THREE-DIMENSIONAL KINEMATIC ANALYSIS OF TANGENT-PLANE MOTION

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CHAPTER I

INTRODUCTION

A mechanism consisting of links and kinematic pairs can generally be synthesized to produce a desired motion with reasonable accuracy. The prescribed path to be generated by a mechanism may be described either by a series of finitely or infinitesimally separated positions of a point. When such infinitesimally separated positions of a point-path are to be satisfied, there exists a need to develop mechanism synthesis theory that takes into consideration the higher order properties of the point path. A curve or a surface may be generated in general by a point, a line, or a plane moving with the coupler-link of a mechanism with four or more links. For planar motion, a line connected to the coupler-link of a mechanism will generatre an envelope. The line is called the tangent-line. For space motion, a plane connected to the coupler link of a mechanism will envelop a surface and the plane is called the tangent-plane.

The present work will develop the curvature theory and its application in mechanism synthesis to generate with a prescribed degree of accuracy an enveloping surface drawn by a plane executing space motion.

1.1 Background Review

We note that in recent years there appears to be considerable interest in the development of higher order curvature theory. The classical curvature theory, also known as the infinitesimal circular Burmester theory, provides the necessary tools to synthesize a planar mechanism for the generation of circular arc and straight-line segment in a coupler curve. This circular Burmester theory was generalized by Freudenstein [12]. His contribution was to characterize a planar curve to the nth order within stretch rotation by (n - 2) dimensionless characteristic numbers. The characteristic equations were derived to locate on the moving plane those points whose trajectories have the same characteristic numbers. The well known inflection circle, cubic of stationary curvature and so on of the infinitesimal Burmester theory, are special cases of the generalized theory developed by Freudenstein. The importance of this generalized theory was highly stressed by Veldkamp [51] who examined the problem using instantaneous invariants. The concept of instantaneous invariants to study the infinitesimal motion of a rigid body was first introduced by Bottema [7]. Veldkamp [48, 50] elaborated on it and extended this concept to study three-dimensional motion. According to the concept of instantaneous invariants, a set of numbers are used to characterize an instantaneous motion. Any mechanism reproducing a sufficient number of these characteristic numbers closely approximates the reference motion. Hence, the use of instantaneous invariants is recognized as one of the most efficient tools to characterize a rigid body motion and is frequently used in curvature theories. For example, using instantaneous invariants, Kamphuis [22], Roth and Yang [54] developed the curvature theory of point-path in spherical motion. Using the analogous approach, Veldkamp [47], Siddhanty and

Soni [39], and Hsia [16] developed the point-path curvature theory for the general space motion. The curvature and torsion including their derivatives were used to characterize a twisted curve. The characteristic equations were derived in terms of instantaneous invariants, and some results similar to the infinitesimal Burmester theory of planar motion were obtained. Veldkamp [47] pointed out that equivalence to Ball's point and Burmester's point do not exist for space motion generally.

A first study of line trajectories in space motion appears to be due to Disteli [10]. He showed that the line trajectories were the spatial analogy of planar path trajectories. Yang, Roth and Kirson [24,59] used dual vectors and the principle of transference to dualize some of the results from spherical curvature theory to spatial curvature theory of ruled surfaces. McCarthy and Roth [29] later reexamined this problem without using the principle of transference.

Just as a point traces a path, a line in planar motion envelops a curve. This line is called the tangent-line. The significant contributions in tangent-line envelope curvature theory appear to be due to Allievi [1] and Bereis [5]. Using instantaneous invariants, Soni, Siddhanty, and Ting [41] developed a tangent-line envelope curvature theory. The approach used in the development of this theory is analogous to that developed by Freudenstein. For the tangent-line envelope curvature theory, a family of lines can be found to envelop cusps and a tangent-line can be located to envelop a double cusp. These properties are analogous to the inflection circle and Ball's point of Burmester theory. However, nothing similar to the Burmester point can be found.

A point and a line are dual concepts in planar geometry. For a planar figure consisting of a number of points and lines, a dual configuration can be formed by replacing every point by a line, and every line by a point. Corresponding to a point tracing a path, a moving tangent-line envelops a curve. Hence, the dual to a point trajectory is a family of tangent-lines which envelop a curve. Using line coordinates, Hunt and Fichter [18] derived the equation of the tangent-line envelop of a four-bar linkage and showed the similarities between the tangent-line envelope and point-trajectory theories.

In space geometry, a point and a plane form the duality, and a line is a self-dual construct in the sense that the system of points on a line is dual to the system of planes through a line. For a geometrical configuration consisting of points, lines, and planes, the dual configuration is formed by replacing every point by a plane, every line by a line, and every plane by a point. Hence, we may expect that these will be analogy existing between the locus of a point and a family of tangent-planes which envelop a surface. From theoretical point of view, it is just as important to study the path of a plane as it is to study the path of a point. For example, a surface requiring a high degree of accuracy in a localized region may be generated by an envelope of a moving plane guided by a coupler-link of a mechanism. This kinematic importance of the tangent-plane envelope has been recognized by Bottema and Roth [8] and Hunt [19].

1.2 Organization

For the reader's convenience, in the present chapter, some background material is summarized. Chapter II introduces the subjects of dual vector calculus. Dual vectors are extremely convenient to describe rigid body space motion. They are especially useful in describing the motion of a straight line. For more information, References [9] and [57] are recommended.

Chapter III describes the instantaneous invariants of a general rigid body. In the derivation of the instantaneous invariants, both ordinary vectors and dual vectors are used frequently and the principle of transference between them is emphasized. This chapter provides insight into the usefulness of the instantaneous invariants and may be suplemented with Reference [24].

The kinematics of a moving plane are investigated in Chapter IV. It describes the general theory of tangent-plane envelopes. The higher order properties of a developable are explicitly described and the characteristic equations are derived up to third order.

Chapter V discusses some special cases of a tangent-plane motion. Such special cases are basically the analogy of the subjects in the infinitesimal Burmester theory.

The tangent-plane envelope curvature theory in spherical kinematics is presented in Chapter VI. The results of the spherical tangent-plane envelope curvature theory are new. Compared to the complexity of general space motions, these results promise to have significant importance in understanding spherical motion of a tangent-plane.

In Chapter VII, a special motion and the general synthesis procedure of a developable are briefly presented.

CHAPTER II

DUAL VECTORS

2.1 Dual Numbers

A dual number $\hat{a}$ is an ordered pair of real numbers associated with an operator ε which has the property $\varepsilon^2 = 0$:

$$\hat{a} = a + \varepsilon a^\circ \tag{2.1}$$

where a is the real part and a° the dual part of the dual number $\hat{a}$. When $a \neq 0$, $\hat{a}$ is a proper dual and when $a = 0$, $\hat{a}$ becomes a pure dual. Any real number is a dual number with a diminished dual part.

The equality of dual numbers is similar to that of complex numbers. Let $\hat{b} = b + \varepsilon b^\circ$ be another dual number.

$$\hat{a} = \hat{b} \quad \text{only when} \quad a = b \text{ and } a^\circ = b^\circ \tag{2.2}$$

The operations of dual numbers are the same as in the ordinary algebra followed by setting $\varepsilon^n = 0$ $(n = 2, 3, \ldots)$. Therefore, we have

$$\left.\begin{aligned} \hat{a} \pm \hat{b} &= (a \pm B) + \varepsilon\,(a^\circ \pm b^\circ) \\ \hat{a}\hat{b} &= ab + \varepsilon\,(a^\circ b + ab^\circ) \\ \hat{a}/\hat{b} &= \frac{a + \varepsilon a^\circ}{b + \varepsilon b^\circ} \cdot \frac{b - \varepsilon b^\circ}{b - \varepsilon b^\circ} = \frac{a}{b} + \varepsilon\frac{a^\circ b - ab^\circ}{b^2} \end{aligned}\right\} \tag{2.3}$$

It is observed that division by a pure dual is not defined.

The expansion of a function follows the Taylor's series expansion.

$$f(\hat{a}) = f(a + \varepsilon a^\circ) = f(a) + \varepsilon a^\circ df(a)/da \tag{2.4}$$

for example,

$$\left.\begin{aligned} e^{\hat{a}} &= e^{(a + \varepsilon a^\circ)} = e^{a} + \varepsilon a^\circ e^{a} \\ a &= \sqrt{(a + \ a^\circ)} = \sqrt{a} \ + \ a^\circ/2\sqrt{a} \end{aligned}\right\} \qquad (2.5)$$

An example of a dual number is the dual angle subtended by two lines in space. In Figure 1 the dual angle between the two straight lines is $\hat{\theta}$,

$$\hat{\theta} = \theta + \varepsilon\theta^\circ \qquad (2.6)$$

where θ is the projected angle between two lines and θ° is the shortest distance between them. In the case of two parallel lines, we have $\theta = 0$ and the dual angle between them is a pure dual. If two lines intersect, we have $\theta^\circ = 0$ and the dual angle is a real number.

The trigonometric functions of dual angels can be obtained by following the Taylor's series expansion of Equation (2.4). For example,

$$\left.\begin{aligned} \sin\hat{\theta} &= \sin(\theta + \varepsilon\theta^\circ) = \sin\theta + \varepsilon\theta^\circ\cos\theta \\ \cos\hat{\theta} &= \cos(\theta + \varepsilon\theta^\circ) = \cos\theta - \varepsilon\theta^\circ\sin\theta \\ \tan\hat{\theta} &= \tan(\theta + \varepsilon\theta^\circ) = \tan\theta + \varepsilon\theta^\circ\sec\theta \end{aligned}\right\} \qquad (2.7)$$

All identities for ordinary trigonometry hold true for dual angles.

2.2 Sliding Vectors

A vector implies a quantity defined by direction and magnitude, but is never restricted in position in any way. A sliding vector is a vector confined to a line in space. It can be specified by its vector $\underline{A}$ and its $\underline{A}_p$ with respect to a point P (the origin, generally). $\underline{A}$ and $\underline{A}_p$ are the Plücker vectors of the sliding vector (Figure 2). With the operator $\varepsilon(\varepsilon^2 = 0)$, the sliding vector may be expressed by a dual vector $\hat{\underline{A}}$.

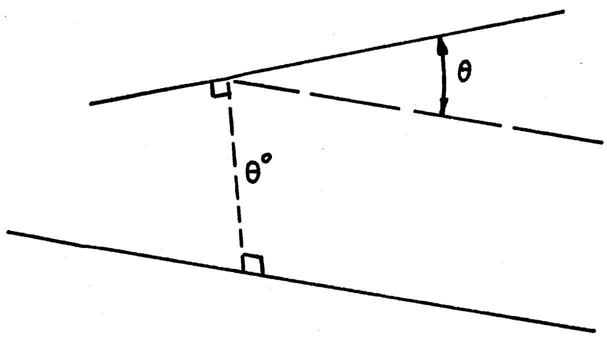


Figure 1. Dual angle $\hat{\theta} = \theta + \varepsilon\theta°$

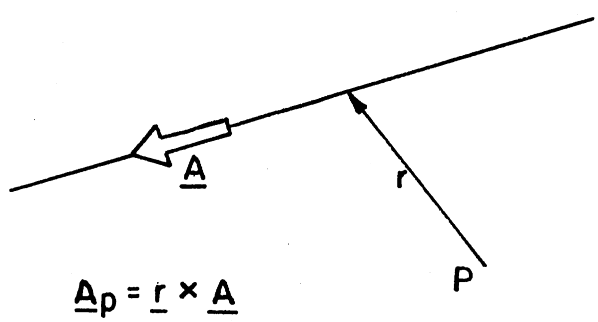


Figure 2. Sliding Vector $\hat{\underline{A}} = \underline{A} + \varepsilon\underline{A}_p$

$$\hat{\underline{A}} = \underline{A} + \varepsilon \underline{A}_p \quad (\underline{A} \cdot \underline{A}_p = 0) \tag{2.8}$$

The primary part $\underline{A}$ of the sliding vector gives the direction of the sliding vector and the dual part $\underline{A}_p$ specifies the location of the sliding vector. A dual vector is always referred to a point and one may indicate its reference point with a suffix at the dual part. For a general dual vector, the restriction $\underline{A} \cdot \underline{A}_p = 0$ does not need to be true.

Let the moment of the same sliding vector be taken with respect to another point Q. The dual vector becomes

$$\hat{\underline{A}} = \underline{A} + \varepsilon \underline{A}_q \tag{2.9}$$

Assume $\underline{X}$ is any point on the line of the sliding vector. The moments of the sliding vector with respect to P and Q are respectively,

$$\underline{A}_p = \underline{PX} \times \underline{A}$$

$$\underline{A}_q = \underline{QX} \times A$$

Hence, we have

$$\underline{A}_p - \underline{A}_q = (\underline{PX} - \underline{QX}) \times \underline{A} = \underline{PQ} \times \underline{A}$$

or

$$\underline{A}_p = \underline{A}_q + \underline{PQ} \times \underline{A} \tag{2.10}$$

From equations (2.8) to (2.10), we have

$$\hat{\underline{A}} = \underline{A} + \varepsilon\,(A_q + PQ \times A) \tag{2.11}$$

$$= \underline{A} + \varepsilon \underline{A}_p$$

Equation (2.11) gives the relationship of two dual vectors which present the same sliding vector but are referred to different points P and Q.

In case that the reference point is on the line of the sliding vector, the dual part of the representing dual vector becomes zero and the dual vector has the form of an ordinary vector. However, it is still a dual vector, since its zero dual part indicates the location of the sliding vector.

In Equation (2.8), if $\underline{A}$ is a unit vector ($\underline{A} \cdot \underline{A} = 1$), then $\hat{\underline{A}}$ represents a unit sliding vector. The dual vector of a unit sliding vector will be utilized to represent a unique straight line in space and it is also called unit screw.

The operations of dual vectors are summarized in the next section.

2.3 Dual Vectors

A dual vector is always referred to a point. It is an ordered pair of vectors associated with the dual operator $\varepsilon(\varepsilon^2 = 0)$. Let $\hat{\underline{A}}$ be a dual vector referred to the point O. We have

$$\hat{\underline{A}} = \underline{A} + \varepsilon\underline{A}_o \tag{2.12}$$

in which $\underline{A}$ is the primary part and is independent of the reference point; $\underline{A}_o$ is the dual part. In Equation (2.12), there is no restriction $\underline{A} \cdot \underline{A}_o = 0$ as in the sliding vector. When the reference point is shifted from O to P, the dual vector becomes

$$\hat{\underline{A}} = \underline{A} + \varepsilon(\underline{A}_o + \underline{PO} \times A) = \underline{A} + \varepsilon\underline{A}_p \tag{2.13}$$

This transformation is called "reduction." Since

$$\underline{A}_p = \underline{A}_o + \underline{PO} \times \underline{A} \tag{2.14}$$

we may obtain

$$\underline{A} \cdot \underline{A}_p = \underline{A} \cdot (\underline{A}_o + \underline{PO} \times \underline{A}) = \underline{A} \cdot \underline{A}_o \tag{2.15}$$

Equation (2.15) shows that the scalar product of the primary part and the dual part is independent of the reference point.

Since a dual vector is the combination of two ordinary vectors, the rules for vector albgebra are also valid for dual vector algebra by setting $\varepsilon^2 = 0$. The principle of transference of dual vector algebra states that all vector identities of ordinary vector algebra are also valid for dual vectors if all the vectors and real numbers are replaced by dual vectors and dual numbers, respectively. Some rules which may be used later are presented in the following.

Let the dual vector $\underline{A} = A + \varepsilon\underline{A}°$ be referred to the origin of the coordinate system.

1. Dual Components: The two vectors $\underline{A}$ and $\underline{A}°$ of the primary and dual parts may be expressed as

$$\left.\begin{aligned} \underline{A} &= a_1\underline{I} + a_2\underline{J} + a_3\underline{K} \\ \underline{A}° &= a_1^\circ\,\underline{I} + a_2^\circ\underline{J} + a_3^\circ\underline{K} \end{aligned}\right\} \tag{2.16}$$

where $\underline{I}$, $\underline{J}$, and $\underline{K}$ are the unit vectors along the three axes of the coordinate system. The dual vector $\underline{A}$ can be written as

$$\hat{\underline{A}} = \hat{a}_1\,\hat{\underline{I}} + \hat{a}_2\hat{\underline{J}} + \hat{a}_3\hat{\underline{K}} \tag{2.17}$$

where $\hat{a}_i = a_i + \varepsilon a°_i$ $(i = 1, 2, 3)$ are the dual components of $\hat{\underline{A}}$. $\hat{\underline{I}}$, $\hat{\underline{J}}$, and $\hat{\underline{K}}$ are the unit sliding vectors of three coordinates axes with reference to the origin.

2. Unit Screw: The unit dual vector of $\hat{\underline{A}}$ is

$$\hat{a} = \hat{\underline{A}}\,/\hat{\alpha} \tag{2.18}$$

where $\hat{\alpha} = |\hat{\underline{A}}| = (\hat{a}_1{}^2 + \hat{a}_2{}^2 + \hat{a}_3{}^2)^{\frac{1}{2}}$ is the dual length of the dual vector $\hat{\underline{A}}$. Let $\hat{\alpha} = \alpha + \varepsilon\alpha°$. From Equations (2.18) and (2.5), we may obtain

$$\left.\begin{aligned} \alpha &= (a_1{}^2 + a_2{}^2 + a_3{}^2)^{\frac{1}{2}} = |\underline{A}| \\ \alpha° &= (a_1a°_1 + a_2a°_2 + a_3a°_3)/\alpha = \underline{A}\cdot\underline{A}_o/\alpha \end{aligned}\right\} \tag{2.19}$$

One may observe that since $\underline{A} \cdot \underline{A}_0$ is independent of the reference point of the dual vector, the dual length of a dual vector is invariant to the reference point.

From Equations (2.18) and (2.19), we have the unit dual vector $\hat{\underline{a}}$,

$$\hat{\underline{a}} = (\underline{A} + \varepsilon\underline{A}^\circ)/(\alpha + \varepsilon\alpha^\circ)$$

$$= \frac{\underline{A} + \varepsilon\underline{A}^\circ}{\alpha + \varepsilon\alpha^\circ} \cdot \frac{\alpha - \varepsilon\alpha^\circ}{\alpha - \varepsilon\alpha^\circ}$$

$$= \underline{A}/\alpha + \varepsilon\,(\alpha\underline{A}^\circ - \alpha^\circ\underline{A})/\alpha^2$$

Let

$$\hat{\underline{a}} = \underline{a} + \varepsilon\underline{a}^\circ$$

then

$$\left.\begin{aligned} \underline{a} &= \underline{A}/\alpha \\ \underline{a}^\circ &= (\alpha\underline{A}^\circ - \alpha^\circ\underline{A})/\alpha^2 \end{aligned}\right\} \qquad (2.20)$$

from Equations (2.19), we may obtain

$$\underline{a} \cdot \underline{a}^\circ = 0$$

Therefore, a unit dual vector is a unit sliding vector representing a line in space. A unit dual vector or a unit sliding vector shall be called a unit screw. A unit screw $\hat{\underline{a}}$ has the properties,

$$\left.\begin{aligned} \underline{a} \cdot \underline{a} &= 1 \\ \underline{a} \cdot \underline{a}^\circ &= 0 \end{aligned}\right\} \qquad (2.21a)$$

and

$$|\hat{\underline{a}}| = 1 \qquad (2.21b)$$

From Equation (2.18), we may express the dual vector $\hat{\underline{A}}$ in terms of its dual length and unit screw. Thus,

$$\hat{\underline{A}} = \hat{\alpha}\hat{\underline{a}} = (\alpha + \varepsilon\alpha^\circ)\hat{\underline{a}} \qquad (2.22)$$

Let

$$\sigma = \alpha^{\circ}/\alpha$$

we have

$$\underline{\hat{A}} = \alpha(1 + \varepsilon\sigma)\ \underline{\hat{a}} \tag{2.23}$$

σ is the pitch of the dual vector $\underline{\hat{A}}$.

3. <u>The Products of Dual Vectors</u>: Let $\underline{\hat{B}} = \hat{\beta}\underline{\hat{b}}$ be another dual vectors with dual length $\hat{\beta}$ and unit screw $\underline{\hat{b}}$.

The scalar product of $\underline{\hat{A}}$ and $\underline{\hat{B}}$ is

$$\underline{\hat{A}} \cdot \underline{\hat{B}} = (\hat{\alpha}\underline{\hat{a}}) \cdot (\hat{\beta}\underline{\hat{b}}) = \hat{\alpha}\hat{\beta}(\underline{\hat{a}} \cdot \underline{\hat{b}}) = \hat{\alpha}\hat{\beta} \cos \hat{\theta} \tag{2.24}$$

where $\hat{\theta}$ is the dual angle between the lines of unit screws $\underline{\hat{a}}$ and $\underline{\hat{b}}$.

The vector product of $\underline{\hat{A}}$ and $\underline{\hat{B}}$ is

$$\underline{\hat{A}} \times \underline{\hat{B}} = (\hat{\alpha}\underline{\hat{a}}) \times (\hat{\beta}\underline{\hat{b}}) = \hat{\alpha}\hat{\beta}\ (\underline{\hat{a}} \times \underline{\hat{b}}) = (\hat{\alpha}\hat{\beta} \sin\hat{\theta})\underline{\hat{c}} \tag{2.25}$$

where $\underline{\hat{c}}$ is the unit screw of the common perpendicular of $\underline{\hat{a}}$ and $\underline{\hat{b}}$ and its direction is defined by following the right hand rule as in ordinary vector product.

If more information is needed, the reader is suggested to refer to References [9] and [57].

CHAPTER III

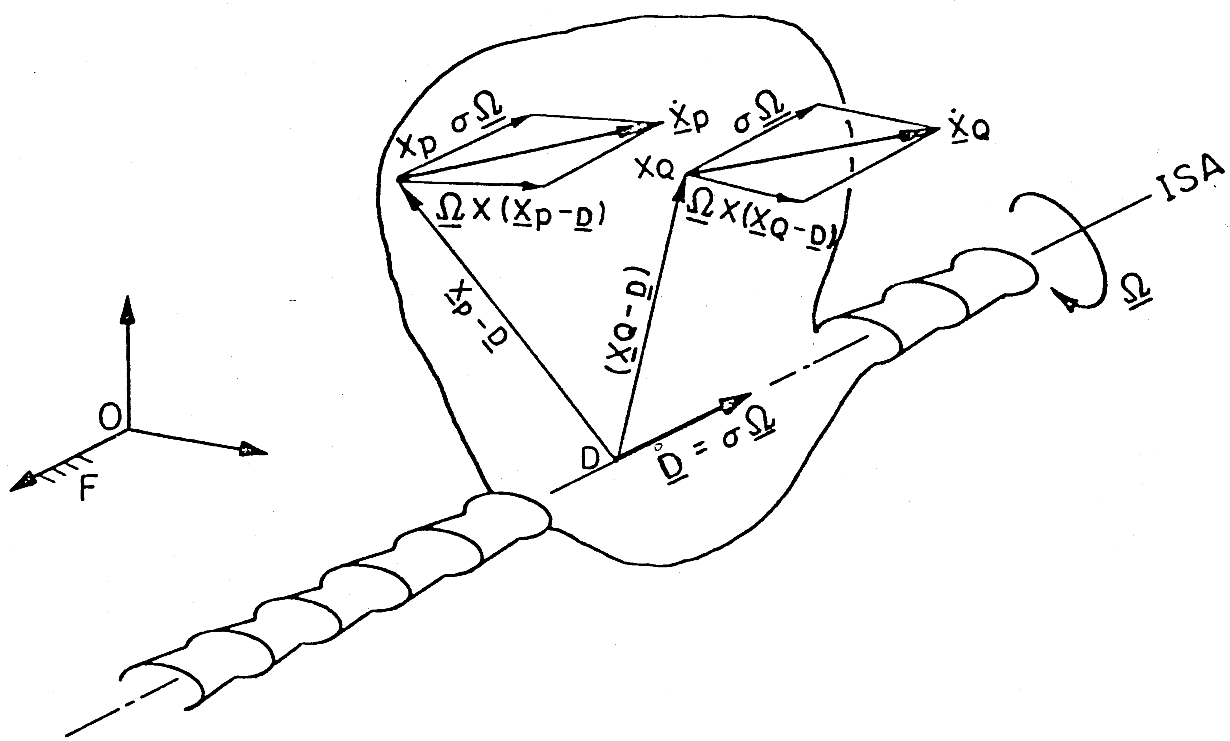
INSTANTANEOUS KINEMATICS OF A RIGID BODY MOTION

For a rigid body motion, time is generally selected as the independent parameter and the position, velocity, and acceleration of the rigid body are utilized to describe the motion at a certain moment. There are cases that two rigid bodies move through the same path with different velocities and accelerations. Obviously, there exist common characteristics between these two motions, such characteristics may be discribed in terms of the geometric properties of the motion.

In this chapter, the geometry of a rigid body moion will be studied. The concept of screw motion will be utilized to describe the geometry of a general rigid body motion and the motion itself will be characterized by a set of instantaneous invariants. Any motion reproducing a sufficient number of these instantaneous invariants closely approximates the reference motion.

3.1 Generalized Screw Motion

A space rigid body motion may be regarded as a generalized screw motion in which the screw axis may not be stationary and the geometry of the motion is represented by its pitch and the location of the screw axis (Figure 3).



$$\dot{\underline{X}}p = \sigma\underline{\Omega} + \underline{\Omega} \times (\underline{X}p - D)$$

Figure 3. A Screw Motion

Let a moving coordinate system M be attached to a moving body. The motion of the rigid body with respect to a fixed coordinate system F may be expressed by

$$\underline{X} = [A]\,\underline{x} + \underline{D} \tag{3.1}$$

where [A] is the orthogonal matrix relating the directions of the axes of both coordinate systems; $\underline{D}$ is the position vector of the origin of the moving system M; and $\underline{X}$ and $\underline{x}$ are the positions, in systems F and M, respectively, of the same point P on the moving body (Figure 4). In Equation (3.1) and the following, a position vector such as $\underline{X}$, $\underline{x}$, and $\underline{D}$ also represents a column matrix. The orthogonal matrix [A] represents the pure rotational motion of the rigid body and the column vector $\underline{D}$ represents the pure translational motion. Both [A] and $\underline{D}$ are functions of a motion parameter such as time.

From Equation (3.1), the inverse motion, the motion of F relative to M, is expressed as

$$\underline{x} = [A]^{-1}\,(\underline{X} - \underline{D}) \tag{3.2}$$

Since [A] is orthogonal, we have

$$[A]^{-1} = [A]^T$$

Differentiating Equation (3.1) with respect to time, we obtain the velocity of point P.

$$\dot{\underline{X}} = [\dot{A}]\underline{x} + \dot{\underline{D}} \tag{3.3}$$

Substituting Equations (3.2) into (3.3), we have

$$\dot{\underline{X}} = [\dot{A}][A]^T\,(\underline{X} - \underline{D}) + \dot{\underline{D}} \tag{3.4}$$

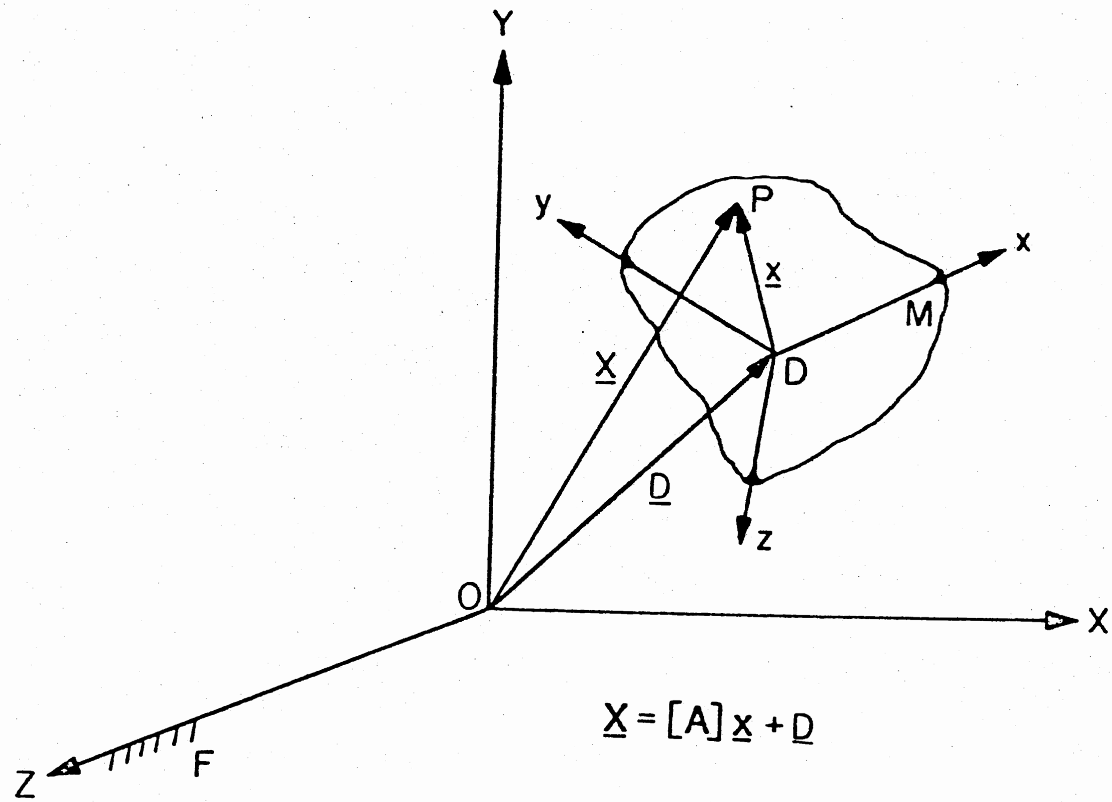


Figure 4. The Fixed System F and the Moving System M.

Since $[A]\,[A]^T = [I]$, differentiating it, we have

$$[\dot{A}]\,[A]^T + [A]\,[\dot{A}]^T = 0$$

or

$$[\dot{A}][A]^T + ([\dot{A}]\,[A]^T)^T = 0 \tag{3.5}$$

Therefore, $[A][A]^T$ is a skew matrix. Let

$$[\dot{A}][A]^T = [\Omega] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \tag{3.6}$$

Equation (3.4) becomes

$$\dot{\underline{X}} = [\Omega]\,(\underline{X} - \underline{D}) + \dot{\underline{D}} \tag{3.7}$$

This equation is equivalent to the vector equation,

$$\dot{\underline{X}} = \underline{\Omega} \times (\underline{X} - \underline{D}) + \dot{\underline{D}} \tag{3.8}$$

where the vector, $\underline{\Omega} = (\omega_1, \omega_2, \omega_3)$, is the angular velocity of the moving body and $\dot{\underline{D}}$ is the linear velocity of the origin of the moving system M.

In planar motion, the linear velocity $\dot{\underline{D}}$ is always orthogonal to the angular velocity $\underline{\Omega}$. Hence, the velocity pole $\underline{X}_v$ satisfying

$$\dot{\underline{X}}_v = 0 = \underline{\Omega} \times (\underline{X}_v - \underline{D}) + \dot{\underline{D}} \tag{3.9}$$

can be found.

In a general space motion, the velocity $\dot{\underline{D}}$ is generally not orthogonal to $\underline{\Omega}$ and no velocity pole can be found. For a general space motion, there are points whose velocities are parallel to the angular velocity. Let these velocities be $\sigma\underline{\Omega}$, where σ is to be determined. From Equation (3.8), the locus of these points can be obtained by solving

$$\dot{\underline{X}} = \sigma\underline{\Omega} = \underline{\Omega} \times (\underline{X} - \underline{D}) + \dot{\underline{D}} \tag{3.10}$$

In Equation (3.10), taking vector product with $\underline{\Omega}$, we have

$$\underline{\Omega} \times [\underline{\Omega} \times (\underline{X} - \underline{D})] = - \underline{\Omega} \times \dot{\underline{D}} \tag{3.11}$$

It can be rewritten as

Eqn of a st. line passing thru $\bar{a}$ & ‖el to $\bar{u}$

$\bar{r} = \bar{a} + t\,\bar{u}$

$$\begin{aligned} \underline{X} = {} & [(\underline{\Omega} \cdot \underline{\Omega})^{-1} \underline{\Omega} \times \dot{\underline{D}} + \underline{D}\,] + \\ & (\underline{\Omega} \cdot \underline{\Omega})^{-1}[\underline{\Omega} \cdot (\underline{X} - \underline{D})]\, \underline{\Omega} \end{aligned} \tag{3.12}$$

It is obvious that Equation (3.10) gives tow linear independent equations and its solution set is on a straight line. From Equation (3.12), we know that this straight line must pass through the point Q with position vector,

REFER (1-24) OF NAYAK.

$$\underline{Q} = (\underline{\Omega} \cdot \underline{\Omega})^{-1}\, \underline{\Omega} \times \dot{\underline{D}} + \underline{D} \tag{3.13}$$

and have the direction of $\underline{\Omega}$. Hence, the solution to Equation (3.10) can be expressed by

$$\underline{X} = \underline{Q} + e\underline{\Omega} \tag{3.14}$$

where e is any real number.

By taking scalar product with $\underline{\Omega}$, Equation (3.10) becomes

$$\sigma\underline{\Omega} \cdot \underline{\Omega} = \underline{\Omega} \cdot \dot{\underline{D}}$$

or

$$\sigma = \underline{\Omega} \cdot \dot{\underline{D}}/\underline{\Omega} \cdot \underline{\Omega} \tag{3.15}$$

From Equation (3.10), (3.14), and (3.15), we know that for a general rigid body motion ($\Omega \neq 0$), there exists a straight line in the direction of the angular velocity and all points on it have the same linear velocity $\sigma\underline{\Omega}$. This straight line is the central axis of the system.

If the origin $\underline{D}$ of the moving system M is on the central axis, from Equations (3.10), (3.14), and (3.15), we have

$$\underline{D} = \underline{Q} + e\underline{\Omega}$$

∴ the origin is on the ISA, $\bar{X} \leftarrow \bar{D}$) from (3.14)
$\underline{X} \leftarrow$ any pt on the ISA

and

$$\dot{\underline{D}} = \sigma\underline{\Omega}$$

← ∴ any pt on the ISA

Substituting them into Equation (3.8), we obtain

$$\dot{\underline{X}} = \underline{\Omega} \times (\underline{X} - \underline{Q}) + \sigma\underline{\Omega} \tag{3.16}$$

where $\underline{Q}$ is the position vector of a point on the central axis. Equation (3.16) shows that the velocity of any point on the moving body contains two components: $\sigma\underline{\Omega}$ is the component in the direction of the central axis and it is an invariant for all points on the moving body; the other component, $\underline{\Omega} \times (\underline{X} - \underline{Q})$ is orthogonal to the central axis and its magnitude is proportional to the distance to the central axis. The velocity distribution (the velocities of all points) of a moving body described in Equation (3.16) is exactly the same as that of a rotating screw (Figure 3).

Therefore, a general rigid body motion may be regarded as a generalized screw motion with the central axis as the instantaneous screw axis (ISA) and σ as the instantaneous pitch. The ISA, σ and $\underline{\Omega}$ are all functions of time and they specify the first order motion of the rigid body. Under the rigid body motion, the ISA traces a ruled surface in both fixed and moving systems. In the fixed system, this ruled surface is called fixed axode and in the moving system, it is called moving axode.

By using dual vector, the ISA can be represented by a unit screw. Since $\underline{Q}$ is the position vector of a point on the ISA, the unit screw of the ISA referred to the origin of the fixed system is

$$\hat{\underline{K}} = \underline{K} + \varepsilon\underline{Q} \times \underline{K} \tag{3.17}$$

where $\underline{K} = \underline{\Omega}/\Omega$, $\Omega = |\underline{\Omega}|$ and $\underline{Q}$ is defined in Equation (3.13). The ISA, $\hat{\underline{K}}$ is determined if the angular velocity $\underline{\Omega}$ and the linear velocity of any point are known.

If the origin of the reference system is on the ISA, $\underline{Q} \times \underline{K}$ diminishes and the unit screw $\hat{\underline{K}}$ becomes

$$\hat{\underline{K}} = \underline{K} \tag{3.18}$$

3.2 Some Geometry in Ruled Suface

A ruled suface is generated by the motion of a straight line. The infinitude of straight lines which lie on the surface are called its genrators. If the consecutive generators intersect, a ruled surface becomes a developable and if all generators intersect at a point, it becomes a cone.

Let $\underline{K}$ and $\underline{K}'$ be the unit vectors along two consecutive generators of a ruled surface (Figure 5). We may write

$$\underline{K}' = \underline{K} + d\underline{K} \tag{3.19}$$

where $\underline{K} \cdot \underline{K} = 1$ and $\underline{K} \cdot d\underline{K} = 0$. $d\underline{K}$ is orthogonal to $\underline{K}$. Take

$$\underline{I} = d\underline{K}/|d\underline{K}| \tag{3.20}$$

as the unit vector along the direction of $d\underline{K}$. The unit vector along the common perpendicular of the two consecutive generators is

$$\underline{J} = \underline{K} \times \underline{K}'/|\underline{K} \times \underline{K}'| \tag{3.21}$$

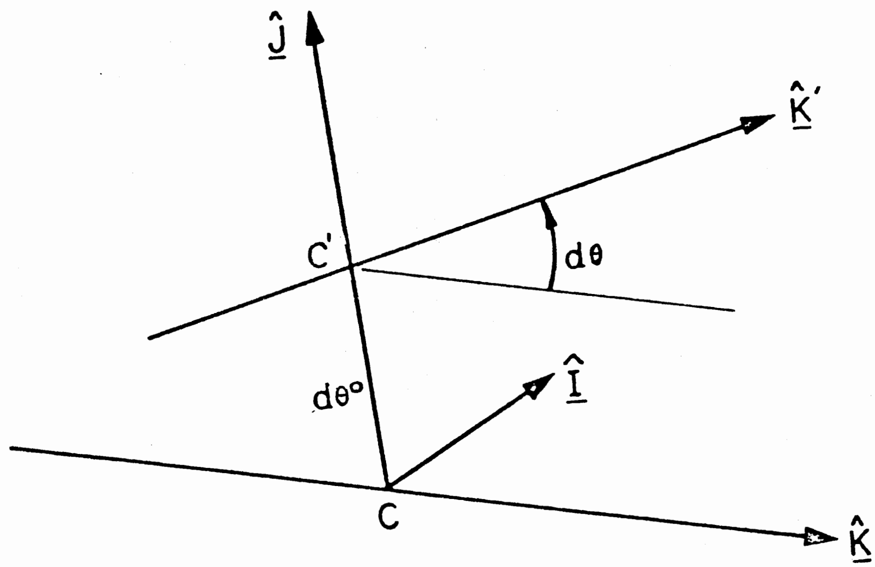


Figure 5. Two Consecutive Generators on a Ruled Surface

From Equations (3.19) to (3.20), the above equation may be rewritten as

$$\underline{J} = \underline{K} \times \underline{I} \tag{3.22}$$

Let $d\theta$ denote the angular displacement from $\underline{K}$ to $\underline{K}'$. We have $|d\underline{K}| = d\theta$ and the following relationships may be obtained [9].

$$\left.\begin{aligned} d\underline{K}/d\theta &= \underline{I} \\ d\underline{I}/d\theta &= \gamma\underline{J} - \underline{K} \\ d\underline{J}/d\theta &= -\gamma\underline{I} \end{aligned}\right\} \tag{3.23}$$

where the unit vectors $\underline{I}$, $\underline{J}$, and $\underline{K}$ are perpendicular to each other. If all the generators intersect at a point, $\underline{I}$, $\underline{J}$, and $\underline{K}$ form a perpendicular trihedron at the vertex of the cone. what cone?

In general, consecutive generators do not intersect. Let CC' be the shortest distance between the consecutive generators as in Figure 5. Point C is the central point (or striction point). A perpendicular trihedron with the three axes $\underline{I}$, $\underline{J}$, and $\underline{K}$ meet at the central point may be identified. These three axes may be identified by the unit screw $\hat{\underline{I}}$, $\hat{\underline{J}}$, and $\hat{\underline{K}}$, with reference to the origin. $\hat{\underline{K}}$ is the generator, $\hat{\underline{J}}$ is the common perpendicular from $\hat{\underline{K}}$ to the consecutive generator and $\hat{\underline{I}} = d\hat{\underline{K}}/|d\hat{\underline{K}}|$. $\hat{\underline{I}}$, $\hat{\underline{J}}$, and $\hat{\underline{K}}$ have the same directions of the unit vectors $\underline{I}$, $\underline{J}$, and $\underline{K}$ correspondingly and the following relationships.

$$\hat{\underline{J}} = \hat{\underline{K}} \times \hat{\underline{I}} \quad \text{and} \quad \hat{\underline{I}} = \hat{\underline{J}} \times \hat{\underline{K}} \tag{3.24}$$

The following relationships can be obtained directly from Equations (3.23) through the principle of transference [9, 24].

$$\left.\begin{aligned} d\hat{\underline{K}}/d\hat{\theta} &= \hat{\underline{I}} \\ d\hat{\underline{I}}/d\hat{\theta} &= \hat{\gamma}\hat{\underline{J}} - \hat{\underline{K}} \end{aligned}\right\} \tag{3.25}$$

$$d\hat{\underline{J}}/d\hat{\theta} = -\hat{\gamma}\hat{\underline{I}} \tag{3.25}$$

where

$$d\hat{\theta} = d\theta + \varepsilon d\theta^{\circ} \tag{3.26}$$

is the dual angle between $\hat{\underline{K}}$ and the consecutive generator. We note that $d\hat{\theta} = |d\hat{\underline{K}}|$ and the detail derivation of Equation (3.25) is exactly the same as Equations (3.23) except that dual vectors and dual numbers are used instead of ordinary vectors and real numbers through the whole process.

Equations (~~2.35~~ 3.25) are equivalent to (3.23) if all the dual parts diminish.

3.3 Central Point

On the axode traced by the ISA, the central point is the intersection point of the ISA and its common perpendicular with the consecutive ISA. On the moving body the central point has the minimum acceleration among all points on the ISA.

In Equations (3.13) and (3.14), the ISA is determined through the angular velocity and the linear velocity of any point on the moving body. To determine the central point on the ISA, both angular acceleration and the linear acceleration are also needed.

From Equation (3.14), any point X on the ISA is associated with a unique number e and its position vector is

$$\underline{X} = \underline{Q} + \underline{QX} \tag{3.27}$$

where

$$\underline{QX} = e\underline{\Omega} \ .$$

The velocity and acceleration of the point corresponding to e are

$$\dot{\underline{X}} = \dot{\underline{Q}} + \underline{\Omega} \times \underline{QX}$$

ORIGIN AT Q

$$\ddot{\underline{X}} = \ddot{\underline{Q}} + \dot{\underline{\Omega}} \times \underline{QX} + \underline{\Omega} \times (\underline{\Omega} \times \underline{QX})$$

or

$$\dot{\underline{X}} = \dot{\underline{Q}} \tag{3.28a}$$

$$\ddot{\underline{X}} = \ddot{\underline{Q}} + e\dot{\underline{\Omega}} \times \underline{\Omega} \tag{3.28b}$$

$\because \underline{QX} = e\underline{\Omega}$

where $\dot{\underline{Q}}$ and $\ddot{\underline{Q}}$ are the linear velocity and acceleration of the point determined in Equation (3.13). From Equation (3.13), let

$$\underline{DQ} = \underline{Q} - \underline{D} = \underline{\Omega} \times \dot{\underline{D}}/\Omega^2 \tag{3.29}$$

we may have

$$\underline{Q} = \underline{D} + \underline{DQ}$$

$$\dot{\underline{Q}} = \dot{\underline{D}} + \underline{\Omega} \times \underline{DQ}$$

ORIGIN AT D.

and

$$\ddot{\underline{Q}} = \ddot{\underline{D}} + \dot{\underline{\Omega}} \times \underline{DQ} + \underline{\Omega} \times (\underline{\Omega} \times \underline{DQ}) \tag{3.30}$$

Substituting Equation (3.29) into Equation (3.30). $\ddot{\underline{Q}}$ can be obtained as

$$\ddot{\underline{Q}} = \ddot{\underline{D}} + \dot{\underline{\Omega}} \times (\underline{\Omega} \times \dot{\underline{D}})/\Omega^2 - \underline{\Omega} \times \dot{\underline{D}} \tag{3.31}$$

The central point, which has the minimum acceleration among all points on the ISA, is determined by solving e in the equation

$$\partial(\ddot{\underline{X}} \cdot \ddot{\underline{X}})/\partial e = 0 \tag{3.32}$$

From Equations (3.28) and (3.32), we obtain

$$\ddot{\underline{Q}} \cdot (\dot{\underline{\Omega}} \times \underline{\Omega}) + e(\dot{\underline{\Omega}} \times \underline{\Omega}) \cdot (\dot{\underline{\Omega}} \times \underline{\Omega}) = 0 \tag{3.33}$$

In general space rigid body motion, $\underline{\dot{\Omega}} \times \underline{\Omega} \neq 0$. Therefore, the number e associated with the central point is

$$e_c = - \underline{\ddot{Q}} \cdot (\underline{\dot{\Omega}} \times \underline{\Omega})/[(\underline{\dot{\Omega}} \times \underline{\Omega}) \cdot (\underline{\dot{\Omega}} \times \underline{\Omega})] \tag{3.34}$$

where $\underline{\ddot{Q}}$ is given in Equation (3.31) and $\underline{\dot{\Omega}}$ is the angular acceleration of the moving body. Hence, the position vector of the central point is

$$\underline{C} = \underline{Q} + e_c\underline{\Omega} \tag{3.35}$$

Let $\underline{DC} = \underline{C} - \underline{D}$. From Equations (3.13) and (3.35), we have

$$\underline{DC} = (\underline{\Omega} \times \underline{\dot{D}})/\Omega^2 + e_c\underline{\Omega} \tag{3.36}$$

The linear motion at the central point can be computed

$$\underline{\dot{C}} = \underline{\dot{D}} + \underline{\Omega} \times \underline{DC} = \sigma\underline{\Omega} \tag{3.37a}$$

$$\underline{\ddot{C}} = \underline{\ddot{D}} + \underline{\dot{\Omega}} \times \underline{DC} + \underline{\Omega} \times (\underline{\Omega} \times \underline{DC}) \tag{3.37b}$$

$$\underline{\dddot{C}} = \underline{\dddot{D}} + \underline{\ddot{\Omega}} \times \underline{DC} + 2\underline{\dot{\Omega}} \times (\underline{\Omega} \times \underline{DC})$$

$$+ \underline{\Omega} \times (\underline{\dot{\Omega}} \times \underline{DC}) + \underline{\Omega} \times [\underline{\Omega} \times (\underline{\Omega} \times \underline{DC})] \tag{3.37c}$$

and so on. Since

$$\underline{\ddot{C}} = \underline{\ddot{Q}} + \underline{\dot{\Omega}} \times \underline{QC} + \underline{\Omega} \times (\underline{\Omega} \times \underline{QC})$$ ⇐ Q AS ORIGIN

and $\underline{QC} = \underline{C} - \underline{Q} = e_c\underline{\Omega}$, $\underline{\ddot{C}}$ may also be computed through

$$\underline{\ddot{C}} = \underline{\ddot{Q}} + e_c\underline{\dot{\Omega}} \times \underline{\Omega} \tag{3.38}$$

where $\underline{\ddot{Q}}$ and e_c are given in Equations (3.31) and (3.34).

The relative position of $\underline{Q}$, $\underline{D}$, and $\underline{C}$ are shown in Figure 6.

3.4 Velocity Screw

A velocity screw is a dual vector referred to a point on a moving body, in which the primary part represents the angular velocity of the

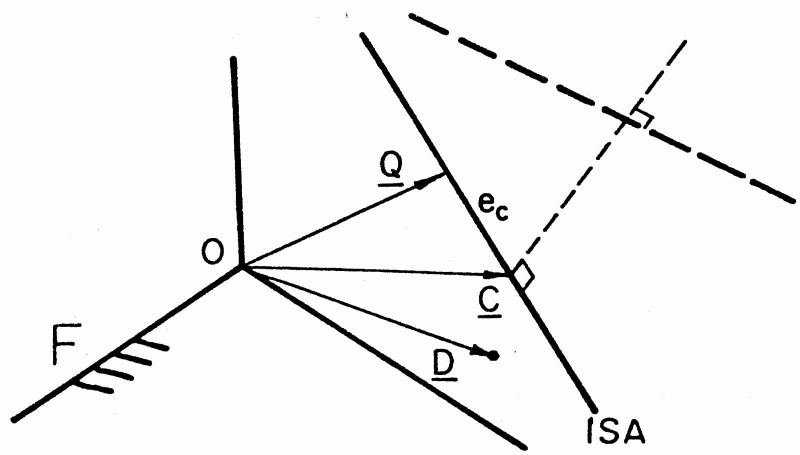


Figure 6. The Relative Position of Points D, Q, and the Central Point C.

moving body and the dual part represents the linear velocity of the reference point. Since every point on the ISA has the velocity $\sigma\underline{\Omega}$, the velocity screw referred to any point X on the ISA is

$$\hat{\underline{\Omega}} = \underline{\Omega} + \varepsilon\sigma\underline{\Omega} \qquad \text{see } (5\text{-}1) \tag{3.39}$$

If $\underline{V}_p$ designates the linear velocity of an arbitrary point P, the velocity screw referred to point P can be written as

$$\hat{\underline{\Omega}} = \underline{\Omega} + \varepsilon\underline{V}_p \qquad \text{see } (5\text{-}2) \;\&\; (5\text{-}3) \tag{3.40}$$

$\underline{V}_P = (\sigma\Omega)\bar{\xi}^* + \Omega\bar{\xi}^* \times \underline{XP}$

where $\underline{V}_p = \sigma\underline{\Omega} + \cancel{\varepsilon}\underline{\Omega} \times \underline{XP}$ and $\underline{\Omega} \times \underline{XP}$ is the relative velocity of point P to any point X on the ISA.

Let the unit screw $\hat{\underline{K}}$ represent the ISA of the moving body. The velocity screw may be written as

$$\hat{\underline{\Omega}} = \Omega(1 + \varepsilon\sigma)\hat{\underline{K}} \tag{3.41}$$

If the unit screw $\hat{\underline{K}}$ is referred to a point on the ISA, then $\hat{\underline{K}} = \underline{K}$ is a degenerated unit screw with zero dual part and the above equation becomes

a wrong conclusion, ∵ ε is still present which is obvious from (3.42)

$$\hat{\underline{\Omega}} = \Omega(1 + \varepsilon\sigma)\underline{K} \tag{3.42}$$

(TRUE ⇒ the unit screw $\underline{K}$ is degenerate

which is equivalent to Equation (3.39). If $\hat{\underline{K}}$ relates to any other point P, then

$$\hat{\underline{K}} = \underline{K} + \varepsilon\underline{PX} \times \underline{K} \tag{3.43}$$

where the dual part $\underline{PX} \times \underline{K}$ is the moment of the sliding $\underline{K}$, along the ISA, with respect to P. Substituting it into Equation (3.41), we have

$$\begin{aligned} \hat{\underline{\Omega}} &= \Omega(1 + \varepsilon\sigma)(\underline{K} + \varepsilon\underline{PX} \times \hat{\underline{K}}) \\ &= \Omega\underline{K} + \varepsilon(\sigma\Omega\,\underline{K} + \underline{PX} \times \Omega\hat{\underline{K}}) \end{aligned} \tag{3.44}$$

$$= \underline{\Omega} + \varepsilon(\sigma\underline{\Omega} + \underline{\Omega} \times \underline{XP}) \tag{3.44}$$

It is equivalent to the velocity screw of Equation (3.40). Hence, the velocity screw of a moving body can be represented by Equation (3.41) in which the dual part represents the linear velocity of the reference point of the unit screw $\hat{\underline{K}}$.

Until now, the rigid body motion is considered as the function of time. However, we are primarily interested in the geometry of the motion which is time independent. Since a general rigid body is a generalized screw motion, the angular displacement ϕ of the rigid body about the screw axis is a geometrical parameter and can be selected as the independent motion parameter to provide a base for the comparison of the geometry of motions. In other words, the angular velocity about the ISA is normalized and remains unity at any time. Thus, $\underline{\Omega} = d\phi/d\phi = d\phi/dt = 1$ and from Equation (3.41), the normalized velocity screw is

$$\hat{\underline{W}} = (1 + \varepsilon\sigma)\hat{\underline{K}} \tag{3.45a}$$

where σ and $\hat{\underline{K}}$ are functions of ϕ. In Equation (3.45a), only the instantaneous pitch σ and the ISA, $\hat{\underline{K}}$ are needed to describe the geometry of the first order general rigid body motion. In case of a spherical motion, the pitch σ remains zero and the ISA always passes through the center of rotation. Therefore, only the unit vector $\underline{K}$ along the rotational axis is needed and the normalized angular velocity is

$$\underline{W} = \underline{K} \tag{3.45b}$$

Holds only if the center of rotation is taken as the origin.

3.5 Higher Order Rigid Body Motion

On the ruled surface traced by the ISA of a rigid body motion, the generator trihedron formed by the three perpendicular axes $\hat{\underline{I}}$, $\hat{\underline{J}}$, and $\hat{\underline{K}}$

is defined in Section 3.2. These three axes meet at the central point and from Equations (3.25), we may obtain

$$\frac{d\hat{\underline{K}}}{dt} = \frac{d\hat{\underline{K}}}{d\hat{\theta}}\frac{d\hat{\theta}}{dt} = \hat{\alpha}\hat{\underline{I}}$$

$$\frac{d\hat{\underline{I}}}{dt} = \frac{d\hat{\underline{I}}}{d\hat{\theta}}\frac{d\hat{\theta}}{dt} = \hat{\alpha}\hat{\gamma}\hat{\underline{J}} - \hat{\alpha}\hat{\underline{K}} = \hat{\beta}\hat{\underline{J}} - \hat{\alpha}\hat{\underline{K}} \tag{3.46}$$

$$\frac{d\hat{\underline{J}}}{dt} = \frac{d\hat{\underline{J}}}{d\hat{\theta}}\frac{d\hat{\theta}}{dt} = -\hat{\alpha}\hat{\gamma}\hat{\underline{I}} = -\hat{\beta}\hat{\underline{I}}$$

in which $\hat{\beta} = \hat{\alpha}\hat{\gamma}$ and $\hat{\alpha} = \frac{d\hat{\theta}}{dt}$. Equations (3.46) may also be written as

$$\frac{d\hat{\underline{K}}}{dt} = \hat{\underline{\pi}} \times \hat{\underline{K}}$$

$$\frac{d\hat{\underline{I}}}{dt} = \hat{\underline{\pi}} \times \hat{\underline{I}} \tag{3.47}$$

$$\frac{d\hat{\underline{J}}}{dt} = \hat{\underline{\pi}} \times \hat{\underline{J}}$$

where $\hat{\underline{\pi}} = \hat{\alpha}\hat{\underline{J}} + \hat{\beta}\hat{\underline{K}}$ is the Darboux screw.

Let us consider a spherical motion. The instantaneous pitch σ remains zero and all the instantaneous screw axis pass through the center of rotation. If the center of rotation be selected as the reference point of the unit screws. Equations (3.46) and (3.47) become

$$\frac{d\underline{K}}{dt} = \underline{\pi} \times \underline{K} = \alpha\underline{I}$$

$$\frac{d\underline{I}}{dt} = \underline{\pi} \times \underline{I} = \beta\underline{J} - \alpha\underline{K} \tag{3.48}$$

$$\frac{d\underline{J}}{dt} = \underline{\pi} \times \underline{J} = -\beta\underline{I}$$

In Equation (3.48), all the dual parts are zero and $\underline{\pi} = \alpha\underline{J} + \beta\underline{K}$ is the Darboux Vector which is the angular velocity of the generator trihedron formed by $\underline{I}$, $\underline{J}$, and $\underline{K}$ at the center of rotation.

Assume that $\underline{\Omega}$ is the angular velocity of a rigid body executing a spherical motion. Since $\sigma \equiv 0$, from Equation (3.42), we have

$$\underline{\Omega} = \Omega\underline{K} \tag{3.49}$$

where Ω and $\underline{K}$ are functions of time t. Let the instantaneous motion to be investigated be at zero position, at which $t = \phi = 0$, and the subscript i associated with any quantity such as $\underline{\Omega}_i$, Ω_i or α_i denote the ith derivative with respect to t at t = 0. From Equations (3.48) and (3.49) we obtain

$$\begin{aligned}
\underline{\Omega}_0 &= \Omega_0\underline{K} \\
\underline{\Omega}_1 &= \alpha_0\Omega_0\underline{I} + \Omega_1\underline{K} \\
\underline{\Omega}_2 &= (2\alpha_0\Omega_1 + \alpha_1\Omega_0)\underline{I} + \alpha_0\beta_0\Omega_0\underline{J} + (\Omega_2 - \alpha_0^2\,\Omega_0)\underline{K} \\
\underline{\Omega}_3 &= (3\alpha_0\Omega_2 + 3\alpha_1\Omega_1 + \alpha_2\Omega_0 - \alpha_0^{\,3}\Omega_0 - \alpha_0\beta_0^2\,\Omega_0)\underline{I} \\
&\quad + (2\alpha_1\beta_0\Omega_0 + \alpha_0\beta_1\Omega_0 + 3\alpha_0\beta_0\Omega_1)\underline{J} \\
&\quad + (\Omega_3 - 3\alpha_0\alpha_1\Omega_0 - 3\alpha_0^2\,\Omega_1)\underline{K}
\end{aligned} \tag{3.50}$$

Each parameter in Equation (3.50) can be determined if $\underline{\Omega}_i$ (i = 0, 3) are known.

$$\begin{aligned}
\alpha_0 &= \underline{\Omega}_1 \cdot \underline{I}/\Omega_0 \\
\beta_0 &= \underline{\Omega}_2 \cdot \underline{J}/(\alpha_0\Omega_0) \\
\alpha_1 &= (\underline{\Omega}_2 \cdot \underline{I} - 2\alpha_0\Omega_1)/\Omega_0
\end{aligned} \tag{3.51}$$

$$\beta_1 = (\underline{\Omega}_3 \cdot \underline{J} - 2\alpha_1\beta_0\Omega_0 - 3\alpha_0\beta_0\Omega_1)/(\alpha_0\Omega_0)$$

$$\alpha_2 = (\underline{\Omega}_3 \cdot \underline{I} - 3\alpha_0\Omega_2 - 3\alpha_1\Omega_1 + \alpha_0^3\Omega_0 + \alpha_0\beta_0^2\Omega_0)/\Omega_0 \qquad (3.51)$$

where

$$\Omega_0 = |\underline{\Omega}_0|$$

$$\Omega_1 = \underline{\Omega}_1 \cdot \underline{K}$$

$$\Omega_2 = \underline{\Omega}_2 \cdot \underline{K} + \alpha_0^2\,\Omega_0$$

$$\Omega_3 = \underline{\Omega}_3 \cdot \underline{K} + 3\,\alpha_0\alpha_1\Omega_0 + 3\alpha_0^2\,\Omega_1$$

and (3.52)

$$\underline{K} = \underline{\Omega}_0/\Omega_0$$

$$\underline{I} = (\underline{\Omega}_1 - \Omega_1\underline{K})/|\underline{\Omega}_1 - \Omega_1\underline{K}|$$

$$\underline{J} = \underline{K} \times \underline{I}$$

Consider a general rigid body motion, the first order motion can be represented by the velocity screw of Equation (3.41) which may be rewritten as

$$\hat{\underline{\Omega}} = \hat{\Omega}\hat{\underline{K}} \qquad (3.53)$$

where $\hat{\Omega} = \Omega(1 + \varepsilon\sigma)$. From Equations (3.46) and (3.53), we may have

$$\hat{\underline{\Omega}}_0 = \hat{\Omega}_0\,\hat{\underline{K}}$$

$$\hat{\underline{\Omega}}_1 = \hat{\alpha}_0\hat{\Omega}_0\hat{\underline{I}} + \hat{\Omega}_1\hat{\underline{K}} \qquad (3.54)$$

$$\hat{\underline{\Omega}}_2 = (2\hat{\alpha}_0\hat{\Omega}_1 + \hat{\alpha}_1\hat{\Omega}_0)\hat{\underline{I}} + \hat{\alpha}_0\hat{\beta}_0\hat{\Omega}_0\hat{\underline{J}} + (\hat{\Omega}_2 - \hat{\alpha}_0^2\hat{\Omega}_0)\hat{\underline{K}}$$

$$\hat{\underline{\Omega}}_3 = (3\hat{\alpha}_0\hat{\Omega}_2 + 3\hat{\alpha}_1\hat{\Omega}_1 + \hat{\alpha}_2\hat{\Omega}_0 - \hat{\alpha}_0^3\hat{\Omega}_0 - \hat{\alpha}_0\hat{\beta}_0^2\hat{\Omega}_0)\hat{\underline{I}}$$

$$+(2\hat{\alpha}_1\hat{\beta}_0\hat{\Omega}_0 + \hat{\alpha}_0\hat{\beta}_1\hat{\Omega}_0 + 3\hat{\alpha}_0\hat{\beta}_0\hat{\Omega}_1)\hat{\underline{J}}$$
$$+(\hat{\Omega}_3 - 3\hat{\alpha}_0\hat{\beta}_1\hat{\Omega}_0 - 3\hat{\alpha}_0^2\hat{\alpha}_1)\hat{\underline{K}} \tag{3.54}$$

where

$$\hat{\Omega}_0 = \Omega_0(1 + \varepsilon\sigma_0)$$

$$\hat{\Omega}_1 = \Omega_1 + \varepsilon(\sigma_0\Omega_1 + \sigma_1\Omega_0)$$

$$\hat{\Omega}_2 = \Omega_2 + \varepsilon(\sigma_0\Omega_2 + 2\sigma_1\Omega_1 + \sigma_2\Omega_0)$$

$$\hat{\Omega}_3 = \Omega_3 + \varepsilon(\sigma_0\Omega_3 + 3\sigma_1\Omega_2 + 3\sigma_2\Omega_1 + \sigma_3\Omega_0)$$

and the subscript i denotes the ith derivative of the associated quantity with respect to t at zero position. Equations (3.54) and (3.50) are equivalent if the dual notation are disregarded.

We may separate Equations (3.54) into primary parts and dual parts: The primary parts represent the rotational motion of the rigid body, i. e., angular velocity, acceleration and jerk; the dual parts represent the linear velocity, acceleration, and jerk of the reference point of the unit screws $\hat{\underline{I}}$, $\hat{\underline{J}}$, and $\hat{\underline{K}}$. Let these unit screws be referred to the central point which is their common point. Then, the dual parts of these unit screws diminish and we have

$$\hat{\underline{I}} = \underline{I} \quad , \quad \hat{\underline{J}} = \underline{J} \quad \text{and} \quad \hat{\underline{K}} = \underline{K} \tag{3.55}$$

Let

$$\left.\begin{aligned} \hat{\alpha}_i &= \alpha_i + \varepsilon\alpha^\circ_i \\ \hat{\beta}_i &= \beta_i + \varepsilon\beta^\circ_i \end{aligned}\right\} \quad (i = 0, 1, 2, \ldots) \tag{3.56}$$

Substituting Equations (3.55) and (3.56) into (3.54) and separating the primary and dual parts, we have

$$
\begin{aligned}
\hat{\underline{\Omega}}_0 &= \underline{\Omega}_0 + \varepsilon \underline{C}_1 \\
\hat{\underline{\Omega}}_1 &= \underline{\Omega}_1 + \varepsilon \underline{C}_2 \\
\hat{\underline{\Omega}}_2 &= \underline{\Omega}_2 + \varepsilon \underline{C}_3 \\
\hat{\underline{\Omega}}_3 &= \underline{\Omega}_3 + \varepsilon \underline{C}_4
\end{aligned}
\tag{3.57}
$$

where $\underline{C}_i$ (i = 1, 4) represents the linear motion at the central point. The velocity, acceleration and jerk at the central point are shown below.

$$\underline{C}_1 = \sigma_0 \Omega_0 \underline{K} \tag{3.58a}$$

$$\underline{C}_2 = (\mathring{\alpha}_0 \Omega_0 + \alpha_0 \sigma_0 \Omega_0)\underline{I} + (\sigma_0 \Omega_1 + \sigma_1 \Omega_0)\underline{K} \tag{3.58b}$$

$$
\begin{aligned}
\underline{C}_3 = {} & [2\alpha_0(\sigma_0 \Omega_1 + \sigma_1 \Omega_0) + 2\mathring{\alpha}_0 \Omega_1 + \alpha_1 \sigma_0 \Omega_0 + \mathring{\alpha}_1 \Omega_0]\underline{I} \\
& + \Omega_0(\alpha_0 \beta_0 \sigma_0 + \mathring{\alpha}_0 \beta_0 + \alpha_0 \mathring{\beta}_0)\underline{J} \\
& + (\sigma_0 \Omega_2 + \sigma_2 \Omega_0 + 2\sigma_1 \Omega_1 - 2\alpha_0 \mathring{\alpha}_0 \Omega_0 - \mathring{\alpha}_0 \sigma_0 \Omega_0)\underline{K}
\end{aligned}
\tag{3.58c}
$$

$\underline{C}_4$ can also be obtained similarly. In the above equations, the primary part are exactly the same as Equation (3.50). This result is consistent with the principle of transference and α_i, β_i, Ω_i can be obtained from Equations (3.51) and (3.52). Knowing the angular motion of the rigid body and the linear motion at any point, we can compute $\underline{C}_i$ from Equations (3.37) and (3.38). The instantaneous pitch σ_0 of the rigid body motion can be determined by Equation (3.15) or

$$\sigma_0 = \overset{C}{\cancel{\underline{D}}}_1 \cdot \underline{K}/\Omega_0 \qquad \text{from 3.58a)} \tag{3.59}$$

where $\underline{D}_1$ is the velocity at any point. From Equations (3.58), the following parameters can be calculated.

$$\begin{aligned}
\alpha_0^\circ &= \underline{C}_2 \cdot \underline{I}/\Omega_0 - \alpha_0\sigma_0 \\
\sigma_1 &= (\underline{C}_2 \cdot \underline{K} - \sigma_0\Omega_1)/\Omega_0 \\
\beta_0^\circ &= (\underline{C}_3 \cdot \underline{J}/\Omega_0 - \alpha_0\beta_0\sigma_0 - \alpha_0^\circ\beta_0)/\alpha_0 \\
\sigma_2 &= (\underline{C}_3 \cdot \underline{K} - \sigma_0\Omega_2 - 2\sigma_1\Omega_1 + 2\alpha_0\alpha_0^\circ\Omega_0 + \alpha_0^\circ\sigma_0\Omega_0)/\Omega_0 \\
\alpha_1^\circ &= (\underline{C}_3 \cdot \underline{I} - 2\alpha_0(\sigma_0\Omega_1 + \sigma_1\Omega_0) - 2\alpha_0^\circ\Omega_1 - \alpha_1\sigma_0\Omega_0)/\Omega_0
\end{aligned} \tag{3.60}$$

For fourth order motion β_1°, σ_3, and α_2° can be obtained in a similar manner if $\underline{C}_4$ is known. In Equations (3.59) and (3.60), Ω_i, β_i, α_i and the unit vectors $\underline{I}$, $\underline{J}$, and $\underline{K}$ are given in Equations (3.51) and (3.52)

3.6 Geometry of Rigid Body Motion

We will now study the geometry of motion of a mechanism. For this purpose, the rigid body motion will be treated as a normalized motion with a constant unit angular velocity. Such a normalized motion is equivalent to a motion with $\phi = t$, where ϕ is the angular displacement about the ISA. To avoid confusion ϕ will be used as the independent motion parameter. Then, the motion described is equivalent to a time-based motion with $\Omega_0 = 1$ and $\Omega_1 = \Omega_2 = \ldots = 0$. All the single subscripts i (i = 0, 1, 2, . . .) in the following denote the ith derivatives of the associates quantities with respect to ϕ at zero position $\phi = 0$.

Let us consider a spherical motion. The normalized angular velocity is given in Equation (3.45b). Referring to Equation (3.50) with $\Omega_0 = 1$ and $\Omega_1 = \Omega_2 = . . . = 0$ or differentiating Equation (3.45b) with respect to ϕ and referring to Equation (3.48), we have

$$\underline{W}_0 = \underline{K}$$

$$\underline{W}_1 = d\underline{K}/d\phi = W_{11}\underline{I}$$

$$\underline{W}_2 = d^2\underline{K}/d\phi^2 = W_{12}\underline{I} + W_{22}\underline{J} - W_{11}^2\underline{K}$$

$$\underline{W}_3 = d^3\underline{K}/d\phi^3 = W_{13}\underline{I} + W_{23}\underline{J} - 3W_{11}W_{12}\underline{K}$$

where

$$W_{11} = \alpha_0$$

$$W_{12} = \alpha_1; \qquad W_{22} = \alpha_0\beta_0 \tag{3.62}$$

$$W_{13} = \alpha_2 - \alpha_0^3 - \alpha_0\beta_0^2; \quad W_{23} = 2\alpha_1\beta_0 + \alpha_0\beta_1$$

α_i, β_i are determined in Equation (3.51) and they are the quantities caused by the rotation of the ISA. We note that in Equation (3.61) the geometry of a spherical motion is characterized by α_i and β_i ($i = 0, 1, 2, . . .$) and $(2n - 3)$ numbers are needed to characterize a nth order spherical motion ($n \geq 2$).

For a general rigid body motion, the normalized velocity screw is given in Equation (3.45a). Let the central point be taken as the reference point of the unit screw $\hat{\underline{I}}$, $\hat{\underline{J}}$, and $\hat{\underline{K}}$. With a normalized constant unit angular velocity, we may have, from Equations (3.54) or (3.57) and (3.58),

$$\hat{\underline{W}}_0 = \underline{W}_0 + \varepsilon d_1\underline{K}$$

Refer (3.56)

$$\hat{\underline{W}}_1 = \underline{W}_1 + \varepsilon(d_{12}\underline{I} + d_{32}\underline{K}) \tag{3.63}$$

$$\hat{\underline{W}}_2 = \underline{W}_2 + \varepsilon(d_{13}\underline{I} + d_{23}\underline{J} + d_{33}\underline{K})$$

$$\hat{\underline{W}}_3 = \underline{W}_3 + \varepsilon(d_{14}\underline{I} + d_{24}\underline{J} + d_{34}\underline{K}) \tag{3.63}$$

and

$$d_1 = \sigma_0 \tag{3.64a}$$

$$d_{12} = \alpha_0\,\sigma_0 + \alpha_0^\circ\,; \qquad d_{32} = \sigma_1 \tag{3.64b}$$

$$d_{13} = 2\alpha_0\sigma_1 + \alpha_1\sigma_0 + \alpha_1^\circ \qquad d_{23} = \alpha_0\beta_0\sigma_0 + \alpha_0^\circ\beta_0 + \alpha_0\beta_0^\circ \tag{3.64c}$$

$$d_{33} = \sigma_2 - \alpha_0^2\sigma_1 - 2\alpha_0\alpha_0^\circ \tag{3.64d}$$

$$d_{14} = 3\alpha_0\sigma_2 - \alpha_0^3\sigma_0 - 3\alpha_0^2\alpha_0^\circ + 3\alpha_1\sigma_1 + \alpha_2\sigma_0 + \alpha_2^\circ \tag{3.64e}$$
$$-\alpha_0\beta_0^2\sigma_0 - \alpha_0^\circ\beta_0^2 - 2\alpha_0\beta_0\beta_0^\circ$$

$$d_{24} = 3\alpha_0\beta_0\sigma_1 + 2\alpha_1\beta_0\sigma_0 + 2\alpha_1^\circ\beta_0 + 2\alpha_1\beta_0^\circ \tag{3.64f}$$
$$+ \alpha_0\beta_1\sigma_0 + \beta_1\alpha_0^\circ + \alpha_0\beta_1^\circ$$

$$d_{34} = \sigma_3 - 3\alpha_0^2\sigma_1 - 3\alpha_0\alpha_1\sigma_0 - 3\alpha_0^\circ\alpha_1 - 3\alpha_0\alpha_1^\circ \tag{3.64g}$$

where the primary parts $\underline{W}_i$ are identical to Equation (3.61). Equation (3.63) may also be obtained by differentiating Equation (3.45a) directly.

We note that in Equation (3.63) and (3.64), the geometry of a rigid body motion is characterized by $\hat{\alpha}_i$, $\hat{\beta}_i$, and σ_i at zero position. $(i = 1,2,3)$ For a nth order motion ($n \geq 3$), five real numbers $(\hat{\alpha}_{n-2}, \hat{\beta}_{n-3}, \sigma_{n-1})$ are needed to characterize each additional order of motion. Thus by the concept of a generalized screw motion, a general rigid body motion is completely specified by the Darboux screw $(\hat{\alpha}\hat{\underline{j}} + \hat{\beta}\hat{\underline{k}})$, the pitch (σ) and their derivatives. Such concept may be helpful in the visualization of the motion of any geometrical element.

3.7 Canonical Systems and Instantaneous Invariants

To treat the motion of a rigid body, Equation (3.1) may be more convenient to use. The higher order motion of a rigid body can be described conveniently by the matrix [A], the column vector $\underline{D}$ and their derivatives with respect to . However, such description varies with the coordinate systems utilized. It is desirable to use the coordinate systems which have geometrical significance and can be duplicated universally. It is also desired that the coordinate systems are capable to provide the simplest description of the motion.

Let the instantaneous motion to be investigated be at the zero position, the three axes $\underline{\hat{i}}$, $\underline{\hat{j}}$, and $\underline{\hat{k}}$ of Equation (3.54) serve as a good choice for the three perpendicular axes of the two coincident coordinate systems. They are the axes of the generator trihedron of the ruled surface traced by the ISA. The unit screw $\underline{k}$ is the generator of the ruled surface at zero position, $\underline{i}$ is the central tangent and $\underline{j}$ the central normal to the surface.

Let us select the canonical systems for the moving and the fixed systems for the moving and the fixed system M and F in Equation (3.1). Being canonical systems (Figure 7) M and F, at zero position are coincident and have $\underline{\hat{I}}$, $\underline{\hat{J}}$, and $\underline{\hat{K}}$ as the three mutual perpendicular axes. We note that the central point is their common origin. From Equations (3.63) and (3.61) we have

$$\left.\begin{aligned} \underline{W}_0 &= (0,\ 0,\ 1) \\ \underline{W}_1 &= (W_{11},\ 0,\ 0) \\ \underline{W}_2 &= (W_{12},\ W_{22},\ -\ W_{11}^2) \end{aligned}\right\} \qquad (3.65)$$

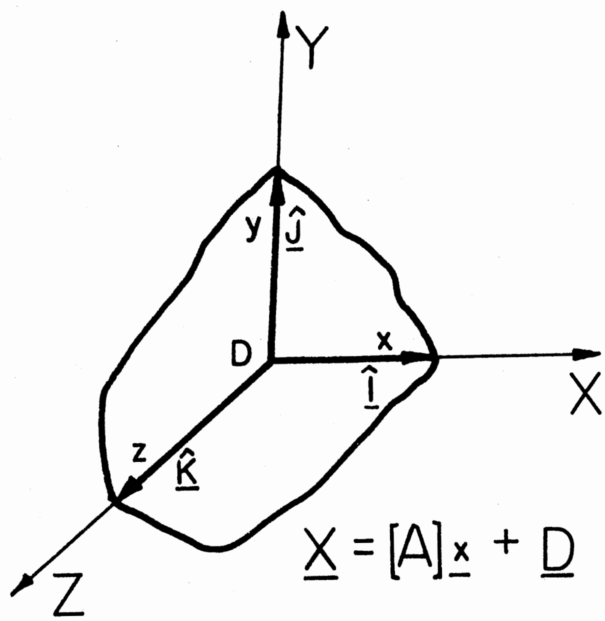


Figure 7. The Canonical Systems

$$\underline{W}_3 = (W_{13}, W_{23}, -3W_{11}W_{12}) \tag{3.65}$$

Since the origin $\underline{D}$ of the moving canoncial system is at the central point, we have from Equation (3.63),

$$\underline{D}_0 = 0$$

$$\underline{D}_1 = (0, 0, d_1)$$

$$\underline{D}_2 = (d_{12}, 0, d_{32})$$

$$\underline{D}_3 = (d_{13}, d_{23}, d_{33})$$

$$\underline{D}_4 = (d_{14}, d_{24}, d_{34})$$

With the normalized motion, we have $\underline{\Omega} = \underline{W}$ and from Equation (3.6),

$$[W] = [\dot{A}]\,[A]^T \tag{3.67}$$

Let the subscript i (i = 0, 1, 2, . . .) denotes the ith derivative of the associated quantity with respect to ϕ at zero position. We have from the above equation,

$$[W_0] = [A_1][A_0]^T$$

$$[W_1] = [A_2][A_0]^T + [A_1][A_1]^T$$

$$[W_2] = [A_3][A_0]^T + 2[A_2][A_1]^T + [A_1][A_2]^T$$

$$[W_3] = [A_4]\,[A_0]^T + 3[A_3][A_1]^T + 3[A_2][A_2]^T + [A_1][A_3]^T$$

Since $[A_0] = [A_0]^T = [I]$, the above equation becomes

$$[A_1] = [W_0]$$

$$[A_2] = [W_1] - [A_1]\,[A_1]T \tag{3.68}$$

$$[A_3] = [W_2] - 2[A_2][A_1]^T - [A_1][A_2]^T$$

$$[A_4] = [W_3] - 3[A_3][A_1]^T - 3[A_2][A_2]^T - [A_1][A_3]^T$$

With the notation used in Equation (3.6), we have from Equation (3.65)

$$[W_0] = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.69}$$

$$[W_1] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -W_{11} \\ 0 & W_{11} & 0 \end{bmatrix} \tag{3.70}$$

$$[W_2] = \begin{bmatrix} 0 & W_{11}^2 & W_{22} \\ -W_{11}^2 & 0 & -W_{12} \\ -W_{22} & W_{12} & 0 \end{bmatrix} \tag{3.71}$$

$$[W_3] = \begin{bmatrix} 0 & 3W_{11}W_{12} & W_{23} \\ -3W_{11}W_{12} & 0 & -W_{13} \\ -W_{23} & W_{13} & 0 \end{bmatrix} \tag{3.72}$$

Substituting them into Equation (3.68), we obtain

$$[A_0] = [I] \tag{3.73}$$

$$[A_1] = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.74}$$

$$[A_2] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & -w_{11} \\ 0 & w_{11} & 0 \end{bmatrix} \tag{3.75}$$

$$[A_3] = \begin{bmatrix} 0 & 1 + w_{11}^2 & w_{11} + w_{22} \\ -(1 + w_{11}^2) & 0 & -w_{12} \\ -w_{22} + 2w_{11} & w_{12} & 0 \end{bmatrix} \tag{3.76}$$

$$[A_4] = \begin{bmatrix} 1 + 4w_{11}^2 & 3w_{11}w_{12} & w_{23} + w_{12} \\ -3w_{11}w_{12} & 1 + w_{11}^2 & -w_{13} + w_{22} + w_{11} \\ -w_{23} + 3w_{12} & w_{13} + 3w_{22} - 3w_{11} & -3w_{11}^2 \end{bmatrix} \tag{3.77}$$

Differenting Equation (3.1) we may have

$$\underline{X}_i = [A_i]\,\underline{x} + \underline{D}_i \tag{3.78}$$

$[A_i]$ and $\underline{D}_i$ $(i = 0, 1, \ldots n)$ describe the instantaneous motion of the rigid body to nth order. In the canonical systems, the elements of $[A_i]$ and $\underline{D}_i$ are called the ith order instantaneous invariants of the rigid body motion. They represent the geometry of a rigid body motion and can be used as the base for the comparison of motions. Any motion having the common instantaneous invariants to nth order with a reference motion approximates that motion to the same order.

In any spherical motion, the pitch σ is always zero and the central point is a fixed point. Thus, we have $\underline{D} \equiv 0$ in the canonical systems and Equation (3.77) becomes

$$\underline{X}_i = [A_i]\,\underline{x}$$

The elements of $[A_i]$ are the ith order instantaneous invariants of a spherical motion, which appear exactly the same as Equations (3.73) to (3.77).

CHAPTER IV

KINEMATICS OF A MOVING PLANE

For one-parameter space motion, a tangent-plane embedded in the moving body envelops a developable. In this chapter, the characteristics of developable surfaces will be defined. The characteristic equations will be derived to locate those tangent-planes whose envelops have the same characteristic numbers. Since the geometry of the motion is of our concern, the space motion is assumed to have the normalized unit angular velocity at any time and for such a case, the angular displacement ϕ about the ISA is considered as the independent motion parameter.

4.1 Tangent-Plane Envelope

A developable is a surface with a single-infinity of lines and each line intersects its neighbor at a point. These lines are the generating lines of the developable. By a succession of small successive rotations about the generating lines, the developable can be developed into a plane without any stretching or tearing.

Let

$$f(\underline{X}, \phi) = 0 \tag{4.1}$$

represent a single-infinity of planes with ϕ as the parameter. The plane (or called tangent-plane) intersects its consecutive plane at a line and the family of these intersecting lines for all values of ϕ is the envelope of these planes. This envelope is a developable and these lines are its generating lines (or generators). Any two consecutive generators intersect at a point; the locus of the intersection point is the edge of regression of the developable. Thus, from Equation (4.1), one obtains

$$\partial f(\phi)/\partial\phi = 0 \tag{4.2}$$

and

$$\partial^2 f(\phi)/\partial\phi^2 = 0 \tag{4.3}$$

The function f and its derivative are assumed to be continuous at ϕ.

Equations (4.1) and (4.2) represent the tangent-plane envelope and (4.1) through (4.3) represent the edge of regression. The planes of Equation (4.1) are the tangent-planes (or osculating planes) of the edge of regression, and the enveloped surface is the osculating developable. Each generating line of the developable is tangent to the edge of regression. Knowing the tangent-planes, the developable or the edge of regression, the other two are determined (Figure 8).

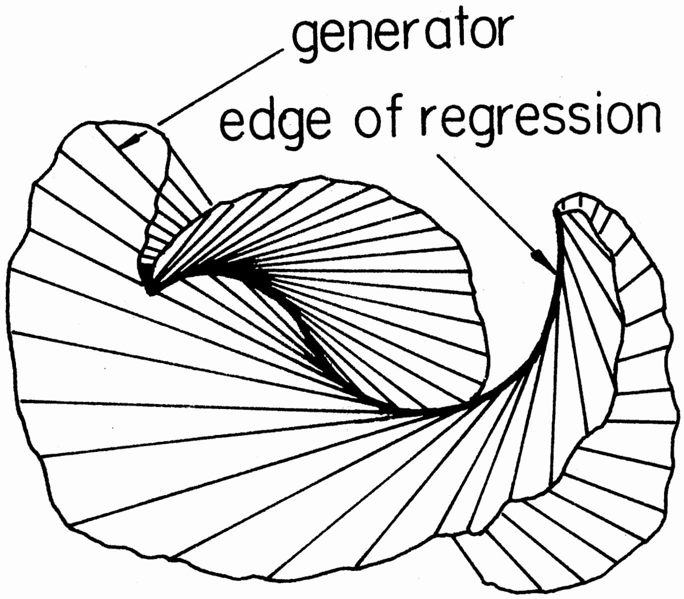
4.2 Motion of a Plane

In the moving canonical system, a plane is expressed uniquely by

$$\underline{\ell} \cdot \underline{x} + p = 0 \tag{4.4}$$

where $\underline{\ell} = (\ell, m, n)$ and $\underline{\ell} \cdot \underline{\ell} = 1$. (ℓ, m, n, p) is the homogeneous coordinates of the plane in the moving system (Figure 9). Substituted by Equation (3.2), the same plane is expressed in the fixed canonical system by

$$\underline{\ell} \cdot [A]^T (\underline{X} - \underline{D}) + p = 0$$



Every two consecutive generators intersect at a point. The locus of such point is the edge of regression.

Figure 8. A Developable

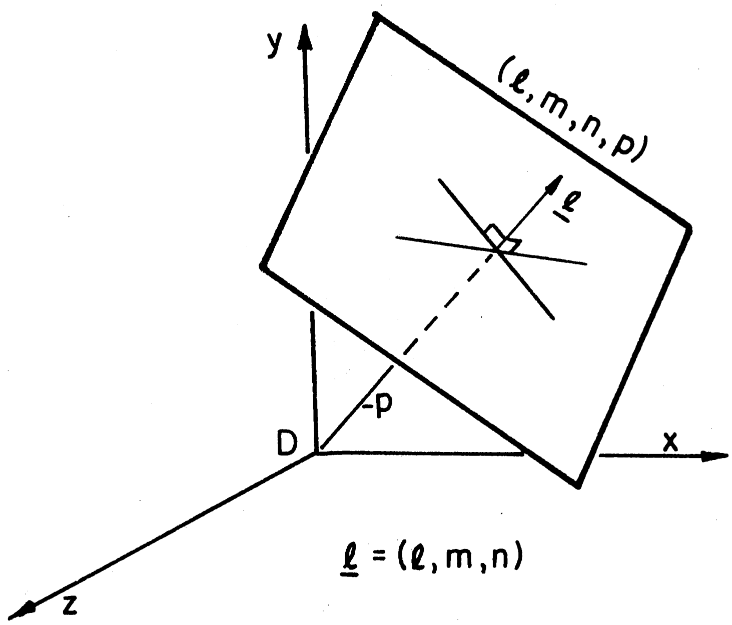


Figure 9. Plane Coordinates (ℓ, m, n, p)

or

$$[A]\,\underline{\ell} \cdot (\underline{X} - \underline{D}) + p = 0 \tag{4.5}$$

where [A] and $\underline{D}$ are functions of ϕ. For all values of ϕ, Equation (4.5) represents a single-infinity of tangent-planes in the fixed system. Let the subscript i (i = 0, 1, 2, . . .) denote the ith derivative of the associated quantity with respect to ϕ. The following plane equations may be obtained.

$$\underline{L}_i \cdot \underline{X} + P_i = 0 \quad (i = 0, 1, 2, \ldots) \tag{4.6}$$

in which

$$\underline{L}_i = [A_i]\,\underline{\ell} \tag{4.7}$$

and

$$\begin{aligned}
P_0 &= P - \underline{L}_0 \cdot \underline{D}_0 \\
P_1 &= -\underline{L}_1 \cdot \underline{D}_0 - \underline{L}_0 \cdot \underline{D}_1 \\
P_2 &= -\underline{L}_2 \cdot \underline{D}_0 - 2\underline{L}_1 \cdot \underline{D}_1 - \underline{L}_0 \cdot \underline{D}_2 \\
P_3 &= -\underline{L}_3 \cdot \underline{D}_0 - 3\underline{L}_2 \cdot \underline{D}_1 - 3\underline{L}_1 \cdot \underline{D}_2 - \underline{L}_0 \cdot \underline{D}_3 \\
P_4 &= -\underline{L}_4 \cdot \underline{D}_0 - 4\underline{L}_3 \cdot D_1 - 6\underline{L}_2 \cdot \underline{D}_2 - 4\underline{L}_1 \cdot \underline{D}_3 - \underline{L}_0 \cdot \underline{D}_4
\end{aligned} \tag{4.8}$$

In Equation (4.6), if $\underline{L}_0 \times \underline{L}_1 \neq 0$, the first two planes intersect at a line which is the generator of the tangent-plane envelope and can be expressed by the dual vector,

$$\hat{\underline{R}} = \underline{L}_0 \times \underline{L}_1 + \varepsilon(\underline{X}) \times (\underline{L}_0 \times \underline{L}_1)$$

where $\underline{X}$ is the position vector of a point common to the two planes and

$\varepsilon^2 = 0$. Since

$$\underline{X} \times (\underline{L}_0 \times \underline{L}_1) = (\underline{L}_1 \cdot \underline{X})\, \underline{L}_0 - (\underline{L}_0 \cdot \underline{X})\, \underline{L}_1 = -P_1\, \underline{L}_0 + P_0\, \underline{L}_1 \tag{4.9}$$

we may write

$$\hat{\underline{R}} = \underline{L}_0 \times \underline{L}_1 + \varepsilon(-P_1\, \underline{L}_0 + P_0 \underline{L}_1) \tag{4.10}$$

As a function of ϕ, Equation (4.10) is the parametric equation of the tangent-plane envelope.

From Equation (4.9), we may have

$$\underline{L}_2 \times [(\underline{X}) \times (\underline{L}_0 \times \underline{L}_1)] = L_2 \times (-P_1 \underline{L}_0 + P_0 \underline{L}_1)$$

or

$$(\underline{L}_0 \underline{L}_1 \underline{L}_2)\underline{X} - (\underline{L}_2 \cdot \underline{X})\, (\underline{L}_0 \times \underline{L}_1) = -P_1 \underline{L}_2 \times \underline{L}_0 - P_0 \underline{L}_1 \times \underline{L}_2 \tag{4.11}$$

where $(\underline{L}_0 \underline{L}_1 \underline{L}_2)$ stands for $(\underline{L}_0 \times \underline{L}_1) \cdot \underline{L}_2$. Since

$$\underline{L}_2 \cdot \underline{X} + P_2 = 0$$

if $(\underline{L}_0 \underline{L}_1 \underline{L}_2) \neq 0$, Equation (4.11) can be written as

$$\underline{X} = -(P_0 \underline{L}_1 \times \underline{L}_2 + P_1 \underline{L}_2 \times \underline{L}_0 + P_2 \underline{L}_0 \times \underline{L}_1)/(\underline{L}_0 \underline{L}_1 \underline{L}_2) \tag{4.12}$$

Equation (4.12) gives the position vector of a point common to the three planes ($i = 0, 1, 2$) in Equation (4.6). This point is the intersection point of the three consecutive planes expressed in Equation (4.5). As a function of , Equation (4.12) is the parametric equation of the edge of regression of the tangent-plane envelope and the line expressed in Equation (4.10) is tangent to this curve.

Substituting Equations (3.65) and (3.73) through (3.77) into (4.7) and (4.8), we have, at zero position,

$$\underline{L}_0 = (\ell,\ m,\ n)$$

$$\underline{L}_1 = (-m,\ \ell,\ 0)$$

$$\underline{L}_2 = (-\ell,\ -m - w_{11}n,\ w_{11}m) \tag{4.13}$$

$$\underline{L}_3 = ((1 + w_{11}^2)m + (w_{11} + w_{22})n - (1 + w_{11}^2)\ \ell - w_{12}n,$$
$$(-w_{22} + 2w_{11})\ell + w_{12}m)$$

$$\underline{L}_4 = ((1 + 4w_{11}^2)\ell + 3w_{11}w_{12}m + (w_{23} + w_{12})n,$$
$$-3w_{11}w_{12}\ell + (1 + w_{11}^2)m + (-w_{13} + w_{22} + w_{11})n, \tag{4.13}$$
$$(-w_{23} + 3w_{12})\ell + (w_{13} + 3w_{22} - 3w_{11})m - 3w_{11}^2 n)$$

and

$$P_0 = p$$

$$P_1 = -d_1 n$$

$$P_2 = -d_{12}\ell - d_{32}n \tag{4.14}$$

$$P_3 = d_{13}\ell + (3d_{12} - 3w_{11}d_1 - d_{23})m - d_{33}n$$

$$P_4 = e_1\ell + e_2 m + e_3 n$$

where

$$e_1 = 6d_{12} - 4d_{23} - d_{14} - 4(2w_{11} - w_{22})d_1$$

$$e_2 = 4d_{13} - d_{24} - 6w_{11}d_{32} - 4w_{12}d_1$$

$$e_3 = -d_{34}$$

Thus, from Equation (4.10) and (4.12), at zero positions, the generator of the envelope of the tangent-plane (ℓ, m, n, p) is

$$\underline{\hat{R}}(0) = (-\ell n, -mn, \ell^2 + m^2) + \varepsilon(d_1 \ell n - mp, d_1 mn + \ell p, d_1 n^2) \qquad (4.15)$$

and the point on the edge of regression is

$$\underline{X}(0) = -(E/H, F/H, G/H) \qquad (4.16)$$

where

$$\begin{aligned} E &= w_{11}\ell mp + d_1 mn^2 + w_{11}d_1\, n(m^2+n^2) + d_{12}\ell^2 n + d_{32}\, \ell n^2 \\ F &= w_{11}m^2 p - (w_{11}d_1 - d_{12})\ell mn - d_1 \ell n^2 + d_{32}\, mn^2 \\ G &= w_{11}mnp - w_{11}d_1 \ell n^2 + (p - d_{12}\ell - d_{32}n)\,(\ell^2 + m^2) \\ H &= (\underline{L}_0 \underline{L}_1 \underline{L}_2) = w_{11}m(\ell^2 + m^2 + n^2) + n(\ell^2 + m^2) \end{aligned} \qquad (4.17)$$

and $H \neq 0$. The point in Equation (4.16) may be expressed using homogeneous coordinates and we have

$$\underline{X}\,(0) = (E, F, G, -H) \qquad (4.18)$$

in which each coordinate is a cubic polynomial in terms of the coordinates (ℓ, m, n, p) of the tangent-plane.

4.3 Edge of Regression and Its Spherical Indicatrix

Let $\underline{X}$ be the position vector of a point on the edge of regresssion associated to a developable and s be the arc-length. A unit tangent to the edge of regression can be expressed as

$$\underline{t} = d\underline{X}/ds \qquad (4.19)$$

Any tangent to the edge of regression is a generator of the developable.

From the well-known Serret-Frenet formulae, we have

$$\frac{d\underline{t}}{ds} = k\underline{n}$$

$$\frac{d\underline{n}}{ds} = \tau\underline{b} - k\underline{t} \tag{4.20}$$

$$\frac{d\underline{b}}{ds} = -\tau\underline{n}$$

in which k and τ are the curvature and torsion of the edge of regression and $\underline{n}$, $\underline{b}$ are its unit normal and binormal, repsectively. The unit vectors $\underline{t}$, $\underline{n}$, and $\underline{b}$ are perpedicular to each other and have the relationship,

$$\underline{t} \times \underline{n} = \underline{b}$$

The locus of a point, whose position vector is the unit binormal $\underline{b}$ of a curve, is called the spherical indicatrix of the binormal to the curve. Such a locus lies on the surface of a unit sphere. Let the suffix s be used to distinguish quantities belonging to this locus. The equation to the spherical indicatrix of the binormal to the edge of regression is

$$\underline{X}_s = \underline{b} \tag{4.21}$$

Therefore, from Equations (4.20), we have

$$\underline{t}_s = \frac{d\underline{x}_s}{ds_s} = \frac{d\underline{b}}{ds}\Big/\frac{ds_s}{ds} = -\tau\underline{n}\Big/\frac{ds_s}{ds} \tag{4.22}$$

We may make the unit vectors,

$$\underline{t}_s = -\underline{n} \tag{4.23}$$

and obtain

$$\tau = ds_s \Big/ ds \tag{4.24}$$

From the above equations, the following relationships can be obtained.

$$\frac{d\underline{X}}{ds_s} = \frac{d\underline{X}}{ds} \Bigg/ \frac{ds_s}{ds} = \frac{1}{\tau}\ \underline{t} \tag{4.25}$$

$$\frac{d\underline{t}}{ds_s} = \frac{d\underline{t}}{ds} \Bigg/ \frac{ds_s}{ds} = \frac{k}{\tau}\underline{n} = \nu\underline{n}$$

$$\frac{d\underline{n}}{ds_s} = \frac{d\underline{n}}{ds} \Bigg/ \frac{ds_s}{ds} = b - \frac{k}{\tau}\underline{t} = \underline{b} - \nu\underline{t} \tag{4.25}$$

and

$$\frac{d\underline{b}}{ds_s} = \frac{d\underline{b}}{ds} \Bigg/ \frac{ds_s}{ds} = -\underline{n}$$

in which $\nu = k/\tau$.

4.4 Contact of Developables

In the line-trajectory curvature theory, Roth, Yang, and Kirson [24, 60] defined the mth-order contact of ruled surfaces as having (m + 1) common consecutive lines. The order of contract of developable surfaces may be defined in the same way. Let $\hat{R}(s)$ and $\hat{R}^*(s^*)$, in a unit screw form, be the parametric equations of two developables surfaces, where s and s* are the suitably chosen geometrical parameters of the developable surfaces. For each value s or s*, the unit screw $\hat{\underline{R}}$ (s) or $\hat{R}^*(s^*)$ represents a generator of the developable.

Two developables, $\underline{\hat{R}}(s)$ and $\hat{R}*(s*)$ have exactly mth order contact, if and only if

$$\frac{d^k\underline{\hat{R}}(s)}{ds^k} = \frac{d^k\underline{\hat{R}}*(s*)}{ds*^k} \quad (k = 0, 1, 2, . . . m)$$

and

$$\frac{d^{m+1}\underline{\hat{R}}}{ds^{m+1}} = \frac{d^{m+1}\underline{\hat{R}}*}{ds*^{m+1}} \tag{4.26}$$

Since the generator of the tangent-plane envelope are tangnet to the edge of regression, they can be represented by the unit screw,

$$\underline{\hat{R}} = \underline{t} + \varepsilon\underline{X} \times \underline{t} = \underline{\hat{t}} \tag{4.27}$$

where $\underline{X}$ is the position vector of a point on the edge of regression.

The edge of regression and its osculating developable are enveloped by the tangnet-plane which has the direction $\underline{b}$. The edge of regression and the developable are functions of the arc-length s_s of the spherical indicatrix, $\underline{X}_s = \underline{b}$. From Equation (4.27), we have

$$d\underline{\hat{R}}\Big/ds_s = k\ (\underline{n} + \underline{X} \times \underline{n})/\tau = \nu\underline{\hat{n}} \tag{4.28}$$

where $\nu = k/\tau$ is a dimensionless number representing the first-order intrinsic property of a developable.

Referring to Equations (4.26), (4.27), and (4.28), any two developables having first-order contact requires that their edges of regression have the same property ν and common tangent and normal (hence binormal too). However, for any two developables, the tangents, normal and binormal, of their edges of regression can be brought to become coincident correspondingly through proper rotation. Therefore, developables will be said to have first-order contact (or first-order contact within rotation strictly) at the contact generator if they have common property ν.

Differentiating Equation (4.28) again, we obtain

$$d^2\underline{\hat{R}}/ds_s^2 = \nu' \underline{\hat{n}} + \nu\{(\underline{b} - \nu\underline{t}) + \varepsilon [\underline{b}/\tau + \underline{X} \times (\underline{b} - \nu\underline{t})]\} \tag{4.29}$$

where $\nu' = d\nu/ds_s$ is a dimensionless number and τ is the torsion of the edge of regression. ν' and τ are the second-order intrinsic properties of the tangnet-plane envelope. Any two developables with the same values of ν, ν', and τ have second-order contact (contact at three infinitesimal separated generators) along the contact generator.

4.5 Contact of Developables Within Stretch Rotation

If the developables are stretched proportionally with the scales "e" equal to the torsion τ of their own edges of regression, the developable is normalized to have unit torsion and the generator in Equation (4.27) becomes

$$\underline{\hat{R}}_n = \underline{t} + \varepsilon\underline{X}_n \times \underline{t} = \underline{\hat{t}}_n \tag{4.30}$$

where

$$\underline{X}_n = e\underline{X} \tag{4.31}$$

Since $ds_s = \tau ds$, we have

$$d\underline{X}_n/ds_s = ed\underline{X}/\tau ds = (e/\tau)\underline{t} \tag{4.32}$$

and referring to Equations (4.25), we obtain

$$d\hat{\underline{R}}_n/ds_s = (K/\tau)(\underline{n} + \varepsilon \underline{X}_n \times \underline{n}) = \nu \hat{\underline{n}}_n \tag{4.33}$$

$$d^2\hat{\underline{R}}_n/ds_s = \nu' \hat{\underline{n}}_n + \hat{\underline{n}}'_n \tag{4.34}$$

$$d^3\hat{\underline{R}}_n/ds_s = \nu'' \hat{\underline{n}}_n + 2\nu' \hat{\underline{n}}'_n + \nu \hat{\underline{n}}''_n \tag{4.35}$$

in which $\underline{n}_n$ is the unit screw representing the normal to the edge of regression $\underline{X}_n$ of the normalized developable and

$$\hat{\underline{n}}'_n = \underline{b} - \nu \underline{t} + \varepsilon[e/\tau\ \underline{b} + \underline{X}_n \times (\underline{b} - \nu t)] \tag{4.36}$$

$$\hat{\underline{n}}''_n = -\nu' \underline{t} - (1 + \nu^2)\underline{n} - \varepsilon\{(e\tau'/\tau^2)\underline{b} + 2(e/\tau)\underline{n} + \underline{X}_n \times [\nu' \underline{t} + (1 + \nu^2)\underline{n}]\} \tag{4.37}$$

Let $e = \tau$. The above equations become

$$\hat{\underline{n}}'_n = \underline{b} - \nu \underline{t} + \varepsilon[\underline{b} + \underline{X}_n \times (\underline{b} - \nu \underline{t})] \tag{4.38}$$

$$\hat{\underline{n}}''_n = -\nu' \underline{t} - (1 + \nu^2)\underline{n} - \varepsilon\{u\underline{b} + 2\underline{n} + \underline{X}_n \times [\nu' \underline{t} + (1 + \nu^2)\underline{n}]\} \tag{4.39}$$

where $u = \tau'/\tau$ and $\tau' = d\tau/ds_s$.

In Equations (4.33) through (4.39), four dimensionless numbers ν, ν', ν'' and u are obtained. They are the first order (ν), the second order (ν'), and the third order (u and ν'') characteristic numbers of a developable. Any two developables with the common characteristic numbers ν and ν' have second-order contact within stretch rotation. For the third-order contact of developables within stretch rotation, they should have four common characteristics numbers ν, ν', ν'', and u. With each additional order of contact, two more common characteristic numbers are needed.

4.6 First-Order Characteristic Equation

In Equation (4.6), $\underline{L}_0$ is the unit normal to the moving plane. Since the moving plane is tangent to the edge of regression of its envelope, $\underline{L}_0$ is also the unit binormal to the edge of regression. Hence, the spherical indicatrix of the binormal to the edge of regression is

$$\underline{X}_s = \underline{b} = \underline{L}_0 \tag{4.40}$$

Differentiating it with respect to ϕ and letting the suffix s distinguish the quantities belonging to the spherical indicatrix, we have by Equations (4.25)

$$\begin{aligned} d\underline{X}_s/d\phi &= (d\underline{X}_s/ds_s)(ds_s/d\phi) = s_{s1}\underline{t}_s \\ &= d\underline{b}/d\phi = (d\underline{b}/ds_s)(ds_s/d\phi) = (-\tau\underline{n})s_{s1} \\ &= \underline{L}_1 \end{aligned} \tag{4.41}$$

where the subscript 1 denotes the first derivative of the associated quantity with respect to ϕ. From Equations (4.41), we may obtain

$$\underline{t}_s = \underline{L}_1/s_{s1} \tag{4.42}$$

and

$$s_{s1}^2 = \underline{L}_1 \cdot \underline{L}_1 \tag{4.43}$$

We note that $s_{s1} = ds_s/d\phi$ is related to the change of the orientation of the tangent-plane. Referring to Equation (4.23), one obtains

$$\underline{n} = -\underline{L}_1/s_{s1} \tag{4.44}$$

and

$$\tau = s_{s1}/s_1 \tag{4.45}$$

The unit tangent to the edge of regression is then obtained

$$\underline{t} = \underline{n} \times \underline{b} = \underline{L}_0 \times \underline{L}_1/s_{s1} \tag{4.46}$$

and we have

$$d\underline{t}/ds_s = (s_{s1}\underline{L}_0 \times L_2 - s_{s2}\underline{L}_0 \times L_1)/s_{s1}^3 \tag{4.47}$$

We note that all subscripts i (i - 0, 1, 2, . . .) denote the ith derivatives of the associated quantities with respect to ϕ. From equations (4.25) and (4.44), we obtain

$$\nu = (d\underline{t}/ds_s) \cdot \underline{n} = (\underline{L}_0\underline{L}_1\underline{L}_2)/s_{s1}^3 \tag{4.48}$$

One should note that ν is the geodesic curvature of the spherical indicatrix. Substituting Equations (4.13) into (4.43) and (4.48), we have

$$s_{s1}^2 = \ell^2 + m^2 \tag{4.49}$$

and

$$H - \nu(\ell^2 + m^2)^{3/2} = 0 \tag{4.50}$$

where H is given in Equation (4.17).

Equation (4.50) is the first-order characteristic equation of tangent-plane envelopes. Since (ℓ, m, n) is the common normal to all parallel tangent-planes, Equation (4.50) shows that all parallel-tangent-plane envelopes have the common characteristic number ν and hence have first-order contact.

4.7 Second-Order Characteristic Equation

The second-order characteristic equation is obtained from Equation (4.48)

$$\nu' = [s_{s1}^2\,(\underline{L}_0\underline{L}_1\underline{L}_3) - 3s_{s1}s_{s2}(\underline{L}_0\underline{L}_1\underline{L}_2)]/s_{s1}^6 \qquad (4.51)$$

where

$$s_{s1}s_{s2} = \underline{L}_1 \cdot \underline{L}_2 \qquad (4.52)$$

is obtained from Equation (4.43). Substituting Equations (4.13) into them, we have the second-order characteristic equation,

$$\begin{aligned}&(\ell^2 + m^2)\,[(2w_{11} - w_{22})\ell + w_{12}m] +\\ &3w_{11}^2\ell mn - \nu'(\ell^2 + m^2)^3 = 0\end{aligned} \qquad (4.53)$$

Since Equations (4.50) and (4.53) contain variables ℓ, m, and n only and all parallel tangent-planes in the moving system have the common normal (ℓ, m, n), therefore parallel-tangent-plane envelopes have the common characteristic number ν and ν' and hence have second-order contact within stretch rotation. Equation (4.10) or (4.15) also shows that the generators of these parallel-tangent-plane envelopes are parallel to each other.

The intrinsic property τ of a tangent-plane envelope can be found from the following derivation. By the edge of regression in Equation (4.12), we have

$$\begin{aligned}d\underline{X}/d\phi = \; &\{-(\underline{L}_0\underline{L}_1\underline{L}_2)(P_0\underline{L}_1 \times \underline{L}_3 + P_1\underline{L}_3 \times \underline{L}_0 + P_3\underline{L}_0 \times \underline{L}_1) +\\ &(\underline{L}_0\underline{L}_1\underline{L}_3)(P_0\underline{L}_1 \times \underline{L}_2 + P_1\underline{L}_2 \times \underline{L}_0 + P_2(\underline{L}_0 \times \underline{L}_1\,))\}\,/\\ &(\underline{L}_0\underline{L}_1\underline{L}_2)^2\end{aligned} \qquad (4.54)$$

which can be simplified to

$$\underline{X}_1 = s_1\underline{t} = U(\underline{L}_0 \times \underline{L}_1)/(\underline{L}_0\underline{L}_1\underline{L}_2)^2 \tag{4.55}$$

where

$$U = P_0(\underline{L}_1\underline{L}_2\underline{L}_3) - P_1(\underline{L}_0\underline{L}_2\underline{L}_3) + P_2(\underline{L}_0\underline{L}_1\underline{L}_3) - P_3(\underline{L}_0\underline{L}_1\underline{L}_2) \tag{4.56}$$

and s_1 is the first derivative of the arc-length of the edge of regression with respect to ϕ. By Equation (4.46), we may obtain

$$s_1 = \underline{X}_1 \cdot \underline{t} = U(\underline{L}_0 \times \underline{L}_1) \cdot (\underline{L}_0 \times \underline{L}_1)/[s_{s1}(\underline{L}_0\underline{L}_1\underline{L}_2)^2] \tag{4.57}$$

Since $\underline{L}_0 \cdot \underline{L}_0 = 1$, we have, at zero position,

$$\underline{L}_0 \cdot \underline{L}_1 = 0 \tag{4.58}$$

and therefore,

$$(\underline{L}_0 \times \underline{L}_1) \cdot (\underline{L}_0 \times \underline{L}_1) = (\underline{L}_0 \cdot \underline{L}_0)(\underline{L}_1 \cdot \underline{L}_1) = s_{s1}^2 \tag{4.59}$$

Thus, we have

$$s_1 = Us_{s1}/(\underline{L}_0\underline{L}_1\underline{L}_2)^2 \tag{4.60}$$

and

$$\tau = ds_s/ds = s_{s1}/s_1 = (\underline{L}_0\underline{L}_1\underline{L}_2)^2/U \tag{4.61}$$

With $\nu = k/\tau$, we can also find

$$k = \nu\tau = (\underline{L}_0\underline{L}_1\underline{L}_2)^3/[s_{s_1}{}^3\, U] \tag{4.62}$$

Substituting Equations (4.13) and (4.14) into Equations (4.61) and (4.62), we have the properties τ and k of the tangent-plane envelope at zero position.

$$\tau = [w_{11}m + n\ (\ell^2 + m^2)]^2/U \tag{4.63}$$

$$k = [w_{11}m + n\ (\ell^2 + m^2)]^3/[U(\ell^2 + m^2)^{3/2}] \tag{4.64}$$

where

$$\begin{aligned} U = {} & c_1\ell^4 + c_2(\ell^2 + m^2 + n^2)\ell m + (c_3\ell n + f_7\ell p + c_5 mn + f_8 mp) \\ & (\ell^2 + m^2) + (c_4m^2 + c_6n^2)\ell^2 + c_7\ell mnp + (c_8\ell + f_2m + f_3n)n^3 \\ & + c_9m^4 + f_1m^2n^2 \end{aligned} \tag{4.65}$$

in which

$$\begin{aligned}
c_1 &= -f_7\ d_{12} \\
c_2 &= -w_{12}d_{12} + w_{11}d_{13} \\
c_3 &= -f_7\ d_{32} + d_{13} \\
c_4 &= c_1 + c_9 \\
c_5 &= f_5 - f_4 \\
c_6 &= f_6 - w_{11}\ f_7\ d_1 + (w_{11} + w_{22})\ d_{12} \\
c_7 &= 3\ w_{11}^2 \\
c_8 &= w_{12}\ d_1 + (w_{11} + w_{22})\ d_{32} \\
c_9 &= -w_{11}\ f_4 \\
f_1 &= w_{11}\ (w_{11} + w_{22})\ d_1 + f_6 + c_9 \\
f_2 &= (w_{11} + w_{22})\ d_1 + f_5 \\
f_3 &= w_{11}\ (w_{11} + w_{22})\ d_1 \\
f_4 &= 3\ d_{12} - 3\ w_{11}\ d_1 - d_{23} \\
f_5 &= w_{11}\ f_6 - w_{12}\ d_{32} \\
f_6 &= (1 + w_{11}^2)\ d_1 + d_{33} \\
f_7 &= 2\ w_{11} - w_{22} \\
f_8 &= w_{12}
\end{aligned}$$

Equations (4.63) and (4.64) are functions of the plane coordinates (ℓ, m, n, p).

4.8 Third-Order Characteristic Equations

The two third-order characteristic numbers are

$$\nu'' = d^2\nu/ds_s{}^2 \quad \text{and} \quad u = d\tau/\tau ds_s$$

Therefore, from Equations (4.61) and (4.48), we may have

$$\nu'' = d\ \nu'/ds_s = A/s_{s1}^9 \tag{4.66}$$

$$u = [2(\underline{L}_0\underline{L}_1\underline{L}_3)\ U - (\underline{L}_0\underline{L}_1\underline{L}_2)\ U_1]/[\underline{L}_1 \cdot \underline{L}_1)^{1/2}\ (\underline{L}_0\underline{L}_1\underline{L}_2)U] \tag{4.67}$$

where U is defined in Equation (4.56) and

$$A = (\underline{L}_1 \cdot \underline{L}_1)^2[(\underline{L}_0\underline{L}_2\underline{L}_3) + (\underline{L}_0\underline{L}_1\underline{L}_4)] - 7\ (\underline{L}_1 \cdot \underline{L}_1)(\underline{L}_1 \cdot \underline{L}_2)$$
$$(\underline{L}_0\underline{L}_1\underline{L}_3) + (\underline{L}_0\underline{L}_1\underline{L}_2)\ [18(\underline{L}_1 \cdot \underline{L}_2)^2 - 3(\underline{L}_1 \cdot \underline{L}_1)$$
$$(\underline{L}_2 \cdot \underline{L}_2 + \underline{L}_1 \cdot L_3)] \tag{4.68}$$

$$U = P_0(\underline{L}_1\underline{L}_2\underline{L}_4) - P_1(\underline{L}_0\underline{L}_2\underline{L}_4) + P_2(\underline{L}_0\underline{L}_1\underline{L}_4) - P_4(\underline{L}_0\underline{L}_1\underline{L}_2) \tag{4.69}$$

Substituting Equations (4.13) and (4.14), Equations (4.66) and (4.67) can be expressed in terms of the plane coordinates (ℓ, m, n, p).

Just as the ν and ν' -characteristic equations, Equation (4.66) contains only three variables, ℓ, m, and n. We note that all parallel-tangent-plane envelopes have the common characteristic numbers ν, ν', ν'' . . . $\nu^{(n)}$($= d\nu^n/ds_s^n$).

Equation (4.67) contians ℓ, m, n, and p of the plane coordinates. In a family of parallel-tangent-plane envelopes, all the envelopes generally have different characteristic number u. Among the parallel-tangent-planes with the common normal (ℓ, m, n), the only tangent-plane whose envelope has the characteristic number u may be found from Equation (4.67). Since $P_0 = p$, if $C \neq 0$, Equation (4.67) may be rewritten as

$$p = B/C \tag{4.70}$$

where

$$\begin{aligned} B = & -2(\underline{L}_0\underline{L}_1\underline{L}_3)\,[-P_1(\underline{L}_0\underline{L}_2\underline{L}_3) + P_2(\underline{L}_0\underline{L}_1\underline{L}_3) - P_3(\underline{L}_0\underline{L}_1\underline{L}_2)] \\ & + (\underline{L}_0\underline{L}_1\underline{L}_2)\,[-P_1(\underline{L}_0\underline{L}_2\underline{L}_4) + P_2(\underline{L}_0\underline{L}_1\underline{L}_4) - P_4(\underline{L}_0\underline{L}_1\underline{L}_2)] \\ & + u(\underline{L}_1 \cdot \underline{L}_1)^{1/2}\,(\underline{L}_0\underline{L}_1\underline{L}_2)\,[-P_1(\underline{L}_0\underline{L}_2\underline{L}_3) + P_2(\underline{L}_0\underline{L}_1\underline{L}_3) - P_3(\underline{L}_0\underline{L}_1\underline{L}_2)] \end{aligned}$$

$$C = 2(\underline{L}_0\underline{L}_1\underline{L}_3)(\underline{L}_1\underline{L}_2\underline{L}_3) - (\underline{L}_0\underline{L}_1\underline{L}_2)\,[(\underline{L}_1\underline{L}_2\underline{L}_4) + u(\underline{L}_1 \cdot \underline{L}_1)^{1/2}(\underline{L}_1\underline{L}_2\underline{L}_3)]$$

CHAPTER V

SPECIAL CASES

In the planar point-path curvature theory, the various cases of singularities such as inflection circle, Ball's point have been studied extensively and applied in mechanism synthesis. In space point-path, the analogy of these singularities leads to the generation of point paths with flex (zero curvature), planar flex (zero torsion) and stationary curvature. The motion of a point and a tangent-plane are somewhat analogous. In the following, some special cases analogous to these singularities are discussed.

5.1 Stationary Planes

For the motion of a plane and a point, a stationary plane and a cusp point (stationary point) are dual configurations. It is obvious that in a planar motion, the point at the instant center is stationary and is a cusp point. In a general space motion, the points on the ISA have the minimum velocity among all points on the moving body. Therefore, one can assure that the existence of a stationary point is possible only when the instantaneous pitch is zero and the locus of stationary points is the ISA, which in the canonical system is

$$x = y = 0 \quad (d_1 = 0 \text{ or } \sigma_0 = 0) \tag{5.1}$$

Apparently, the points on the ISA of a moving body executing spherical motion are all stationary points.

A stationary point becomes a double stationary point if the following condition is also satisfied.

$$\underline{X}_2 = [A_2]\,\underline{x} + \underline{D}_2$$

$$= (-x + d_{12},\ -y - w_{11}z,\ w_{11}y + d_{32}) = 0$$

This leads to

$$d_{12} = d_{32} = 0 \qquad (\text{or } \alpha_0^\circ = \sigma_1 = 0) \tag{5.2}$$

If $w_{11} \neq 0$, the central point (0, 0, 0) is the only stationary point. If $w_{11} = 0$, then the ISA becomes stationary and all points on it are double stationary points.

In a general space motion, the above motion conditions are usually not satisfied and hence no stationary point exists.

Let (ℓ, m, n, p) be a plane on the moving body. From Equation (4.6) the plane is stationary if $\underline{L}_1 = 0$ and $P_1 = 0$. This requires that

$$\ell = m = 0 \tag{5.3}$$

and

$$d_1 = 0 \quad (\text{or } \sigma_0 = 0) \tag{5.4}$$

Equations (5.3) and (5.4) represent the planes perpendicular to the ISA of the moving body which has zero instantaneous pitch (Figure 10).

A double stationary plane exists if $\underline{L}_2 = 0$ and $P_2 = 0$ in Equation (4.6). From Equations (4.13) and (4.14), this leads to

$$w_{11} = d_{32} = 0$$

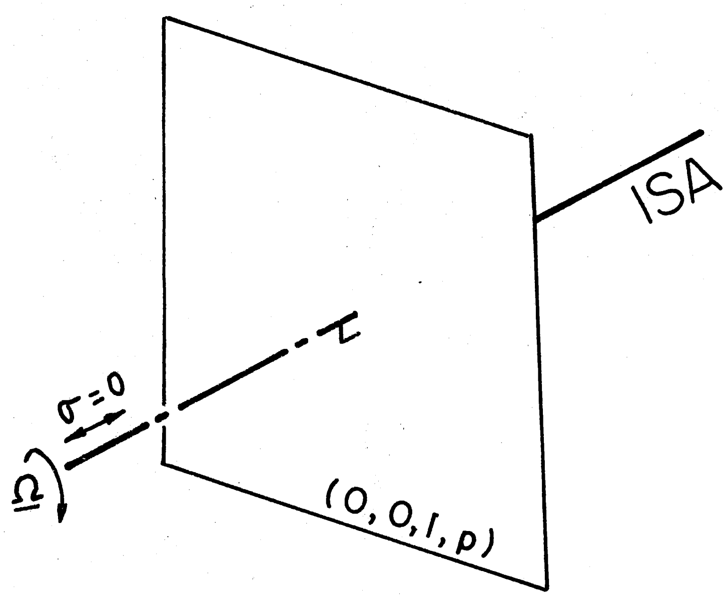


Figure 10. A Stationary Plane

Therefore, for the existence of a double stationary plane, the moving body should have

$$w_{11} = d_1 = d_{32} = 0 \qquad (\text{or} \ \alpha_0 = \sigma_0 = \sigma_1 = 0) \tag{5.5}$$

and the double stationary planes are those planes orthogonal to the ISA.

5.2 Stationary Generators

Under one-parameter motion, a plane generally intersects its consecutive plane at a straight line, which is the generator of the tangent-plane envelope surface. For three consecutive planes, two generators may be obtained. However, there are cases that three consecutive planes intersect at a common line which becomes a stationary generator of the enveloped surface.

Equation (4.4) represents a plane in the moving system. If the three consecutive planes intersect at a common straight line, the coefficient matrix of the three plane equations (i = 0, 1, 2) in Equation (4.6),

$$\begin{bmatrix} \ell & m & n & p \\ -m & \ell & 0 & -d_1 n \\ -\ell & -m-w_{11}n & w_{11}m & -d_{12}\ell - d_{32}n \end{bmatrix} \tag{5.6}$$

should have rank less than 3. This condition is satisfied if

$$\begin{vmatrix} \ell & m & p \\ -m & \ell & -d_1 n \\ -\ell & -m-w_{11}n & -d_{12}\ell - d_{32}n \end{vmatrix} \tag{5.7}$$

$$= w_{11}mnp - w_{11}d_1\ell n^2 + (p - d_{12}\ell - d_{32}n)\,(\ell^2 + m^2) = 0$$

and

$$(\underline{L}_0\underline{L}_1\underline{L}_2) = w_{11}m(\ell^2 + m^2 + n^2) + n(\ell^2 + m^2) = 0 \tag{5.8}$$

These are third degree homogeneous equations in terms of the plane coordinates. The locus of the planes (ℓ, m, n, p) satisfying Equations (5.7), (5.8), and the constraint,

$$\ell^2 + m^2 + n^2 = 1 \tag{5.9}$$

is a single-infinity of planes and each plane envelopes a stationary generator at zero position.

Excluding the case $\ell^2 + m^2 = 0$, the following cases are distinguished.

1. $w_{11} \neq 0$

From Equations (5.7) to (5.8), with n as the parameter, we have

$$\left.\begin{aligned} m &= -n(1 - n^2)/w_{11} \\ \ell^2 &= (1-n^2)\ [w_{11}{}^2 - n^2(1-n^2)]/w_{11}{}^2 \\ p &= [w_{11}d_1\ell n^2 + (d_{12}\ell + d_{32}n)(1-n^2)]/(1-n^2)^2 \end{aligned}\right\} \tag{5.10}$$

For any value n, two tangent-planes may be found to envelop a stationary generator.

2. $w_{11} = 0$

From Equations (5.7) and (5.8), we have

$$n = 0$$

$$\ell^2 + m^2 = 1$$

and (5.11)

$$p = d_{12}\ell$$

The above equations represent the family of tangent-planes which are parallel and have a distance $|d_{12}\ell|$ to the ISA.

We note that there exists analogy between point-path and tangent-plane motion. On a moving body, there exist points which are colinear with their two consecutive points and on the point pahts, points of inflection appear. For tangent-plane motion, we have from the above discussion, tangent-planes which have common straight lines with their two consecutive planes and on the tangent-plane envelopes, stationary generators appear. Such analogy also exists in the following double stationary generator and the straight line segment traced by a Ball's point. Since $(\underline{L}_0\underline{L}_1\underline{L}_2) = 0$, the point on the edge of regression cannot be determined. For any plane cutting through the stationary generator, the intersection curve with the tangent-plane envelope has a double point or a cusp at the stationary generator.

5.3 Double Stationary Generators

A double stationary generator exists when four consecutive planes intersect at a common straight line. In such a case, the coefficient matrix.

$$\begin{bmatrix} \ell & m & n & p \\ -m & \ell & 0 & -d_1 m \\ -\ell & -m-w_{11}n & w_{11}m & -d_{12}\ell-d_{32}n \\ (1+w_{11}^2)m+(w_{11}+w_{22})n & -(1+w_{11}^2)\ell-w_{12}n & (2w_{11}-w_{22})\ell+w_{12}m & P_3 \end{bmatrix} \quad (5.12)$$

of the plane equations (i = 0, 1, 2, 3) in Equations (4.6) should have rank less than three. Thus in addition to Equations (4.7) and (5.8), the following two more conditions must be satisfied.

$$(\underline{L}_0\underline{L}_1\underline{L}_3)=\begin{vmatrix} \ell & m & n \\ -m & \ell & 0 \\ (1+w_{11}^2)m+(w_{11}+w_{22})n & -(1+w_{11}^2)\ -w_{12}n & (2w_{11}-w_{22})\ +w_{12}m \end{vmatrix}$$

$$= [(2w_{11} - w_{22})\ \ell + w_{12}m]\ (\ell^2 + m^2 + n^2) - 3w_{11}\ell n = 0 \qquad (5.13)$$

$$\begin{vmatrix} \ell & m & p \\ -m & \ell & -d_1n \\ (1+w_{11}^2)m+(w_{11}+w_{22})n & -(1+w_{11}^2)\ell - w_{12}n & P_3 \end{vmatrix}$$

$$= [w_{12}m - (w_{11} + w_{22})n]np - [(w_{11} + w_{22})m + w_{12}\ell]d_1n^2$$

$$+ \{(3d_{12} - 3w_{11}d_1 - d_{23})m - [(1 - w_{11}^2)d_1 + d_{33}]n - d_{13}\ell\}$$

$$(\ell^2 + m^2) = 0 \qquad (5.14)$$

where P_3 is given in Equation (4.14). A tangent-plane is determined by only three independent data. For a general space motion, we are generally not able to find a plane whose coordinates satisfy Equations (5.7), (5.8), (5.13), and (5.14) simultaneously and hence in general, a doulbe stationary generator is not available.

A double stationary generator may exist in some special motions. Assume w_{11} has a non-zero value. Eliminating ℓ and m from Equations (5.8), (5.13), and (5.9) and excluding the case $n^2 = 1$, we have

$$B_1n^8 + B_2n^6 + B_3n^4 + B_4n^2 + B_5 = 0 \qquad (5.15)$$

where

$$B_1 = 9w_{11}^2$$

$$B_2 = -3w_{11}[3w_{11} + 2(2w_{11} - w_{22})]$$

$$B_3 = 9w_{11}^4 + (2w_{11} - w_{22})^2 + w_{12}^2 + 6w_{11}(2w_{11} - w_{22})$$

$$B_4 = -(2w_{11} - w_{22})^2 - 6w_{11}^3 (2w_{11} - w_{22}) - w_{12}^2$$

$$B_5 = w_{11}^2 (2w_{11} - w_{22})^2$$

This is an eighth degree polynomial. For each existing solution $n(|n| < 1)$, we may find ℓ and m from the following equation.

$$\left.\begin{aligned} & w_{11}m + n(1 - n^2) = 0 \\ & w_{11} [(2w_{11} - w_{22}) - 3w_{11}n^2]\ell = w_{12}n (1 - n^2) \end{aligned}\right\} \quad (5.16)$$

Once ℓ, m, and n are known, a double stationary generator may be found if a value of p exists to satisfy Equation (5.7) and (5.14) simultaneously. However, such a solution (ℓ, m, n, p) is unlikely to exist in a general space motion. Hence, in general, a double stationary generator does not exist. In a spherical motion, we can always find $p = 0$ satisfying Equations (5.7) and (5.14) and therefore, from Equation (5.15), at most eight tangent-planes may be found to envelop doulbe stationary generators.

If $w_{11} = 0$, we have also $w_{22} = 0$. Thus, from Equation (5.7), (5.8), (5.13), or (5.14) and (5.9), we may obtain

$$\left.\begin{aligned} & n = 0 \\ & w_{12}m = 0 \\ & p = d_{12}\ell \\ & \ell^2 + m^2 = 1 \end{aligned}\right\} \quad (5.17)$$

and

$$d_{13}\ell = (3d_{12} - d_{23})m \tag{5.17}$$

The following cases may be distinguished.

1. $w_{12} \neq 0$ and $d_{13} \neq 0$

   No double stationary generator exists.

2. $w_{12} \neq 0$ and $d_{13} = 0$

   $(1, 0, 0, d_{12})$ is the only tangent-plane to envelop a double stationary generator

3. $w_{12} = 0$ and $d_{13} \neq 0$ (i. e. $\alpha_0 = \alpha_1 = 0$, $\alpha_1^\circ \neq 0$)

   From Equations (5.17), we have

   $m = \{d_{13}^2/[d_{13}^2 + (3d_{12} - d_{23})^2]\}^{1/2}$

   $\ell = (3d_{12} - d_{23})m/d_{13}$

   $n = 0$

   $p = d_{12}\ell$

Hence, a tangent-plane exists to envelop a double stationary generator.

4. $w_{12} = d_{13} = 3d_{12} - d_{23} = 0$ (i. e. $\alpha_0 = \hat{\alpha}_1 = \alpha_0^\circ (3 - \beta_0) = 0$)

We have,

$$\left.\begin{aligned} n &= 0 \\ p &= d_{12}\ell \\ \ell^2 + m^2 &= 1 \end{aligned}\right\} \tag{5.18}$$

Any tangent-plane (ℓ, m, n, p) satisfying the above equations envelopes a double stationary generator. Equations (5.18) represent a family of tangent-planes which are parallel and have the distance $|d_{12}\ell|$ to the ISA.

5. $w_{12} = d_{13} = 0$ and $3d_{12} - d_{23} \neq 0$

The plane $(1, 0, 0, d_{12})$ envelopes a double stationary generator.

From Equations (5.8) and (5.13) we have $(\underline{L}_0\underline{L}_1\underline{L}_2) = (\underline{L}_0\underline{L}_1\underline{L}_3) = 0$ Hence, the characteristic numbers $\nu = \nu' = 0$. With the consecutive generators being coincident, the point on the edge of regression is indeterminate. However, for any plane cutting through the double stationary generator, the intersection curve with the tangent-plane envelope has a triple point or a double cusp at the double stationary generator.

5.4 Stationary Points

The stationary point of a tangent-plane envelope is a cusp point on the edge of regression. It is a point common to four consecutive planes and may be considered as the analogy of the planar flex ($\tau = 0$, $k \neq 0$) [47] which has four consecutive points on the same plane.

For a tangent-plane envelope a stationary point exists if the change of the arc-length of the associated edge of regression is zero or the determinant of the coefficient matrix in Equation (5.12) is zero. This leads to

$$U = 0 \tag{5.19}$$

where U is given in Equation (4.65).

Let us exclude the cases of stationary plane and stationary generator from Equation (4.19) and consider only the situation,

$$\ell^2 + m^2 \neq 0 \text{ and } (\underline{L}_0\underline{L}_1\underline{L}_2) \neq 0$$

in the rest of this section. A stationary point on the edge of regression is a cusp point which as shown in Equations (4.63) and (4.64) has the properties

$$k^{-1} = \tau^{-1} = 0$$

A double stationary point exist if five consecutive planes intersect at a common point. Thus, on the edge of regression, a double cusp appears and the second order derivative s_2 of the arc-length of the edge of regression becomes zero. Differentiating Equation (4.60) with respect to ϕ and with U = 0, we have

$$U_1 = 0 \tag{5.20}$$

where U_1 is given in Equation (4.69). Substituting Equations (4.13) and (4.14), Equation (5.20) becomes

$$\begin{aligned} U_1 = {} & g_1\ell^4 + g_2\ell^3m + g_3\ell^3n + g_4\ell^2m^2 + g_5\ell^2mn + g_6\ell^2n^2 \\ & +g_7 m^3 + g_8\ell m^2n + g_9\ell mn^2 + g_{10}\ell n^3 + g_{11}m^4 + g_{12}m^3n \\ & +g_{13}m^2n^2 + g_{14}mn^3 + g_{15}n^4 + p(a_3\ell^3 + h_1\ell^2m + c_3\ell^2n + h_2\ell m^2 \\ & +h_3\ell mn + h_4m^3 + h_5m^2n + h_6mn^2) \end{aligned} \tag{5.21}$$

in which

$$g_1 = -a_3d_{12}$$

$$g_2 = -e_1w_{11} - b_3d_{12}$$

$$g_3 = -e_1 - a_3d_{32} - (c_3 - a_1)d_{12}$$

$$g_4 = -e_2w_{11} + g_1$$

$$g_5 = -e_3w_{11} - e_2 - b_3d_{32} - a_2w_{11}d_1$$

$$g_6 = -(a_2 + a_3w_{11})d_1 + c_1d_{12} - (c_3 - a_1)d_{32} - e_3$$

$$g_7 = g_2$$

$$g_8 = (a_1w_{11} - b_2w_{11})d_1 + a_1d_{12} - a_3d_{32} - e_1$$

$$g_9 = (a_1 - b_2 - b_3w_{11} - c_2w_{11})d_1 + c_2d_{12} - e_1w_{11}$$

$$g_{10} = c_1d_{32} + (-c_2 + a_1w_{11} - c_3w_{11})d_1$$

$$g_{11} = -e_2w_{11}$$

$$g_{12} = b_1w_{11}d_1 - b_3d_{32} - e_3w_{11} - e_2$$

$$g_{13} = (b_1 + c_1w_{11})d_1 - (c_3 - b_2)\,d_{32} - e_2w_{11} - e_3$$

$$g_{14} = (b_1w_{11} + c_1)d_1 + c_2d_{32} - e_3w_{11}$$

$$g_{15} = c_1w_{11}d_1$$

$$h_1 = b_3 + a_1w_{11}$$

$$h_2 = a_3 + a_2w_{11} + b_1w_{11}$$

$$h_3 = (a_3 + c_1)w_{11}$$

$$h_4 = b_3 + b_2w_{11}$$

$$h_5 = c_3 + b_3w11 + c_2w_{11}$$

$$h_6 = c_3w_{11}$$

where a_i, b_i, and c_i are the fourth order instantaneous invariants,

$$[A_4] = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

and

$$(e_1, e_2, e_3) = (6d_{12} - 4d_{23} - d_{14} - 4(2w_{11} - w_{22})d_1,$$

$$4d_{13} - d_{24} - 6w_{11}d_{32} - 4w_{12}d_1, - d_{34})$$

in which $[A_4]$ is given in Equation (3.77).

Equations (5.19) and (5.20) represent on the moving system, a single-infinity of tangent-planes which envelop double stationary points at zero position. A triple stationary point may be obtained with the additional conditions $s_3 = 0$ or $U_2 = 0$, where

$$U_2 = P_0\,(\underline{L}_1\underline{L}_3\underline{L}_4) - P_1\,(\underline{L}_0\underline{L}_3\underline{L}_4) + P_3\,(\underline{L}_0\underline{L}_1\underline{L}_4) -$$

$$P_4\,(\underline{L}_0\underline{L}_1\underline{L}_3) + P_0(\underline{L}_1\underline{L}_2\underline{L}_5) - P_1(\underline{L}_0\underline{L}_2\underline{L}_5) +$$

$$P_2\,(\underline{L}_0\underline{L}_1\underline{L}_5) - P_5\,(\underline{L}_0\underline{L}_1\underline{L}_2) \tag{5.22}$$

5.5 Helical Developables

A curve is a helix if and only if its curvature and torsion are in a constant ratio. On a helical curve, the angle between its tangents and the axis of the helix are constant. A helical developable shall be referred to a developable with a helical edge of regression. It may be called helical osculating developable more properly.

A tangent-plane envelopes a helical developable if the second order property $\nu' = 0$. Hence, from Equation (4.53), we have

$$(\ell^2 + m^2)\ [(2w_{11} - w_{22})\ell + w_{12}m] + 3w_{11}^2 \ell nm = 0 \quad (5.23)$$

Equation (5.23) represents the family of tangent-planes enveloping helical developables.

To synthesize a helical tangent-plane envelope with the specified first order property $\nu(=k/\tau)$, the tangent-plane can be found by solving Equations (4.50) and (5.23) simultaneously. The helical axis is perpendicular to the normal ($\underline{n} = -\underline{L}_1/s_{s1}$) of the curve and has the direction $(\underline{L}_0 \times \underline{L}_1)/s_{s1} + \nu\underline{L}_0$. The angle between the generator of the helical developables and the axis is $\cos^{-1}(1 + \nu^2)^{-1/2}$ [61].

A better helical developable may be obtained if the characteristic number $\nu'' = 0$ is also satisfied. From Equation (4.66), we obtain A = 0 which with $\nu' = 0$ in Equation (4.51), may be written as

$$(\underline{L}_1 \cdot \underline{L}_1)^2\ [(\underline{L}_0\underline{L}_2\underline{L}_3)+(\underline{L}_0\underline{L}_1\underline{L}_4)] - 3[(\underline{L}_1 \cdot \underline{L}_2)^2 + (\underline{L}_1\cdot \underline{L}_1)(\underline{L}_2\cdot \underline{L}_2 + \underline{L}_1 \cdot \underline{L}_3)](\underline{L}_0\underline{L}_1\underline{L}_2) = 0 \quad (5.24)$$

From Equations (4.13), Equation (5.24) can be expressed in terms of ℓ, m, and n. For a tangent-plane satisfying Equations (5.23) and (5.24), the envelope has the properties $\nu' = \nu'' = 0$. Once the normal (ℓ, m, n) of the tangent-planes is determined, one can generally find the tangent-plane whose envelope has the desired characteristic number u from Equation (4.70).

5.6 Cylindrical Developables

From Equation (4.46), we have

$$\underline{L}_0 \times \underline{L}_1 = s_{s1}\underline{t} \quad (5.25)$$

where $\underline{t}$ is the unit vector along the generator of a tangent-plane envelope.

If the consecutive generator also has the same direction, then the two consecutive generators are parallel and the tangent-plane envelope is a first-order cylindrical surface. In such a case, the derivative of $\underline{t}$ is zero. Thus, from the above equation, we obtain

$$\underline{L}_0 \times \underline{L}_2 = s_{s2}\underline{t} \tag{5.26}$$

and

$$(\underline{L}_0 \times \underline{L}_1) \times (\underline{L}_0 \times \underline{L}_2) = s_{s1}s_{s2}\ \underline{t} \times \underline{t} = 0$$

or

$$(\underline{L}_0\underline{L}_1\underline{L}_2)\underline{L}_0 = 0$$

Since $\underline{L}_0 \neq 0$, we have

$$(\underline{L}_0\underline{L}_1\underline{L}_2) = 0 \tag{5.27}$$

which is the condition for a tangent-plane to envelop a first-order cylindrical surface (Figure 11).

If one more consecutive generator also has the direction $\underline{t}$, then by differentiating Equation (5.26), we have

$$(\underline{L}_0 \times \underline{L}_3) + (\underline{L}_1 \times \underline{L}_2) = s_{s3}\underline{t} \tag{5.28}$$

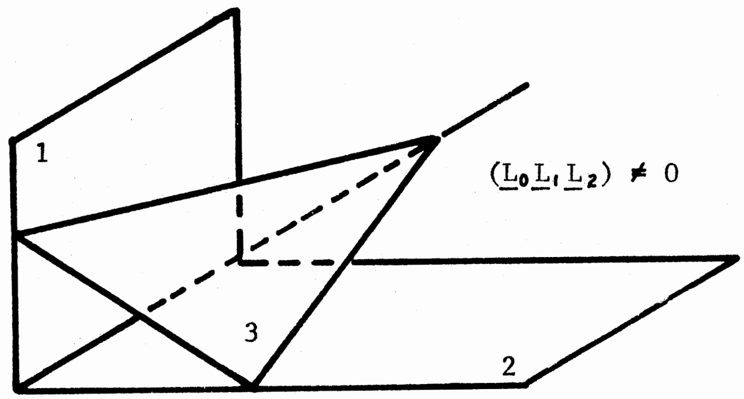
Taking a vector product with Equation (5.25), we obtain

$$(\underline{L}_0 \times \underline{L}_1) \times (\underline{L}_0 \times \underline{L}_3 + \underline{L}_1 \times \underline{L}_2) = 0$$

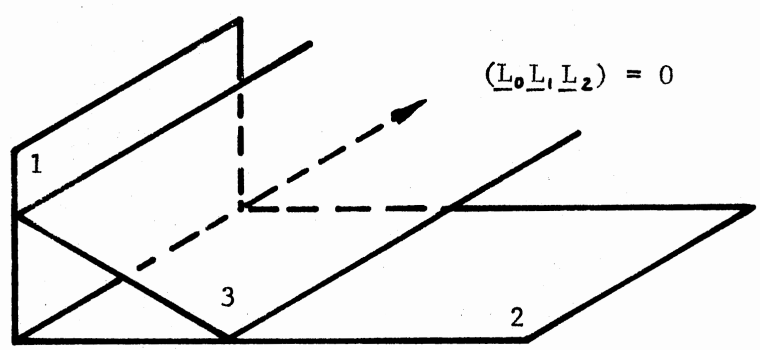
or

$$(\underline{L}_0\underline{L}_1\underline{L}_3)\underline{L}_0 + (\underline{L}_0\underline{L}_1\underline{L}_2)\underline{L}_1 = 0 \tag{5.29}$$

Since $\ell^2 + m^2 \neq 0$, $\underline{L}_0$ and $\underline{L}_1$ are nonzero vectors. From Equation (5.27) and (5.29), we have the following cases.



a) A General Case



b) First Order Cylindrical Developable

Figure 11. The Intersection of Three Consecutive Planes.

(a) $(\underline{L}_0\underline{L}_1\underline{L}_2) = 0, (\underline{L}_0\underline{L}_1\underline{L}_3) \neq 0$

The first two consecutive generators are parallel and the tangent-plane envelope is a first-order cylindrical surface. Equation (5.8) gives the family of tangent-planes enveloping first-order cylindrical surfaces.

If Equation (5.7) is also satisfied, the two parallel cylindrical generators are coincident and become a stationary generator.

(b) $(\underline{L}_0\underline{L}_1\underline{L}_2) = (\underline{L}_0\underline{L}_1\underline{L}_3) = 0$

The first three consecutive generators are parallel and the tangent-plane envelope is a second-order cylindrical surface. From Equations (5.8) and (5.13), the following cases may be distinguished.

(b.1) $\omega_{11} \neq 0$

Equations (5.15) and (5.16) give the tangent-planes enveloping second-order cylindrical surface.

(b.2) $\omega_{11} = \omega_{22} = 0, \omega_{12} \neq 0$ (i. e. $\alpha_0 = 0, \alpha_1 \neq 0$)

Any tangent-plane with the normal (1, 0, 0) envelops a second-order cylindrical surface.

(b.3) $\omega_{11} = \omega_{22} = \omega_{12} = 0$ (i. e. $\alpha_0 = \alpha_1 = 0$)

We have from Equations (5.8) and (5.13),

$$n = 0, \ \ell^2 + m^2 = 1$$

Thus, any tangent-plane parallel to the ISA envelops a second-order cylindrical surface.

In the above cases, if Equations (5.7) and (5.14) are also satisfied, the three consecutive cylindrical generators are coincident and become a double stationary generator.

CHAPTER VI

SPHERICAL TANGENT-PLANE MOTION

A spherical tangent-plane motion is basically a special case of a general space motion discussed in the previous chapters. However, it possesses some distinct properties and deserved special attention. In this chapter, a tangent-plane executing a spherical motion is analyzed. One may also find the analogy between the spherical point path and tangent-plane envelope curvature theories.

6.1 Characteristic Equations

In a space rigid body motion, if the instantaneous pitch is zero, the motion is a spherical one and all the translational instantaneous invariants diminish (Figure 12). Thus, in Equation (4.14), we have,

$$P_0 = p, \qquad P_i = 0 \qquad (i > 0) \tag{6.1}$$

Let (ℓ, m, n, p) be a tangent-plane on a moving system. The edge of regression of its envelope is

$$\underline{X} = -p\underline{L}_1 \times \underline{L}_2/(\underline{L}_0\underline{L}_1\underline{L}_2) \tag{6.2}$$

and the point on the edge of regression at zero position is

$$\underline{X} = (-w_{11}\ell mp/H,\ -w_{11}m^2P/H,\ -(w_{11}mn + \ell^2 + m^2)P/H) \tag{6.3}$$

where

$$H = (\underline{L}_0\underline{L}_1\underline{L}_2) = w_{11}m + n(\ell^2 + m^2) \tag{6.4}$$

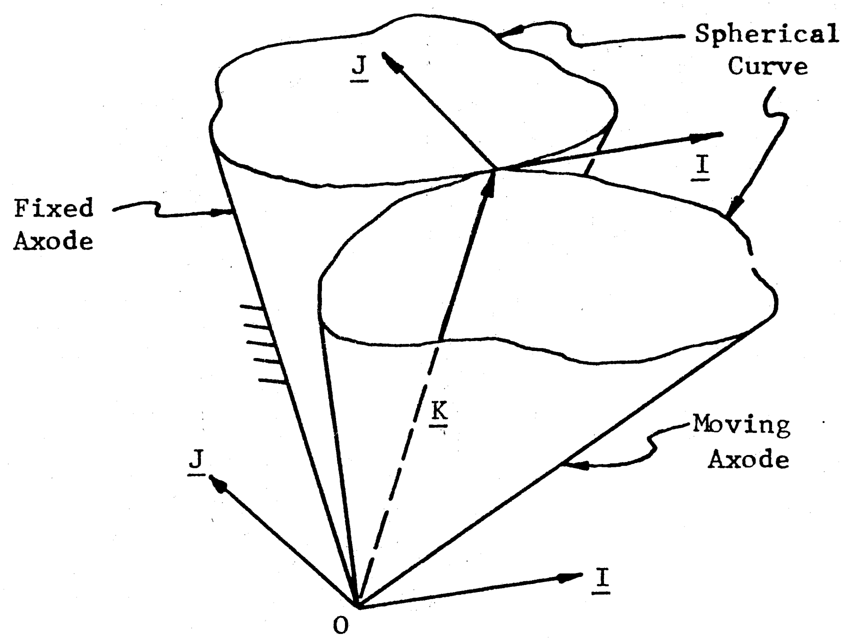


Figure 12. A Spherical Motion

The generator of the tangent-plane envelope can be obtained from Equation (4.10)

$$\underline{\hat{R}} = \underline{L}_0 \times \underline{L}_1 + \varepsilon p \underline{L}_1 \tag{6.5}$$

Substituted by Equation (4.13), the generator at zero position is

$$\underline{\hat{R}}(0) = (-\ell n,\ -mn,\ \ell^2 + m^2) + \varepsilon(-mp,\ \ell p,\ 0) \tag{6.6}$$

The characteristic numbers ν and its derivatives ν', ν'' with respect to s_s are independent of the translational motion of the general space motion. Hence, we can have the following first- and second-order characteristic equations which are identical to Equations (4.50) and (4.53).

$$w_{11}m + n(\ell^2 + m^2) - \nu(\ell^2 + m^2)^{3/2} = 0 \tag{6.7}$$

$$(\ell^2 + m^2)\ [2w_{11} - w_{22})\ell + w_{12}m] + 3w_{11}^2\ mn$$

$$-\nu'(\ell^2 + m^2)^3 = 0 \tag{6.8}$$

The second-order intrinsic property of the tangent-plane envelope is found from Equation (4.63).

$$\tau = (\underline{L}_0\underline{L}_1\underline{L}_2)^2/U \tag{6.9}$$

where

$$U = p(\underline{L}_1\underline{L}_2\underline{L}_3) \tag{6.10}$$

Substituted by Equations (4.13), the above equations become

$$\tau = [w_{11}m + n(\ell^2 + m^2)]^2/U \tag{6.11}$$

where

$$U = [(2w_{11} - w_{22})\ell + w_{12}m]\ (\ell^2 + m^2)\ p + 3w_{11}{}^2\ell mnp \tag{6.12}$$

With $k = \nu\tau$, the curvature of the edge of regression can be compued from Equations (6.7) and (6.11).

$$k = [w_{11}m + n(\ell^2 + m^2)]^3/[U(\ell^2 + m^2)^{3/2}] \qquad (6.13)$$

The third-order ν'' characteristic equation is identical to Equation (4.66). Since from Equation (4.69), we have

$$U_1 = p(\underline{L}_1\underline{L}_2\underline{L}_4) \qquad (6.14)$$

the u-characteristic Equation (4.67) can be rewritten as

$$u = [2(\underline{L}_1\underline{L}_2\underline{L}_3)(\underline{L}_0\underline{L}_1\underline{L}_3) - (\underline{L}_1\underline{L}_2\underline{L}_4)(\underline{L}_0\underline{L}_1\underline{L}_2)]/$$

$$[\underline{L}_1 \cdot \underline{L}_1)^{1/2} (\underline{L}_1\underline{L}_2\underline{L}_3)(\underline{L}_0\underline{L}_1\underline{L}_2)] \qquad (6.15)$$

From the above characteristic equations, one may find that for parallel-tangent-plane envelopes, their generators are parallel at any moment and all of them have common characteristic numbers. These envelopes may be termed as "parallel envelopes." However, the radii of torsion and the radii of curvature, τ^{-1} and k^{-1}, of the edges of regression of these parallel envelopes are proportional to the distance, $|p|$, between tangent-plane and the origin if $(\underline{L}_1\underline{L}_2\underline{L}_3) \neq 0$.

With the constraint, $\ell^2 + m^2 + n^2 = 1$, we note that there are only two independent variables in each characteristic equation. Hence, tangent-plane envelopes can only be synthesized through Equations (6.7) and (6.8) to have second-order contact within stretch rotation with a reference developable. Assume the edge of regression of the reference developable has torsion and curvature, τ_d and k_d, respectively. For the tangenplane envelope, τ and k are calculated from Equations (6.11) and (6.13) If the tangent-plane envelope and its associated system were stretched with the scale τ/τ_d, the tangent-plane envelope and the reference developable would have the same geometrical properties, ν, ν', τ_d, and k_d, to second order and hence a second-order contact (three generator contact) would be achieved.

6.2 Special Cases

In the following, the special cases of tangent-plane envelopes are investigated.

6.2.1. Stationary Planes

In Equation (4.6), we know that for a plane to remain stationary, $\underline{L}_1$ must be zero. This leads to

$$\ell = m = 0 \tag{6.16}$$

which represents the parallel planes orthogonal to the axis of rotation.

The stationary planes become double stationary planes (second-order stationary planes) if $\underline{L}_2$ is also equal to zero, that is, $w_{11} = 0$ (i. e. $\alpha_0 = 0$).

In a spherical motion all the points along the rotational axis are stationary, which are the duals to the stationary planes.

6.2.2 Stationary Generators

From Equations (5.7) through (5.10), a tangent-plane (ℓ, m, n, p), ($\ell^2 + m^2 \neq 0$), envelopes a stationary generator if

$$\begin{aligned} &\ell^2 + m^2 + n^2 = 1 \\ &w_{11}m + n(1 - n^2) = 0 \end{aligned} \tag{6.17}$$

and

$$p = 0 \tag{6.17}$$

Equations (6.17) represent in the moving system the family of tangent-planes which envelop stationary generators. Since $p = 0$, the tangent-plane envelope is a cone with vertex at the center of rotation.

if $w_{11} \neq 0$, Equation (6.17) represents an assemblage of tangent-plane, tangent to a cone with vertex at the center of rotation. The direction vector at zero-position is

$$\underline{L}_0 \times \underline{L}_1 = (-\ell n, -mn, \ell^2 + m^2) \tag{6.18}$$

If $w_{11} = 0$, we have, from Equation (6.17),

$$\left.\begin{aligned} &\ell^2 + m^2 = 1 \\ \text{and} \quad & \\ &n = p = 0 \end{aligned}\right\} \tag{6.19}$$

which represent a pencil of planes passing through the axis of rotation and the stationary generator is coincident with the axis of rotation.

The analogy between the motion of a point and of a plane has been pointed out in Section 5.6. It is interesting to see that for a spherical motion, the locus of the point tracing an inflection point is the inflection cone [22, 55] while the family of tangent-plane enveloping stationary generators is an assemblage of planes tangent to a cone. On the other hand, we note that to envelop a stationary generator, the tangent-plane always passes through the center of rotation and the edge of regression is degenerated to a point, while in point path, a point on the inflection curve may not move on the same plane except for some special motions.

6.2.3 Double Stationary Generators

With the additional condition $(\underline{L}_0\underline{L}_1\underline{L}_3) = 0$ in Equation (5.13) satisfied, a stationary generator becomes a double stationary generator. The family of tangent-planes which envelop double stationary generators is distinguished in the following.

(a) $w_{11} \neq 0$ (i. e. $\alpha_0 \neq 0$)

We have

$$B_1n^8 + B_2n^6 + B_3n^4 + B_4n^2 + B_5 = 0 \tag{6.20a}$$

$$m = -n\,(1 - n^2)/w_{11} \tag{6.20b}$$

$$w_{11}[(2w_{11} - w_{22}) - 3w_{11}n^2]\,\ell = w_{12}n\,(1 - n^2) \tag{6.20c}$$

$$p = 0 \tag{6.20d}$$

where B_i (i = 1, 5) are given in Equation (5.15).

There exists at most eight tangent-planes which envelop double stationary generators.

(b) $w_{11} = w_{22} = 0$ (i. e. $\alpha_0 = 0$)

If $w_{12} = 0$ (i. e. $\alpha_1 = 0$), the same tangent-planes described in Equation (6.19) envelop double stationary generators.

If $W_{12} \neq 0$, the tangent-plane (1, 0, 0, 0) is the only one to envelop a double statonary generator which is the axis of rotation.

6.2.4 Stationary Points

The statinary point of a tangent-plane envelope appears as a cusp point on the edge of regression. It is a point common to four consecutime planes.

For a tangent-plane envelope, a stationary point exists if on the edge of regression,

$$s_1 = 0$$

From Equations (4.56) and (6.10), this leads to

$$U = p(\underline{L}_1\underline{L}_2\underline{L}_3) = 0 \tag{6.21}$$

It is obvious that any plane passing through the center of rotation has a fixed point on it. Excluding the case p = 0. We have from Equation (6.12)

$$(\underline{L}_1\underline{L}_2\underline{L}_3) = [(2w_{11} - w_{22})\ \ell + w_{12}m](\ell^2 + m^2) + 3w_{11}^2\ell mn = 0 \tag{6.22}$$

There exists an infinite number of solutions in Equation (6.22). Each solution ℓ:m:n represents a family of parallel planes and all these parallel tangent-planes envelop stationary points. We note that at a stationary point $k^{-1} = \tau^{-1} = 0$.

A double stationary point may be obtained if the tangent-plane also satisfies $U_1 = 0$ or

$$\begin{aligned}(\underline{L}_1\underline{L}_2\underline{L}_4) = {} & (3w_{12} - w_{23})\ell^3 + [w_{11}(1 + 4w_{11}^2) + (w_{13} + 3w_{22} - 3w_{11})]\ell^2 m \\ & - 3w_{11}^2\ell^2 n + (-w_{23} + 3w_{12})\ell m^2 + w_{11}(w_{23} + w_{12})\ell mn \\ & + [w_{11}(1 + w_{11}^2) + (w_{13} + 3w_{22} - 3w_{11})]m^3 \\ & + [w_{11}(-w_{13} + w_{22} + w_{11}) - 3w_{11}^2]m^2 n = 0\end{aligned} \tag{6.23}$$

Equations (6.22) and (6.23) are cubic homogeneous equations. Any solution (ℓ:m:n) represents a family of parallel tangent-planes and all these planes envelop double stationary points at zero position.

In general, a triple stationary point cannot be obtained.

6.2.5 Helical Developables

The characteristic equations related to the characteristic number ν, ν', ν'' and so on are identical to those of general space motion. Therefore, the conditions for the existence of helical developables enveloped by tangent-planes in spherical motion are the same as the conditions described in Section 5.5 However, the characteristic number u is independent of p of the tangnet-plane coordinates. Hence, one can only find tangent-plane envelopes with two prescribed characteristic numbers among ν, ν', ν'', and u.

6.2.6. Cylindrical Developables

The conditions for a tangent-plane envelope to envelop a first-or second-order cylindrical surfaces are identical to that in Section 5.6. Among all these first- or second-order cylindrical surfaces, the one which is enveloped by the tangent-plane passing through the center of rotation has a stationary generator or a double stationary generator.

CHAPTER VII

EXAMPLES

In this chapter, two examples are presented. The first one shows that a tangent-plane on a rigid body executing the Darboux motion envelops a helical developable. This is another case showing that a tangent-plane under various special motions may provide distinct features. The other example demonstrates the synthesis procedure or a tangent plane envelope in a general space motion.

7.1 Darboux Motion

In the following, a tangent-plane embedded in a rigid body executing Darboux motion is discussed. For a rigid body executing Darboux motion, every point on the moving body traces a planar path. A trivial example is the planar motion in which all the planes of the paths are parallel. In the canonical systems, a general space Darboux motion can be expressed as Equation (3.1) with

$$[A] = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{7.1}$$

and $\underline{D} = (f(1 - \cos\phi), 0, g\sin\phi + h\,(1 - \cos\phi))$ (7.2)

where $g^2 + h^2 \neq 0$ and ϕ is the angular displacement about the instantaneous screw axis [8, 48].

Differentiating Equations (7.1) and (7.2) with respect to ϕ, we have

$$\left.\begin{array}{l}
[A_1] = \begin{bmatrix} -\sin\phi & -\cos\phi & 0 \\ \cos\phi & -\sin\phi & 0 \\ 0 & 0 & 0 \end{bmatrix} \\[2em]
[A_2] = \begin{bmatrix} -\cos\phi & \sin\phi & 0 \\ -\sin\phi & -\cos\phi & 0 \\ 0 & 0 & 0 \end{bmatrix} \\[2em]
[A_{i+2}] = -[A_i] \qquad\qquad (i = 1, 2, \ldots)
\end{array}\right\} \qquad (7.3)$$

and

$$\left.\begin{array}{l}
\underline{D}_1 = (f \sin\phi,\ 0,\ g\cos\phi + h\sin\phi) \\
\underline{D}_2 = (f\cos\phi,\ 0,\ -g\sin\phi + h\cos\phi) \\
\underline{D}_{i+2} = -\underline{D}_i \qquad\qquad (i = 1, 2, \ldots)
\end{array}\right\} \qquad (7.4)$$

We observe that the rotational matrix [A] is exactly the same as that of a planar. The instantaneous screw axis of a Darboux Motion is always in the same direction and translates in a parallel direction.

Since $\underline{L}_i=[A_i]\ \underline{\ell}$ (referring to the above equations) we may have the parametric equation of the tangent-plane envelope from Equations (4.10) and (4.12).

At zero position $\phi = 0$, the motion is characterized by the instantaneous invariants as shown in Equations (3.64) through (3.77) in which

$$w_{ij} = 0 \qquad (i, j = 1, 2 \ldots) \qquad (7.5)$$

$$\underline{D}_0 = 0 \qquad (7.6)$$

$$\left.\begin{aligned}
\underline{D}_1 &= (0,\ 0,\ d_1) && = (0,\ 0,\ g) \\
\underline{D}_2 &= (d_{12},\ 0,\ d_{32}) && = (f,\ 0,\ h) \\
\underline{D}_3 &= (d_{13},\ d_{23},\ d_{33}) && = (0,\ 0,\ -g) \\
\underline{D}_4 &= (d_{14},\ d_{24},\ d_{34}) && = (-f,\ 0,\ -h)
\end{aligned}\right\} \qquad (7.6)$$

and so on.

For a tangent-plane (ℓ, m, n, p) in the moving system, if $n(\ell^2 + m^2) \neq 0$, the point on the edge of regression and the generator of the tangent-plane envelope at zero position are, respectively,

$$X(0) = -(E/H,\ F/H,\ G/H) \qquad (7.7)$$

and

$$\underline{\hat{R}}\ (0) = (-\ell n,\ -mn,\ \ell^2 + m^2) + \varepsilon(g\ell n - mp,\ gmn + \ell p,\ gn^2) \qquad (7.8)$$

where

$$E = gmn^2 + f\ell^2 n + h\ell n^2$$

$$F = f\ell mn - g\ell n^2 + hmn^2$$

$$G = (p - f\,\ell - hn)(\ell^2 + m^2)$$

$$H = n(\ell^2 + m^2)$$

The above equations are obtained by substituting Equations (7.5) and (7.6) into (4.15) and (4.16).

By substituting Equations (7.3) and (7.4) into (4.7) and (4.8), Equation (4.50) becomes

$$\nu = n(\ell^2 + m^2)^{1/2} \qquad (7.9)$$

Equation (7.9) is the characteristic equation at any position and hence,

$$\nu' = \nu'' = \ldots = 0 \tag{7.10}$$

If $n(\ell^2 + m^2) \neq 0$, Equations (7.11) shows that the edge of regression of any tangent-plane envelope is a helix and all the tangent-plane envelopes are helical developables (Figure 13).

The torsion of the edge of regression of a tangent-plane envelope at zero position may be obtained by substituting Equations (7.5) and (7.6) into Equation (4.63). Thus, we have

$$\tau = -n(\ell^2 + m^2)/3fm \tag{7.11}$$

and the curvature k,

$$k = \nu\tau = -n^2(\ell^2 + m^2)^{1/2}/3fm \tag{7.12}$$

Similarly, the third order characteristic numbers may be obtained from Equation (4.67)

$$u = -2\ell/[m(\ell^2 + m^2)^{1/2}] \tag{7.13}$$

We note that Equations (7.9) to (7.13) contain only two independent variables. In fact, all parallel tangent-planes in the rigid body executing Darboux motion envelop identical parallel helical developables. Therefore, in a system executing Darboux motion. Equations (7.9) and (7.11) are the only two characteristic equations.

For the Darboux motion, each point traces a planar path but any tangent-plane other than $n(\ell^2 + m^2) = 0$ envelops a helical developable and does not always pass through a fixed point. Hence, it is dubious to relate the motion, of a point and a plane through the duality. However, in such a motion each plane in the fixed system does always pass through the same point in the moving system.

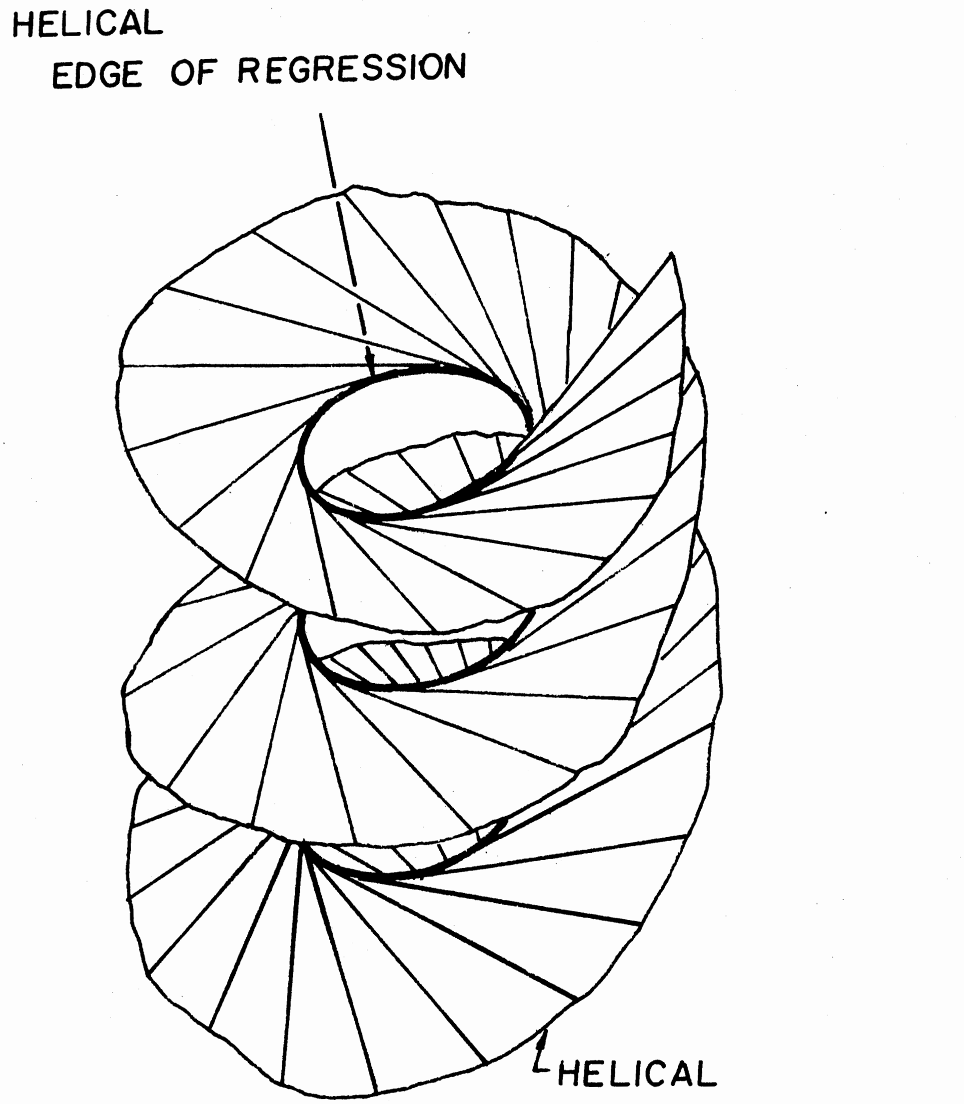


Figure 13. Part of a Helical Developable Surface

7.2 Synthesis of Developables

In a general space motion, Equations (4.50) and (4.53) are the two synthesis equations for a second order developables. Eliminating ℓ and m from them, with the constraint

$$\ell^2 + m^2 + n^2 = 1 \tag{7.14}$$

we can obtain the following sixteenth degree polynomial in terms of the n-coordinate of a tangent-plane.

$$\sum_{i=1}^{16} E_i n^i = 0 \tag{7.15}$$

where

$$E_0 = C_0^2 - D_0^2$$

$$E_1 = 2(C_1C_0 - D_1D_0)$$

$$E_2 = C_1^2 + D_0^2 - D_1^2 + 2(C_2C_0 - D_2D_0)$$

$$E_3 = 2(C_3C_0 + C_2C_1 + D_1D_0 - D_3D_0 - D_2D_1)$$

$$E_4 = C_2^2 + D_1^2 + 2(C_4C_0 + C_3C_1 + D_2D_0 - D_4D_0 - D_3D_1) - D_2^2$$

$$E_5 = 2(C_5C_0 + C_4C_1 + C_3C_2 + D_3D_0 - D_2D_1 - D_5D_0 - D_4D_1 - D_3D_2)$$

$$E_6 = C_3^2 + D_2^2 - D_3^2 + 2(C_6C_0 + C_5C_1 + C_4C_2 + D_4D_0 + D_3D_1 - D_5D_1 - D_4D_2)$$

$$E_7 = 2(C_6C_1 + C_5C_2 + C_4C_3 + D_5D_0 + D_4D_1 + D_3D_2 - D_7D_0 - D_5D_2 - D_4D_3)$$

$$E_8 = C_4^2 + D_3^2 - D_4^2 + 2(C_8C_0 + C_6C_2 + C_5C_3 + D_5D_1 + D_4D_2 - D_7D_1 - D_5D_3)$$

$$E_9 = 2(C_8C_1 + C_6C_3 + C_5C_4 + D_7D_0 + D_5D_2 + D_4D_3 - D_7D_2 - D_5D_4)$$

$$E_{10} = C_5^2 + D_4^2 - D_5^2 + 2(C_8C_2 + C_6C_4 - D_7D_1 - D_5D_3 - D_7D_3)$$

$$E_{11} = 2(C_8C_3 + C_6C_5 + D_7D_2 + D_5D_4 - D_7D_4)$$

$$E_{12} = C_6^2 + D_5^2 - D_6^2 + 2(C_8C_4 + D_7D_3 - D_7D_5)$$

$$E_{13} = 2(C_8C_5 + D_7D_4)$$

$$E_{14} = -D_7^2 + 2(C_8C_6 + D_7D_5)$$

$$E_{15} = 0$$

$$E_{16} = C_8^2 + D_7^2$$

$$C_8 = -9\, w_{11}^2(\nu^4 - 6\nu^2 + 1)$$

$$C_6 = w_{11}^2\, (27\nu^4 - \nu^2 - 108\nu^2 + 9) + 6Bw_{11}(1 - 3\nu^2)$$

$$C_5 = 2\nu' w_{11}w_{12}$$

$$C_4 = 3w_{11}^2(-9\nu^4 + 18\nu^2 + \nu'^2) + 6Bw_{11}(6\nu^2 - 1) + (\nu^2 - 1)(9w_{11}^4 + w_{12}^2 + B^2)$$

$$C_3 = -2C_5$$

$$C_2 = 3w_{11}^2(3\nu^4 - \nu'^2 - 3\nu^2 w_{11}^2) + 6Bw_{11}(w_{11}^2 - 3\nu^2) + (1 - 2\nu^2)(w_{12}^2 + B^2)$$

$$C_1 = C_5$$

$$C_0 = w_{11}^2(\nu'^2 - B^2) + \nu^2(w_{12}^2 + B^2)$$

$$D_7 = 36\nu w_{11}^2(\nu^2 - 1)$$

$$D_5 = 36\nu w_{11}^2\ (1 - 2\nu^2) + 6\nu B w_{11}(3 - \nu^2)$$

$$D_4 = \nu C_5$$

$$D_3 = -2\nu(B^2 + w_{12}^2) + 12\nu^3 w_{11}(3w_{11} + B) - 18\nu w_{11}(B + w_{11}^3)$$

$$D_2 = -2D_4$$

$$D_1 = 2\nu(B^2 + w_{12}^2) + 6\nu B w_{11}(w_{11}^2 - \nu^2)$$

$$D_0 = D_4$$

and

$$B = 2w_{11} - w_{22}$$

The coefficients of the above equation are functions of the characteristic numbers ν, ν', and the rotational instantaneous invariants up to third order. Knowing n-coordinate of a tangent-plane from Equation (7.15), we may obtain ℓ and m from Equations (4.50) and (4.53) which may be expressed explicity by

$$m = [\nu(1 - n^2)^{3/2} - n(1 - n^2)]/w_{11} \tag{7.16}$$

and

$$\ell = [\nu'(1 - n^2)^3 - w_{12}m(1 - n^2)/[(2w_{11} - w_{22})(1 - n^2)+ 3w_{11}^2 mn] \tag{7.17}$$

The p-coordinate of the tangent-plane may be selected arbitrarily or determined through Equation (4.70) if the characteristic number u is prescribed. For the obtained tangent-plane (ℓ, m, n, p), the property τ of its envelop can be calculated from Equation (4.63).

Let the desired torsion be τ_d. The tangent-plane and the associated mechanism can be stretched (or scaled) proportionally by the scale $e = \tau/\tau_d$. In the stretched mechanism the tangent-plane will envelop a developable with the prescribed properties ν, ν', (and u if any). A simple example is given below.

Let an RCCC mechanism be used to synthesize a second order helical developable with the properties $\nu = 0.9$, $\nu' = 0$ and $\tau = 0.2$. The RCCC mechanism has correspondingly the input, coupler, output and fixed links:

$$\hat{\alpha} = 30° + \varepsilon 2 \qquad \hat{\beta} = 55° + \varepsilon 4$$

$$\hat{\Upsilon} = 45° + \varepsilon 3 \qquad \hat{\delta} = 60° + \varepsilon 5$$

The zero position is assumed at the input link position $\hat{\phi} = 60°$. From Reference [16], we have the instantaneous invariants in the canonical systems.

$$w_{11} = 1.83$$

$$w_{12} = 0.38 \qquad 2w_{11} - w_{22} = 2.26 \qquad w_{11} + w_{22} = 3.23$$

$$d_1 = 4.58$$

$$d_{12} = 9.96 \qquad d_{32} = -5.22$$

$$d_{13} = -14.21 \qquad d_{23} = -9.93 \qquad d_{33} = -14.45$$

Using Equations (7.15) to (7.17), we obtain four solution sets $\underline{\ell}$ =

(ℓ, m, n):

$$\underline{\ell}_1 = (-0.7560, 0.5758, -0.3083)$$

$$\underline{\ell}_2 = (\ 0.7293, 0.5662, -0.3832)$$

$$\underline{\ell}_3 = (-0.3858, -0.0548, 0.8882)$$

$$\underline{\ell}_4 = (\ 0.3941, -0.0556, 0.8813)$$

Let us take the solution set $\underline{\ell}_1$ and assume $p = 2.0$. $(\ell, m, n, p) = (-0.7560, 0.1578, -0.3083, 2.0)$ is the homogeneous coordinates of a tangent-plane in the canonical system. We note that p can be calculated from Equation (4.76) if the property u is prescribed. The property τ of the envelope is found from Equation (4.63) and we have

$$\tau = 0.05$$

and

$$e = \tau/\tau_d = 0.2497$$

After being stretched with the scale "e", the RCCC mechanism has the links:

$$\hat{\alpha} = 30° + \varepsilon 0.4994 \qquad \hat{\beta} = 55° + \varepsilon 0.9988$$

$$\hat{\Upsilon} = 45° + \varepsilon 0.7491 \qquad \hat{\delta} = 60° + \varepsilon 1.2485$$

and in the associated moving canonical system on the coupler link, the tangent-plane is located by the homogeneous coordinates,

$$(\ell, m, n, p) = (-0.7560, 0.5758, -0.3083, 0.4993)$$

whose envelope has the prescribed properties $\nu = 0.9$, $\nu' = 0$ and $\tau = 0.2$.

CHAPTER VIII

SUMMARY AND CONCLUSION

This dissertation represents the first extensive study on the motion of a plane. It extends the curvature theory from the traditional point path to tangent-plane envelope. Although the motion of a plane is complicated, this dissertation provides a handy tool to analyze and synthesize a tangent-plane motion.

A rigid body motion is considered as a screw motion and the canonical systems related to the motion of the screw axis are established to simplify the description of motion. A series of numbers, such as ν, ν', ν'', u and so on were defined to characterize a tangent-plane envelope. The family of tangent-planes whose envelopes have the common characteristic numbers are located through the characteristic equations. These characteristic equations can be used for synthesis purpose.

It was found that parallel-tangent-plane envelopes have not only parallel generators but also common characteristic number ν and its derivatives. Parallel-tangent-plane envelopes have second order contact within stretch rotation with each other. For a general motion, one may find tangent-plane envelops which have at most three common characteristic numbers, say ν, ν' an u, with a reference developable. For a spherical motion, all the dimensionless characteristic numbers depend only on the orientation of the tangent-plane and the best tangent-plane envelope one may find is a second order developable.

Several special cases were investigated. No stationary plane can be found unless the motion is spherical or have zero instantaneous pitch. For a space motion, one may find eight families of parallel-tangent-planes which envelop second order cylindrical surfaces and generally only first order stationary generator can be obtained. In a spherical motion, among all the second order cylindrical surfaces, the one with tangent-plane passing through the central point has a double stationary generator. For the stationary point which is a cusp point on the edge of regresssion, one may find tangent-planes associated with edges of regression having triple cusps. However, in spherical motion excluding the fixed point, we can find only double stationary point or double cusp. The generation of helical developables is also investigated. In Darboux motion, any plane not parallel or perpendicular to the direction of the angular velocity envelopes a helical developable.

The analogy between the motion of a point and that of a tangent-plane has been stressed. Such analogy may stimulate one fo find, in the tangent-plane envelopes, the properties or applications analogous to those of point trajectories. However, it is improper to use duality to relate point trajector and tangent-plane envelope. For example, in Darboux motion, any point traces a planar path but any tangent-plane, which is dual to a point, not parallel or perpendicular to the ISA envelops a helical developable instead of passing through a fixed point. A contrary situation to such duality is also exhibited in the Ball's point and the tangent-plane which envelops a stationary generator.

The theory developed here can be applied to any one-parameter space motion. In planar motion, a similar approach can be utilized to develop a tangent-line envelope curvature theory. A general case in in tangent-plane envelope curvature is a two-parameter motion. For a two-parameter motion, an ordinary suface can be generated through a tangent-plane envelope and one may synthesize any surface with high accuracy in the vicinity of a point. Such a study, however, is considered beyond the scope of the present dissertation.

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2

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