) of the process $\underline{x}(t)$ can be derived from (3.6) in the following manner (Cumming, 1967). Let

$$G(\underline{x},t) = x_1^{n_1} x_2^{n_2} \dots x_n^{n_n} \quad , \quad n_1 + n_2 + \dots + n_n = N \quad (3.7)$$

Then the initial moments of order N are

$$\alpha_{n_1 n_2 \dots n_n}(t) = E\{G(\underline{x},t)\} \quad (3.8)$$

Equations (3.6) are then used to obtain the differential equations for the initial moments of any order N. Note that in this case G is a function of \underline{x} alone, that is, $G_t = 0$. Following are the differential equations for the first four orders.

i. First order initial moments

$$\dot{\alpha}_{n_i=1}(t) = E\{A_i\}$$

ii. Second order initial moment

$$\dot{\alpha}_{\substack{n_i=1 \\ n_j=1}}(t) = E\{x_j A_i + x_i A_j\} + E\{\beta_{ij}\}$$

iii. Third order initial moments

$$\dot{\alpha}_{\substack{n_i=1 \\ n_j=1 \\ n_k=1}}(t) = E\{x_j x_k A_i + x_i x_k A_j + x_i x_j A_k\} \\ + E\{x_k \beta_{ij} + x_i \beta_{jk} + x_j \beta_{ik}\}$$

iv. Fourth order initial moments

$$\dot{\alpha}_{\substack{n_i=1 \\ n_j=1 \\ n_k=1 \\ n_l=1}}(t) = E\{x_j x_k x_l A_i + x_i x_k x_l A_j + x_i x_j x_l A_k \\ + x_i x_j x_k A_l\} + E\{x_k x_l \beta_{ij} + x_j x_k \beta_{il} \\ + x_j x_l \beta_{ik} + x_i x_l \beta_{jk} + x_i x_k \beta_{jl} + x_i x_j \beta_{kl}\}$$

When B is not a function of \underline{x} the equations take a simpler form.

The number of differential equations for the moments of order k is given by

$$M_k = \frac{n(n+1)(n+2)\dots(n+k-1)}{k!}$$

where n is the order of the system in (3.1)

For processes with symmetrical distributions the number of equations for the moments may be reduced by working with the central moments (moments about the mean) of the process. The differential equations for the central moments of $\underline{x}(t)$ can be obtained from (3.6) in a similar manner as the equations for the initial moments. Thus, letting

$$G(\underline{x}, t) = (x_1^o)^{n_1} (x_2^o)^{n_2} \dots (x_n^o)^{n_n} \quad (3.9)$$

where,

$$x_i^o = x_i - \alpha_{n_i=1} \quad \text{and} \quad n_1 + n_2 + \dots + n_n = N$$

Then the central moments of order N are defined by

$$\mu_{n_1 n_2 \dots n_n} (t) = E\{G(\underline{x}, t)\} \quad (3.10)$$

Equations (3.6) are then used to derive the differential equations for the central moments. The equations for the central moments of the first four orders are as follows

i. Second order central moments

$$\dot{\mu}_{\substack{n_i=1 \\ n_j=1}} (t) = E\{x_j^0 A_i + x_i^0 A_j\} + E\{\beta_{ij}\}$$

ii. Third order central moments

$$\begin{aligned} \dot{\mu}_{\substack{n_i=1 \\ n_j=1 \\ n_k=1}} (t) = E \left\{ \left[x_j^0 x_k^0 - \mu_{\substack{n_i=1 \\ n_k=1}} \right] A_i + \left[x_i^0 x_k^0 - \mu_{\substack{n_i=1 \\ n_k=1}} \right] A_j \right. \\ \left. + \left[x_i^0 x_j^0 - \mu_{\substack{n_i=1 \\ n_j=1}} \right] A_k \right\} + E \{ x_k^0 \beta_{ij} \\ + x_i^0 \beta_{jk} + x_j^0 \beta_{ik} \} \end{aligned}$$

iii. Fourth order central moments

$$\begin{aligned} \dot{\mu}_{\substack{n_i=1 \\ n_j=1 \\ n_k=1 \\ n_l=1}} (t) = E \left\{ \left[x_j^0 x_k^0 x_l^0 - \mu_{\substack{n_j=1 \\ n_k=1 \\ n_l=1}} \right] A_i + \left[x_i^0 x_k^0 x_l^0 - \mu_{\substack{n_i=1 \\ n_k=1 \\ n_l=1}} \right] A_j \right. \\ \left. + \left[x_i^0 x_j^0 x_l^0 - \mu_{\substack{n_i=1 \\ n_j=1 \\ n_l=1}} \right] A_k + \left[x_i^0 x_j^0 x_k^0 - \mu_{\substack{n_i=1 \\ n_j=1 \\ n_k=1}} \right] A_l \right\} \\ + E \{ x_k^0 x_l^0 \beta_{ij} + x_j^0 x_l^0 \beta_{ik} + x_j^0 x_k^0 \beta_{il} + x_i^0 x_l^0 \beta_{jk} + \\ x_i^0 x_k^0 \beta_{jl} + x_i^0 x_j^0 \beta_{kl} \} \end{aligned}$$

For symmetrical distributions all odd central moments are zero. This will reduce the number of equations to be solved. Thus for a first order system there will be two equations instead of four; for a second order system eight instead of fourteen; and so on.

Equations for the Characteristic

Functions

The differential equations for the characteristic function of the process $\underline{x}(t)$ can also be derived from (3.6). Let

$$G(\underline{x}, t) = \exp(i \underline{z}^T \underline{x}), \quad \underline{z} = \{z_i\}, \quad i=1, 2, \dots, n \quad (3.11)$$

The characteristic function is then given by

$$f(\underline{z}, t) = E\{G(\underline{x}, t)\} \quad (3.12)$$

From equation (3.6), it follows that

$$\frac{d}{dt} f(\underline{z}, t) = iE(\underline{Gz}^T \underline{A}) - \frac{1}{2}E(\underline{Gz}^T \underline{B} \underline{B}^T \underline{z}) \quad (3.13a)$$

or in component form

$$\frac{d}{dt} f(\underline{z}, t) = iE \left\{ e^{i \sum_{j=1}^n z_j x_j} \sum_{j=1}^n z_j A_j \right\} - E \left\{ \sum_{j,k=1}^n (\beta_{jk}) z_j z_k e^{i \sum_{j=1}^n z_j x_j} \right\} \quad (3.13b)$$

The log characteristic function is defined by

$$g(\underline{z}, t) = \ln f(\underline{z}, t) \quad (3.14)$$

from which follows

$$f(\underline{z}, t) = \exp g(\underline{z}, t) \quad (3.15)$$

The differential equation for the log characteristic function is obtained by differentiating both sides of (3.14) with respect to t and considering (3.15). Thus

$$\frac{d}{dt} g(\underline{z}, t) = \frac{1}{f(\underline{z}, t)} \frac{d}{dt} f(\underline{z}, t)$$

Using (3.13) and (3.15) we get

$$\frac{d}{dt} g(\underline{z}, t) = \{iE(\underline{z}^T \underline{A} \underline{G}) - 1/2E(\underline{z}^T \underline{B} \underline{B}^T \underline{z} \underline{G})\} e^{-g(\underline{z}, t)} \quad (3.16a)$$

or in component form

$$\begin{aligned} \frac{d}{dt} g(\underline{z}, t) = & \{iE(e^{i \sum_{j=1}^n z_j x_j} \sum_{j=1}^n z_j A_j) \\ & - 1/2E(\sum_{j,k=1}^n \beta_{jk} z_j z_k e^{i \sum_{j=1}^n z_j x_j})\} e^{-g(\underline{z}, t)} \end{aligned} \quad (3.16b)$$

It may be pointed out that the differential equations for the initial moments can also be obtained from (3.13) since the N th order moment is given by

$$\alpha_{n_1 n_2 \dots n_n} (t) = (-i)^N \frac{\partial^N f(\underline{z}, t)}{\partial z_1^{n_1} \partial z_2^{n_2} \dots \partial z_n^{n_n}} \Big|_{\underline{z} = \underline{0}} \quad (3.17)$$

Thus by differentiating (3.13) repeatedly with respect to z_i and evaluating at $\underline{z}=\underline{0}$, the differential equations for the moments of various orders are obtained.

The Semi-invariants Equations

The semi-invariants of a distribution are obtained from the log characteristic function just as the initial moments are obtained from the characteristic function. The semi-invariants of order N are given by

$$\lambda_{n_1 n_2 \dots n_n} (t) = (-i)^N \frac{\partial^N g(\underline{z}, t)}{\partial z_1^{n_1} \partial z_2^{n_2} \dots \partial z_n^{n_n}} \Big|_{\underline{z} = \underline{0}} \quad (3.18)$$

The differential equations for the semi-invariants of order N are thus obtained by differentiating (3.16) repeatedly with respect to z_i a total of N times and evaluating at $\underline{z}=\underline{0}$. The equations for the semi-invariants of the first four orders are as follows

i. First order semi-invariants

$$\dot{\lambda}_{n_i=1} (t) = E\{A_i\}$$

ii. Second order semi-invariants

$$\dot{\lambda}_{\substack{n_i=1 \\ n_j=1}} (t) = E\{x_{j i}^o A_i + x_{i j}^o A_j\} + E\{\beta_{ij}\}$$

iii. Third order semi-invariants

$$\begin{aligned} \dot{\lambda}_{\substack{n_k=1 \\ n_j=1 \\ n_k=1}}(t) = E \left[\{x_j^{\circ} x_k^{\circ} - \lambda_{\substack{n_j=1 \\ n_k=1}}\} A_i + \{x_i^{\circ} x_k^{\circ} - \lambda_{\substack{n_i=1 \\ n_k=1}}\} A_j \right. \\ \left. + \{x_i^{\circ} x_j^{\circ} - \lambda_{\substack{n_i=1 \\ n_j=1}}\} A_k \right] + E \{x_i^{\circ} \beta_{jk} + x_j^{\circ} \beta_{ik} + x_k^{\circ} \beta_{ij}\} \end{aligned}$$

iv. Fourth order semi-invariants

$$\begin{aligned} \dot{\lambda}_{\substack{n_k=1 \\ n_k=1 \\ n_j=1 \\ n_k=1}}(t) = E \left[\{x_j^{\circ} x_k^{\circ} x_{\ell}^{\circ} - x_j^{\circ} \lambda_{\substack{n_k=1 \\ n_{\ell}=1}} - x_k^{\circ} \lambda_{\substack{n_j=1 \\ n_{\ell}=1}} - x_{\ell}^{\circ} \lambda_{\substack{n_j=1 \\ n_k=1}} \right. \\ - \lambda_{\substack{n_j=1 \\ n_j=1 \\ n_k=1}}\} A_i + \{x_j^{\circ} x_k^{\circ} x_{\ell}^{\circ} - x_i^{\circ} \lambda_{\substack{n_k=1 \\ n_{\ell}=1}} - x_k^{\circ} \lambda_{\substack{n_i=1 \\ n_{\ell}=1}} \\ - x_{\ell}^{\circ} \lambda_{\substack{n_i=1 \\ n_k=1}} - \lambda_{\substack{n_i=1 \\ n_i=1 \\ n_k=1}}\} A_j + \{x_i^{\circ} x_j^{\circ} x_{\ell}^{\circ} - x_i^{\circ} \lambda_{\substack{n_j=1 \\ n_{\ell}=1}} \\ - x_j^{\circ} \lambda_{\substack{n_i=1 \\ n_{\ell}=1}} - x_{\ell}^{\circ} \lambda_{\substack{n_i=1 \\ n_j=1}} - \lambda_{\substack{n_i=1 \\ n_i=1 \\ n_j=1}}\} A_k + \{x_i^{\circ} x_j^{\circ} x_k^{\circ} \\ - x_i^{\circ} \lambda_{\substack{n_j=1 \\ n_{\ell}=1}} - x_j^{\circ} \lambda_{\substack{n_i=1 \\ n_j=1}} - x_{\ell}^{\circ} \lambda_{\substack{n_i=1 \\ n_j=1}} - \lambda_{\substack{n_i=1 \\ n_i=1 \\ n_j=1}}\} A_{\ell} \left. \right] \\ + E \left[\{x_k^{\circ} x_{\ell}^{\circ} - \lambda_{\substack{n_k=1 \\ n_{\ell}=1}}\} \beta_{ij} + \{x_j^{\circ} x_{\ell}^{\circ} - \lambda_{\substack{n_j=1 \\ n_{\ell}=1}}\} \beta_{ik} \right. \\ \left. + \{x_j^{\circ} x_k^{\circ} - \lambda_{\substack{n_j=1 \\ n_k=1}}\} \beta_{i\ell} + \{x_i^{\circ} x_{\ell}^{\circ} - \lambda_{\substack{n_i=1 \\ n_{\ell}=1}}\} \beta_{jk} \right] \end{aligned}$$

$$+ \left[\{x_i^o x_k^o - \lambda_{\substack{n_i=1 \\ n_k=1}}\} \beta_{j\ell} + \{x_i^o x_j^o - \lambda_{\substack{n_i=1 \\ n_j=1}}\} \beta_{k\ell} \right]$$

The Correlation Function Equations

The equations for the correlation functions are derived from basic relations in what follows. For convenience, let B be a function of t only. Then the system equation can be written in the form

$$\dot{\underline{x}}(t) = \underline{A}(\underline{x}, t) + B(t) \underline{v}(t) \quad (3.19)$$

where $\underline{v}(t)$ is now a white noise vector with zero mean vector and intensity equal to the identity matrix. The correlation functions of $\underline{x}(t)$ are defined as

$$R(t, s) = E\{\underline{x}^o(t) \underline{x}^{oT}(s)\} \quad (3.20)$$

Differentiating with respect to t

$$\frac{\partial R(t, s)}{\partial t} = E\{\dot{\underline{x}}^o(t) \underline{x}^{oT}(s)\} \quad (3.21)$$

where from the system equation

$$\dot{\underline{x}}^o(t) = \underline{A}^o(\underline{x}, t) + B(t) \underline{v}(t) \quad (3.22)$$

where

$$\underline{A}^o(\underline{x}, t) = \underline{A}(\underline{x}, t) - E\{\underline{A}(\underline{x}, t)\}$$

Thus

$$\frac{\partial R(t,s)}{\partial t} = E\{\underline{A}^0(\underline{x},t) \underline{x}^0{}^T(s)\} + E\{B(t) \underline{v}(t) \underline{x}^0{}^T(s)\} \quad (3.23)$$

$\underline{x}^0(s)$ is the solution of the system equation, i.e.

$$\underline{x}^0(s) = G(t_0, s) \underline{x}^0(t_0) + \int_{t_0}^s G(t, \tau) B(\tau) \underline{v}(\tau) d\tau \quad (3.24)$$

where $G(t,s)$ is the transition matrix.

Substituting for $\underline{x}^0{}^T(s)$ in the second term of equation (3.23), we have

$$\begin{aligned} \frac{\partial R(t,s)}{\partial t} &= E\{\underline{A}^0(\underline{x},t) \underline{x}^0{}^T(s)\} + E\{B(t) \underline{v}(t) \cdot \\ &\quad \underline{x}^0{}^T(t_0) G^T(t_0, s) + B(t) \underline{v}(t) \cdot \int_{t_0}^s \underline{v}^T(\tau) B^T(\tau) \\ &\quad G^T(t, \tau) d\tau\} = \{ \underline{A}^0(\underline{x},t) \underline{x}^0{}^T(s) \} + \int_{t_0}^s B(t) B^T(\tau) \cdot \\ &\quad G^T(t, \tau) \delta(t-\tau) d\tau \\ &= E\{\underline{A}^0(\underline{x},t) \underline{x}^0{}^T(s)\} + B(t) B^T(s). \\ &G(t,s) \mathbb{1}(s-t) \end{aligned} \quad (3.25)$$

The unit function is included to emphasize that

$$G(t,s) = 0 \text{ for } t > s.$$

This equation is solved with the initial conditions

$$R(t,t) = P(s)$$

where P is the covariance matrix (matrix of second order central moments) of the process $\underline{x}(t)$.

Similarly, with respect to s , we get

$$\begin{aligned} \frac{\partial R(t,s)}{\partial t} &= E\{\underline{x}^o(t) \underline{A}^{oT}(\underline{x},s)\} + B(t) B^T(s). \\ G(t,s) &= 1 \cdot (t-s), \quad G(t,s) = 0, \quad s > t \end{aligned} \quad (3.26)$$

and the initial conditions

$$R(s,s) = P(t)$$

From above, we finally have

$$\begin{aligned} \frac{\partial R(t,s)}{\partial t} &= E\{\underline{A}^o(\underline{x},t) \underline{x}^{oT}(s)\}, \quad t > s \\ R(t,t) &= P(s) \end{aligned} \quad (3.27)$$

and

$$\begin{aligned} \frac{\partial R(t,s)}{\partial t} &= E\{\underline{x}^o(t) \underline{A}^{oT}(\underline{x},s)\}, \quad s > t \\ R(s,s) &= P(t) \end{aligned} \quad (3.28)$$

Integrating the first equation for fixed values of s , we obtain a number of sections of the correlation functions parallel to the t -axis and lying in the region $t > s$. Similarly, integrating the second equation for fixed values of t , we obtain sections parallel to the s -axis and lying in the region $s > t$.

If the system is stationary, the correlation functions will satisfy the following equation

$$\begin{aligned} \frac{\partial R(\tau)}{\partial \tau} &= E\{\underline{A}(\underline{x},t) \underline{x}^{oT}(t+\tau)\}, \quad \tau = t-s \\ R(0) &= P \end{aligned} \quad (3.29)$$

Note that in order to calculate the correlation functions the second order central moments must be known. This is achieved by solving the set of equations for the central moments first then using them in the solution of the correlation functions equations.

Computation of the System Statistics

Now consideration is given to methods of solving the differential equations for the statistics. It is pointed out that the three types of statistics considered here are related by definite relationships (Kendall and Stuart, 1969). Thus one can calculate the statistics which involve less effort then, if necessary, obtain the other types of statistics from the relations between the members of the different types.

When $\underline{A}(\underline{x}, t)$ is linear in $\underline{x}(t)$ and B is a function of time only, the process $\underline{x}(t)$ is Gaussian. It is completely specified by its mean vector and correlation matrix. The equations for these statistics are solved easily.

If \underline{A} is linear in \underline{x} and B is also linear in \underline{x} , the process $\underline{x}(t)$ is not Gaussian. However, the differential equations for the statistics of order N will involve statistics of orders equal to and less than N . A closed set of differential equations can be obtained for any order and solved for statistics up to that order. Of course, an arbitrary (non-Gaussian) Markov process $\underline{x}(t)$ is specified by its infinity (countable) statistics. However, experience has shown that statistics up to the fourth order, inclusive, describe the process with sufficient accuracy. This is beside the fact that higher order statistics, if they exist, may involve errors. We recall that the physical process is not actually a Markov process, strictly speaking. Its high order statistics

computed on the basis of a Markov process will not be the actual statistics. Errors due to the approximation accumulate and render the higher order statistics unreliable.

If $\underline{A}(\underline{x}, t)$ is nonlinear in \underline{x} then $\underline{x}(t)$ is non-Gaussian whether B is a function of \underline{x} or not. In order to solve the resulting set of differential equations for the statistics, the right hand sides of the equations must be expressed explicitly in terms of these statistics. If the nonlinear functions are in the form of polynomials in \underline{x} , with time-varying coefficients, it is possible to express the right hand sides of the equations explicitly in terms of the statistics. However, the differential equations for the statistics of order N will now include statistics of orders higher than N. A more complicated situation arises when the nonlinear characteristics cannot be written as polynomials in \underline{x} . Such case includes essentially nonlinear characteristics like relays. The calculation of the statistics in these last two cases is a main contribution of the method to be adopted in this work.

In all cases, if only the steady-state values of the statistics are needed, the equations reduce to a set of algebraic equations. Steady state values are independent of the initial conditions.

In order to solve the equations for the statistics, a closed set of equations must be found, or formulated. When the nonlinear functions are expressed as polynomials in \underline{x} a closed set of equations for the semi-invariants up to a certain order N can be obtained by setting equal to zero all semi-invariants of orders higher than N. It is well known that the semi-invariants above the second order of a Gaussian process are all zero. The semi-invariants of an arbitrary process $\underline{x}(t)$ of higher orders

are not zero, in general, nor do they always decrease in value with increase in order. However, the contribution of higher order semi-invariants to the specification of a random process becomes less important with increasing order. Having in mind processes which are common in control systems and the normalization of processes by the system inertias the above suggestion can be justified. The same cannot be said of the moments. It is known that even the moments of a Gaussian process are not zero above a certain order and may very well be large quantities.

When the nonlinear characteristics cannot be written as polynomials in \underline{x} a way must be found to express the right hand sides of the equations explicitly in terms of the statistics. Also needed is a way of obtaining a closed set of equations for the statistics up to a certain order.

We have found that the best way to accomplish the above needs lies in an appropriate analytic representation of the probability density function of the process $\underline{x}(t)$. Such a representation will have the generality of application and has been found to give results with satisfactory accuracy for engineering purposes. It will take care of all types of nonlinearities in (3.1) or (3.2). The right hand sides of the equations are easily expressed in terms of the statistics up to a desired order and a closed set of equations for statistics up to that order may be obtained. The representation problem is described in the appendix.

A Computational Aspect

Investigation of the equations for the statistics and consideration of the representation of $p(\underline{x})$ as in the appendix reveal that we

will be evaluating integrals of the form

$$(I_{n_1 n_2 \dots n_n})_i = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \left(\frac{x_1^{-m}}{\sigma_{x_1}} \right) \dots \left(\frac{x_n^{-m}}{\sigma_{x_n}} \right)_{n_n}.$$

$$A_i(\underline{x}, t) p^*(\underline{x}) dx_1 \dots dx_n$$

$$i = 1, 2, \dots, n \quad n_1, \dots, n_n = 0, 1, 2, \dots \quad (3.30)$$

where $p^*(\underline{x})$ is the n -dimensional normal density function for independent elements. Similar integrals involving B_{ij} may also occur. These integrals can be evaluated beforehand for various types of nonlinearities. Tables of such integrals for various nonlinear characteristics can be prepared. The set of differential equations can also be written in terms of these integrals for repeated use.

CHAPTER IV

APPLICATIONS

Introduction

The method developed in Chapter III is now applied to the analysis of nonlinear feedback systems. Two forms of systems will be considered according to the type of the nonlinear characteristic involved. The first is that of systems containing odd symmetric nonlinearities. In many such systems, once the transients have disappeared, the mean value of the input can be tracked by the output with zero-mean error. Thus no generality is lost if zero-mean inputs are considered. In this case, the output will also be zero-mean and the system is a "zero-mean" system. The second form is that of systems containing other types of nonlinearities, for example, odd nonsymmetric and even nonlinearities.

Application to Zero-Mean Systems

Figure 5a shows a simple nonlinear feedback system of a general form. The block diagram is easily transformed to the form in Figure 5b. An accurate expression for the spectral density function of $x(t)$ can be obtained when $z(t)$ is a stationary process (Pyatnitskii, 1960). The development is based on the assumption that $x(t)$ is Gaussian. The first term in the expression for the spectral density function has been found to coincide with the expression obtained by the BMSL (based on the minimum mean-square-error criterion). This fact is very interesting since

it disproves the long-held belief that the BMSL is different in kind from the powerful functional approach to systems analysis. As a matter of fact, it is easily shown that if the density function of $x(t)$ is normal then all the relations obtained in Chapter III reduce to those of the BMSL. This is a theoretical justification of the BMSL. It also shows that the BMSL is a first approximation to the accurate solution of the problem. Of course, we still have with us the assumption that $x(t)$ is Gaussian, the effect of which must be investigated.

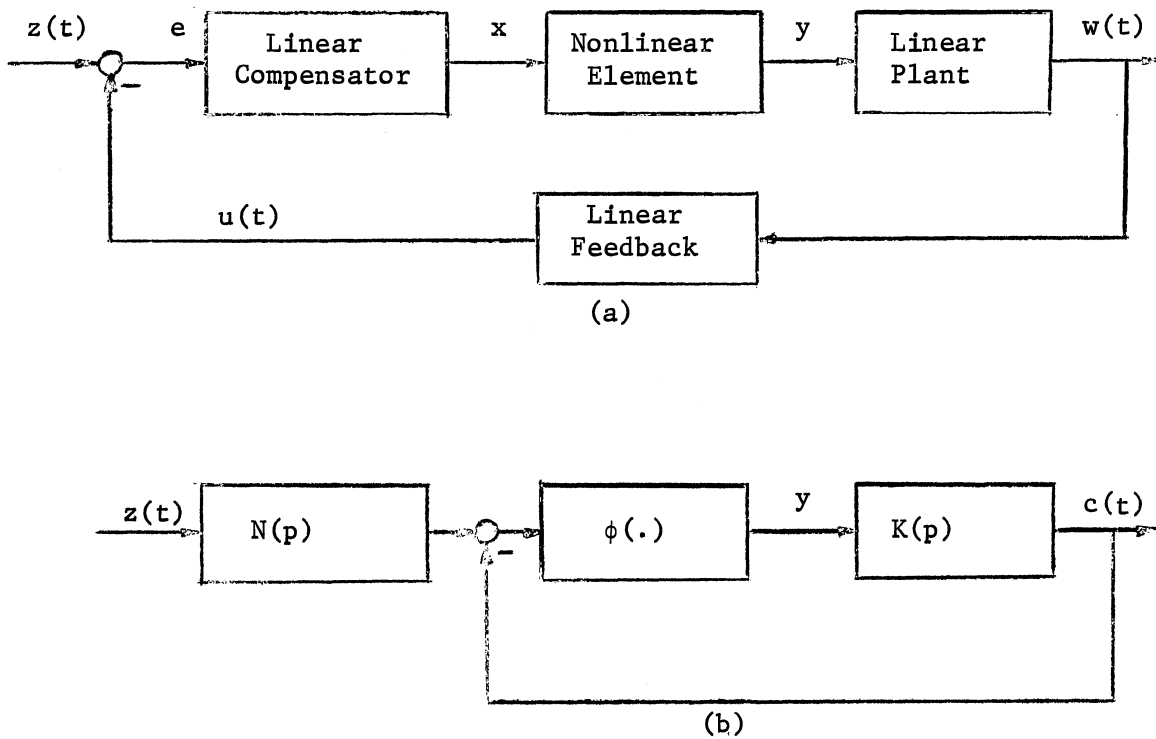


Figure 5. (a) A Simple Nonlinear Feedback System
(b) The Equivalent Unit Feedback System

Theoretical Investigation

In Chapter III an approximate method of analysis was developed which accounts for the nonlinear effects in the system and which uses an expansion for the probability density of $x(t)$, as compared with a normal $x(t)$ in the BMSL. In this section, exact solutions known in the literature, for simple problems, are used to examine both the accuracy of the approximate method of Chapter III and that of the BMSL.

Consider the simple first order system in Figure 6 which is described by the first order differential equation

$$dx = -\phi(x) dt + dW(t) \quad (4.1)$$

where $W(t)$ is a Wiener process with the incremental properties

$$E\{dW(t)\} = 0 \text{ and } E\{dW(t) dW(t)\} = 2b^2 dt$$

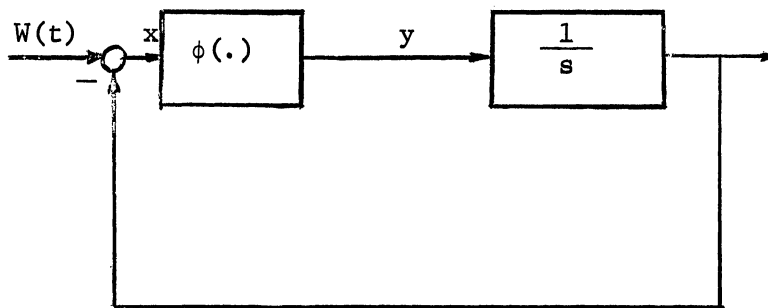


Figure 6. A Simple First Order Nonlinear System

Equation (4.1) is written in the more convenient form

$$dx = -\phi(x) dt + b\sqrt{2} dw(t) \quad (4.2)$$

where $w(t)$ is a normalized Wiener process. Since $b\sqrt{2}$ in (4.2) is not a function of time, (4.2) may be written, using the white noise concept, as

$$\dot{x} = -\phi(x) + b\sqrt{2} v(t)$$

where $v(t)$ is white noise with unit intensity.

Using the BMSL we should be able to determine the mean-square values of $x(t)$. By the method of Chapter III statistics of any order can be calculated. On the other hand, the exact probability density functions are determined by solving the Fokker-Plank-Kolmogorov equation directly. Three nonlinear characteristics, shown in Figure 7, are considered.

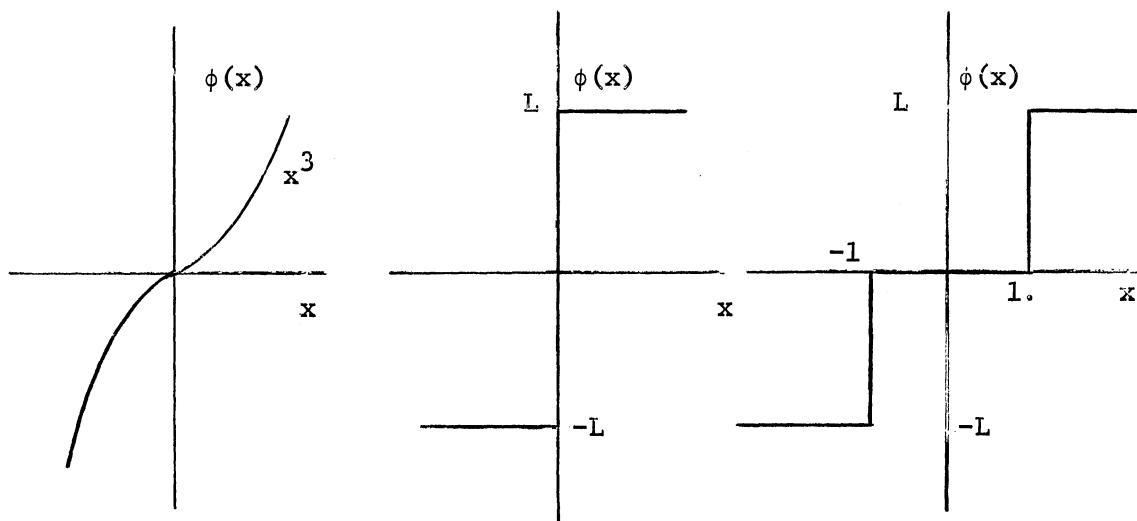


Figure 7. Nonlinear Characteristics Used with the First Order System with White Noise Input

First, consider the cubic nonlinearity. In this case and according to (3.2),

$$A(x,t) = -x^3, \quad B(x,t) = b\sqrt{2}$$

From symmetry, we have

$$\lambda_k = 0 \text{ for all odd } k$$

The differential equations for λ_2 and λ_4 are

$$\dot{\lambda}_2(t) = -6\lambda_2^2 - 2\lambda_4 + 2b^2 \quad (4.3)$$

$$\dot{\lambda}_4(t) = -24\lambda_2^3 - 48\lambda_2\lambda_4 \quad (4.4)$$

Giving the steady-state values

$$\lambda_2 = \sigma_x^2 = .632b \quad (4.5)$$

$$\lambda_4 = - .2b^2 \quad (4.6)$$

The differential equations for λ_2 corresponding to the BMSL is obtained from (4.3) by setting λ_4 equal to zero,

$$\dot{\lambda}_2(t) = -6\lambda_2^2 + 2b^2 \quad (4.7)$$

from which we get

$$\lambda_2 = \sigma_x^2 = .577b \quad (4.8)$$

The exact probability density function is given by

$$p(x) = \frac{\sqrt{2}}{b^{1/2} \Gamma(1/4)} \exp(-x^4/4b^2) \quad (4.9)$$

from which we have

$$\lambda_2 = .677b \quad (4.10)$$

Figure 8 shows the curves for the root mean-square values of the error and input signals corresponding to the three methods of calculation. Figure 9 shows the corresponding probability density functions for $b=.5$. For this system, the root mean-square value of the input equals b .

Similar calculations are carried out for the other two nonlinearities. The results are included below and shown in Figures 10 to 13.

For the ideal relay,

Exact:

$$\lambda_2 = 2 b^4/L^2 = 8 \times 10^{-4} b^4 \quad (4.11)$$

Approximate:

$$\lambda_2 = 1.99 b^4/L^2 = 7.98 \times 10^{-4} b^4 \quad (4.12)$$

BMSL:

$$\lambda_2 = (\pi/2) b^4/L^2 = 6.27 \times 10^{-4} b^4 \quad (4.13)$$

The exact density function is given by

$$p(x) = L/2b^2 \exp(-L/b^2|x|) = 25/b^2 \exp(-50/b^2|x|) \quad (4.14)$$

For the relay with dead zone,

Exact:

$$\lambda_2 = \frac{1}{1+.02b^2} (.333 + .02b^2 + 8 \times 10^{-4}b^4 + 16 \times 10^{-6}b^6) \quad (4.15)$$

The equations corresponding to the approximate method for λ_2 and λ_4 , in the steady state, are

$$0 = -100 \sigma_x / 24 \lambda_2^2 \{ 24 \lambda_2^2 f_o + \lambda_4 (-f_o - 2f_2 + f_4) \} + 2b^2 \quad (4.16)$$

$$0 = 24 \lambda_2^2 (-f_o + f_2) + \lambda_4 (9f_o + 9f_2 - 3f_4 + f_6) \quad (4.17)$$

$$f_k = 2/\sqrt{2\pi} (1/\sigma_x)^k e^{-1/2\sigma_x^2}$$

λ_2 corresponding to the BMSL is the solution of the algebraic equation

$$0 = -100 \sigma_x f_o + 2b^2, \quad \lambda_2 = \sigma_x^2 \quad (4.18)$$

The correlation function of $x(t)$ satisfies the differential equation

$$\frac{d}{d\tau} R(\tau) = -E\{\phi[x(t)] x^o(t+\tau)\} \quad (4.19)$$

Letting $x_1 = x(t)$, $x_2 = x(t+\tau)$ and using the two-dimensional Edgeworth expansion for the joint density function of x_1 and x_2 we have

$$\frac{dR}{d\tau} = -1/\sigma_x \{ I_1 + \lambda_4 / 24 \lambda_2^2 (I_5 - 6I_3 + 3I_1) \} R(\tau) \quad (4.20)$$

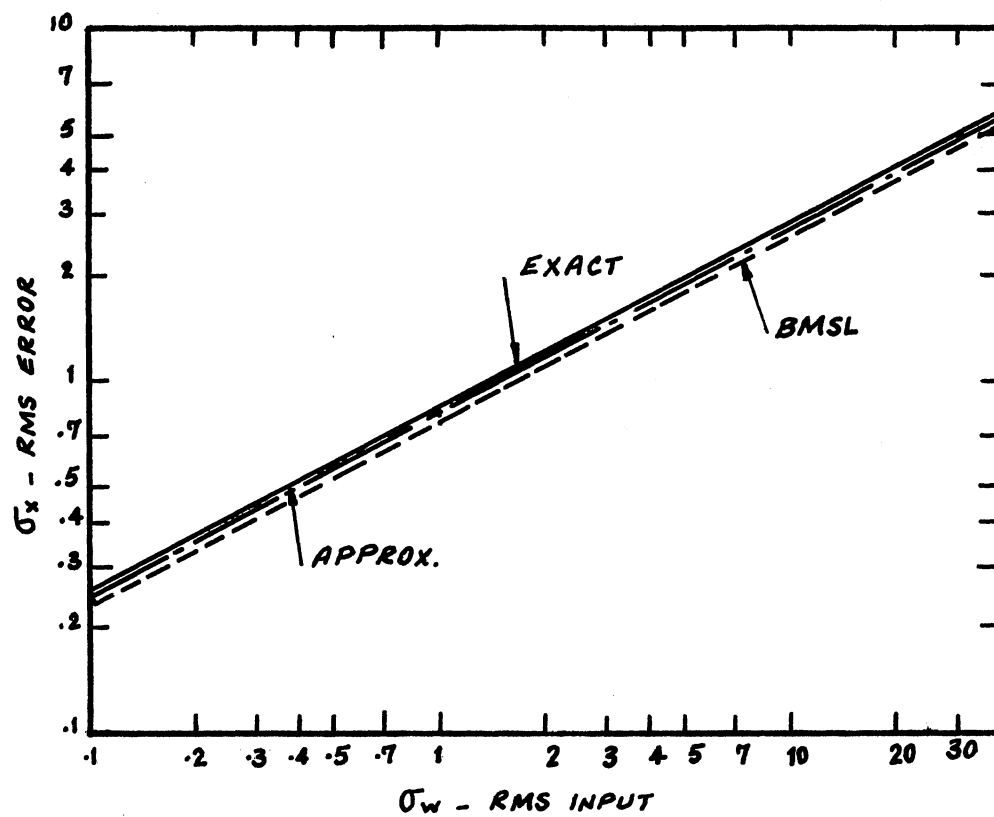


Figure 8. Input-Error Curves. First Order System with White Noise Input- Cubic

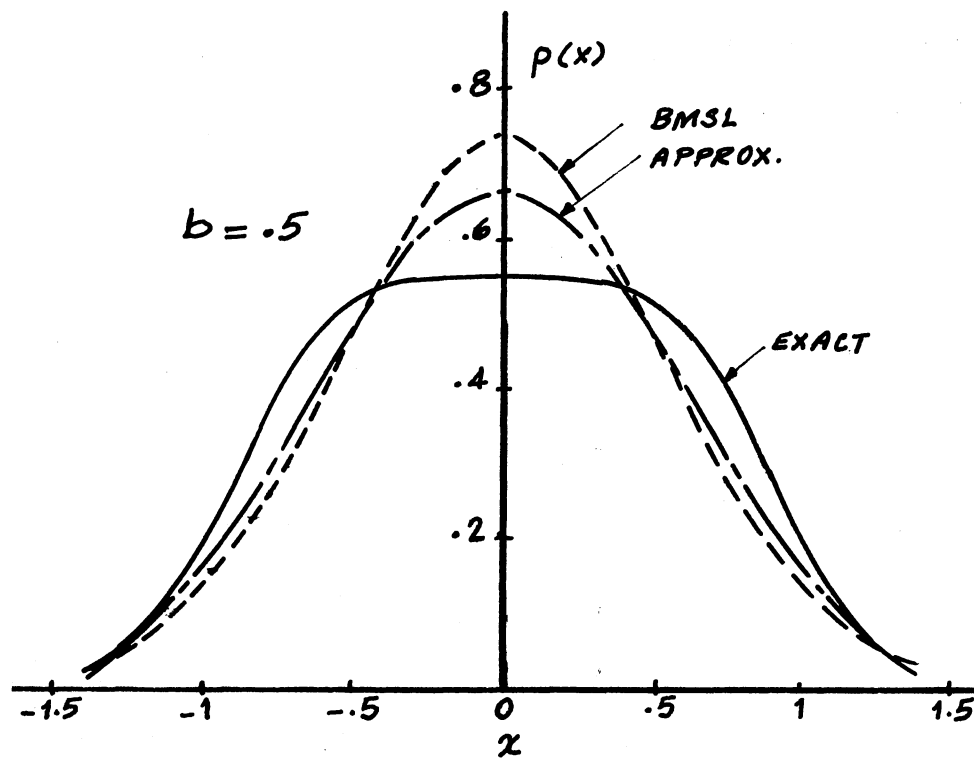


Figure 9. Amplitude Probability Density Functions of the Error Signal in First Order System with White Noise Input-Cubic Nonlinearity

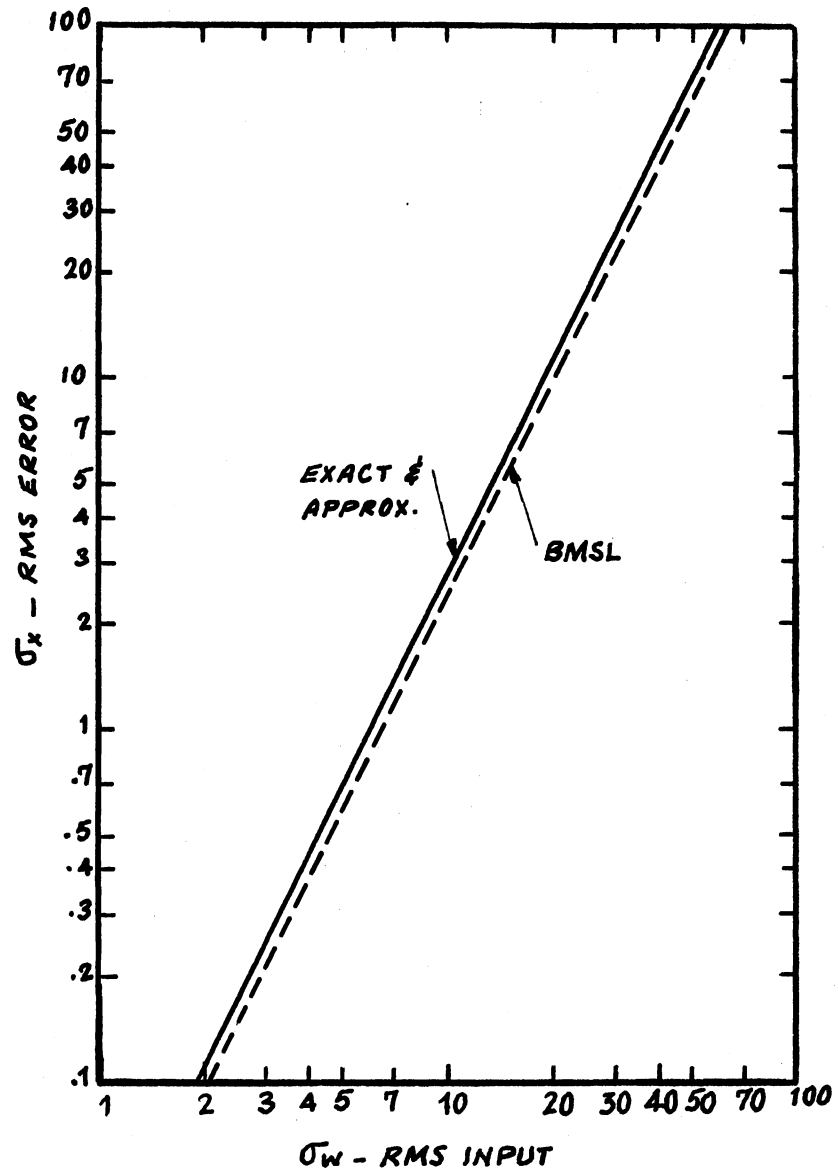


Figure 10. Input-Error Curves. First Order System with White Noise Input-Ideal Relay

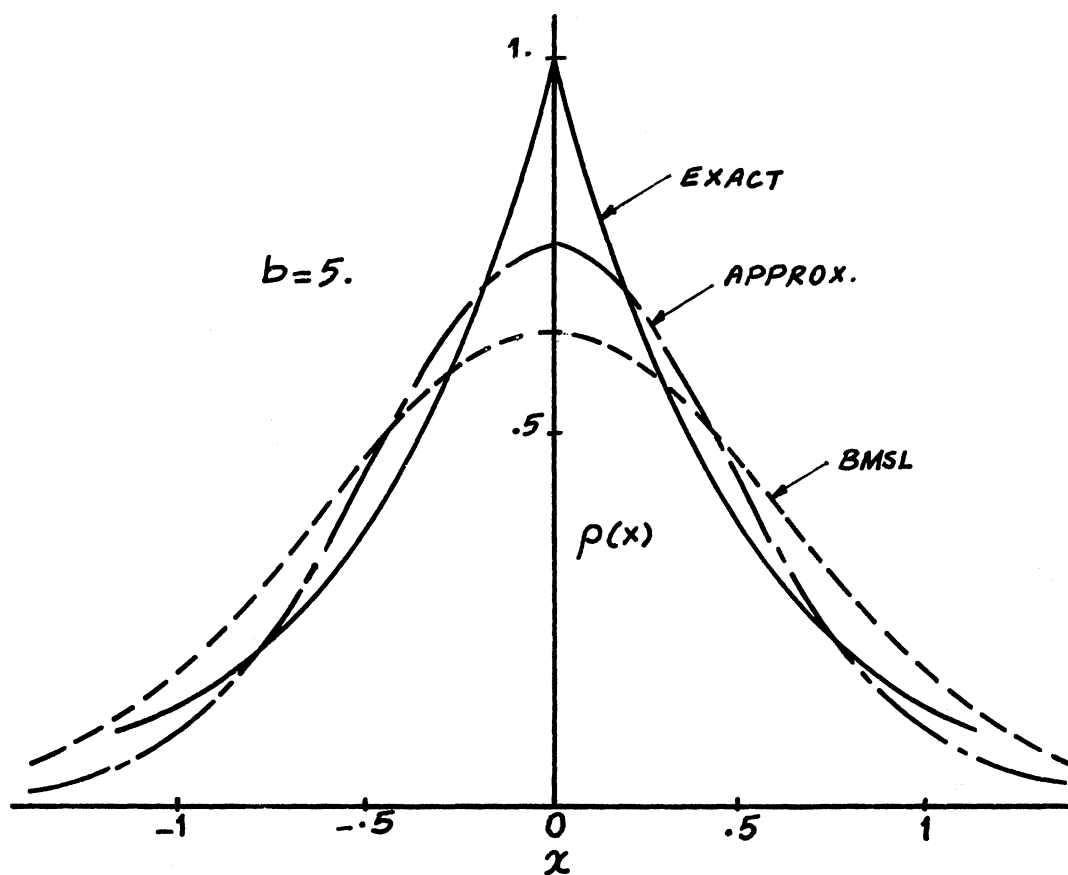


Figure 11. Amplitude Probability Density Functions of the Error Signal in First Order System with White Noise Input-Ideal Relay

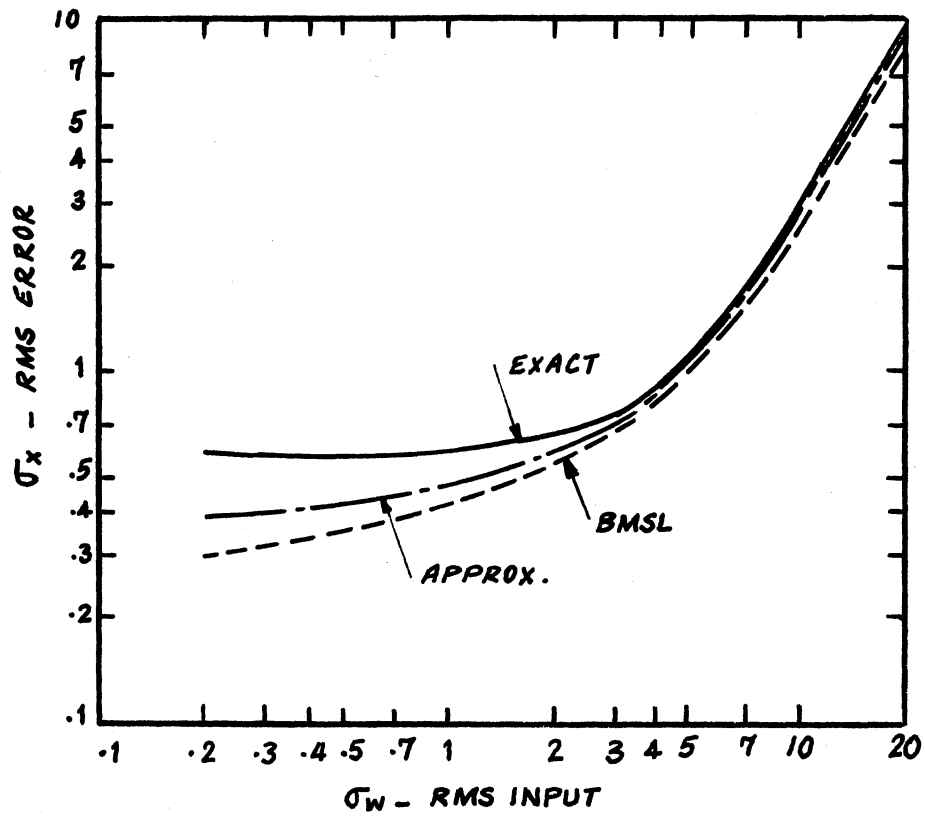


Figure 12. Input-Error Curves. First Order System with White Noise Input- Relay with Dead Zone

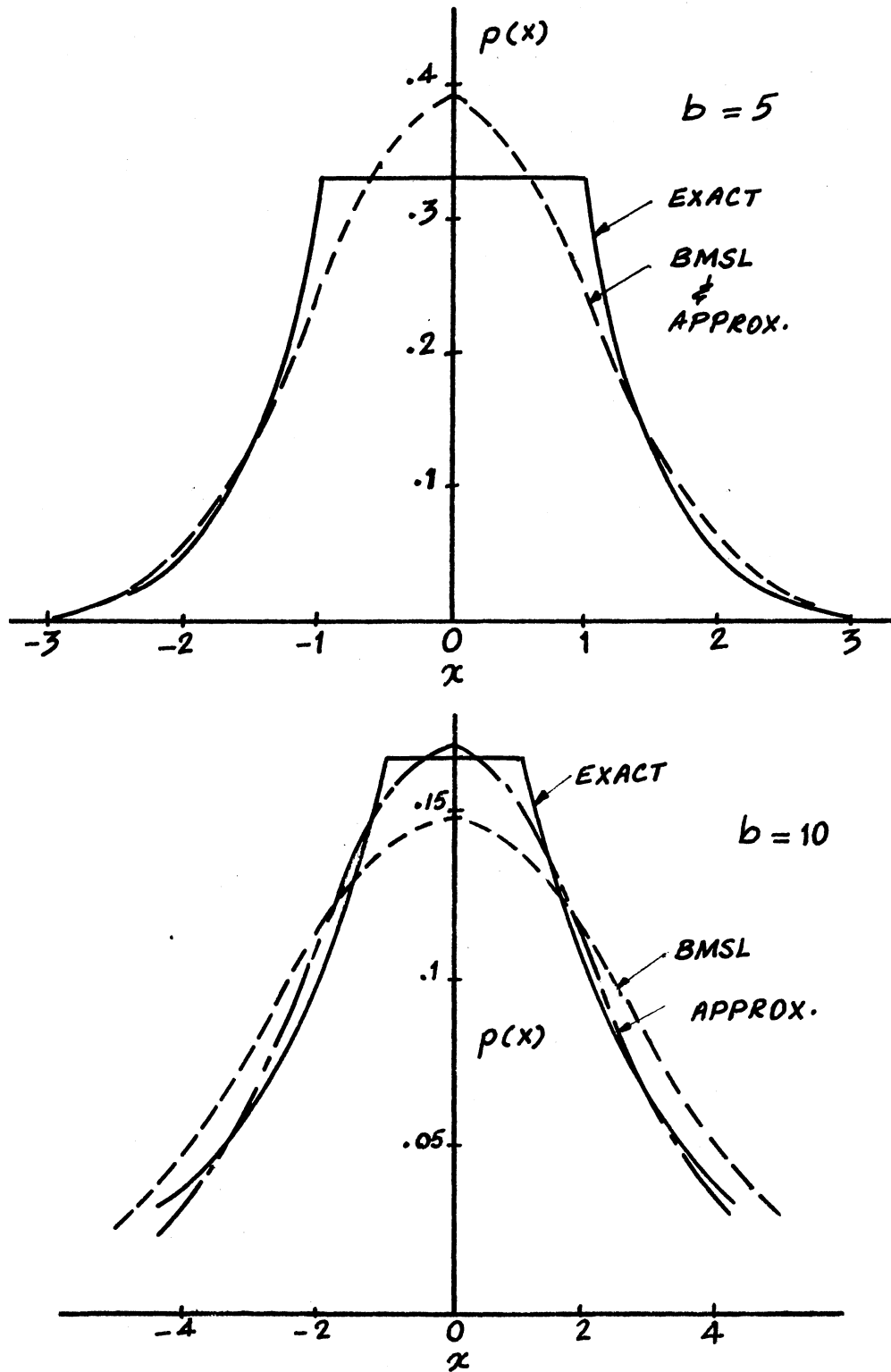


Figure 13. Amplitude Probability Density Functions of Error Signal in First Order System with White Noise Input-Relay with Dead Zone.
 (a) $b=5$ (b) $b=10$

where I_k has been defined in Chapter III. The equation corresponding to the BMSL is

$$\frac{dR}{d\tau} = -1/\sigma_x I_1 R(\tau) \quad (4.21)$$

Substituting for σ_x , λ_2 , λ_4 and I_k for the different non-linearities and carrying out the integration, expressions for $R(\tau)$ are easily obtained from (4.20) and (4.21).

The spectral density functions are then obtained as the Fourier transforms of the correlation functions. The final results are as follows:

i. Cubic nonlinearity

Approximate:

$$R(\tau) = .632b e^{-1.58b\tau} \quad (4.22)$$

$$S(\omega) = .1275 \frac{1}{1 + (\omega/1.58b)^2} \quad (4.23)$$

BMSL:

$$R(\tau) = .577b e^{-\sqrt{3}b\tau} \quad (4.24)$$

$$S(\omega) = .106 \frac{1}{1 + (\omega/\sqrt{3}b)^2} \quad (4.25)$$

ii. Ideal relay. The relations for the BMSL are

$$R(\tau) = \lambda_2 e^{-(b^2/\lambda_2)\tau} \quad (4.26)$$

$$S(\omega) = \frac{\lambda_2^2}{4\pi b^2} \frac{1}{1 + \{\omega/(2b^2/\lambda_2)\}^2} \quad (4.27)$$

where λ_2 is given by (4.13).

The approximate method gives the same expressions except that λ_2 is determined from (4.12).

iii. Relay with dead zone. The same expressions in (4.26) and (4.27) apply except that λ_2 is given by (4.18) for the BMSL and by (4.16) and (4.17) for the approximate method.

Figure 14 shows the spectral densities for the ideal relay system corresponding to the BMSL and the approximate method. Also shown, for comparison purposes, is the exact spectral density function. The exact spectral density function is obtained by the direct solution of the Fokker-Plank-Kolmogorov equation for the transient density function and the steady-state density function. It is given by

$$S(\omega) = b^2/\pi \left\{ 1/\omega^2 + L^4/2b^4\omega^4 \left[1 - \sqrt{\frac{1 + (1+16b^4\omega^4/L^4)^{1/2}}{2}} \right] \right\} \quad (4.28)$$

where

$$S(0) = \frac{5}{\pi} b^6/L^4 \quad (4.29)$$

obtained by choosing the proper sign of the radical.

Inspection of the results shows that the approximate method of Chapter III is consistently more accurate than the BMSL. This is expected since the excess of the actual density function as compared to the normal has been taken into account by the λ_4 term in the density function expansion. The root mean-square values from the approximate method are very close to the exact although only one correction term

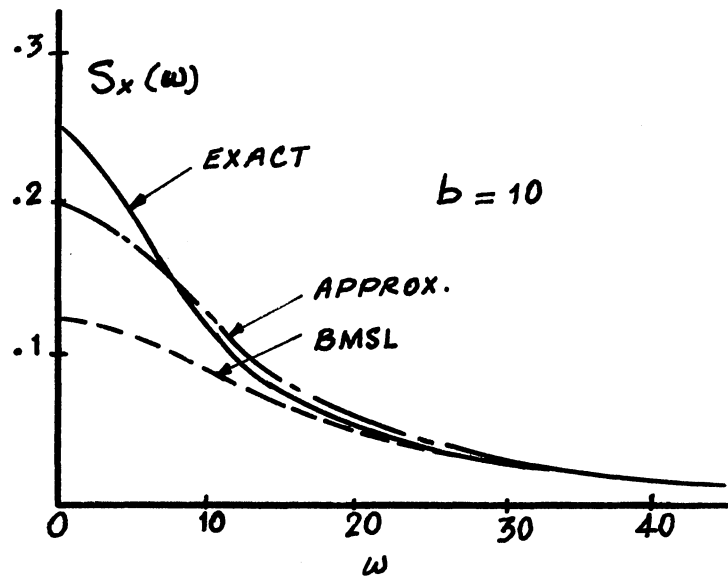


Figure 14. Spectral Density Functions of Error Signal in First Order System with White Noise Input-Ideal Relay

has been used in the expansion for the probability density function. The results can, of course, be made more accurate by including one or more additional correction terms. It is also expected that λ_4 involves some error which can be reduced by such additional corrections. As for the density function, the same remarks are also true and the approximate density function approaches the exact with the inclusion of additional correction terms. The error in the approximation being of the order of the first neglected term in the Edgeworth (asymptotic) expansion.

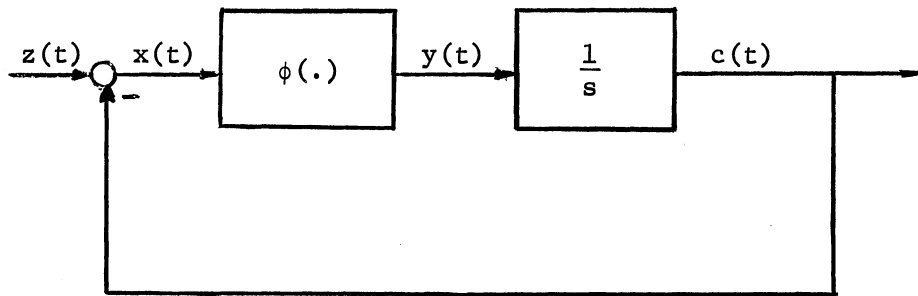
Experimental Investigation

Smith (1966) conducted an experimental study of the two systems shown in Figure 15 where three nonlinearities have been used, one at a time, with each system. These are shown in Figure 16. The input was filtered white noise. The filter transfer function was

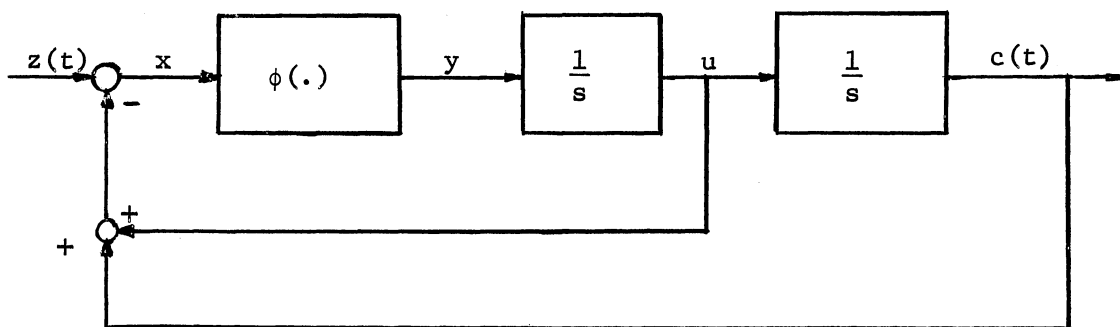
$$G(s) = \frac{K}{s + a}$$

In that study, root mean-square values of the error signal $x(t)$ were measured for a wide range of input root mean-square values and for three filter bandwidths, namely, $a=1$, $a=11$, $a=51$ radians/second. For certain selected cases, the amplitude probability density functions were also measured. Since the systems in Figure 15 are not amenable to exact calculations, the results of Smith's experimental study are used as the basis for the investigation.

A Markov Process Representation of the Input. The spectral density function of $z(t)$ is given by



(a)



(b)

Figure 15. Nonlinear Feedback Systems Used in Experimental Investigation. (a) First Order (b) Second Order

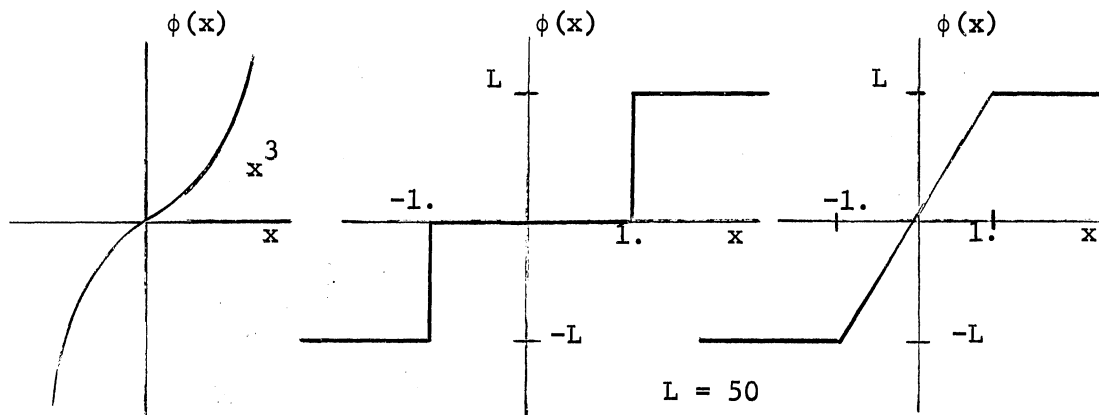


Figure 16. Nonlinear Characteristics Used in Experimental Investigation

$$S_z(\omega) = K^2 \left| \frac{1}{j\omega + a} \right|^2$$

K^2 may be considered as the intensity ($2b^2$) of the white noise at the input of a filter with a transfer function

$$\frac{1}{s + a}$$

or we may consider a unit intensity white noise at the input of a filter with a transfer function

$$\frac{K}{s + a}$$

Both representations are equivalent as far as $z(t)$ is concerned. In either case, K may be expressed in terms of b as

$$K^2 = 2b^2 \quad \text{or} \quad K = b\sqrt{2}$$

Thus

$$S_z(\omega) = 2b^2 \left| \frac{1}{j\omega + a} \right|^2$$

from which

$$\sigma_z^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_z(\omega) d\omega = b^2/a \quad (4.30)$$

The input process $z(t)$ is then expressed as a Markov process as follows

$$dz(t) = -az(t) dt + b\sqrt{2} dw(t) \quad (4.31)$$

where $w(t)$ is a normalized Wiener process.

A Markov Process Representation of the First Order System. The error signal $x(t)$ in Figure 15a is written as an element of a two-dimensional Markov process as follows

Let

$$x_1 = c(t) \quad , \quad x_2 = x(t) \quad (4.32)$$

then the process $X(t) = \{x_1, x_2\}$ is a Markov process satisfying the following stochastic differential equations

$$dx_1(t) = \phi(x_2) dt \quad (4.33)$$

$$dx_2(t) = -\{a(x_1 + x_2) + \phi(x_2)\} dt + b\sqrt{2} dw(t) \quad (4.34)$$

From the nature of the input and the symmetry of the nonlinear characteristic it is clear that all odd semi-invariants of the process $X(t)$ are zero, i.e.

$$\lambda_{ij}(t) = 0 \quad \text{for } i+j \text{ odd} \quad (4.35)$$

The differential equations for the even semi-invariants are easily obtained by the procedure described in Chapter III. For the cubic nonlinearity, these equations are

$$\dot{\lambda}_{20}(t) = 2(\lambda_{13} + 3\lambda_{02}\lambda_{11})$$

$$\begin{aligned}
\dot{\lambda}_{11}(t) &= \lambda_{04} + 3\lambda_{02}^2 - \lambda_{13} - 3\lambda_{02}\lambda_{11} - a(\lambda_{20} + \lambda_{11}) \\
\dot{\lambda}_{02}(t) &= 2b^2 - 2\{\lambda_{04} + 3\lambda_{02}^2 + a(\lambda_{11} + \lambda_{02})\} \\
\dot{\lambda}_{40}(t) &= 4\{3\lambda_{31}\lambda_{02} + 9\lambda_{22}\lambda_{11} + 6\lambda_{11}^3\} \\
\dot{\lambda}_{04}(t) &= -4\{12\lambda_{04}\lambda_{02} + 6\lambda_{02}^3 + a(\lambda_{13} + \lambda_{04})\} \\
\dot{\lambda}_{31}(t) &= 18\lambda_{13}\lambda_{11} + 18\lambda_{22}\lambda_{02} + 18\lambda_{02}^2\lambda_{11} \\
&\quad - 3\lambda_{31}\lambda_{02} - 9\lambda_{22}\lambda_{11} - 6\lambda_{11}^3 - a(\lambda_{40} + \lambda_{31}) \\
\dot{\lambda}_{13}(t) &= 12\lambda_{04}\lambda_{02} + 6\lambda_{02}^3 - 9\lambda_{04}\lambda_{11} - 27\lambda_{13}\lambda_{02} \\
&\quad - 18\lambda_{02}^2\lambda_{11} - 3a(\lambda_{22} + \lambda_{13}) \\
\dot{\lambda}_{22}(t) &= 2\{3\lambda_{04}\lambda_{11} + 9\lambda_{13}\lambda_{02} + 6\lambda_{02}^2\lambda_{11} \\
&\quad - 6\lambda_{13}\lambda_{11} - 6\lambda_{22}\lambda_{02} - 6\lambda_{02}^2\lambda_{11} \\
&\quad - a(\lambda_{31} + \lambda_{22})\}
\end{aligned} \tag{4.36}$$

The equations corresponding to the BMSL are obtained from the first three equations of the set (4.36) by setting to zero all semi-invariants of order greater than the second. Thus we have

$$\begin{aligned}
\dot{\lambda}_{20}(t) &= 6\lambda_{02}\lambda_{11} \\
\dot{\lambda}_{11}(t) &= 3\lambda_{02}^2 - 3\lambda_{02}\lambda_{11} - a(\lambda_{20} + \lambda_{11}) \\
\dot{\lambda}_{02}(t) &= 2b^2 - 6\lambda_{02}^2 - 2a(\lambda_{11} + \lambda_{02})
\end{aligned} \tag{4.37}$$

The steady state values are obtained by solving the set of algebraic equations formed by setting to zero all the derivatives. Those corresponding to (4.37) are obtained by inspection,

$$\begin{aligned}
\lambda_{11} &= 0 \quad , \quad \lambda_{20} = 3\lambda_{02}^2/a \\
\lambda_{02} &= -a/6 + 1/6 \sqrt{a^2 + 12b^2}
\end{aligned} \tag{4.38}$$

The set corresponding to (4.36) must be solved numerically. The root-mean-square values of the error signal $x(t)$, are plotted versus those of the input $z(t)$ in Figure 17 for $a=1, 11, \text{ and } 51$ radians/second. Also shown in the figure are the experimental results obtained by Smith for the same system. The histograms for the probability density obtained by Smith and the probability density functions corresponding to the BMSL and the approximate method are plotted for six selected points in Figure 18.

Similar calculations are carried out for the other two nonlinearities. The results are shown in Figure 19 thru 22.

The correlation functions matrix satisfies the differential equation

$$\frac{dR(\tau)}{dt} = \underline{A}(\underline{x}, t) \underline{x}^{oT}(t + \tau)$$

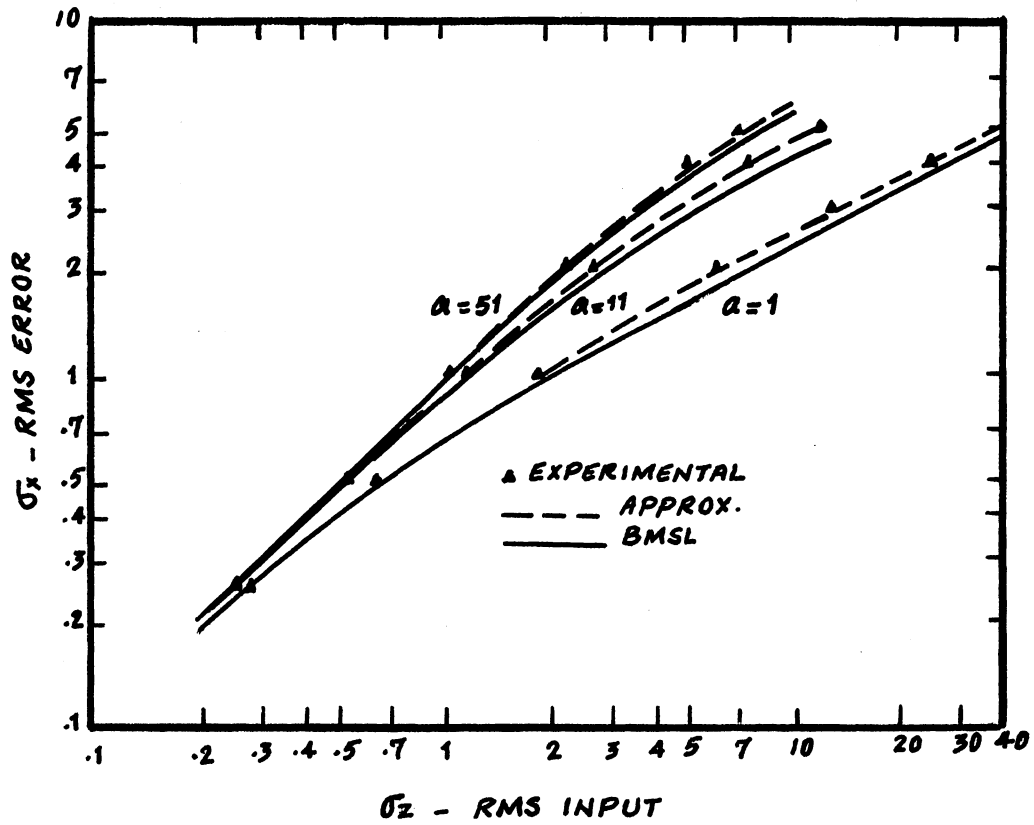


Figure 17. Input-Error Curves. First Order System with Filtered White Noise Input- Cubic Nonlinearity

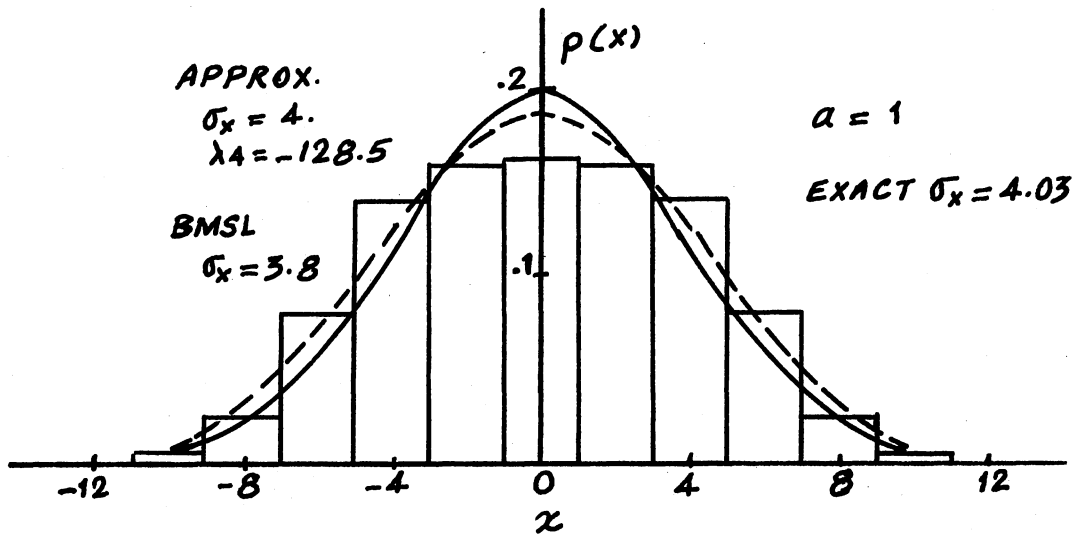
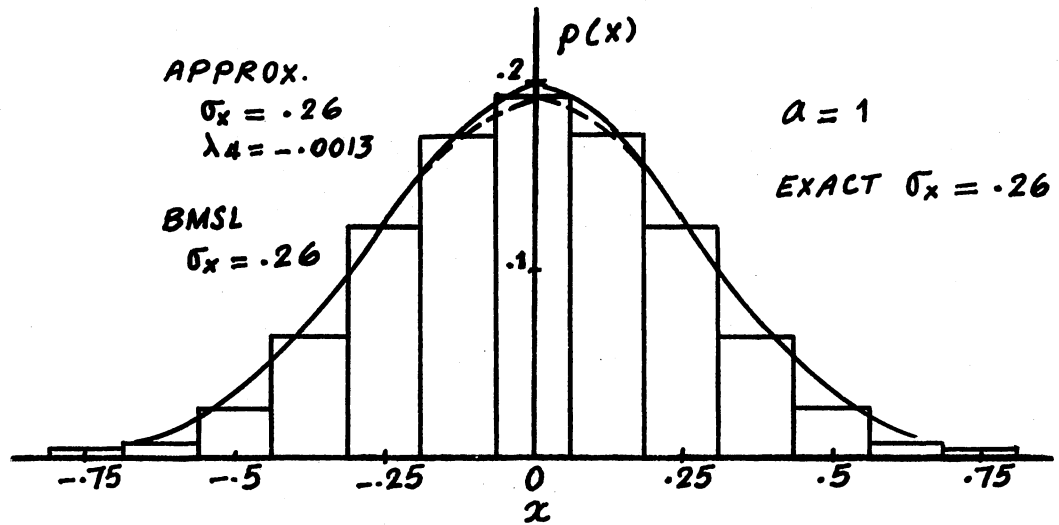


Figure 18. Amplitude Probability Density Functions and Histograms for Error Signal in the First Order System with Filtered White Noise Input. Cubic Nonlinearity
 ——— BMSL, - - - - - Approximate.

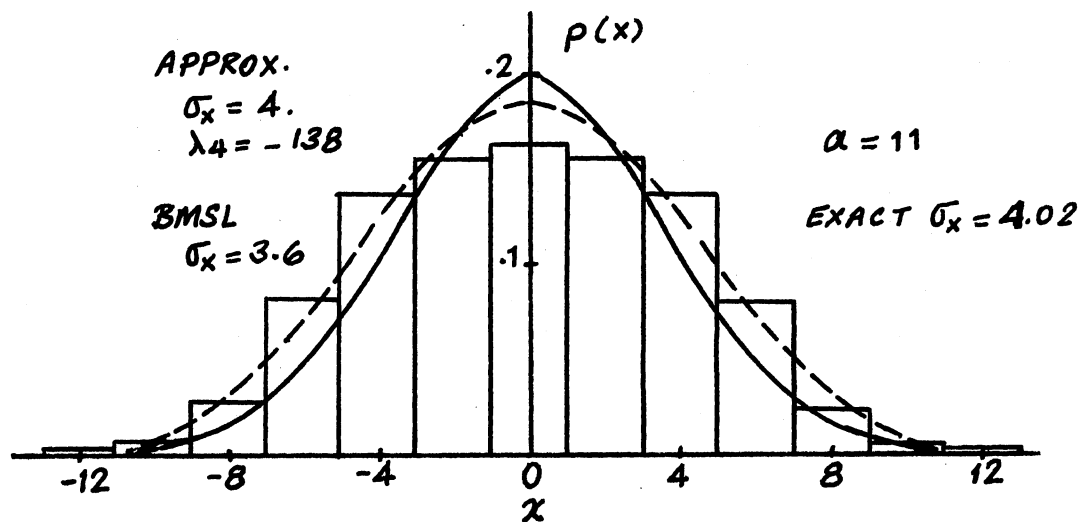
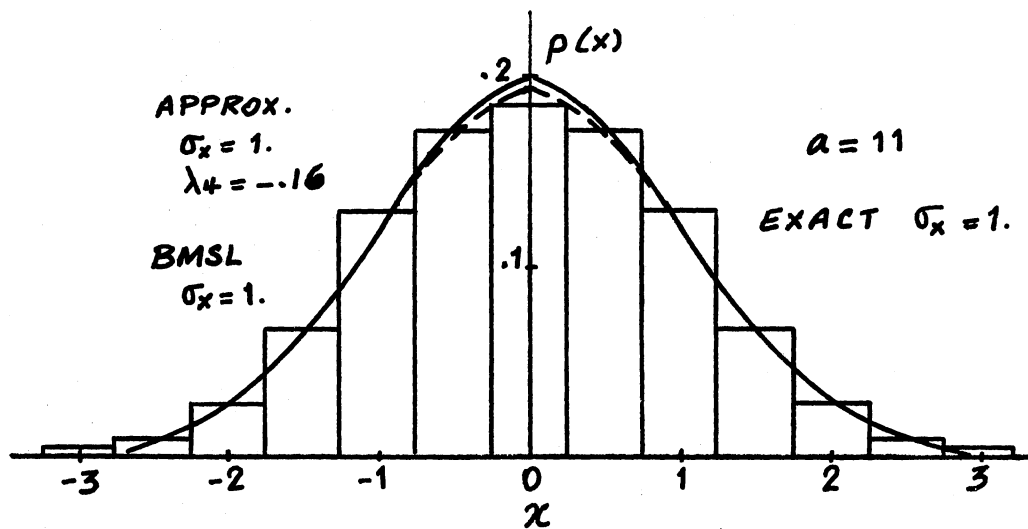


Figure 18 (Continued)

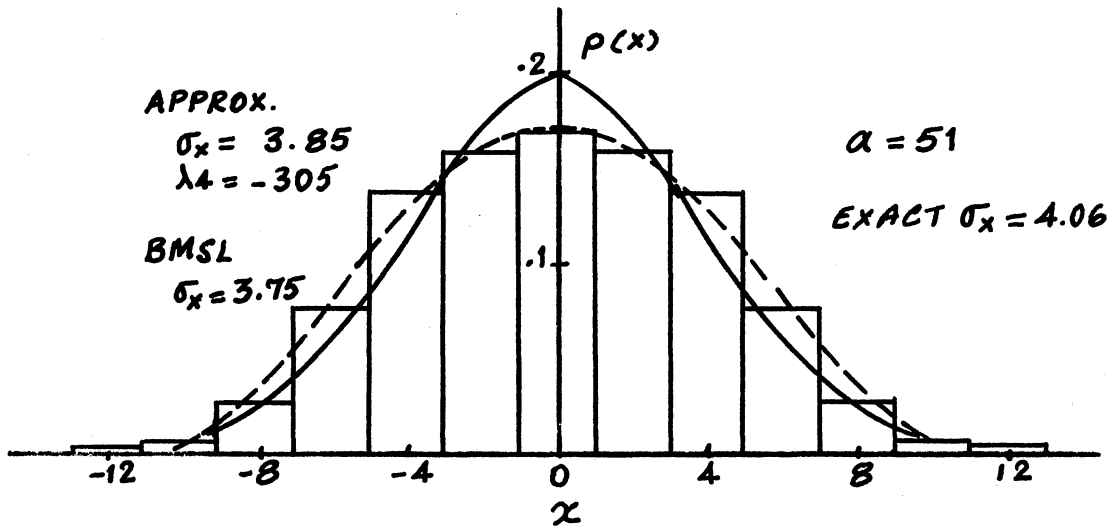
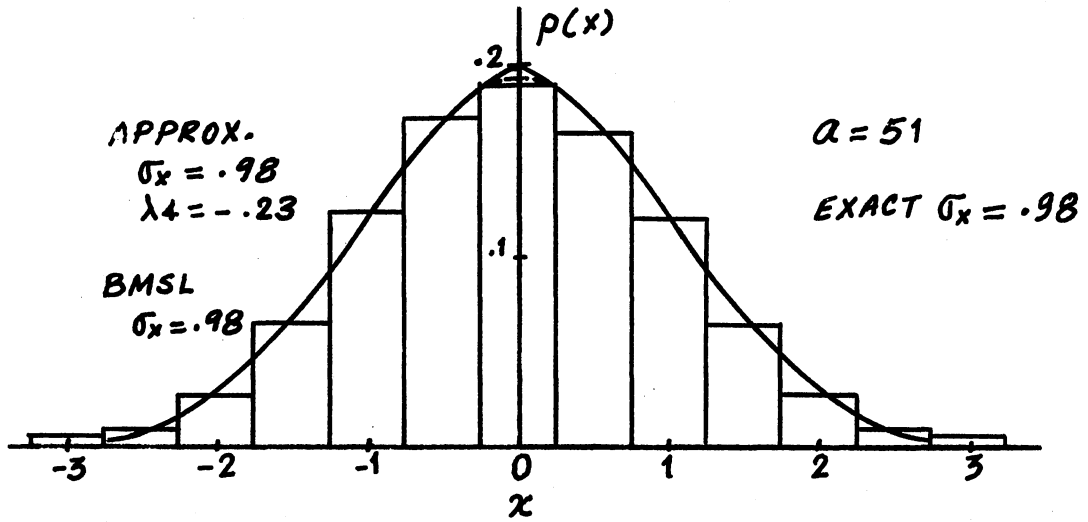


Figure 18 (Continued)

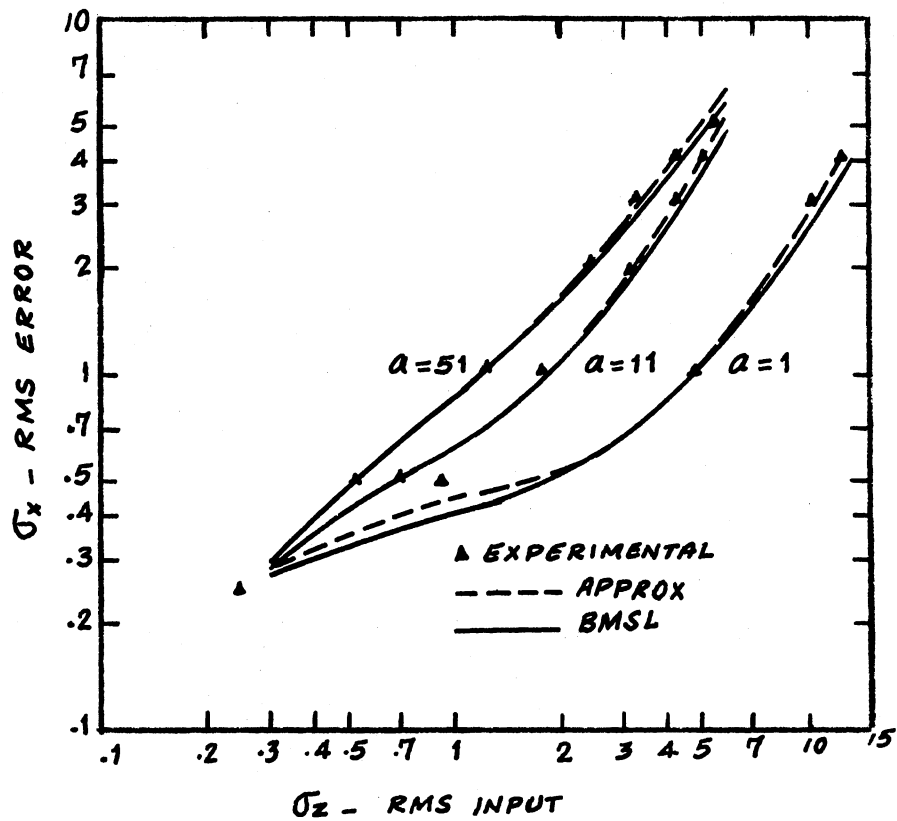


Figure 19. Input-Error Curves. First Order System with Filtered White Noise Input-Relay with Dead Zone

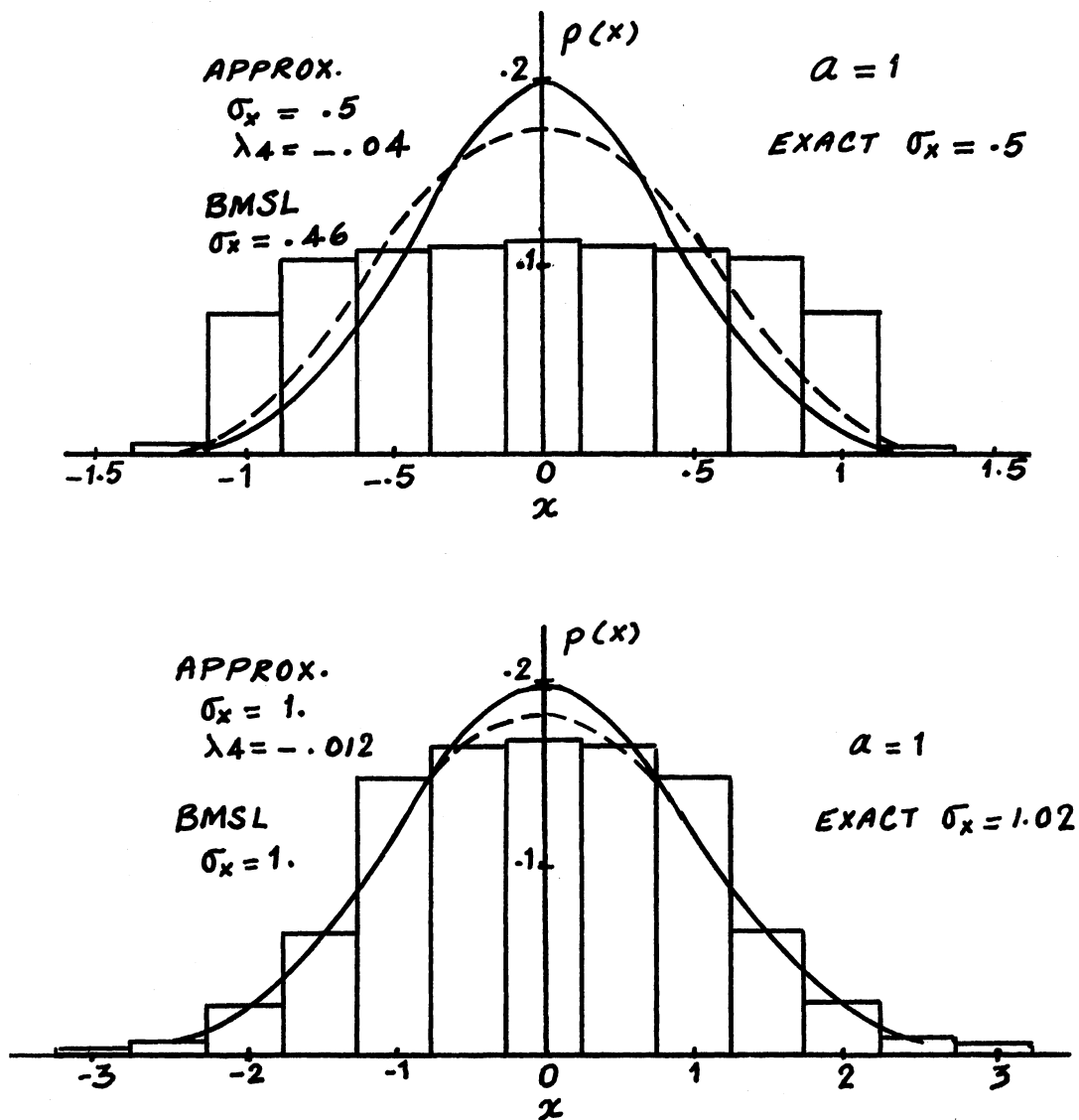


Figure 20. Amplitude Probability Density Functions and Histograms of Error Signal in First Order System with Filtered White Noise Input- Relay with Dead Zone
 — BMSL, - - - - - Approximate.

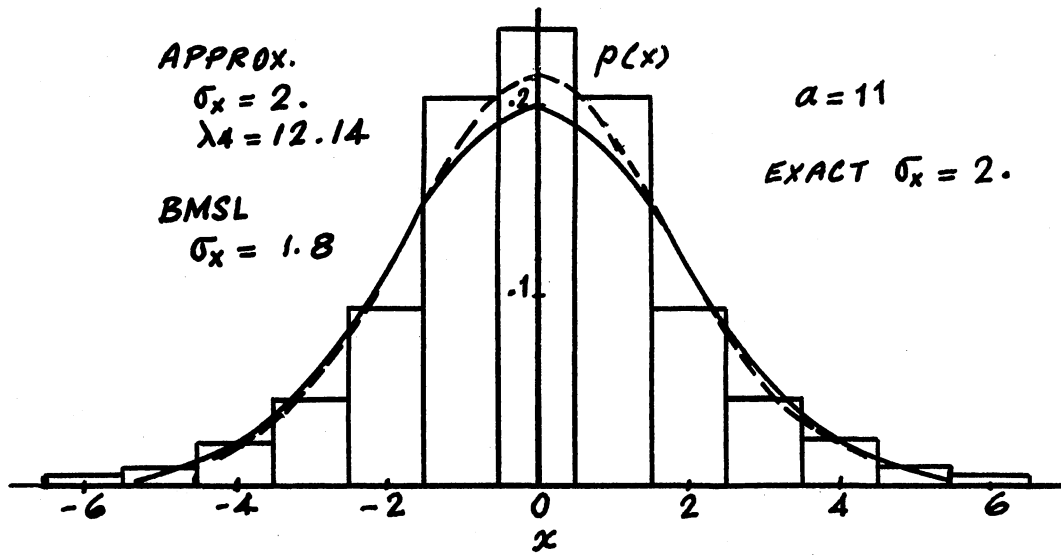
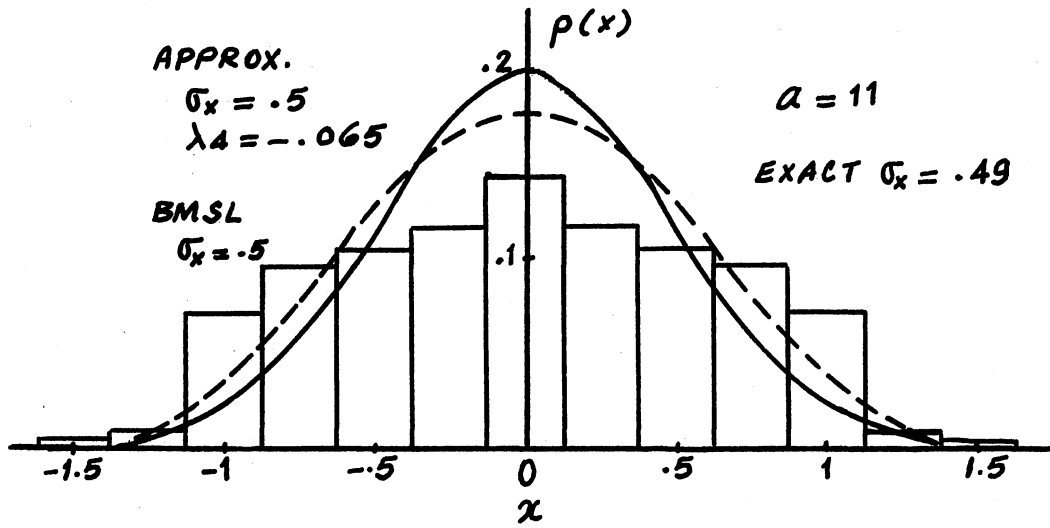


Figure 20 (Continued)

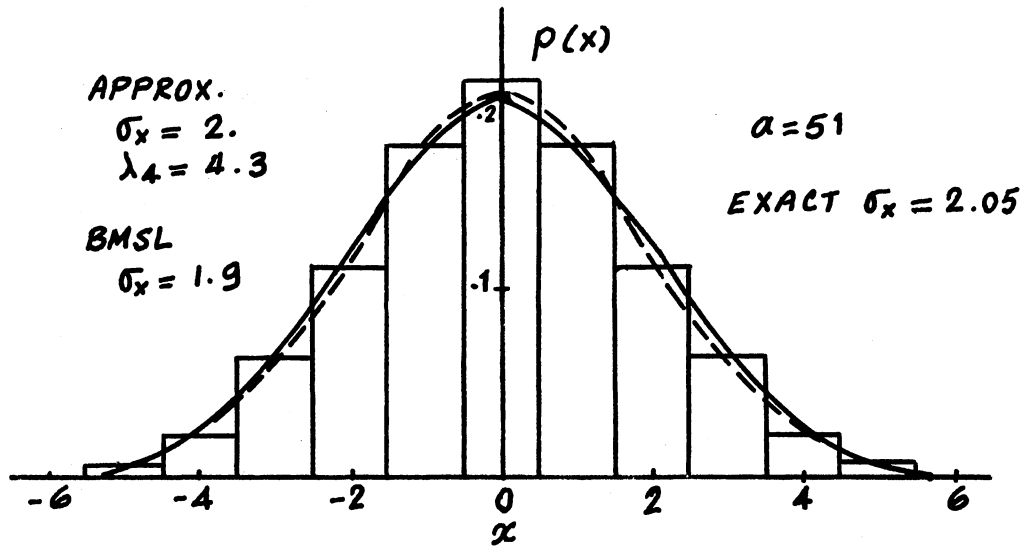
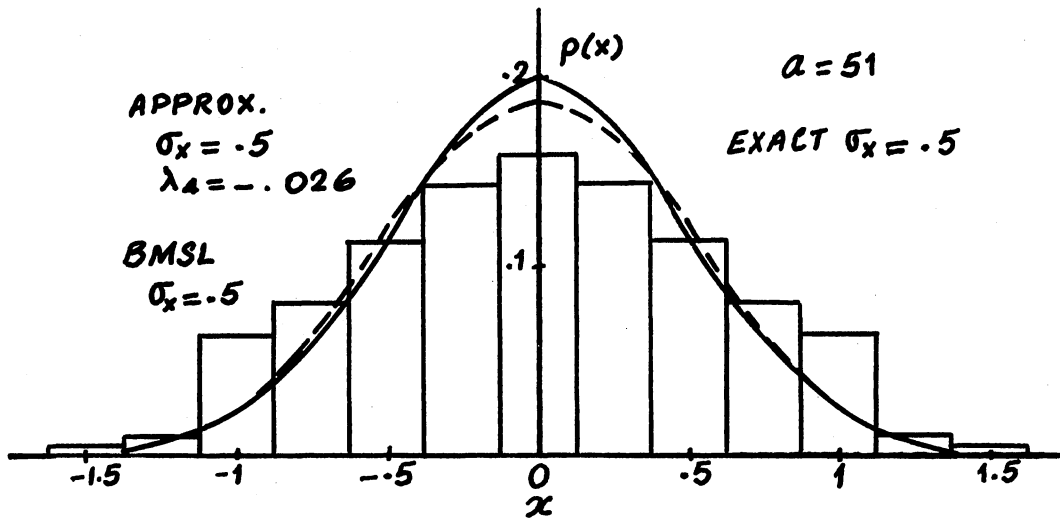


Figure 20 (Continued)

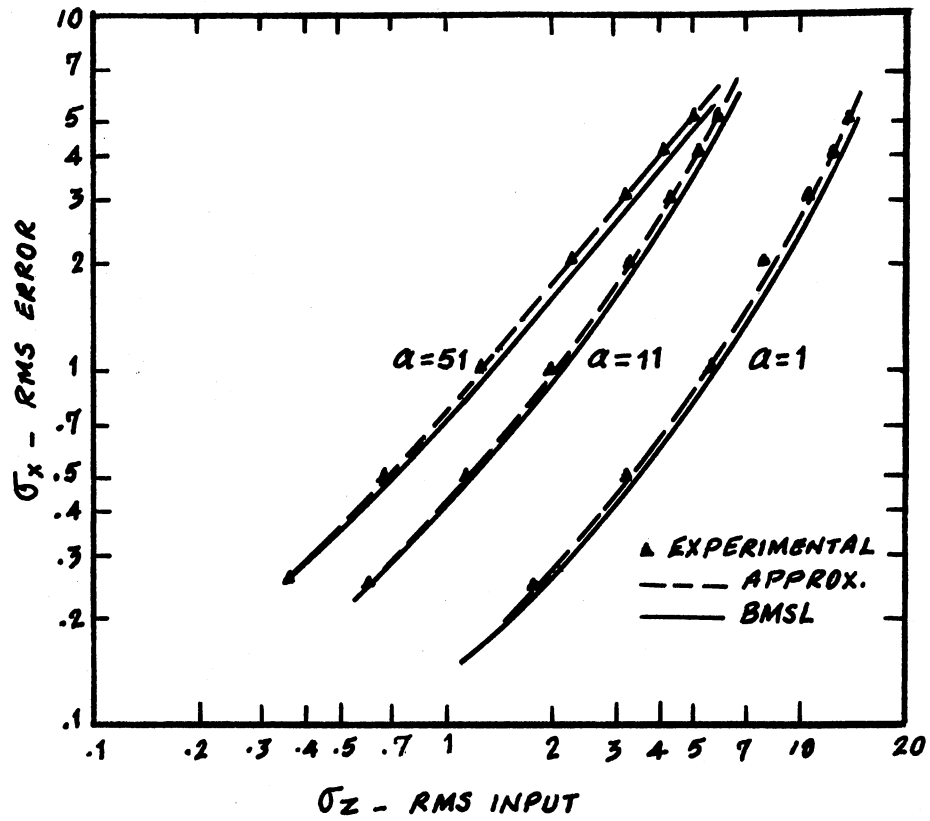


Figure 21. Input-Error Curves. First Order System with Filtered White Noise Input-Limiter Nonlinearity

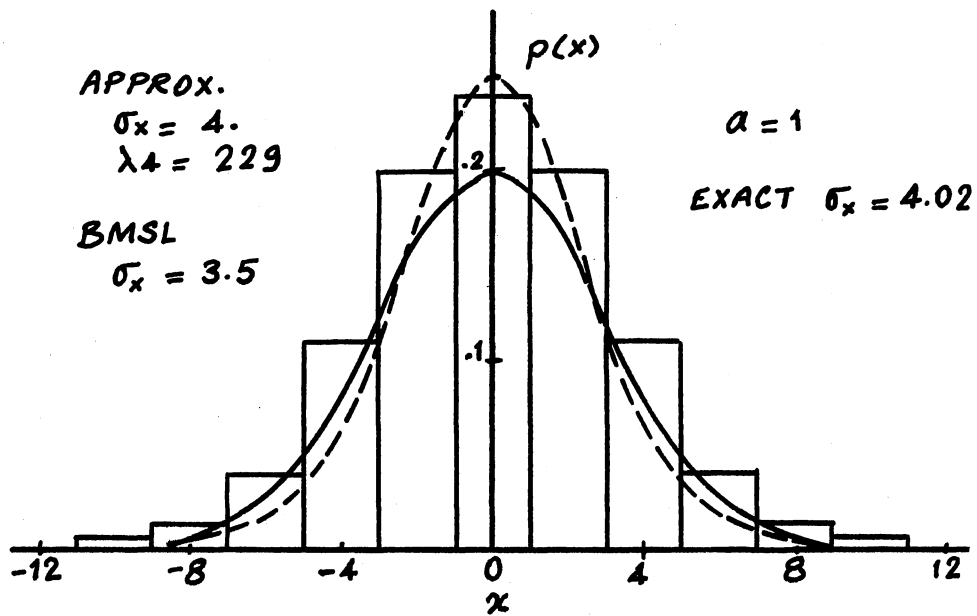
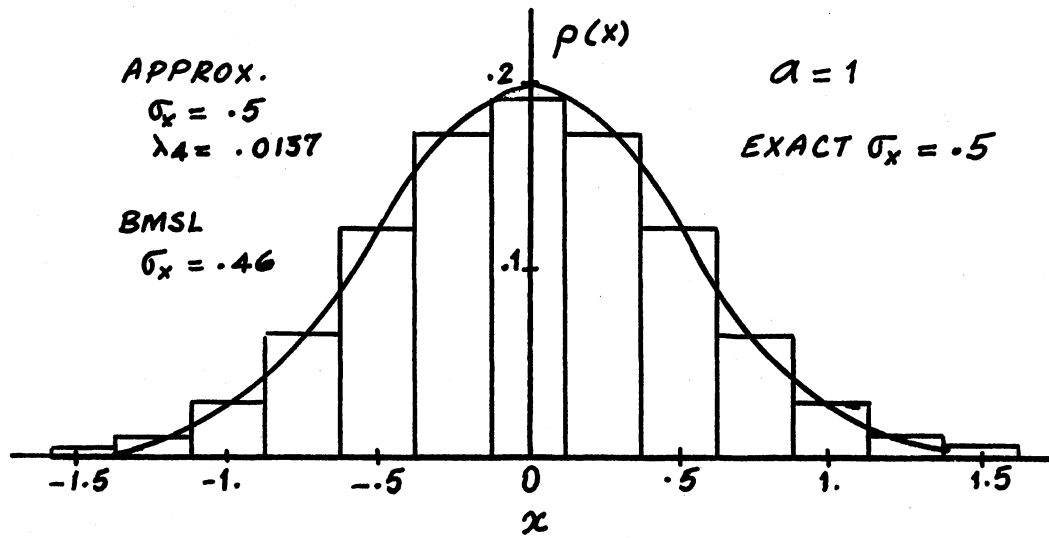


Figure 22. Amplitude Probability Density Functions and Histograms of Error Signal in First Order System with Filtered White Noise Input-Limiter Nonlinearity.

———— BMSL, - - - - - Approximate

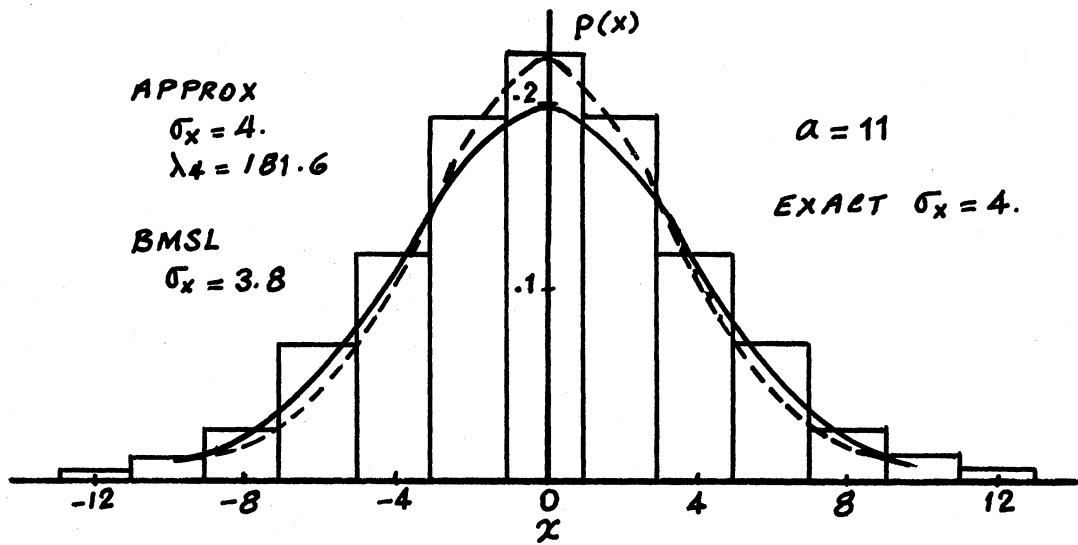
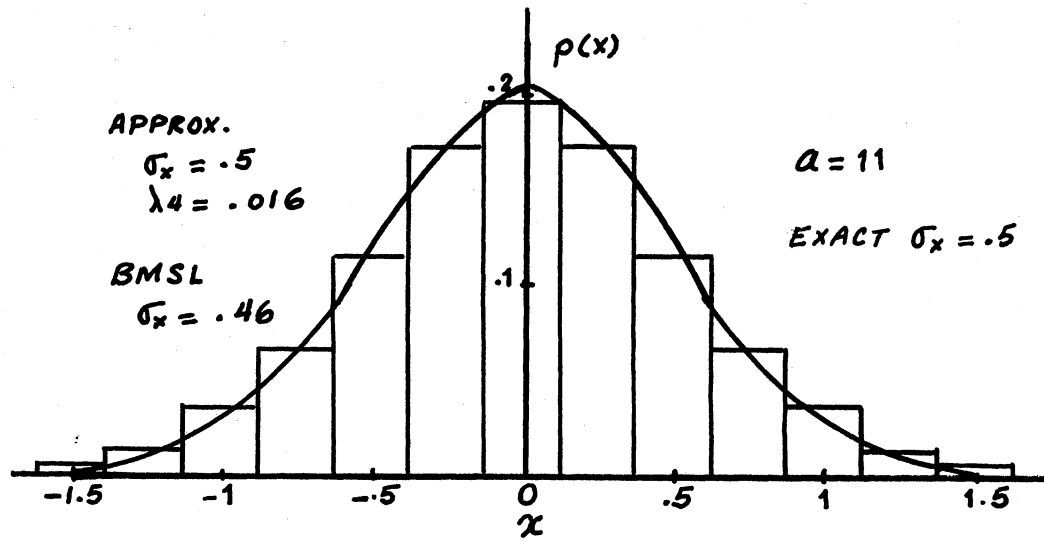


Figure 22 (Continued)

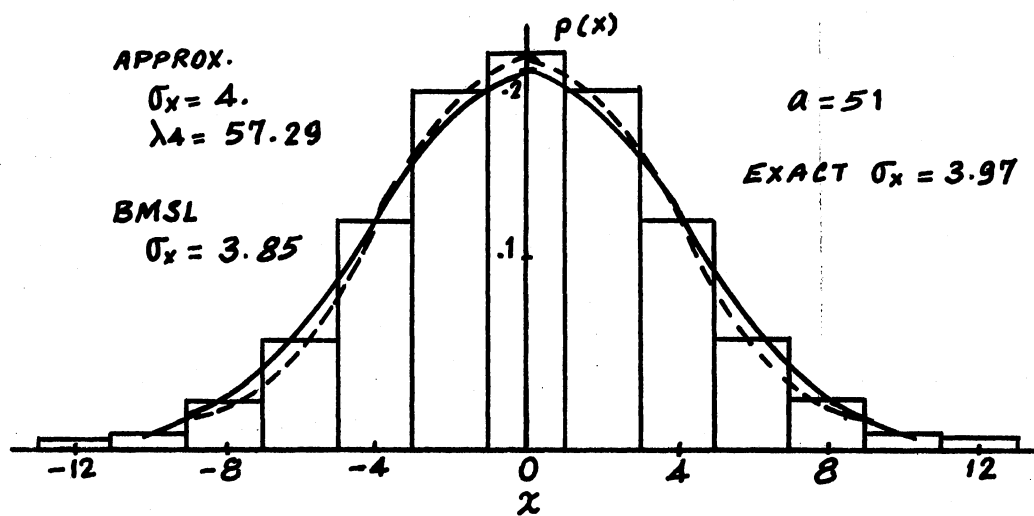
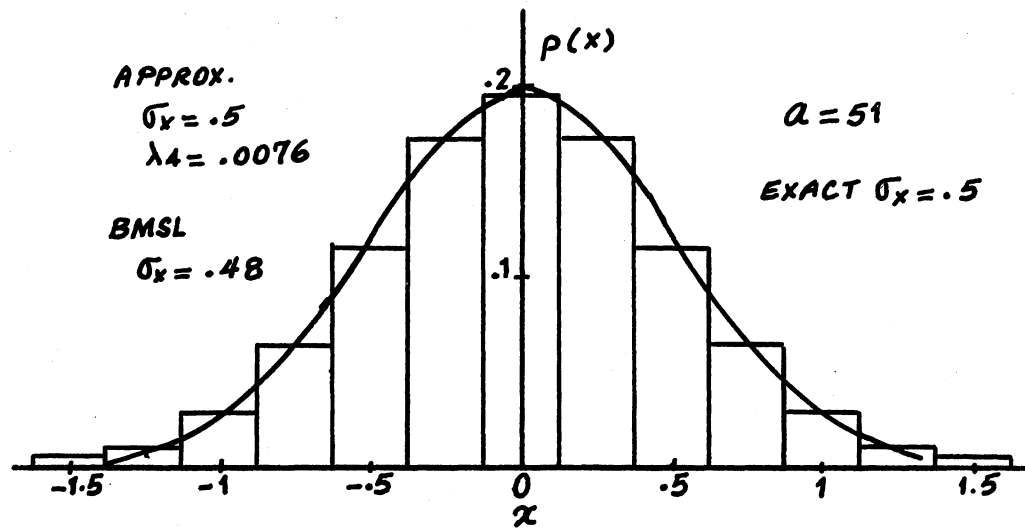


Figure 22 (Continued)

where

$$A_1 = \phi(x_2) \quad , \quad A_2 = -\{a(x_1 + x_2) + \phi(x_2)\}$$

Again, using the two-dimensional Edgeworth expansion for the joint density functions, we obtain

$$\frac{dR_{11}}{d\tau} = c/\sigma_{x_2} R_{21}(\tau) \quad , \quad R_{11}(0) = \lambda_{20}$$

$$\frac{dR_{12}}{d\tau} = c/\sigma_{x_2} R_{22}(\tau) \quad , \quad R_{12}(0) = \lambda_{11}$$

$$\frac{dR_{21}}{d\tau} = -\{a R_{11}(\tau) + a R_{21}(\tau) + c/\sigma_{x_2} R_{21}(\tau)\} \quad ,$$

$$R_{21}(0) = \lambda_{11}$$

$$\frac{dR_{22}}{d\tau} = -\{a R_{12}(\tau) + a R_{22}(\tau) + c/\sigma_{x_2} R_{22}(\tau)\} \quad ,$$

$$R_{22}(0) = \lambda_{02}$$

where

$$c = I_{01} + \lambda_{04}/24\lambda_2^2 (I_{05} - 6I_{03} + 3I_{01})$$

and

$$I_{0k} = \int_{-\infty}^{\infty} (x_2^0/\sigma_{x_2})^k \phi(x_2) p_0(x_2) dx_2$$

as defined in Chapter III.

The equations corresponding to the BMSL are obtained from the above set of equations with

$$c = I_{01}$$

only.

A Markov Process Representation of the Second Order System. In a similar manner, the error signal $x(t)$ in Figure 15b can be represented as an element of a three-dimensional Markov process. Let

$$x_1 = c(t) \quad , \quad x_2 = u(t) \quad , \quad x_3 = x(t)$$

then the process $X = \{ x_1, x_2, x_3 \}$ is a Markov process satisfying the following differential equations

$$dx_1 = x_2 dt$$

$$dx_2 = \phi(x_3) dt$$

$$dx_3 = -\{ ax_1 + (1+a)x_2 + ax_3 + \phi(x_3) \} dt + b\sqrt{2} dw \quad (4.39)$$

The differential equations are obtained in the same way as those for the first order system. Only the even semi-invariants are needed since the odd ones vanish. The results are shown in Figures 23 thru 28. Similarly, the equations for the correlation functions are obtained in the same way.

Application to Non-Zero-Mean Systems

An interesting feature of non-zero-mean systems is that a mean component appears at the output of the nonlinearity even when there was

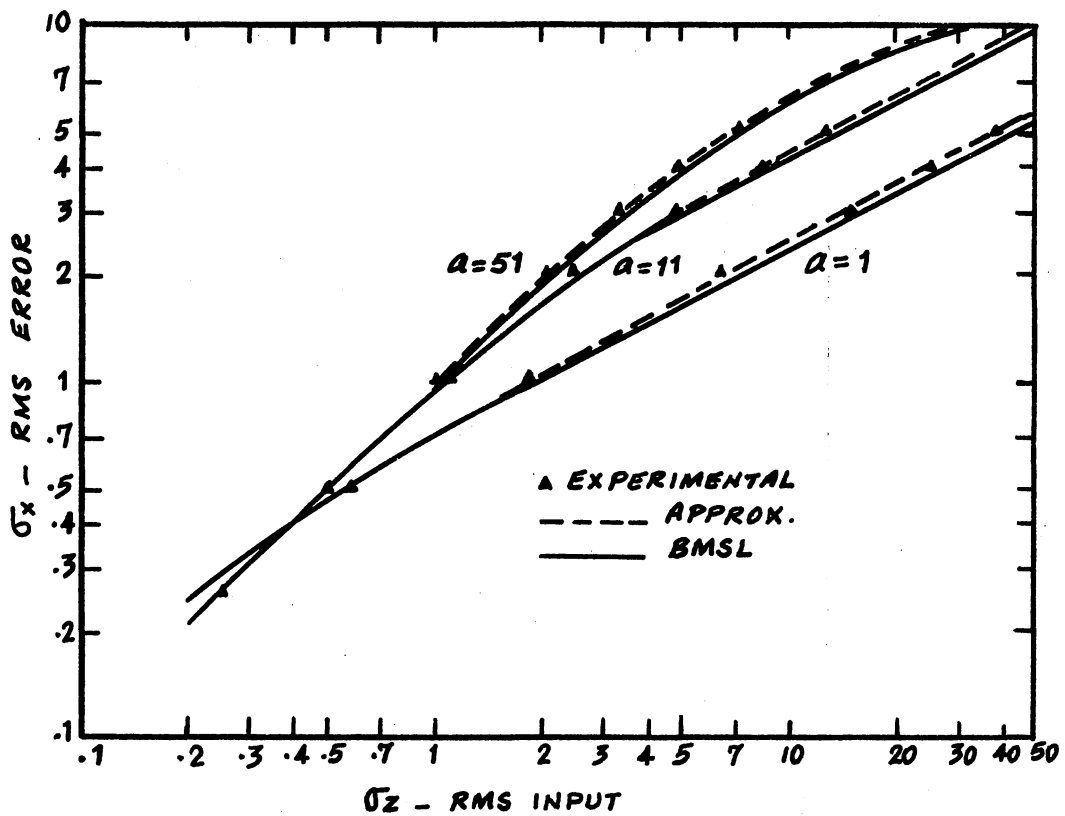


Figure 23. Input-Error Curves. Second Order System with Filtered White Noise Input- Cubic Nonlinearity

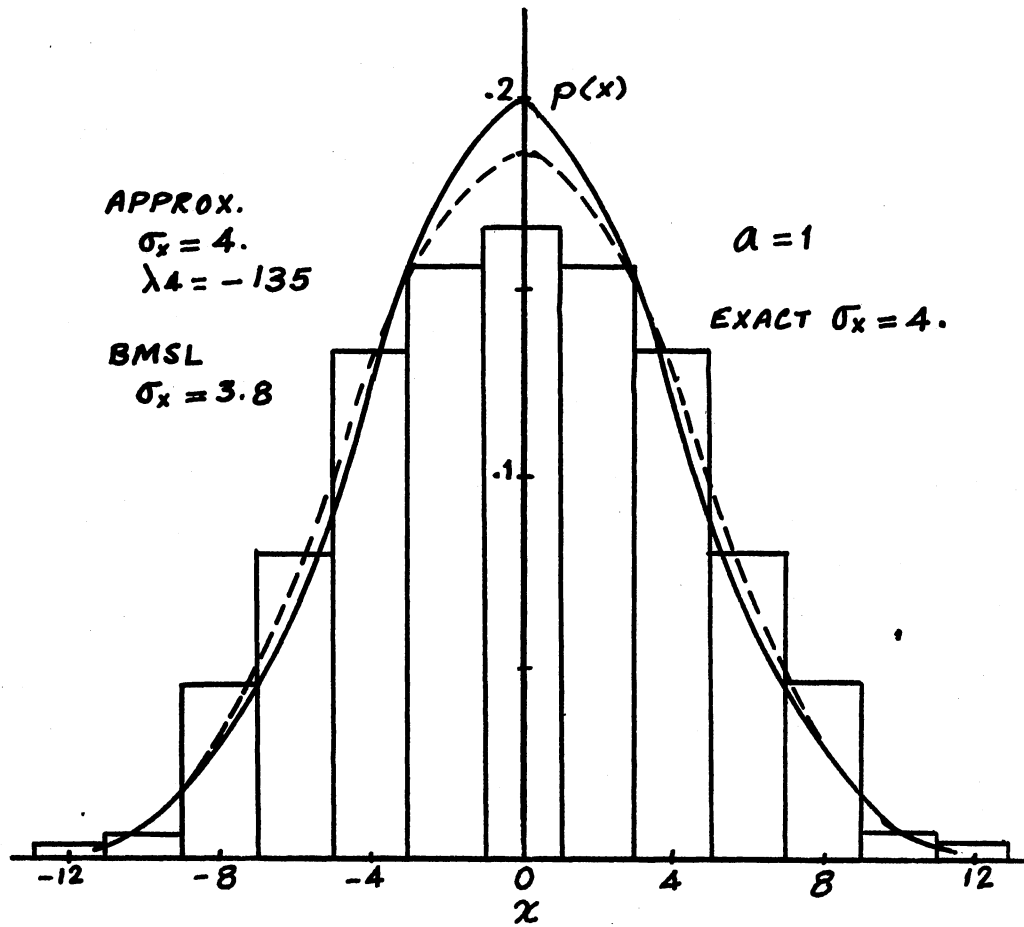


Figure 24. Amplitude Probability Density Functions and Histogram of Error Signal in Second Order System with Filtered White Noise Input-Cubic Nonlinearity.

———— BMSL, - - - - - Approximate

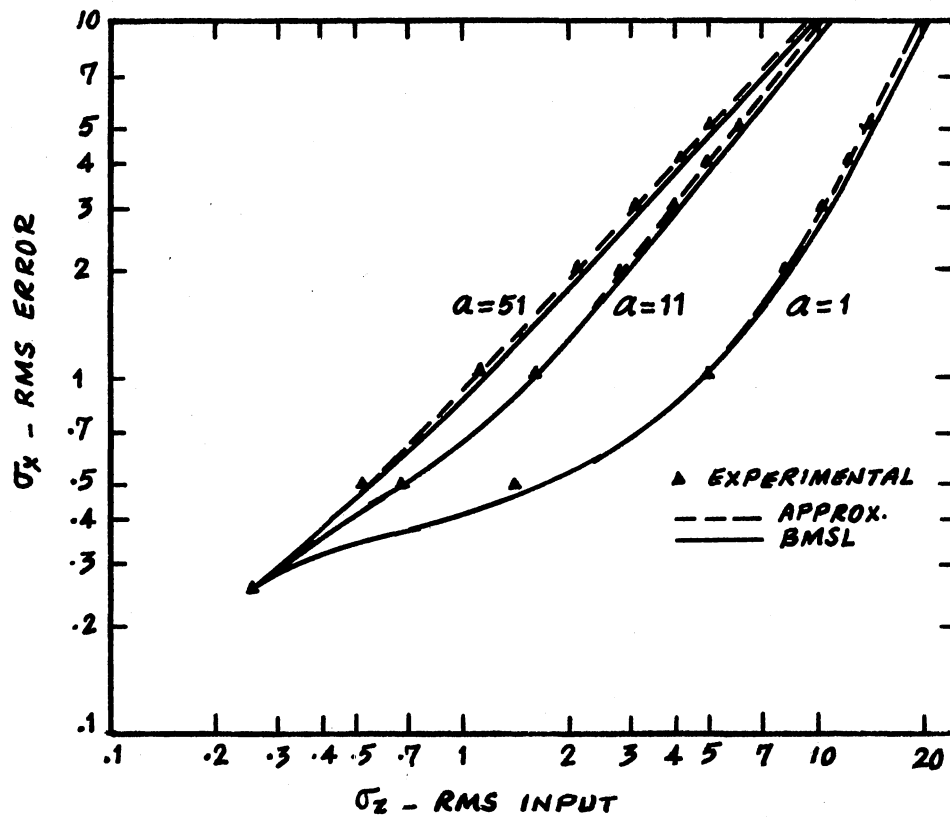


Figure 25. Input-Error Curves. Second Order System with Filtered White Noise Input-Relay with Dead Zone

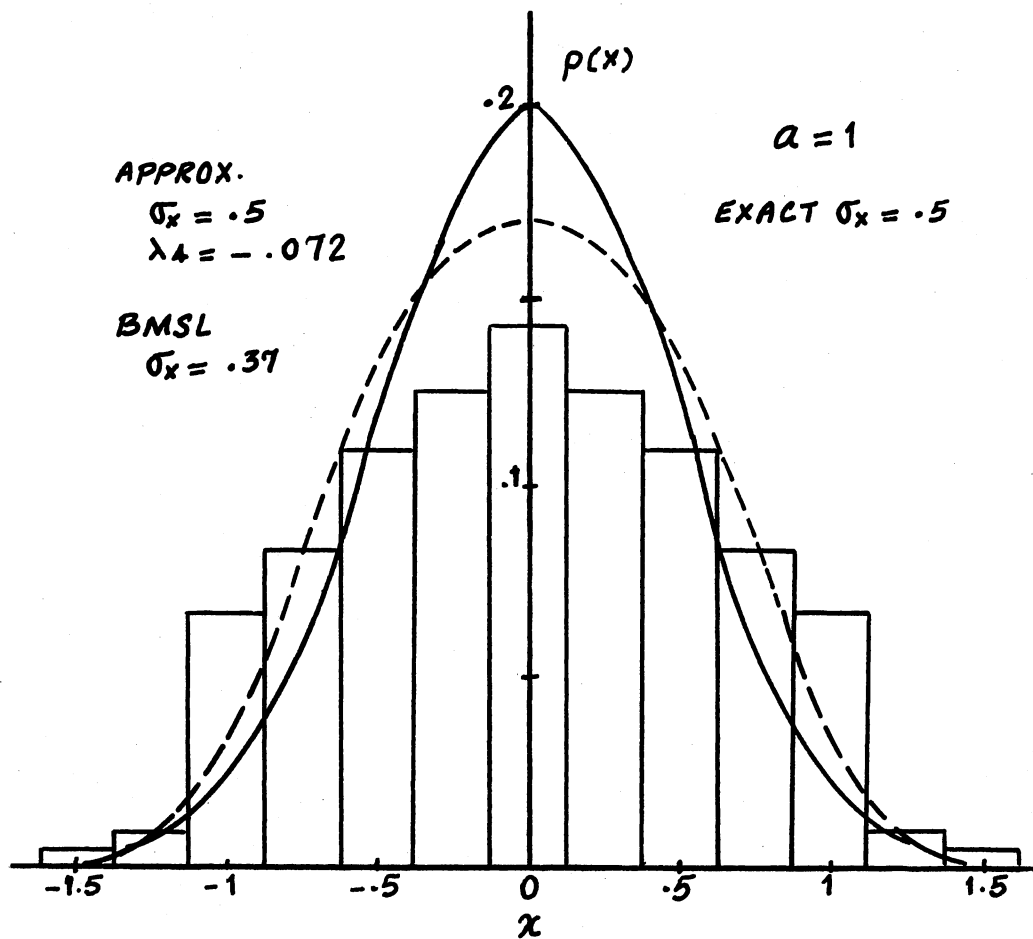


Figure 26. Amplitude Probability Density Functions and Histogram of Error Signal in Second Order System with Filtered White Noise Input- Relay with Dead Zone
 ——— BMSL, - - - - - Approximate

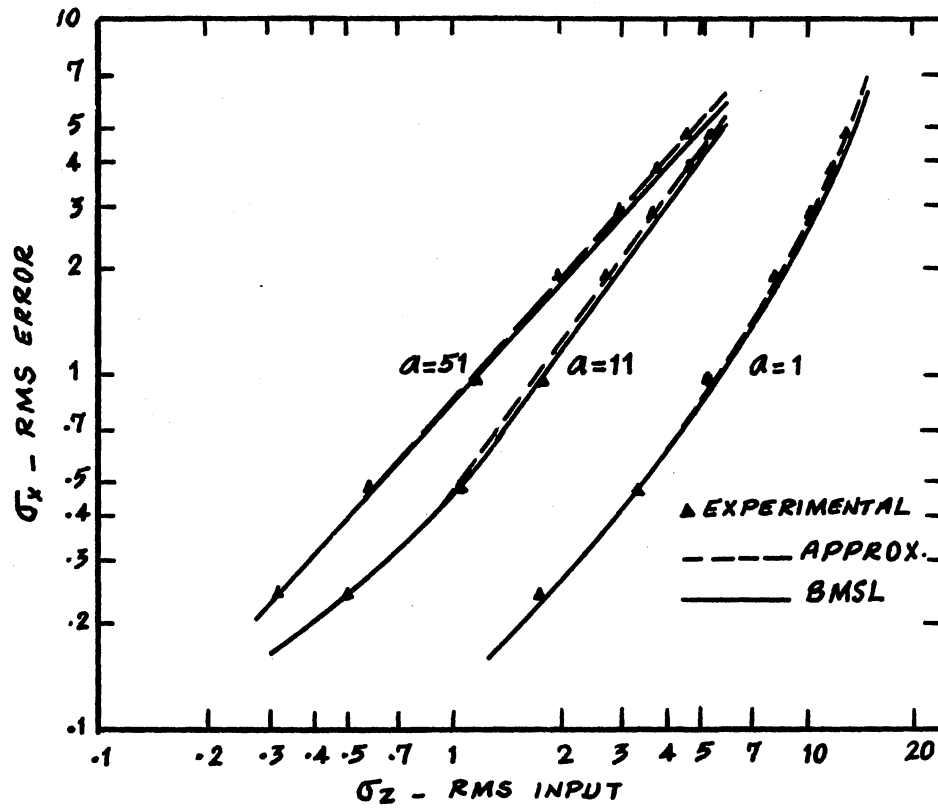


Figure 27. Input-Error Curves. Second Order System with Filtered White Noise Inputs-Limiter Nonlinearity

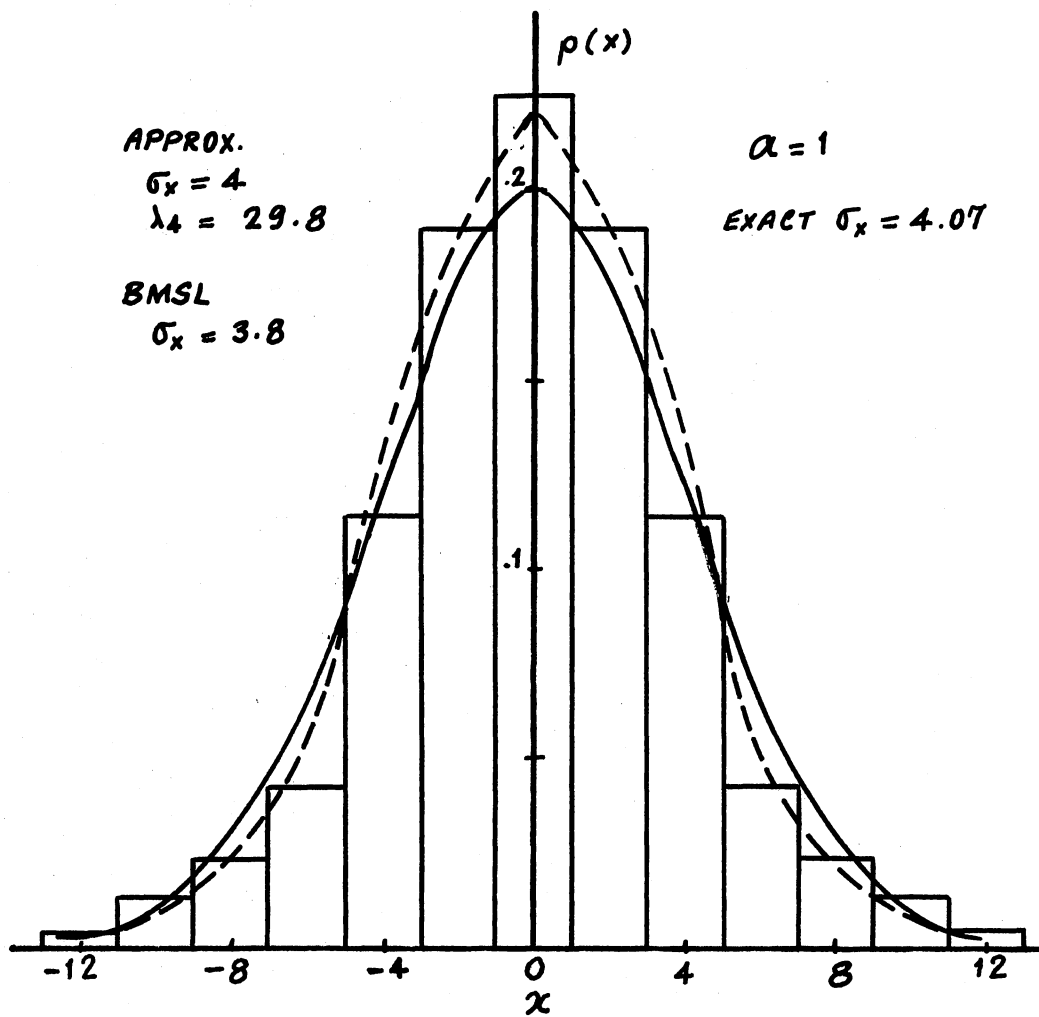


Figure 28. Amplitude Probability Density Functions and Histogram of Error Signal in Second Order System with Filtered White Noise Input- Limiter Nonlinearity
 ——— BMSL, - - - - - Approximate

no such component at the input, which is where the term "non-zero-mean" came from. A typical non-zero-mean system is one with a nonsymmetric nonlinear characteristic. A nonlinear transformation in such systems makes it possible to detect the random component of the input signal.

The application of the methods of analysis to non-zero-mean systems proceeds in the same manner as the zero-mean systems. Here, however, all semi-invariants are of interest since both odd and even statistics are usually non-zero. The number of equations to be solved is more than that for zero-mean systems. But the procedure is exactly the same. Therefore, only a simple system is investigated here to show the effectiveness of the approximate method. In addition to the excess of the distribution a skewness factor enters, in analyzing these systems, which was absent in symmetric systems. This is expected to be another factor contributing to the inaccuracy in the BMSL. At the same time it shows how the approximate method is still well-suited for analysis even with the presence of considerable skewness.

Consider the scheme shown in Figure 29.

Let,

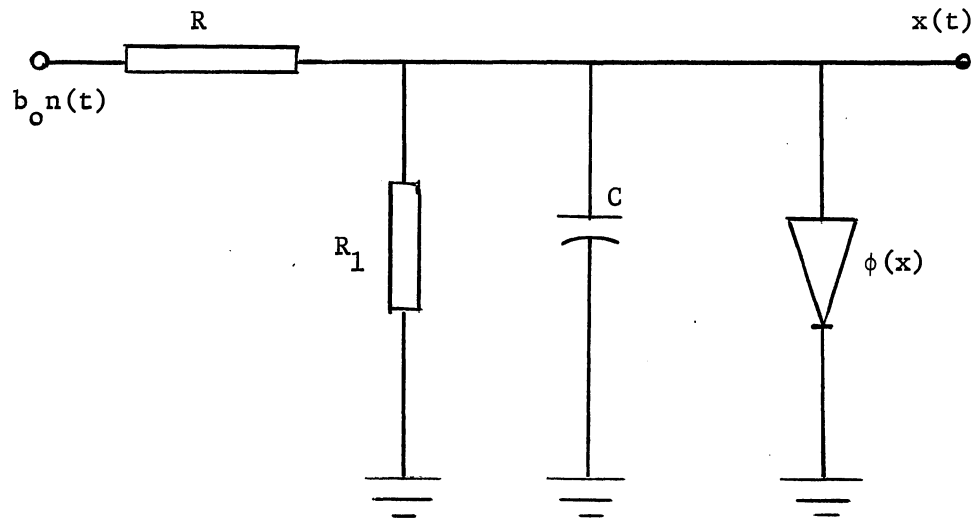
$$R = R_1 = 100 \text{ kilo ohms}$$

$$\text{Diode forward resistance, } r_f = 2 \text{ Kilo-ohms}$$

$$\text{Diode backward resistance, } r_b = 12 \text{ Mega-ohms}$$

The differential equation for $x(t)$ is

$$\frac{dx}{dt} = -1/RC(1 + R/R_1 + R/r) x(t) + b_o/RC n(t) \quad (4.40)$$



$$\beta_1 = (1 + R/R_1 + R/r_b)$$

$$\beta_2 = (1 + R/R_1 + R/r_f)$$

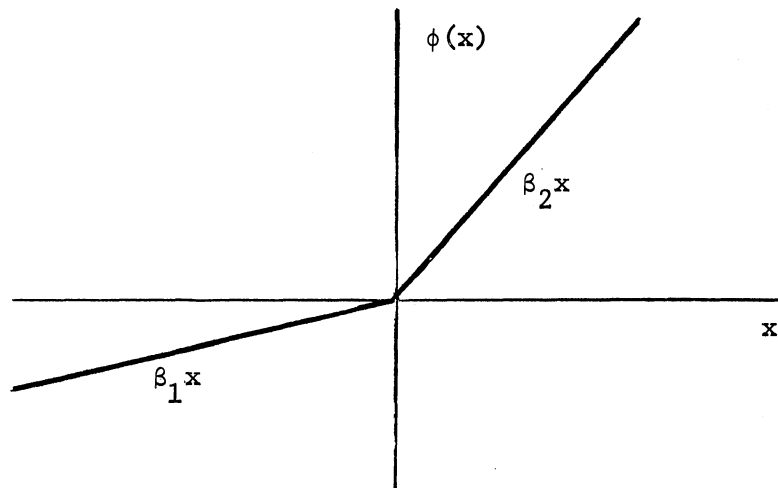


Figure 29. A Simple First Order Non-Zero System with White Noise Input

where

$$r = \begin{cases} r_f & x \geq 0 \\ r_b & x < 0 \end{cases}$$

and $n(t)$ is a unit intensity white noise.

Or,

$$\frac{dx}{dt} = A(x,t) + b n(t) \quad (4.41)$$

where

$$A(x,t) = \begin{cases} -\beta_1 x & x < 0 \\ -\beta_2 x & x \geq 0 \end{cases}, \text{ and } b = b_o/RC$$

The exact density function is obtained by the direct solution of the Fokker-Plank-Kolmogorov equation, in the steady state. It is given by

$$p(x) = \frac{2}{b} \frac{1}{\sqrt{\pi}(1/\sqrt{\beta_1} + 1/\sqrt{\beta_2})} e^{-\beta x^2/b^2} \quad (4.42)$$

where

$$\beta = \begin{cases} \beta_1 & x < 0 \\ \beta_2 & x \geq 0 \end{cases}$$

From (4.42) it is easy to calculate the statistics of $x(t)$. In particular,

$$\lambda_1 = -.322 b_o/\sqrt{RC}, \quad \lambda_2 = .1079 b_o^2/RC$$

The BMSL gives

$$\lambda_1 = -.33 b_o / \sqrt{RC} \quad , \quad \lambda_2 = .069 b_o^2 / RC$$

while the approximate method gives,

$$\lambda_1 = -.33 b_o / \sqrt{RC} \quad , \quad \lambda_2 = .1043 b_o^2 / RC$$

$$\lambda_3 / \sigma_x^3 = -.782 \quad , \quad \lambda_4 / 24 \lambda_2^2 = .62$$

Figure 30 shows the dependence of the root mean-square output on b_o , for $RC=1$, corresponding to the three methods of analysis. Figure 31 shows the corresponding density functions for the output process $x(t)$.

The correlation function of $x(t)$ satisfies the equation

$$dR/d\tau = E \{ A(x, t) x^0(t, \tau) \}$$

Using the two-dimensional Edgeworth expansion we get,

$$dR/d\tau = - c R(\tau)$$

where

$$c = 1/\sigma_x \{ I_1 + \lambda_3/6\sigma_x^3 (I_4 - 3I_2) + \lambda_4/24\lambda_2^2 (I_5 - 6I_3 + 3I_1) + \lambda_3^2/72\lambda_2^3 (I_7 - 15I_5 + 45I_3 - 15I_1) \}$$

from which

$$R(\tau) = \lambda_2 e^{-c\tau} \quad , \quad S(\omega) = (\lambda_2/2\pi) \frac{2c}{\omega^2 + c^2}$$

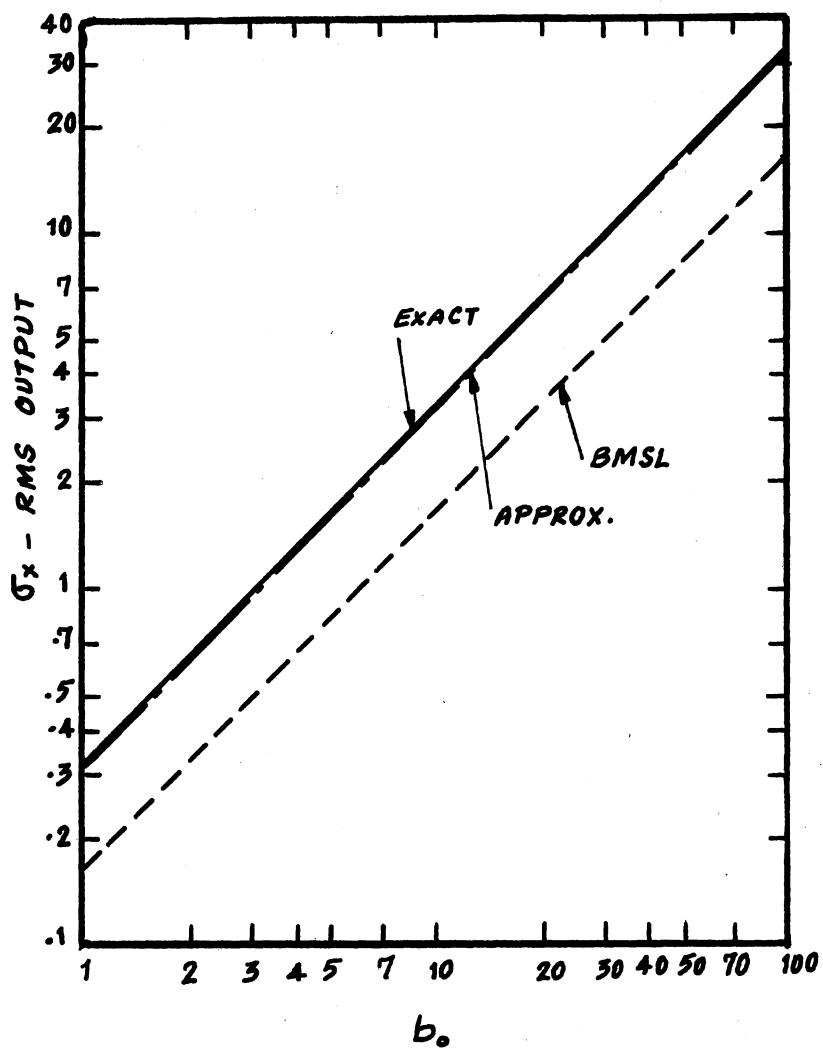


Figure 30. Input-Output Curves. First Order Non-Zero Mean System with White Noise Input- Diode Nonlinearity

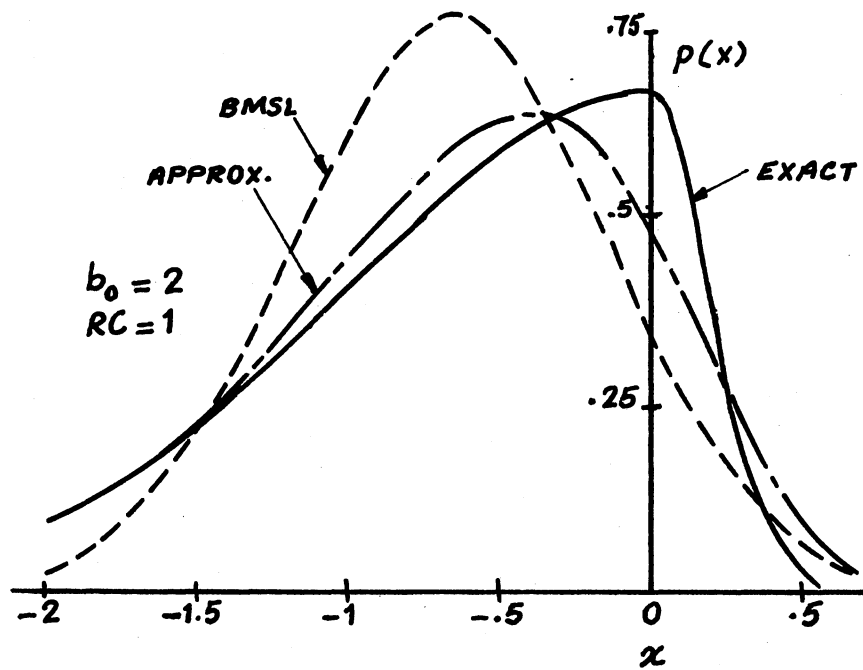


Figure 31. Amplitude Probability Density Functions of Output Signal in First Order Non-Zero Mean System with White Noise Input- Diode Nonlinearity

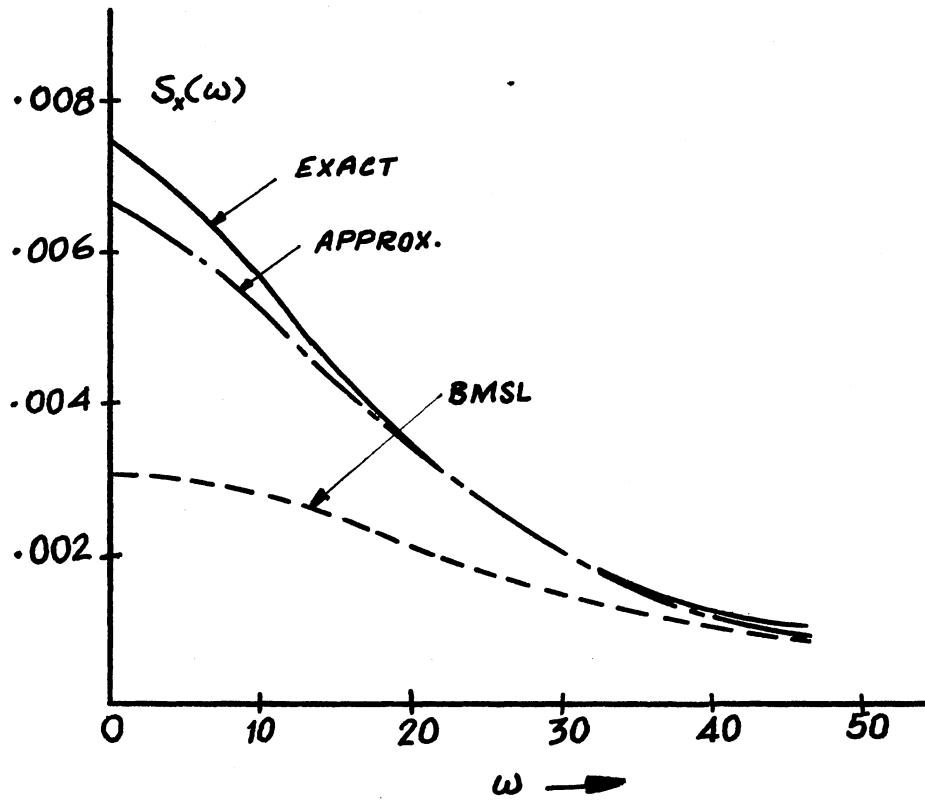


Figure 32. Spectral Density Functions of Output Signal in Non-Zero Mean First Order System with White Noise Input-Diode Nonlinearity

The BMSL gives the same expressions with $c = I_1/\sigma_x$. The spectral density functions are shown in Figure 32. The exact spectral density function is also shown for comparison.

Upon examination of these results, the following observations are noted.

1. The approximate method developed in Chapter III is consistently more accurate than the BMSL. The root-mean-square values obtained by this method, for the first order system, involve errors of less than five per cent when only one correction term is included in the expansion of the probability density. For the second order system the predicted results practically coincide with measured data.

2. The results obtained by the BMSL involve errors ranging from five to 20 per cent. Accuracy of the method is excellent for rms error signals which are of the order of magnitude of the rms inputs. This occurs mostly for the wide-band inputs. When σ_x is much smaller than σ_z the accuracy decreases. The limiter nonlinearity may still give good accuracy in this case if the element is not driven to saturation. So, in addition to the order of magnitude of the rms error the distortion effects must also be considerable for the accuracy to decrease appreciably.

The effect of distortion is manifested in the density function curves and in the spectral density curves. However, it is observed that in cases where the histograms are quite different from the normal curves having the same mean-square values, the root-mean-square values have been predicted within 10 per cent of the measured values. This indicates that the root-mean-square values of $x(t)$ are not very sensitive to the distribution of the signal. This is in favor of the BMSL.

3. Both the approximate method and the BMSL give better results for the second order system than for the first order system, under the same input conditions. The methods are expected to give better results for higher order systems. This is due to the severe attenuation of the distortion effects by the system dynamics and the normalization of the signal at the input of the nonlinearity.

4. The accuracy of the BMSL is less for non-zero-mean systems than for zero-mean systems. This is due to the fact that the BMSL does not take into account any skewness in the distribution. It assumes a symmetric (about the mean) distribution. On the other hand, the approximate method is as valid for these systems as for the zero-mean systems. This is due to the fact that the skewness is accounted for by the λ_3 term in the probability density expansion. Moreover, the λ_3^2 term in this expansion accounts for any additional excess in the distribution. This term when added to the λ_4 term may even improve the accuracy of the method. The improvement depends on the relative magnitude of this additional term.

The approximate method has thus been found to be of sufficient accuracy for practical purposes when statistics up to the fourth order only are employed in the probability density expansion. The accuracy of the BMSL can therefore be checked by this approximate method. Once the BMSL is found to be of sufficient accuracy for a particular problem analysis proceeds according to linear theory using the linearized model. Otherwise, either the linearized model is improved and the accuracy reassessed or the method is rejected as a means of analysis. In either case, the BMSL can be considered as a first step towards a more thorough

analysis by other elaborate (experimental) procedures. This is still a great help in many instances.

CHAPTER V

APPLICABILITY AND DESIGN APPLICATIONS OF THE BMSL

Introduction

A method of assessing the accuracy of the BMSL, in a particular application, is now available. In this chapter we proceed to investigate the applicability of the BMSL based on information gained about its accuracy by the approximate method. A modified version of the MSL is described for use in exceptional cases when the BMSL is in error or when the Markov assumption of Chapter III is not valid. The application of the BMSL to the design problem is also outlined.

Applicability of the BMSL

It is desirable, from the standpoint of the design engineer, to have a guide that indicates the sort of feedback systems to which the BMSL may be applied with confidence.

It has been shown experimentally that the accuracy of the BMSL decreases whenever the effects of nonlinear distortions are large. The accurate computation of these effects is then the basis for testing the applicability of the method. A coarse estimate of these effects may also help as a first step, rule of thumb, towards a more accurate assessment of the applicability of the method.

When $x(t)$ is normal, this means that the distortion effects have been greatly attenuated by the system inertias. The BMSL has been found to be quite satisfactory in this case. The inaccuracies, which are mainly due to unfiltered distortion, can be estimated.

When $x(t)$ deviates considerably from the normal, this means that the distortion effects are large and the BMSL will involve greater errors. Calculation of the distortion effects, in this case, on the basis of a normal $x(t)$ does not reflect the true process which is taking place in the system. Smith (1966) used such calculations as a "confidence test", that is, if a large amount of distortion is found on this basis the BMSL is suspected to yield inaccurate results; otherwise, it is applicable. While this argument is not unreasonable, it cannot be proved. A more accurate assessment of the actual distortion effects present is still needed. The method of calculating the correlation functions (and therefore the spectral densities) described in Chapter III provides such accurate assessment. The method is not based on any assumptions concerning the distribution of $x(t)$. It takes the nonlinearity into consideration and it can be made arbitrarily accurate.

Nonlinear distortion effects have been studied by many workers, both in the open-loop and closed-loop configurations (West, Douce and Leary, 1960) (Pyatnitskii, 1960) (Smith, 1966). All the investigations have been carried out with the assumption that $x(t)$ is normal. In most of these works, the distortion effects have been approximated by a white noise component whose intensity is equal to the amount of distortion at zero frequency. This approximation is therefore valid only at low frequencies. Pyatnitskii's result is a more true assessment of the distortion effects on this basis and his expressions are amenable to

computations. Calculations based on the method of Chapter III have been carried out in Chapter IV. The procedure is quite simple and applies to systems of any order and containing any type of nonlinear characteristics. This is quite important if one realizes that exact calculations have been possible only for first order systems with piecewise-linear nonlinearities, and with great difficulty.

From the preceding, it is observed that the assumption that $x(t)$ is normal has been the only limitation which prevented the MSL from "turning" into a precise method. It may be useful to study the implications of this assumption.

The distribution at the output of the linear part of the system is essentially given by the frequency characteristics of the linear part and by the frequency characteristics of the signal at the input of the nonlinear element. The linear part of a feedback system can be regarded as a lowpass filter with a passband over the range $0 \leq \omega \leq \omega_c$, where ω_c is the cutoff frequency. The relationship between the passband of the linear part of the system and the band of effective frequencies in the spectrum of the input signal can take on several forms.

1. A wide-band signal. The highest frequency of the signal is much greater than ω_c . The output of the system will be normalized and $x(t)$ will have a normal distribution. The BMSL can be safely applied in this case.

2. A narrow-band signal. Here we distinguish among three cases. The first case is that when the spectral density of the signal is appreciable only at low frequencies. In this case, the system as a whole may be considered lagless. The problem is reduced to that of a lagless nonlinear transformation which is not difficult to handle. Here, $x(t)$ is

not normal. The second case is that when the frequency band of the input signal lies above the passband of the linear part of the system. In this case, the system turns out to be open and may be handled without much difficulty. The third case is when the frequency band of the input signal lies within the passband of the linear part of the system. In this case, if $z(t)$ is a stationary Gaussian process, the BMSL may be applied together with the method of harmonic linearization.

A Modified MSL

The assumption of a normal $x(t)$ in the BMSL is obviously a convenience which made it a simple problem to calculate the static gains of the equivalent linear model. Neither $x(t)$ nor $z(t)$ have to be Gaussian for a linear model to be formulated. This is clear from the general formulation of the MSL in Chapter II. The density function $p(x)$ needed for the calculation of the static gains may be taken as the Edgeworth expansion, again. It is straightforward then to calculate the static gains on this basis. This will reflect more accurately the actual nature of the distribution of $x(t)$. Thus we write,

$$y(t) = \phi_0 + k_1 x^0 \quad (5.1)$$

where

$$\phi_0 = \int_{-\infty}^{\infty} \phi(x) p(x) dx \quad (5.2)$$

$$k_1^{(1)} = \left[\frac{\int_{-\infty}^{\infty} \phi^2(x) p(x) dx - \phi_0^2}{\sigma_x^2} \right]^{1/2} \quad (5.3)$$

$$k_1^{(2)} = \frac{1}{\sigma_x^2} \int_{-\infty}^{\infty} \phi(x) x^2 p(x) dx \quad (5.4)$$

and

$$p(x) = p_0(x) \sum_E b_k H_k(x^0/\sigma_x) \quad (5.5)$$

where \sum_E denotes the summation of terms in the Edgeworth series manner, b_k are the coefficients in the Edgeworth expansion.

Substituting (5.5) in (5.2), (5.3) and (5.4), we have

$$\phi_0 = \phi_{00}(m_x, \sigma_x) + \sum_E b_k \phi_{0k}(m_x, \sigma_x) \quad (5.60)$$

where

$$\phi_{00}(m_x, \sigma_x) = \int_{-\infty}^{\infty} \phi(x) p_0(x) dx = a_0 \quad (5.7)$$

corresponding to ϕ_0 of the BMSL, and

$$\phi_{0k}(m_x, \sigma_x) = \int_{-\infty}^{\infty} \phi(x) H_k(x^0/\sigma_x) p_0(x) dx = a_k \quad (5.8)$$

Similarly,

$$k_1^{(1)} = k_{10}^{(1)}(m_x, \sigma_x) + \sum_E b_k k_{1k}^{(1)}(m_x, \sigma_x) \quad (5.9)$$

where

$$k_{10}^{(1)}(m_x, \sigma_x) = \frac{\int_{-\infty}^{\infty} \phi^2(x) p_0(x) dx - \phi_{00}^2}{\sigma_x^2} \quad (5.10)$$

$$k_{1k}^{(1)}(m_x, \sigma_x) = \frac{1}{2} k_{10}^{(1)}(m_x, \sigma_x) \left\{ \frac{- \int_{-\infty}^{\infty} x^2 H_k(x/\sigma_x) p_0(x) dx}{\sigma_x^2} + \frac{\int_{-\infty}^{\infty} \phi^2(x) H_k(x/\sigma_x) p_0(x) dx - 2\phi_{00}\phi_{0k}}{\int_{-\infty}^{\infty} \phi^2(x) p_0(x) dx} \right\}^{\frac{1}{2}} \quad (5.11)$$

and

$$k_1^{(2)} = k_{10}^{(2)}(m_x, \sigma_x) + \sum_E b_k k_{1k}^{(2)}(m_x, \sigma_x) \quad (5.12)$$

where

$$k_{10}^{(2)}(m_x, \sigma_x) = 1/\sigma_x^2 \int_{-\infty}^{\infty} \phi(x) x^0 p_0(x) dx = a_1/\sigma_x \quad (5.13)$$

$$k_{1k}^{(2)}(m_x, \sigma_x) = 1/\sigma_x^2 \int_{-\infty}^{\infty} \phi(x) x^0 H_k(x/\sigma_x) p_0(x) dx - k_{10}^{(2)}/\sigma_x^2 \int_{-\infty}^{\infty} x^2 H_k(x/\sigma_x) p_0(x) dx \quad (5.14)$$

Considering the mean-square error criterion and using the Edgeworth series expansion for $p(x)$, we have

$$\phi_0 = a_0 + \gamma_1/3! a_3 + \gamma_2/4! a_4 + \gamma_1^2/72 a_6 \quad (5.15)$$

and

$$k_1^{(2)} = 1/\sigma_x \{a_1 + \gamma_1/3!(a_4 + 3a_2) + \gamma_4/4! (a_5 + 4a_3) + \gamma_1^2/72 (a_7 + 6a_5)\} \quad (5.16)$$

Note that in this formulation two more parameters, γ_1 and γ_2 , of the distribution have been used, which must be determined in addition to m_x and σ_x in order to determine the static gains of the linear model.

Consider again Figure 4. The system equations are repeated here for reference.

$$Q(p) Y(t) + R(p) X(t) = S(p) Z(t) \quad (5.17)$$

$$Y = \phi(X)$$

The mean values are related by

$$Q(0) m_y + R(0) m_x = S(0) m_z \quad m_y = \phi_0 \quad (5.18)$$

and the centered components satisfy the relation

$$Q(p) Y^0(t) + R(p) X^0(t) = S(p) Z^0(t) \quad Y^0 = k_1 X^0 \quad (5.19)$$

From (5.19) it is easy to write the equations for σ_x , γ_1 and γ_2 . They are

$$\sigma_x^2 = 1/2\pi \int_{-\infty}^{\infty} |G(j\omega)|^2 S_z(\omega) d\omega \quad (5.20)$$

$$\gamma_1 = \lambda_3 / \sigma_x^3 = \frac{1}{(2\pi)^2 \sigma_x^3} \int_{-\infty}^{\infty} G(j\omega_1) G(j\omega_2) G(-j\omega_1, -j\omega_2) S_z(\omega_1, \omega_2) d\omega_1 d\omega_2 \quad (5.21)$$

$$\gamma_2 = \lambda_4 / \sigma_x^4 = -3 + \frac{1}{(2\pi)^3 \sigma_x^4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(j\omega_1) G(j\omega_2) G(j\omega_3) G(-j\omega_1, -j\omega_2, -j\omega_3) S_z(\omega_1, \omega_2, \omega_3) d\omega_1 d\omega_2 d\omega_3 \quad (5.22)$$

$$G(p) = \frac{S(p)}{Q(p) k_1 + R(p)}$$

The integral (5.20) is easy to evaluate. Integrals (5.21) and (5.22) are more difficult especially with high order systems. As in the BMSL, the parameters are determined by iterative procedures.

The formulation just described is useful in two ways. First, if $x(t)$ is Gaussian and the accuracy of the BMSL is found to be unsatisfactory, for a certain problem, then (5.15) and (5.16) provide accurate expressions for the statistical gains. In this case, the parameter values obtained by the approximate method are used in evaluating the static gains from (5.15) and (5.16). Second, if $Z(t)$ is not a Gaussian process, then the above formulation becomes the only practical way of checking and, if necessary, improving the accuracy of the BMSL.

Finally, we point out that if the above formulation had to be resorted to due to gross inaccuracies in the BMSL, then it is recommended that a dynamic model be used, in this case, for the nonlinearity.

Design Applications

So far we have been concerned with the analysis problem. We now turn to the synthesis, or design, problem. Here, the characteristics of the input signal are given. The desired characteristic of the output signal is also given. It is required to find the form of the compensation network, out of a class of compensation networks, which will make the transformation approximate the desired output optimally (in some given sense). We consider in what follows two problems to the solution of which the BMSL can be useful.

Optimum Linear Compensation

One of the classical design problems is the Wiener filter problem. The extension of the solution of this problem to the design of optimum linear compensation for a system with fixed linear and nonlinear parts is almost trivial using the statistical linearization of the nonlinear characteristic. Smith (1966) described and illustrated the appropriate procedure. The conditions of the problem are as follows.

1. The system responds to an input which consists of a signal and noise, both of which are considered stationary Gaussian random processes with given power spectral density function.

2. The desired output of the system is a function derived from the signal component of the input by some linear invariant operation, not necessarily physically realizable.

3. The criterion for optimality is minimum steady-state mean-square error, which is the difference between the actual output of the system and the desired output.

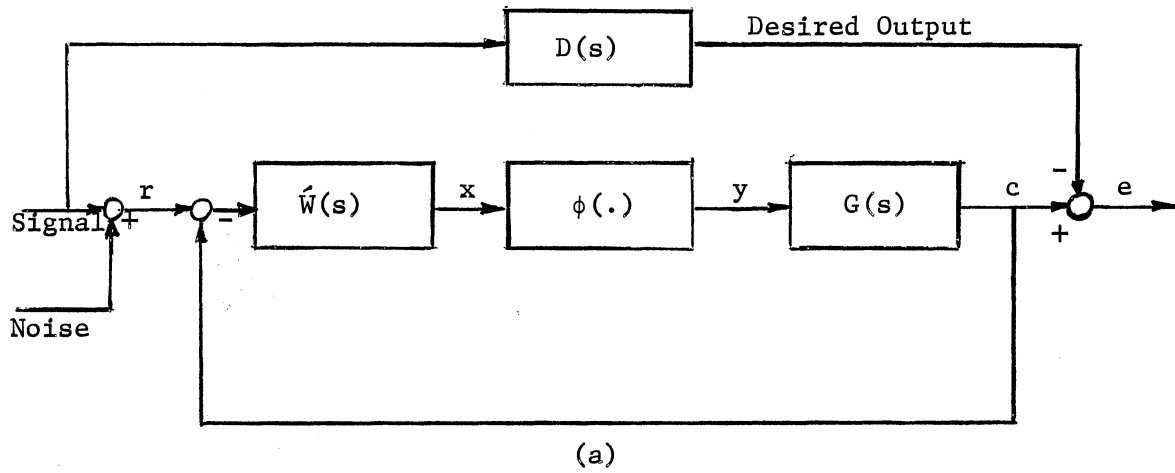
4. The system contains an invariant stable linear part and a static single-valued nonlinear part which are considered fixed. The optimum linear, physically realizable compensation is to be designed for the system.

The system configuration is shown in Figure 33a. $\phi(\cdot)$ and $G(s)$ characterize the fixed nonlinear and linear parts of the system. $\hat{W}(s)$ is the compensation to be designed. Figure 33b is an open-loop equivalent configuration of the original system. This is the configuration to which the familiar filter theory is applicable were it not for the nonlinear element in the loop.

The nonlinearity is characterized by the statistical gain k_1 which is a function of m_x and σ_x . Since m_x and σ_x are unknown until $W(s)$ is determined, which in turn depends on k_1 , the constrained-optimum design technique given by Newton, Gould and Kaisar (1957) provides the appropriate procedure for the solution of the problem. Assuming a value for m_x and σ_x , k_1 is determined and the constrained-optimum design solution gives $W(s)$. The solution is repeated for different values of m_x and σ_x . Every time the mean-square error is noted and the solution which gives the least mean-square error is the optimum.

Nonstationary systems are more conveniently treated in the state space formulation. Having a statistical linearization description of the nonlinear system, linear filter theory in the form of the Kalman filter is employed directly to obtain approximations of the first- and second-order statistics which are in this case time varying.

The procedure described above may be laborious but it requires no new concepts and provides a quite satisfying solution to a rather complex



$$W(s) = \frac{\hat{W}(s)}{1 + \hat{W}(s) k G(s)}$$

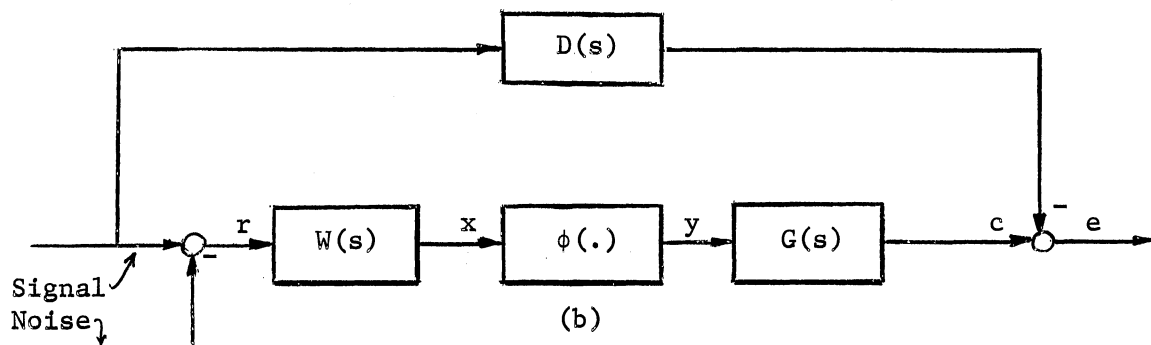


Figure 33. (a) Block Diagram of Original System for the Linear Compensation Problem
 (b) Equivalent Open-Loop Configuration

problem. It provides a framework for practical design by trial-and-error methods particularly for nonlinear systems, where past experience with linear systems is a doubtful guide, the procedure is of considerable assistance to the engineer.

Optimum Nonlinear Compensation

A somewhat different problem is presented by a system having a given fixed part which is just asked to follow a random input as well as possible in the least mean-square-error sense. The system should respond optimally to inputs of various levels. For each input level there will be a different optimum value of forward gain if the system fixed part contains a nonlinearity. What is needed then is a forward gain which depends on its input rms value, that is, a nonlinear element. Using the statistical linearization theory, it is possible to design a nonlinear compensator in an optimal way.

The configuration of a typical system is shown in Figure 34. The input signal is a stationary process with a power spectral density of prescribed shape but with a variable level. $\phi_2(\cdot)$ is to be designed to yield a minimum mean-square value of the error $x_1(t)$ for a range of rms values of the input. The solution of this problem by the BMSL is straightforward, though tedious. Note that the output of the $\phi_2(\cdot)$ is fed to the input of $\phi_1(\cdot)$ without the benefit of the heavy filtering by the system dynamics. However, in many cases, $\phi_1(\cdot)$ is not strongly dependent on the distribution of the input, which in this case will not be normal. Here, we have in mind limiter-type characteristics which are commonly used for $\phi_1(\cdot)$.

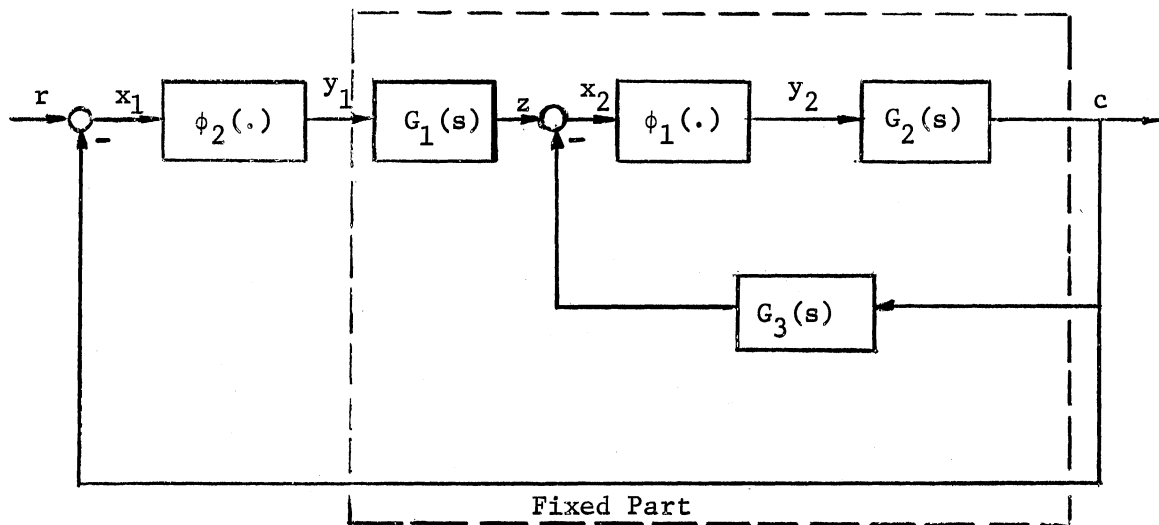


Figure 34. System Configuration for Nonlinear Compensation Problem

Other Applications

The impressive feature of the statistical linearization theory is the breadth of the range of complex and practically significant system situations in which this theory can be applied with reasonable facility. It is difficult to list all the variations of problems that can be treated. An important problem which occurs in tracking applications where the input to the nonlinear system consists of a systematic signal corrupted by a wideband random noise can also be treated by this theory. Smith described and illustrated the procedure for this problem. Evlanov and Kazakov (1972) used the theory for identification of nonlinear elements in multidimensional systems.

CHAPTER VI

CONCLUSION

Conclusions

The objective of this investigation was stated in Chapter I to be the development of a method of assessing the accuracy of the BMSL. The applicability of the BMSL was then to be investigated on the light of information, gained about its accuracy, from this method. The method was to be easy to apply and at the same time accurate enough to be taken as a reference for judging the accuracy of the BMSL. Such a method was developed in Chapter III and was examined in Chapter IV. The method has great generality and, although it can be made arbitrarily accurate, at the expense of added labor, it has been found sufficiently accurate if only statistics up to the fourth order are used in characterizing the probability density function of the process involved. It is recognized that the BMSL is the simplest and most practical method of nonlinear systems analysis and design now available. It is also expected to remain so for quite some time. The main need is then to ascertain that the results obtained by this method are satisfactorily accurate. This is mainly what the approximate method developed in Chapter III was developed for. Being accurate enough, this approximate method was used in Chapter IV to investigate the various aspects of the BMSL.

Following are the conclusions arrived at from this investigation.

1. It is a known result in nonlinear transformations of random processes that their statistics transform linearly. This is true whether the statistics are the moments functions, the semi-invariants functions or the quasi-moments functions. The coefficients of the linear transformation for a particular order statistic, however, may depend on input statistics of higher orders. This is characteristic of nonlinear transformations of random processes. In the BMSL, the statistics transform linearly too. However, in this case, the coefficients of the linear transformations are dependent on the input's mean value and standard deviation only. This is due to the assumption that the input to the non-linearity is Gaussian. Thus, the BMSL is truly a well-based theory, contrary to the belief that it lacked such a basis. It is really a first approximation, and very often a good one, to the actual nonlinear transformation.

2. Any improvement on the BMSL should be in such a way as to reflect the dependence of the statistics of the transformation on the higher order statistics of the input to the nonlinear element. This is what was done in developing the modified version of the MSL in Chapter V.

3. The approximate method developed in Chapter III has been found to possess adequate simplicity of application, generality and accuracy. The method can therefore be used as a test for the accuracy of the BMSL in an effort to determine the latter's applicability to a particular problem.

4. Theoretical as well as experimental analysis revealed that the BMSL is accurate for practical purposes when the basic assumption of a Gaussian input to the nonlinear element holds. This occurs when nonlinear distortion effects are either originally small or have been attenuated

by the plant dynamics. The method involved errors in excess of 20 per cent when the assumption does not hold. This occurs when adequate filtering of the distortion effects does not take place before the signal is fed into the nonlinear element. In most modern dynamic systems, linear compensations are often inserted in the loop to improve system performance. These compensators often have lowpass characteristics. This may help the normalization process of the signal fed to the nonlinear element and increases the chances for acceptable results from the BMSL.

5. Limited evidence indicates the preference of the linear model based on the minimum mean-square error criterion of statistical equivalence. It has already been observed that the method of analysis based on this model is a special case of the more general method developed in Chapter III and that if $x(t)$ is truly Gaussian then both methods coincide.

6. The BMSL may be applied with confidence to systems subjected to wideband inputs and where the range of useful frequencies in the input signal spectrum extends well above the effective bandwidth of the system under consideration. In these cases the method gave second order statistics of five to 10 per cent accuracy.

7. The accuracy of the BMSL is expected to decrease when low order systems under narrow-band inputs are considered. It has been found that when the input signals have power spectra that are different from zero only for low frequencies the basic assumption of a normalized signal does not hold. The same applies to input signals with considerable power spectra for frequencies within the passband of the system. In these cases, the BMSL is only a rough approximation which can still be quite useful. However, more accurate analysis of the systems in such cases is not

difficult since the problems reduce to lagless transformations which can be handled by direct nonlinear transformation methods.

8. The BMSL has been found useful in design problems like the Wiener filter problem, for stationary systems, and the Kalman filter problem for time-varying systems. The conventional description of the input processes in such problems is that of a narrow-band signal corrupted with a wideband random noise. The BMSL has been found to give good results in these cases.

9. The BMSL has been found to be less accurate when applied to non-zero-mean systems than when applied to zero-mean systems of the same order and under the same input conditions. This is again due to the assumption of a Gaussian signal at the input of the nonlinear element. The distribution of such a signal is symmetric (about the mean). The BMSL thus neglects any skewness that may exist in the actual distribution of the signal. The decrease in accuracy, however, depends on the relative magnitude of the skewness effects. These are usually small in systems common in engineering applications.

10. The approximate method developed in this work is valid for skew distributions just as it is valid for symmetric distributions and with the same accuracy. This promotes its use as a valid method of checking the results obtained by the BMSL in all cases and thus fulfills the objective of this research.

Problems for Further Investigation

1. The investigation in this work has been carried out using zero-memory nonlinearities. The application of the BMSL to nonlinearities with memory is possible on the basis of the general theory of

multi-dimensional nonlinearities described in Chapter II. Here, the multi-valued nonlinearity is reduced to a multi-dimensional single-valued characteristic which can be handled by the BMSL. The approximate method of Chapter III is extended to this case by the use of a multidimensional Edgeworth expansion for the probability density as usual. Investigation of this problem will be of great help in extending the knowledge concerning the behavior of the BMSL in these cases. The same remark also applied to nonlinearities which depend on random parameters.

2. In Chapter III the assumption of a Markov process was made, for the state vector of the nonlinear system, which made it possible to develop the method of nonlinear analysis. Although the assumption has been found to be a good approximation in the majority of problems occurring in practice, there are still few instances in which the processes involved are quite different from a Markov process. This suggests a need for the development of accurate methods of analysis for such problems. Just as the method developed here, these methods, if developed, will only serve as a device for checking the accuracy of the MSL in the BMSL form or as modified in Chapter V. It will be very difficult to use a method of nonlinear analysis of any generality in the study of system performance or in designing systems containing nonlinear elements. However, the information obtained from these methods will be of great assistance in establishing the accuracy of the MSL which may then be used without difficulty for detailed analysis or design of a system. The density function representation used here may be used in conjunction with the conventional methods of functional analysis, for this purpose. However, it is recommended that the functional formulation be made with respect to the signal statistics and not the signals themselves. Limited investigation

by the author indicates that such a development is possible with the risk that one may become involved with multiple integrals. The density function representation in an Edgeworth expansion will definitely simplify the problem somewhat. A further simplification might be obtained by using an orthogonal expansion for the nonlinear characteristics. An orthogonal expansion will represent the nonlinear characteristic satisfactorily only if enough terms are taken in the expansion. The number of terms needed will depend on the level of the input to the system. Fewer terms are needed for a high level input than for a small random input. However, working with the statistics of the signal, instead of the signal itself, may make the number of terms in the expansion immaterial since all but a few of these terms will cancel due to the orthogonality of the expansion. If used with the Edgeworth representation of the probability density, further simplification may result. A thorough investigation of the problem is however necessary before any commitment is made as to its potential success.

3. Finally, and very importantly, a need exists for methods of rapid computations. It has been observed in the development of a modified version of the MSL that although that version is an improvement on the BMSL, yet one has to evaluate multiple integrals in order to determine the parameters needed. For systems with high dimensions, this is still a difficult task. A greater need for such methods will be encountered in the development of the methods proposed in (2) above.

A SELECTED BIBLIOGRAPHY

Barrett, J.F. and J.F. Coales.

- 1955 "An Introduction to the Analysis of Nonlinear Control Systems with Random Inputs." Proc. IEE Vol. C-103, No. 3 (November), 190-199.

Barton, D.E. and K.E. Dennis.

- 1952 "The Conditions Under Which Gram-Charlier and Edgeworth Curves Are Positive Definite and Unimodal." Biometrika, 39, 425-427.

Bharucha-Reid, A.T.

- 1960 Elements of the Theory of Markov Processes and Their Applications. New York: McGraw Hill Book Co., Inc.

Booton, R.

- 1953 "The Analysis of Nonlinear Control Systems with Random Inputs." Proceedings of the Symposium of Nonlinear Circuit Analysis. Brooklyn, N.Y.: Polytechnic Institute of Brooklyn.

Brown, J.A.

- 1964 "Linearization of Backlash in Control Systems with Random Inputs." Proc. IEE, Vol. 111, No. 2 (February), 379-386.

Cramer, H.

- 1946 Mathematical Methods of Statistics. Princeton: Princeton University Press.

Cumming, I.G.

- 1967 "Derivation of the Moments of A Continuous Stochastic System." Int. J. Control, Vol. 5, No. 1, 85-90

Doob, J.L.

- 1953 Stochastic Processes. New York: John Wiley & Sons, Inc.

Douce, J.L. and P.D. Roberts.

- 1963 "Effect of Distortion on the Performance of Nonlinear Control Systems." Proc. IEE, Vol. 110, No. 8, 1497-1502

Evlanov, L.G. and I.E. Kazakov.

- 1969 "Statistical Investigation of Nonlinear Oscillatory Systems in Steady-State Modes." Avtom. i Telem., Eng. Trans., No. 12, 1902-1909.

Evlanov, L.G. and I.E. Kazakov.

- 1970 "Statistical Investigation of Nonlinear Auto-Oscillatory Systems with Non-Steady-State Modes." Avtom. i Telem., Eng. Trans., No. 1, 37-44.

- 1972 "Correlation Analysis of Systems with Stochastic Inertialess Elements." Avtom. i Telem., Eng. Trans., No. 10, 1606-1610.

Kazakov, I.E.

- 1954 "Approximate Method for the Statistical Study of Nonlinear Systems." Trans. Zhukovsky All-Union Higher Aviation Inst., 394.

- 1965 "Generalization of the Method of Statistical Linearization to Multidimensional Systems." Avtom. i Telem., Eng. Trans., Vol. 26, No. 7, 1201-1206.

Kendall, M. and A. Stuart.

- 1969 The Advanced Theory of Statistics. Vol. 1 (Distribution Theory). 3rd. ed. London: Charles Griffin and Co. Ltd.

Kolovskii, M.Z.

- 1966 "Estimating the Accuracy of Solutions Obtained by the Method of Statistical Linearization." Avtom. i Telem., Eng. Trans. Vol. 27, 1692-1701.

Kuznetsov, R.L., V.I. Stratonovich and V.I. Tikhonov.

- 1953 "The Passage of Random Functions Through Nonlinear Systems." Avtom. i Telem., Vol. 14, No. 4, 375-385.

Leland, H.R.

- 1960 "Input-Output Cross-Correlation Functions for Some Memory-Type Nonlinear Systems with Gaussian Inputs." Proc. AIEE, 219-223.

Newton, G.C., L.A. Gould and J.F. Kaiser.

- 1957 Analytic Design for Linear Feedback Controls. New York: John Wiley & Sons, Inc.

Pugachev, V. S.

- 1965 Theory of Random Functions and its Application to Control Problems. London: Pergamon Press.

Pupkov, K.A.

- 1960 "Method of Investigating the Accuracy of Essentially Nonlinear Automatic Control Systems by Means of Equivalent Transfer Functions." Avtom. i Telem., Eng. Trans., Vol. 21, No. 2, 126-140.

Pyatnitskii, G.I.

- 1960 "The Effect of Stationary Random Processes on Automatic Control Systems Containing Essentially Nonlinear Elements." Avtom. i Telem., Eng. Trans., Vol. 21, No. 4, 328-332.

Smith, H.W.

- 1966 Approximate Analysis of Randomly Excited Nonlinear Controls. M.I.T. Monograph No.34. Cambridge, Mass.: The M.I.T. Press.

Stratonovich, R.L.

- 1968 Conditional Markov Processes and Their Application to the Theory of Optimal Control. New York: American Elsevier Publishing Co.

West, J.C., J.L. Douce and B.G. Leary.

- 1960 "Frequency Spectrum Distortion of Random Signals in Nonlinear Feedback Systems." Proc. IEE, Vol. 108C, 259.

Wiener, N.

- 1958 Nonlinear Problems in the Theory of Random Processes. New York: Technology Press and John Wiley & Sons, Inc.

APPENDIX

APPROXIMATE ANALYTIC REPRESENTATION

OF PROBABILITY DENSITY

FUNCTIONS

Introduction

The majority of the frequency distributions encountered in practice possess a high degree of regularity. The forms of the frequency polygons, or histograms, obtained in practice suggest, almost inevitably, that data are approximations to distributions which can be specified by smooth curves and simple mathematical expressions.

From the variety of forms assumed by the frequency distributions in practice, it is evident that a flexible system would be required to describe them all in mathematical terms. Three approaches have been used in the statistical literature for this purpose. The first, due to Karl Pearson, seeks to ascertain a family of curves which will satisfactorily represent practical distributions. The second, due to Bruns, Gram, Charlier and Edgeworth, seeks to represent a given frequency (density) function as a series of derivatives of the normal density function. The third, due to Edgeworth and later writers, seeks a transformation of the variate which will throw the distribution at least approximately into a known form.

Of these three approaches, the second has been found, by the author, to be most promising for use in complicated operations. It has been

found that this approach has the simplicity and, above all, the generality required when the resulting expressions for the approximate density functions are to be used in further computational operations. As distinguished from the other two approaches, the series representation approach is extended to multidimensional distributions in a workable manner. The representation has been found very helpful in statistical systems analysis. It is subjected here to a thorough investigation in a unified form not available before.

Univariate Distributions

Univariate distributions are the most studied distributions in the literature. The following treatment is based on material available in the published literature. It is, however, organized in a manner suitable for the extension to multivariate distributions which are treated under a later subtitle. Few remarks derived from the author's own experience with this type of representation are also included.

Representation by a Series

Let $p(x)$ be the density function which we require to represent analytically, $p_0(x)$ be a "standard" probability density function and $Q_0(x), Q_1(x), \dots, Q_n(x), \dots$ be a set of polynomials in x which are orthogonal with respect to $p_0(x)$, i.e.

$$\int_{-\infty}^{\infty} Q_n(x) Q_m(x) p_0(x) dx = \begin{cases} 0 & , m \neq n \\ 1 & , m = n \end{cases} \quad (\text{A.1})$$

The set of polynomials $Q_n(x)$ which satisfy (A.1) may be found by orthogonalizing the set of step functions $1, x, x^2, \dots$ and using them as

the initial set of functions. This procedure causes Q_n to be a polynomial of degree n in x and Q_0 will be equal to unity,

$$Q_0(x) = 1, \quad Q_n(x) = \sum_{k=0}^n a_{nk} x^k, \quad n = 1, 2, \dots \quad (\text{A.2})$$

We seek a formal series expansion of the probability density $p(x)$ in the form

$$p(x) = p_0(x) \sum_{n=0}^{\infty} C_n Q_n(x) \quad (\text{A.3})$$

where

$$C_n = \int_{-\infty}^{\infty} Q_n(x) p(x) dx, \quad n=0, 1, 2, \dots \quad (\text{A.4})$$

Substituting Q_n from (A.2), we get

$$C_0 = 1, \quad C_n = \sum_{k=0}^n a_{nk} \alpha_k, \quad n=1, 2, \dots \quad (\text{A.5})$$

where α_k is the k th initial moment (moment about $x=0$) of the random variable x whose density function is $p(x)$. Thus if the moments of x are known we can determine the coefficients in (A.3) without difficulty. Of course, the existence of finite values of all moments used is essential.

Before using (A.3) to represent the density function of an arbitrary random variable x it is convenient to normalize x and instead use a variable $u = (x - m_x) / \sigma_x$. This new random variable has a zero mean and a unit dispersion and is subject to the same distribution law as x . Therefore, for the random variable u , we have $\alpha_1 = 0$, $\alpha_2 = 1$ and

$$c_0 = 1, \quad c_1 = a_{10}, \quad c_2 = a_{20} + a_{22}$$

$$c_n = a_{n0} + a_{n2} + \sum_{k=3}^n a_{nk} \alpha_k, \quad n=3,4,5,\dots \quad (\text{A.6})$$

Series (A.3) may be based on different "standard" densities $p_0(x)$. Examples of standard density functions are those of the normal, gamma, and beta distributions. The corresponding polynomials are those of Chebychev-Hermite, Laguerre, and Jacobi, respectively. The choice of $p_0(x)$ depends to a large extent on the nature of the function $p(x)$. Simplicity and accuracy in representing $p(x)$ by the first few terms of (A.3) depend on the apt choice of $p_0(x)$. In engineering applications, it is common to take the normal law as the basis for the expansion (A.3) which, in the standardized form, is written as

$$p_0(u) = \frac{1}{\sqrt{2\pi}} \exp(-u^2/2), \quad (\text{A.7})$$

The polynomials Q_n become the Chebychev-Hermite (Ch-H) polynomials $H_n(u)$.

The one-dimensional Ch-H polynomials are defined by

$$H_n(u) = (-1)^n \exp(u^2/2) \frac{d^n}{du^n} \exp(-u^2/2) \quad (\text{A.8})$$

These polynomials have a few attractive properties, especially the orthogonality property, i.e.

$$\frac{1}{n!} \int_{-\infty}^{\infty} H_m(u) H_n(u) p_0(u) du = \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases}$$

By convention, $H_0(u) = 1$. The next ten polynomials are

$$\begin{aligned}
 H_1(u) &= u \\
 H_2(u) &= u^2 - 1 \\
 H_3(u) &= u^3 - 3u \\
 H_4(u) &= u^4 - 6u^2 + 3 \\
 H_5(u) &= u^5 - 10u^3 + 15u \\
 H_6(u) &= u^6 - 15u^4 + 45u^2 - 15 \\
 H_7(u) &= u^7 - 21u^5 + 105u^3 - 105u \\
 H_8(u) &= u^8 - 28u^6 + 210u^4 - 420u^2 + 105 \\
 H_9(u) &= u^9 - 36u^7 + 378u^5 - 1260u^3 + 945u \\
 H_{10}(u) &= u^{10} - 45u^8 + 630u^6 - 3150u^4 + 4725u^2 - 945
 \end{aligned}$$

In this case, formulas (A.6) yield

$$\begin{aligned}
 C_0 &= 1, C_1 = C_2 = 0, C_3 = \alpha_3/3!, C_4 = (\alpha_4 - 3)/4! \\
 C_5 &= (\alpha_5 - 10\alpha_3)/5!, C_6 = (\alpha_6 - 10\alpha_4 + 30)/6! \quad (A.9a)
 \end{aligned}$$

and so on.

In terms of Thiele's semi-invariants (cumulants), these become

$$\begin{aligned}
 C_0 &= 1, C_1 = C_2 = 0, C_3 = \lambda_3/3!, C_4 = \lambda_4/4! \\
 C_5 &= \lambda_5/5!, C_6 = (\lambda_6 - 10\lambda_3^2)/6! \quad (A.9b)
 \end{aligned}$$

and so on.

Series (A.3) becomes

$$p(u) = \frac{1}{\sqrt{2\pi}} \exp(-u^2/2) \left\{ 1 + \sum_{n=3}^{\infty} C_n H_n(u) \right\} \quad (\text{A.10})$$

This is the Gram-Charlier (GC) type-A series. It is an orthogonal expansion for $p(x)$.

Since for any random variable x the k th moment of the standardized variable u is μ_k/μ_x^k , where μ_k is the k th order central moment (moment about the mean) of the random variable x and σ_x is its standard deviation, it follows that for any arbitrary random variable x , having a standard deviation σ_x and a nonzero mean m_x , the values of α_k in (A.6) and (A.9a) must be replaced by the corresponding quantities μ_k/σ_x^k . Similarly, the λ_k in (A.9b) are replaced by λ_k/σ_x^k . As a result, formulas (A.9a) become

$$\begin{aligned} c_0 &= 1, \quad c_1 = c_2 = 0, \quad c_3 = \frac{1}{3!} \mu_3/\sigma^3 \\ c_4 &= \frac{1}{4!} (\mu_4/\sigma^4 - 3), \quad c_5 = \frac{1}{5!} (\mu_5/\sigma^5 - 10\mu_3/\sigma^3) \\ c_6 &= \frac{1}{6!} (\mu_6/\sigma^6 - 15\mu_4/\sigma^4 + 30) \\ c_7 &= \frac{1}{7!} (\mu_7/\sigma^7 - 21\mu_5/\sigma^5 + 105\mu_3/\sigma^3) \\ c_8 &= \frac{1}{8!} (\mu_8/\sigma^8 - 28\mu_6/\sigma^6 + 210\mu_4/\sigma^4 - 315) \\ c_9 &= \frac{1}{9!} (\mu_9/\sigma^9 - 36\mu_7/\sigma^7 + 378\mu_5/\sigma^5 - 1260\mu_3/\sigma^3) \\ c_{10} &= \frac{1}{10!} (\mu_{10}/\sigma^{10} - 45\mu_8/\sigma^8 + 630\mu_6/\sigma^6 \\ &\quad - 3150\mu_4/\sigma^4 + 3780) \end{aligned} \quad (\text{A.11a})$$

and (A.9b) and (A.10) become

$$\begin{aligned}
c_0 &= 1 \quad , \quad c_1 = c_2 = 0 \quad , \quad c_3 = \frac{1}{3!} \lambda_3 / \sigma^3 \quad , \quad c_4 = \frac{1}{4!} \lambda_4 / \sigma^4 \\
c_5 &= \frac{1}{5!} \lambda_5 / \sigma^5 \quad , \quad c_6 = \frac{1}{6!} (\lambda_6 / \sigma^6 + 10 \lambda_3^2 / \sigma^6) \\
c_7 &= \frac{1}{7!} (\lambda_7 / \sigma^7 + 35 \lambda_3 \lambda_4 / \sigma^7) \\
c_8 &= \frac{1}{8!} (\lambda_8 / \sigma^8 + 56 \lambda_3 \lambda_5 / \sigma^8 + 35 \lambda_4^2 / \sigma^8) \\
c_9 &= \frac{1}{9!} (\lambda_9 / \sigma^9 + 84 \lambda_3 \lambda_6 / \sigma^9 + 126 \lambda_3 \lambda_6 / \sigma^9 + 280 \lambda_3^3 / \sigma^9) \\
c_{10} &= \frac{1}{10!} (\lambda_{10} / \sigma^{10} + 120 \lambda_3 \lambda_7 / \sigma^{10} + 210 \lambda_4 \lambda_6 / \sigma^{10} + \\
&\quad 126 \lambda_5^2 / \sigma^{10} + 2100 \lambda_3^2 \lambda_4 / \sigma^{10}) \tag{A.11b}
\end{aligned}$$

$$p(x) = p_0(x) \left[1 + \sum_{n=3}^{\infty} c_n H_n \left(\frac{x - m_x}{\sigma_x} \right) \right] \tag{A.12}$$

where

$$p_0(x) = \frac{1}{\sigma_x \sqrt{2\pi}} \exp \left[-1/2 \left(\frac{x - m_x}{\sigma_x} \right)^2 \right] \tag{A.13}$$

and C_n are those in (A.11). C_3 and C_4 have a special significance in that they account for most of the skewness and excess of $p(x)$ as compared to $p_0(x)$. For this purpose we introduce the skewness coefficient

$$\gamma_1 = \mu_3 / \sigma_x^3 = \lambda_3 / \sigma_x^3 \tag{A.14}$$

and the excess coefficient

$$\gamma_2 = \mu_4/\sigma_x^4 - 3 = \lambda_4/\sigma_x^4 \quad (\text{A.15})$$

All random variables - continuous, discrete, or mixed - may be represented by the expansion (A.3). Naturally, to obtain a good approximation one must select the appropriate standard density function $p_0(x)$.

But one must not forget that the above representation is only a formal one, and we do not know that it is valid. To prove its validity we must first show that the series is convergent and secondly that it actually represents $p(x)$, for all values of x .

This is by no means a simple task and it cannot be done by elementary methods. A Russian mathematician, Vera Myller-Lebedeff has, however, given an elegant solution by means of some well-known theorems from the Fredholm integral equations. She has proved, among other things, the following criterion: "Every function $p(u)$ which together with its first two derivatives is finite and continuous in the interval $-\infty$ to $+\infty$ and which vanishes together with its derivatives at $x = \pm\infty$ can be developed into an infinite series of the form

$$p(u) = \sum_{i=0}^{\infty} C_i H_i(u) \exp(-u^2/2) "$$

Under certain conditions the series (A.3) converges to $p(x)$ and yields an analytic representation of arbitrary accuracy for $p(x)$. On the other hand, it can be shown by examples (Cramer, 1946 p. 258) that, if these conditions are not satisfied, the series may be divergent. Thus, it is in reality only for a comparatively small class of distributions that we can assert the validity of the series (A.3). In fact, the

majority of the important distributions dealt with in practice are not included in this class. However, in practical application it is only of little value to know the convergence properties of the series. What we really want to know is whether a small number of terms - usually two or three - suffice to give a good approximation to $p(x)$. If we know this to be the case, it does not concern us much whether the infinite series is convergent or divergent. And conversely, if we know that one of the series is convergent, this knowledge is of little practical value if it will be necessary to evaluate a large number of coefficients C_n in order to have the sum of the series determined to a reasonable approximation. Besides, only a few of the initial moments of a random variable are usually known with any degree of accuracy. It is not even known whether the others exist. Thus, regardless of convergence, expansion (A.3) is used.

Derivation of Gram's Series. Gram's problem in a somewhat modified form may briefly be stated as follows: Given an arbitrary density function $p(x)$, continuous and finite in the interval $-\infty$ to $+\infty$ and vanishes at $x = \pm\infty$, it is required to determine the constants C_0, C_1, \dots in such a way that the series

$$1/\sqrt{p_0(x)} \sum_{i=1}^n C_i H_i(x) p_0(x)$$

gives the best approximation to the quantity $p(x)/\sqrt{p_0(x)}$ in the least squares sense. That is to say we wish to determine the constant coefficients C_i in such a manner that the sum of the squares of the differences between the function and the approximate series becomes a minimum. This means that the expression

$$I = \int_{-\infty}^{\infty} \left[p(x)/\sqrt{p_0(x)} - \sum_{i=1}^n C_i H_i(x) \sqrt{p_0(x)} \right]^2 dx$$

must be a minimum.

Differentiating with respect to C_i and setting the derivative equal to zero we have for the extremum condition

$$C_i = \frac{1}{i!} \int_{-\infty}^{\infty} p(x) H_i(x) dx \quad i = 0, 1, 2, \dots$$

From the sign of the second derivatives it is found that the extremum is a minimum. The solution is gotten by the introduction of $\sqrt{p_0(x)}$ which serves to make all terms of the form $C_i H_i(x) \sqrt{p_0(x)}$ orthogonal to each other in the interval $-\infty$ to $+\infty$.

In a series representation the determination of the unknown coefficients or parameters can be looked at from two points of view. We may either consider the series as infinite in which case the determination of the coefficients becomes a problem in the theory of functions; or we may decide to consider a finite number of terms in the series and determine the coefficients so that the sum of the squares of the deviations of the resulting function from the actual function becomes a minimum in the sense of the method of least squares. In the latter case, the coefficients and not the moments or semi-invariants are representative of the observations. This latter method is the classical method as used by Gram in his fundamental research on the expansion of the density functions in series. A statement of the essential differences of the two methods may, however, be of advantage.

The method of moments requires that the areas of the definite integrals of the form

$$\int_{-\infty}^{\infty} x^r p(x) dx$$

must equal the areas of the observations which are expressed as power sums of the form

$$\sum x^r p(x) ,$$

while the method of least squares requires that

$$\int_{-\infty}^{\infty} \left\{ p(x) - \sum_{i=0}^n C_i H_i(x) p_0(x) \right\}^2 dx$$

must equal a minimum but does not necessarily impose any restrictions as to the condition of equality of the observed and computed areas derived from the mathematical formula.

An Alternative Derivation of the A-series. The A-series can also be obtained by the powerful theory of characteristic functions. The characteristic function is a very important device in probability theory and has a number of properties which are important in obtaining many useful results. The characteristic function of a random variable is defined as

$$f(z) = E\{\exp (izx)\} \tag{A.16}$$

where z is a real constant. The theory of characteristic functions is well covered in the basic literature.

It is known that if the central moments μ_r up to order $n+1$ exist then $f(z)$ can be expanded in terms of μ_r in a Taylor series about a small z ,

$$f(z) = e^{izm_x} \left\{ 1 + \sum_{r=2}^n \frac{i^r \mu_r}{r!} z^r + 0 \left| \frac{i^{n+1} \mu_{n+1}}{(n+1)!} z^{n+1} \right| \right\}$$

where we have used the familiar relation

$$\mu_r = (i)^{-r} \left\{ \frac{d^r}{dz^r} e^{-izm_x} f(z) \right\} \Big|_{z=0}$$

which is the moment generating property of the characteristic function.

The log characteristic function is defined as

$$g(z) = \ln f(z) \tag{A.17}$$

$g(z)$ is also called the cumulant (semi-invariant) - generating function since the cumulant λ_r is obtained by the relation

$$\lambda_r = (i)^r \frac{d^r}{dz^r} g(z) \Big|_{z=0} .$$

$g(z)$ can also be expanded in terms of λ_r in a Taylor series of the form

$$g(z) = \sum_{r=1}^n \lambda_r \frac{(iz)^r}{r!} + 0 \left| \frac{(iz)^{n+1}}{(n+1)!} \right|$$

For the standardized random variable u , we have

$$\lambda_1 = 0 \quad , \quad \lambda_2 = 1$$

Thus

$$g(z) = (iz)^2/2 + (iz)^3\lambda_3/3! + (iz)^4\lambda_4/4! + \dots$$

or

$$f(z) = \{\exp(-z^2/2)\} \exp\{-iz^3\lambda_3/3! + z^4\lambda_4/4! + \dots\}$$

Expanding the second exponent according to the familiar expansion

$$\exp(t) = 1 + t + t^2/2 + t^3/3! + \dots$$

we have

$$f(z) = e^{-z^2/2} \{1 - i\lambda_3 z^3/3! + \lambda_4 z^4/4! + \dots\}$$

It is known that the density function $p(u)$ is the Fourier transform of $f(z)$. Thus

$$p(u) = 1/2\pi \int_{-\infty}^{\infty} f(z) e^{-izu} dz$$

To perform the integration we note that

$$1/2\pi \int_{-\infty}^{\infty} (-iz)^r \exp(-z^2/2 - izu) dz = (-1)^r H_r(u) p_0(u)$$

From this follows directly the series

$$p(u) = p_0(u) \{1 + \lambda_3/3! H_3(u) + \lambda_4/4! H_4(u) + \lambda_5/5! H_5(u) + 10\lambda_3^2/6! H_6(u) + \dots\}$$

which is the same as (A.10). All the discussion that followed (A.10) applies also here.

Asymptotic Series Expansion. Cramer (1946) investigated the order of magnitude of the coefficients C_n in an expansion similar to (A.10). In that case the expansion was for the density function of the sum of N independent random variables. The orders of magnitude of the first few C_n are given below for reference.

$n = 3$	$N^{-1/2}$
4 , 6	N^{-1}
5 , 7 , 9	$N^{-3/2}$
8 , 10 , 12	N^{-2}
11 , 13 , 15	$N^{-5/2}$

and so on.

We are not dealing here with sums of independent random variables but we still can use a parameter N as a means of grouping terms according to order of magnitude. Extensive work with the A-series verified the arrangement shown above for the general case we are considering here. Thus the order of magnitude of the terms in the A-series is not steadily decreasing with increasing n .

Based on the results obtained by Cramer we assume that the cumulant λ_n is of order of magnitude $N^{-n/2+1}$ for $n \geq 3$. Suppose we take few terms in the expansion such that we include all terms of order of magnitude equal to N^{-1} . It then follows from the above values that we must consider the terms up to $n=6$ inclusive. Thus we require the calculation of λ_n up to the sixth order. But the contributions to the order of magnitude N^{-1} really do not contain any cumulant of order higher than the fourth, according to the above assumption. If we proceed further and

include terms of orders of magnitude $n^{-3/2}$, N^{-2} , etc. it is easily seen that we shall encounter similar inadequacies.

Thus the A-series cannot be considered as a satisfactory solution to the expansion problem for $p(x)$. We want, in fact, a series which gives a straightforward expansion in powers of $N^{-1/2}$, and is such that the calculation of the terms up to a certain order does not require the knowledge of any statistics that are not really required. These conditions are satisfied by the Edgeworth series. This series is an asymptotic expansion. It is obtained by arranging the series in (A.10) in a decreasing order of magnitude according to the assumption made above concerning the order of magnitude of λ_n for $n \geq 3$. The resulting expansion is as follows

$$\begin{aligned}
 p(x) = p_0(x) & \left[\{1\} + \left\{ \frac{1}{3!} \frac{\lambda_3}{\sigma^3} H_3 \left(\frac{x - m_x}{\sigma} \right) \right\} \right. \\
 & + \left\{ \frac{1}{4!} \frac{\lambda_4}{\sigma^4} H_4 \left(\frac{x - m_x}{\sigma} \right) + \frac{10}{6!} \frac{\lambda_3^2}{\sigma^6} H_6 \left(\frac{x - m_x}{\sigma} \right) \right\} \\
 & + \left\{ \frac{1}{5!} \frac{\lambda_5}{\sigma^5} H_5 \left(\frac{x - m_x}{\sigma} \right) + \frac{35}{7!} \frac{\lambda_3 \lambda_4}{\sigma^7} H_7 \left(\frac{x - m_x}{\sigma} \right) \right. \\
 & \quad \left. + \frac{280}{9!} \frac{\lambda_3^3}{\sigma^9} H_9 \left(\frac{x - m_x}{\sigma} \right) \right\} \\
 & + \left\{ \frac{1}{6!} \frac{\lambda_6}{\sigma^6} H_6 \left(\frac{x - m_x}{\sigma} \right) + \frac{35}{8!} \frac{\lambda_4^2}{\sigma^8} H_8 \left(\frac{x - m_x}{\sigma} \right) \right. \\
 & \quad \left. + \frac{56}{8!} \frac{\lambda_3 \lambda_5}{\sigma^8} H_8 \left(\frac{x - m_x}{\sigma} \right) + \frac{210}{10!} \frac{\lambda_3^2 \lambda_4}{\sigma^{10}} \right. \\
 & \quad \left. H_{10} \left(\frac{x - m_x}{\sigma} \right) + \frac{15400}{12!} \frac{\lambda_3^4}{\sigma^{12}} H_{12} \left(\frac{x - m_x}{\sigma} \right) \right\} + \dots \left. \right] \quad (A.18a)
 \end{aligned}$$

where the terms inside each pair of brackets are of the same order of magnitude. Or, in terms of the central moments, we have

$$\begin{aligned}
 p(x) = p_0(x) & \left[\{1\} + \left\{ \frac{1}{3!} \frac{\mu_3}{\sigma^3} H_3 \left(\frac{x - m_x}{\sigma_x} \right) \right\} \right. \\
 & + \left\{ \frac{1}{4!} \left(\frac{\mu_4}{\sigma^4} - 3 \right) H_4 \left(\frac{x - m_x}{\sigma_x} \right) + \frac{10}{6!} \frac{\mu_3^2}{\sigma^6} H_6 \left(\frac{x - m_x}{\sigma_x} \right) \right\} \\
 & + \left\{ \frac{1}{5!} \left(\frac{\mu_5}{\sigma^5} - 10 \frac{\mu_3}{\sigma^3} \right) H_5 \left(\frac{x - m_x}{\sigma_x} \right) \right. \\
 & + \frac{35}{7!} \frac{\mu_3}{\sigma^3} \left(\frac{\mu_4}{\sigma^4} - 3 \right) H_7 \left(\frac{x - m_x}{\sigma_x} \right) \\
 & \left. + \frac{280}{9!} \left(\frac{\mu_3}{\sigma^3} \right)^3 H_9 \left(\frac{x - m_x}{\sigma_x} \right) \right\} + \dots \dots \dots \left. \right] \quad (A.18b)
 \end{aligned}$$

The terms of order of magnitude $N^{-n/2}$ in the Edgeworth series contain the moments μ_3, \dots, μ_{n+2} , which are the moments necessarily required for an approximation to this order. In practice, it is usually not advisable to go beyond the fourth moment.

Illustrative Examples. A few examples have been prepared to illustrate the previous treatment. The density functions used are known functions from the literature. The examples are shown in Figures 35 to 38, and include functions of shapes encountered in practice. From these examples one can see the advantage gained by including the λ_3^2 terms. GC in these examples refers to the GC series terminated at the λ_4 term. These examples give some idea about the degree of representation to be expected from an Edgeworth series expansion.

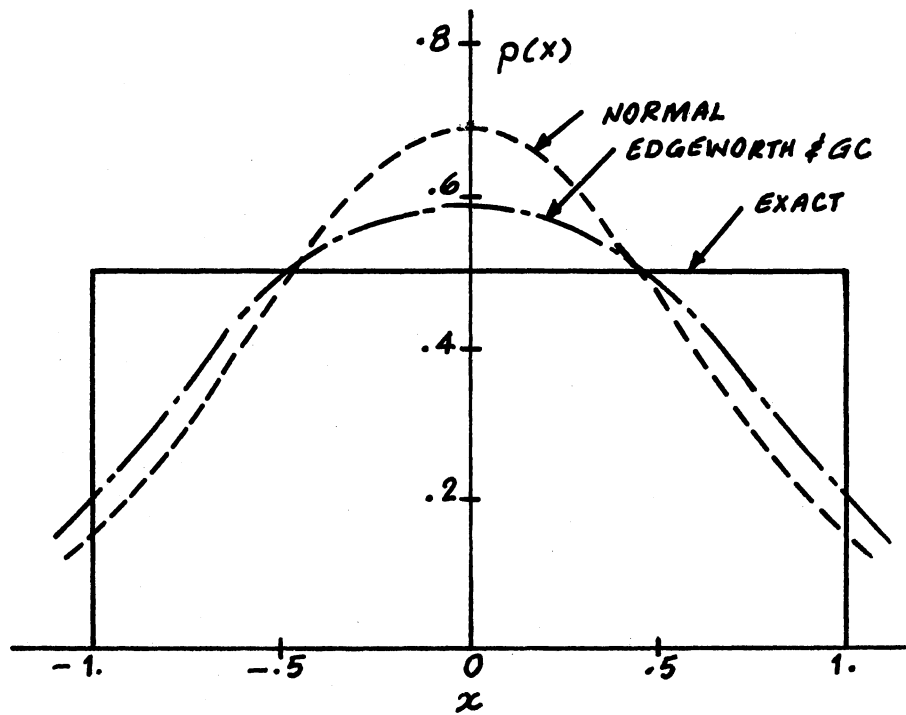


Figure 35. Exact and Approximate Density Functions for A Uniform Distribution over $(-1,1)$

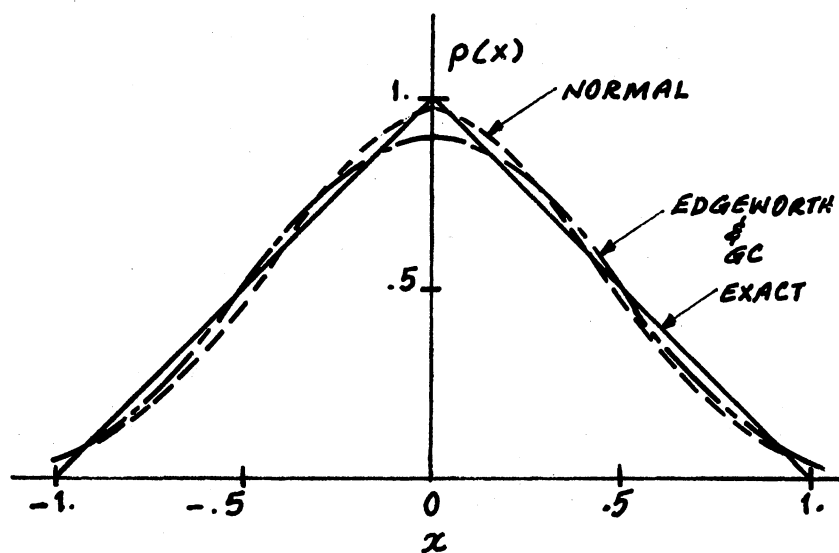
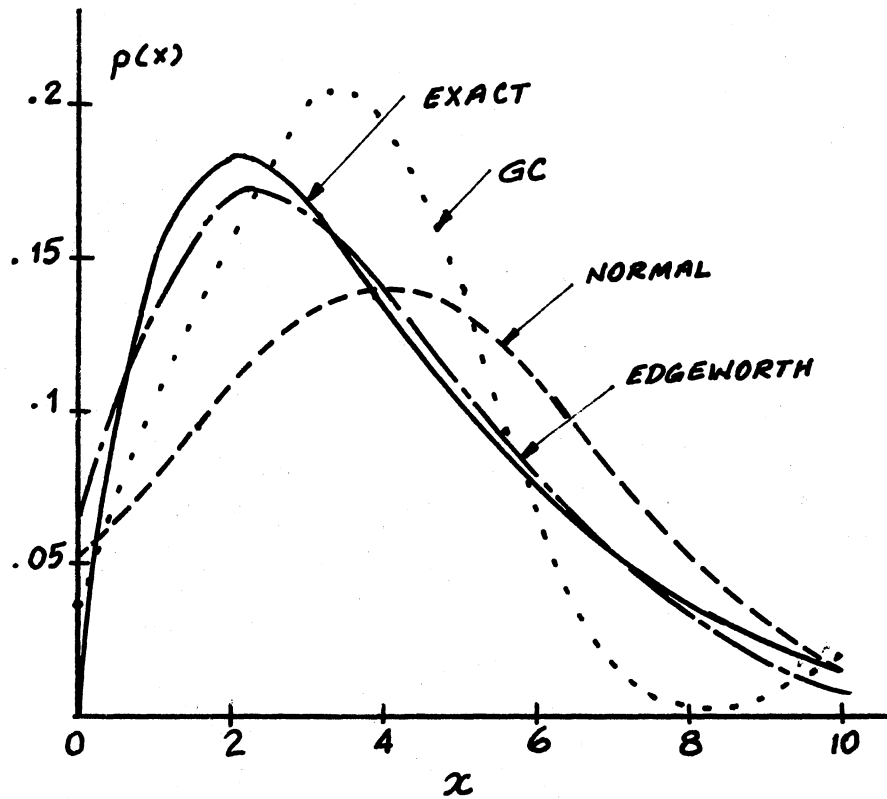
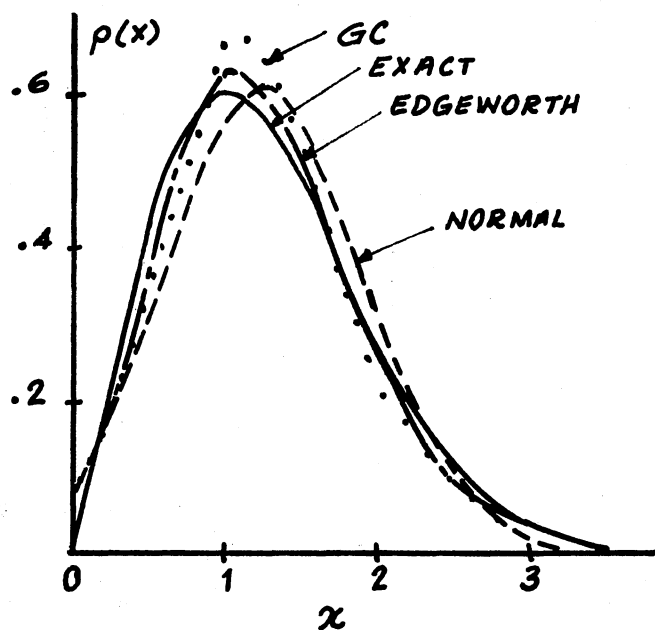


Figure 36. Exact and Approximate Density Functions for A triangular Distribution over $(-1,1)$



$$p(x) = \frac{x^{r/2-1} e^{-x/2}}{\Gamma(r/2) 2^{r/2}} \quad \text{EXACT}$$

Figure 37. Exact and Approximate Density Functions for A Chi-Square Distribution, $r=4$



$$p(x) = \frac{x}{b} e^{-x^2/2b}, \quad x > 0 \quad \text{EXACT}$$

Figure 38. Exact and Approximate Density Functions for A Rayleigh Distribution, $b=1$

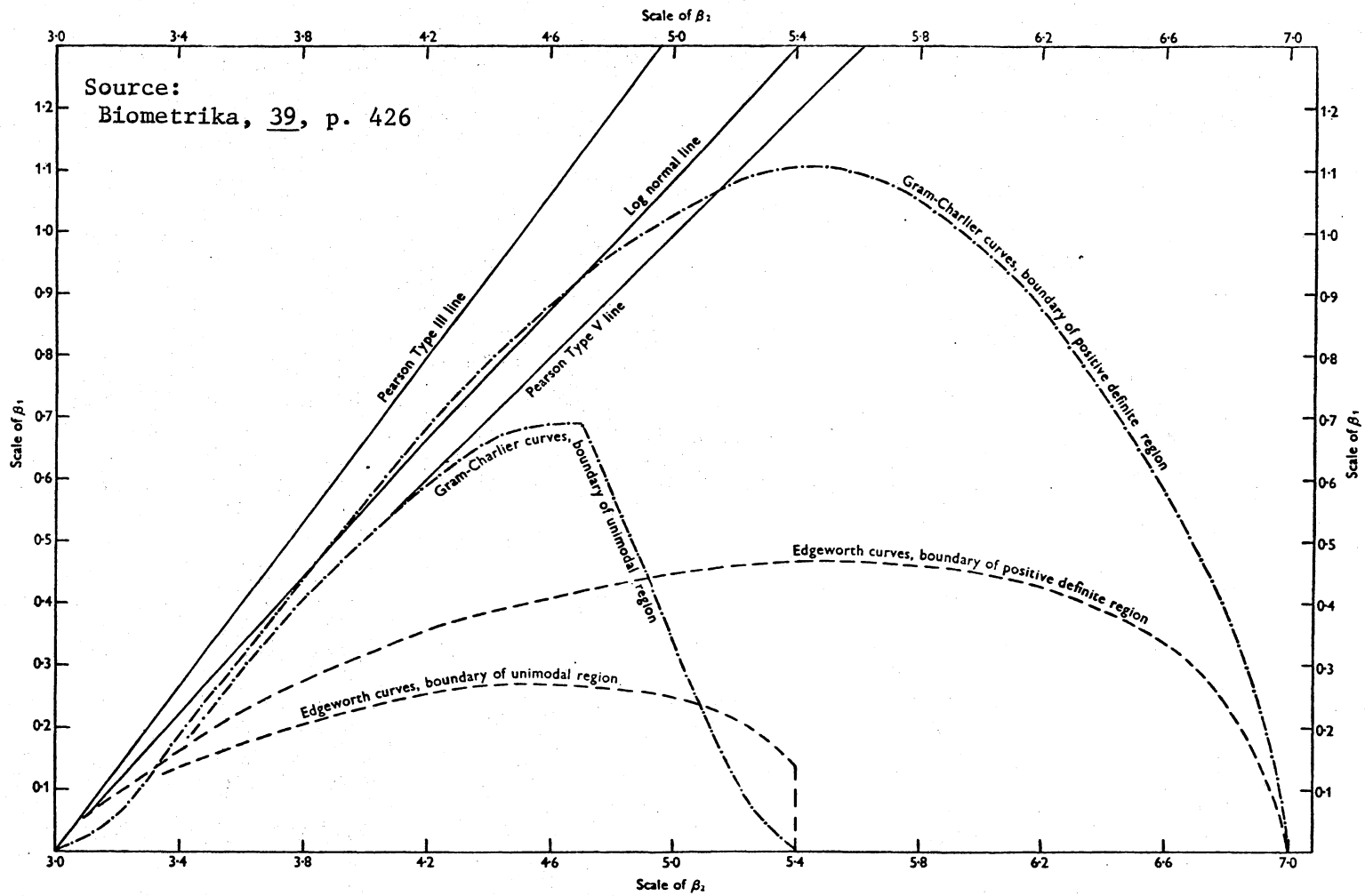


Figure 39. Regions of Unimodal Curves and Regions of Non-Negative Ordinates

Concluding Remarks. It should be noted that (A.18) is not a proper density function in the sense that for large x we may encounter negative values for the approximate $p(x)$. This is expected when truncating the series. From purely practical considerations this counts little, because the observations at the extremities are very few in number. We usually argue that it is the main body of the density function which is important and the tails, where negative values may occur, do not contribute much to the description of the variable. It is possible to modify the series so that it has the properties of the density function in this sense, but this is not necessary for our purposes here. Parameter variations in engineering systems are usually of the order of one to two standard deviations for which values of x the series is well-behaved. Statisticians are often concerned with points of significance in the tails of the distributions. This is why they approach this type of representation with some caution. Figure 39 shows the behavior of the series expansion in the β_1, β_2 plane (Barton and Dennis, 1952). β_1 and β_2 are defined as

$$\beta_1 = \gamma_1^2, \quad \beta_2 = \gamma_2 + 3$$

The Edgeworth series has been found satisfactory for cases with a considerable amount of skewness. For decidedly skew distributions, Edgeworth suggested a simple transformation of the variable. Instead of working with x one works with $\log x$. It is known that $\log x$ has the same distribution as x .

Finally, both Pearson's system and the series representation characterize a distribution by its moments and in particular the first few of them. A question may now be raised as to the extent that moments

represent a distribution and whether the representation is unique. It is known that the characteristic function specifies a distribution uniquely. And it is known that the characteristic function determines the moments. It does not, however, follow that the moments completely determine a distribution, even when the moments of all orders exist. Only under certain conditions will a set of moments determine a distribution uniquely, but, fortunately for systems engineers, those conditions are obeyed by all the distributions arising in practical stochastic systems. For all ordinary processes, therefore, a knowledge of the moments, when they exist, is equivalent to a knowledge of the distribution law: equivalent, that is, in the sense that it should be possible 'theoretically' to exhibit all the properties of a distribution in terms of the moments.

The values of the moments give information about the shape of the density function. The higher the order of the moment, the greater is the contribution to it by the tails of the density function, hence, the more information it gives about the nature of the tails. The even centered moments make no distinction between the two tails of the density function, so they give information about the width of the distribution, roughly speaking. The odd central moments all vanish for a symmetric distribution, so they give information about the lack of symmetry present.

The width of the distribution is readily indicated by the second moment. This information is removed from the higher moments by non-dimensionalizing by the standard deviation. The skewness is negative if the tail on the left of the mean is larger than the tail on the right.

The excess coefficient is larger as the tails of the distribution for values of the variate above a standard deviation are larger.

Multivariate Distributions

No completely satisfactory method has yet been found of setting up families of bivariate frequency functions, much less multivariate frequency functions, in extension of the families of distributions obtained by the three approaches mentioned at the beginning of this treatment, for the univariate case.

Consideration will be limited in this work to the series type of representation. The following treatment is the product of an effort by the author to extend the univariate treatment to the multidimensional case both for random variables and random processes. Interest in this development has been motivated by the need for an appropriate representation of the density functions for use in complex operations involving integrations. The sought representation must have generality and must be suitable for various manipulations. The immediate purpose is for use in evaluating various statistical characteristics of dynamic systems with stochastic disturbances whether as inputs or as perturbations in the system parameters. The development will thus be made with this particular objective in mind. The development will also be useful for other purposes since, it is recognized, that an appropriate representation of the density functions has always been the major obstacle in the various approaches to systems analysis and synthesis.

Series Representation of Multivariate

Density Functions

The series (A.10) for the univariate density function is extended to the multivariate density function in a straight-forward manner. The series corresponding to an n-dimensional density function is written in the following manner

$$p(\underline{u}) = p_0(\underline{u}) \sum_{n_1=0}^{\infty} \cdots \sum_{n_n=0}^{\infty} C_{n_1 \dots n_n} H_{(n_1 \dots n_n)} \quad (\text{A.19})$$

where,

$$\underline{u} = \{u_i\}, \quad i = 1, 2, \dots, n$$

$H_{(n_1 \dots n_n)}(\underline{u})$ are the ordinary multidimensional Ch-H polynomials defined by

$$H_{(n_1 \dots n_n)}(\underline{u}) = (-1)^N e^{\phi(\underline{u})} \frac{\partial^N}{\partial u_1^{n_1} \dots \partial u_n^{n_n}} e^{-\phi(\underline{u})}$$

$$N = n_1 + \dots + n_n \quad (\text{A.20})$$

where,

$$\phi(\underline{u}) = 1/2 \underline{u}^T \mathbf{a} \underline{u} \quad (\text{A.21})$$

$$\mathbf{a} = \{a_{ij}\} = \rho^{-1} = \{\rho_{ij}\}^{-1}, \quad i, j = 1, \dots, n$$

and ρ is the correlation matrix of the standardized vector \underline{u} ,

$$\rho_{ij} = E(u_i u_j) \quad (\text{A.22})$$

and $p_0(\underline{u})$ is the standardized multidimensional density function given by

$$p_0(\underline{u}) = \frac{1}{(2\pi)^{n/2} |\rho|^{1/2}} \exp(-\frac{1}{2} \underline{u}^T \rho^{-1} \underline{u}) \quad (\text{A.23})$$

The coefficients $C_{n_1 \dots n_n}$ are obtained in the same way as C_n in the univariate case, using $p_0(\underline{u})$ and $H_{(n_1 \dots n_n)}(\underline{u})$.

To illustrate, consider the two-dimensional (bivariate) density function. Here, (A.19) takes the following form

$$p(u_1, u_2) = p_0(u_1, u_2) \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} C_{n_1 n_2} H_{(n_1 n_2)}(u_1, u_2) \quad (\text{A.24})$$

where

$$p_0(u_1, u_2) = \frac{1}{2\pi(1-\rho_{12}^2)^{1/2}} \exp \frac{-(u_1^2 - 2\rho_{12}u_1u_2 + u_2^2)}{2(1-\rho_{12}^2)} \quad (\text{A.25})$$

The first few values of $C_{n_1 n_2}$ are

$$\begin{aligned} C_{00} &= 1, \quad C_{ij} = 0 \quad i+j = 1, 2 \\ C_{30} &= 1/3! \mu_{30}, \quad C_{03} = 1/3! \mu_{03} \\ C_{21} &= 1/2 \mu_{21}, \quad C_{12} = 1/2 \mu_{12} \\ C_{40} &= 1/4! (\mu_{40}^{-3}), \quad C_{04} = 1/4! (\mu_{04}^{-3}) \\ C_{22} &= 1/4 (\mu_{22}^{-1} \mu_{20}^{-1} \mu_{02}^{-1} - 2\mu_{12}^2 - 1) \\ C_{31} &= 1/3! (\mu_{31}^{-3} \mu_{11}), \quad C_{13} = 1/3! (\mu_{13}^{-3} \mu_{11}) \\ C_{60} &= 10/6! \mu_{30}^2, \quad C_{06} = 10/6! \mu_{03}^2 \\ C_{51} &= 10/6! \mu_{30} \mu_{21}, \quad C_{15} = 10/6! \mu_{03} \mu_{12} \\ C_{42} &= 1/6! (4\mu_{30} \mu_{12} + 6\mu_{21}^2), \\ C_{24} &= 1/6! (4\mu_{03} \mu_{21} + 6\mu_{12}^2), \quad C_{33} = 1/6! (\mu_{30} \mu_{03} + 9\mu_{21} \mu_{12}) \end{aligned} \quad (\text{A.26})$$

The first few Ch-H polynomials are obtained from the definition above.

If we define

$$y_i = \sum_{j=1}^2 a_{ij} u_j$$

where for this case,

$$a_{11} = (1 - \rho_{12}^2)^{-1}, \quad a_{22} = a_{11}$$

$$a_{21} = a_{12} = -\rho_{12} (1 - \rho_{12}^2)^{-1}$$

then,

$$H_{(00)}(y_1, y_2) = 1$$

$$H_{(10)}(y_1, y_2) = y_1, \quad H_{(01)}(y_1, y_2) = y_2$$

$$H_{(20)}(y_1, y_2) = y_1^2 - a_{11}, \quad H_{(02)}(y_1, y_2) = y_2^2 - a_{22}$$

$$H_{(11)}(y_1, y_2) = y_1 y_2 - a_{12}$$

$$H_{(30)}(y_1, y_2) = y_1^3 - 3a_{11}y_1, \quad H_{(03)}(y_1, y_2) = y_2^3 - 3a_{22}y_2$$

$$H_{(21)}(y_1, y_2) = y_1^2 y_2 - 2a_{12}y_1 - a_{11}y_2$$

$$H_{(12)}(y_1, y_2) = y_1 y_2^2 - 2a_{12}y_2 - a_{22}y_1$$

$$H_{(40)}(y_1, y_2) = y_1^4 - 6a_{11}y_1^2 + 3a_{11}^2$$

$$H_{(04)}(y_1, y_2) = y_2^4 - 6a_{22}y_2^2 + 3a_{22}^2$$

$$H_{(31)}(y_1, y_2) = y_1^3 y_2 - 3a_{12}y_1^2 - 3a_{11}y_1 y_2 + 3a_{11}a_{12}$$

$$H_{(13)}(y_1, y_2) = y_1 y_2^3 - 3a_{12}y_2^2 - 3a_{22}y_1 y_2 + 3a_{22}a_{21}$$

$$H_{(22)}(y_1, y_2) = y_1^2 y_2^2 - a_{22}y_1^2 - a_{11}y_2^2 - 4a_{12}y_1 y_2 + a_{11}a_{22} + 2a_{12}^2$$

$$\begin{aligned}
H_{(60)}(y_1, y_2) &= y_1^6 - 15a_{11}y_1^4 + 45a_{11}^2y_1^2 - 15a_{11}^3 \\
H_{(51)}(y_1, y_2) &= y_1^5y_2 - 5a_{12}y_1^4 - 10a_{11}y_1^3y_2 + 30a_{12}a_{11}y_1^2 \\
&\quad + 15a_{11}^2y_1y_2 - 15a_{12}a_{11}^2 \\
H_{(42)}(y_1, y_2) &= y_1^4y_2^2 - a_{22}y_1^4 - 8a_{12}y_1^3y_2 - 6a_{11}y_1^2y_2^2 \\
&\quad + 3a_{11}^2y_2^2 + 24a_{12}a_{11}y_1y_2 + 6a_{11}a_{22} \\
&\quad (1 + 2\rho_{12}^2)y_1^2 - 3a_{11}^2a_{22}(1 + 4\rho^2) \\
H_{(33)}(y_1, y_2) &= y_1^3y_2^3 - 3a_{11}y_1y_2^3 - 3a_{22}y_1^3y_2 - 9a_{12}y_1^2y_2^2 \\
&\quad + 9a_{12}a_{22}y_1^2 + 9a_{12}a_{11}y_2^2 + 3a_{11}a_{22} \\
&\quad (3 + 5\rho_{12}^2)y_1y_2 - 3a_{12}a_{11}a_{22}(3 + 2\rho_{12}^2)
\end{aligned} \tag{A.26}$$

The density function for the arbitrary random vector: \underline{x} is obtained by replacing u_i by $(x_i - m_i)/\sigma_i$, and μ_{ij} by $\mu_{ij}/\sigma_{x_1}^i \sigma_{x_2}^j$.

Similarly, for a trivariate density function, we have,

$$\begin{aligned}
p(u_1, u_2, u_3) &= p_0(u_1, u_2, u_3) \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} C_{n_1 n_2 n_3} \\
&\quad H_{(n_1 n_2 n_3)}(u_1, u_2, u_3)
\end{aligned} \tag{A.27}$$

where,

$$p_0(\underline{u}) = \frac{1}{(2\pi)^{3/2} |\Delta|^{1/2}} \exp - \frac{1}{2} \underline{u}^T \mathbf{a} \underline{u} \tag{A.28}$$

$$\Delta = 1 - \rho_{23}^2 - \rho_{13}^2 - \rho_{12}^2 - \rho_{12}\rho_{13}\rho_{23}$$

$$\begin{aligned}
a &= \{a_{ij}\} \quad , \quad i, j = 1, 2, 3 \\
a_{11} &= \frac{1}{\Delta} (1 - \rho_{23}^2) \quad , \quad a_{22} = \frac{1}{\Delta} (1 - \rho_{13}^2) \\
a_{33} &= \frac{1}{\Delta} (1 - \rho_{12}^2) \quad , \quad a_{12} = a_{21} = \frac{1}{\Delta} (\rho_{13}\rho_{23} - \rho_{12}) \\
a_{13} &= a_{31} = \frac{1}{\Delta} (\rho_{12}\rho_{23} - \rho_{13}) \quad , \\
a_{23} &= a_{32} = \frac{1}{\Delta} (\rho_{12}\rho_{13} - \rho_{23})
\end{aligned}$$

and,

$$\begin{aligned}
C_{000} &= 1 \quad , \quad C_{n_1 n_2 n_3} = 0 \quad \text{for } n_1 + n_2 + n_3 = 1, 2 \\
C_{300} &= 1/3! \mu_{300} \quad , \quad C_{210} = 1/2! \mu_{210} \quad , \quad C_{111} = \mu_{111} \\
C_{220} &= 1/4 (\mu_{220} - \mu_{200} - \mu_{020} - 2\rho_{12}^2 - 1) \quad , \\
C_{400} &= 1/4! (\mu_{400} - 3) \quad , \quad C_{310} = 1/3! (\mu_{310} - 3\mu_{110}) \\
C_{211} &= 1/2 (\mu_{211} - \rho_{23}\mu_{200} - \rho_{13}\mu_{110} - 2\rho_{12}\mu_{101}) \quad (A.29)
\end{aligned}$$

If we use the definition

$$y_i = \sum_{j=0}^3 a_{ij} u_j \quad \quad i = 1, 2, 3$$

then the first few Ch-H polynomials are,

$$\begin{aligned}
H_{(000)}(\underline{u}) &= 1 \quad , \quad \underline{u} = \{u_1, u_2, u_3\} \\
H_{(100)}(\underline{u}) &= y_1 \quad , \quad H_{(200)}(\underline{u}) = y_1^2 - a_{11} \\
H_{(110)}(\underline{u}) &= y_1 y_2 - a_{12} \quad , \quad H_{(300)}(\underline{u}) = y_1^3 - 3a_{11} y_1 \\
H_{(210)}(\underline{u}) &= y_1^2 y_2 - 2a_{12} y_1 - a_{11} y_2 \\
H_{(111)}(\underline{u}) &= y_1 y_2 y_3 - a_{23} y_1 - a_{13} y_2 - a_{12} y_3
\end{aligned}$$

$$\begin{aligned}
H_{(400)}(\underline{u}) &= y_1^4 - 6a_{11}y_1^2 + 3a_{11}^2 \\
H_{(310)}(\underline{u}) &= y_1^3y_2 - 3a_{12}y_1^2 - 3a_{11}y_1y_2 + 3a_{11}a_{12} \\
H_{(220)}(\underline{u}) &= y_1^2y_2^2 - a_{22}y_1^2 - a_{11}y_2^2 - 4a_{12}y_1y_2 + a_{11}a_{22} + 2a_{12}^2 \\
H_{(211)}(\underline{u}) &= y_1^2y_2y_3 - a_{23}y_1^2 - 2a_{13}y_1y_2 - 2a_{12}y_1y_3 \\
&\quad - a_{11}y_2y_3 + a_{11}a_{23} + 2a_{12}a_{13}
\end{aligned} \tag{A.30}$$

Again, we point out that (A.19) can be obtained from the characteristic function as in the univariate case. The same discussion and criticism given the A-series in the univariate case apply to the multidimensional case in (A.19).

The multivariate Edgeworth asymptotic expansion is developed in a manner similar to that in the univariate case. As an example, consider the two-dimensional problem. The bivariate Edgeworth series is written as

$$\begin{aligned}
p(x_1, x_2) &= p_0(x_1, x_2) \left\{ 1 + \sum_{r+s=3} \frac{1}{r!s!} \frac{\lambda_{rs}}{\sigma_1^r \sigma_2^s} H_{(rs)}(x_1, x_2) \right. \\
&\quad + \sum_{r+s=4} \frac{1}{r!s!} \frac{\lambda_{rs}}{\sigma_1^r \sigma_2^s} H_{(rs)}(x_1, x_2) \\
&\quad + 1/2 \sum_{\substack{r+s=3 \\ p+q=3}} \frac{1}{r!s!p!q!} \frac{\lambda_{rs} \lambda_{pq}}{\sigma_1^{r+p} \sigma_2^{s+q}} \\
&\quad \left. H_{(r+p \ s+q)}(x_1, x_2) + \dots \right\}
\end{aligned} \tag{A.31}$$

where,

$$p_o(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2(1-\rho_{12}^2)^{\frac{1}{2}}} \exp \left\{ -\frac{1}{2(1-\rho_{12}^2)} \left[\left(\frac{x_1-m_1}{\sigma_1}\right)^2 - 2\rho_{12} \left(\frac{x_1-m_1}{\sigma_1}\right) \left(\frac{x_2-m_2}{\sigma_2}\right) + \left(\frac{x_2-m_2}{\sigma_2}\right)^2 \right] \right\}$$

Again, as in the univariate case, the approximate density function will not be a proper density function and regions can be defined where the approximate expression is non-negative and unimodal.

It is now remarked that, in the multivariate case, the expansions already obtained are useful for the purpose of approximating density functions of random vectors as an end result. However, if the resulting approximation is to be used in operations of some complexity the above forms of expansions result in cumbersome computational operations which will be difficult to perform. Therefore, it becomes necessary to present the above expansions in more attractive forms as far as computations are concerned so that they can be used in performing various transformations of random vectors.

The main complication results from the correlation matrix in $p_o(\underline{x})$. For, in a multiple integration operation the integrals will be coupled and a simple analytic expression will not be possible in such an operation. Since we will be needing the expansions mostly in integration operations governing the transformations of signals or signal statistics, it is necessary to simplify the form of these expansions.

The first step in this direction is to work on $p_o(\underline{x})$. We attempt to express the general form of $p_o(\underline{x})$ in terms of $p_o^*(\underline{x})$, the density function for independent elements. This is achieved by working with the

characteristic function. The development is carried out in three steps to clarify the details involved.

i. Consider first the two-dimensional density function. It is known that the density function is the Fourier transform of the characteristic function.

$$p_o(x_1, x_2) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_o(z_1, z_2) e^{-i \sum_{j=1}^2 z_j x_j} dz_1 dz_2$$

where

$$f_o(z_1, z_2) = e^{i \sum_{j=1}^2 m_{x_j} z_j - \frac{1}{2} \sigma_1 \sigma_2 (z_1^2 + 2\rho_{12} z_1 z_2 + z_2^2)}$$

Substituting for $f_o(\underline{z})$, and expanding the exponential cofactor

$$\exp(-\sigma_1 \sigma_2 \rho_{12} z_1 z_2)$$

in a power series of z_1 and z_2 , we get the following relation

$$\begin{aligned} p_o(x_1, x_2) &= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left\{-i \sum_{j=1}^2 z_j (x_j - m_j) - \right. \\ &\quad \left. \sigma_1 \sigma_2 (z_1^2 + z_2^2) \right\} \left\{ \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} (\sigma_1 \sigma_2 \rho_{12} z_1 z_2)^k \right\} \\ &\quad dz_1 dz_2 \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\rho_{12}^k}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left\{-i \sum_{j=1}^2 z_j (x_j - m_j) \right. \\ &\quad \left. - \sigma_1 \sigma_2 / 2 (z_1^2 + z_2^2) \right\} z_1^k z_2^k \sigma_1^k \sigma_2^k dz_1 dz_2 \end{aligned}$$

Simplifying and carrying out the integrations, we get

$$p_0(x_1, x_2) = p_0^*(x_1, x_2) \sum_{k=0}^{\infty} \frac{\rho_{12}^k}{k!} H_k(x_1^0/\sigma_1) H_k(x_2^0/\sigma_2) \quad (\text{A.32})$$

ii. Next, consider the trivariate density function. The development proceeds similar to the bivariate case. In this case, the characteristic function of a trivariate normal vector is given by

$$f_0(z_1, z_2, z_3) = \exp\left\{i \sum_{j=1}^3 m_{x_j} z_j - \frac{1}{2} \sum_{j,k=1}^3 \sigma_j \sigma_k \rho_{jk} z_j z_k\right\}$$

Using this expression and expanding the cofactors in powers of $z_i z_j$, $i, j = 1, 2, 3$, we have

$$\begin{aligned} p_0(x_1, x_2, x_3) &= \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left\{-i \left[(x_1 - m_1) z_1 + \right. \right. \\ &\quad \left. \left. (x_2 - m_2) z_2 + (x_3 - m_3) z_3 \right] - 1/2 \left[\sigma_1^2 z_1^2 + \right. \right. \\ &\quad \left. \left. \sigma_2^2 z_2^2 + \sigma_3^2 z_3^2 \right] \right\} \sum_{\ell, m, n=0}^{\infty} (-1)^{\ell+m+n} \frac{(\sigma_1 \sigma_2 \rho_{12} z_1 z_2)^{\ell}}{\ell!} \\ &\quad \frac{(\sigma_2 \sigma_3 \rho_{23} z_2 z_3)^m}{m!} \cdot \frac{(\sigma_1 \sigma_3 \rho_{13} z_1 z_2)^n}{n!} dz_1 dz_2 dz_3 \\ &= \sum_{\ell, m, n=0}^{\infty} \frac{\rho_{12}^{\ell} \rho_{23}^m \rho_{13}^n}{\ell! m! n!} (-1)^{\ell+m+n} \end{aligned}$$

$$\frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \{-i [(x_1 - m_1) z_1 + (x_2 - m_2) z_2 + (x_3 - m_3) z_3] - 1/2 [\sigma_1^2 z_1^2 + \sigma_2^2 z_2^2 + \sigma_3^2 z_3^2]\} (\sigma_1 z_1)^{\ell+n} (\sigma_2 z_2)^{\ell+m} (\sigma_3 z_3)^{m+n} dz_1 dz_2 dz_3$$

$$\rho_1 = \rho_{12} \quad , \quad \rho_2 = \rho_{23} \quad , \quad \rho_3 = \rho_{13}$$

which, after simplification, reduces to

$$p_0(x_1, x_2, x_3) = p_0^*(x_1, x_2, x_3) \sum_{\ell, m, n=0}^{\infty} \frac{\rho_1^{\ell} \rho_2^m \rho_3^n}{\ell! m! n!} H_{\ell_1}^0(x_1/\sigma_1) H_{\ell_2}^0(x_2/\sigma_2) H_{\ell_3}^0(x_3/\sigma_3)$$

$$\ell_1 = \ell+n \quad , \quad \ell_2 = \ell+m \quad , \quad \ell_3 = m+n \quad (A.33)$$

Note that ℓ_1, ℓ_2 and ℓ_3 are determined by summing up the powers of ρ_i where x_1, x_2, x_3 are involved, respectively.

iii. Finally, the expression for the n-dimensional density function is obtained in the same manner. The final result is given by

$$p_0(x_1, \dots, x_n) = p_0^*(x_1, \dots, x_n) \sum_{k_1=0}^{\infty} \dots \sum_{k_m=0}^{\infty}$$

$$\frac{\rho_1^{k_1} \dots \rho_m^{k_m}}{k_1! \dots k_m!} H_{\ell_1}^0(x_1/\sigma_1) \dots H_{\ell_n}^0(x_n/\sigma_n) \quad (A.34)$$

ρ_i are the correlation coefficients between the different pairs of the elements of \underline{x} . The number of these pairs is

$$m = \frac{n}{2(n-2)} \quad (\text{A.35})$$

ρ_i are obtained by summing up the powers of the ρ_i 's where x_i are involved.

Note that now in any integration operation involving $p_0(\underline{x})$ the integrals will be separate with respect to each x_i . Note also that the series expansion becomes more convergent with the increase in the order of the system by virtue of $\rho_i^{k_i}/k_i!$.

Now the expansion (A.34) replaces $p_0(\underline{x})$ in the expansions obtained previously to result in a modified form of expansion which will be suitable for the purposes discussed above.

As an illustration consider the relation (A.31). Using the expression in (A.32) the following modified form results

$$\begin{aligned} p_0(x_1, x_2) = p_0^*(x_1, x_2) & \left\{ \sum_{k=0}^{\infty} \frac{\rho_{12}^k}{k!} H_k(x_1^0/\sigma_1) H_k(x_2^0/\sigma_2) \right. \\ & + \sum_{r+s=3} \frac{1}{r!s!} \frac{\lambda rs}{\sigma_1 \sigma_2} \sum_{k=0}^{\infty} \frac{\rho_{12}^k}{k!} H_{k+r}(x_1^0/\sigma_1) H_{k+s}(x_2^0/\sigma_2) \\ & + \sum_{r+s=4} \frac{1}{r!s!} \frac{\lambda rs}{\sigma_1 \sigma_2} \sum_{k=0}^{\infty} \frac{\rho_{12}^k}{k!} H_{k+r}(x_1^0/\sigma_1) H_{k+s}(x_2^0/\sigma_2) \\ & \left. + 1/2 \sum_{\substack{r+s=3 \\ p+q=3}} \frac{1}{r!s!p!q!} \frac{\lambda rs \lambda pq}{\sigma_1^{r+p} \sigma_2^{s+q}} \right. \end{aligned}$$

$$\left. \begin{aligned} & \sum_{k=0}^{\infty} \frac{\rho_{12}^k}{k!} H_{k+r+p} (x_1^0/\sigma_1) H_{k+s+q} (x_2^0/\sigma_2) \\ & + \dots \end{aligned} \right\} \quad (\text{A.36})$$

The extension to higher order density functions is now obvious.

The relations (A.18) and A.36) were used extensively by the author and have been found of extreme help in many otherwise complicated situations. This experience has led to the search for further simplifications. One such attempt was to obtain a series expansion, for the arbitrary density function, much like the expansion for the multidimensional normal density function just presented. In other words, we seek an expansion for $p(x_1, \dots, x_n)$ of the form

$$p(x_1, \dots, x_n) = p_0^*(x_1, \dots, x_n) \sum_{k_1=0}^{\infty} \dots \sum_{k_n=0}^{\infty} \Theta_{k_1 \dots k_n} \cdot H_{k_1} (x_1^0/\sigma_1) \dots H_{k_n} (x_n^0/\sigma_n) \quad (\text{A.37})$$

To define the coefficients $\Theta_{k_1 \dots k_n}$ we multiply (A.37) by $H_{i_1} (x_1^0/\sigma_1) \dots H_{i_n} (x_n^0/\sigma_n)$ and integrate with respect to all variables from $-\infty$ to $+\infty$,

$$\begin{aligned} & \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} p_n(x_1, \dots, x_n) H_{i_1} (x_1^0/\sigma_1) \dots H_{i_n} (x_n^0/\sigma_n) dx_1 \dots dx_n \\ & = \sum_{k_1=0}^{\infty} \dots \sum_{k_n=0}^{\infty} \Theta_{k_1 \dots k_n} \int_{-\infty}^{\infty} H_{k_1} (x_1^0/\sigma_1) H_{i_1} (x_1^0/\sigma_1) \dots \end{aligned}$$

$$p_0(x_1) dx_1 \dots \int_{-\infty}^{\infty} H_{k_n}(x_n^0/\sigma_n) H_{i_n}(x_n^0/\sigma_n) p_0(x_n) dx_n$$

Note how the integral decomposes into separate single integrals. Keeping in mind the orthogonality of the Ch-H polynomials, we get

$$\Theta_{k_1 \dots k_n} = \frac{1}{k_1! \dots k_n!} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} p_n(x_1, \dots, x_n) \cdot H_{k_1}(x_1^0/\sigma_1) \dots H_{k_n}(x_n^0/\sigma_n) dx_1 \dots dx_n \quad (\text{A.38})$$

$\Theta_{k_1 \dots k_n}$ may thus be expressed in terms of the moments of x . For this purpose we use the familiar relation

$$H_n(z)/n! = \sum_{\nu=0}^{\{n/2\}} \frac{(-1)^\nu z^{(n-2\nu)}}{2^\nu (n-2\nu)! \nu!} = \sum_{\nu=0}^{\{n/2\}} \frac{(-1)^\nu (2\nu-1)!!}{(2\nu)!} \cdot z^{(n-2\nu)} / (n-2\nu)! \quad (\text{A.39})$$

where $\{n/2\}$ is the greatest integer $\leq n/2$. From this follows

$$\Theta_{k_1 \dots k_n} = \sum_{\nu_1=0}^{\{k_1/2\}} \dots \sum_{\nu_n=0}^{\{k_n/2\}} \frac{(-1)^\nu (2\nu_1-1)!! \dots (2\nu_n-1)!!}{(2\nu_1)! \dots (2\nu_n)!} \cdot \frac{\mu(k_1-2\nu_1) \dots (2k_n-\nu_n)}{(k_1-2\nu_1)! \sigma_1^{(k_1-2\nu_1)} \dots (k_n-2\nu_n)! \sigma_n^{(k_n-2\nu_n)}} \quad (\text{A.40})$$

Once the coefficients in (A.37) are determined, the series terms must be rearranged to form an Edgeworth expansion. The advantage gained by this representation is, again, simplicity.

The coefficients $\theta_{n_1} \dots \theta_{n_n}$ will be called "generalized quasi-moments." Generalized, that is, to distinguish them from the ordinary quasi-moments introduced by Kuznetsov and his associates (1953). The ordinary quasi-moments will be described shortly in connection with random processes. It has been found that the type introduced above is more useful than the ordinary quasi-moments in computational operations.

Probability Density Function of A

Random Process

For a given argument t , the value of a random process $x(t)$ is generally a random variable. Therefore, to give a complete probabilistic characterization of $x(t)$ we have to give the distribution of the values of the random variable $x(t)$ for every t . These are the one-dimensional distributions of the random process $x(t)$. In general, the one-dimensional distributions of $x(t)$ depend on t as a parameter. These are a sufficient characterization of the random process in all problems where the values of the random process for particular arguments are considered in isolation. To solve problems where the values of the random process have to be considered together with two or more arguments, we have to introduce the joint distribution for random processes with several arguments. We characterize the n -dimensional distribution of the random process $x(t)$ by the n -dimensional probability density function $p(x_1, \dots, x_n; t_1, \dots, t_n)$ which will, in general, depend on t_1, \dots, t_n as parameters. However, no finite dimensional distribution is capable of characterizing a random

process exhaustively, in the general case. In many instances a partial characterization is sufficient, however.

The development already presented for univariate and multivariate density functions of random variables carries over to the case of random processes and vectors of random processes. Moments and semi-invariants have been used with the expansions of the density functions. For a random process these are the moment functions and the semi-invariant functions. Since the quasi-moments have not been used previously, their counterparts, the quasi-moment functions, are chosen for the development here.

Consider a one-dimensional distribution. Let the random variable $x(t)$ have the probability density $p(x)$. If all the moments of the quantity x are finite then $p(x)$ can be associated with the following formal series in the sense discussed previously.

$$p(x) = p_0(x) \sum_{k=0}^{\infty} \frac{1}{k!} \beta_k / \sigma_x^k H_k(x^0 / \sigma_x) \quad (\text{A.41a})$$

or equivalently,

$$p(x) = p_0(x) \sum_{k=0}^{\infty} \frac{1}{k!} \beta_k H_k(x^0) \quad (\text{A.41b})$$

The equivalence follows directly from the definition of the one-dimensional Ch-H polynomials. β_k are the quasi-moments. From the orthogonality of the Ch-H polynomials, it follows that

$$\beta_k = E\{H_k(x^0)\} \quad (\text{A.42a})$$

or

$$\beta_k = \sigma^k E\{H_k(x^0/\sigma^k)\} \quad (\text{A.42b})$$

From (A.42) it is evident that $\beta_0=1$, $\beta_1 = \beta_2 = 0$. Also, for the Gaussian $x(t)$, $\beta_k=0$ for all $k>0$. For an arbitrary $x(t)$, it is easy to show that the following relations are true.

$$\begin{aligned} \beta_3 &= \mu_3, \quad \beta_4 = \mu_4 - 3\mu_2^2, \quad \beta_5 = \mu_5 - 10\mu_3\mu_2 \\ \beta_6 &= \mu_6 - 15\mu_4\mu_2 + 30\mu_2^3 \\ \beta_7 &= \mu_7 - 21\mu_5\mu_2 + 105\mu_2^2\mu_3 \\ \beta_8 &= \mu_8 - 28\mu_6\mu_2 + 210\mu_4\mu_2^2 - 315\mu_2^4 \\ \beta_9 &= \mu_9 - 36\mu_7\mu_2 + 378\mu_5\mu_2^2 - 1260\mu_3\mu_2^3 \\ \beta_{10} &= \mu_{10} - 45\mu_8\mu_2 + 630\mu_6\mu_2^2 - 3150\mu_4\mu_2^3 + 3780\mu_2^5 \end{aligned} \quad (\text{A.43})$$

Or, in terms of the semi-invariants,

$$\begin{aligned} \beta_3 &= \lambda_3, \quad \beta_4 = \lambda_4, \quad \beta_5 = \lambda_5, \quad \beta_6 = \lambda_6 + 10\lambda_3^2 \\ \beta_7 &= \lambda_7 + 35\lambda_3\lambda_4, \quad \beta_8 = \lambda_8 + 56\lambda_3\lambda_5 + 35\lambda_4^2 \\ \beta_9 &= \lambda_9 + 84\lambda_3\lambda_6 + 126\lambda_4\lambda_5 + 280\lambda_3^3 \\ \beta_{10} &= \lambda_{10} + 120\lambda_3\lambda_7 + 210\lambda_4\lambda_6 + 126\lambda_5^2 + 2100\lambda_4\lambda_3^2 \end{aligned} \quad (\text{A.44})$$

Note that (A.41) is not different from the A-series (A.10). The β_k are easily related to C_k in (A.10) by

$$C_k = 1/k! \beta_k / \sigma^k$$

So, actually, the notion of quasi-moments is only artificial. It may simplify the appearance of the expansion but does nothing to promote the application of the original expansion.

The characteristic function corresponding to $p(x)$ in (A.41) is written as

$$f(x) = f_0(z) \sum_{k=0}^{\infty} \frac{i^k}{k!} \beta_k z^k \quad (\text{A.45})$$

where

$$f_0(z) = \exp\{ i m_x z - 1/2 \sigma_x^2 z^2 \}$$

If we expand the exponent in $f_0(z)$ in a Taylor series and carry out the multiplication we can readily relate β_k to the initial moments α_k by comparing with the series

$$f(z) = \sum_{k=0}^{\infty} i^k / k! \alpha_k z^k$$

Similarly, we write the characteristic function of a two-dimensional process as

$$f(z_1, z_2; t_1, t_2) = f_0(z_1, z_2; t_1, t_2) \left\{ 1 + \sum_{j,k,\ell=1}^2 \frac{i^3}{3!} \beta_3(t_j, t_k, t_\ell) z_j z_k z_\ell + \dots \right\} \quad (\text{A.46})$$

where,

$$f_o(z_1, z_2; t_1, t_2) = \exp \left[i \sum_{j=1}^2 m_x(t_j) z_j - 1/2 \sum_{j,k=1}^2 R(t_j, t_k) z_j z_k \right]$$

with

$$R(t_j, t_k) = E\{x^o(t_j) x^o(t_k)\}$$

is the characteristic function corresponding to the two-dimensional normal density function $p_o(x_1, x_2)$, and where we have used the fact that

$$\beta_1 = \beta_2 = 0.$$

In the same way, we write the n-dimensional characteristic function as

$$f(z_1, \dots, z_n; t_1, \dots, t_n) = f_o(z_1, \dots, z_n; t_1, \dots, t_n) \cdot$$

$$\left\{ 1 + \sum_{s=3}^{\infty} \frac{i^s}{s!} \sum_{w_1, \dots, w_s=1}^n \beta(t_{w_1}, \dots, t_{w_s}) z_{w_1} \dots z_{w_s} \right\}$$

$$f_o(\cdot) = \exp \left\{ i \sum_{j=1}^n m_x(t_j) z_j - 1/2 \sum_{j,k=1}^n R(t_j, t_k) z_j z_k \right\} \quad (\text{A.47})$$

If we consider the symmetry of the quasi-moment functions and combine similar terms, we can write

$$f(\underline{z}; \underline{t}) = f_o(\underline{z}; \underline{t}) \sum_{k=0}^{\infty} \sum_{m_1 + \dots + m_k = k} \frac{i^k}{m_1! \dots m_k!} \cdot$$

$$\beta_k(t_1^{(m_1)}, \dots, t_n^{(m_n)}) z_1^{m_1} \dots z_n^{m_n}$$

$$m_i = 0, 1, \dots, k \quad (\text{A.48})$$

where $t_i^{(m_i)}$ means t_i repeated m_i times.

To determine the expression for the n -dimensional density function, we take the Fourier transform of the expression for the characteristic function. Thus corresponding to (A.47) we have,

$$p(\underline{x}; \underline{t}) = \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \exp \left\{ -i \sum_{k=1}^n x_k z_k \right\} \cdot$$

$$f(\underline{z}; \underline{t}) dz_1 \dots dz_n \quad (\text{A.49})$$

Interchanging the operation of summation with respect to s and the integration, we can write

$$p(\underline{x}; \underline{t}) = \left\{ 1 + \sum_{s=3}^{\infty} \frac{1}{s!} \sum_{\omega_1, \dots, \omega_s=1}^n \beta_s(t_{\omega_1}, \dots, t_{\omega_s}) \cdot \right.$$

$$\left. (-\partial/\partial t(\omega_1)) \dots (-\partial/\partial t(\omega_s)) \right\} p_0(\underline{x}; \underline{t}) \quad (\text{A.50})$$

In terms of the Ch-H polynomials this becomes

$$p(\underline{x}; \underline{t}) = p_0(\underline{x}; \underline{t}) \left\{ 1 + \sum_{s=3}^{\infty} \frac{1}{s!} \sum_{\omega_1, \dots, \omega_s=1}^n \beta_s(t_{\omega_1}, \dots, t_{\omega_s}) H_{\omega_1} \dots H_{\omega_s} \left(\underline{x} - \frac{\underline{m}}{\underline{x}} \right) \right\} \quad (\text{A.51})$$

where we have used a form of Ch-H polynomials compatible with the representation in (A.47). The relation of these polynomials to the ordinary Ch-H polynomials previously introduced is clear from the following argument. If we compare $H_{\omega_1 \dots \omega_s}(\cdot)$ with $H_{(m_1 \dots m_n)}(\underline{x})$ and imagine the subscript ω_1 repeated m_1 times, ω_2 repeated m_2 times, etc. such that $m_1 + \dots + m_n = s$, then the two forms are equivalent. Thus, in two dimensions

$$H_{(m_1 m_2)} = H_{\underbrace{1 \dots 1}_{m_1}, \underbrace{2 \dots 2}_{m_2}}$$

The probability density function corresponding to (A.48) is written as

$$p(\underline{x}; \underline{t}) = p_0(\underline{x}; \underline{t}) \sum_{k=0}^{\infty} \sum_{m_1 + \dots + m_n = k} \frac{1}{m_1! \dots m_n!} \beta_k(t_1^{(m_1)}, \dots, t_n^{(m_n)}) H_{m_1 \dots m_n}(\underline{x}^0) \quad (\text{A.52})$$

Finally, we write the density function in terms of still another type of Ch-H polynomials as

$$p(\underline{x}; \underline{t}) = p_0(\underline{x}; \underline{t}) \sum_{k=0}^{\infty} 1/k! \sum_{\ell_1, \dots, \ell_k=1}^n \sum_{m_1, \dots, m_k=1}^n \beta_k(t_{\ell_1} \dots t_{\ell_k}) a_{\ell_1 m_1} \dots a_{\ell_k m_k} \mathcal{H}_{m_1 \dots m_k}^{(A)}(y_{m_1}, \dots, y_{m_k}) \quad (\text{A.53})$$

where $A = R^{-1}$ as before. The $\mathcal{H}_{m_1 \dots m_k}^{(A)}$ are the generalized Ch-H polynomials,

$$\mathcal{H}_{m_1 \dots m_k}^{(A)}(y_{m_1}, \dots, y_{m_k}) = (-1)^k e^{\phi(\underline{x})} \frac{\partial^k}{\partial x_{m_1} \dots \partial x_{m_k}} e^{-\phi(\underline{x})}$$

where $\phi(x_1, \dots, x_n)$ and y_i have been defined previously.

Equations (A.51) and A.53) differ in an advantageous manner from (A.52) in that the number of arguments of the functions β_k and $H_{\omega_1 \dots \omega_s}$ does not change (remains equal to s) when one increases the number n of the time instants which are selected.

We have examined more than one way of representation for the multi-dimensional density function. Attention has been mainly directed towards a form suitable for machine calculations. We have examined three types of Ch-H polynomials which could be easily related to each other. The last type is a general way of writing the Ch-H polynomials from which the other two forms can be obtained. To illustrate, consider the first few orders of the generalized polynomials

$$\mathcal{H}_i^{(A)}(y_i) = y_i, \quad i = 1, 2, \dots, n$$

$$\mathcal{H}_{ij}^{(A)}(y_i, y_j) = y_i y_j - a_{ij}, \quad i, j = 1, 2, \dots, n$$

$$\mathcal{H}_{i,j,k}^{(A)}(y_i, y_j, y_k) = y_k y_j y_i - y_i a_{jk} - y_j a_{ik} - y_k a_{ij}$$

Applying these expressions to a two-dimensional vector $\{y_1, y_2\}$, it is easily shown by permutation of the subscripts that

$$\mathcal{H}_{11}^{(A)}(y_1, y_1) = H_{11} = H_{(20)} = y_1^2 - a_{11}$$

$$\mathcal{H}_{12}^{(A)}(y_1, y_2) = H_{12} = H_{(11)} = y_1 y_2 - a_{12}$$

$$\mathcal{H}_{22}^{(A)}(y_2, y_2) = H_{22} = H_{(02)} = y_2^2 - a_{22}$$

$$\mathcal{H}_{111}^{(A)}(y_1, y_1, y_1) = H_{111} = H_{(30)} = y_1^3 - 3a_{11}y_1$$

$$\mathcal{H}_{112}^{(A)}(y_1, y_1, y_2) = H_{112} = H_{(21)} = y_1^2 y_2 - 2a_{12}y_1 - a_{11}y_2$$

$$\mathcal{H}_{122}^{(A)}(y_1, y_2, y_2) = H_{122} = H_{(12)} = y_1 y_2^2 - 2a_{12}y_2 - a_{22}y_1$$

$$\mathcal{H}_{222}^{(A)}(y_2, y_2, y_2) = H_{222} = H_{(03)} = y_2^3 - 3a_{22}y_2$$

No matter how convenient any of the above forms is for the computation of the Ch-H polynomials none of them is well suited for use in other operations due to the dependence on the correlation coefficients involved in the polynomials. We also note that all the expansions are orthogonal expansions which must be reordered to form the corresponding Edgeworth asymptotic expansions. To make the above expansions suitable for other operations we expand $p_o(\underline{x})$ in terms of $p_o^*(\underline{x})$ in the same way as with random vectors. To illustrate, consider relation (A.43).

Using the expansion for $p_o(\underline{x})$ we get

$$p(\underline{x}) = p_o^*(\underline{x}) \sum_{k=0}^{\infty} \sum_{m_1 + \dots + m_n = k} \frac{\beta_k(t_1^{m_1}, \dots, t_n^{m_n})}{m_1! \sigma_1^{m_1} \dots m_n! \sigma_n^{m_n}}$$

$$\sum_{k_1=0}^{\infty} \dots \sum_{k_m=0}^{\infty} \frac{\rho_1^{k_1} \dots \rho_m^{k_m}}{k_1! \dots k_m!}$$

$$H_{\ell_1 + m_1}(x_1^o / \sigma_1) \dots H_{\ell_n + m_n}(x_n^o / \sigma_n)$$

l_i are found from combinations of k_i as discussed previously. To form the Edgeworth series we reorder the terms as usual to get

$$\begin{aligned}
 p(\underline{x}) = p_o^*(\underline{x}) & \sum_{k_1=0}^{\infty} \dots \sum_{k_m=0}^{\infty} \frac{\rho_1^{k_1} \dots \rho_m^{k_m}}{k_1! \dots k_m!} H_{\ell_1} (x_1^o/\sigma_1) \dots \\
 & \dots H_{\ell_n} (x_n^o/\sigma_n) + \sum_{m_1+\dots+m_n=3} \frac{1}{\sigma^3} \dots \\
 & \frac{\beta_3(t_1^{m_1}, \dots, t_n^{m_n})}{m_1! \dots m_n!} \sum_{k_1=0}^{\infty} \dots \sum_{k_m=0}^{\infty} \frac{\rho_1^{k_1} \dots \rho_m^{k_m}}{k_1! \dots k_m!} \dots \\
 & H_{\ell_1+m_1} (x_1^o/\sigma) \dots H_{\ell_n+m_n} (x_n^o/\sigma) + \sum_{m_1+\dots+m_n=4} \frac{1}{\sigma^4} \dots \\
 & \frac{\beta_4(\cdot)}{m_1! \dots m_n!} \sum_{k_1=0}^{\infty} \dots \sum_{k_m=0}^{\infty} \frac{\rho_1^{k_1} \dots \rho_m^{k_m}}{k_1! \dots k_m!} H_{\ell_1+m_1} (\cdot) \dots \\
 & H_{\ell_n+m_n} (\cdot) + \sum_{\substack{m_1+\dots+m_n=3 \\ p_1+\dots+p_n=3}} \dots + \dots \}
 \end{aligned}$$

Evidently, an expansion can also be obtained using the notation in (A.42). The extension to the case of a random process is now clear.

Probability Density Function of A Vector
of Random Processes

The joint probability density function for several random processes is treated in just the same way. Using notation compatible with the ordinary Ch-H polynomials, consider two random processes $x(t)$ and $y(t)$. Associated with them is the vector $\underline{m}_x(t)$ and the matrix $R_x(t_1, t_2)$ for x ; $\underline{m}_y(t)$, $R_y(t_1, t_2)$ for $y(t)$ and $R_{xy}(t_1, t_2)$ for the correlation between x and y . The mixed quasi-moment functions are

$$\beta_{(p)x, (q)y}(t_1, \dots, t_p; t'_1, \dots, t'_q) ; p, q = 0, 1, 2, \dots$$

The characteristic function of the multidimensional distribution of the quantities $x(t_1), \dots, x(t_p); y(t'_1), \dots, y(t'_q)$ is written in terms of the quasi-moment functions as follows

$$\begin{aligned} & f(z_1, \dots, z_p; w_1, \dots, w_q; t_1, \dots, t_p; t'_1, \dots, t'_q) \\ &= \exp \left\{ i \sum_{j=1}^p m_x(t_j) z_j + i \sum_{j=1}^q m_y(t'_j) w_j + \right. \\ & \quad i^2/2 \sum_{j,k=1}^p R_x(t_j, t_k) z_j z_k + i^2 \sum_{j,k=1}^{p,q} R_{xy}(t_j, t'_k) z_j w_k \\ & \quad \left. + i^2/2 \sum_{j,k=1}^q R_y(t'_j, t'_k) w_j w_k \right\} \sum_{r,s} \frac{i^{r+s}}{r!s!} \sum_{\nu, \mu} \beta_{(r)_x, (s)_y} \\ & \quad (t_{\nu_1}, \dots, t_{\nu_r}; t'_{\mu_1}, \dots, t'_{\mu_s}) z_{\nu_1} \dots z_{\nu_r} w_{\mu_1} \dots w_{\mu_s} \end{aligned}$$

To this characteristic function corresponds the probability density

$$p(\underline{x}, \underline{y}; \underline{t}, \underline{t}') = \frac{(\det A)^{1/2}}{(2\pi)^{(p+q)/2}} e^{-\frac{1}{2} \sum_{i,j} a_{ij} u_i u_j} .$$

$$\sum_{r,s} \sum_{\substack{\mu_1 + \dots + \mu_p = r \\ \nu_1 + \dots + \nu_q = s}} \frac{\beta_{(r)\underline{x}, (s)\underline{y}}(\mu_1, \dots, \mu_p; \nu_1, \dots, \nu_q)}{\mu_1! \dots \mu_p! \nu_1! \dots \nu_q!} .$$

$$H_{(\mu_1, \dots, \mu_p, \nu_1, \dots, \nu_q)}(u_1, \dots, u_{p+q})$$

where,

$$A = \{a_{ij}\} = \begin{bmatrix} \{R_x(t_\alpha, t_\gamma)\} & \{R_{xy}(t_\alpha, t'_\sigma)\} \\ \{R_{yx}(t'_\sigma, t_\gamma)\} & \{R_y(t'_\sigma, t'_\rho)\} \end{bmatrix}^{-1}$$

$$u_i = \begin{cases} x_i^o & i \leq p \\ y_i^o & i > p \end{cases} , \quad \begin{matrix} i, j = 1, \dots, p+q \\ \alpha, \gamma = 1, \dots, p \\ \sigma, \delta = 1, \dots, q \end{matrix}$$

In terms of the generalized Ch-H polynomials the expression for the density function becomes

$$p(\underline{x}, \underline{y}; \underline{t}, \underline{t}') = \frac{(\det A)^{1/2}}{(2\pi)^{(q+p)/2}} e^{-\frac{1}{2} \sum_{j=1}^p x_j^o Q_j - \frac{1}{2} \sum_{j=1}^q y_j^o T_j} .$$

$$\sum_{k, \ell=0}^{\infty} \frac{1}{k! \ell!} \sum_{\nu_1, \dots, \nu_k=1}^p \sum_{\mu_1, \dots, \mu_\ell=1}^q \beta_{(k)\underline{x}, (\ell)\underline{y}}(\underline{t}, \underline{t}') .$$

$$\mathcal{H}_{\nu_1 \dots \nu_k \mu_1 \dots \mu_k}^{(A)}(Q_{\nu_1}, \dots, Q_{\nu_k}; T_{\mu_1}, \dots, T_{\mu_k})$$

where,

$$Q_i = \sum_{j=1}^p a_{ij} x_j^o + \sum_{\mu=1}^q a_{i,p+\mu} y_\mu^o$$

$$T_\mu = \sum_{j=1}^p a_{p+\mu,j} x_j^o + \sum_{\delta=1}^q a_{p+\mu,q+\delta} y_\delta^o$$

Again, the formulas must be simplified by expanding the multidimensional normal density function and must be rearranged to form the Edgeworth series. The working details of these operations are now clear. The extension to more than two random processes is also obvious.

Summary and Conclusions

In this appendix the problem of analytic approximate representation of the probability density function has been treated in some detail. Emphasis has been on the series type of representation. This type of representation is quite general especially as extended to the multidimensional functions. The Edgeworth series form of this type of representation has been found quite satisfactory for all practical purposes. A potential application of the representation is in evaluating mathematical expectation operations of nonlinear functions where the exact density function is unknown. Another application which is recommended for further investigation is in connection with nonlinear estimation and control problems. A representation like this may be employed for the

density function in the Bayesian approach to the estimation problem.

It is known that the Bayesian approach is quite powerful. The only difficulty is a proper characterization of the density function to which this type of representation may prove a satisfactory solution. The same remark applies also to the optimal nonlinear control problem. Of course, the representation is approximate. So the results of its application to optimal problems will not be the optimal results. However, these near-optimal results may be brought as close as desired to the optimal.

Finally, the method may also be used in connection with the functional approach to nonlinear systems analysis as a first step towards the development of that approach. Actually, a suitable representation of the density function has always been a main difficulty in all three main methods of nonlinear analysis known today, namely, the functional theory of Pugachev and Wiener, the Markov theory and the linearization theory. The feasibility of applying the representation described above to these basic methods of analysis is recommended for further investigation.

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