INVENTORY SYSTEMS OF AGE-INDEPENDENT PERISHABLE ITEMS SUBJECT TO ON-GOING DETERIORATION

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CHAPTER I

INTRODUCTION

An important area in the field of Management Science is Inventory Theory. During the last seventy years many journal articles and textbooks have been written on this subject matter. Indeed, the field of inventory theory has been a fertile area of research by mathematicians, engineers, economists, and computer scientists. Since there are many different definitions for inventory theory in the literature, it is necessary to define the inventory systems that will be dealt with in this dissertation. Paraphrasing Naddor [61]

> An inventory system is a system in which only the following costs are significant, and in which any two or more are subject to the control of the decision maker:
> 1. The cost of carrying or holding inventories.
> 2. The cost of incurring shortages.
> 3. The cost of replenshing the inventory system.
> 4. The cost of deterioration/perishing of inventories.

There are various types of costs or expenses that are classified into the above four major cost categories. The first category includes the cost of carrying or holding inventories in any inventory or production system. It includes:

1. The cost of capital tied up in inventories.
2. The marginal cost of storage space.
3. The marginal cost of insurance and taxes.
4. The marginal cost of handling equipment in the warehouse.

The second category is the cost attributed to shortages when there is customer demand. This may include:

1. The cost in the loss of sales.
2. The cost in the loss of good will.
3. The cost of special administrative efforts.

The third category is the cost incurred in the replenishment of inventories. This may include one or more of the following:

1. The cost of ordering a new lot size.
2. The cost of machine set-ups for new production runs.
3. The cost of handling and shipments.
4. The cost of receiving and inspection.

The last category is the cost associated with perishability or deterioration of items while in stock. Although many authors have included this cost as a part of carrying or holding cost, this is conceptually and realistically incorrect. Hadley and Whitin [43] have stated that, wherever applicable, this cost must be considered explicitly. Deterioration or perishability is defined as decay, damage, spoilage, or obsolescence that prevents the item from being used for its original intended purpose. This may include:

1. The cost of actual physical depletion of volatile liquids such as gasoline and alcohol.
2. The cost of spoilage.
3. The cost of obsolescence.
4. The cost of damage and pilferage.

The sum of these four cost categories will be referred to as the "total cost".

In any inventory problem, decisions are usually made in terms of time and quantity. These are the basic controllable variables in any inventory system; that is, the decision maker must specify one or both of the following:

1. When should the order be placed?
2. How much should be added to inventory?

The objective of an inventory problem is to determine the value of the controllable variable(s) that will minimize the total inventory cost.

Most research in the mathematical inventory models to date have made the implicit assumption that inventory was "non-perishable"; that is, the units, once in stock, could be used at anytime to satisfy demand. Almost all items deteriorate over time; if the rate of deterioration is very slow, its effect can be ignored; otherwise the units in stock might have deteriorated to the point that they may no longer be able to satisfy demand. The loss due to perishability is quite important, and there are various contexts in which it could provide valuable insights into inventory decision making. This effect is so vital in many inventory systems that it cannot be lightly disregarded. For example, in the field of perishable foods, especially fruits and dairy products, one must always consider the effect of spoilage, because not only do these types of goods become spoiled, but most likely they lose their value as time passes on. Another example is the case of physical depletion of volatile liquids such as alcohol and turpentine in the chemical industry. The effect of deterioration plays a significant role in other areas such as production and inventory of photographic films, radioactive substances, nuclear material processing, pharmaceutical drugs, and electronic components.

1.1 Background

In order to have a clear picture of the effect of perishability on inventory, consider the following example which is basically due to Nahmias [62].

Consider a simple EOQ (Economic Order Quantity) Model. This is a continuous time model where demand is assumed to be constant, and the following costs to be significant:

i. C_0 = unit cost (charge for each unit purchased)

ii. C_1 = carrying cost (cost of holding a unit of inventory for a unit of time)

iii. C_3 = replenishment cost (fixed cost for placing an order)

iv. r = constant demand rate

Let q_0 be the optimal lot size which is received into inventory when the stock level is zero. The following relation then holds:

$$q_0 = \quad 2C_3r/C_1; \; t_0 = q_0/r.$$

Schematically this is depicted in Figure 1.

Now, assume that all unused products will perish at some time t' after receipt. If $t' > t_0$, then q_0 is the optimal lot size and the problem remains the same. But, if $t' < t_0$, and q_0 is ordered, a number of items in stock will no longer be in their useful state. Schematically this may be presented in Figure 2.

Therefore, q_p is the amount of product that perishes every t' units of time. However if q' is chosen such that $q' = q_0 - q_p$ (this is equivalent to $q' = rt'$), Figure 2 can be modified into Figure 3.

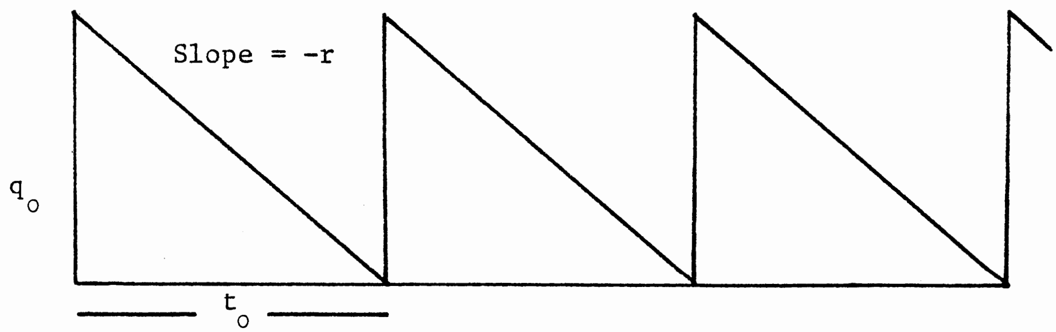


Figure 1. Inventory Level with No Deterioration

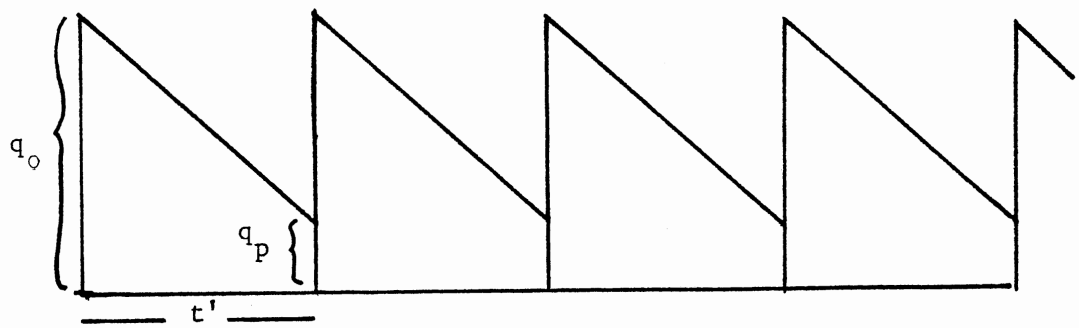


Figure 2. Inventory Level with Deterioration

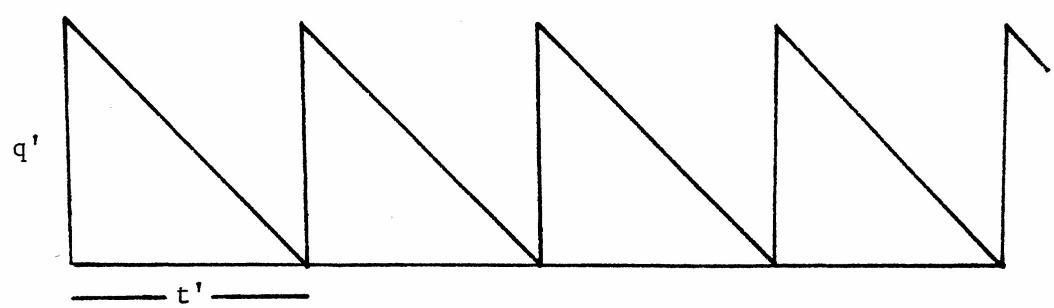


Figure 3. Adjusted Inventory Level

In this case no inventory item deteriorates; therefore, the total inventory cost will be lower than using q_0 policy; q' is the optimal policy, and it is always optimal to order in such a way that no inventory item deteriorates.

In recent years, efforts in analyzing mathematical models in which items deteriorate while in storage have drawn attention of various researchers. However, there are many areas that require additional exploration, elaboration, and extensions. The purpose of this study is to undertake such a task.

1.2 Research Objectives

The objective of this dissertation is to derive and present mathematical inventory models that include the assumption of deterioration for various classes of inventories which will be useful in the broad range of real life inventory situations. To fulfill this objective a number of models have been developed to gain additional insight and to incorporate realism into the existing body of inventory systems. Specifically, the following inventory models are investigated:

1. Lot size inventory systems incorporating various types of perishability rate function.
2. Order level inventory systems incorporating constant perishability rate and various types of demand rate functions.
3. Probabilistic inventory systems incorporating constant perishability rate.

1.3 Summary of Results

The objectives of this research have been met. Numerous models are developed which are useful in determination of the optimal replenishment size, or inventory cycle time. The Models are:

1. The inventory characteristics for Lot-size inventory Models (Model I) incorporating various perishability rate functions have been determined. The inventory fluctuatons of this model are illustrated by Figure 4. In this model the inventory holding cost is charged on all the units that remain in inventory, whether perished or not; and perishability costs are charged at the end of the inventory cycle time. The perishability rate functions that are utilized for this analysis are: constant, linear, and exponential funtions. Depending on how a perishability rate function is applied to the basic lot size system, various submodels are developed and analyzed. Figure 5 depicts these subsets of Model I lot size system.

2. The inventory characteristics for Lot size Models (Model II) incorporating various underlying perishability distribution functions have been determined. The inventory fluctuation of this model is illustrated by Figure 6. In this model the inventory carrying cost is charged on all the non-perished units in inventory, and the perishability costs are charged whenever a unit perishes. The underlying perishability distribution of items in inventory that are utilized for this analysis are: exponential, Weibull, and Rayleigh distribution functions. These correspond to constant, general, and linear rates of perishability respectively. Figure 7 depicts the models of lot size system-Model II. (Model II and Model I differ fundamentally in the method of analysis of calculating the average

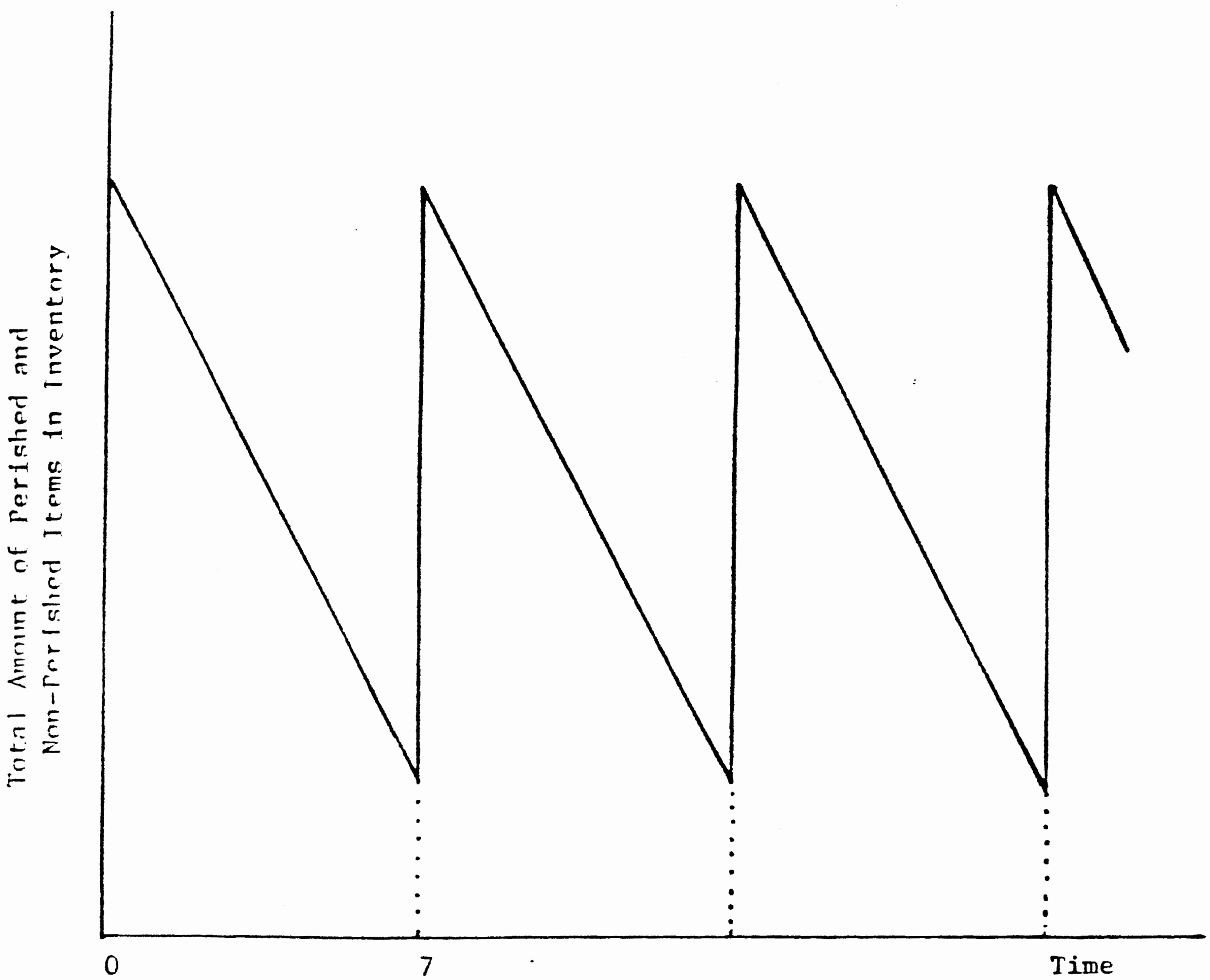


Figure 4. Lot Size System--Model I--Inventory Depletion Pattern

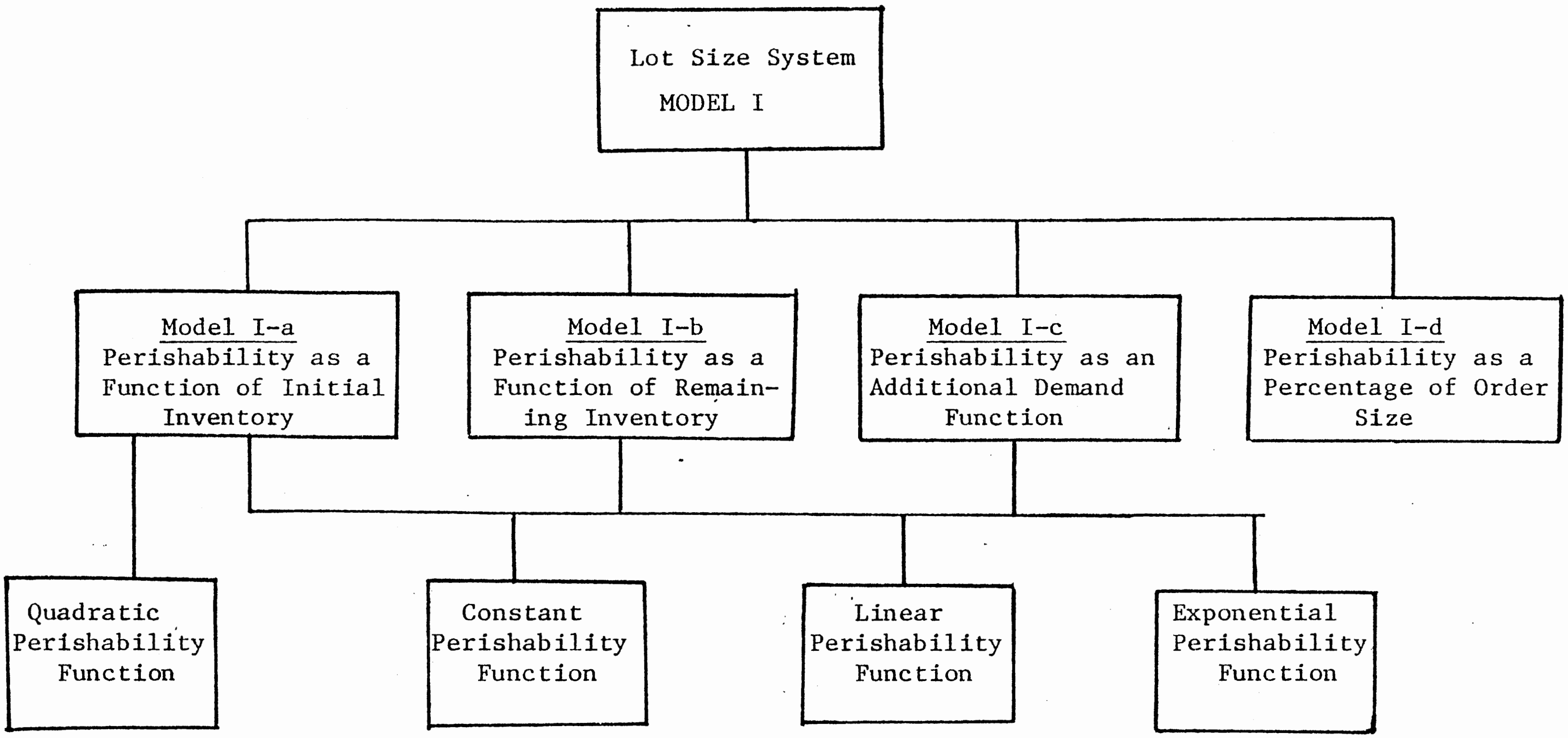


Figure 5. Lot Size System--Model I

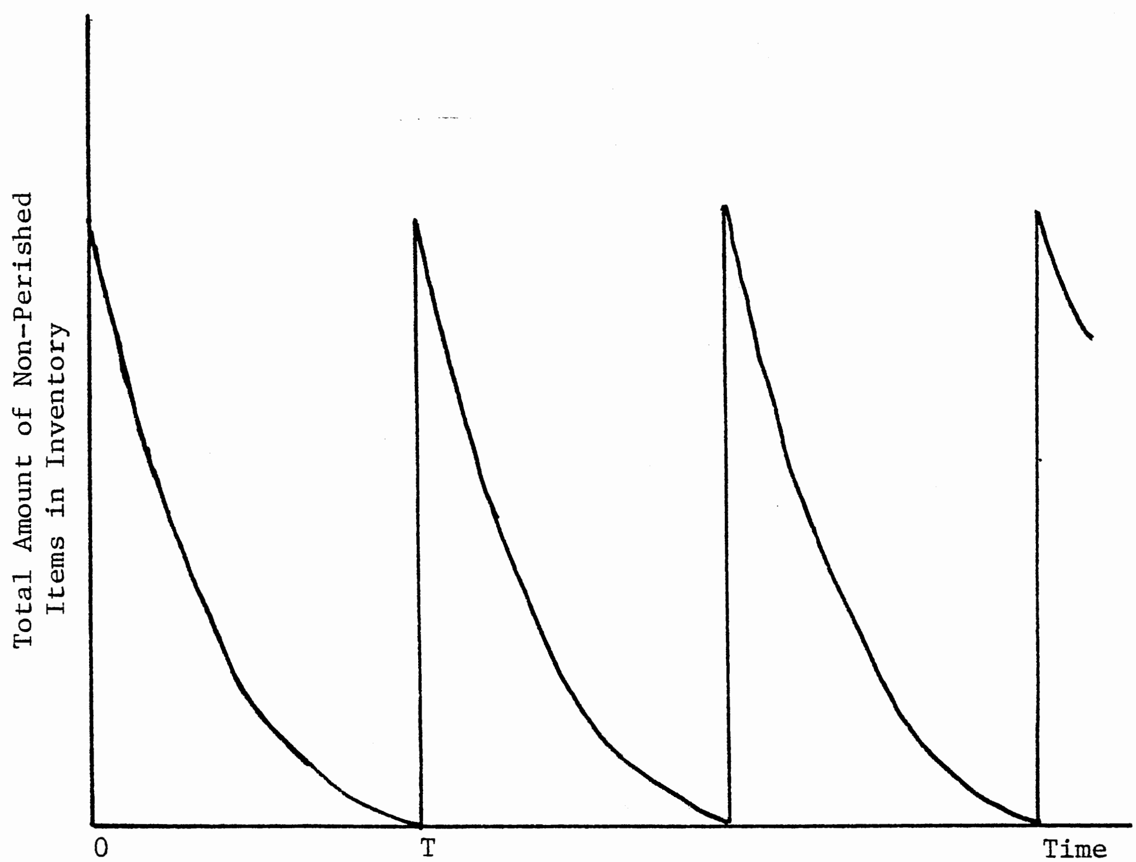


Figure 6. Lot Size System--Model II--Inventory Depletion Pattern

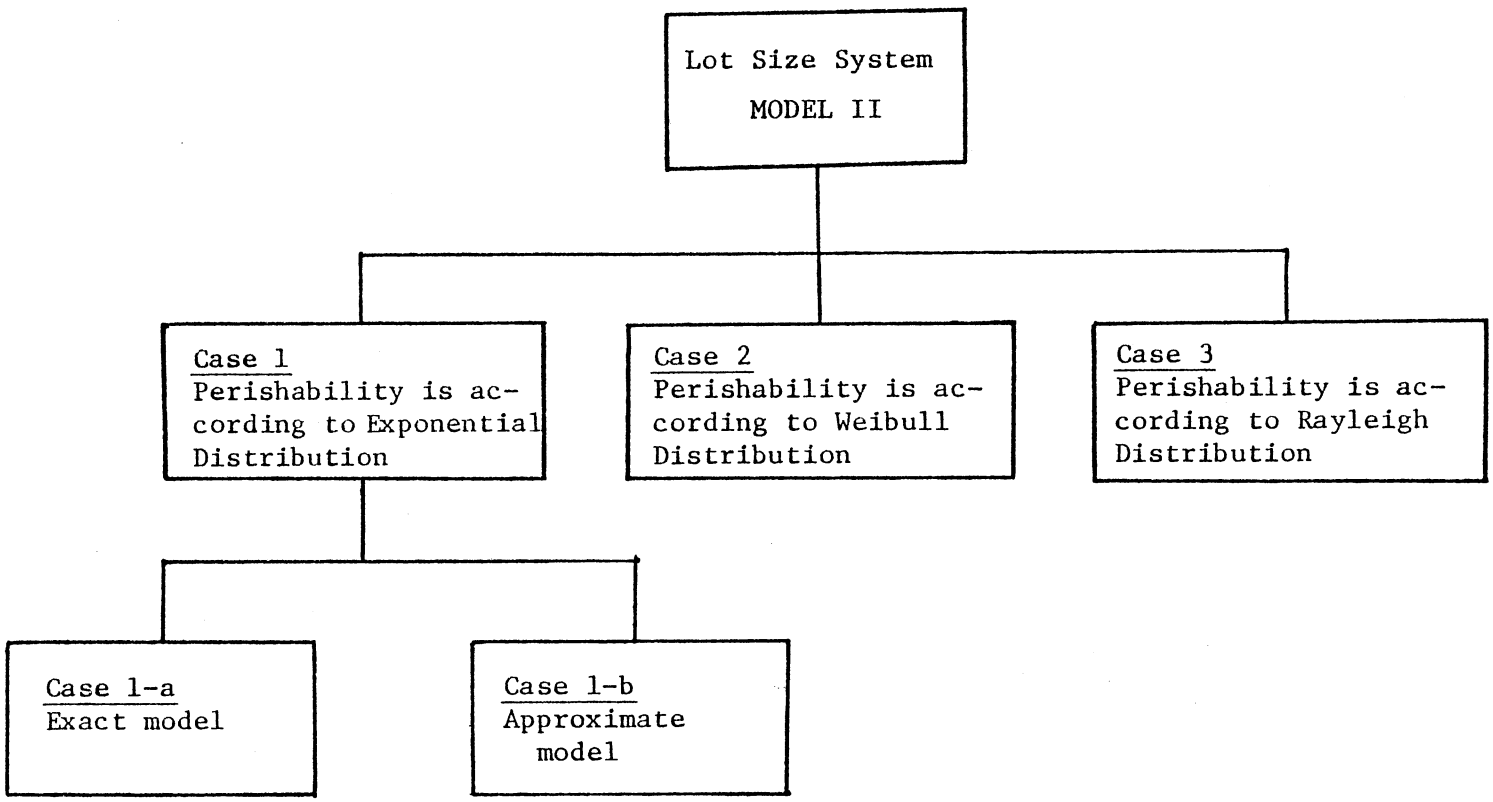
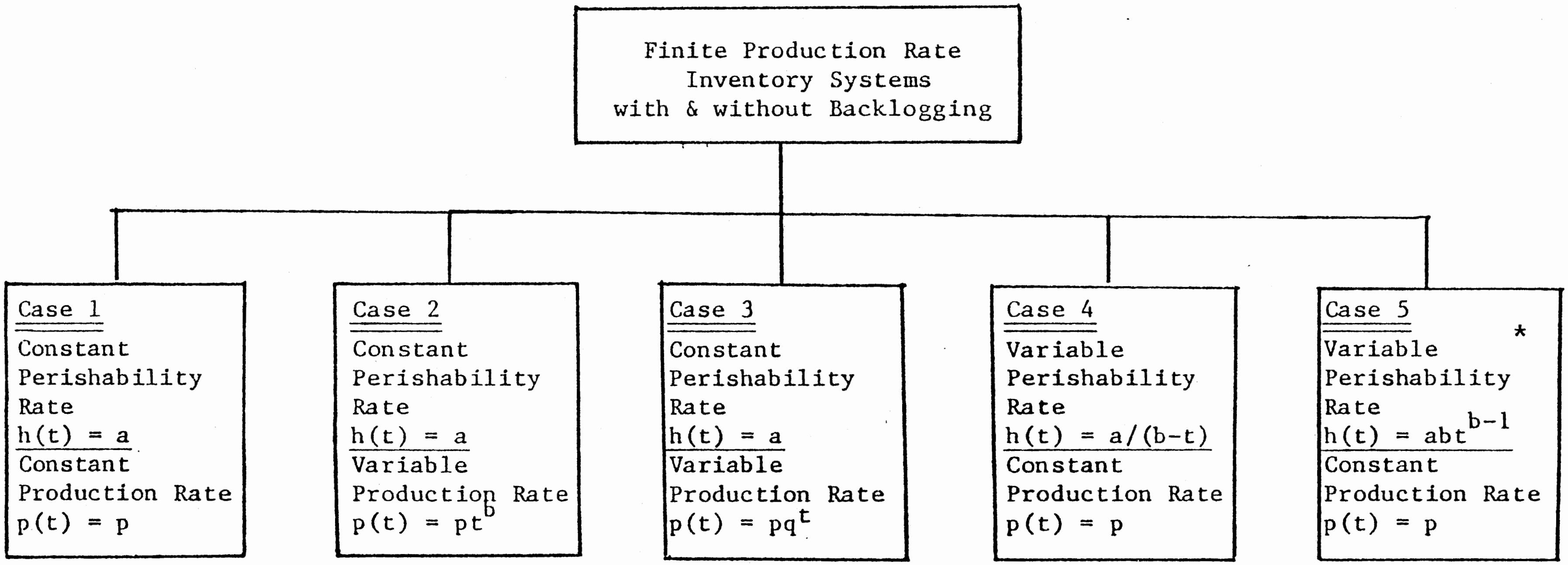


Figure 7. Lot Size System--Model II

inventory. Due to their assumptions, the analysis of Model I is based on techniques of differential calculus, and Model II is based on solving a set of differential equations for determining the total carrying inventory. In both Model I and Model II, carrying costs are balanced against replenishment and perishing costs.)

3. The inventory characteristic for finite production rate inventory models with and without backlogging have been determined by incorporating various perishability rates and number of production rate functions. The demand is assumed to occur at a constant rate. Figure 8 depicts the various Models of this inventory system.
4. The inventory characteristics for the order level inventory systems incorporating constant perishability rate (i.e. perishability according to exponential distribution) and various demand patterns have been determined. The case of lost sales model has also been considered. In addition, a model of discrete-in-time order level inventory model for non-constant demand is discussed.
5. The inventory characteristics of finite-horizon, increasing demand models with constant perishability rate are determined. In these models, the total demand over a given time horizon is fixed, however, the demand is low in the beginning and increases as time passes on.
6. Considerations of quantity discounts when constant rate of perishability is present have been evaluated.
7. Inventory characteristics of single period and multiperiod



h(t) = Perishability Rate Function
p(t) = Production Rate Function
* The backlogging case not considered

Figure 8. Finite Production Rate Inventory System

inventory systems with power demand pattern and constant perishability are determined.

8. Inventory characteristics of two probablistic inventory systems; scheduling period system, and order level system with instantaneous demand, with constant rate of perishability have been determined.

1.4 Contributions

This research has made a number of contributions. In addition to the compilation of comprehensive bibliographic material on age independent perishable items and related topics, and presentations of various types of perishability concepts and assumptions, they include:

1. The general solution methodology emphasized in this paper is deriviation and determination of exact total cost equations of various inventory systems that are subject to perishability. By knowing this exact cost function, then it is possible to utilize a number of computer search techniques, specifically Fibonacci or Hooke and Jeeves' search methods, or a numerical technique to obtain the optimal solution to the various parameters of the inventory system. This approach, while in someways similar to that of other researchers in this area, is different in the following way: Typically to obtain "optimal" results in perishable inventory models approximations have to be made in the determination of carrying inventory. Also, an additional number of other numerical approximations are necessary in the course of development of the "optimal" cost equation. Then, by using analytical and numerical techniques the "optimal"

results are obtained. No search technique has been utilized. However, because of this research it is now possible to obtain the optimal result more directly and more accurately. So, instead of utilizing optimizing techniques on the "approximate" cost equation to obtain the optimal solution, one can obtain the "optimal" solution within a predetermined interval of uncertainty. When using the Fibonacci search technique this interval, for all practical purposes, could be assumed negligible for most typical problems.

2. It has been shown that in the scheduling order level inventory systems the time when the inventory level reaches zero is not a function of demand pattern, but is a function of carrying cost, ordering cost, perishability cost, and perishability rate.

3. It has been shown in a fixed horizon and constant perishability rate inventory model where demand is increasing, the total number of items perishing is not a function of replenishment and can be calculated directly.

1.5 Organization of Chapters

Chapter II is devoted to the discussion of the relevent literature in the area of perishable inventories. It contains various concepts and classifications of perishability. Chapter III explains the terminology, and the general notation that is used throughout this research. The work is organized in such a way that those who are not interested in the details of various models may read only the third chapter, and then go directly to the desired model for its assumptions and derivations.

Chapter IV describes the inventory lot size models with functional deterioration (Model I); Chapter V presents the inventory lot size models with deterioration as a function of inventory level (Model II). Chapters VI and VII discuss the finite production rate inventory system without and with backlogging respectively. Chapter VIII is devoted to various order level inventory models. Chapter IX presents the finite horizon inventory model and considers two different demand patterns. Chapter X discusses cases of quantity price discounts. In Chapter XI power demand pattern inventory models for single and multiple periods are presented. Chapters XII and XIII are devoted to probabilistic inventory models, and alternative probability distributions have been considered. Chapter XIV contains the summary and conclusions as well as implications for future research in the area of perishable inventory items.

CHAPTER II

LITERATURE REVIEW

This chapter reviews the development of inventory theory and describes the inventory models relevant to the study of the deteriorating items.

Mathematical modeling of inventory systems dates back to 1915 when Harris [43] first published the classical lot size formula

$$q_0 = \sqrt{2rC_3 \ C_1}$$

where q_0 is the optimal lot size, r is the rate of demand per time unit, C_1 is the carrying cost of one unit in inventory per unit time, and C_3 is the reorder cost. This is also known as Wilson's formula. In 1926 Cooper [61] analyzed an inventory system with finite rate of production, and in 1928 Fry [4] studied some probabilistic inventory systems. However, Raymond [82] was the first to attempt to deal with a large variety of inventory systems and to present the beginnings of the theory of inventory systems. He summarized the work in this area prior to 1931.

Interest in the study of inventory systems has increased tremendously since World War II. Arrow et al. [4] published their classical paper "Optimal Inventory Policy" in 1951, and Dvoretsky et al. [29, 30], "The Inventory Problem" in 1952. These works mark the beginning of modern analysis of inventory systems. An excellent review and summary of the models and systems which were studied from 1923-1951 is presented by Whitin [100]. (In a later edition he updated his bibliography to the year 1956.) Veinott [99] published a detailed summary of inventory research up to 1965, and Fortuin [35] published the latest summary of inventory studies up to 1976. As far as the author is aware, there has not been any major published summary of inventory research since 1976. However, there is a great deal of information and updated bibliographic references in inventory theory which can be readily obtained through various on-line computer terminals.

It should be mentioned that one of the basic, implicit assumptions of most of these inventory models has been the infinite shelf-life of products while in stock; that is, a product remains unchanged for the purpose of satisfying demand as long as it is in the "warehouse." As has been previously mentioned, this assumption is not valid for a number of very important situations; therefore, it is the purpose of this research to explicitly analyze the effects of perishability in the various inventory models.

2.1 Perishability Classification

The analysis of inventories that are subject to perishability involves different concepts of deterioration. Cohen [18] made the following distinction. First, there are those problems in which all items in the inventory become simultaneously obsolete at some fiscal point in time; that is, all the units remaining in inventory at the end of the planning horizon become useless. This is the case of the style goods inventory, like fashion merchandising. Second are those problems in which the items deteriorate throughout the planning horizon. This latter category is broken into two classes: (1) items whose rate of deterioration is age dependent, e.g., inventory items with fixed life such as blood, and (2) items whose rate of decay is independent of their age, e.g., volatile liquids such as gasoline, radioactive chemicals, etc.

Deteriorating items could also be classified as to their utility as a function of time. Constant utility perishable goods undergo age dependent decay and face no decrease in value during their useable lifetime, e.g., prescription drugs. Decreasing utility goods undergo age dependent decay and lose value throughout their lifetime, e.g., fruits such as berries. Increasing utility goods undergo age dependent decay and increase in value, e.g., some wines appreciate in value. Naddor [61] briefly mentioned these types of perishability and gave some general cost equations.

The change in utility for an age independent inventory item is usually a function of total inventory on hand. These items are usually grouped together in the inventory for the purpose of determining the amount of decay during the planning horizon. The above classification is depicted in Figure 9.

2.2 Age-Dependent Perishable Inventories

Significant research has been done to describe the optimal stocking policies for items with a fixed life time. In these cases when demand is deterministic, the problem has a trivial solution; that is, one places an order so that no item perishes. When demand is random, the solution becomes very complex.

Most researchers in this area have considered simultaneous obsolescence: i.e., all units remaining in inventory at the end of planning horizon become useless, e.g., style-goods merchandise. The time horizon

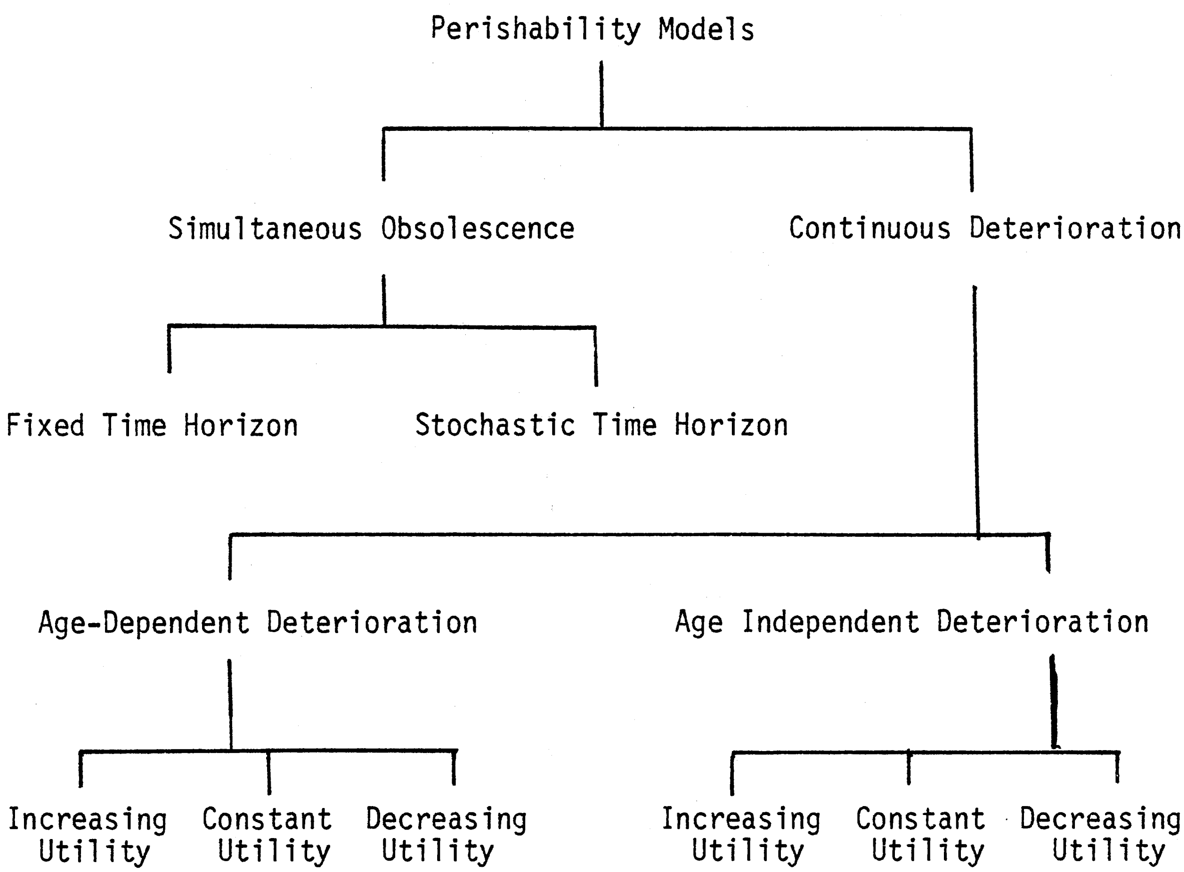


Figure 9. Classification of Perishability Model

may be fixed, (Whitin [100], Hadley and Whitin [41], Murray and Silver [60], Ravindran [83]) or stochastic (Hadley and Whitin [42], Barankin and Denny [7], Brown, Lu and Wolfson [14], Pierskalla [80]).

Bulinskaya [15] in a part of his paper used a dynamic programming approach to obtain optimum policies for inventories that have a high rate of obsolescence which can be used for exactly one period. His approach is a generalization of the "Newsboy Problem," which is a single period inventory model in which the item has a lifetime of one period. Van Zyl [98] analyzed a general model where an item has a lifetime of two periods. His model does not include a perishability cost; only ordering and penalty costs due to lost sales are included. Fries [36, 37], Nahmias [62, 65], and Nahmias and Pierskalla [71, 72] extended the results of Van Zyl. That is, the model for an item with a lifetime of two periods has been extended for a product with arbitrary but fixed lifetime under the assumption of FIFO issuing policy and fresh supply. Their cost structure is also more general, and they included perishability as well as holding cost, lost sales, backlogging, and salvage cost. These studies basically rely on the analysis of an appropriate dynamic program functional equation. Cohen [18] has extended the above works, using similar approaches, and has applied the results to the area of blood inventory management. Many authors using the above concepts and incorporating various issuing policies have published a number of papers dealing with blood inventory management and its specific requirements. Pegal et al. [77] and Chazan [17] used Markov chains to determine the issuing policy. Jennings [49] and Brodheim et al. [13] discussed various aspects of blood bank inventory systems. Additional references in this area may be obtained from the above publications.

It should be mentioned that there is a class of inventory problems that also deals somewhat with the problem of obsolescence, but in a different context than that used in this study. This class of work has to do basically with finding the optimal issuing policies which maximize the total field life for a stockpile; Derman and Klien [26], Lieberman [56], and Zehna [104] have given conditions as to when to use FIFO or LIFO, when the utility characteristics of an item are changing with time. For example, according to them, if the utility function of an item is increasing, LIFO is best, and if the utility function is decreasing, FIFO is best. Eilon [31] discussed the relationship between field life and issuing policies when items in inventory begin to deteriorate. Pierskalla [79], Pierskalla and Roach [81] also described the optimal issuing policies and proved that FIFO is optimal policy for an inventory where all issued stock is consumed; Klein and Rosenberg [54] discussed optimal issuing policies, and they used inspection sampling to maintain a prescribed level of stored goods. Thorburn [97] solved an inventory problem and showed that in order to maximize the total field life a LIFO issuing policy should be used.

In this area of age-dependent perishable inventory, there are a number of specific papers that have been published specially by Nahmias [63-69]. Others include Cohen [19], Friedman and Hock [36], Nahmias and Pierskalla [71, 72], Nahmias and Wang [73, 74], Smith [93], and Weiss [101].

2.3 Age-Independent Perishable Inventories

This study is primarily concerned with inventory models of age-independent, constant utility, perishable items subject to ongoing deterioration; therefore some specifically related literature will be presented in this part.

The earliest work in this area is due to Ghare and Schrader [36]. They assumed a constant rate of deterioration in the face of constant demand and derived a relation for optimum cycle time. In their model the carrying cost is assessed on the average initial and ending inventory. The optimum cycle time is obtained iteratively by solving the following equation:

$$\frac{CRa}{2} + \frac{C_1R}{2} + \frac{C_1RTa}{2} + \frac{C_3}{T^2} = 0$$

Where a is the constant rate of deterioration, R is the constant demand rate, T is the inventory cycle time, and C, C_1, C_3 are cost of purchase price, carrying cost, and ordering cost, respectively. Van Zyle [98] in his thesis, briefly formulated a general age-independent perishable model in which a fixed or stochastic amount of product, depending on the total inventory, deteriorates.

Emmons [33, 34] considered a problem of exponential decay when one product decayed at one rate into a new product which decayed at a second rate. His model was used specifically for radioactive nuclide generators which are used for diagnosis and treatment of patients. His models are very useful in inventories of radioactive materials. Covert and Philip [21] obtained an Economic Order Quantity (EOQ) model for a variable rate of deterioration assuming a two parameter Weibull distribution for the deterioration time of items in stock. This permits already deteriorated items to be received by an inventory system as well as items which may start deteriorating in the future. Shah [87] generalized the previous works and assumed a general deterioration distribution and included backlogging under the condition of immediate replenishment. Aggarwal [3] evaluated Shah's paper and made a few corrections. In these two papers, cases of exponential decay and Weibull decay are explicitly treated but not others. Tadikamalla [94] in his paper assumed a gamma distribution for the deterioration time of items in stock. In comparing Weibull and gamma distributions for variable rates of deterioration, he observed that even when these two distributions have similar shapes, their instantaneous failure (decay) functions are significantly different. Therefore it is quite essential that the underlying decay distribution be known. Misra [58] developed a production lot size model for both a constant and variable rate of deterioration using a two parameter weibull distribution. In his model he did not allow for shortages and backlogging. This paper is an extension of the Ghare-Schrader and Cover-Philip models by assuming a finite production rate. Extension of the Misra paper was attempted by an anonymous author [1] to include shortages and backlogging assuming an exponential decay distribution.

Shah and Jaiswal [89] developed an order level model under constant and probabilistic demand assuming instantaneous delivery and constant rate of deterioration. In their model, scheduling period T is a prescribed constant and lead time is zero. Shah and Jaiswal extended the probabilistic periodic inventory model to include the effect of deterioration for constant and variable deterioration rate. In this model no shortages are allowed, and the review period is a prescribed constant.

Jani et al. [48] developed a probabilistic reorder point inventory model with constant rate of deterioration. In this model, the lot size and review period are assumed to be constant and the reorder point is the decision variable. Lead time is zero and shortages are made up as soon as new orders arrive. Shah and Jaiswal [91] also considered a probabilistic scheduling inventory model. Dave [23] discussed an order level inventory model where time is considered as discrete units. In the model he assumes constant failure rate, and allows for shortages. Demand is assumed to be constant, lead time is zero, and lot-size is a prescribed constant. This is a rather interesting paper, because the method of solution is different from that of the previously mentioned articles. Dave [21, 22] used the calculus of finite differences to solve some of the equations, instead of differential equations, because of the discrete nature of a time variable in the model. Dave [25], in addition to the above articles, developed an inventory model for deteriorating items that operates for exactly m-scheduling periods, under the assumption of probabilistic demand and constant decay rate. In the same line, Dave and Jaiswal [24] generalized the previous discrete-in-time models to a probabilistic inventory model and presented some sensitivity analyses.

In addition to the above articles, there are several papers that incorporate the age-independent decaying inventory models into their own specific field of interest. For example, Cohen [20] considered the problem of joint ordering and pricing for an exponentially decaying product under known demand. Nahmias and Wang [73] calculated the expected number of shortages during the lead time for an exponentially decaying product. Also, Nahmias and Wang [74] developed a heuristic lot-size reorder point model for decaying inventories.

In summary, the existing literature on perishable inventory has been gathered and the more relevant papers have been briefly discussed. It is apparent that a number of extensions and additions to the perishable inventory models are possible. Specifically age-independent perishable inventory models are considered.

CHAPTER III

INVENTORY TERMINOLOGY, NOTATION AND PROPERTIES

The purpose of the following material is to represent the notation that is used throughout this research, along with the basic building blocks of the various inventory properties that are included in some of these models. These objectives are accomplished by presenting the basic definitions and by analyzing the various types of replenishment, demands, and inventory models. However, it must be noted that the author does not claim any originality in these definitions and terminologies. Some of these definitions have been modified and rephrased for purposes of this treatise.

3.1 Definitions and Terminology

According to Naddor [61] for an inventory system to be analyzed, its characteristic properties must be considered. He suggests that all the properties of an inventory system can be classified into four categories: (1) demand properties, (2) replenishment properties, (3) cost properties, and (4) constraints properties. Each of these classifications of properties is discussed in detail.

3.1.1 Demand Properties

The demand properties involve information regarding the nature, size, timing, and the pattern of demand occurences. For example, when the demand size is the same from period to period, it is referred to as a constant (uniform) demand; otherwise, it is called a variable demand. If the demand size is known in advance with certainty, it is referred to as a deterministic demand, otherwise it will be referred to as a probabilistic demand. If the demand occurs at a known and non-uniform rate, it may be classified as a patterned demand.

3.1.1.1 Demand Pattern

There are numerous ways by which stocks can be taken out of the inventory to satisfy demand. There are essentially five cases that can be considered, and they are depicted in Figure 10. In all five patterns the inventory level at the beginning of the demand period is S, the demand period is T time units, and the total demand during the period is x number of units. The general equation of the quantity in inventory at time t during the demand period T is given by

$$Q(t) = S - x \sqrt[n]{\frac{t}{T}} \; ; \qquad o \leq t \leq T \qquad (3\text{-}1)$$

where Q(t) is the inventory level at time t, and n is the demand pattern index. This equation is due to Naddor [61].

Of course, the demand pattern can be described explicitly, rather than in the terms of inventory level. For example

$$d(t) = rt^n \; ; \qquad 0 \leq t \leq T, \; r > 0 \qquad (3\text{-}2)$$

or

$$d(t) = pq^t \; ; \qquad 0 \leq t \leq T, \; p > 0, \; q \geqslant 1 \qquad (3\text{-}3)$$

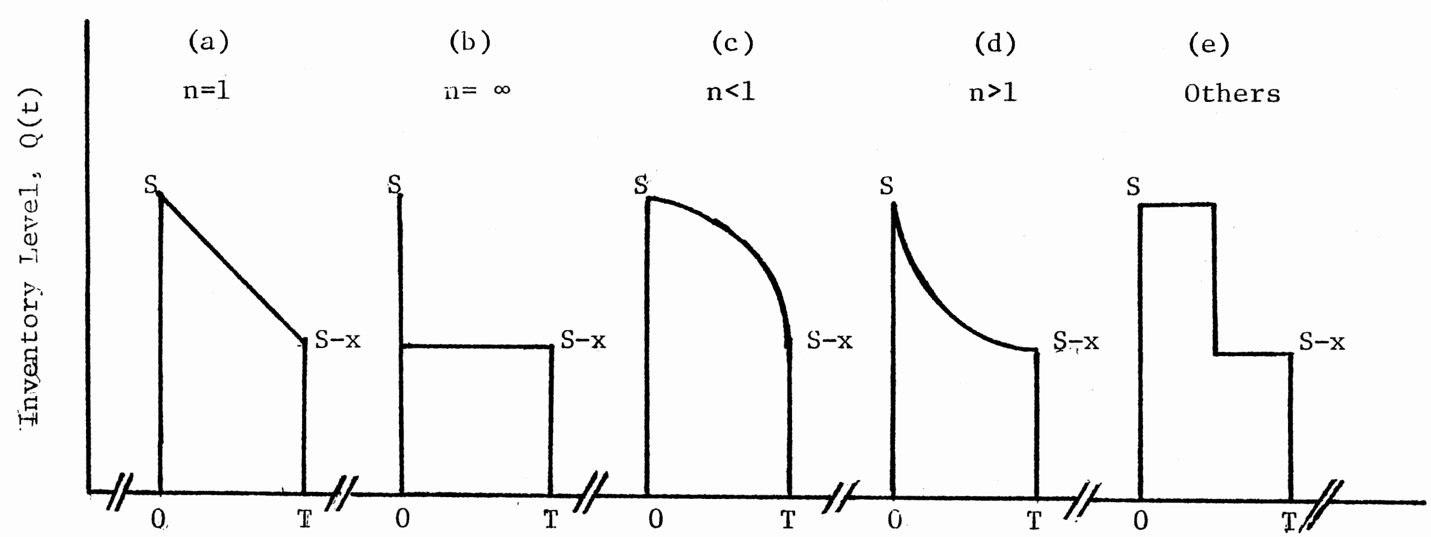


Figure 10. Power Demand Patterns

where r, n, p, and q, are the demand parameters. Note, however, that the total demand is still a known quantity given by x; that is:

$$x = \frac{r}{n+1} T^{n+1} \tag{3-4}$$

for the demand of equation (3-2), or

$$x = - \frac{p}{\ln(q)} (1-pq^{T}) \tag{3-5}$$

for the demand of equation (3-3)

3.1.2 Replenishment Properties

The replenishment properties involve information regarding the nature, timing, quantity and pattern of replenishment activity. In general, the replenishment properties can be controlled by the decision maker, and hence, can be used to reduce the total inventory costs. One important property of replenishment is the lead time factor. Lead time is defined as the length of time between placing an order and the actual addition of the order quantity to inventory. In general, lead time is assumed to be known and constant, or insignificant for deterministic inventory systems. Lead time becomes important in the probabilistic models. Similar to the case of demand patterns, replenishments also have patterns that must be taken into account when analyzing inventory models. This pattern may be the result of production and shipment methods.

3.1.2.1 Replenishment Pattern

Consider a period of time over which the replenishment lot size Q is being added to the inventory level. Again, there are five cases that can be considered, and they are depicted in Figure 11. In all five patterns, the replenishment period is T_r time units, and the replenishment lot size is Q number of units. The general equation of inventory level at time t during the replenishment period T is given by

$$Q(t) = Q\sqrt[m]{\frac{t}{T_r}} \qquad (3\text{-}6)$$

where m is the replenishment pattern index.

By letting p(t) represent the replenishment rate, the replenishment pattern can also be described explicitly. For example:

$$p(t) = \lim_{T_r \to 0} \frac{Q}{T_r} = \infty \qquad 0 \leq t \leq T_r$$

indicates instantaneous replenishment which is shown in Figure 11-b, and

$$p(t) = \frac{Q}{T_r} = \text{constant}$$

which is the case for uniform replenishment, Figure 11-a.

By using some other functions to represent p(t), other cases can also be analyzed. Two functions which have practical usefulness in this context are:

$$p(t) = bt^m \qquad \begin{matrix} 0 \leq t \leq T_r \\ b > 0 \end{matrix} \qquad (3\text{-}7)$$

and

$$p(t) = pq^t \qquad \begin{matrix} 0 \leq t \leq T_r \\ p > 0 \\ q \geq 1 \end{matrix} \qquad (3\text{-}8)$$

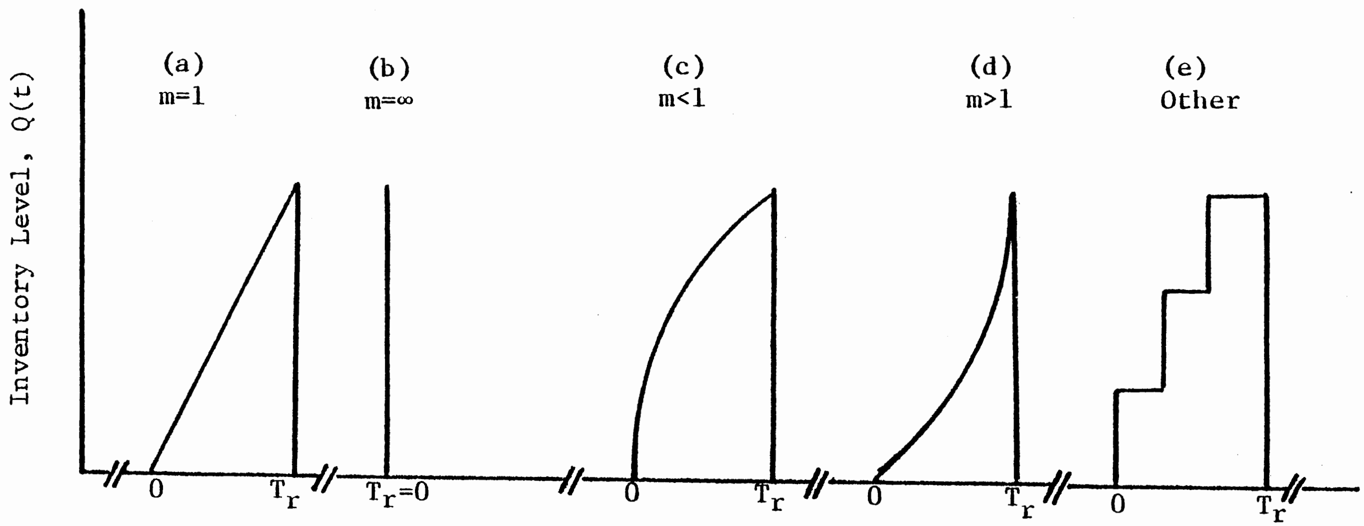


Figure 11. Power Replenishment Pattern

where b, p, q, and m are the replenishment parameters. Again, note that the total ordered quantity must be equal to

$$Q = \frac{b}{m+1} T_r^{m+1} \qquad (3\text{-}9)$$

for the replenishment of equations (3-7), and

$$Q = \frac{p}{\ln(q)}(1-pq^T r) \qquad (3\text{-}10)$$

for the replenishment of equation (3-8).

3.1.3 Cost Properties

Costs are measures of performance of controllable variables in an inventory system. The various categories of costs which are of interest to this treatise have already been mentioned in Chapter I and will not be repeated. However, the respective symbols and their associated dimensions will be given in section 3.2

3.1.4 Constraint Properties

The constraint properties involve placing limitations upon any pertinent factor of the inventory system which have been mentioned to this point, such as cost constraints, replenishment constraints, demand constraints, etc.

3.1.5 Additional Definitions

The following terminology is used throughout this treatise; therefore, these definitions are given at this point for the sake of clarity and completeness:

Scheduling Period: The scheduling period is the length of time, measured in time units, between consecutive decisions with respect to replenishments. When the scheduling period is prescribed, the decision maker cannot control it, hence it must be treated as an inventory parameter. In most situations, scheduling periods are assumed to be equal. A constant scheduling period is denoted by T.

Reorder Point: Reorder point refers to a specific amount in inventory. An order is placed when the inventory on hand is equal to or below the reorder point.

Reviewing Period: Reviewing period is the time interval between consecutive review of the inventory level.

Order Level: Order level refers to a specific amount in inventory and it is used as a benchmark for ordering the amount of required replenishment.

Optimal Inventory Policy: The set of decision rules which optimizes the performance criteria are referred to as the optimal inventory policy. These decision rules are obtained by analyzing mathematical models of the inventory situation.

3.2 Notation

The following notations are used throughout this treatise with no change in meaning or assumption.

C_1 = The unit inventory carrying cost (\$/item/unit time).

C_2 = The unit inventory shortage cost (back order: \$/item/unit time; lost sale: \$/unit).

C_3 = The unit replenishment cost or set up cost (\$/cycle).

C_4 = The unit deterioration/perishability cost (\$/unit). [Purchase price + disposal cost - salvage value].

$K(.)$ = Total expect cost.

$C(.)$	=	Total expected cost per unit time.
t, T	=	Cycle time or scheduling period.
$I_1(.)$	=	Average carrying inventory per time unit.
$I_2(.)$	=	Average shortage in an inventory per time unit.
$I_3(.)$	=	Average number of replenishment per time unit.
$I_4(.)$	=	Average number of items perishing per time unit.
I_1	=	Total carrying inventory in a cycle.
I_2	=	Total shortage in an inventory cycle.
$D(.)$	=	The expected total number of units deteriorating during a given cycle.
$Q(t)$	=	The inventory level at time t.
Q, q	=	Order quantity/replenishment size.
Q_{max}	=	The maximum inventory level.
Q_{min}	=	The minimum inventory level.
$r, R, d(.)$	=	The demand rate.
$p(.)$	=	The replenishment rate.
$h(t)$	=	The perishability rate.

3.3 Assumptions

In the analysis of mathematical models of this dissertation, replenishment cycle t rather than the customary replenishment quantity Q is used for calculating the optimal cost. The replenishment cycle is preferable because of the time dependent nature of perishable inventories.

The following assumptions are implicit in all of the inventory models that are presented in this treatise.

i. There is no repair or replacement of any deteriorated items during a given inventory cycle.

ii. Lead time is zero.

iii. Production rate is greater than the demand rate (applicable models).

iv. Deterioration begins only after the items are received into inventory.

v. Items or products are treated as continuous units.

vi. Replenishment lot size is fixed and will not vary from one cycle to another.

vii. Infinite demand horizons.

CHAPTER IV

LOT SIZE SYSTEM--MODEL I

In this model the inventory holding cost is charged on all the units that remain in inventory, whether perished or not; and perishability costs are charged at the end of the inventory cycle. This is a very reasonable assumption since from an accounting point of view one can write off (down) the costs of all the units in inventory that are no longer in their original useful state.

Model I is a lot size inventory model for a system of perishable units and has been developed by rectifying the error in Thomopoulos and Lehman's [96] analysis in calculating the average inventory holding cost. This model is more general than their constant demand inventory model, and provides additional insight into selected inventory policies. In their paper, they present an inventory situation in the context of obsolescence, that is, the more an item remains in stock, the more likely that it becomes useless. They propose to show this behavior as a probability function $P(T) = kT^2$; $(T = 1,2 \dots)$, where $P(T)$ is the probability of obsolescence at the storage time T and k is a constant which is determined for a given situation.

4.1 Lot Size Model I-a

In this model, the optimal inventory characteristics will be analyzed as a function of the initial inventory level, in the subsequent Model I-b, the analysis will be preformed on the basis of the remaining inventory, that is, the inventory level after the demand has been satisfied.

Let $h(t)$ be a monotonically increasing probability function of perishability at time t. Therefore, in time interval $(t, t + \Delta t)$, $\Delta t h(t)$ indicates the probability that the items in the inventory will deteriorate. Depending on the shape of $h(t)$, the characteristics of the inventory model will change, but the underlying method of analysis stays the same. The whole objective here is to determine the minimum average total inventory cost. This can be obtained by balancing the inventory carrying cost, replenishment cost and perishability cost. Now, for various functions of $h(t)$, this system will be analyzed.

4.1.1 Case 1: Constant Perishability Rate

Let $h(t)$ be equal to a constant perishability function, that is;

$$h(t) = a. \qquad (4.1.1\text{-}0)$$

For this case Figures 12 and 13 depict the inventory situation. Therefore, the probability of perishing is determined by

$$p = \int_0^t a d\tau = at \qquad (4.1.1\text{-}1)$$

If t^* is optimal value of inventory cycle time, t, then a $1/t^*$.

The average amount of inventory, average number of replenishments, and average number of items perishing are given by $I_1(t)$, $I_3(t)$, and $I_4(t)$.

$$I_1(t) = \frac{1}{t}\left[\frac{Rt^2}{2} + Rt * at\right] \qquad (4.1.1\text{-}2)$$

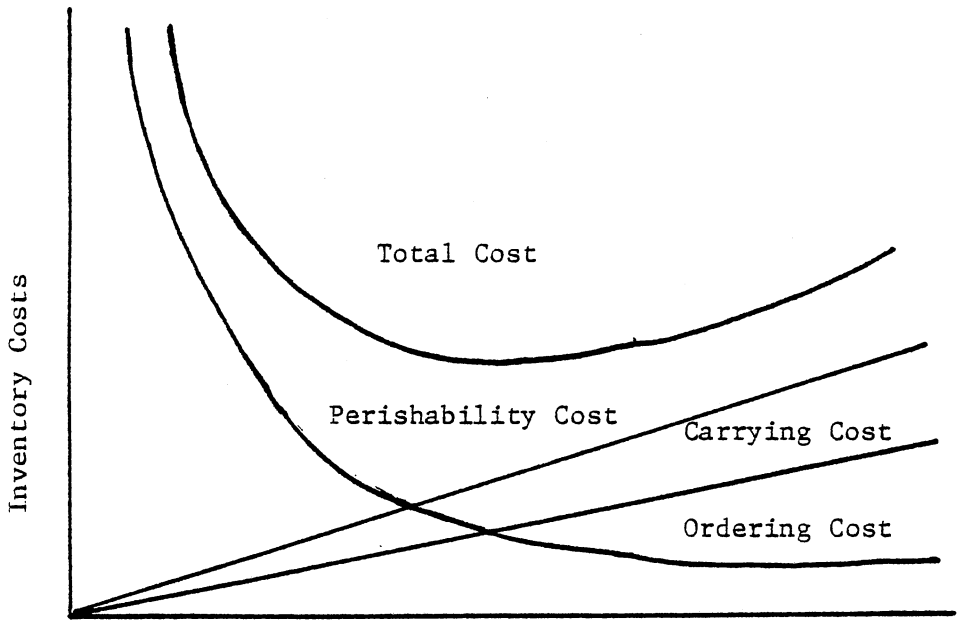


Figure 12. Cost Function of Constant Perishability Lot Size System

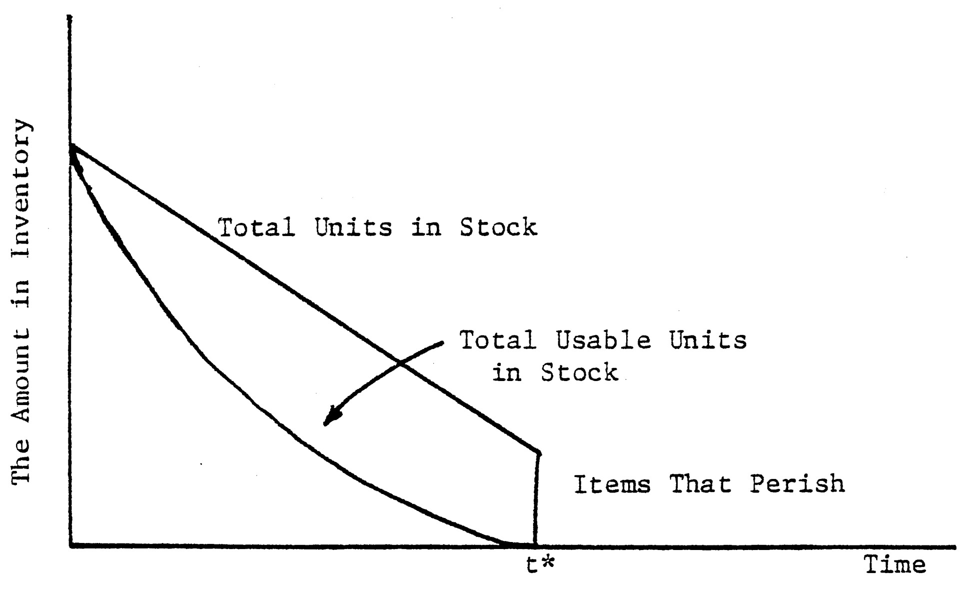


Figure 13. Inventory Level of Constant Perishability Lot Size System

$$I_3(t) = \frac{1}{t} \tag{4.1.1-3}$$

$$I_4(t) = \frac{1}{t} \; [Rt * at] \tag{4.1.1-4}$$

The total per unit time cost function is given by

$$C(t) = C_1 I_1(t) + C_3 I_3(t) + C_4 I_4(t) \tag{4.1.1-5}$$

which for this particular case can be written as

$$C(t) = \frac{C_1}{2} Rt + \frac{C_3}{t} + (C_1 + C_4) \; Rat \tag{4.1.1-6}$$

Since $C(t)$ is a continuous, differentiable, and convex function, the optimal cycle time can be obtained by differentiating $C(t)$, setting the result equal to zero, and then solving for t^*:

$$\frac{d}{dt} C(t) = C_1 R/2 - \frac{C_3}{t^2} + (C_1 + C_4)Ra \tag{4.1.1-7}$$

$$t^* = \left[\frac{C_3}{\frac{C_1 R}{2} + (C_1 + C_4)Ra} \right]^{\frac{1}{2}} \tag{4.1.1-8}$$

If $a = 0$, then $t^* = \sqrt{2C_3/C_1R}$, which is the result of the classical Economic Lot Size (EOQ) Model.

The lot size in this situation is not equal to Rt, but is equal to

$$Q = Rt + (Rt * at) \tag{4.1.1-9}$$

The optimal lot size then can be written as:

$$Q^* = Rt^* \; (1 + at^* \;). \tag{4.1.1-10}$$

By substituting equation (4.1.1-8) into equation (4.1.1-6) the optimal cost is obtained.

$$C^*(t) = [2C_1C_3R + 4C_3Ra(C_1 + C_4)]^{\frac{1}{2}} \qquad (4.1.1\text{-}11)$$

As a matter of comparison, the optimal classical EOQ cost is given by

$$C^*(t) = (2C_1C_3R)^{\frac{1}{2}} \qquad (4.1.1\text{-}12)$$

For this particular case then, the additional cost due to perishability can readily be observed.

4.1.2 Case 2: Linear Perishability Rate

Let h(t) be equal to a linear perishability function, that is;

$$h(t) = a + bt \qquad (4.1.2\text{-}0)$$

For this case the graph of the inventory level also resembles Figure 13. Therefore, by utilizing equation (4.1.2-0) the probability of perishing can be determined by

$$p = \int_0^t (a + b\Upsilon)d\Upsilon = at + \frac{bt^2}{2} \qquad (4.1.2.1)$$

If t^* is the optimal value of the inventory cycle time, then:

i) if $a \geqslant 0$, $b > 0$; then at t^*, $a \ll \frac{1}{t^*}$ and $b \ll \frac{2(1 - 2at^*)}{t^{*2}}$,

ii) if $a \geqslant 0$, $b < 0$; then at t^*, $a \ll \frac{1}{t^*}$ and $-b \ll \frac{2(1-2at^*)}{t^{*2}}$,

iii) if $a < 0$, $b > 0$ then the restrictions of (i) apply to the integral of equation (4.1.2-1) with the limit of integration from b/a to t instead of 0 to t.

The average amount of inventory, average number of replenishment, and average number of items perishing can be written as

$$I_1(t) = \frac{1}{t}\left[\frac{Rt^2}{2} + Rt^2a + \frac{Rt^3b}{2}\right] \qquad (4.1.2\text{-}2)$$

$$I_3(t) = \frac{1}{t} \qquad (4.1.2\text{-}3)$$

$$I_4(t) = \frac{1}{t}\left[Rt^2a + \frac{Rt^3b}{2}\right] \qquad (4.1.2\text{-}4)$$

By using equation (4.1.1-5) the total average cost is determined

$$C(t) = \frac{C_1Rt}{2} + \frac{C_3}{t} + (C_1 + C_4)\ (Rt)\ (a + bt/2) \qquad (4.1.2\text{-}5)$$

Again, by differentiating C(t) with respect to t, optimal t^* may be obtained as

$$t^* = \left[\frac{C_3}{[.50\ C_1R + (C_1 + C_4)\quad R\ (a + bt)]}\right]^{\frac{1}{2}} \qquad (4.1.2\text{-}6)$$

Though it is possible to rewrite this equation as a cubic equation and solve for t^* analytically, it would be more efficient to use a numerical technique such as Newton's Method or Secant Method to solve for t^*. An algorithm is presented later to solve this class of equations.

The lot size in this situation is given by

$$Q = Rt + (Rta + Rt^2b/2) \qquad (4.1.2\text{-}7)$$

which for optimal lot size is written as

$$Q^* = Rt^* \ (1 + at^* + b/2\ t^{*2}) \qquad (4.1.2\text{-}8)$$

Since t^* cannot be obtained in a closed form solution, after obtaining t^* numerically, $C^*(t)$ may then be determined by substituting t^* in equation (4.1.2-5).

4.1.3 Case 3: Quadratic Perishability Rate

Let h(t) be equal to a quadratic perishability function, that is;

$$h(t) = a + bt + ct^2 \tag{4.1.3-0}$$

Therefore, the probability of perishing is given by

$$p = \int_0^t (a + b\tau + c\tau^2)d\tau = at + \frac{bt^2}{2} + \frac{ct^3}{3} \tag{4.1.3-1}$$

Restrictions on a and b hold as in Case 2 and at t^*,

$$c \ll \frac{3(1 - at^* - bt^{*2})}{t^{*3}} .$$

The average carrying inventory, the average replenishment, and the average perishing is given by

$$I_1(t) = \frac{1}{t}\left[\frac{Rt^2}{2} + Rt^2a + \frac{Rt^3b}{2} + \frac{Rt^4c}{3}\right] \tag{4.1.3-2}$$

$$I_3(t) = \frac{1}{t} \tag{4.1.3-3}$$

$$I_4(t) = \frac{1}{t}\left[Rt^2a + \frac{Rt^3b}{2} + \frac{Rt^4C}{3}\right] \tag{4.1.3-4}$$

By using equation (4.1.1-5) the total average cost function is determined.

$$C(t) = \frac{C_1Rt}{2} + \frac{C_3}{t} + (C_1 + C_4) * Rt * (a + \frac{tb}{2} + \frac{t^2c}{3}) \tag{4.1.3-5}$$

By differentiating C(t) with respect to t, optimal t^* may be obtained as

$$t^* = \left[\frac{C_3}{[.50C_1R + (C_1 + C_4)Ra + (C_1 + C_4)Rbt + (C_1 + C_4)Rct^2]}\right]^{\frac{1}{2}} \quad (4.1.3\text{-}6)$$

By using a numerical technique, t^* can be determined. The optimal lot size is given by

$$Q^* = Rt^* \quad 1 + at^* + \frac{bt^{*2}}{2} + \frac{ct^{*3}}{3} \quad (4.1.3\text{-}8)$$

The optimal cost, $C^*(t)$ can be determined by substituting t^* into equation (4.1.3-5).

4.1.4 Case 4: Polynomial Perishability Rate

Let h(t) be equal to some monotonic polynomial perishability function, that is,

$$h(t) = a_0 + \ldots + a_nt^n \quad (4.1.4\text{-}0)$$

For this general case, the procedure in the previous sections indicates that, optimal t^*, and Q^* can be determined by using the following general equation

$$t^* = \left[\frac{C_3}{.50C_1R + (C_1 + C_4) * R * [a_0 + \ldots + a_nt^n]}\right]^{\frac{1}{2}} \quad (4.1.4\text{-}1)$$

$$Q^* = Rt^* \left(1 + a_0t^* + \ldots + \frac{a_nt^{*n+1}}{n+1}\right). \quad (4.1.4\text{-}2)$$

4.1.5 Case 5: Exponential Perishability Rate

Let $h(t)$ be equal to an exponential perishability function, that is,

$$h(t) = ae^{bt} \tag{4.1.5-0}$$

Therefore, the probability of perishing is given by

$$p = \int_0^t ae^{b\tau} d\tau = a/b\ [e^{bt} - 1] \tag{4.1.5-1}$$

Where a and b are positive real numbers.

The average amount of inventory, the average replenishment, and the average perishability can be written as:

$$I_1(t) = \frac{1}{t}\left[\frac{Rt^2}{2} + \frac{Rta}{b}e^{bt} - \frac{Rta}{b}\right] \tag{4.1.5-2}$$

$$I_3(t) = \frac{1}{t} \tag{4.1.5-3}$$

$$I_4(t) = \frac{1}{t}\left[\frac{Rta}{b}e^{bt} - \frac{Rta}{b}\right] \tag{4.1.5-4}$$

The average total cost is

$$C(t)\ \frac{C_1Rt}{2} + \frac{C_3}{t} + \frac{(C_1 + C_4)Ra}{b}\ (e^{bt} - 1). \tag{4.1.5-5}$$

Optimal inventory cycle time is determined by differentiating $C(t)$ with respect to t,

$$\frac{d}{dt}\ C(t) = \frac{C_1R}{2} - \frac{C_3}{t^2} + (C_1 + C_4)Rae^{bt} \tag{4.1.5-6}$$

By setting equation (4.1.5-6) equal to zero and solving for t, one obtains

$$t^* = \left[\frac{C_3}{(.50C_1R + (C_1 + C_4)Rae^{bt})}\right]^{\frac{1}{2}} \tag{4.1.5-7}$$

The value of t^* can be obtained numerically, and from it, the optimal lot size can be determined.

$$Q^* = Rt^* \ ((b-a) + ae^{bt^*})/b \tag{4.1.5-8}$$

The optimal cost is obtained by substituting t^* into equation (4.1.5-5).

4.2 Lot Size Model I-b

This model is exactly similar to Model I-a with the exception that the perishability occurs as a function of the remaining items in inventory after the demand has been satisfied. Let D(t) be the number of items that deteriorate during the inventory cycle t. If t^* is the optimal t, then the initial inventory should be $Rt^* + D(t^*)$. Assuming the same perishability rate functions, the same cases are analyzed under this new assumption.

4.2.1 Case 1: Constant Perishability Rate

In this, and subsequent cases, the function D(t) must be determined. Function D(t) can be written implicitly as:

$$D(t) = \int_0^t [(Rt + D(t)) - R\tau]\, h(\tau)d\tau \tag{4.2.1-1}$$

which upon simplification becomes

$$D(t) = [(Rt + D(t)]\int_0^t h(\tau)d\tau - \int_0^t R\tau h(\tau)d\tau \qquad (4.2.1\text{-}2)$$

For a constant perishability rate function, equation (4.2.1-2) can be written as

$$D(t) = [Rt + D(t)]at - \frac{Rat^2}{2} \qquad (4.2.1\text{-}3)$$

which simplifies further to

$$D(t) = \frac{Rat^2}{2(1 - at)} \qquad (4.2.1\text{-}5)$$

Now, the average carrying inventory, the average replenishment, and the average perishability may be written as:

$$I_1(t) = \frac{1}{t}\left[\frac{Rt^2}{2} + \frac{Rat^2}{2(1 - at)}\right] \qquad (4.2.1\text{-}5)$$

$$I_3(t) = \frac{1}{t} \qquad (4.2.1\text{-}6)$$

$$I_4(t) = \frac{1}{t}\left[\frac{Rat^2}{2(1 - at)}\right]; \qquad (4.2.1\text{-}7)$$

and the average total cost function as:

$$C(t) = \frac{C_1Rt}{2} + \frac{C_3}{t} + \frac{(C_1 + C_4)Rat}{2(1 - at)}. \qquad (4.2.1\text{-}8)$$

To find t^*, $C(t)$ is differentiated with respect to t, and is set equal to zero.

$$\frac{d}{dt}C(t) = \frac{C_1R}{2} - \frac{C_3}{t^2} + \frac{Ra(C_1 + C_4)}{2(1 - at)^2} = 0 \qquad (4.2.1\text{-}9)$$

which reduces to

$$t^* = \left[\frac{2C_3}{C_1R + Ra\,(C_1 + C_4)/(1 - at)^2}\right]^{\frac{1}{2}}, \qquad (4.1.1\text{-}10)$$

Consequently, the optimal lot size, Q^*, can now be determined as:

$$Q^* = Rt^*\left[1 + \frac{at^*}{2(1 - at^*)}\right]. \qquad (4.2.1\text{-}11)$$

4.2.2 Case 2: Linear Perishability Rate

By utilizing equation (4.2.1-2) and substituting a linear perishability rate function, the following can be written:

$$D(t) = [Rt + D(t)]\int_0^t (a + b\Upsilon)\, d\Upsilon - \int_0^t R\Upsilon(a + b\Upsilon)d\Upsilon \qquad (4.2.2\text{-}1)$$

which reduces to

$$D(t) = \frac{Rat^2}{2} + \frac{Rbt^3}{6} + \left(at + \frac{bt^2}{2}\right) D(t) \qquad (4.2.2\text{-}2)$$

and upon further simplification to

$$D(t) = \left(\frac{3Rat^2 + Rbt^3}{6}\right) \Big/ \left(\frac{2 - 2at - bt^2}{2}\right) \qquad (4.2.2\text{-}4)$$

The average carrying inventory, the average replenishment, and the average perishability may then be written as

$$I_1(t) = \frac{1}{t}\left[\frac{Rt^2}{2} + \frac{Rt^2\,(3a + bt)}{6 - 6at - 3bt^2}\right] \qquad (4.2.2\text{-}4)$$

$$I_3(t) = \frac{1}{t} \qquad (4.2.2\text{-}5)$$

$$I_4(t) = \frac{1}{t}\left[\frac{Rt^2(3a + bt)}{3(2 - 2at - bt^2)}\right]; \qquad (4.2.2\text{-}6)$$

and the average total cost function as:

$$C(t) = \frac{C_1Rt}{2} + \left[\frac{(C_1 + C_4)}{3}\right]Rt(3a + bt)/(2 - 2at - bt^2) + \frac{C_3}{t} \qquad (4.2.2\text{-}7)$$

By differentiating C(t) with respect to t, setting it equal to zero, and solving for t, the optimal inventory cycle time can be determined. That is,

$$\frac{d}{dt}C(t) = \frac{C_1R}{2} - \frac{C_3}{t^2} + \frac{R\,(C_1 + C_4)\,(6a + 4bt + abt^2)}{3(2 - 2at - bt^2)^2} = 0 \qquad (4.2.2\text{-}8)$$

$$t^* = \left[\frac{C_3}{.50\;C_1R + [R(C_1 + C_4)(6a + 4bt + abt^2)\;/\;(3(2 - 2at - bt^2)^2)]}\right]^{\frac{1}{2}}$$

(4.2.2-9)

After obtaining t^*, the optimal lot size Q^* is determined by

$$Q^* = Rt^*\left(1 + \frac{3at^* + bt^{*2}}{6 - 6at^* - 3bt^{*2}}\right) \quad . \qquad (4.2.2\text{-}10)$$

4.2.3 Case 3: Exponential Perishability Rate

Again, by utilizing equation (4.2.1-2) and substituting an exponential perishability rate function the following can be written:

$$D(t) = [Rt + D(t)]\int_0^t ae^{b\Upsilon}\,d\Upsilon - \int_0^t R\Upsilon ae^{b\Upsilon}\,d\Upsilon \qquad (4.2.3\text{-}1)$$

which reduces to

$$D(t) = \frac{a}{b}\;[Rt + D(t)]\;[e^{bt} - 1] - Ra\,\frac{ebt}{b^2}\,[bt - 1] \qquad (4.2.3\text{-}2)$$

and upon further simplification

$$D(t) = \frac{Ra[e^{bt} - bt]}{ab + b^2 - abe^{bt}} \qquad (4.2.3\text{-}3)$$

The average carrying inventory, the average replenishment, and the average perishability may then be written as:

$$I_1(t) = \frac{1}{t}\left[\frac{Rt^2}{2} + \frac{Ra[ebt-bt]}{ab + b^2 - abe^{bt}}\right] \tag{4.2.3-4}$$

$$I_3(t) = \frac{1}{t} \tag{4.2.3-5}$$

$$I_4(t) = \frac{1}{t}\left[\frac{(Ra/b)[ebt - bt]}{(a + b - ae^{bt})}\right] \quad ; \tag{4.2.3-6}$$

and the average total cost function as:

$$C(t) = \frac{C_1Rt}{2} + \frac{C_3}{t} + \frac{(C_1 + C_4)Ra[e^{bt} - bt]}{t(ab + b^2 - abe^{bt})} \tag{4.2.3-7}$$

The optimal t^* can be determined by differentiating $C(t)$ with respect to t, and setting it equal to zero.

$$\frac{d}{dt}C(t) = \frac{C_1R}{2} - \frac{C_3}{t^2} \tag{4.2.3-8}$$

$$+ \frac{(C_1 + C_4)(Ra/b)(e^{bt})[tab + tb^2 - a - b + ae^{bt} - ab^2t^2]}{(ta + tb - tae^{bt})^2} = 0$$

$$t^* = \left[\frac{C_3}{.5C_1R + \dfrac{(C_1 + C_4)(Ra/b)(e^{bt})(tab - tb^2 - a - b + ae^{bt} - ab^2t^2)}{(ta + tb - tae^{bt})^2}}\right]^{\frac{1}{2}} \tag{4.2.3-9}$$

The optimal lot size is given by

$$Q^* = Rt^*\ [1 + \frac{(Ra/b)(e^{bt^*} - bt^*)}{(a + bt^* + ae^{bt})}] \tag{4.2.3-10}$$

As is evident, some of these equations are quite involved and too complex. But, from the practical stand point, these equations can be readily programmed and analyzed. Because of the flexibility that is available in the selection of values for the perishability parameters (or even costs), adequate "near optimal" results can be obtained for most situations. An interactive Fortran Program has been developed (see Appendix A) to provide a simple means of comparing the various perishable inventory models. In addition, these programs permit a parametric analysis of the perishability coefficients.

4.3 Algorithm for Determining t*

Because of the nature of the functions in equations (4.1.2-6, 4.1.3-6, 4.1.4-7) of Model I-a and (4.2.1-10, 4.2.2-9, 4.2.3-8) of Model I-b, a simple algorithm is devised which guarantees a rapid convergence to t^* in only a few iterations. Figure 14 depicts this algorithm.

To solve for t^* it is necessary to find the intersection of two functions, $Y_1 = t^2$, which is a pure quadratic; and $Y_2 = \dfrac{1}{A + f(t)}$, which is a monotonicly decreasing function. Y_2 is the square of the right hand side of the above mentioned equations. Since the optimal t obtained from the classical EOQ formulation is always larger than t^*, thus by using this value as an initial solution, t^* can be obtained rather rapidly using the following steps:

1. Determine $t_0 = \sqrt{2C_3/(C_1R)}$
2. Evaluate Y_2 at t_0, assume the value is F_0
3. Determine the new t, $t_1 = \sqrt{F_0}$
4. Evaluate Y_2 at t_1, assume the value is F_1
5. If $|F_1 - F_0| \leq \epsilon$ or $|t_1 - t_0| \leq \delta$ stop. Otherwise set $t_0 = t_1$, $F_0 = F_1$ and go the step 2.

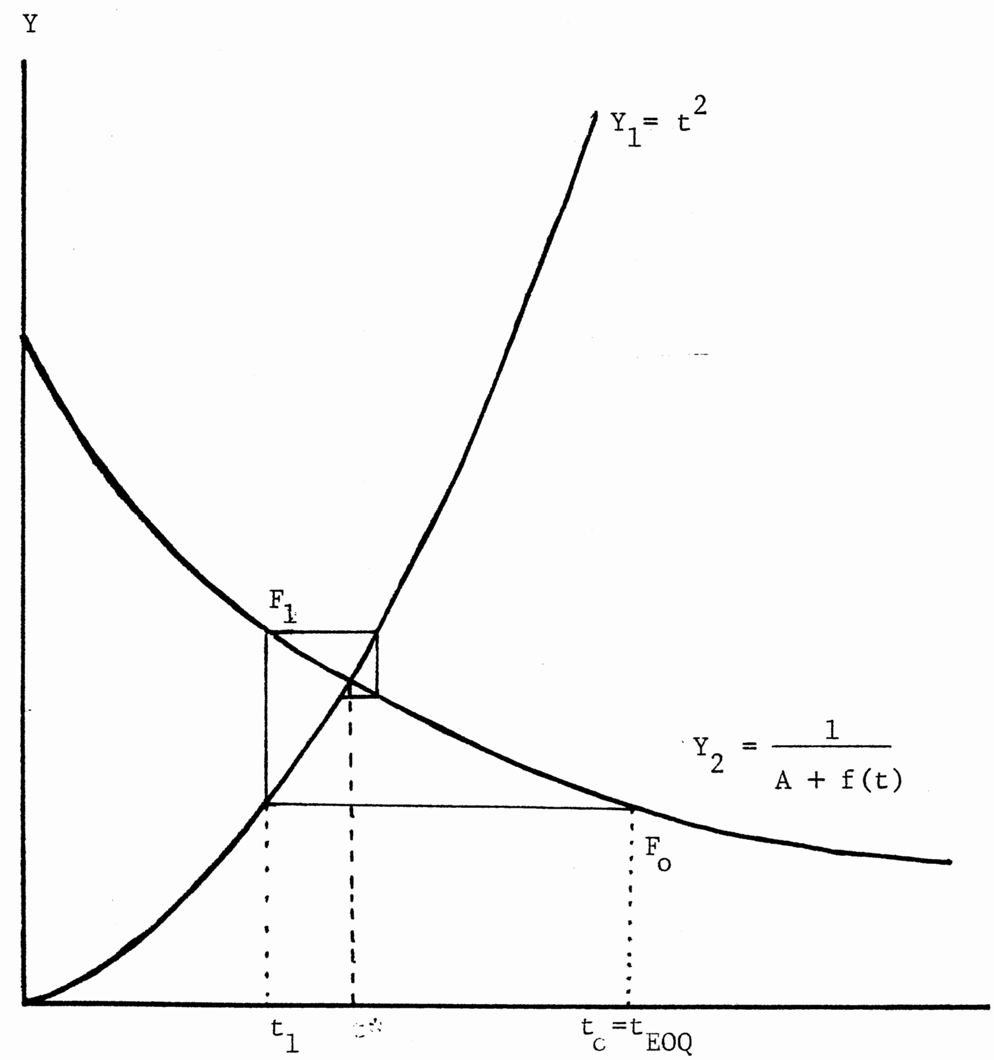


Figure 14. The Graph of Algorithm for Determining t*

(ϵ, δ are arbitrary small numbers, i.e., $.001 \simeq \epsilon$, $\delta \simeq .005$).

4.4 Lot Size Model I-C

Another method for analyzing perishable inventories, (keeping the assumptions of the previous cases) is to regard perishability as an addition to regular demand. For the various models the following results are obtained.

4.4.1 Case 1: Constant Perishability Rate

Let a be the number of units perishing per unit of time. Then the total demand on inventory can be written as:

$$R' = R + a \qquad (4.4.1\text{-}1)$$

where R' now is the new demand rate. The average total cost of this model can be written as

$$C(t) = \frac{C_1 R' t}{2} + \frac{C_3}{t} + C_4 a \qquad (4.4.1\text{-}2)$$

or

$$C(t) = \frac{C_1 R t}{2} + \frac{C_3}{t} + \frac{C_1 a t}{2} + C_4 a \ . \qquad (4.4.1\text{-}3)$$

Now, by taking the derivative of C(t) and setting it equal to zero, t^* and hence Q^* can be determined.

$$\frac{d}{dt} C(t) = \frac{C_1 R}{2} - \frac{C_3}{t^2} + .5C_1 a = 0 \qquad (4.4.1\text{-}4)$$

$$t^* = \left[\frac{2\ C_3}{C_1 R + C_1 a} \right]^{\frac{1}{2}} \qquad (4.1.1\text{-}5)$$

The optimal order size is equal to

$$Q^* = R't^* = Rt^* + at^* \qquad (4.4.1\text{-}6)$$

In inventory literature, the perishability cost is usually included in the inventory carrying cost C_1. Recall for the standard EOQ model, the optimal inventory cycle time is give by $\sqrt{2C_3/C_1R}$. This implies that the C_1 used in the EOQ analysis is $(1 + a/R)$ times larger than the C_1 used in equation (4.1.1-5). Therefore, this explains the reason for the inclusion of perishability cost into inventory carrying cost, of an EOQ Model whenever this class of models is considered for the inventory of perishable items.

4.4.2 Case 2: Linear Perishability Rate

Let $h(t)$ be a linear function indicating number of units perishing per unit of time during the inventory replenishment cycle. Therefore, R' is equal to

$$R' = R + a + bt = (R + a) + bt \qquad (4.4.2\text{-}1)$$

In this case the new demand rate is increasing linearly; however, note that this does not increase indefinitely in time, and it ceases at the end of each inventory cycle. By utilizing equation (4.4.2-1) the replenishment size can be determined by

$$Q = \int_0^t [(R + a) + b\Upsilon]d\Upsilon \qquad (4.4.2\text{-}2)$$

which simplifies to

$$Q = (R+a)t + \frac{bt^2}{2} \qquad (4.4.2\text{-}3)$$

The average carrying inventory is evaluated from the following equation:

$$I_1(t) = \frac{1}{t}\int_0^t \left\{ \left[(R + a)t + \frac{bt^2}{2} \right] - [(R + a) + b\Upsilon] \right\} d\Upsilon \qquad (4.4.2\text{-}4)$$

which reduces to

$$I_1(t) = \frac{(R+a)t}{2} + \frac{bt^2}{6} \, . \qquad (4.4.2\text{-}5)$$

The average number of items perishing is given by

$$I_4(t) = \frac{1}{t}\int_0^t (a + b\Upsilon)\, d\Upsilon \qquad (4.4.2\text{-}6)$$

which simplifies to

$$I_4(t) = a + bt/2 \quad . \qquad (4.4.2\text{-}7)$$

The average total cost equation is equal to

$$C(t) = \frac{C_1Rt}{2} + \frac{C_3}{t} + \frac{(C_1t + 2C_4)a}{2} + \frac{bt \quad (C_1t + 3C_4)}{6} \quad . \qquad (4.4.2\text{-}8)$$

By differentiating $C(t)$ and setting it equal to zero, t^* can be determined.

$$\frac{d}{dt}C(t) = \frac{C_1R}{2} - \frac{C_3}{t^2} + \frac{aC_1}{2} + \frac{b(2C_1t + 3C_4)}{6} \qquad (4.4.2\text{-}9)$$

$$t^* = \left[\frac{2C_3}{C_1R + aC_1 + \frac{b}{3}(2C_1t + 3C_4)} \right]^{\frac{1}{2}} \qquad (4.4.2\text{-}10)$$

By substituting t^* into equation (4.4.2-3) the optimal order size is determined.

4.4.3 Case 3: Exponential Perishability Rate

Let h(t) be an exponential function indicating number of units perishing per unit of time during the inventory replenishment cycle. Therefore, R' is equal to

$$R' = R + ae^{bt} \tag{4.4.3-1}$$

By utilizing equation (4.4.3-1) the replenishment size can be determined by:

$$Q = \int_0^t (R + ae^{b\Upsilon})\, d\Upsilon = Rt + \frac{a}{b}[e^{bt} - 1] \quad . \tag{4.4.3-2}$$

The average carrying inventory is given by:

$$I_1(t) = \frac{1}{t}\int_0^t \left[Rt + \frac{a}{b}[e^{bt} - 1] - [(R + ae^{b\Upsilon})\Upsilon] \right] d\Upsilon \tag{4.4.3-3}$$

which simplifies to

$$I_1(t) = \frac{Rt}{2} - \frac{a}{b} + \frac{ae^{bt}}{tb^2} \quad . \tag{4.4.3-4}$$

The average replenishment size is

$$I_3(t) = \frac{1}{t} \quad ; \tag{4.4.3-5}$$

The average number of units perishing is

$$I_4(t) = \frac{1}{t}\int_0^t ae^{b\tau}\, d\tau = \frac{a}{bt}[e^{bt} - 1] \;; \tag{4.4.3-6}$$

and the average total cost equation is

$$C(t) = \frac{C_1Rt}{2} + \frac{C_1ae^{bt}}{tb^2} - \frac{C_1a}{b} + \frac{C_3}{t} + \frac{C_4ae^{bt}}{bt} - \frac{C_4a}{b^2t} \;. \tag{4.4.3-7}$$

This euqation can also be written as:

$$C(t) = \frac{C_1Rt}{2} + \frac{C_3}{t} + \frac{(C_1 + bC_4)ae^{bt}}{tb^2} - \frac{C_4a}{bt} - \frac{C_1a}{b} \,. \tag{4.4.3-8}$$

Again by differentiating C(t) optimal t can be found.

$$\frac{d}{dt}C(t) = \frac{C_1R}{2} - \frac{C_3}{t^2} + \frac{C_4a}{bt^2} + \frac{a\,(C_1 + bC_4)e^{bt}[bt-1]}{t^2b^2} = 0 \tag{4.4.3-9}$$

$$t^* = \left[\frac{C_3}{.5C_1R + C_4abe^{bt} + a(C_1 + bC_4)e^{bt}\,[bt-1]/t^2b^2}\right]^{\frac{1}{2}} \tag{4.4.3-10}$$

The optimal replenishment size, and the optimal cost can be determined by substituting t^* into equations (4.4.3-2) and (4.4.3-8) respectively.

4.4.4 Case 4: Quadratic Perishability Rate

Let h(t) be a quadratic function indicating number of units perishing per unit of time during the inventory replenishment cycle. Therefore, R' is equal to:

$$R' = R + a + bt + ct^2 \tag{4.4.4-1}$$

The equation for replenishment size then becomes equal to

$$Q = \int_0^t R' d\tau = Rt + at + \frac{bt^2}{2} + \frac{ct^3}{3} \tag{4.4.4-2}$$

The average carrying inventory is obtained by

$$I_1(t) = \frac{1}{t}\int_0^t \left[\left[Rt + at + \frac{bt^2}{2} + \frac{ct^3}{3}\right] - R'\right] d\tau \tag{4.4.4-3}$$

which simplifies to

$$I_1(t) = \frac{Rt}{2} + \frac{at}{2} + \frac{bt^2}{6} + \frac{ct^3}{12} \quad . \tag{4.4.4-4}$$

The average number of replenishments is

$$I_3(t) = \frac{1}{t} , \tag{4.4.4-5}$$

the average number of units perishing is

$$I_4(t) = \frac{1}{t}\int_0^t (a + b\tau + c\tau^2)\, d\tau = a + \frac{bt}{2} + \frac{ct^2}{3} \quad , \tag{4.4.4-6}$$

and the average total cost equation is

$$C(t) = \frac{C_1Rt}{2} + \frac{C_3}{t} + \frac{[C_1t + 2C_4]\,a}{2} + \frac{[C_1t + 3C_4]\,bt}{6} + \frac{[C_1t + 4C_4]\,ct^2}{12} \quad . \tag{4.4.4-7}$$

Optimal inventory cycle time can now be determined by differentiating $C(t)$ and setting it equal to zero.

$$\frac{d}{dt}\,C(t) = \frac{C_1R}{2} - \frac{C_3}{t^2} + \frac{aC_1}{2} + \frac{(2C_1t + 3C_4)b}{3} + \frac{(3C_1t - 8C_4)ct}{4}$$

(4.4.4-8)

$$t^* = \left[\frac{2C_3}{C_1R + aC_1 + \frac{b(C_1t + 3C_4)}{3} + \frac{ct(3C_1t + 8C_4)}{6}}\right]^{1/2}$$

(4.4.4-9)

Q^* may be found by substituting t^* from the above equation into equation (4.4.4-2).

4.5 Lot Size Model I-d

By adopting and modifying a recent inventory model by Shih [102], an interesting lot size perishable inventory model is obtained. He combined the EOQ System with probabilistic percentage defective items in the lot.

Assume a percentage of the order quantity perishes during the inventory cycle time. This would be similar in concept as Model I-C, Case 1. In order to avoid shortages, one must order a large enough quantity so that the demand can be met during the inventory cycle time. For this model assume a periodic inventory model so that the perishing costs will be calculated at the end of the cycle.

The average carrying inventory during an inventory cycle is equal to

$$(1 - a)\,q/2 + aq = (1 + a)q/2. \qquad (4.5\text{-}1)$$

The first term is the average carrying inventory of non-perished items, and the second, the average carrying inventory of perished units. The length of an inventory cycle is given by

$$[(1-a)q]/R \tag{4.5-2}$$

and hence, the holding cost per inventory cycle is given by

$$\frac{C_1}{2R}(1 + a)(q)(1 - a)(q) = \frac{C_1}{2R} q^2 (1 - a^2) \tag{4.5-3}$$

Replenishment cost is equal to C_3, and perishing cost is equal to

$$C_4aq \tag{4.5-4}$$

The total inventory cycle cost function is therefore equal to

$$K(q) = \frac{C_1}{2R} q^2 (1 - a^2) + C_3 + C_4aq \tag{4.5-5}$$

In order to obtain the total cost per unit time C(q), one must divide K(q), the total cost per cycle, by the mean of the inventory cycle time. Assume a is a random variable with probability distribution function of g(a). Then by utilizing equation (4.5-2), the mean cycle length can be determined.

$$\text{Mean cycle length} = \int_0^1 [(1 - a)q/R]g(a)da \tag{4.5-6}$$

$$= q/R\,[1 - \bar{a}],$$

where $\bar{a}$ is the mean of a.

The average total inventory cost can now be determined by dividing K(q) by equation (4.5-6).

$$C(q) = \frac{C_1R}{2R}q^2\left[\frac{1 - a^2}{q(1 - \bar{a})}\right] + \frac{C_3R}{q(1 - \bar{a})} + \frac{C_4aR}{q(1 - \bar{a})} \tag{4.5-7}$$

C(q) is also a random variable, since a is a random variable. Therefore, the expected average total cost for this model is given by

$$E[C(q)] = \frac{R}{q(1 - a)}\int_0^1\left[\frac{C_1q^2}{2R}(1 - a^2) + C_3 + C_4a\right]g(a)da \tag{4.5-8}$$

which simplifies to

$$E[C(q)] = \frac{C_3R}{q(1 - \bar{a})} + \frac{C_4R\bar{a}}{q(1 - \bar{a})} + \frac{RC_1q^2}{2Rq(1 - \bar{a})}\int_0^1 (1-a^2)g(a)da \tag{4.5-9}$$

By differentiating equation (4.5.9) with respect to q, and setting it to zero, the optimal lot size can be determined.

$$\frac{d}{dq}E[C(q)] = -\frac{C_3R}{(1 - \bar{a})q^2} - \frac{C_4R\bar{a}}{(1 - \bar{a})q^2} + \frac{C_1\int_0^1 {}_0(1 - a^2)g(a)da}{2(1 - \bar{a})} = 0 \tag{4.5-10}$$

$$q^* = \left[\frac{2R(C_3 + C_4\bar{a})}{C_1} \Big/ \int_0^1 (1 - a^2)g(a)da\right]^{\frac{1}{2}} \tag{4.5-11}$$

The second derivative of E [C(q)] with respect to q is given by:

$$\frac{d^2}{dq^2} E\,[C(q)] = \frac{2C_3R}{(1 - \bar{a})q^3} + \frac{2C_4R\bar{a}}{(1 - \bar{a})q^3} \tag{4.5-12}$$

which is positive for every q. Hence, the expected average total cost will be at the minimum for $q = q^*$.

Now, if a is a given constant, then

$$q^* = \left[\frac{2R(C_3 + aC_4)}{C_1(1 - a^2)}\right]^{\frac{1}{2}}. \tag{4.5-13}$$

If a is beta distributed with parameters m, and n, then g(a) and $\bar{a}$ can be written as:

$$g(a) = \frac{(m + n + 1)!}{m!n!} a^m(1-a)^n \tag{4.5-14}$$

$$\bar{a} = \frac{m+1}{m+n+2} \tag{4.5-15}$$

Then by substituting these into the denominator of equation (4.5-11), that is,

$$\int_0^1 (1-a^2)\, g(a)da = 1 - \frac{(m+1)(m+2)}{(m+n+2)(m+n+3)} \tag{4.5-16}$$

The optimal q is obtained

$$q^* = \left[\frac{2R(C_3+\bar{a}C_4)/C_1}{1 - \frac{(m+2)\bar{a}}{(m+n+3)}}\right]^{\frac{1}{2}} \tag{4.5-17}$$

In this chapter various lot size Models for perishable items have been considered. The inventory carrying cost has been applied to all the units in stock, whether perished or not, in addition to the perishability cost which is charged to the individual items that perish at the end of the inventory cycle time. In most of the models it is necessary to obtain the optimal inventory cycle time through a numerical technique. An algorithm is devised to accomplish this requirement. In this class of models, when perishability rate is constant, by adjusting the inventory carrying cost, one can obtain the equivalent results by utilizing the EOQ analysis.

CHAPTER V

LOT SIZE SYSTEM--MODEL II

In this model the inventory carrying cost is charged on all the non-perished units in inventory, and the perishability costs are charged whenever a unit perishes. This model may be looked upon as a "continuous review" inventory model, as to the "periodic review" of model I. This class of models is the one that has been addressed more extensively in the literature, and the methodology used in this section is due primarily to Shah [87] and Aggarwal [3]. In this model, inventory items perish (deteriorate) continuously in time in accordance with some probability distribution function f(t). All the assumptions of Model I holds except for the definition of h(t). For this model h(t) is defined as

$$h(t) = \frac{f(t)}{1 - F(t)} \quad ; \quad t \geqslant 0 \tag{5-1}$$

where F(t) is the cumulative distribution function of f(t).

In this instance h(t) is the instantaneous or age-specific deterioration rate function of an item. This means that h(t)dt indicates the probability of perishability of an item during the period (t, t + dt), given that it has not failed prior to t. h(t) is called the hazard function in reliabiliy terminology. The cumulative deterioration rate function is given by

$$H(t) = \int_0^t h(\tau)d\tau = -\ln (1 - F(t)). \tag{5-2}$$

Solving for $F(t)$,

$$F(t) = 1 - e^{-H(t)} \qquad (5.3)$$

Therefore, the total percentage of items deteriorating by a given time t, can be determined through equation (5-3).

An inventory cycle for Model II inventory systems is depicted in Figure 6. The order quantity at the beginning of the inventory cycle must be sufficient for the real demand, RT, plus the amount of items that will perish during T.

Let $Q(t)$ denote the inventory level of the system at time t ($0 \leq t \leq T$). $Q(t)$ is a function of the demand rate R and the deterioration function $h(t)$. The differential equation that describes the instantaneous states of $Q(t)$ over the inventory cycle $(0,T)$ is given by

$$\frac{d}{dt} Q(t) = -Q(t)\, h(t) - R \; ; \qquad 0 \leq t \leq T \qquad (5\text{-}4)$$

Equation (5-4) indicates that during a small interval of time dt, the level of inventory will decrease by an amount equal to the sum of real demand and the number of units that perish. The number of units that perish is a function of the level of non-perished units in inventory at time t. Now, by rewriting equation (5-4) as

$$Q'(t) + h(t)Q(t) = -R, \qquad 0 \leq t \leq T \qquad (5\text{-}5)$$

and letting

$$u(t) = \exp\,[\quad h(t)dt] \quad . \qquad t \geq 0 \qquad (5\text{-}6)$$

equation (5-4) can be solved for $Q(t)$,

$$Q(t) = \frac{1}{u(t)} \left[\int_0^t -Ru(x)dx + k \right] \quad , \qquad 0 \leq t \leq T \qquad (5\text{-}7)$$

where k is a constant of integration. Let

$$U(t) = \int_0^t u(x)dx \; ; \tag{5-8}$$

Equation (5-7) may now be written as

$$Q(t) = \frac{1}{u(t)} \left[-RU(t) + k \right] \qquad 0 \leq t \leq T \tag{5-9}$$

The value of the constant of integration, k, can be found at boundary conditions. That is, when t = 0, Q(t) = Q, the initial inventory (lot size); and when t = T, Q(t) = 0, since by definition the inventory is depleted. Therefore, the initial inventory is equal to

$$Q = Q(0) = RU(T) = R \int_0^T u(t)dt. \tag{5-10}$$

Equation (5-9) can now be written as

$$Q(t) = \frac{R}{u(t)} \quad [U(T) - U(t)] \qquad 0 \leq t \leq T. \tag{5-11}$$

By utilizing equation (5-11) the average inventory can be calculated by

$$I_1(T) = \frac{1}{t} \int_0^T Q(t)dt \tag{5-12}$$

which reduces to

$$I_1(T) = \frac{R}{t} \int_0^T \frac{U(T) - U(t)}{u(t)} dt \; . \tag{5-13}$$

The average number of items perishing is given by

$$I_4(T) = (Q - RT)/T. \tag{5-14}$$

Thus the total average cost for this model is obtained by

$$K(T) = \frac{C_1R}{T} \int_0^T \left[\frac{U(T)-U(t)}{u(t)} \right] dt + \frac{C_3}{T} + \frac{C_4R}{T} U\,(T) - C_4R \tag{5-15}$$

The optimal inventory cycle time, can be found by differentiating K(T) with respect to T, and setting the result equal to zero.

$$\frac{d}{dT} K(T) = -\frac{C_1R}{T^2} \int_0^T \left[\frac{U(T)-U(t)}{u(t)}\right]dt + \frac{C_1R}{T} \int_0^T \frac{u(T)}{u(t)} dt - \frac{C3}{T^2}$$

$$+ \frac{C_4}{T} u(T) - \frac{C_4R}{T^2} U(T) = 0 \qquad (5\text{-}16)$$

Rewriting this equation, by multiplying by $(-T^2/R)$ yields:

$$C_1 \int_0^T \left[\frac{U(T) - U(t) - Tu(T)}{u(t)}\right]dt + C_4 [U(T) - Tu(T)] + \frac{C_3}{R} = 0. \quad (5\text{-}17)$$

The optimal value of T, can be determined iteratively for various functions of h(t), by utilizing equation (5-17). Several functions of h(t) will now be considered in the following sections.

5.1 Case 1-a: Constant Perishability Rate Function h(t) = a

Since h(t) is constant, by definition of perishability equation (5-1), this implies that f(t) is exponentially distributed. That is,

$$f(t) = ae^{-at}; \text{ and } F(t) = 1 - e^{-at} \qquad t \geqslant 0.$$

Using equations (5-6) and (5-8) the following can be written:

$$u(t) = \exp [at] \qquad (5.1\text{-}1a)$$

$$U(t) = \frac{1}{a} [\exp(at) - 1] \qquad (5.1\text{-}2a)$$

Hence, the average carrying inventory can be calculated by using equation (5-13).

$$I_1(T) = \frac{R}{aT} \int_0^T \left[\frac{\exp(aT) - \exp(at)}{\exp(at)}\right]dt \qquad (5.1\text{-}3a)$$

which simplifies to:

$$I_1(T) = \frac{R}{aT} \int_0^T \left[\exp[a(T - t)] - 1\right] dt = \frac{R}{a^2T} [e^{aT} - aT - 1] . \qquad (5.1\text{-}a)$$

The inventory lot size, Q, can be obtained readily through equations (5-10) and (5.1-2a); and the average total perishing per cycle is obtained through equation (5-14). The total average cost function then becomes

$$K(T) = \frac{C_1R}{a^2T}[e^{aT} - aT - 1] + \frac{C_4Ra}{a^2T}[e^{aT} - aT - 1] + \frac{C_3}{T} \qquad (5.1\text{-}5a)$$

which simplifies to

$$K(T) = \left(\frac{C_1 + aC_4}{a^2}\right)\left(\frac{R}{T}\right)(e^{aT} - aT - 1) + \frac{C_3}{T} \qquad (5.1\text{-}6a)$$

By using equation (5-17) or differentiating (5.1-6a) with respect to T, the optimal inventory cycle time, T^* can be determined.

$$\frac{d}{dT}K(T) = \left(\frac{C_1 + aC_4}{a^2}\right)\left[\frac{-R}{T^2}(e^{aT} - aT - 1) + \frac{R}{T}(ae^{aT} - a)\right] - \frac{C_3}{T^2} = 0. \qquad (5.1\text{-}7a)$$

Rewriting equation (5.1-7a) as:

$$R\left(\frac{C_1 + aC_4}{a^2}\right)(aTe^{aT} - e^{aT} - 1) = C_3\ , \qquad (5.1\text{-}8a)$$

and simplifying

$$e^{aT}(aT - 1) = \frac{a^2C_3}{R(C_1 + aC_4)} - 1\ . \qquad (5.1\text{-}9a)$$

An implicit function of T can be written as

$$T = \frac{1}{a}\left\{\left[\frac{a^2C_3}{R(C_1 + aC_4)} - 1\right]e^{-aT} + 1\right\} \qquad (5.1\text{-}10a)$$

The form of this function is given in Figure 15, where f(T) is equal to the right hand side of equation (5.1-10a).

If in equation (5.1-9a), e^{aT} is approximated by $1 + aT + \frac{a^2T^2}{2}$,

then by rewriting equation (5.1-9a) the following simple equation is obtained:

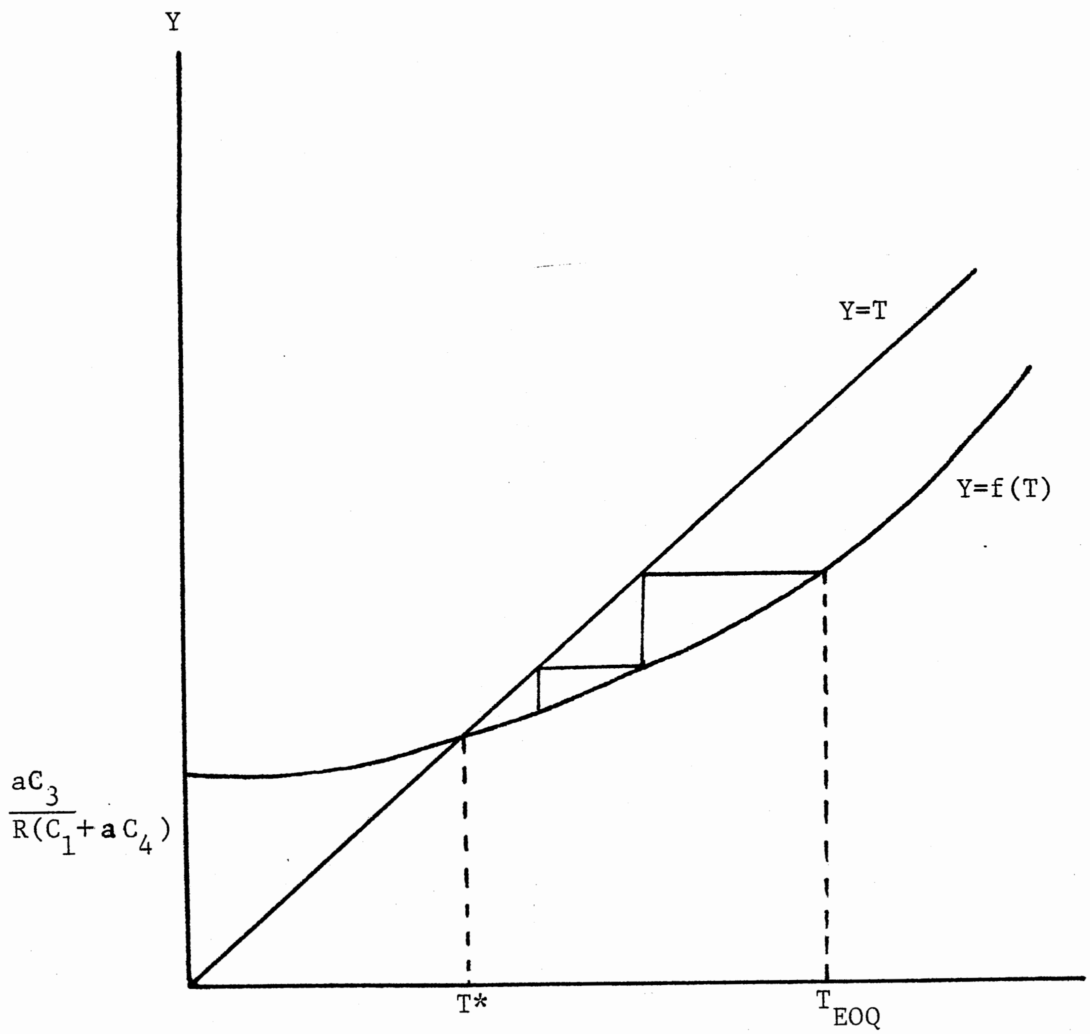


Figure 15. Inventory Cycle Function of a Lot-Size Model with Constant Perishability Rate

$$T^* = \left[\frac{2C_3}{R(C_1 + aC_4)} - aT^3\right]^{\frac{1}{2}} \qquad (5.1\text{-}11a)$$

Note that if a = 0 in (5.1-11a) the classical Wilson's formula for T^* is obtained. Equation (5.1-11a) is much simpler than equation (5.1-10a) and can be easily remembered. T^* can be initially approximated as

$$T^* = \left[\frac{2C_3}{RC_1+RC_4a}\right]^{\frac{1}{2}} \qquad (5.1\text{-}12a)$$

The inventory lot size and the total number of units perishing are given by:

$$Q^* = R/a\ [\exp(aT) - 1], \qquad (5.1\text{-}13a)$$

and

$$D(T) = R/a[\exp(aT)-aT-1] \qquad (5.1\text{-}14a)$$

5.1' Case 1-b: Constant Perishability Rate Function h(t) = a (approximate model)

In some models in the literature the average carrying inventory is approximated by Q(0)/2. By using equation (5.1-13a) the average inventory then is equal to:

$$I_1(T) = \frac{Q(0)}{2} = \frac{R}{2a}[\exp(aT) - 1]; \qquad (5.1'.1\text{-}b)$$

and the total number of units perishing in an inventory cycle is

$$D(T) = R/a[\exp(aT)-1]-RT. \qquad (5.1'.2\text{-}b)$$

The total average cost function may now be written as:

$$K(T) = C_1I_1(T) + \left[C_3 + C_4D(T)\right]/T \qquad (5.1'.3\text{-}b)$$

which for this case is equal to

$$K(T) = \frac{C_1R}{2a}\,[e^{aT} - 1] + \left\{ C_3 + C_4\left[\frac{R}{a}\,(e^{aT} - 1) - RT\right]\right\}/T \qquad (5.1'.4\text{-b})$$

Now by taking the derivative of K(T) and setting it equal to zero, the optimal T can be obtained.

$$\frac{d}{dt}\,K(T) = \frac{C_1R}{2}e^{aT} - \frac{C_3}{T^2} - \frac{C_4R}{aT^2}\,[eaT - 1] + \frac{C_4R}{T}e^{aT} = 0 \qquad (5.1'.5\text{-b})$$

This equation can be simplified and rewritten as

$$\frac{C_1R}{2}\,T^2 + (C_4R/a)(e^{aT} + aT - 1) - C_3\,e^{-aT} = 0 \qquad (5.1'.6\text{-b})$$

from which an implicit function of T may be obtained,

$$T^* = \left[\frac{C_3e^{-aT} + (C_4R/a)(1-aT-e^{-aT})}{.5C_1R}\right]^{\frac{1}{2}} \qquad (5.1'.7\text{-b})$$

By approximating the exponential in equation (5.1'.6-b), the following equation is obtained which is much simpler than equation (5.1'.7-b).

$$T^* = \left[\frac{2C_3e^{-aT}}{R(C_1+aC_4)}\right]^{\frac{1}{2}} \qquad (5.1.8\text{-b})$$

Note as a approaches zero, $T^* = \sqrt{2C_3/C_1R}$ which is the classical result for the EOQ Model.

5.2 Case 2: Weibull Distribution Deterioration -- $h(t) = abt^{b-1}$

Covert and Philip [21] in the development of their model assumed arbitrarily that the average inventory on hand is equal to one half of the initial level of inventory although they recognized that the inventory depletion curve is not a straight line. Aggarwal [3] also recognizes the same problem in the analysis of Shah's model [87] without actually deriving the exact equations. Following Covert and Philip [21], ignoring their assumption of linearity, and utilizing equations (5-6) and (5-8) the following equations may be written:

$$u(t) = \exp(at^{b}) \tag{5.2-1}$$

$$U(t) = \int_0^t e^{ax^{b}} dx = \int_0^t \sum_{n=o}^{\infty} \frac{a^n x^{nb}}{n!} dx \tag{5.2-2}$$

Interchanging the order of integration and summation

$$U(t) = \sum_{n=o}^{\infty} \int_0^t \frac{a^n x^{nb}}{n!} dx \tag{5.2-3}$$

and then integrating

$$U(t) = \sum_{n=o}^{\infty} \frac{a^n t^{(nb + 1)}}{(nb + 1)n!} = W(a,b,t) \tag{5.2-4}$$

The inventory lot size using equation (5-10) is then equal to

$$Q = Q(0) = RU(T) = RW(a,b,T) \tag{5.2-5}$$

and hence the total number of units perishing is determined by

$$D(T) = Q - RT \tag{5.2-6}$$

The average carrying inventory may be obtained by utilizing equation (5-13), that is,

$$I_1(T) = \frac{R}{T} \int_0^T \left[\frac{W(a,b,T) - W(a,b,t)}{e^{at^{b}}} \right] dt \tag{5.2-7}$$

This can be written as:

$$I_1(T) = \frac{R}{T} \left\{ W(a,b,T)\ W(-a,b,T) - \int_0^T \left[\sum_{n=o}^{\infty} \frac{a^n t^{(nb + 1)}}{(nb + 1)n!} \right] \left[\sum_{k=o}^{\infty} \frac{(-1)^k a^k t^{kb}}{k!} \right] dt \right\} \tag{5.2-8}$$

The integral can be written as:

$$\int_0^T \sum_{n=o}^{\infty} \sum_{k=o}^{\infty} \frac{(-1)^k a^{(n + k)} t^{(nb + kb + 1)}}{(nb + 1)\ n!\ k!} dt . \tag{5.2-9}$$

Interchanging the integration and the summations, and then performing integration, the following equation is obtained.

$$\sum_{n=o}^{\infty} \sum_{k=o}^{\infty} \frac{(-1)^k a^{(n+k)} T^{nb+kb+2}}{(nb+kb+2)(nb+1)\, n!\, k!} \qquad (5.2\text{-}10)$$

Now, by substituting the above equation and equation (5.2-4) into equation (5.2-8) one obtains

$$I_1(T) = \frac{R}{T}\left\{\sum_{n=o}^{\infty} \sum_{k=o}^{\infty} \frac{(-1)^k a^{n+k} T^{nb+kb+2}}{(nb+1)(kb+1)\, n!\, k!} - \frac{(-1)^k a^{n+k} T^{nb+kb+2}}{(nb+1)(nb+kb+2)\, n!\, k!}\right\}$$

(5.2-11)

which simplifies to

$$I_1(T) = \frac{R}{T}\left\{\sum_{n=o}^{\infty} \sum_{k=o}^{\infty} \frac{(-1)^k a^{n+k} T^{nb+kb+2}}{(kb+1)(kb+nb+2)\, n!\, k!}\right\}. \qquad (5.2\text{-}12)$$

By expanding the terms of the double summation and adding the terms of similar exponents, equation (5.2-12) simplifies to

$$I_1(T) = \frac{R}{T}\left\{\sum_{m=o}^{\infty} \prod_{j=o}^{m} \frac{(ab)^m T^{mb+2}}{(mb+2)(mb-j)}\right\}; \qquad (5.2\text{-}13)$$

The cost function may now be written as:

$$K(T) = C_1 I_1(T) + \frac{C_3}{T} + \frac{C_4}{T} W(a,b,T) - C_4 R \qquad (5.2\text{-}14)$$

To find T^*, equation (5.2-14) may be differentiated or the respective formulas may be substituted into equation (5-16).

$$\frac{d}{dT} K(T) = C_1 R \sum_{m=o}^{\infty} \prod_{j=o}^{m} \frac{(mb+1)(ab)^m T^{mb}}{(mb+2)(mb-j)} - \frac{C3}{T^2}$$

$$+ C_4 R \sum_{n=o}^{\infty} \frac{a^n (nb) T^{nb-1}}{(nb+1) n!} = 0 \qquad (5.2\text{-}15)$$

Thus, the optimal T is equal to

$$T = \left[\frac{C_3}{C1R\left[\sum_{m=o}^{\infty} \prod_{j=o}^{m} \frac{(mb+1)(ab)^m T^{mb}}{(mb+2)(mb-j)}\right] + C_4R\left[\sum_{n=0}^{\infty} \frac{a^n (nb) T^{nb-1}}{(nb+1)n!}\right]} \right]^{\frac{1}{2}}$$

(5.2-16)

T^* can now be determined numerically as accurately as desired and thereby the optimal inventory lot size can be determined through equation (5-10).

5.3 Case 3: Rayleigh Distribution Deterioration -- $h(t) = at$

A special case of the Weibull distribution is the Rayleigh distribution which has a linearly increasing perishability rate. Using the same methodology for this case as before, the following can be written immediately:

$$u(t) = \exp(at^2/2) \tag{5.3-1}$$

Let $A=a/2$, and rewrite the equation (5.3-1) as

$$u(t) = \exp(At^2) \tag{5.3-1'}$$

Equation (5-8) can now be written as

$$U(t) = \int_o^t e^{Ax^2}dx = \int_o^t \sum_{n=o}^{\infty} \frac{A^n x^{2n}}{n!} dx \tag{5.3-2}$$

which reduces to

$$U(t) = \sum_{n=o}^{\infty} \frac{A^n t^{2n+1}}{(2n+1)n!} = W(A,2,t) \tag{5.3-3}$$

The inventory lot size then is equal to

$$Q = Q(0) = RU(T) = RW(A,2,T) \tag{5.3-4}$$

which is obtained by utilizing equation (5.2-5). The total number of units perishing is given by

$$D(T) = Q-RT \tag{5.3-5}$$

By utilizing equation (5.2-13), the average carrying inventory becomes

$$I_1(T) = \frac{R}{T}\left\{ \sum_{m=o}^{\infty} \prod_{j=o}^{m} \frac{A^m T^{2m+2}}{(2m+2)(mb-j)} \right\} \tag{5.3-6}$$

The cost function for this case is equal to:

$$K(T) = C_1 I_1(T) + \frac{C_3}{T} + \frac{C_4}{T} \; W(A,2,T) - C_4 R \tag{5.3-7}$$

and the optimal T is given by:

$$T^* = \left[\frac{C_3}{C_1 R \left(\sum_{m=o}^{\infty} \prod_{j=o}^{m} \frac{(2m) A^m \; T^{2m}}{(2m+2)(2m-j)} \right) + C_4 R \left(\sum_{n=0}^{\infty} \frac{A^n (2n) T^{2n-1}}{(2n+1) n!} \right)} \right]^{\frac{1}{2}} \tag{5.3-8}$$

By finding T^* from (5.3-8), the optimal lot size and optimal cost can be determined by utilizing equations (5.3-4) and (5.3-7).

In this chapter a general methodology is developed for determining the optimal inventory characteristics of items that are subject to a given perishability distribution function. Corrections are made in the determination of the total carrying inventory with respect to the models of other researchers. Results are also obtained for the case of the Rayleigh distribution, which can be considered as a special case of a Weibull distribution function. The Rayleigh distribution has a special property of having a linearly increasing perishability function.

CHAPTER VI

FINITE PRODUCTION RATE INVENTORY CONTROL SYSTEM

The behavior of the inventory level for a finite production rate model is depicted in Figure 16. The inventory level at the beginning, and the end of the inventory cycle is zero. Following Misra [58], and Shah and Jaiswal [88], let T be the inventory cycle length, then the inventory cycle will consist of two segments.

During the first segment $(0,T_1)$, the production occurs at a rate of p(t) units per time unit, and demand occurs at a rate of d(t) units per time unit. In the second segment (T_1,T) there is no production and demand is satisfied at a rate of d(t) from the inventory.

Let h(t) be the instantaneous deterioration rate function for the items in inventory, and let Q(t) be the inventory level at time t(0 t T). The change in the inventory level during a small interval of time can be represented mathematically as:

$$-d\,Q(t) = Q(t)h(t)dt + d(t)dt-p(t)dt, \qquad 0 \leq t \leq T_1 \qquad (6\text{-}1)$$

and

$$-d\,Q(t) = Q(t)\,h(t)dt + d(t)dt \qquad T_1 \leq t \leq T \qquad (6\text{-}2)$$

These equations can be rewritten as

$$\frac{d}{dt}\,Q_1(t) + h(t)\,Q_1(t) = (p(t) - d(t)), \qquad 0 \leq t \leq T_1 \qquad (6\text{-}3)$$

$$\frac{d}{dt}\,Q_2(t) + h(t)\,Q_2(t) = -d(t) \qquad T_1 \leq t \leq T \qquad (6\text{-}4)$$

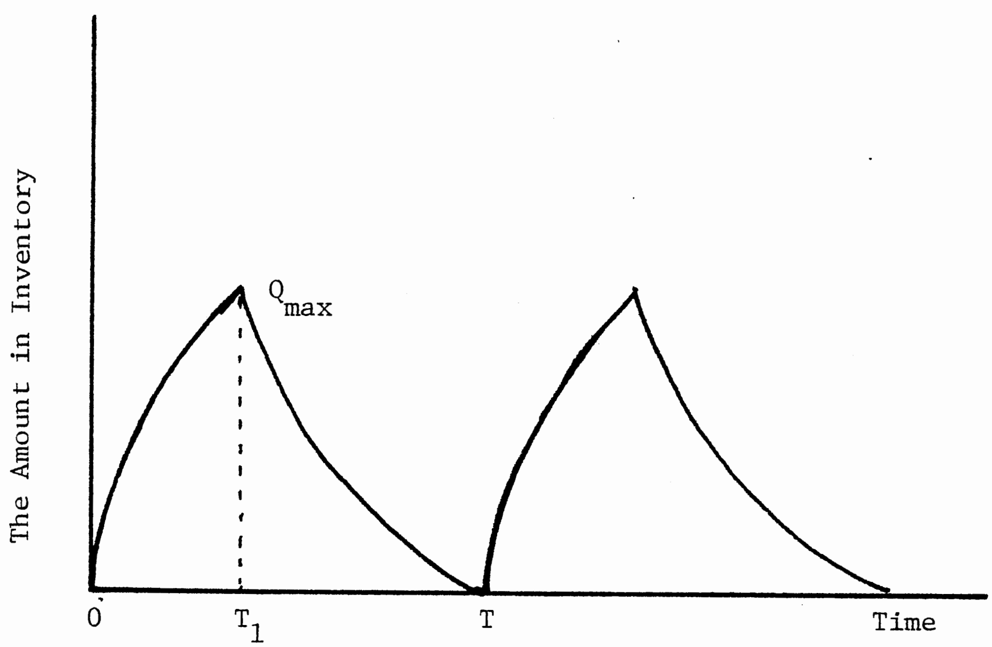


Figure 16. Finite Production Rate Inventory Model with No Shortages

The solutions to the general first order linear differential equation is given by Boyce and DiPrima [12]. For the above equations they are:

$$Q_1(t_1) = \frac{\int_o^{t_1} [(p(t)-d(t))]\exp\left(\int h(t)dt\right)dt+k_1}{\exp\left(\int_0^{t_1} h(t)dt\right)} ; \qquad 0 \leq t_1 \leq T_1 \qquad (6\text{-}5)$$

and

$$Q_2(t_2)= \frac{\int_{T_1}^{t_2} (-d(t)) \exp\left(\int h(t)d(t)\right) dt+k_2}{\exp\left(\int_{T_1}^{t_2} h(t)dt\right)} ; \qquad T_1 \leq t_2 \leq T \qquad (6\text{-}6)$$

The values for the constants of integration k_1 and k_2 are determined by using the boundary conditions. That is, at $t_1=0$, $Q_1(0)=0$, and at $t_2=T_1$, $Q_2(T_1)=Q_{max}$. Applying these boundary conditions results in $k_1=0$, and $k_2=Q_{max}$. Equations (6-5) and (6-6) can be rewritten as:

$$Q_1(t_1) = \frac{\int_0^{t_1} (p(t)-d(t)) \exp\left(\int h(t)dt\right)dt}{\exp\left(\int_o^{t_1} h(t)dt\right)} \qquad 0 \leq t \leq T_1 \quad (6\text{-}7)$$

$$Q_2(t_2) = \frac{\int_{T_1}^{t_2} (-d(t)) \exp\left(\quad h(t)dt\right) dt+Q_{max}}{\exp\left(\int_{T_1}^{t_2} h(t)dt\right)} \qquad T_1 \leq t_2 \leq T \quad (6\text{-}8)$$

In order to evaluate equations (6-7) and (6-8), specific cases must be considered.

6.1 Case 1: $p(t)$, $d(t)$ and $h(t)$ Are Constants

In this case the production rate and demand rate are constant, and items in inventory are being perished according to the exponential distribution function. Let $p(t)=p$, $d(t)=d$, and $h(t)=h$. Then substituting these values into equations (6-7) and (6-8) yields:

$$Q_1(t_1) = \frac{\int_0^{t_1} (p-d) \exp\left(\int hdt\right) dt}{\exp\left(\int_0^{t_1} hdt\right)} ; \quad 0 \leq t_1 \leq T_1 \qquad (6.1\text{-}1)$$

$$Q_2(t_2) = \frac{\int_T^{t_2} (-d) \exp\left(\int hdt\right) dt + Q_{max}}{\exp\left(\int_{T_1}^{t_2} hdt\right)} ; \quad T_1 \leq t_2 \leq T \qquad (6.1\text{-}2)$$

which can be solved and simplified further as:

$$Q_1(t_1) = \frac{p-d}{h}\left[1-\exp(-ht_1)\right] \qquad 0 \leq t_1 \leq T_1 \qquad (6.1\text{-}3)$$

$$Q_2(t_2) = \frac{-d}{h} + Q_{max}\, e^{h(T_1-t_2)} \qquad T_1 \leq t_2 \leq T \qquad (6.1\text{-}4)$$

Since at $t_2=T$, $Q_2(T)=0$, this implies that

$$Q_{max} = \frac{d}{h} e^{h(T-T_1)}, \qquad (6.1\text{-}5)$$

and equation (6.1-4) becomes:

$$Q_2(t_2) = \frac{d}{h}\left\{\exp[h(T-t_2)]-1\right\} \qquad T_1 \leq t_2 \leq T \qquad (6.1\text{-}6)$$

The inventory level $Q(t)$ at the termination of the first segment of the inventory cycle is equal to the initial inventory level of segment two. Therefore, by using equations (6.1-3) and (6.1-6), T_1 can be determined;

$$\frac{p-d}{h}\left[1-\exp(-hT_1)\right] = \frac{d}{h}\left\{\exp[h(T-T_1)]-1\right\} \qquad (6.1\text{-}7)$$

$$T_1 = \frac{1}{h} \ln\left[1 + \frac{d}{p}\left(e^{hT} - 1\right)\right] \qquad (6.1\text{-}8)$$

Since production rate is a constant rate of p, the production lot size and hence the number of items that deteriorate is determined readily. Let q be production lot size, then

$$q = pT_1 = \frac{p}{h} \ln \left[1 + \frac{d}{p} (e^{hT} - 1)\right] \quad . \qquad (6.1\text{-}9)$$

The number of items perishing, D(t), during the cycle time T is determined by:

$$D(T) = q - dT = \frac{p}{h} \ln \left[1 + \frac{d}{p} (e^{hT} - 1)\right] - dT \ . \qquad (6.1\text{-}10)$$

In order to determine the optimal T for this case two subcases must be analyzed: (a) carrying cost is applied to the average total number of units in stock; (b) carrying cost is applied only to the nonperished items in stock. These subcases are illustrated in Figure 17. To determine the inventory carrying cost during a cycle for each subcase the areas under the curves must be calculated.

6.1.1 Subcase a. Inventory Carrying Cost on Total Units in Inventory

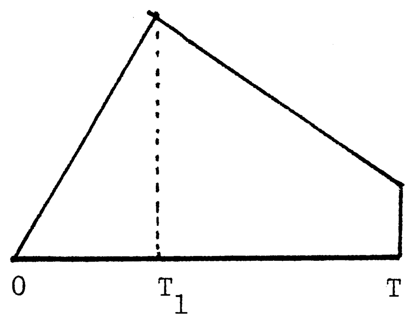
Total carrying inventory during the cycle is given by

$$I_1 = \tfrac{1}{2}(p-d)\, T_1 + \tfrac{1}{2}(d)(T-T_1) + D(T)(T-T_1). \qquad (6.1.1\text{-}1)$$

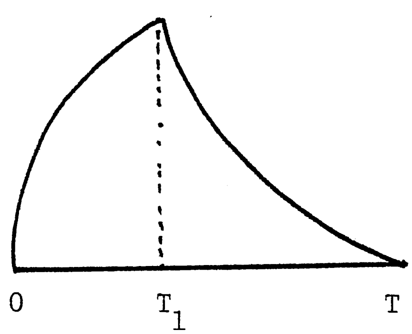
which can be written as

$$I_1 = \frac{dT}{2} - dT^2 + T_1 \left\{ pT + dT + \frac{p}{2} - d - pT_1 \right\} . \qquad (6.1.1\text{-}2)$$

By substituting equation (6.1-8) into (6.1.1-2), the total carrying inventory is obtained as a function of T, that is,



(a) Total Number of Units in Stock



(b) Non Perished Number of Units in Stock

Figure 17. Inventory Level of a Finite Production Model

$$I_1 = \frac{dT}{2} - dT^2 + \frac{1}{h}\left\{\ln[1 + \frac{d}{p}(e^{hT} - 1)]\right\}\left\{pT + dT + \frac{p}{2} - d\right.$$

$$\left. -p\left[\frac{1}{h} \ln[1 + \frac{d}{p}(e^{hT} - 1)]\right]\right\}. \qquad (6.1.1\text{-}3)$$

The total average cost per unit time equation of the system is:

$$C(T) = \frac{C_1 I_1}{T} + \frac{C_3}{T} + \frac{C_4}{T} D(T) \qquad (6.1.1\text{-}4)$$

Since C(T) is a convex function the optimal cycle time T* can be determined analytically. However, because of the form of the equation an easier method of finding T^* is to use the Fibonacci Search Technique. By determining T^*; T_1 and q can be readily calculated utilizing equations (6.1-8) and (6.1-9).

6.1.2 Subcase b. Inventory Carrying Cost on Non-Perished Units in Inventory

The total carrying inventory during the cycle is given by integrating equations (6.1-3) and (6.1-6), that is,

$$I_1 = \int_0^{T_1} [\frac{p-d}{h}(1-\exp(-ht))]\,dt + \int_{T_1}^{T} [\frac{d}{h}(\exp(h(T-t))-1]dt \qquad (6.1.2\text{-}1)$$

Integrating and collecting similar terms results in:

$$I_1 = \frac{pT_1}{h} - \frac{d}{h}T - \frac{P}{h^2} + \frac{p-d}{h^2} e^{-hT_1} + \frac{d}{h^2} e^{h(T-T_1)} \qquad (6.1.2\text{-}2)$$

Since the value of T_1 as a function of T is known from equation (6.1-8), equation (6.1.2-2) can be rewritten as a function of T only. Equation (6.1.1-4) remains the same for the average total cost function equation of this subcase, and only an appropriate I_1 must be utilized in the equation.

The results that have been derived for this case are equivalent to those of Misra [58] if the series form of the exponential is used, and terms with second and higher powers of h are ignored in the above equations. The optimal cost and cycle time should be obtained through a search technique. Misra [58], through exponential approximation of equivalent of equation (6.1-7), has tried to establish an analytical relationship between T_1 and $(T-T_1)$. His approximations for the resulting quadratic equation implicitly assumes that $h(T-T_1)^2/2$ is approximately zero. By substituting these results into the cost equation, he differentiates and then further simplifies to obtain T_1^* and Q^*. The results that he has obtained for an example problem are incorrect; however, if the proper calculations are made, the results would be "good" or "close" only if T 1. That is, one must normalize inventory parameters in such a way that this condition would hold.

6.1.3 Approximation of Optimal Q

Using the series form of the exponential, logarithms, and ignoring terms with second and higher powers of h under the assumption that $1/h \ll T$, the following results can be obtained. First, rewrite equation (6.1-8) as:

$$T_1 = \frac{1}{h} \ln \left[1 + \frac{d}{p} \left(hT + \frac{h^2T^2}{2}\right)\right] \qquad (6.1.3\text{-}1)$$

Then, expand the logarithm as

$$T_1 = \left(\frac{2}{h}\right) \left[\frac{\frac{d}{p} \left(hT + \frac{h^2T^2}{2}\right)}{2 + \frac{d}{p} \left(hT + \frac{h^2T^2}{2}\right)}\right] \qquad (6.1.3\text{-}2)$$

the approximate optimal T_1^* is then equal to

$$T_1^* = \left[\frac{(2/h)}{1 + \dfrac{2}{\frac{d}{p}\left(T + \frac{hT^2}{2}\right)}}\right] \qquad (6.1.3\text{-}3)$$

The optimal order quantity, Q*, can be determined by:

$$Q^* = pT_1^*. \qquad (6.1.3\text{-}4)$$

6.2 Case 2: d(t) and h(t) are constant, $p(t) = pt^b$

In this case the production rate is increasing according to a polynomial function, demand rate is constant, and perishability of items in inventory are characterized by the exponential distribution. Let $d(t)=d$, $h(t)=h$, and $p(t) = pt^b$, where b is a nonnegative integer. Substituting these values into equations (6-7) and (6-8) and simplifying yields:

$$Q_1(t_1) = \left(\frac{p}{h}\right)\left[\sum_{i=0}^{b} (-1^i) \frac{b!(t_1)^{b-i}}{(b-i)!h^i}\right] - \left(\frac{d}{h}\right)[1-e^{-ht_1}]; \qquad 0 \leq t_1 \leq T_1 \qquad (6.2\text{-}1)$$

$$Q_2(t_2) = \frac{d}{h}\left\{\exp[h(T-t_2)]-1\right\} \qquad T_1 \leq t_2 \leq T \qquad (6.2\text{-}2)$$

Equation (6.2-2) is the same as equation (6.1-6) of the previous model.

Again at $t_1=T_1$ and $t_2=T_1$, equations (6.2-1) and (6.2-2) are equal, therefore:

$$\left(\frac{p}{h}\right)\left[\sum_{i=0}^{b} (-1)^i \frac{b! \quad T_1^{(b-i)}}{(b-i)!h^i}\right] - \left(\frac{d}{h}\right)[1-e^{-hT_1}] = \left(\frac{d}{h}\right)[e^{h(T-T_1)}-1] \qquad (6.2\text{-}3)$$

This simplifies further to:

$$\left(\frac{pe^{hT_1}}{d}\right)\left[\sum_{i=0}^{b} \frac{(-1)ib! \quad T_1(b-i)}{(b-i)! \; h^i}\right] = e^{hT}-1 \tag{6.2-4}$$

Solving equation (6.2-4) for T in terms of T_1 yields

$$T = \frac{1}{h} \ln \left\{ 1 + \left(\frac{pe^{hT_1}}{d}\right)\left[\sum_{i=0}^{b} (-1)^i \frac{b!T_1^{(b-i)}}{(b-i)!h^i}\right] \right\} \tag{6.2-5}$$

In production situations where learning curve effect is present, the replenishment rate p(t) can be approximated closely by adjusting p and b. By utilizing equation (6.2-5), the problem can be solved as a function of T_1 rather than T. Special subcases of interest would be when b=1, or b=2, that is when the production rate is increasing linearly or quadratically. In these special cases equation (6.2-5) reduces to:

$$T = \frac{1}{h}\left\{\ln \; 1 + \left(\frac{pe^{hT_1}}{d}\right)\left(T_1 - \frac{1}{h}\right)\right\} \quad ; \qquad \text{for } b = 1 \tag{6.2-6}$$

$$T = \frac{1}{h} \ln\left\{1 + \left(\frac{p \; e^{hT_1}}{d}\right)\left(T_1^2 - \frac{2T_1}{h} + \frac{2}{h^2}\right)\right\}; \qquad \text{for } b = 2 \tag{6.2-6}$$

Now, the total carrying inventory can be calculated for the respective values of b. For b=1,

$$I_1 = \frac{p}{h^2}\int_0^{T_1}\left(t_1 - \frac{1}{n}\right) dt_1 \quad - \frac{d}{h}\int_0^{T_1} (1-e^{-ht_1}) \, dt_1 +$$

$$\int_{T_1}^{T} \left(\frac{d}{h}\right)(e^{h(T-t2)} - 1) \;\; dt_2 \, . \tag{6.2-7}$$

This simplifies to:

$$I_1 = \frac{p}{h}\left[\frac{T^2}{2} - \frac{T_1}{h}\right] - \left(\frac{d}{h}\right)\left[T_1 + \frac{1}{h}\,(e^{-hT_1} - 1)\right]$$

$$+\left(\frac{d}{h}\right)\left[\frac{e^{h(T-T)}}{h} - \frac{1}{h} - (T-T_1)\right] \tag{6.2-8}$$

By regrouping the similar terms

$$I_1 = \frac{p}{2h} T_1^2 - \frac{pT_1}{h^2} - \frac{d}{h} T + \frac{d}{h^2}\left[e^{-hT_1} [1-e^{hT}]\right] \qquad (6.2\text{-}9)$$

Since the value of T can be determined from (6.2-6), equation (6.2-9) can be solved explicitly as a function of T_1.

The amount of deterioration is given by:

$$D(T_1) = \frac{pT_1^{\,b+1}}{b+1} - dT \qquad (6.2\text{-}10)$$

Hence, the average total cost equation can be determined as a function of T_1; that is,

$$C(T_1) = \frac{C_1 I_1}{T} + \frac{C_3}{T} + \frac{C_4}{T} D(T_1) \qquad (6.2\text{-}11)$$

When b=2, the total carrying inventory is equal to:

$$I_1 - \frac{p}{h} \int_0^{T_1} (t^2 - \frac{2t}{h} + \frac{2}{h^2})\, dt - \frac{d}{h} \int_0^{T_1} (1-e^{-ht})\, dt$$

$$+ \frac{d}{h}\left\{\int_{T_1}^{T} (e^{h(T-t)} - 1)\, dt\right\} \qquad (6.2\text{-}12)$$

This reduces to:

$$I_1 = \frac{p}{h}\left\{\frac{T_1^3}{3} - \frac{T_1^2}{h} + \frac{2T_1}{h^2}\right\} + \frac{d}{h^2} e^{-hT_1} (e^{hT} - 1) - \frac{dT}{h} \qquad (6.2\text{-}13)$$

The deterioration and cost functions remain the same as (6.2-10) and (6.2-11).

6.3 Case 3: d(t) and h(t) are Constant; $p(t) = pq^t$

In this case, the production rate is increasing as an exponential function of time, demand rate is constant, and perishability is according to an exponential distribution function. Let $d(t)=d$, $h(t)=h$, and $p(t)=pq^t$; $p>0$, $q\geqslant 1$. Substituting these values into equations (6-7) and (6-8) and simplifying yields:

$$Q_1(t_1) = \frac{(pe^{-ht_1})}{(h+\ln q)} \, [e^{(h+\ln q)t_2} - 1] - \frac{d}{h}(1-e^{-ht_1}) \quad 0 \leq t_1 \leq T_1 \tag{6.3-1}$$

$$Q_2(t_2) = \frac{d}{h}\left[\exp\,[h(T-t_2)] - 1\right] \qquad T_1 \leq t_2 \leq T \tag{6.3-2}$$

At $t_1=T_1$ and at $t_2=T_1$ equations (6.3-1) and (6.3-2) are equal; therefore, after simplification the following relationship is obtained.

$$T = \frac{1}{h}\ln\left\{\frac{ph}{d(h+\ln q)}(q^{T_1}e^{hT_1} - 1) + 1\right\} \tag{6.3-3}$$

Again, total carrying inventory can be calculated by integrating equations (6.3-1) and (6.3-2), that is:

$$I_1 = \frac{p}{(\ln q+h)}\left\{\frac{1}{\ln q}(e^{\ln qT_1} - 1) + \frac{1}{h}(e^{-hT_1} - 1)\right\} - \frac{d}{h}\left\{T_1 + \frac{1}{h}(e^{-hT_1} - 1)\right\} + \frac{d}{h}\left\{-\frac{e^{hT}}{h}(e^{-hT} - e^{-hT_1}) - (T-T_1)\right\} \tag{6.3-4}$$

which reduces to

$$I_1 = \frac{p}{(\ln q+h)}\left\{\frac{1}{\ln q}(T_1 - 1) + \frac{1}{h}(e^{-hT_1} - 1)\right\} + \frac{d}{h^2}e^{-hT_1}(e^{hT} - 1) - \frac{dT}{h} \tag{6.3-5}$$

By substituting equation (6.3-3) into equation (6.3-5), the total carrying inventory function is obtained as a function of T_1. Equation (6.2-1) for the average total cost function will be used again for deteriming T^*. The optimal production lot size, Q^*, and deterioration, $D^*(T)$ can be determined by:

$$Q^* = \int_{o}^{T^*_1} pq^t dt = \frac{pq^{T_1^*}}{\ln q} \qquad (6.3\text{-}6)$$

$$D(T) = Q^* - dT^* \qquad (6.3\text{-}7)$$

6.4 Case 4: d(t) and p(t) are Constant, $h(t) = \frac{a}{b-t}$

An interesting result is obtained if it is assumed that demand and production rate are costant, and perishability rate function is a specific increasing function. Let $d(t)=d$, $p(t)=p$, and $h(t)= \frac{a}{b-t}$, where a and b are positive; and b is greater than T_{EOQ}. The shape of the deterioration rate function of this case is given in Figure 18.

The integral of h(t) is given by:

$$\int h(t)dt = \int \frac{a}{b-t} dt = -a \ln (b-t) \qquad 0 \leq t \leq b \qquad (6.4\text{-}0)$$

Now, by substituting the respective values for d(t), p(t), and h(t) into equations (6.7) and (6.8) the following equations are obtained.

$$Q_1(t_1) = \frac{(p-d)\ [b^{(1-a)} - (b-t_1)^{(1-a)}]}{(1-a)(b-t_1)^{-a}} \qquad 0 \leq t_1 \leq T_1 \qquad (6.4\text{-}1)$$

$$Q_2(t_2) = \frac{\frac{-d}{1-a}\ [(b-t_2)^{1-a} - (b-T_1)^{1-a}] + Q_{max}}{(b-t_2)^{-a}\ (b-T_1)^{a}} \qquad T_1 \leq t_2 \leq T_2 \qquad (6.4\text{-}2)$$

Since, at $t_2=T$, $Q_2(T)=0$, this implies Q_{max} is equal to:

$$Q_{max} = -\frac{d}{1-a}\ [(b-T)^{1-a} - (b-T_1)^{1-a}] \qquad (6.4\text{-}3)$$

Substituting this equation into equation (6.4-2), and simplifying yields:

$$Q_2(t_2) = \frac{\left(\frac{-d}{1-a}\right)[(b-t_2)^{1-a} - (b-T)^{1-a}]}{(b-t_2)^{-a}\ (b-T_1)^{a}} \qquad T_1 \leq t_2 \leq T \qquad (6.4\text{-}4)$$

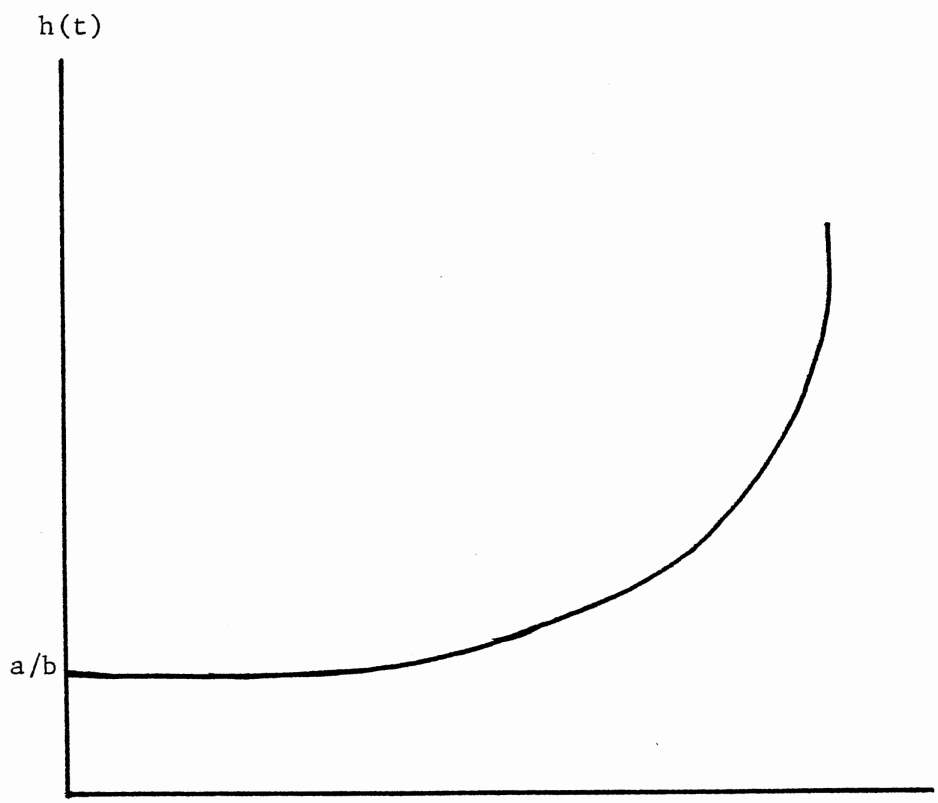


Figure 18. Deterioration Rate When $h(t) = \frac{a}{b-t}$

At $t_1=T_1$ and $t_2=T_1$, equations (6.4-1) and (6.4-4) are equal, therefore T, or T_1 may be determined explicitly, that is,

$$\frac{(p-d)\,[b^{1-a} - (b-T_1)^{1-a}]}{(1-a)(b-T_1)^{-a}} = \frac{(-d)\,[(b-T_1)^{1-a} - (b-T)^{1-a}]}{(1-a)(b-T_1)^{-a}(b-T_1)^{a}}. \quad (6.4\text{-}5)$$

Simplifying further and Solving for T yields

$$T = b - \left\{ (b-T_1)^{1-a} + \frac{(p-d)}{d}\;[b^{1-a}\,(b-T_1)^{a} - (b-T_1)]\right\}^{1/(1-a)} . \quad (6.4\text{-}6)$$

Now, all that remains to be determined is the total carrying inventory, which is obtained by integrating equations (6.4-1) and (6.4-4):

$$I_1 = \int_0^{T_1} \left\{ \left(\frac{p-d}{1-a}\right)[(b-t)^a b^{(1-a)} - (b-t)] \right\} dt$$

$$+ \int_{T_1}^{T} \left\{ \left(\frac{-d}{(1-a)(b-T_1)^{a}}\right)[(b-t) - (b-T)^{1-a}(b-t)^a] \right\} dt \quad (6.4\text{-}7)$$

This reduces to:

$$I_1 = \left[\frac{(p-d)}{(1-a)}\right] \left\{ \left[\frac{b^{(1-a)}}{(1+a)}\right] \left[b^{(1+a)} - (b-T_1)^{(1+a)}\right] - (bT_1 - T_1^2/2) \right\} +$$

$$\left[\left(\frac{d}{1-a}\right)(b-T_1)^{-a}\right] \left\{ \tfrac{1}{2}[(b-T)^2 - (b-T_1)^2] + \frac{(b-T)^{(1-a)}}{1+a}[(b-T)^{(1+a)} - (b-T_1)^{(1+a)}] \right\} \quad (6.4\text{-}8)$$

Again, by substituting T of equation (6.4-6) into this equation and utilizing the cost equation (6.2-11), the optimal T^*_1, can be found. Using this result, T^* can be determined.

6.5 Case 5: d(t) and p(t) are Constant, $h(t) = abt^{b-1}$

In this case the production and demand rate are constant, and the items in inventory perish according to a Weibull distribution function. Let $p(t)=p$, $d(t)=d$, and $h(t)=abt^{b-1}$, where a and b are positive numbers. This is the case of varying rate of deterioration, and the solution is obtained similar to previous cases; therefore:

$$Q_1(t_1) = \frac{\int_o^{t_1} (p-d) \exp (at^b)dt}{\exp (at_1^{\ b})} \quad ; \quad 0 \leq t_1 \leq T_1 \qquad (6.5\text{-}1)$$

$$Q_2(t_2) = \frac{\int_{T_1}^{t_2} -d \exp (at^b)\, dt + Q_{max}}{\exp [a(t^b{}_2 - T_1^{\ b})]} \quad ; \quad T_1 \leq t_2 \leq T \qquad (6.5\text{-}2)$$

at $t_2=T$, $Q_2(t_2)$ in equation (6.5-2) is equal to zero, hence

$$Q_{max} = \int_{T_1}^{T} d \exp (at^b)\, dt \qquad (6.5\text{-}3)$$

Substituting this in equation (6.5-2) yields:

$$Q_2(t_2) = \frac{\int_{T_1}^{t_2} -d \exp (at^b)\, dt + \int_{T_1}^{T} d \exp (at^b)\, dt}{\exp [a(t^b{}_2 - T_1^{\ b})]} \quad ; \quad T_1 \leq t_2 \leq T \qquad (6.5\text{-}4)$$

At $t_1=T_1$, and $t_2=T_1$, equations (6.5-1) and (6.5-4) are equal and hence the following relationship exists:

$$\frac{\int_o^{T_1} (p-d) \exp (at^b)\, dt}{\exp (aT_1^{\ b})} = \int_{T_1}^{T} d \exp (at^b)\, dt \qquad (6.5\text{-}5)$$

Because of the difficulty in integration, equation (6.5-5) cannot be simplified any further.

By integrating equations (6.5-1) and (6.5-4), the total carrying inventory can be obtained, and hence the average total cost equation becomes:

$$C(T,T_1) = \frac{C_1}{T} \left[\int_o^{T_1} Q_1(t)dt + \int_{T_1}^{T} Q_2(t)dt \right] + \frac{C_3}{T} + \frac{C_4}{T} (pT_1 - dT) \tag{6.5-6}$$

Equation (6.5-6) is a function of two variables, T_1 and T; however, these two variables are not independent and are related by equation (6.5-5). Theoretically, it should be possible to solve for T or T_1 of equation (6.5-5) and substitute the value in the cost equation of (6.5-6); then, by differentiating, the optimal T^* or T^*_1, would be determined. Unfortunately, this is not practical because the value of T or T_1 can not be determined explicitly; and in addition, the integrals in equation (6.5-6) cannot be integrated analytically. To solve these equations, one must expand the exponential terms into their respective series form, and then use numerical techniques to obtain the solution.

In this chapter the methodology of Chapter V is extended to encompass the finite production rate inventory systems. In addition to the models that have been developed in the literature, various cases of non-constant production rate have been considered. These models are especially significant in situations where the production rate is increasing during the inventory cycle time, or where learning curve effect is present. They are also useful in such areas as nuclide pharmacuetical drugs production where the half life of various isotopes have a major bearing on the amount and the rate of production.

Also a case of an increasing perishability rate function with interesting analytical characteristics has been considered.

CHAPTER VII

FINITE PRODUCTION RATE INVENTORY CONTROL SYSTEM WITH BACKLOGGING

These models are a further generalization of the previous models of Chapter VI, and Figure 19 depicts the inventory level of this system of inventory. In these models, the inventory cycle consists of four phases: 1) $(0,T_1)$ production and demand taking place simultaneously; 2) (T_1,T_2) demand is satisfied from the stock; 3) (T_2,T_3) demand is being backlogged; and 4) (T_3,T) production and demand taking place simultaneously with the reduction of backlogs. There is no deterioration in phases three and four, and deterioration is only taking place in the first two phases.

As in the models of Chapter VI, the differential equations that describe the system are given by:

$$\frac{d}{dt} Q(t) + h(t)\, Q(t) = p(t) - d(t) \qquad 0 \leq t \leq T_1 \qquad (7\text{-}1)$$

$$\frac{d}{dt} Q(t) + h(t)\, Q(t) = -d(t) \qquad T_1 \leq t \leq T_2 \qquad (7\text{-}2)$$

$$\frac{d}{dt} Q(t) = -d(t) \qquad T_2 \leq t \leq T_3 \qquad (7\text{-}3)$$

$$\frac{d}{dt} Q(t) = p(t)-d(t) \qquad T_3 \leq t \leq T \qquad (7\text{-}4)$$

Solution of these equations are:

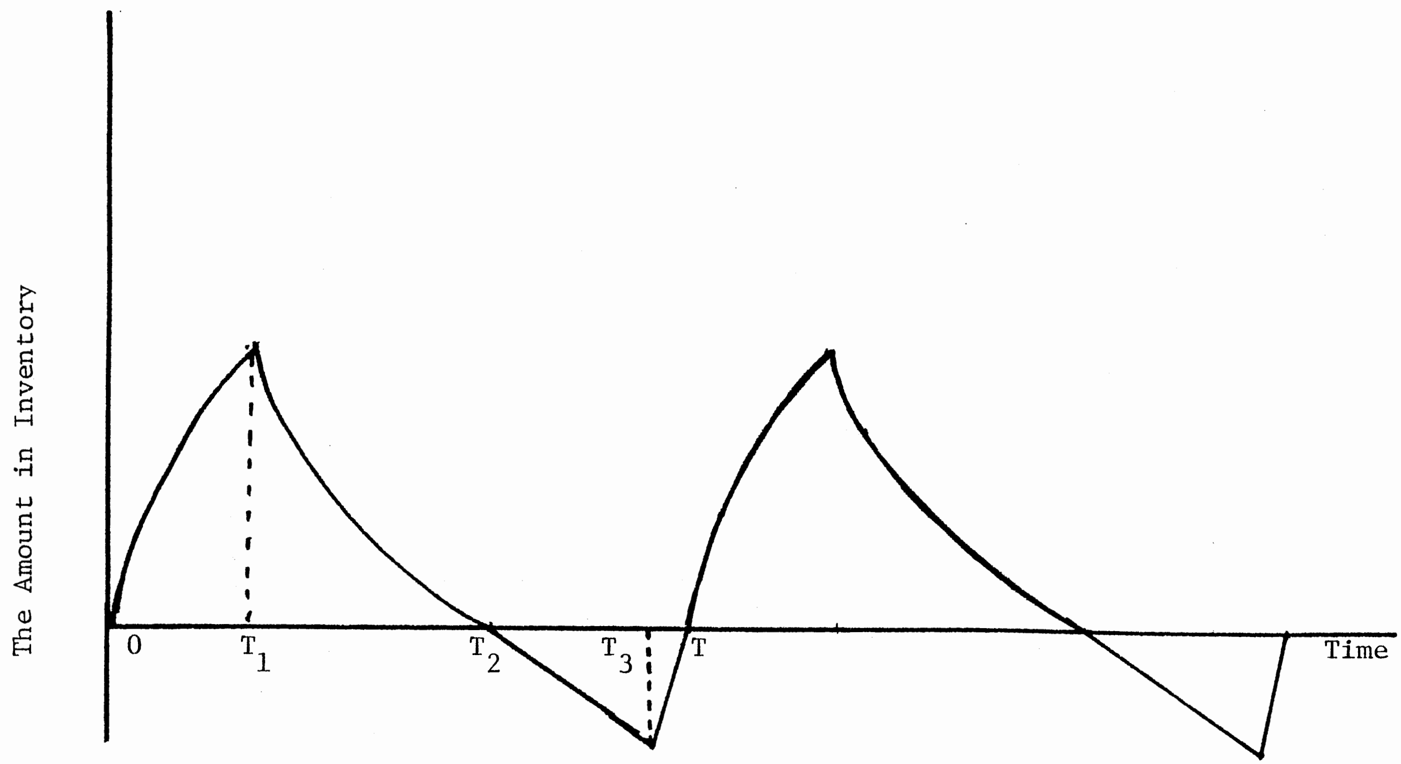


Figure 19. Finite Production Rate Inventory Model with Shortages

$$Q_1(t) = \frac{\int_0^t [(p(t_1) - d(t_1)) \exp (\int h(t)dt)]dt_1 + K_1}{\exp (\int_0^t h(t)dt)} \; ; \quad 0 \leq t \leq T_1 \qquad (7\text{-}5)$$

$$Q_2(t) = \frac{\int_{T_1}^t [(-d(t_2) \exp (\int h(t)dt)]dt_2 + K_2}{\exp (\int_{T_1}^t h(t)dt)} ; \quad T_1 \leq t \leq T_2 \qquad (7\text{-}6)$$

$$Q_3(t) = \int_{T_2}^t -d(t_3)dt_3 + K_3 \; ; \qquad T_2 \leq t \leq T_3 \qquad (7\text{-}7)$$

$$Q_4(t) = \int_{T_3}^t (p(t_4) - d(t_4))\, dt_4 + K_4; \qquad T_3 \leq t \leq T \qquad (7\text{-}8)$$

The values of constants of integrations are obtained using the boundary conditions. Therefore, $K_1=0$, $K_2=Q_{max}$, $K_3=0$, $K_4=Q_{min}$. Now, specific cases will be considered.

7.1 Case 1: p(t), d(t), and h(t) are Constant

In this case the production and demand rate are constant and items in inventory are being perished according to an exponential distribution function. Let $p(t)=p$, $d(t)=d$, and $h(t)=h$. Substituting these values in equations (7-5) through (7-8), and utilizing the results of Chapter VI the following equations are obtained.

$$Q_1(t) = \frac{p-d}{h} \; [1-\exp(-ht)] \; ; \qquad 0 \leq t \leq T_1 \qquad (7.1\text{-}1)$$

$$Q_2(t) = \frac{-d}{h} + [Q_{max} \exp (h(T_1-t))]; \qquad T_1 \leq t \leq T_2 \qquad (7.1\text{-}2)$$

$$Q_3(t) = -d(t-T_2); \qquad T_2 \leq t \leq T_3 \qquad (7.1\text{-}3)$$

$$Q_4(t) = (p-d)\ (t-T_3) + Q_{min}; \qquad T_3 \leq t \leq T \qquad (7.1\text{-}4)$$

At $t=T$, $Q_4(t)=0$; therefore, $Q_{min}=-(p-d)(T-T_3)$. At $t=T_2$, $Q_2(t)=0$; therefore, $Q_{max} = \frac{d}{h} \exp\ (h(T_2-T_1))$. So,

$$Q_4(t)=(p-d)(t-T). \qquad (7.1\text{-}5)$$

At $t=T_1$, equations (7.1-1) and (7.1-2) are equal; and hence,

$$T_1 = \frac{1}{h} \ln \left[1 + \frac{d}{p} (\exp\ (hT_2) - 1)\right]. \qquad (7.1\text{-}6)$$

At $t=T_3$, equations (7.1-3) and (7.1-4) are equal; therefore,

$$T_3 = \frac{d}{p} T_2 + \frac{(p-d)}{p} T \qquad (7.1\text{-}7)$$

Production lot size, Q, is obtained by:

$$Q = pT_1 + p(T-T_3) = p(T+T_1-T_3), \qquad (7.1\text{-}8)$$

which can be rewritten as:

$$Q = d(T-T_2) + \frac{p}{h} \ln\ [1 + (d/p)\ (\exp(hT_2) -1)]\ . \qquad (7.1\text{-}9)$$

Note that this is a function of T_2 and T.

The total number of items that deteriorate during a cycle is equal to the difference of total production and total demand during the inventory cycle time, that is,

$$D(T,T_2) = Q - dT. \qquad (7.1\text{-}10)$$

Now, all that remains to be calculated is the total carrying inventory and the total backlogs. By utilizing equation (6.1.1-2) of Chapter VI models, the total carrying inventory can be written as:

$$I_1 = \frac{pT_1}{h} - \frac{d}{h} T_2 - \frac{p}{h^2} + \left(\frac{p-d}{h^2}\right) e^{-hT_1} + \frac{d}{h^2}\ e^{h(T_2-T_1)} \qquad (7.1\text{-}11)$$

To calculate the total backlog, equations (7.1-3) and (7.1-5) are integrated, that is,

$$I_2 = \int_{T_2}^{T_3} - d(T_2-t)\,dt + \int_{T_3}^{T} - (p-d)(t-T)\,dt \qquad (7.1\text{-}12)$$

By simplifying and substituting into equation (7.1-7) the following equation is obtained:

$$I_2 = \frac{d}{2}(T-T_2)^2\,(1-d/p) \qquad (7.1\text{-}13)$$

The average total cost function can be written as a function of T and T_2, that is,

$$C(T_2,T) = (C_1I_1 + C_2I_2 + C_3 + C_4\,D(T_2,T))/T \qquad (7.1\text{-}14)$$

A search technique such as Hooke and Jeeves may now be used to find the optimal T^* and T^*_1. Of course, one could take the partial of $C(T_2,T)$ with respect to T_2 and T; however, this process is too cumbersome and inefficient for obtaining the optimal results.

7.2 Case 2: d(t) and h(t) are Constant $p(t) = bt^b$

In this case the production rate is increasing according to a polynomical function, demand rate is constant, and perishability of items in inventory are characterized by an exponential distribution. Let d(t)=d, h(t)=h, and $p(t)=pt^b$, where b is a nonnegative integer. Again, using the results of Chapter VI, that is, utilizing equation (6.2-5), T_2 can be determined.

$$T_2 = \frac{1}{h}\ln\left\{1 + \left(\frac{pe^{hT_1}}{d}\right)\sum_{i=0}^{b} (-1)i\,\frac{b!T_1^{(b-i)}}{(b-i)!h^i}\right\} \qquad (7.2\text{-}1)$$

Using equations (7-7) and (7-8) and noting that at $t=T_3$ they are equal, the following is obtained.

$$T_3 = \left[T^{b+1} - \frac{d(b+1)}{p}(T-T_2)\right]^{1/b+1} \qquad (7.2\text{-}2)$$

The production lot size is then obtained by integrating the production rate in phase one and phase four, that is,

$$Q = \frac{p}{b+1}\left(T_1^{b+1} + T^{b+1} - T_3^{b+1}\right) \qquad (7.2\text{-}3)$$

Total backlogging is obtained by integrating $-Q(t)$ between T_2 and T.

$$I_2 = \frac{-p}{(b+1)(b+2)}\left(T^{b+2} - T_3^{b+2}\right) + \frac{pTb+1}{(b+1)}[T-T_3] + \frac{d}{2}[T-T_3]^2$$

$$+ \frac{d}{2}(T_3-T_2)^2 \qquad (7.2\text{-}4)$$

The total carrying inventory and the total backlog can now be stated for specific values of b.

For b=1, by utilizing equations(6.2-9) and (7.2-4) the following can be written:

$$I_1 = \frac{p}{2h}T_1^2 - \frac{pT_1}{h^2} - \frac{d}{h}T + \frac{d}{h^2}[(e^{-hT}1)(1-e^{hT}2)] \qquad (7.2\text{-}5)$$

$$I_2 = \frac{-p}{6}(T^3-T_3^3) + \frac{pT^2}{3}(T-T_3) + (d/2)(T-T_3)^2 + \frac{d}{2}(T_3-T_2)^2 \qquad (7.2\text{-}6)$$

The production lot size is then equal to:

$$Q = (p/2)\,(T_1^2 + T^2 - T_3^2) \qquad (7.2\text{-}7)$$

For b=2, equations (6.2-13) and (7.2-4) are utilized to obtain the total carrying inventory and the total backlog.

$$I_1 = p\left[\frac{T_1^3}{h} - \frac{T_1^2}{3} + \frac{2T_1}{h}\right] + \frac{d}{h^2}[(e^{-hT}1)(e^{hT}2 - 1)] - \frac{dT_2}{h} \qquad (7.2\text{-}8)$$

$$I_2 = \frac{p}{2}(T^4-T_3^4) + \frac{pT^3}{3}(T-T_3) + \frac{d}{2}(T-T_3)^2 + \frac{d}{2}(T_3-T_2)^2, \qquad (7.2\text{-}9)$$

The production lot size for this case is then equal to:

$$Q = \frac{p}{3}(T_1^3+T^3-T_3^3) \quad . \qquad (7.2\text{-}10)$$

The amount of deterioration is given by

$$D(T,T_1) = Q-dT; \qquad (7.2\text{-}11)$$

where Q is given by equation (7.2-7) or (7.2-10). The cost equation (7.1-14), is then used to find the optimal T_2^* and T^*.

7.3 Case 3: d(t), h(t) are Constant. $p(t) = pq^t$

In this case the production rate is increasing as an exponential function of time, demand rate is constant, and perishability is according to an exponential distribution function. Let d(t)=d, h(t)=h, and $p(t)=pq^t$; p 0, q 1. By utilizing equation (6.3-3) the equation for T_2 can be written as:

$$T_2 = \frac{1}{h} \ln \left\{ \frac{ph}{d(h+\ln q)} \left[(q^{T_1} e^{hT_1}) - 1 \right] + 1 \right\} . \qquad (7.3\text{-}1)$$

By utilizing equations (7-7) and (7-8), and noting that at $t=T_3$ they are equal; the value of T_3 can be determined.

$$T_3 = \frac{1}{\ln q} \ln \left[-\frac{d}{p} \ln q \, (T-T_2) + q^T \right] \qquad (7.3\text{-}2)$$

The total carrying inventory is obtained by using equation (6.3-5),

$$I_1 = \frac{p}{(\ln q+h)} \left\{ \frac{1}{\ln q} (T_1-1) + \frac{1}{h} (e^{-hT_1}-1) \right\} + \frac{d}{h^2} \left[e^{hT_1} (e^{hT_2}-1) \right] - \frac{dT_2}{h} \qquad (7.3\text{-}3)$$

and the total backlogging is obtained by integrating -Q(t) in phase three and phase four.

$$I_2 = \frac{d}{2} (T_3-T_2)^2 + (T-T_3) \left(\frac{p}{\ln q} q^T - dT \right) + (d/2) (T-T_3)^2 - \left[\frac{p}{(\ln q)^2} \right] (q^T - q^{T_3}) \qquad (7.3\text{-}4)$$

The production lot size is equal to:

$$Q = \int_0^{T_1} pq^t \, dt + \int_{T_3}^{T} pq^t \, dt, \qquad (7.3\text{-}5)$$

which simplifies to

$$Q = \frac{p}{\ln q}\left(q^{T_1} + q^{T} - q^{T_3} - 1\right) \tag{7.3-6}$$

The total deterioration function, and the average total cost function are the same as equations (7.1-14) and (7.2-11).

7.4 Case 4: d(t), p(t) are Constant $h(t) = \frac{a}{b-t}$

In this case the production and demand rates are constant, however, the perishability of items in inventory is according to a specific perishability rate function. Let d(t)=d, p(t)=p, and $h(t)= \frac{a}{b-t}$, where

a and b are positive; and b is greater than T_{EOQ}. By using equation (6.4-6) value of T_2 can be written as:

$$T_2 = b - \left\{(b-T_1)^{1-a} + \frac{(p-d)}{d}\left[b^{1-a}(b-T_1)^{a} - (b-T_1)\right]\right\}^{1/(1-a)} \tag{7.4-1}$$

Again, by utilizing equations (7-7) and (7-8), the value of T_3 can be written as:

$$T_3 = T - \frac{d}{p}(T-T_2) \quad . \tag{7.4-2}$$

The total carrying inventory is obtained by utilizing equation (6.4-8).

$$I_1 = \left[\frac{(p-d)}{(1-a)}\right]\left\{\left[\frac{b^{(1-a)}}{(1+a)}\right]\left[b^{(1+a)} - (b-T_1)^{(1+a)}\right] - (bT_1 - T_1^2/2)\right\}$$
$$+ \left[\left(\frac{d}{1-a}\right)(b-T_1)^{-a}\right]\left\{+\tfrac{1}{2}\left[(b-T_2)^2-(b-T_1)^2\right] + \frac{(b-T_2)^{(1-a)}}{1+a}\left[(b-T_2)^{(1+a)} - (b-T_1)^{(1+a)}\right]\right\} \tag{7.4-3}$$

The total backlog is obtained by integrating -Q(t) in phase three and phase four of the inventory cycle, that is,

$$I_2 = (d/2)\ (T-T_2)^2\ (1-d/p). \tag{7.4-4}$$

The production lot size is equal to

$$Q = p(T+T_1-T_3)\quad , \tag{7.4-5}$$

and the perishability function is the same as equation (7.2-11)

The average total cost function is

$$C(T,T_1) = \frac{1}{T}\left\{C_1\ I_1 + C_2I_2 + C_4D(T,T_1) + C_3\right\}. \tag{7.4-7}$$

In this chapter, the results of Chapter VI are extended to include the case of backlogging of demand. In addition to the case of constant production rate, various nonconstant production rates have been analyzed. The average total cost equations of these models are essentially functions of two independent variables; the inventory cycle time, T, and the production cycle time, T_1. To determine the optimal inventory characteristics, one can resort to and benefit from search techniques. The Hooke and Jeeve's search technique seems especially appropriate for this class of problems.

CHAPTER VIII

ORDER-LEVEL INVENTORY SYSTEMS

Order level inventory systems are deterministic systems in which the carrying costs are balanced against shortage costs. The only variable subject to control is the order level S. The scheduling period is a prescribed constant. Figure 20 depicts the inventory situation of this system. Two cases will be analyzed; constant demand and pattern demand under the assumption of constant deterioration rate.

8.1 Case 1: Constant Demand and Constant Deterioration Rate

For this case, Figure 20 will be used to describe the model derivation process. Following Shah and Jaiswal [89], at time t=0 of an inventory cycle, a replenishment of size q enters the inventory system from which (q-S) units are used to satisfy the backlog, leaving a remainder of S units as the initial inventory level. As time passes, the inventory level decreases due to demands and deterioration up to time T_1. From time T_1 to T, demands are then backlogged. Replenishment size, q, is given by:

$$q = S + R\,(T-T_1) = S + RT - RT_1 \quad . \qquad (8.1\text{-}1)$$

The differential equations describing the inventory level are:

$$\frac{d}{dt}\, Q(t) + aQ(t) = -R \qquad 0 \leq t \leq T_1 \qquad (8.1\text{-}2)$$

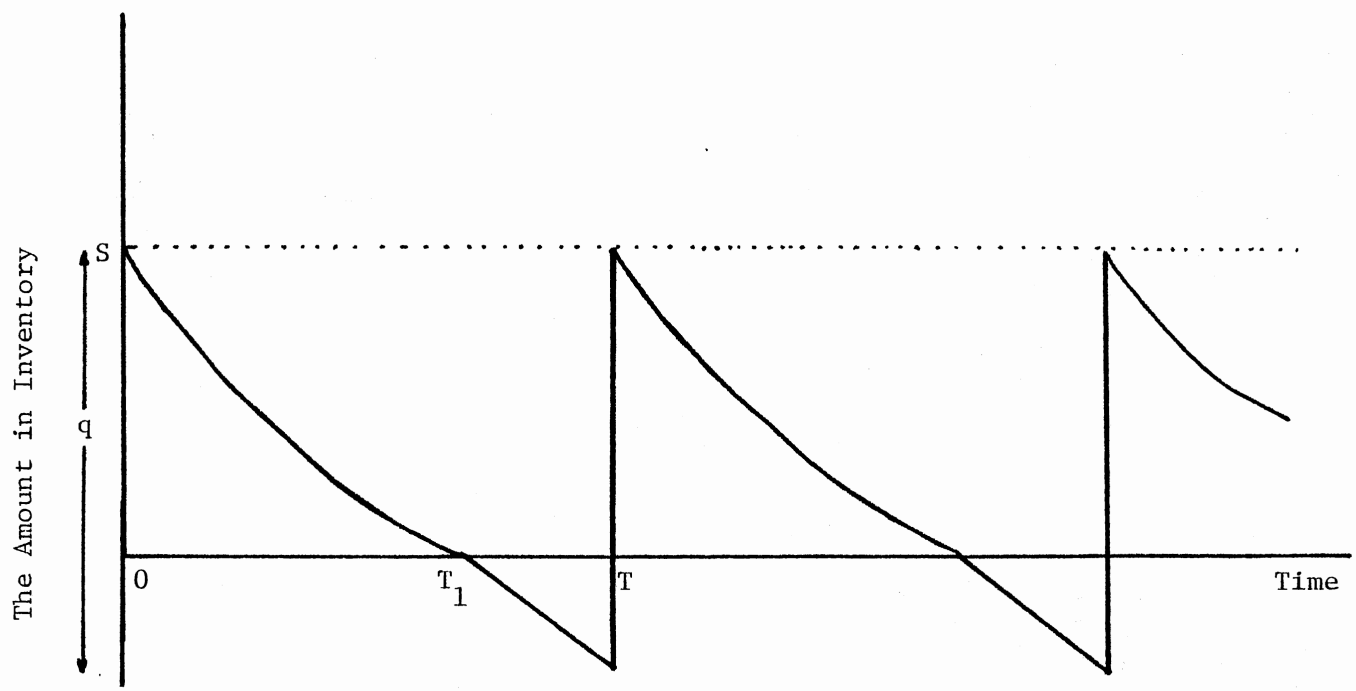


Figure 20. The Order-Level Inventory Model for Perishable Items

$$\frac{d}{dt}Q(t) = -R \qquad T_1 \leq t \leq T \qquad (8.1\text{-}3)$$

The solutions to the above equations are determined to be

$$Q(t) = \left(-\frac{R}{a}\right)(e^{-at})(e^{at}-1) + k_1 e^{-at} \qquad 0 \leq t \leq T_1 \qquad (8.1\text{-}4)$$

$$Q(t) = -R(t-T_1) + k_2 \qquad T_1 \leq t \leq T \qquad (8.1\text{-}5)$$

Since at $t=0$, $Q(t)=S$, and at $t=T_1$, $Q(t)=0$, one can solve for the constants of integration, that is, $k_1=S$, and $k_2=0$. Rewriting equations (8.1-4) and (8.1-5) as:

$$Q(t) = \left[S - \frac{R}{a}(e^{at}-1)\right]e^{-at} \qquad 0 \leq t \leq T_1 \qquad (8.1\text{-}6)$$

$$Q(t) = RT_1 - Rt \qquad T_1 \leq t \leq T \qquad (8.1\text{-}7)$$

one can find the following useful identities:

$$S = \left(\frac{R}{a}\right)(e^{aT_1} - 1) \qquad (8.1\text{-}8)$$

$$T_1 = \frac{1}{a}\ln\left(1 + \frac{aS}{R}\right) \qquad (8.1\text{-}9)$$

Now, by substituting equation (8.1-9) into equation (8.1-1), one can determine q as a function of S,

$$q = S - \frac{R}{a}\ln\left(1 + \frac{aS}{R}\right) + RT \ . \qquad (8.1\text{-}10)$$

The total carrying inventory is determined by integrating equation (8.1-6), that is,

$$I_1 = \left(-\frac{S}{a}\right)(e^{-aT_1}-1) - \left(\frac{R}{a}\right)T_1 - \left(\frac{R}{a^2}\right)[e^{-aT_1}-1] \qquad (8.1\text{-}11)$$

This equation can be further simplified by using equation (8.1-8),

$$I_1 = \frac{R}{a^2}(e^{aT_1}-1) - \frac{R}{a}T_1 \qquad (8.1\text{-}12)$$

or can be written as:

$$I_1 = S/a - (R/a^2)\ln\left(1 + \frac{aS}{R}\right) \qquad (8.1\text{-}13)$$

Equations (8.1-12) and (8.1-13) are exact calculations for the carrying inventory as opposed to the calculation of Shah and Jaiswal [89]. The total number of items deteriorating is given by:

$$D = q - RT = S - \frac{R}{a} \ln\left(1 + \frac{aS}{R}\right), \tag{8.1-14}$$

or

$$D = \frac{R}{a}\left(e^{aT_1} - 1\right) - RT_1 \tag{8.1-15}$$

The total backlog is calculated by integrating equation (8.1-7), that is,

$$I_2 = \frac{R}{2}(T-T_1)^2. \tag{8.1-16}$$

The average total cost equation of the system is:

$$C(T_1) = [C_1 I_1 + C_2 I_2 + C_4 D]/T \quad , \tag{8.1-17}$$

which can be written as

$$C(T_1) = \left[\frac{C_1 R}{a^2}\left(e^{aT_1}-1\right) - \frac{C_1 R}{a} T_1 + \frac{C_2 R}{2}(T-T_1)^2 + \frac{C_4 R}{a}\left(e^{aT_1}-1\right) - C_4 R T_1\right]/T \tag{8.1-18}$$

Equation (8.1-18) can also be written as a function of S, that is,

$$C(S) = \left[\frac{C_1 S}{a} - \frac{C_1 R}{a^2} \ln\left(1 + \frac{aS}{R}\right) + \frac{C_2 R}{2}\left(T - \frac{1}{a} \ln\left(1 + \frac{aS}{R}\right)\right)^2 + C_4 S - \frac{C_4 R}{a} \ln\left(1 + \frac{aS}{R}\right)\right]/T \tag{8.1-19}$$

For optimum value of T_1 or S, equations (8.1-18) or (8.1-19) can be differentiated and set equal to zero, that is;

$$\frac{d}{dS} C(S) = (C_1/a) - \left(\frac{C_1 R}{a^2}\right)\left(\frac{1}{1 + \frac{aS}{R}}\right)\left(\frac{a}{R}\right) + C_2 R \left(T - \frac{1}{a} \ln\left(1 + \frac{aS}{R}\right)\right) \left(\frac{-1}{a}\right)\left(\frac{1}{1 + \frac{aS}{R}}\right)\left(\frac{a}{R}\right) + C_4 - \frac{C_4 R}{a}\left(\frac{1}{1 + \frac{aS}{R}}\right)\left(\frac{a}{R}\right) = 0 \tag{8.1-20}$$

which upon further simplification becomes

$$S(C_1+aC_4) + \left(\frac{C_2R}{a}\right) \ln\left(1 + \frac{aS}{R}\right) - C_2RT = 0 \qquad (8.1\text{-}21)$$

Though the calculation for the total carrying inventory has been exact rather than approximate, the resulting equation (8.1-21) is "simpler" than the one derived by Shah and Jaiswal [89]. Their equation is

$$S(C_1+2aC_4) + \frac{1}{a}(C_1R+2C_2R+C_1Sa)\left[\ln\left[1 + \frac{aS}{R}\right]\right] - 2C_2RT = 0 \qquad (8.1\text{-}22)$$

The second derivative of equation (8.1-21) is positive for positive S values and therefore optimal S^* can be determined by using a search or a numerical technique on equation (8.1-21). Note, when a=0 in equation (8.1-21), meaning no deterioration, one can solve for S and obtain the order level system for non-deteriorating items, which is:

$$S = \frac{C_2}{C_1 + C_2} RT \qquad (8.1\text{-}23)$$

The following relationship is used to obtain the above equation.

$$\lim_{a \to 0} \frac{1}{a} \ln\left(1 + \frac{aS}{R}\right) = \frac{S}{R} \qquad (8.1\text{-}24)$$

Some approximate results can be obtained using assumptions $a \ll 1/T$, and $aS/R < 1$. By using series form of logarithmic and exponential terms, and ignoring second and higher order terms of a, the following equations can be written.

$$T_1 = \frac{S}{R}\left(1 - \frac{aS}{2R}\right) \qquad (8.1\text{-}25)$$

$$S = RT_1 + \frac{RaT_1^2}{2} \qquad (8.1\text{-}26)$$

$$q = RT + \frac{RT_1{}^2}{2} \qquad (8.1\text{-}27)$$

or

$$q = RT + \frac{aS^2}{2R} \qquad (8.1\text{-}28)$$

The average total cost function is then equal to:

$$C\ (S) = \frac{S^2(C_1+aC_4)}{2RT} + \frac{C_2R}{2T}\ \ T - \frac{S}{R} - \frac{aS^2}{2R^2}\ ^2\ . \qquad (8.1\text{-}29)$$

Also, equation (8.1-21) can be written as:

$$S(C_1+C_2+aC_4) - \frac{C_2aS^2}{2R} - C_2RT = 0 \qquad (8.1\text{-}30)$$

Equation (8.1-30) is a "better" approximation than the one which is obtained by differentiating equation (8.1-24). Similar derivations for this particular case were also obtained by Aggarwal [2].

8.2 Case 2: Non-Linear Deterministic Demand and Constant Deterioration Rate

Let the demand rate $r = pq^t$. The differential equations for this model are:

$$\frac{d}{dt} Q(t) + a\ Q(t) = -pq^t \qquad 0 \leq t \leq T_1 \qquad (8.2\text{-}1)$$

$$\frac{d}{dt} Q(t) = -pq^t \qquad T_1 \leq t \leq T \qquad (8.2\text{-}2)$$

The solutions of these equations are:

$$Q(t) = \frac{-p}{\ln q + a}\ (q^t e^{-at}) + K_1 e^{-at} \qquad 0 \leq t \leq T_1 \qquad (8.2\text{-}3)$$

$$Q(t) = \left(\frac{-p}{\ln q}\right)\left(q^t - q^{T_1}\right) + K_2 \qquad T_1 \leq t \leq T \qquad (8.2\text{-}4)$$

Solving for constants of integration and noting that at $t=0$, $Q(t)=S$, and at $t=T_1$, $Q(t)=0$,

$$Q(t) = \left[S - \left(\frac{p}{\ln q + a}\right)\left(e^{(a+\ln q)t} - 1\right)\right] e^{-at} \qquad 0 \leq t \leq T_1 \tag{8.2-5}$$

$$Q(t) = \frac{p}{\ln q} \quad (q^{T_1} - q^t) \qquad T_1 \leq t \leq T \tag{8.2-6}$$

From the above equations, the following useful identities are obtained.

$$S = \frac{p}{\ln q + a} \quad (e^{aT_1} q^{T_1} - 1) \tag{8.2-7}$$

$$T_1 = \frac{1}{\ln q + a} \ln\left(1 + \frac{S \ln q}{p} + \frac{aS}{p}\right) \tag{8.2-8}$$

The inventory lot size, q, can now be determined.

$$q = S + \int_{T_1}^{T} pq^t dt = S + \left(\frac{p}{\ln q}\right)\left(q^T - q^{T_1}\right) . \tag{8.2-9}$$

The total carrying inventory is determined by integrating equation (8.2-5).

$$I_1 = \left(-\frac{S}{a}\right) (e^{-aT_1} - 1) - \left(\frac{p}{\ln q + a}\right)\left[\frac{q^{T_1} - 1}{\ln q} + \frac{e^{-aT_1} - 1}{a}\right] \tag{8.2-10}$$

Substituting for S and simplifying this equation further, one obtains:

$$I_1 = \left(\frac{p/a}{\ln q + a}\right) q^{T_1}(e^{aT_1} - 1) - \frac{pq^{T_1} - p}{\ln q(\ln q + a)} \tag{8.2-11}$$

which can also be written as:

$$I_1 = \left(\frac{p}{\ln q + a}\right) q^{T_1} \left[\frac{(e^{aT_1} - 1)}{a} - \frac{1}{\ln q}\right] + \frac{p}{\ln q(\ln q + a)} \tag{8.2-12}$$

The total number of items that deteriorate is given by:

$$D = S - \frac{p}{\ln q}\left(q^{T_1} - 1\right) = \left(\frac{p}{\ln q + a}\right)\left(e^{aT_1} q^{T_1} - 1\right) - \frac{p}{\ln q} (q^{T_1} - 1) , \tag{8.2-13}$$

and the total backlog is given by:

$$I_2 = \frac{p}{(\ln q)^2} \quad (q^T - q^{T_1}) + (T_1 - T) \frac{p}{\ln q} \quad q^{T_1} \quad . \tag{8.2-14}$$

The average total cost function is the same as equation (8.1-17), therefore,

$$C(T_1) = \left\{ \left[C_1 \frac{p/a}{\ln q + a} \quad q^{T_1} \left(e^{aT_1}-1\right) - \frac{C_1 p q^{T_1} - C_1 p}{\ln q(\ln q + a)} \right] + \right.$$
$$\left[C_2 \frac{p}{(\ln q)^2} \left(q^{T} - q^{T_1}\right) + C_2(T_1 - T) \frac{p}{\ln q} q^{T_1} \right] +$$
$$\left. \left[C_4 \frac{p}{\ln q + a} \left(e^{aT_1} q^{T_1} - 1\right) - C_4 \left(\frac{p}{\ln q}\right)\left(q^{T_1}-1\right) \right] \right\} / T \tag{8.2-15}$$

By differentiating equation (8.2-15) with respect to T_1 and setting it equal to zero, the following equation is obtained:

$$(C_1 + aC_4) \frac{e^{aT_1}-1}{a} + C_2(T_1 - T) = 0 \tag{8.2-16}$$

Note

$$\lim_{a \to 0} \frac{e^{aT_1}-1}{a} = T_1 \ . \tag{8.2-17}$$

Using equation (8.2-17) in equation (8.2-16) will result in

$$T_1 = \frac{C_2}{C_1 + C_2} \quad T \tag{8.2-18}$$

which is the standard result for non-perishable items. An interesting point of observation is that T_1 is not a function of the demand pattern and is only a function of C_1, C_2, C_4, and a.

8.3 Case 3: Nonlinear Deterministic Demand, $r=bt^n$ and Constant Deterioration Rate

Let the demand rate $r=bt^n$, where b and n are positive constants. The differential equations of this model are:

$$\frac{d}{dt} Q(t) + a\, Q(t) = -bt^n \qquad 0 \leq t \leq T_1 \tag{8.3-1}$$

$$\frac{d}{dt}\, Q(t) = -bt^n \qquad T_1 \leq t \leq T \qquad (8.3\text{-}2)$$

The solution of these equations are:

$$Q(t) = -be^{-at}t^{n+1} \sum_{i=0}^{\infty} \frac{(at)i}{(i+n+1)i} + k_1 e^{-at} \qquad 0 \leq t \leq T_1 \qquad (8.3\text{-}3)$$

$$Q(t) = -\frac{b}{n+1}(t^{n+1} - T_1^{n+1}) + k_2 \qquad T_1 \leq t \leq T \qquad (8.3\text{-}4)$$

Again, at t=0, Q(t)=S, and at T_1, Q(t)=0, therefore; the values for the constants of integration, k_1 and k_2 can be determined.

$$Q(t) = \left[S - bt^{n+1} \sum_{i=0}^{\infty} \frac{(at)^i}{(i+n+1)i!} \right] e^{-at} \qquad 0 \leq t \leq T_1 \qquad (8.3\text{-}5)$$

$$Q(t) = \frac{b}{n+1}\,(T_1^{n+1} - t^{n+1}) \qquad T_1 \leq t \leq T \qquad (8.3\text{-}6)$$

The order level S can now be determined.

$$S = bT_1^{n+1} \sum_{i=0}^{\infty} \left(\frac{(aT_1)i}{(i+n+1)i!} \right) \qquad (8.3\text{-}7)$$

The replenishment, q, is determined by:

$$q = S + \int_{T_1}^{T} bt^n\, dt = (bT_1^{n+1}) \sum_{i=0}^{\infty} \frac{(aT_1)^i}{(i+n+1)i!} + \frac{b}{n+1}\,(T^{n+1} - T_1^{n+1}) \qquad (8.3\text{-}8)$$

The total carrying inventory is determined by integrating equation (8.3-5), that is,

$$I_1 = \frac{S}{a}(e^{-at_1} - 1) - b \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \frac{(-1)^j\,(a)^{i-j}\, T_1^{i+j+n+2}}{(i+j+n+2)(i+n+1)i!j!} \qquad (8.3\text{-}10)$$

Deterioration is given by:

$$D = S - \int_0^{T_1} bt^n dt = \left(bT_1^{n+1}\right) \left[\sum_{i=0}^{\infty} \frac{(aT_1)^i}{(i+n+1)i!} - \frac{1}{n+1} \right] \qquad (8.3\text{-}11)$$

And the total backlog is equal to

$$I_2 = \frac{-b}{n+1}\, T_1^{n+1}\,(T - T_1) + \frac{b}{(n+1)(n+2)} \left[T^{n+2} - T_1^{n+2} \right] \qquad (8.3\text{-}12)$$

The average total cost function then becomes:

$$C(T_1) = \left\{ C_1 I_1 + C_2 I_2 + C_4 D \right\} / T \qquad (8.3\text{-}13)$$

By differentiating equation (8.3-13) with respect to T_1 and setting it equal to zero, the following equation is obtained, through which optimal T_1 can be determined.

$$(C_1 + aC_4)\left(\frac{e^{aT_1} - 1}{a}\right) - C_2\ (T-T_1) = 0 \qquad (8.3\text{-}14)$$

Note that this equation is again the same as equation (8.2-16). Therefore, it can be concluded that for power demand, determination of T_1 is not a function of demand but only the function of its cost parameters and the deterioration rate.

8.4 Lost Sales

The system to be considered here is an extension of the order-level system when the cost of shortage depends only on the quantity short and not on the duration of shortages. The dimension of the shortage cost is \$/unit instead of the usual \$/unit/unit-time. Therefore, if there are X number of units short at the end of the prescribed scheduling period, the shortage cost would be C_2X.

Applying this cost measure to the results obtained in this chapter, one finds that the optimal T_1 is also independent of the demand pattern. The cost structure of this system differs from the previous section due to the shortage cost dimension. For the following demand patterns the expected total shortage per inventory cycle is:

$I_2\ (T_1) = R\ (T - T_1)$	Demand Rate Constant(R)	(8.4-1)
$I_2\ (T_1) = b/2\ (T^2 - T_1{}^2)$	Demand Rate Linear (bt)	(8.4-2)
$I_2\ (T_1) = \frac{p}{\ln q}\ (q^T - q^{T_1})$	Demand Rate Nonlinear (pq^t)	(8.4-3)

By substituting these equations into the respective cost equations of the previous section, one can determine the average total cost equation for the lost sales models. By differentiating and solving for T_1, one obtains:

$$T_1 = (1/a) \ln \left(1 + \frac{aC_2}{C_1 + aC_4}\right). \tag{8.4-4}$$

for all the above cases. Note as a approaches zero, equation (8.4-4) becomes

$$T_1 = \frac{C_2}{C_1} \tag{8.4-5}$$

which is the standard result for the non-perishable inventory systems. If equation (8.4-4) is written in the following format,

$$(C_1 + aC_4) \left(\frac{e^{aT_1} - 1}{a}\right) - C_2 = 0 \tag{8.4-6}$$

one can compare it to the case for backlogging, equation (8.2-16), which is repeated here as:

$$(C_1 + aC_4) \left(\frac{e^{aT_1} - 1}{a}\right) + C_2 (T_1 - T) = 0 \tag{8.4-7}$$

Equations (8.4-6) and (8.4-7) are quite different. Equation (8.4-6) is a function of the cost parameters, and the deterioration rate while equation (8.4-7) also involves the scheduling period T. It is possible to rewrite equations (8.4-6) and (8.4-7) into simple forms if one approximates the exponential by the first two terms of its series expansion, that is,

$$T_1 = \frac{C_2}{C_1 + aC_4} \tag{8.4-6'}$$

and

$$T_1 = \frac{C_2}{C_1 + C_2 + aC_4} T \qquad (8.4\text{-}7')$$

Equations (8.4-6') and (8.4-7') clearly show this structural difference. Therefore, by changing the property of unit cost of shortage, or by having different assumptions of backlogging or lost sales, totally different results are obtained for T_1.

8.5 A Discrete-in-Time Order Level Inventory Model for Non-Constant demand

In this model, it is assumed that time can be treated as discrete points and that the deterioration rate is constant. Scheduling period T, is a given constant. Demand is given by the following equation;

$$r = pq^t \qquad ;p > 0,\ q \geq 1 \qquad (8.5\text{-}1)$$

The difference equation for the inventory level can be written by letting Q(t) be the inventory level at time t, and deterioration rate be a.

$$Q(t+1) = Q(t) - a\,Q(t) - pq^t \qquad (8.5\text{-}2)$$

or

$$\Delta Q(t) + a\,Q(t) = -pq^t \qquad (8.5\text{-}3)$$

Notice that this relation holds only for $t = 0,1,\ldots T_1-1$. The following difference equation describes the inventory level for $t = T_1, \ldots T$.

$$\Delta Q(t) = -pq^t \qquad (8.5\text{-}4)$$

Solving difference equations (8.5-3) and (8.5-4), the following is obtained

$$Q(t) = K_1(1-a)^t - \frac{p}{q+a-1} \quad q^t-(1-a)^t \qquad t = 0,1, \ldots T_1-1 \tag{8-5}$$

$$Q(t) = K_2 - \frac{p}{(q-1)}(q^t-1) \qquad t = T_1, \ldots T \tag{8-6}$$

Where K_1 and K_2 are constants of finite integration. Using the boundary conditions one can solve for K_1 and K_2. At $t = T_1$, $Q(t) = 0$, and therefore

$$K_1 = \left(\frac{p}{q+a-1}\right)(1-a)^{-T_1}\left[q^{T_1} - (1-a)^{T_1}\right] \tag{8.5-7}$$

$$K_2 = \left(\frac{p}{q-1}\right)(q^{T_1}-1) \tag{8.5-8}$$

By substituting for K_1 and K_2 in equations (8.5-5) and (8.5-6) respectively, the following equations are obtained:

$$Q(t) = \left(\frac{p}{q+a-1}\right) \quad (1-a)^{t-T_1}\left[q^{T_1} - (1-a)^{T_1}\right] - \left[q^t - (1-a)^t\right] \tag{8.5-9}$$
$$t = 0, 1, \ldots T_1-1$$

$$Q(t) = \frac{p}{q-1} \quad (q^{T_1} - q^t) \qquad t = T_1, \ldots T \tag{8.5-10}$$

At $t = 0$, $Q(t) = S$, and hence, the order level is equal to

$$S = \left(\frac{p}{q+a-1}\right) q^{T_1}(1-a)^{-T_1} - 1 \,, \tag{8.5-11}$$

and the replenishment size is equal to

$$Q = S + \left(\frac{p}{q-1}\right)\left(q^T - q^{T_1}\right) \quad . \tag{8.5-12}$$

The number of items that deteriorate can now be found as a function of T_1.

$$D(T_1) = Q - p\sum_{i=1}^{T} q^{i-1} \tag{8.5-13}$$

Since the second term on the right hand side is a geometric series, it can be simplified further and be written as:

$$D(T_1) = S - \frac{p}{q-1} \quad (q^{T_1} - 1) \tag{8.5-14}$$

The average total carrying inventory during the cycle is equal to

$$I_1(T_1) = \frac{1}{T+1} \sum_{t=0}^{T_1-1} Q(t) \tag{8.5-15}$$

which upon substitution for Q(t) is equal to

$$I_1(T_1) = \left(\frac{1}{T+1}\right)\left(\frac{p}{q+a-1}\right) \Big\{(-1/a)\ [1-(1-a)^{-T_1}]\ [q^{T_1} - (1-a)^{T_1}] + (-1/a)\ [(1-a)^{T_1} - 1] - \left(\frac{1}{q-1}\right)\ (q^{T_1} - 1)\Big\} \tag{8.5-16}$$

The average total backlog during the inventory cycle is given by

$$I_2(T_1) = \left(\frac{-1}{T+1}\right)\left(\frac{p}{q-1}\right)\left\{q^{T_1}\ (T - T_1 + 1) - \frac{q^T - q^{T_1}}{(q-1)}\right\} \tag{8.5-17}$$

The average total cost as a function of T_1 is given by

$$C(T_1) = C_1 I_1(T_1) + C_2 I_2(T_1) + C_4\ D(T_1)/T \tag{8.5-18}$$

In order to find the optimal value of T_1, it is necessary to determine the first difference of equation (8.5-18), that is,

$$\Delta C(T_1) = C(T_1+1) - C(T_1) \tag{8.5-19}$$

which is equal to

$$\Delta C(T_1) = \frac{pC_1}{a(T+1)(q+a-1)} \left[q^{T_1}(1-q) - q^{T_1}(1-a)^{-T_1-1}\ (q+a-1) - aq^{T_1}\right] - \left(\frac{pC_2}{T+1}\right) \left[(q^{T_1})\ (T-T_1) + \frac{C_4}{T}\ (pq^{T_1})\ [(1-a)^{-T_1-1} - 1]\right] \tag{8.5-20}$$

The second difference of equation (8.5-18) is equal to

$$\Delta^2 C(T_1) = \frac{pC_1}{a(T+1)(q+a-1)} \left[qT_1(1-q)^2 + qT_1(1-a)^{-T_1-2}(q+a-1)^2 - aqT_1(1-q)\right. + \left(\frac{pC_2}{T+1}\right)(q^{T_1})\ [q-(T-T_1)(q-1)] + \left(\frac{C_4 p}{T}\right)\left[q^{T_1}(1-q) + q^{T_1}(1-a)^{-T_1-2}\ (q+a-1)\right] \tag{8.5-21}$$

which for all $T_1 = 0,1,\ldots,T$; is positive, that is

$$\Delta^2 C(T_1) \geqslant 0 \quad \text{for all } T_1 = 0,1,\ldots,T \tag{8.5-22}$$

Therefore, the necessary conditions for optimality is given by:

$$C(T_1^* -1) \quad 0 \quad C(T_1^*) \tag{8.5-23}$$

where T_1^* is the optimal solution. Then using equation (8.5-20) in (8.5-23) the condition for optimality becomes

$$M(T_1^*-1) \qquad \frac{C_2 aT}{C_2 T+C_4 a(T+1)} \qquad M(T_1^*) \tag{8.5-24}$$

where

$$M(T_1) = \frac{[(1-a)^{T_1-1} -1]}{T-T_1} \tag{8.5-25}$$

By knowing T_1^*, the optimal replenishment size Q^* can be determined by utilizing equation (8.5-11) and (8.5-12).

Again, it can be seen as in the continuous order-level Inventory Model, T_1^* is not a function of demand pattern. Note that equations (8.5-24) and (8.5-25) are the same as equations (17 and 18) of Dave's [22] paper for constant demand rate. In the author's opinion, similar results would also hold for linear and other deterministic power demands. The proofs for this conjecture will be left for further research.

In this chapter, the order-level inventory systems are analyzed. Aside from the constant demand rate, two nonconstant deterministic demand rate models are considered. In each model two distinct cases are presented; the case of backlogging, and the case of lost sales. The results of the analysis indicate, although the solution structure of the two cases are different for these models, the point T_1, that is, the point where the inventory level first reaches zero within the inventory cycle time is independent of the demand pattern. This is a rather unexpected result, since one would expect the demand pattern would also be involved. In this chapter a case of discrete-in-time order level inventory system is also analyzed with the similar conclusion.

CHAPTER IX

FINITE HORIZON INCREASING DEMAND MODELS WITH CONSTANT PERISHABILITY RATE

In these models, the total demand required to be satisfied over a given time horizon is fixed with demand low in the beginning and increasing as time passes by. These models would be appropriate, for example, for items that are new to the market, and demand for them increases with time as people become more familiar with them. Note that the rate of demand is changing throughout the horizon, however the total requirement is constant. The basic objective of these inventory models is to find the optimal number of replenishments such that the total cost is minimized throughout the horizon. The inventory situation for these models is depicted in Figure 21.

To develop and present the models the following definitions are in order. Let the number of replenishments be denoted by J (J=1,2,...), the total demand by R, the horizon time by T, and the cycle time by t'. By definition, the following equation can be written.

$$t' = T/J \qquad J = 1, 2, \ldots \qquad (9\text{-}1)$$

and the total number of replenishments then is equal to

$$I_3 = J \qquad (9\text{-}2)$$

Two cases of interest would be considered at this time; linear demand and non-linear demand. Deterioration takes place at a constant rate, a, throughout the horizon.

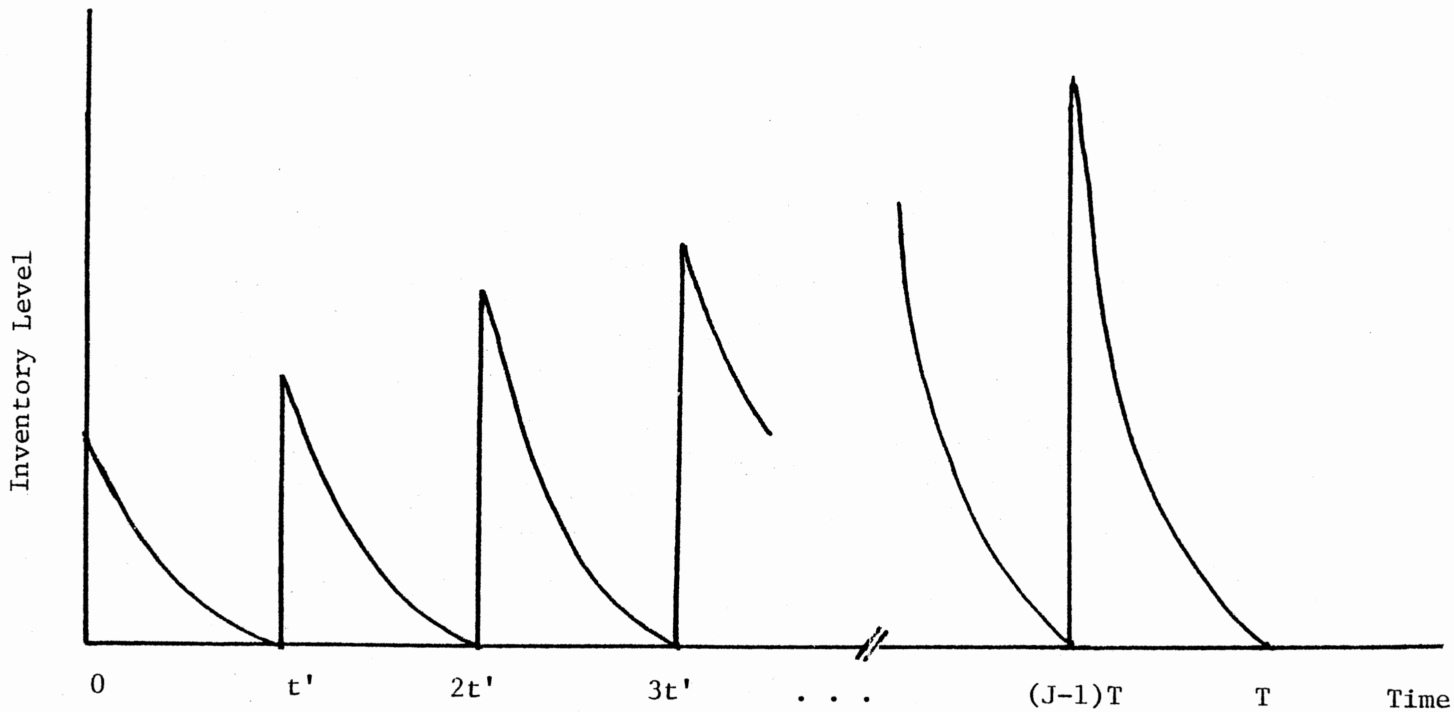


Figure 21. Finite Horizon-Increasing Demand with Constant Perishability Inventory Model

9.1 Case 1: Linear Demand

For this case let the demand rate r be given by

$$r = bt \tag{9.1-1}$$

where b is a positive constant, and t is a point of time within the time horizon. Therefore, the total required demand is equal to

$$R = \int_0^T btdt = \tfrac{1}{2}bT^2 \tag{9.1-2}$$

This implies by knowing or estimating the total demand, R, and the horizon, T, b can be readily calculated.

Let Q_i be the replenshiment size for each inventory cycle (i=0, 1,...J-1). The differential equation describing the inventory level is given by

$$\frac{d}{dt} Q(t) + a\,Q(t) = -bt \qquad it' \leq t \leq (i+1)t' \tag{9.1-3}$$

which upon solving becomes

$$Q(t) = e^{-a(t-it')} [K_i - (b/a) [e^{at} (t-1/a) - e^{ait'} (it'-1/a)] \tag{9.1-4}$$

Assuming at t=(i+1)t', Q(t) is zero or approximately zero, K_i and Q_i can be determined.

$$K_i = Q_i = (b/a)\ (e^{ait'})\ [(it'-1/a)(e^{at'}-1) + t'e^{at'}];\ i=0,\ldots,J-1 \tag{9.1-5}$$

By integrating equation (9.1-4) and using the results of equation (9.1-5), total carrying inventory, I_1, can be determined.

$$I_1 = \sum_{i=0}^{J-1} \left\{ K_i\left(\frac{1-e^{-at'}}{a}\right) - \left(\frac{b}{2a^2}\right)(e^{ait'}) \quad [(2it'-2/a)(at'+e^{-at'}-1)+at'^2] \right. \tag{9.1-6}$$

$$I_1 = (b/a^2)\left\{\frac{(e^{aT}-1)}{(e^{at'}-1)}\left[\frac{(1-e^{at'})(1-at')}{a} + (3/2at'^2-t')\right] - [(e^{at'}+at'-1)(t'e^{at'})]\left[\frac{Je^{aT}(e^{-at'}-1)+(e^{aT}-1)}{(e^{at'}-1)^2}\right]\right\} \qquad (9.1\text{-}7)$$

The total number of items that deteriorate is determined by:

$$D = \sum_{i=0}^{J-1} Q_i - \tfrac{1}{2}bT^2 \qquad (9.1\text{-}8)$$

which reduces to

$$D = (b/a^2)\,[e^{aT}\,(aT-1)+1] - \tfrac{1}{2}bT^2 \qquad (9.1\text{-}9)$$

Note that as a approaches zero, the first term of equation (9.1-9) approach $\frac{1}{2}bT^2$, indicating that there is no deterioration. Also, note that total number of items deteriorating is not a function of replenishment; therefore in order to determine optimal t' or J, one must balance the costs of ordering and replenishments only.

9.2 Case 2: Non-Linear Demand

For this case let the demand rate be given by

$$r = pq^t \qquad (9.2\text{-}1)$$

where p is a positive constant; q has a value greater than or equal to one, and t is a point of time within the time horizon. The total actual demand for the horizon is then given by

$$R = \int_0^T rdt = \frac{p}{\ln q}\,(q^T-1) \qquad (9.2\text{-}2)$$

The differential equation describing the inventory level is

$$\frac{d}{dt}\,Q(t) + a\,Q(t) = -pq^t \qquad it' \leq t \leq (i+1)t' \qquad (9.2\text{-}3)$$

which upon solving becomes

$$Q(t) = e^{-a(t-it')}\left\{K_i - \frac{p}{\ln q+a}\left[e^{(\ln q+a)t} - e^{(\ln q+a)it'}\right]\right\} \qquad (9.2\text{-}4)$$

It is desirable to have zero or negligible amount in inventory at the end of each inventory cycle. Therefore, at $t=(i+1)t'$, $(i=0,1\ldots J-1)$, $Q(t)=0$. By this assumption one can solve for K_i's, which are the constants of integration and are equal to the required replenishment sizes.

$$K_i = Q_i = \left(\frac{p}{\ln q+a}\right)\left[e^{(\ln q+a)it'}\right]\left(q^{t'}e^{at'}-1\right) \qquad (9.2\text{-}5)$$

By substituting equation (9.2-5) into (9.2-4), integrating equation (9.2-4) for $it' \leq t \leq (i+1)t'$ $(i=0,\ldots J-1)$, and summing over i's, one can obtain the total carrying inventory.

$$I_1 = \sum_{i=0}^{J-1} K_i\left(\frac{1-e^{-at'}}{a}\right) - \left(\frac{p}{\ln q+a}\right)\left[\frac{e^{(\ln q+a)(i+1)t'} - e^{(\ln q+a)it'}}{\ln q+a} - t'e^{(\ln q+a)it'}\right]\Bigg\} \qquad (9.2\text{-}6)$$

This simplifies further to

$$I_1 = \left(\frac{p}{\ln q+a}\right)\left(q^Te^{aT}-1\right)\left[\frac{\left(q^{t'}e^{at'}-1\right)\left[(\ln q+a)\left(\frac{1-e^{at'}}{a}\right)-1\right]-(\ln q+a)t'}{\left(qt'e^{at'}-1\right)(\ln q+a)}\right] \qquad (9.2\text{-}7)$$

The total number of items that deteriorate is determined by:

$$D = \sum_{i=0}^{J-1} Q_i - \frac{p}{\ln q}\left(q^T-1\right) \qquad (9.2\text{-}8)$$

which reduces to

$$D = \left[\frac{p}{\ln q+a}\left(q^Te^{aT}-1\right)\right] - \frac{p}{\ln q}\left(q^T-1\right) \qquad (9.2\text{-}9)$$

An interesting point of observation regarding equations (9.2-9) and (9.1-9) is that the total number of items perishing is independent of the number of replenishments. Therefore, perishing cost is a given constant and will not affect the solution of the problem of finding the number of optimal replenishments. Hence, for these models one must balance the costs of carrying and replenishments such that the resulting cost would be minimized. That is,

$$C(t') = C_1 I_1(t') + C_3 I_3(t') \qquad (9.2\text{-}10)$$

where I_1 and I_3 are both functions of t'. Consequently, to minimize the total cost for the above models, one must determine the value of t', and substitute into equation (9.2-10). This procedure must be done iteratively by utilizing equation (9-1). The above cost equation, though, is not a function of C_4; it is a function of perishability rate a.

In this chapter two finite-horizon inventory models with constant deterioration rate are considered. The first model assumes linearly increasing demand, and the second assumes an exponentially increasing demand function during the specified horizon. It is shown that under the assumptions of these models the optimal replenishment and the optimal lot sizes are not a function of deterioration cost, but only a function of inventory carrying cost, replenishment cost and perishability rate function.

CHAPTER X

QUANTITY DISCOUNTS

Inventory systems in which the purchasing price per unit quantity depends on the amount purchased are referred to as systems with quantity discounts. In these systems the unit purchasing price decreases as the quantity purchased increases.

An extension of a lot-size system with quantity discount is considered at this time. Now, instead of having constant costs of purchasing, carrying and perishing, the costs are all non-constant and are a function of the replenishment size. Let q be the size of replenishment, then the purchasing cost for this quantity can be written as qb(q). The function b(·) describes the per unit purchase price whenever a lot size is purchased. Also, let the carrying cost fraction be f, and perishing cost fraction will be g per unit time. Then the unit carrying and perishing cost will be equal to

$$C_1 = f\ b(q) \tag{10-1}$$

$$C_4 = g\ b(q) \tag{10-2}$$

By rewriting equation (5.1-6a) and assuming a constant perishing rate of a, and demand rate of R, the average total cost equation as a function of q can be written as:

$$K(q) = qb\ (q) + b(q)\ \frac{f+ag}{a^2}\ (R)\ (e^{aT}-aT-1) + C_3\ \ /T \tag{10-3}$$

Note however, that q is itself a function of T. Its value is given by

$$q = R/a\ [e^{aT}-1] \qquad (10\text{-}4)$$

In order to obtain the optimal result for equation (10-3), the function b(q) must be transformed and rewritten as a function of inventory cycle, T, so that the entire equation (10-3) would be explicitly a function of T.

Two functional price relationships of interest are: (1) the case of a linearly decreasing per unit cost as a function of the order size; (2) the case of a hyperbolic decrease in unit cost after an initial price break. The equations for these two cases are given as equations (10-5) and (10-6), respectively.

$$b(q) = K-bq \qquad (10\text{-}5)$$

$$b(q) = \begin{cases} K_1 & 0 \leq q \leq q_c \\ K_2 + \dfrac{b}{q} & q_c \leq q < \infty \end{cases} \qquad (10\text{-}6)$$

In the above equations K, K_1, K_2, and b are all positive constants; q_c is an arbitrary quantity defining the quality of the initial price break.

Equations (10-5) and (10-6) can be rewritten as a function of T by utilizing equation (10-4). Let b'(.) be the transformation for b(.). Therefore:

$$b'(T) = K - \frac{bR}{a}(e^{aT}-1) \qquad (10\text{-}7)$$

$$b'(T) = \begin{cases} K_1 & 0 \leq T \leq T_c \\ K_2 + \dfrac{ab}{R(e^{at}-1)} & T_c \leq T < \infty \end{cases} \qquad (10\text{-}8)$$

where $T_c = 1/a \ln\left(1 + \frac{aq_c}{R}\right)$.

Equations for discrete quantity discounts along with the two above mentioned cases will now be presented.

10.1 Case 1: Linear Discount

By substituting equation (10-7) into equation (10-3) the average total cost equation as a function of T is determined.

$$C(T) = \Big\{ [K - \frac{bR}{a} (e^{aT}-1)] \, [(R/a) \, (e^{aT}-1) + (R/a^2) \, (f+ag) \, (e^{aT}-aT-1)] + C_3 \Big\} / T \qquad (10.1\text{-}1)$$

The optimal value for T can be obtained by taking the derivative of C(T), setting it equal to zero and then solving for T* by a numerical method. However by using the Fibonacci search technique one will be able to obtain the same results in a more direct and efficient manner. By knowing T optimal, the optimal order size can be found through equation (10-4).

10.2 Case 2: Hyperbolic Discount

By substituting equation (10-8) into (10-3) the average total cost equation as a function of T is determined.

$$C(T) = K_1 \Big\{ (R/a) \, (e^{aT}-1) + (R/a^2) \, (f+ag) \, (e^{aT}-aT-1) + C_3 \Big\} /T; \quad 0 \leq T \leq T_c \qquad (10.2\text{-}1)$$

$$C(T) = \Big[K_2 + ab \Big] \Big\{ (R/a) \, (e^{aT}-1) + (R/a^2) \, (f+ag)(e^{at}-aT-1)+C_3 \Big\} /T; \quad T_c \leq T < \infty \qquad (10.2\text{-}2)$$

Again by using a Fibonacci search technique the optimal T may be obtained.

10.3 Case 3: Discrete Quantity Discount

In systems with discrete quantity discounts function b(q) is generally given in the following form.

$$b(q) = \begin{cases} b_1 & q_1 \leq q \leq q_2 \\ b_2 & q_2 \leq q \leq q_3 \\ \vdots & \vdots \\ b_{n-1} & q_{n-1} \leq q_0 \leq q_n \\ b_n & q_0 \geq q_n \end{cases} \qquad (10.3\text{-}1)$$

In this case, n prices are specified. Lot sizes smaller than q_1 are not allowed. The quantities $q_1, q_2, \ldots$ are increasing and the prices $b_1, b_2, \ldots$ are decreasing in general. The average total cost of the system can be obtained by utilizing equations (10-3), (10-4), and (10.3-1).

$$C(T) = \left\{\frac{Rb_i}{a}\ (e^{aT}\ -1) + b_i\left[\frac{f+aq}{a}\right]\ (R)\ (e^{aT}-aT-1)+C_3\right\}/T; \qquad T_i \leq T \leq T_{i+1} \qquad (10.3\text{-}2)$$

$$i = 1;\ldots,n-1$$

where T_i is given by

$$T_i = \frac{1}{a} \ln\ (1 + \frac{aq_i}{R}) \qquad (10.3\text{-}3)$$

Now, the solution procedure for this system is exactly the same as for nonperishable items with the exception that the cost function must be evaluated as a function of T rather than q. The procedure is as follows:

1 Let $C(T')$ be the cost of the system for $T'=T_i$, $(i = 1,\ldots,n)$

2 Let T_0' be a specific T' such that $C(T_0') \leq K(T')$

3 Let T_0'' be the largest T for which $T_i = T_0'' \leq T \leq T_{i+1}$ where T is determined through evaluation of equation (10.3-2).

4 Compare $C(T_0')$ and $C(T_0'')$ and select the smaller of the two. By knowing the optimal T, the order quantity can be determined by utilizing equation (10.4)

In this chapter the problem of quantity discounts is discussed when perishability rate is constant. In order to solve this class of problems, one must transform the original price break function into an equivalent equation which is a function of time. The analysis then follows steps similar to those of nonperishable inventory items. In these problems, not only the inventory carrying cost is balanced against the additional price discount, but the perishability cost is also included. Although it is known that a greater number of units will perish and become useless, it is still less costly to order more units because of the discount embedded in total purchasing cost. Moreover, the average number of replenishments decreases as in the case of nonperishable items. However, as the perishability rate increases, the number of replenishments per unit time increases at a decreasing rate. This decrease is due to an increase in inventory cycle time, more units become deteriorated, and therefore, shortening the inventory cycle time is not as cost effective as otherwise might have been anticipated.

CHAPTER XI

POWER DEMAND PATTERN INVENTORY MODELS WITH CONSTANT PERISHABILTY RATE

In these models, though the demand is known during a given inventory cycle time, its rate of occurrence is not constant. For example, this demand pattern exists at many supermarkets, where the demand rate increases at the latter part of the week while the total demand during the week stays relatively constant from week to week. To describe the demand pattern assume that the demand rate is given by the following equation:

$$d(t) = -\frac{X}{n}\left(\frac{t^{n-1}}{T}\right)^{1/n} \qquad (11.1)$$

where X is the demand size during a fixed period T. Note when n=1, that is when demand rate is constant, then X=RT. Equation (11-1) is obtained by differentiating the inventory status equation for non-perishable items which is

$$Q(t) = S-X\sqrt[n]{t/T} \qquad (11\text{-}2)$$

(This is the same equation as equation (3-1)). S is the amount of inventory at the beginning of the inventory cycle and n is the index of the demand pattern.

Perishability of inventory items is a function of the demand-pattern index. Depending on when the inventory items are removed from the stock, the number of items that perish will be affected.

11.1 Case 1: Single Period Inventory Model with Power Demand Pattern

The differential equation describing the inventory level is given as

$$\dot{Q}(t) + (a)Q(t) = \frac{-X}{n}\left(\frac{t^{n-1}}{T}\right)^{1/n} \qquad (11.1\text{-}1)$$

In order to simplify the equation, let

$$A = \frac{X}{nT^{1/n}} \qquad (11.1\text{-}2)$$

Equation (11.1-1) may now be written as:

$$Q'(t) + aQ(t) = -A\, t^{\left(\frac{n-1}{n}\right)}. \qquad (11.1\text{-}3)$$

The solution to this first order linear differential equation is given by:

$$Q(t) = e^{-at}\int_0^t e^{ay}\left[-Ay^{\frac{n-1}{n}}\right] dy + ke^{-at};\ 0 \leq t \leq T \qquad (11.1\text{-}4)$$

where k is the constant of integration. At t=0, Q(t) is equal to the the initial inventory, which is equal to the inventory lot size q_0, that is,

$Q(0) = k = q_0$,

At t = T, it would be desirable to have no item in inventory, i.e.,

$Q(T) = 0$.

By using the above boundary conditions, equation (11.1-4) can be rewritten as

$$0 = e^{-aT}\int_0^T e^{ay}\left[-A\, y^{\frac{n-1}{n}}\right] dy + q_0\, e^{-aT} \qquad (11.1\text{-}5)$$

$$q_0 = A\int_0^T e^{ay}\, y^{\frac{n-1}{n}}\, dy \qquad (11.1\text{-}6)$$

Now, expand the exponential into its series form,

$$q_0 = A \int_0^T y^{\frac{n-1}{n}} \left[\sum_{i=0}^{\infty} \frac{(ay)^i}{i!} \right] dy \quad ; \qquad (11.1\text{-}7)$$

this can be written as:

$$q_0 = A \int_0^T \sum_{i=0}^{\infty} \frac{a^i \, y^{\frac{ni+n-1}{n}}}{i!} dy \qquad (11.1\text{-}8)$$

Interchanging the integration and summation signs,

$$q_0 = A \sum_{i=0}^{\infty} \int_0^T \frac{a^i \, y^{\frac{ni+n-1}{n}}}{i!} dy \qquad (11.1\text{-}9)$$

and then integrating

$$q_0 = A \sum_{i=0}^{\infty} \frac{a^i}{i!} \frac{T^{\frac{ni+2n-1}{n}}}{\frac{ni+2n-1}{n}} \qquad (11.1\text{-}10)$$

Now, substitute for A in equation (11.1-10),

$$q_0 = \left(\frac{X}{n\, T^{1/n}} \right) \sum_{i=0}^{\infty} \left[\frac{a^i}{i!} \frac{T^{\frac{ni+2n-1}{n}}}{(ni+2n-1)/n} \right] \qquad (11.1\text{-}11)$$

which reduces to

$$q_0 = X \sum_{i=0}^{\infty} \left(\frac{a^i}{i!} \frac{T^{\frac{ni+2n-2}{n}}}{ni+2n-1} \right) \qquad (11.1\text{-}12)$$

and upon further simplification, to

$$q_0 = \left(X T^{\frac{2(n-1)}{n}} \right) \sum_{i=0}^{\infty} \frac{(aT)^i}{i!(ni+2n-1)} \qquad (11.1\text{-}13)$$

Note when n=1, that is considering the case of uniform demand, equation (11.1-13) reduces to:

$$q_0 = X \sum_{i=0}^{\infty} \frac{(aT)^i}{i!\ (i+1)} \qquad (11.1\text{-}14)$$

and substituting for X, the value RT,

$$q_0 = \frac{RT}{aT} \sum_{i=0}^{\infty} \frac{(aT)^{i+1}}{(i+1)!} \qquad (11.1\text{-}15)$$

$$q_0 = \frac{R}{a}\,[e^{aT} - 1] \qquad (11.1\text{-}16)$$

which is exactly the same result as of the previously obtained equation (5.1 -13) for the uniform demand and constant perishability rate.

11.2 Multiple Period Inventory Model With Power Demand Pattern

Let the inventory cycle be denoted by T' = mT, (m=1, 2, ...). Figure 22 depicts the inventory level for this case. By using equation (11.1-3) the following can be written:

$$Q(t) = -e^{-at} A \int_{(m-1)T}^{t} e^{ay}\, y^{\left(\frac{n-1}{n}\right)} dy + k_m e^{-at} \quad ; \; (m-1)\, T \leq t \leq mT \qquad (11.2\text{-}1)$$

where k_m is the constant of integration. Note that at t=mT, Q(t)=0; and at t=(m-1)T, Q(t) = q_{m-1}.

$$Q[(m-1)T] = q_{m-1} = k_m e^{-a(m-1)T} \qquad (11.2\text{-}2)$$

Solving for k_m by using equations (11.2-1) and (11.2-2), q_{m-1} is obtained.

$$q_{m-1} = e^{-a(m-1)T} A \int_{(m-1)T}^{mT} e^{ay}\, y^{\left(\frac{n-1}{n}\right)} dy \qquad (11.2\text{-}3)$$

Following the same procedure in the interval, (m-2) T≤t≤(m-1)T, the above process can be repeated.

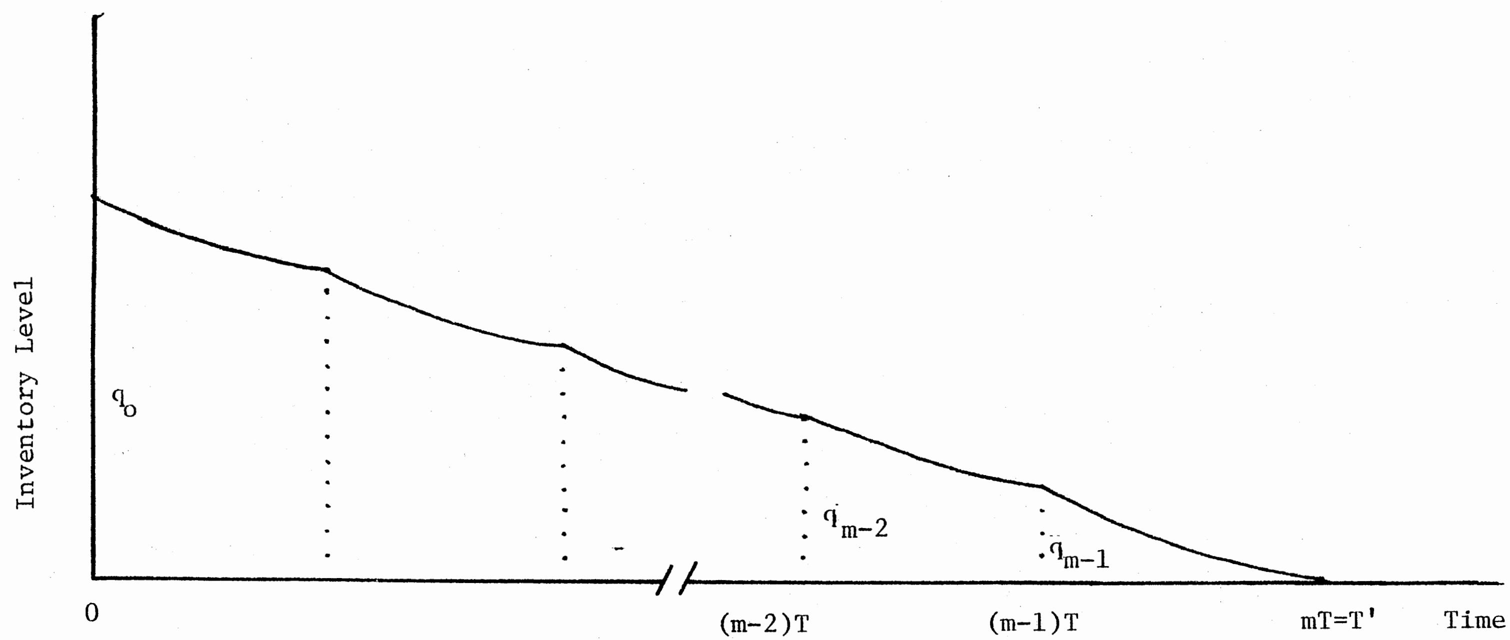


Figure 22. Multiple Period Inventory Model with Power Demand

$$Q\,[(m-1)T] = q_{m-1} = -e^{-a(m-1)T}\,A\int_{(m-2)T}^{(m-1)T} e^{ay}\, y^{\left(\frac{n-1}{n}\right)} dy + k_{m-1}e^{-a(m-1)T} \qquad (11.2\text{-}4)$$

Substituting for q_{m-1} and simplifying

$$k_{m-1} = A\int_{(m-2)T}^{(m-1)T} e^{ay}\, y^{\frac{n-1}{n}} dy + \int_{(m-1)T}^{mT} e^{ay}\, y^{\left(\frac{n-1}{n}\right)} dy \qquad (11.2\text{-}5)$$

which reduces to

$$k_{m-1} = A\int_{(m-2)T}^{mT} e^{ay}\, y^{\left(\frac{n-1}{n}\right)} dy \qquad (11.2\text{-}6)$$

The inventory level and the lot size at $t=(m-2)T$ can now be written as:

$$Q\,[(m-2)T] = q_{m-2} = k_{m-1}e^{-a(m-2)T} \qquad (11.2\text{-}7)$$

$$q_{m-2} = e^{-a(m-2)T}\,A\int_{(m-2)T}^{mT} e^{ay}\, y^{\left(\frac{n-1}{n}\right)} dy \qquad (11.2\text{-}8)$$

By repeating this process q_0 is determined to be

$$q_0 = A\int_0^{mT} e^{ay}\, y^{\left(\frac{n-1}{n}\right)} dy, \qquad (11.2\text{-}9)$$

Substituting for A, and solving the integral by expanding the exponential, the following equation is obtained:

$$q_o = \frac{X}{nT^{1/n}} \sum_{i=0}^{\infty} \frac{a^i \quad (mT)^{\frac{ni+2n-1}{n}}}{i! \quad \frac{ni+2n-1}{n}} \qquad (11.2\text{-}10)$$

which reduces to

$$q_0 = \frac{(2n-1)}{m^n} \times \frac{2(n-1)}{T^n} \sum_{i=0}^{\infty} \frac{(amT)^i}{i!(ni+2n-1)} \qquad (11.2\text{-}11)$$

Again, by substituting one for the value of m, equation (11.2-11) reverts back to equation (11.1-13). As a approaches zero, equations (11.2-9) or (11.2-11) reduces to

$$q_0 = m^{\frac{(2n-1)}{n}} X\ T^{\frac{2(n-1)}{n}} /(2n-1) \qquad (11.2\text{-}12)$$

Note if n is equal to 1, equation (11.2-12) reduces to mX which is the required amount of inventory for m periods of T duration.

If the average total inventory could be approximated by $q_0/2$, the optimal m, that is, the optimal number of periods for which items should be kept in inventory can be determined through the following cost function.

$$C(m) = \frac{C_1 q_0}{2} + \frac{C_3}{mT} + \frac{C_4}{mT}(q_0 - mX) \qquad (11.2\text{-}13)$$

this can be rewritten as

$$C(m) = \frac{C_1}{2} + \frac{C_4}{mT} \quad m^{\frac{(2n-1)}{n}} X\ T^{\frac{2(n-1)}{n}} \sum_{i=0}^{\infty} \frac{(amT)^i}{i!(ni+2n-1)} + \frac{C_3}{mT} - \frac{C_4 X}{T}$$

(11.2-14)

The optimal value of m can now be determined numerically by evaluating c(m) such that the average total cost is minimized.

In this chapter, inventory models with power demand are considered. A model is derived for a single period power demand, and then it is extended to a multiperiod model. This class of models are applicable to situations where the total demand stays relatively constant from one given fixed period to another; however, the demand rate is changing within the prescribed period. Potential savings are possible by considering this demand pattern in many inventory situations.

CHAPTER XII

PROBABILISTIC ORDER-LEVEL SYSTEM WITH INSTANTANEOUS DEMAND AND CONSTANT PERISHABILITY RATE FUNCTION

In this class of inventory systems, the costs of the inventory carrying cost, the shortage cost and the perishability costs are balanced in such a way that the optimal order level S_0 is determined. Scheduling period is a prescribed constant T. Demand occurs instantaneously at the beginning of each scheduling period immediately after the inventory has been raised to the level S. Demand x $(x \geq 0)$ has a probability density function f(x) during the scheduling period T. The inventory fluctuations of this system are described in Figure 23.

Whenever there are stocks in inventory, there will be a certain amount of deterioration that takes place. Thus, one must calculate the expected number of items that deteriorate given a particular demand density during T.

The change of inventory level, right after the demand has been satisfied, given that demand x is less than the order level S is given by:

$$\frac{d}{dt} Q(t) = -aQ(t) \qquad (12\text{-}1)$$

Solving this equation, and considering the boundary conditions, the inventory level during the cycle T is given by

$$Q(t,x) = \begin{cases} S & t=0 \\ (S-x)e^{-at} & 0 \leq t \leq T \end{cases} \qquad x \leq S \qquad (12\text{-}2)$$

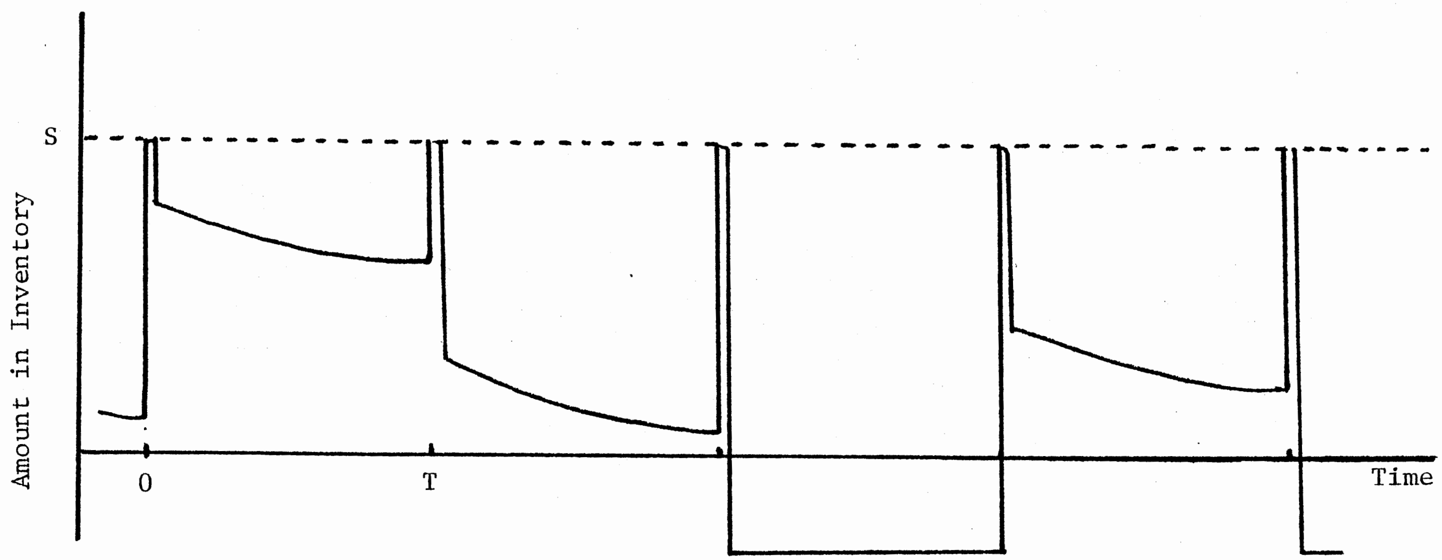


Figure 23. Probabilistic Order-Level System with Instantaneous Demand and Constant Perishability Rate

When the demand exceeds the order level during the cycle time, the equation for inventory level becomes;

$$Q(t,x) = \begin{cases} S & t=0 \\ S-x & 0 \leq t \leq T \end{cases} \qquad x > S \qquad (12\text{-}3)$$

Now, the average carrying inventory and the average shortage can be calculated as a function of demand x.

$$I_1(x) = \frac{1}{T} \int_0^T (S-x)\, e^{-at}\, dt = (S-x)\, {}^{(1-e^{-at})}/(aT) \qquad (12\text{-}4)$$

and

$$I_2(x) = \frac{1}{T} \int_0^T - (S-x)\, dt = x-S \qquad (12\text{-}5)$$

The expected average amount in inventory and the expected shortage can now be determined to be

$$I_1(T) = \int_0^S (S-x)\, \frac{(1-e^{-aT})}{aT}\, f(x)dx \qquad (12\text{-}6)$$

and

$$I_2(T) = \int_S^\infty (x-S)\, f(x)\, dx \qquad (12\text{-}7)$$

The total number of units that deteriorate during the inventory cycle, given that there are units in stock, is given by:

$$D(T,x) = (S-x)\, (1-e^{-aT}) \qquad (12\text{-}8)$$

The expected total number of deteriorating units can be obtained by:

$$D(T) = \int_0^S (S-x)\, (1-e^{-aT})\, f(x)\, dx \qquad (12\text{-}9)$$

The expected average total cost of the system can now be obtained by utilizing equations (12-6), (12-7) and (12-9).

$$C(S) = C_1 \int_0^S (S-x)\ [(1-e^{-aT})/aT]\ f(x)\ dx + C_2 \int_S^{\infty} (x-S)\ f(x)\ dx + \frac{C_4}{T} \int_0^S (S-x)(1-e^{-aT})\ f(x)dx \qquad (12\text{-}10)$$

By differentiating C(S) with respect to S and setting it equal to zero, the optimal S can be determined.

$$\frac{dC(S)}{dS} = C_1[(1-e^{-aT})/aT] \int_0^S f(x)\ dx - C_2 \int_S^{\infty} f(x)\ dx + (C_4 a)\ [(1-e^{-aT})/aT] \int_0^S f(x)\ dx = 0 \qquad (12\text{-}11)$$

Simplifying further, the optimal order level S_0 can be obtained from the following relationship.

$$\int_0^{S_0} f(x)\ dx = \frac{C_2}{(C_1+aC_4)\ [(1-e^{-aT})/(aT)] + C_2} \qquad (12\text{-}12)$$

As a matter of interest, note that as a approaches zero, equation (12-12) reduces to:

$$\int_0^{S_0} f(x)\ dx = \frac{C_2}{C_1 + C_2} \qquad (12\text{-}13)$$

which is the standard formula for finding the optimum value of S for a probabilistic-order-level system for non-deteriorating items, as given by Naddor [61, p. 136].

The right-hand side equation of (12-12) can be readily evaluated, and the result would be equal to some constant g. The difficulty arises in determining the value of S_0. This can be done explicitly only for a limited number of probability distribution functions that have a closed form cumulative distribution function; otherwise S_0 must be determined numerically, though this approach may not be very elegant.

For some specific cases of interest, the value of S_0 is evaluated.

12.1 Case 1: Demand is Exponentially Distributed

Let f(x) be an exponentially distributed density function with parameter (b), then

$f(x) = (1/b)\ e^{-x/b} \quad ; x \quad 0$

By substituting this function into equation (12-12), S_0 can be obtained.

$$S_0 = b \ \ln \left[\frac{(C_1+aC_4)\ [(1-e^{-aT})/(aT)] + C_2}{(C_1+aC_4)\ (1-e^{-aT})/(aT)} \right] \tag{12.1-1}$$

By substituting the above value of S_0 into the equation (12-10), the expected average minimum cost as a function of S_0 is obtained.

$$C(S_0) = (C_1+aC_4)\ [(1-e^{-aT})/(at)]\ S_0 \quad . \tag{12.1-2}$$

Equation (12.1-2) may also be rewritten as:

$$K = [b(C_1+aC_4)\ (1-e^{-aT})/(aT)] \ln \left[\frac{(C_1+aC_4)\ [(1-e^{-aT})/(aT)] + C_2}{(C_1+aC_4)\ (1-e^{-aT})/(aT)} \right] \tag{12.1-3}$$

Note as the perishability rate approaches zero, equation (12.1-3) reduces to:

$$K = b\ C_1 \ln \left[\frac{C_1 + C_2}{C_2} \right] \tag{12.1-4}$$

which is the cost function for nonperishable items as obtained by Naddor [61, p. 137].

12.2 Case 2: Demand is Uniformly Distributed

Let f(x) be a uniform density function with parameter (b), then

$f(x) = 1/b \ ; \quad 0 \leq x \leq b$

Now, by substituting this function into equation (12-12), S_0 can be determined.

$$S_0 = \frac{bC_2}{[(C_1+aC_4)\ (1-e^{-aT})/(aT)] + C_2} \tag{12.2-1}$$

The expected average total cost function as a function of S_0 then can be written by using equation (12-10).

$$C\ (S_0) = [(1-e^{-aT})\ (C_1+aC_4)/(2abT)]\ S_0^2 + \frac{C_2}{2b}\ (b-S_0)^2 \tag{12.2-2}$$

This equation may also be written as:

$$K = \frac{.5b\ C_2\ (C_1+aC_4)(1-e^{-aT})}{(C_1+aC_4)(1-e^{-aT})+aTC_2} \tag{12.2-3}$$

12.3 Case 3: Demand is Weibull Distributed

Let f(x) be a Weibull density function with parameters (b,c), then

$$f(x) = [c/b^c]\ x^{c-1}\ e^{-(x/b)c}\ ; \quad x \geqslant 0$$

Again, by substituting the above equation into equation (12-12), S_0 can be determined.

$$S_0 = b\ \left\{ \ln \left[\frac{(C_1+aC_4)\ [(1-e^{-aT})/(aT)] + C_2}{(C_1+aC_4)\ (1-e^{-aT})/(aT)} \right] \right\}^{1/c} \tag{12.3-1}$$

Note, when c=1, the Weibull distribution is the same as an exponential distribution, and equation (12.3-1) would be the same as equation (12.1-1).

Unfortunately, the total cost function for this distribution cannot be written in a closed form for a general value of parameter c.

13.4 Case 4: Demand is Normally Distributed

Let f(x) be a normal density function with parameters (u,σ). Then

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-u)^2}{2\sigma^2}} \quad ; \quad x \geq 0$$

Cummulative normal distribution cannot be written in a closed form; however, since normal tables are widely available the S_0 can be readily determined. Equation (12-12) can be rewritten as:

$$F(S_0) = g \tag{12.4-1}$$

Where F(x) is a cummulative normal distribution, and g is a constant obtained through the evaluation of the right hand side equation of (12-12). Since g is known, the Z value from the normal table can be readily determined, and hence S_0, i.e.,

$$S_0 = Z\sigma + u \tag{12.4-2}$$

In order to determine the value of the expected average total cost, one must revert totally to a numerical technique for evaluating the integrals of equation (12-10).

In this chapter a probabilistic inventory model with constant perishability rate is developed. The optimal order level has been determined for exponential, uniform, weibull, and normal distribution functions. It is shown that as perishability rate approaches zero, the models reduce to the inventory models of nonperishable items as developed by Naddor [66].

CHAPTER XIII

PROBABILISTIC SCHEDULING SYSTEM WITH CONSTANT PERISHABILITY RATE FUNCTION

In this class of inventory systems, the cost of the inventory carrying cost, perishability cost, and ordering cost are balanced in such a way that the optimal ordering interval, T, is determined. Demand x occurs uniformly during each scheduling period T. Demand x ($x_{min} \leq x \leq x_{max}$) has a probability density function f(x). The replenishment size is a variable quantity ordered at the beginning of every scheduling period so that the inventory level reaches S. In this system no shortages are allowed. The inventory fluctuations of this system are described in Figure 24.

Since no shortage is allowed in this model, the order level must be large enough to satisfy the maximum demand, x_{max}, and the total amount that deteriorates during such demand period. Therefore, using the results of Chapter V models; specifically equation (5.1-13a), the following can be written:

$$S = x_{max}(T)\ [\exp(aT)-1]/(aT) \qquad (13\text{-}1)$$

The differential equation describing the inventory level during the scheduling period is given by:

$$Q'(t,x) + a\ Q(t,x) = \frac{x}{T} \qquad 0 \leq t \leq T \qquad (13\text{-}2)$$

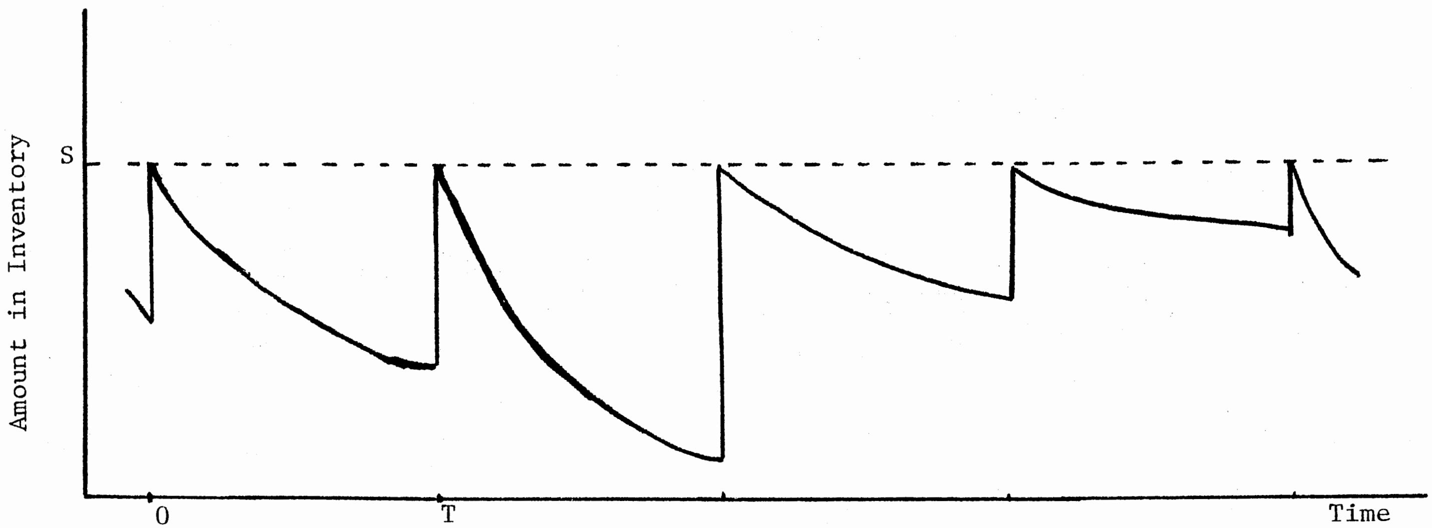


Figure 24. Probabilistic Scheduling Period System with Constant Rate of Perishability

where Q(t,x) denotes the inventory level at time t when the demand of x units occur during the scheduling period T. The solution of this equation is given by:

$$Q(t,x) = [Se^{-at} - \frac{x}{aT}(1-e^{-at})] \qquad 0 \leq t \leq T \tag{13-3}$$

The average carrying inventory during any scheduling period, $I_1(x)$, is then determined by integrating Q(t,x) of equation (13-3).

$$I_1(X) = \frac{1}{T}\int_o^T Q(t,x)dt$$

$$= \frac{1}{T}\left[\left(S + \frac{x}{aT}\right)\left[(1-e^{-aT})/a\right] - (x/a)\right] \tag{13-4}$$

The expected average carrying inventory can now be determined.

$$I_1(T) = \int_{x_{min}}^{x_{max}} I_1(x)f(x)dx$$

$$= \frac{1}{T}\left[\left(S + \frac{\bar{x}(T)}{aT}\right)(1-e^{-aT})/a - x(T)/a\right] \tag{13-5}$$

Where $\bar{x}(T)$ is the mean demand during the scheduling period. Let R be the average rate of demand, therefore,

$$R = x(T)/T \quad . \tag{13-6}$$

The average replenishment is given by

$$I_3(T) = 1/T \quad . \tag{13-7}$$

The average deterioration during any scheduling period is determined by:

$$D(x) = \frac{1}{T}[S - x - Q(T,x)] \quad . \tag{13-8}$$

The expected average deterioration then becomes,

$$D(T) = \int_{x_{min}}^{x_{max}} D(x)\, f(x)dx$$

$$= \frac{1}{T}\left\{S - \bar{x}(T) - [Se^{-aT} - x(T)(1-e^{-aT})/aT]\right\} \tag{13-9}$$

Now, the expected average total cost of this system can be written as a function of T.

$$C(T) = (C_1+aC_4)\left\{[x_{max}(T)(e^{aT}-1)+RT](1-e^{-aT})/(aT)^2 - \frac{R}{a}\right\} + \frac{C_3}{T} \qquad (13\text{-}10)$$

In order to find the optimal schedulng period, it is necessary to know only the function $x_{max}(T)$. Following Naddor [61], let

$$X_{max}(T) = x(T)A(T) = RTA(T) \qquad (13\text{-}11)$$

where A(T) is some function relating the maximum demand during any period T to the average demand during that period. Substituting the value of x_{max} of equation (13-11) into equation (13-10), yields the following useful equation.

$$K(T) = (C_1+aC_4)\left\{RT[1+A(T)e^{aT}-A(T)](1-e^{-aT})/(aT)^2 - R/a\right\} + \frac{C_3}{T} \qquad (13\text{-}12)$$

Two special cases of A(T), will now be considered.

13.1 Case 1: A(T) = k

This is the case when the ratio of maximum demand to the average demand during any period T is assumed to be a constant k. By substituting this value into equation (13-12), one obtains the following equation:

$$C(T) = R\,\frac{(C_1 + aC_4)}{a^2}\left\{\frac{1-aT-e^{-aT}-2k+2k\cosh(aT)}{T}\right\} + \frac{C_3}{T} \qquad (13.1\text{-}1)$$

The optimal inventory cycle time, T^*, can be readily determined using the Fibonacci search technique; knowing T^*, the value of S can be determined by

$$S = \frac{kR}{a}(e^{aT*}-1) \tag{13.1-2}$$

When k=1, that is when demand is deterministic, equation (13.1-1) reverts to the lot size system of Chapter V. (See equation (5.1-6a)).

13.2 Case 2 A(T) = 1 + b/T

This is the more realistic situation where the ratio of the maximum demand to average demand during the inventory cycle, T, is dependent on the value T. In this case b is a positive constant. Substituting this equation into equation (13-12), one obtains the following equation:

$$C(T) = \frac{R(C_1+aC_4)}{a^2}\left[\frac{2b\ \cosh(aT)+TeaT-Tb-aT2-T-b}{T^2}\right]+\frac{C_3}{T} \tag{13.2-1}$$

Similarly optimal inventory cycle Time, T^* can be determined using Fibonacci technique. The value of order level S is given by

$$S = (R/a)(1+b/T^*)(e^{aT*}-1) \tag{13.2-2}$$

In this chapter the objective is to determine the optimal inventory cycle time when the order level is prescribed and demand is probabilistic. For the models developed, it is not necessary to know the probability distribution of demand explicitly, only the functional relationship between the maximum demand and the average demand is all that is required to find the optimal inventory characteristics. As in previous models, as the perishability rate approaches zero, the inventory models of nonperishable items are obtained.

CHAPTER XIV

SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

The purpose of this concluding chater is to summarize the research efforts, draw conclusions, and make recommendations for future research. First, a review and summary of each chapter is presented; then findings and relevant results are discussed; and finally, recommendations for future research are stated.

14.1 Summary

The efforts of this dissertation represent an attempt to develop mathematical inventory models that can be used to obtain optimal replenishment policies for products which are subject to a continuous deterioration (perishability) while in stock. Consideration of perishability in mathematical modeling and management of perishable items is one of the most challenging and potentially fruitful areas of research.

Chapter I serves to introduce the inventory problem. In particular, the general effect of perishability on items while in stock is discussed. Also, the research objectives of the dissertation are stated. Chapter II reviews the pertinent inventory literature on the topic of perishable items and represents the various classifications of perishability. Chapter III defines and explains the specific terminology, notation, and the various common assumptions that are utilized in the development of inventory models in the subsequent chapters.

Chapters IV and V describe the mathematical models of inventory lot size systems with alternative types of perishability functions. The models developed in Chapter IV are better-suited for inventory situations with high obsolescence, especially models I-a and I-b. In contrast, the models derived in Chapter V are more appropriate for items that are subject to physical deterioration. In the case of models of Chapter IV, when the perishability rate is constant, it is shown that similar results can be obtained by using the standard EOQ model with an adjusted inventory carrying cost.

Chapters VI and VII develop the finite production rate inventory system with and without backlogging considerations. A number of production rate functions have been investigated. The analysis of varying production rates is an important factor in determining the optimal inventory characteristics especially when the learning (improvement) curve is present in a production system.

Chapter VIII discusses the order level inventory models with constant perishability rate. Both linear and nonlinear deterministic demand have been examined. A discrete-in-time order level inventory model has also been analyzed through the use of calculus of finite differences.

Chapter IX presents two unique finite horizon inventory models. Linearly increasing, and exponentially increasing demand functions with a constant perishability rate are analyzed. The objective is to determine the optimal number of orderings within the specified time horizon and the corresponding replenishment size.

Chapter X discusses the effect of quantity price discounts on the inventory analysis of perishable items. In order to solve this class of problems, quantity price breaks must be transformed to a new equation which is a function of time. The optimal inventory characteristics then can be solved using steps similar to the nonperishable items.

Chapter XI presents a special case of power demand pattern. The effect of demand pattern and perishability is analyzed for a single period model and then it is extended to a multiperiod model.

Chapters XII and XIII each describes a probabilistic inventory system. For the model developed in Chapter XII, a number of probability distributions are considered for determining the optimal inventory characteristics. For the model of Chapter XIII, no specific knowledge of the demand distribution is required, albeit the relationship between the maximum demand and average demand must be specified. These two models are extensions of nonperishable inventory models as developed by Naddor.

14.2 Conclusions

Based on the results of inventory models derived in this research the following statements can be made:

1. Perishability has a significant economic impact on the optimal inventory cost. The possible savings associated with the models developed in this research as compared to EOQ models is dependent on the cost parameters and the perishability rate of each specific problem. For example, given a set of cost parameters in a constant perishability model (see Appendix A), a cost reduction between .45% to 15.65% occurs as the rate of perishability increases from .1% to 1%. Because the cost savings is chiefly problem-specific, an interactive FORTRAN program is developed to calculate the optimal inventory characteristics

and make comparisons with the EOQ model. The program also conducts a sensitivity analysis for a range of perishability rates. In addition, it furnishes a menu for selecting other perishability models so that the user can decide on the proper model and the proper perishability function.

2. The solution methodology of this paper has emphasized the determination of the exact total cost equation. By obtaining such equations, search techniques can be used to obtain the optimal inventory chracteristics such as optimal replenishment size, cycle time, etc. An example, a problem presented by Shah and Jaiswal [88], is solved using this methodology (see Appendix B). The results indicate that the proposed methodology is superior to the EOQ model, and in comparison to Shah and Jaiswal methodology, cost improvements of .49% to 11.1% are possible depending on the perishability rate (.02 a .10)).
3. In the order level inventory systems, it is shown that the time at which the inventory level reaches zero is not a function of the demand pattern. However, the demand pattern must be considered for determining the optimal replenishment size.
4. The sensitivity analysis of perishable models indicate that as the perishability rate, or perishability cost increases, the replenishment size, q (or order level S), and the inventory cycle time T decrease. But q and T increase, with an increase in replenishment cost. The average total cost, as expected, increases as any of the inventory parameters increases, though it is less sensitive to the inventory holding cost.

14.3 Recommendations

In order to reduce inventory costs, it is expedient for those in charge of inventory management of perishable items to use the findings of this research in their decision making process. For future research, it is recommended that further efforts be devoed to the following areas.

1. Development of mathematical methodology for discrete items that are subject to deterioration.
2. Investigation of Deterministic and Probabilistic Reorder-Point, Order-Level System (i.e., (s,S) inventory policy) for perishable items.
3. Comprehensive sensitivity study of each model and its relation compared to other models developed in this paper, similar to the efforts of Jones [51], for inventory models of perishable items.
4. Investigation of discrete-in-time order-level inventory models for linear and various types of power demand for perishable items.
5. Investigation could be pursued in the analysis of additional demand patters in the order level inventory system. Also, extensions of finite production rate inventory systems are possible by considering additional nonlinear production rate functions.
6. Development of probabilistic inventory models for perishable items may be a rewarding possibility. Excepting the references [48], [90], and [91], there has not been much work done in this area.
7. Application of these models to non-inventory situations, such as financial analysis. Since money can be considered a perishable

item, especially during high inflationary periods and high interest rates. Under these severe economic conditions the methodology and models of this research are even more useful for decision-making purposes than before, especially in banking and fianancial markets.

8. An interesting but probably a difficult generalization would be to allow the items arriving into inventory to have a mixed perishability distribution function. For example, for the simplest case of exponentially distributed deterioration, items are subject to two or more different rates of deterioration due to environmental, manufacturing, or other reasons affecting them.

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APPENDIXES

APPENDIX A

DESCRIPTION AND SOURCE LISTING OF FORTRAN PROGRAM FOR EVALUATING LOT-SIZE - MODEL I

Program Description

This appendix describes the computer program used to evaluate the optimal inventory characteristics of the models developed in Chapter IV. A description is made of the output from the program. A sample of outputs and the listing of the program is presented at the end of this appendix.

The program consists of a short main program and one major subroutine. The main program asks for the inputs consisting of purchasing cost, carrying cost, ordering cost, perishability cost, and the demand rate. The program then provides the user with a menu, so that he may use the desired initial model. Upon selection of a model, the program is transferred to subroutine MODEL I. In this subroutine, appropriate calculations for the chosen model are made, and the program transfers back to the main program. Then, the user is asked if additional analysis with a different model is desired. If yes, he is given an opportunity to change any of his input parameters. Since the program is interactive, there are no user requirements for this program.

Program Output

A definition of the variables in the output tables is given below, with the subsequent example outputs of the program.

1.	A,B,C	parameters of perishability
2.	OPT T*	optimal inventory cycle time
3.	OPT Q*	minimum average total cost
4.	MIN COST	minimum average total cost
5.	OPT PER	optimal number of units perishing

6.	COST-EOQ	average total cost, if an EOQ model is used instead of the perishable model
7.	PER-EOQ	Number of units perishing, if an EOQ model is used instead of the perishable model

LOT SIZE INVENTORY--MODEL I

```
INPUT YOUR UNIT COST,HOLDING COST,ORDERING COST
AND PERISHABILITY COST(THIS MAY BE EQUAL TO THE UNIT COST),
AND YOUR DEMAND RATE IN THE FOLLOWING ORDER;

CO ,C1 ,C3 ,C4 ,R

THEN PRESS THE ENTER KEY

?
.5,.005,50,.5,100

UNIT COST =          0.50
HOLDING COST =        0.0050
ORDERING COST =         50.00
PERISHING COST =          0.50
DEMAND RATE =        100.00


IS EVERYTHING CORRECT ? IF YES, TYPE  1
OTHERWISE TYPE 0 AND PRESS THE ENTER KEY


?
1
SELECT THE DESIRED MODEL FROM THE  MENUE & TYPE IN
ITS RESPECTIVE NUMBER; THEN PRESS THE ENTER KEY
```

```
*****     MODEL I-A      *****

1.   CONSTANT PERISHABILITY RATE

2.   LINEAR PERISHABILITY RATE

3.   QUADRATIC PERISHABILITY RATE

4.   EXPONENTIAL PERISHABILITY RATE

*****      MODEL I-B      *****

5.   CONSTANT PERISHABILITY RATE

6.   LINEAR PERISHABILITY RATE


?
5

TYPE IN YOUR ESTIMATE OF THE VALUE A,
THE PERISHABILITY CONSTANT ( 0 < A <.9 % )

  A =
  ?
  0
```

MODEL I-A--CONSTANT PERISHABILITY RATE

A	OPT T*	OPT Q*	MIN. COST	OPT PER	COST-EOQ	PER-EOQ
.0	14.142	1414.214	7.07	0.0	7.071	0.0
.0010	12.899	1306.558	7.75	16.64	7.785	20.00
.0020	11.935	1222.015	8.38	28.49	8.499	40.00
.0030	11.159	1153.303	8.96	37.36	9.214	60.00
.0040	10.518	1096.005	9.51	44.25	9.928	80.00
.0050	9.975	1047.261	10.02	49.75	10.642	100.00
.0060	9.509	1005.122	10.52	54.25	11.356	120.00
.0070	9.102	968.215	10.99	57.99	12.070	140.00
.0080	8.744	935.534	11.44	61.16	12.784	160.00
.0090	8.425	906.326	11.87	63.87	13.499	180.00
.0100	8.138	880.013	12.29	66.22	14.213	200.00

DO YOU WANT TO TRY ANOTHER VALUE FOR A ?
YES(1)/NO(0)

?
0

DO YOU WANT TO TRY A DIFFERENT MODEL ? IF YES,
TYPE 1, OTHERWISE TYPE 0 AND THEN PRESS THE ENTER KEY

?
1
SELECT THE DESIRED MODEL FROM THE MENU & TYPE IN
ITS RESPECTIVE NUMBER; THEN PRESS THE ENTER KEY

MODEL I-A--LINEAR PERISHABILITY RATE

A	B	OPT T*	OPT Q*	MIN COST	OPT PER	COST-EOQ	PER-EOQ
.0	.0	14.142	1414.214	7.07	0.0	7.071	0.0
.0	.0010	8.560	887.395	9.83	31.36	12.121	141.42
.0	.0020	7.166	753.401	11.36	36.80	17.171	282.84
.0	.0030	6.402	679.542	12.52	39.36	22.221	424.26
.0010	.0	12.899	1306.558	7.75	16.64	7.785	20.00
.0010	.0010	8.327	868.535	10.26	35.81	12.835	161.42
.0010	.0020	7.033	743.056	11.72	39.74	17.885	302.84
.0010	.0030	6.309	672.502	12.84	41.64	22.935	444.26
.0020	.0	11.935	1222.015	8.38	28.49	8.499	40.00
.0020	.0010	8.109	850.672	10.67	39.81	13.549	181.42
.0020	.0020	6.906	733.047	12.07	42.47	18.599	322.84
.0020	.0030	6.219	665.659	13.15	43.80	23.649	464.26
.0030	.0	11.159	1153.303	8.96	37.36	9.214	60.00
.0030	.0010	7.903	833.700	11.08	43.41	14.264	201.42
.0030	.0020	6.783	723.336	12.42	45.01	19.314	342.84
.0030	.0030	6.131	658.926	13.46	45.84	24.364	484.26

DO YOU WANT TO TRY ANOTHER VALUE FOR A & B ?
YES(1)/NO(0)

MODEL I-A--QUADRATIC PERISHABILITY RATE

A	B	C	OPT T*	OPT Q*	MIN COST	OPT PER	COST-EOQ	PER-EOQ
.0	.0	.0	14.142	1414.214	7.07	0.0	7.071	0.0
.0	.0	.0002	7.685	791.744	9.96	23.25	16.593	266.67
.0	.0	.0004	6.630	688.720	11.16	25.76	26.116	533.33
.0	.0	.0006	6.059	632.846	12.01	26.95	35.638	800.00
.0	.0	.0008	5.676	595.280	12.69	27.68	45.161	1066.67
.0	.0010	.0	8.560	887.395	9.83	31.36	12.121	141.42
.0	.0010	.0002	6.841	714.724	11.28	30.61	21.643	408.09
.0	.0010	.0004	6.147	645.323	12.19	30.65	31.166	674.75
.0	.0010	.0006	5.718	602.547	12.89	30.73	40.688	941.42
.0	.0010	.0008	5.412	572.011	13.47	30.80	50.211	1208.09
.0010	.0	.0	12.899	1306.558	7.75	16.64	7.785	20.00
.0010	.0	.0002	7.552	782.600	10.34	27.39	17.308	286.67
.0010	.0	.0004	6.548	683.553	11.49	28.79	26.830	553.33
.0010	.0	.0006	5.998	629.226	12.32	29.47	36.352	820.00
.0010	.0	.0008	5.627	592.583	12.98	29.90	45.875	1086.67
.0010	.0010	.0	8.327	868.535	10.26	35.81	12.835	161.42
.0010	.0010	.0002	6.742	707.854	11.62	33.64	22.358	428.09
.0010	.0010	.0004	6.078	640.918	12.50	33.12	31.880	694.75
.0010	.0010	.0006	5.664	599.238	13.17	32.87	41.402	961.42
.0010	.0010	.0008	5.367	569.456	13.74	32.74	50.925	1228.09

MODEL I-A--EXPONENTIAL PERISHABILITY RATE

A	B	OPT T*	OPT Q*	MIN COST	OPT PER	COST-EOQ	PER-EOQ
.0	.0050	14.142	1414.214	7.07	0.0	7.071	0.0
.0	.0150	14.142	1414.214	7.07	0.0	7.071	0.0
.0	.0250	14.142	1414.214	7.07	0.0	7.071	0.0
.0	.0350	14.142	1414.214	7.07	0.0	7.071	0.0
.0010	.0050	12.828	1299.793	7.77	16.99	7.811	20.72
.0010	.0150	12.678	1285.502	7.82	17.70	7.867	22.28
.0010	.0250	12.518	1270.245	7.87	18.40	7.928	23.99
.0010	.0350	12.350	1254.106	7.92	19.08	7.995	25.88
.0020	.0050	11.832	1212.038	8.41	28.84	8.551	41.45
.0020	.0150	11.621	1191.615	8.49	29.51	8.662	44.56
.0020	.0250	11.406	1170.713	8.57	30.11	8.785	47.98
.0020	.0350	11.189	1149.499	8.65	30.65	8.919	51.75
.0030	.0050	11.042	1141.786	9.01	37.60	9.291	62.17
.0030	.0150	10.806	1118.674	9.11	38.03	9.458	66.84
.0030	.0250	10.572	1095.615	9.21	38.38	9.641	71.97
.0030	.0350	10.341	1072.750	9.31	38.65	9.843	77.63

MODEL I-B --CONSTANT PERISHABILITY RATE

A	OPT T*	OPT Q*	MIN COST	OPT PER	COST-EOQ	PER-EOQ
.0	14.142	1414.214	7.07	0.0	7.071	0.0
.0010	12.899	1298.348	7.42	8.43	7.423	10.14
.0020	11.935	1208.118	7.76	14.59	7.765	20.58
.0030	11.159	1135.270	8.09	19.33	8.097	31.33
.0040	10.518	1074.853	8.40	23.09	8.419	42.40
.0050	9.975	1023.690	8.70	26.18	8.730	53.80
.0060	9.509	979.638	8.99	28.77	9.032	65.56
.0070	9.102	941.190	9.28	30.97	9.323	77.69
.0080	8.744	907.253	9.55	32.88	9.605	90.20
.0090	8.425	877.009	9.81	34.56	9.876	103.13
.0100	8.138	849.834	10.07	36.05	10.137	116.47

MODEL I-B--LINEAR PERISHABILITY RATE

A	B	OPT T*	OPT Q*	MIN COST	OPT PER	COST-EOQ	PER-EOQ
.0	.0	14.142	1414.214	7.07	0.0	7.071	0.0
.0	.0010	14.087	1451.079	8.59	42.38	8.601	42.85
.0	.0020	14.050	1482.178	9.85	77.20	9.877	78.57
.0	.0030	14.024	1508.913	10.91	106.49	10.956	108.78
.0010	.0	14.135	1423.329	7.42	9.85	7.423	9.86
.0010	.0010	14.082	1458.860	8.89	50.71	8.902	51.29
.0010	.0020	14.046	1488.980	10.11	84.40	10.138	85.89
.0010	.0030	14.022	1514.964	11.13	112.81	11.186	115.22
.0020	.0	14.127	1432.112	7.76	19.41	7.766	19.45
.0020	.0010	14.076	1466.392	9.18	58.81	9.196	59.51
.0020	.0020	14.042	1495.591	10.36	91.41	10.393	93.04
.0020	.0030	14.019	1520.872	11.36	118.99	11.411	121.53
.0030	.0	14.119	1440.568	8.10	28.69	8.099	28.78
.0030	.0010	14.070	1473.681	9.47	66.70	9.482	67.52
.0030	.0020	14.038	1502.017	10.61	98.26	10.643	100.03
.0030	.0030	14.016	1526.632	11.58	125.03	11.631	127.69

```
C                    FORTRAN CODES FOR INVENTORY MODELS OF
C                       ITEMS SUBJECT TO DETERIORATION
C
C
C
C                                 PROGRAMMED BY
C                                 FERAIDOON RAAFAT
C
C
C                                 MARCH 1981
C
C
C
C                     ***************************
C                     *                         *
C                     *     LOT SIZE SYSTEMS    *
C                     *                         *
C                     *         MODEL I         *
C                     *                         *
C                     ***************************
C
C
C
C SET THE INPUT-OUTPUT UNIT NUMBERS
C
      IN-5
      IOUT=6
C
C READ THE INVENTORY PARAMETERS- SEE GLOSSARY FOR DEFINITIONS
C
      WRITE(IOUT,100)
  100 FORMAT(1H1,/,16X,'LOT SIZE INVENTORY--MODEL I',//)
  104 WRITE(IOUT,105)
  105 FORMAT(1X,'INPUT YOUR UNIT COST,HOLDING COST,',
     1'ORDERING COST',/1X,'AND PERISHABILITY COST',
     2'(THIS MAY BE EQUAL TO THE UNIT COST),'/1X,
```

```
     3'AND YOUR DEMAND RATE IN THE FOLLOWING ORDER; '/,
     4/' CO ,C1 ,C3 ,C4 ,R '/,
     5/1X,'THEN PRESS THE ENTER KEY'/)
C
      READ(IN,*)CO,C1,C3,C4,R
C
C VALIDATE INPUT DATA(ECHO PRINT)
C
      WRITE(IOUT,112) CO,C1,C3,C4,R
  112 FORMAT(1X/1X,'UNIT COST = ',F10.2,/1X,'HOLD',
     1'ING COST = ',F10.4,/1X,'ORDERING COST = ',F10.2,
     2/1X,'PERISHING COST = ',F10.2,/1X,'DEMAND RATE = ',
     3F10.2//)
C
      WRITE(IOUT,115)
  115 FORMAT(1X,'IS EVERYTHING CORRECT ? IF YES, TYPE',
     1'  1'/1X,'OTHERWISE TYPE 0 AND PRESS THE ENTER KEY'//)
C
C REREAD THE PARAMETERS IF THERE IS ANY ERROR IN INPUT
C
      READ(IN,*)IWORD
      IF(IWORD.NE.1)GO TO 104
C
C THIS IS THE MENUE OF AVAILABLE MODELS
C
  118 WRITE(IOUT,120)
  120 FORMAT(1X,'SELECT THE DESIRED MODEL FROM THE ',
     1' MENUE & TYPE IN'/1X,'ITS RESPECTIVE NUMBER; THEN PRESS',
     3' THE ENTER KEY '//1X,'*****      MODEL I-A       *****'//
     41X,'1.  CONSTANT PERISHABILITY RATE'//
     51X,'2.  LINEAR PERISHABILITY RATE'//
     61X,'3.  QUADRATIC PERISHABILITY RATE'//
     81X,'4.  EXPONENTIAL PERISHABILITY RATE'//
     91X,'*****       MODEL I-B       *****'//
     11X,'5.  CONSTANT PERISHABILITY RATE'//
     21X,'6.  LINEAR PERISHABILITY RATE'//)
```

```
C
C SELECT A MODEL FROM THE MENUE
C
      READ(IN,*)KK
C
C CALL THE SUBRIOUTINE MODELI TO MAKE THE APPROPRIATE
C CALCULATIONS AND PRINT THE RESULTS.
C
      CALL MODELI(IN,IOUT,C0,C1,C3,C4,R,KK)
C
      WRITE(IOUT,125)
  125 FORMAT(1X/1X,'DO YOU WANT TO TRY A DIFFERENT MODEL ? IF YES,
'/
     11X,'TYPE 1, OTHERWISE TYPE 0 AND THEN PRESS THE ENTER KEY'/)
      READ(IN,*)IWORD
      IF(IWORD.EQ.1)GO TO 118
C
      WRITE(IOUT,127)
  127 FORMAT(1X/1X,'DO YOU WANT TO CHANGE YOUR INPUT'/
     1' PARAMETERS ? IF YES TYPE 1, OTHERWISE TYPE 0 '/
     2' AND THEN PRESS THE ENTER KEY'//)
      READ(IN,*)IWORD
      IF(IWORD.EQ.1)GO TO 104
      STOP
      END
C
C               ****************************
C               *                          *
C               *    SUBROUTINE MODELI     *
C               *                          *
C               ****************************
C
      SUBROUTINE MODELI(IN,IOUT,C0,C1,C3,C4,R,KK)
C
```

```
C
C---------->FUNCTION DEFINITION BLOCK
C
C THE ORDER QUNTITY FUNCTIONS OF MODELS 1-6
C
      Q1(T,A)=R*T*(1.+A*T)
      Q2(T,A,B)=R*T*(1.+A*T+.50*B*T**2)
      Q3(T,A,B,C)=R*T*(1.+A*T+.50*B*T**2+C*T**3/3.)
      Q4(T,A,B)=R*T*(1.+A*EXP(B*T)/B-A/B)
      Q5(T,A)=R*T*(1.+.50*A*T/(1.-A*T))
      Q6(T,A,B)=R*T*(1.+(3.*A*T+B*T**2)/(6.+6.*A*T+3.*B*T**2))
C
C THE PERISHABILITY FUNCTIONS OF MODELS 1-6
C
      D1(T,A)=Q1(T,A)-R*T
      D2(T,A,B)=Q2(T,A,B)-R*T
      D3(T,A,B,C)=Q3(T,A,B,C)-R*T
      D4(T,A,B)=Q4(T,A,B)-R*T
      D5(T,A)=Q5(T,A)-R*T
      D6(T,A,B)=Q6(T,A,B)-R*T
C
C THE COST FUNCTIONS OF MODELS 1-6
C
      C1T(T,A)=.50*C1*R*T+C3/T+(C1+C4)*R*A*T
      C2T(T,A,B)=C1T(T,A)+.50*(C1+C4)*R*B*T**2
      C3T(T,A,B,C)=C2T(T,A,B)+(C1+C4)*R*C*T**3/3.
      C4T(T,A,B)=.50*C1*R*T+(C1+C4)*(R*A/B)*(EXP(B*T)-1)+C3/T
      C5T(T,A)=.50*C1*R*T+.50*(C1+C4)*(1.-A*T)*R*A*T+C3/T
      C6T(T,A,B)=.50*C1*R*T+C3/T+((C1+C4)/T)*D6(T,A,B)
C
C THE OPTIMAL CYCLE FUNCTIONS OF MODELS 2-6
C
      Y2(T,A,B)=C3/(.50*C1*R+(C1+C4)*(R*A+R*B*T))
      Y3(T,A,B,C)=C3/(.50*C1*R+(C1+C4)*(R*A+R*B*T+R*C*T**2))
      Y4(T,A,B)=C3/(.50*C1*R+(C1+C4)*R*A*EXP(B*T))
      Y5(T,A,B)=2.*C3/(C1*R+(C1+C4)*R*A/(1.-A*T)**2)
```

```
      Y6(T,A,B)=C3/(.50*C1*R+((C1+C4)*(6.*A+5.*A*B+4.*B*T-10.
     1*A*B*T**2)/(3.*(2.-2.*A*T+B*T**2)**2)))
C------------>
C
C CYCLE TIME IF EOQ MODEL IS USED
C
      TEOQ=SQRT(2.*C3/(C1*R))
      TINV=1./TEOQ
C
      GO TO(10,20,30,40,50,60,70),KK
C
C ------>CONSTANT MODEL I-A
C
   10 WRITE(IOUT,100)
  100 FORMAT(1X/1X,'TYPE IN YOUR ESTIMATE OF THE VALUE A,',
     1/1X,'THE PERISHABILITY CONSTANT ( 0 < A <.9 % ) '//)
      WRITE(IOUT,101)
  101 FORMAT(1X,'A =   ')
      READ(IN,*)A
      IF(A.LE.TINV)GO TO 106
      WRITE(IOUT,107)TINV
  107 FORMAT(1X/1X,'PERISHABILITY COEFFICIENT(S) MUST BE',
     1/1X' SMALLER THAN = ',F5.4,//)
C
C DETERMINE THE RANGE & INCREMENT OF A
C FOR THE PRINTOUT
C
  106 AA=A/2.
      DEL1=.001
C
      WRITE(IOUT,110)
  110 FORMAT(1H1//,17X,'MODEL I-A--CONSTANT PERISHABILITY RATE',
     1//1X,72('-')/3X,'A',9X,'OPT T*',5X,'OPT Q*',4X,
     2'MIN COST',2X,'OPT PER',2X,'COST-EOQ',3X,'PER-EOQ'/
     31X,72('-')/)
```

```
C
C THE FOLLOWING LOOP CALCULATES THE INVENTORY
C CHARACTERISTICS FOR THE VARIOUS VALUES OF A
C
        DO 200 J=1,11
           A=AA
           IF(A.GT.TINV)GO TO 123
           TOPT=SQRT(2.*C3/(C1*R+2.*A*(C1*R+C4*R)))
           QOPT=Q1(TOPT,A)
           COPT=C1T(TOPT,A)
           DOPT=D1(TOPT,A)
           CEOQ=C1T(TEOQ,A)
           DEOQ=D1(TEOQ,A)
           WRITE(IOUT,120)A,TOPT,QOPT,COPT,DOPT,CEOQ,DEOQ
  120      FORMAT(1X,F5.4,3X,F9.3,3X,F9.3,3X,F8.2,3X,F5.2,3X,F9.3,3X
,F7.2)
           AA=AA+DEL1
  200 CONTINUE
C
C CHECK TO SEE IF ANOTHER TRIAL IS REQUIRED
C WITH A DIFFERENT PARAMETER
C
  123 WRITE(IOUT,124)
  124 FORMAT(1X/1X,'DO YOU WANT TO TRY ANOTHER',
     1' VALUE FOR A ?'/,1X,'YES(1)/NO(0)'/)
      READ(IN,*)IWORD
      IF(IWORD.EQ.1)GO TO 10
      RETURN
C
C------>LINEAR MODEL I-A
C
   20 WRITE(IOUT,130)
  130 FORMAT(1X/1X,'TYPE IN YOUR ESTIMATE OF THE VALUES A,',
     1/1X,'AND B THE PERISHABILITY COEFFICIENTS ( A,B  < .9 % 0 )'/)
```

```
      WRITE(IOUT,101)
      READ(IN,*)A
      WRITE(IOUT,102)
 102  FORMAT(1X,'B = ')
      READ(IN,*)B
      IF(A.LE.TINV.OR.B.LE.TINV)GO TO 206
      WRITE(IOUT,107)TINV
C
C DETERMINE THE RANGE & INCREMENT OF A & B
C FOR THE PRINTOUT
C
  206 AA=A/2.
      DEL1=.001
      BB=B/2.
      BT=BB
      DEL2=.001
      WRITE(IOUT,135)
  135 FORMAT(1H1//,17X,'MODEL I-A--LINEAR PERISHABILITY RATE',
     1//1X,80('-')/3X,'A',6X,'B',7X,'OPT T*',5X,'OPT Q*',3X,
     2'MIN COST',2X,'OPT PER',4X,'COST-EOQ',3X,
     3'PER-EOQ'/1X,80('-')/)
C
C THE FOLLOWING LOOP CALCULATES THE INVENTORY
C CHARACTERISTICS FOR THE VARIOUS VALUES OF A
C
      DO 202 J=1,4
         A=AA
         IF(A.GT.TINV)GO TO 223
         DO 204 K =1,4
            B=BB
            IF(B.GT.TINV)GO TO 204
            TO=TEOQ
            FO=Y2(TO,A,B)
  219       T1=SQRT(FO)
            F1=Y2(T1,A,B)
            IF(ABS(F1-FO).LT..001.OR.ABS(T1-TO).LT..001)GO TO 221
```

```
                TO=T1
                FO=F1
                GO TO 219
  221           TOPT=T1
                QOPT=Q2(TOPT,A,B)
                COPT=C2T(TOPT,A,B)
                DOPT=D2(TOPT,A,B)
                CEOQ=C2T(TEOQ,A,B)
                DEOQ=D2(TEOQ,A,B)
                WRITE(IOUT,121)A,B,TOPT,QOPT,COPT,DOPT,CEOQ,DEOQ
   121          FORMAT(1X,F5.4,2X,F5.4,1X,F9.3,3X,F9.3,2X,F8.2,4X,
      1         F7.2,3X,F9.3,3X,F7.2)
                BB=BB+DEL2
   204      CONTINUE
            BB=BT
            AA=AA+DEL1
   202 CONTINUE
C
C CHECK TO SEE IF ANOTHER TRIAL IS REQUIRED
C WITH A DIFFERENT PARAMETER
C
   223 WRITE(IOUT,224)
   224 FORMAT(1X/1X,'DO YOU WANT TO TRY ANOTHER',
      1' VALUE FOR A & B ?'/,1X,'YES(1)/NO(0)'/)
       READ(IN,*)IWORD
       IF(IWORD.EQ.1)GO TO 20
       RETURN
C
C------>QUADRATIC MODEL I-A
C
    30 WRITE(IOUT,132)
   132 FORMAT(1X/1X,'TYPE IN YOUR ESTIMATE OF THE VALUES A,',
      1/1X,'B, AND C THE PERISHABILITY COEFFICIENTS '/
      1'(  A < .9 % , B <.9 % , C < .09 % )  '/)
       WRITE(IOUT,101)
       READ(IN,*)A
```

```
      WRITE(IOUT,102)
      READ(IN,*)B
      WRITE(IOUT,316)
  316 FORMAT(1X,'C = ')
      IF(A.LE.TINV.OR.B.LE.TINV.OR.C.LE.TINV)GO TO 306
      READ(IN,*)C
C
      WRITE(IOUT,107)TINV
  306 AA=A/2.
      DEL1=.001
      BB=B/2.
      BT=BB
      DEL2=.001
      CC=C/2.
      CT=CC
      DEL3=.0002
      WRITE(IOUT,335)
  335 FORMAT(1H1//,17X,'MODEL I-A--QUADRATIC PERISHABILITY RATE',
     1//1X,82('-')/3X,'A',6X,'B',6X,'C',6X,'OPT T*',5X,'OPT Q*',3X
,
     2'MIN COST',2X,'OPT PER',4X,'COST-EOQ',3X,
     3'PER-EOQ'/1X,82('-')/)
C
      DO 303 J=1,2
         A=AA
         IF(A.GT.TINV)GO TO 323
         DO 304 K =1,2
          B=BB
           IF(B.GT.TINV)GO TO 303
           DO 305 I =1,5
            C=CC
            IF(C.GT.TINV)GO TO 304
            TO=TEOQ
            FO=Y3(TO,A,B,C)
  319       T1=SQRT(FO)
            F1=Y3(T1,A,B,C)
```

```
                IF(ABS(F1-F0).LT..001.OR.ABS(T1-T0).LT..001)GO TO 331
                T0=T1
                F0=F1
                GO TO 319
      331       TOPT=T1
                QOPT=Q3(TOPT,A,B,C)
                COPT=C3T(TOPT,A,B,C)
                DOPT=D3(TOPT,A,B,C)
                CEOQ=C3T(TEOQ,A,B,C)
                DEOQ=D3(TEOQ,A,B,C)
                WRITE(IOUT,340)A,B,C,TOPT,QOPT,COPT,DOPT,CEOQ,DEOQ
      340       FORMAT(1X,F5.4,2X,F5.4,2X,F5.4,1X,F8.3,3X,F9.3,2X,
         1      F8.2,4X,F5.2,3X,F9.3,3X,F7.2)
                CC=CC+DEL3
      305     CONTINUE
              CC=CT
              BB=BB+DEL2
      304    CONTINUE
             BB=BT
             AA=AA+DEL1
      303 CONTINUE
C
      323 WRITE(IOUT,324)
      324 FORMAT(1X/1X,'DO YOU WANT TO TRY ANOTHER',
         1' VALUE FOR A & B & C ?'/,1X,'YES(1)/NO(0)'/)
          READ(IN,*)IWORD
          IF(IWORD.EQ.1)GO TO 30
          RETURN
C
C------>EXPONENTIAL MODEL I-A
C
       40 WRITE(IOUT,430)
      430 FORMAT(1X/1X,'TYPE IN YOUR ESTIMATE OF THE VALUES A,',
         1/1X,'AND B THE PERISHABILITY COEFFICIENTS (0 < A <.9 % & B =
10*A',
         1/'  & B NOT EQUAL TO ZERO )'//)
```

```
      WRITE(IOUT,101)
      READ(IN,*)A
      WRITE(IOUT,102)
      READ(IN,*)B
C
      AA=A/2.
      DEL1=.001
      BB=B/2.
      BT=BB
      DEL2=.01
      WRITE(IOUT,435)
  435 FORMAT(1H1//,17X,'MODEL I-A--EXPONENTIAL PERISHABILITY RATE'
,
     1//1X,80('-')/3X,'A',6X,'B',7X,'OPT T*',5X,'OPT Q*',4X,
     2'MIN COST',1X,'OPT PER',4X,'COST-EOQ',3X,
     3'PER-EOQ'/1X,80('-')/)
C
      DO 404 J=1,4
         A=AA
         DO 406 K =1,4
            B=BB
            TO=TEOQ
            FO=Y4(TO,A,B)
  419       T1=SQRT(FO)
            F1=Y4(T1,A,B)
            IF(ABS(F1-FO).LT..001.OR.ABS(T1-TO).LT..001)GO TO 441
            TO=T1
            FO=F1
            GO TO 419
  441       TOPT=T1
            QOPT=Q4(TOPT,A,B)
            COPT=C4T(TOPT,A,B)
            DOPT=D4(TOPT,A,B)
            CEOQ=C4T(TEOQ,A,B)
            DEOQ=D4(TEOQ,A,B)
            WRITE(IOUT,121)A,B,TOPT,QOPT,COPT,DOPT,CEOQ,DEOQ
```

```
                  BB=BB+DEL2
  406       CONTINUE
            BB=BT
            AA=AA+DEL1
  404  CONTINUE
C
       WRITE(IOUT,224)
       READ(IN,*)IWORD
       IF(IWORD.EQ.1)GO TO 40
       RETURN
C
C----->CONSTANT MODEL I-B
C
   50  WRITE(IOUT,100)
       READ(IN,*)A
       IF(A.LE.TINV)GO TO 506
       WRITE(IOUT,107)TINV
  506  AA=A/2.
       DEL1=.001
       WRITE(IOUT,510)
  510  FORMAT(1H1//,17X,'MODEL I-B --CONSTANT PERISHABILITY RATE',
      1//1X,72('-')/3X,'A',9X,'OPT T*',5X,'OPT Q*',5X,
      2'MIN COST',1X,'OPT PER',2X,'COST-EOQ',3X,'PER-EOQ'/
      31X,72('-')/)
       DO 500 J=1,11
          A=AA
          IF(A.GT.TINV)GO TO523
          TOPT=SQRT(2.*C3/(C1*R+2.*A*(C1*R+C4*R)))
          QOPT=Q5(TOPT,A)
          COPT=C5T(TOPT,A)
          DOPT=D5(TOPT,A)
          CEOQ=C5T(TEOQ,A)
          DEOQ=D5(TEOQ,A)
          WRITE(IOUT,120)A,TOPT,QOPT,COPT,DOPT,CEOQ,DEOQ
          AA=AA+DEL1
  500  CONTINUE
```

```
C
  523 WRITE(IOUT,124)
      READ(IN,*)IWORD
      IF(IWORD.EQ.1)GO TO 50
      RETURN
C
C------>LINEAR MODEL I-B
C
   60 WRITE(IOUT,130)
      WRITE(IOUT,101)
      READ(IN,*)A
      WRITE(IOUT,102)
      READ(IN,*)B
      IF(A.LE.TINV.OR.B.LE.TINV)GO TO 606
      WRITE(IOUT,107)TINV
C
C DETERMINE THE RANGE & INCREMENT OF A & B
C FOR THE PRINTOUT
C
  606 AA=A/2.
      DEL1=.001
      BB=B/2.
      BT=BB
      DEL2=.001
      WRITE(IOUT,636)
  636 FORMAT(1H1//,17X,'MODEL I-B--LINEAR PERISHABILITY RATE',
     1//1X,80('-')/3X,'A',6X,'B',7X,'OPT T*',5X,'OPT Q*',3X,
     2'MIN COST',2X,'OPT PER',4X,'COST-EOQ',3X,
     3'PER-EOQ'/1X,80('-')/)
C
C THE FOLLOWING LOOP CALCULATES THE INVENTORY
C CHARACTERISTICS FOR THE VARIOUS VALUES OF A
C
      DO 602 J=1,4
         A=AA
         IF(A.GT.TINV)GO TO 623
```

```
          DO 604 K =1,4
            B=BB
            IF(B.GT.TINV)GO TO 602
            TO=TEOQ
            FO=Y6(TO,A,B)
  619       T1=SQRT(FO)
            F1=Y6(T1,A,B)
            IF(ABS(F1-FO).LT..001.OR.ABS(T1-TO).LT..001)GO TO 661
            TO=T1
            FO=F1
            GO TO 619
  661       TOPT=T1
            QOPT=Q6(TOPT,A,B)
            COPT=C6T(TOPT,A,B)
            DOPT=D6(TOPT,A,B)
            CEOQ=C6T(TEOQ,A,B)
            DEOQ=D6(TEOQ,A,B)
            WRITE(IOUT,121)A,B,TOPT,QOPT,COPT,DOPT,CEOQ,DEOQ
            BB=BB+DEL2
  604     CONTINUE
          BB=BT
          AA=AA+DEL1
  602 CONTINUE
C
  623 WRITE(IOUT,224)
      READ(IN,*)IWORD
      IF(IWORD.EQ.1)GO TO 20
      RETURN
   70 WRITE(IOUT,700)
  700 FORMAT(1X,'YOU HAVE INPUTTED  A WRONG MODEL NUMBER'//)
      RETURN
      END
```

APPENDIX B

TABLES OF COMPARATIVE RESULTS OF AN EXAMPLE PROBLEM

To illustrate the results of the methodology of this paper in comparison to the methods of other researchers and the EOQ analysis, the following finite production rate example which is due to Shah and Jaiswal [88] is considered.

Assume:

p = 625 Items/Month	(Production Rate)
d = 200 Items/Month	(Demand Rate)
C_1 = \$.05/Item/Month	(Carrying Cost)
C_3 = \$50.00/Order	(Replenishment Cost)
C_4 = \$3.00/Item	(Perishing Cost)
a = .02; .05; .10	(Perishability Constant)

The results in the following tables are based on the calculations made on an IBM/3031 computer utilizing Shah and Jaiswal's equations, and the equations derived in this research. (For the case $a = .02$; this author was unable to duplicate the results, as stated in Shah and Jaiswal's paper [88], by utilizing their equations. Their stated results are slightly different from the ones stated in Table I.)

TABLE I

COMPARATIVE RESULTS FOR PERISHABILITY RATE = .02

Inventory Parameters	Results of this Methodology	Results of EOQ Analysis	Results of Shah and Jaiswal
T*(Inventory Cycle)	2.5070	3.8348	2.7946
T_1(Production Cycle)	.8156	1.2594	.9114
q (Replenishment Size)	509.7604	787.1460	569.6040
D_1(Items perishing per year)	40.9876	63.1667	45.8464
D_2(Items perishing per cycle)	8.5629	20.1860	10.6770
C*(Optimal cost)	38.7572	41.9865	38.9484

TABLE II

COMPARATIVE RESULTS FOR PERISHABILITY RATE = .05

Inventory Parameters	Results of this Methodology	Results of EOQ Analysis	Results of Shah and Jaiswal
T*(Inventory Cycle)	1.8900	3.8348	2.4752
T_1(Production Cycle)	.6246	1.3089	.8259
q (Replenishment Size)	390.3659	818.0596	516.1706
D_1(Items perishing per year)	77.9314	159.9028	102.3970
D_2(Items perishing per cycle)	12.2742	51.0996	21.1211
C*(Optimal cost)	52.4239	66.3394	54.3954

TABLE III

COMPARATIVE RESULTS FOR PERISHABILITY RATE = .10

Inventory Parameters	Results of this Methodology	Results of EOQ Analysis	Results of Shah and Jaiswal
T*(Inventory Cycle)	1.8280	3.8348	2.4076
T_1(Production Cycle)	.6221	1.3938	.8352
q (Replenishment Size)	388.8420	871.1329	522.0031
D_1(Items perishing per year)	152.3113	325.9818	201.7890
D_2(Items perishing per cycle)	23.2021	104.1729	40.4856
C*(Optimal cost)	71.7685	108.1165	79.7288

APPENDIX C

DESCRIPTION AND SOURCE LISTING OF FORTRAN PROGRAMS OF SEARCH ROUTINES

Program Description

This appendix describes the computer programs used to evaluate the optimal inventory cycle time through search routines. The two programs that are utilized for this purpose are Fibonacci and Hooke and Jeeves' search routines. The FORTRAN programs are based on the codes of Keuster and Mize [56].

Fibonacci Search Procedure

This program consists of a main program and a user-supplied subroutine, FUNC, which contains the objective function to be optimized. The main program reads the inputs consisting of inventory parameters and specifications for the search routine. The program then provides the user with the optimal inventory cycle time and optimal cost.

User Requirements for Fibonacci Program

The evaluation of the optimal inventory cycle time requires the inputs of the following variables in the format described below:

Card Type	Format	Variables
1	(3E10.4)	ALPHA, A, B
2	(8E10.4)	H, P, D, C, C1, C2, C3, C4

The required variables are defined as follows:

ALPHA -- Desired accuracy specified as a fraction of the original search interval. The recommended value of alpha is .01 or less.

A -- Lower constraint. The recommended value is (B/4).

B -- Upper constraint. The recommended value is equal to (2C3/C1D).

H -- Perishability Constraint

P -- Production Rate

D -- Demand Rate

C -- Purchase Cost

C1 -- Carrying Cost

C2 -- Shortage Cost

C3 -- Replenishment Cost

C4 -- Perishing Cost

The objective function must be set equal to variable Y in the subroutine FUNC.

Hooke and Jeeves' Search Procedure

This program consists of a main program, subroutine Hooke, and user-supplied subroutine OBJECT which contains the objective function to be optimized. The main program reads the inputs consisting of inventory parameters and specification for the search routine, and subroutine Hooke performs all searches and provides the printout.

User Requirements for Hooke and Jeeves' Program

The input variables and the required format for this program are:

Card Type	Format	Variable
1	2I10	ITMAX, NKAT
2	2E10.4	(RK(J), J=1,2)
3	2E10.4	(EPS(J),J=1,2)
4	(3E10.4)	ALPHA, BETA, EPSY
5	(8E10.4)	H,P,D,C,C1,C2,C3,C4

The objective function must be set equal to variable SUMN in the subroutine OBJECT.

The variables are defined as follows:

ITMAX -- Maximum number of times the objective function is called.

NKAT -- Maximum number of times the initial step size is to be reduced.

RK -- Nector of initial guesses for decision variables, that is, the inventory cycle time, and the production cycle time.

EPS -- Vector of initial step size to be used for each of the variables.

ALPHA -- Factor for extending the size of initial setps, greater than 1.0.

BETA -- Factor for reducing the initial step size, greater than zero and less than 1.0.

EPSY -- Error in objective function to be reached before program terminates.

Program Output

These programs provide the user with optimal inventory cycle time and "optimal" inventory cost. In addition, they provide information as to number of function evaluations, and the degree of accuracy at the final stage of calculations.

```
C
C
C
C
C                            ******************
C                            *                *
C                            * SEARCH ROUTINES *
C                            *                *
C                            ******************
C
C
C
C
C
C  THESE ROUTINES ARE BASED LARGELY ON THE CODES PRESENTED
C  IN OPTIMIZATION TECHNIQUES WITH FORTRAN BY KUESTER & MIZE.
C
C
C
C
C---->FIBONACCI SEARCH PROCEDURE
C
C
      DIMENSION FIB(50)
      COMMON/PARAM/H,P,D,C,C1,C2,C3,C4
C
      NI=5
      NO=6
C
      READ (NI,016) ALPHA, A, B
  016 FORMAT (8E10.4)
C
C     READ INVENTORY PARAMETERS
C
      READ (NI,016)H,P,D,C,C1,C2,C3,C4
C
```

```
          DEL = B - A
          WRITE (NO,001)
      001 FORMAT (1H1,10X,35HFIBONACCI SINGLE-VARIABLE PROCEDURE )
C
C         DEFINE THE FIRST THREE FIBONACCI NUMBERS
C
          FIBO = 1.0
          FIB(1) = 1.0
          FIB(2) = 2.0
C
C          CALCULATE THE REMAINING FIBONACCI NUMBERS
C
        5 BB = 1.0/ALPHA
          IF (BB - 2.0)  10, 10, 11
       10 GO TO 14
       11 CONTINUE
          JJ=2
       12 JJ=JJ+1
          FIB(JJ)=FIB(JJ-1)+FIB(JJ-2)
          CC=FIB(JJ)
          IF(CC-BB) 13,15,15
       13 GO TO 12
       14 WRITE (NO,002)
      002 FORMAT (///,10X,42HACCURACY SPECIFIED IN FUNC NOT SUFFICIENT. ,
         1 //,10X,34HPROGRAM RESET ALPHA, ALPHA = 0.005)
          ALPHA = 0.005
          GO TO 5
C
C          FIRST STEP IN THE TABLEAU
C
       15 I=0
          KK=JJ-2
          IK=JJ-2
          BL=B-A
          ALL=FIB(IK)*BL/FIB(JJ)
          W=A+ALL
```

```
      V=B-ALL
      CALL FUNC(W,T)
      CALL FUNC(V,U)
      JK=1
      WRITE (NO,003)
  003 FORMAT (//,1X, 1HK,5X,2HLK,10X,2HAK,11X,2HBK,09X,3HLLK,11X,1HX,
     1 12X,1HY )
      WRITE (NO,004)  JK, BL, A, B, ALL, W, T
      WRITE (NO,006) V, U
  004 FORMAT (/,1X, I1,2X,E11.4,1X,E11.4,2X,E11.4,2X,E11.4,
     1 2X,E11.4,2X,E11.4)
  006 FORMAT(55X,E11.4,2X,E11.4)
C
C     SUCCEEDING STEPS IN THE TABLEAU
C
      IK=IK-1
      JJ=JJ-1
      DO 70 I=1,KK
      IF(U-T) 20,20,22
   20 A=A+ALL
      BL=B-A
      W=V
      CALL FUNC(W,T)
      ALL=FIB(IK)*BL/FIB(JJ)
      V=B-ALL
      CALL FUNC(V,U)
      II=I+1
      IK=IK-1
      JJ=JJ-1
      IF(IK-1) 28,29,29
   28 IK=1
   29 CONTINUE
      WRITE (NO,004) II, BL, A, B, ALL, W, T
      WRITE (NO,006) V, U
      GO TO 70
   22 B=B-ALL
```

```
      BL=B-A
      V=W
      CALL FUNC(V,U)
      ALL=FIB(IK)*BL/FIB(JJ)
      W=A+ALL
      CALL FUNC(W,T)
      II=I+1
      IK=IK-1
      JJ=JJ-1
      IF(IK-1) 30,31,31
   30 IK=1
   31 CONTINUE
      WRITE (NO,004) II, BL, A, B, ALL, V, U
      WRITE (NO,006) W, T
      GO TO 70
   70 CONTINUE
C
C     CALCULATION OF THE FINAL RANGE OF THE DEPENDENT VARIABLE
C
      EPS = 0.001 * W
      DL=W+EPS
      CALL FUNC(DL,YL)
      IF(YL-T) 80,80,81
   80 CALL FUNC(B,BF)
      WRITE (NO,007) W, B
  007 FORMAT(///       26H THE FINAL FEASIBLE REGION,2X,2HX=,
     1 E11.4,2X,2HX=,E11.4)
      WRITE (NO,008) T, BF
  008 FORMAT(/  21H WITH FUNCTION VALUES,7X,2HY=,E11.4,2X,2HY=,E11.4)
      GO TO 87
   81 CALL FUNC(A,AF)
      WRITE (NO,009) W, A
  009 FORMAT(///       26H THE FINAL FEASIBLE REGION,2X,2HX=,
     1 E11.4,2X,2HX=,E11.4)
      WRITE (NO,017) T, AF
  017 FORMAT(/  21H WITH FUNCTION VALUES,7X,2HY=,E11.4,2X,2HY=,E11.4)
```

```
   87 ACC=(W-A)/(DEL)
      WRITE (NO,018) ACC
  018 FORMAT(/    ,    16H THE ACCURACY IS,12X,E11.4)
      WRITE (NO,019) ALPHA
  019 FORMAT(/      26H THE REQUIRED ACCURACY WAS,2X,E11.4)
  999 CONTINUE
      WRITE (NO,001)
      STOP
C
      END
C
C

      SUBROUTINE FUNC(X,Y)
      COMMON/PARAM/H,P,D,C,C1,C2,C3,C4
      T2=X
      T1=(1./H)*ALOG(1.+(D/P)*(EXP(H*T2)-1.))
      XI1=(P*T1-D*T2)/H+(-P+(P-D)*EXP(-H*T1)+D*EXP(H*(T2-T1)))/H**2
      Q=P*T1
      DT=P*T1-D*T2
      Y=(C3+C1*XI1+C4*DT)/T2
      RETURN
C
      END
```

```
C
C
C
C----->HOOKE & JEEVES SEARCH PROCEDURE
C
C
C
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION EPS(2), RK(2), Q(2), QQ(2), W(2)
      COMMON NI,NO
      COMMON/PARAM/H,P,D,C,C1,C2,C3,C4
C
      NI=5
      NO=6
      NSTAGE=2
      IPRINT=0
C
C
    5 CONTINUE
      READ (NI,001,END=4) ITMAX,NKAT
  001 FORMAT (8I10)
      READ (NI,002) (RK(II), II=1,NSTAGE)
  002 FORMAT (8E10.4)
      READ (NI,002) (EPS(JJ), JJ=1,NSTAGE)
C
      READ (NI,003) ALPHA, BETA, EPSY
C
C     READ INVENTORY PARAMETERS
C
      READ (NI,002)H,P,D,C,C1,C2,C3,C4
C
  003 FORMAT (8E10.4)
      QD = 0.0
C
      CALL HOOKE (RK,EPS,NSTAGE,ITMAX,NKAT,EPSY,ALPHA,BETA,QD,Q,QQ,W,
     1 IPRINT)
```

```
C
      GO TO 5
    4 STOP
      END
      SUBROUTINE HOOKE (RK,EPS,NSTAGE,MAXK,NKAT,EPSY,ALPHA,BETA,QD,
     1 Q,QQ,W,IPRINT)
C
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION RK(NSTAGE), EPS(NSTAGE), Q(NSTAGE), QQ(NSTAGE),
     1 W(NSTAGE)
      COMMON/PARAM/H,P,D,C,C1,C2,C3,C4
      COMMON NI,NO
C
C
      WRITE (NO,001)
  001 FORMAT (1H1,10X,37HHOOKE AND JEEVES OPTIMIZATION ROUTINE)
      WRITE (NO,002) ALPHA, BETA, MAXK, NKAT
  002 FORMAT (//,2X,10HPARAMETERS,/,2X,8HALPHA = ,F5.2,4X,
     1 7HBETA = ,F5.2,4X,8HITMAX = ,I4,4X,7HNKAT = ,I3)
      WRITE (NO,003) NSTAGE
  003 FORMAT (/,2X,22HNUMBER OF VARIABLES = ,I3)
      WRITE (NO,004)
  004 FORMAT (/,2X,18HINITIAL STEP SIZES)
      DO 6  I=1,NSTAGE
      WRITE (NO,005)  I, EPS(I)
  005 FORMAT (/,2X,4HEPS(,I2,4H) = ,E16.8)
    6 CONTINUE
      WRITE (NO,007) EPSY
  007 FORMAT (/,2X,43HERROR IN FUNCTION VALUES FOR CONVERGENCE = ,E16.8)
      KFLAG = 0
      DO 601 I=1,NSTAGE
      Q(I) =RK(I)
      W(I) = 0.0
  601 CONTINUE
      KAT =0.0
```

```
      KK1 =0
   70 KCOUNT =0
      WBEST = W(NSTAGE)
      CALL OBJECT (SUM,RK,NSTAGE)
      KK1= KK1+ 1
      BO =SUM
      IF (KK1.EQ. 1)  QD = SUM
      IF (KK1.EQ. 1)   GO TO 201
      IF(BO.GT.QD)   KFLAG = 1
      IF (BO.LT.QD)  QD = BO
C
C        ESTABLISHING THE SEARCH PATTERN
C
  201 DO   55    I = 1,NSTAGE
      QQ(I)=RK(I)
      TSRK = RK(I)
      RK(I) = RK(I) + EPS(I)
      CALL OBJECT (SUM,RK,NSTAGE)
      KK1= KK1+ 1
      W(I) = SUM
      IF (W(I) .LT.QD)   GO TO 58
      RK(I) = RK(I) - 2.0*EPS(I)
      CALL OBJECT (SUM,RK,NSTAGE)
      KK1= KK1+ 1
      W(I) = SUM
      IF (W(I) .LT.QD)   GO TO 58
      RK(I) = TSRK
      IF (I.EQ. 1) GO TO 513
      W(I) =W(I-1)
      GO TO 613
  513 W(I) =BO
  613 CONTINUE
      KCOUNT =1+ KCOUNT
      GO TO 55
   58 QD= W(I)
      QQ(I) =RK(I)
```

```
   55 CONTINUE
      IF (IPRINT) 60, 65, 60
   60 WRITE (NO,100) KK1
C
C        RECORD RESPONSES AND LOCATION
C
      WRITE(NO,102)
      WRITE(NO,207) (RK(I), I=1,NSTAGE), QD
C
C        TEST TO DETERMINE TERMINATION OF PROGRAM
C
   65 IF (KK1.GT.MAXK) GO TO 94
      IF (KAT .GE. NKAT)   GO TO 94
      IF(DABS(W(NSTAGE)-WBEST).LE.EPSY) GO TO 94
C
C        IF ALL AXES FAIL REDUCE STEP SIZE
C
      IF (KCOUNT .GE. NSTAGE ) GO TO 28
      DO 26    I = 1,NSTAGE
      RK(I) =RK(I) + ALPHA*(RK(I) - Q(I))
   26 CONTINUE
      DO 25     I = 1,NSTAGE
      Q(I) =QQ(I)
   25 CONTINUE
      GO TO 70
C
C        REDUCE STEP SIZE
C
   28 KAT = KAT + 1
      IF (KFLAG .EQ. 1)    GO TO 202
      GO TO 204
  202 KFLAG = 0
      DO 203    I = 1,NSTAGE
      RK(I) = Q(I)
  203 CONTINUE
```

```
  204 DO 80     I=1,NSTAGE
      EPS(I) =EPS(I) *BETA
   80 CONTINUE
      IF (IPRINT) 85, 70, 85
   85 WRITE (NO,101) KAT
      GO TO 70
   94 WRITE (NO,460) (EPS(I), I=1,NSTAGE)
      WRITE (NO,461) (RK(I), I=1,NSTAGE)
      WRITE (NO,462) QD
      DO 104 I=1,NSTAGE
  104 WRITE (NO,103) I, RK(I)
      WRITE (NO,100) KK1
  100 FORMAT (//,2X,33HNUMBER OF FUNCTION EVALUATIONS = ,I8)
  101 FORMAT (/,2X,18HSTEP SIZE REDUCED ,I2,6H TIMES)
  102 FORMAT(1X,26HEND OF EACH PATTERN SEARCH/)
  103 FORMAT (//,2X,8HFINAL X(,I2,4H) = ,1PE16.8)
  207 FORMAT(1X,18HVARIABLES AND SUMN,3X,9E12.4//)
  460 FORMAT(1X, 18H THE FINAL EPS ARE,    4F20.8/)
  461 FORMAT (1X, 18H THE FINAL RK ARE ,    5F20.8/)
  462 FORMAT (1X, 24H THE MINIMUM RESPONSE IS,     F20.8/)
      WRITE(NO,1)
      RETURN
      END

      SUBROUTINE OBJECT (SUMN,AKE,NSTAGE)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION AKE(NSTAGE)
      COMMON/PARAM/H,P,D,C,C1,C2,C3,C4
      T2=AKE(1)
      T=AKE(2)
      T1=(1./H)*DLOG(1.+(D/P)*(DEXP(H*T2)-1.))
      XI1=(P*T1-D*T2)/H+(-P+(P-D)*DEXP(-H*T1)+D*DEXP(H*(T2-T1)))/H**2
      XI2=(D/2)*(T-T2)**2*(1.-(D/P))
      Q=P*T1+D*(T-T2)
      DT=Q-D*T
      Y=(C3+C1*XI1+C4*DT+C2*XI2)/T
```

```
        SUMN=Y
  C
        RETURN
        END
```

VITA

Feraidoon Raafat

Candidate for Degree of

Doctor of Philosophy

Thesis: ANALYSIS OF AGE-INDEPENDENT PERISHABLE ITEMS SUBJECT TO ON-GOING DETERIORATION INVENTORY SYSTEM

Major Field: Industrial Engineering and Management

Biographical:

Personal Data: Born in Tehran, Iran, January 5, 1953, the son of Manucher and Parvaneh Raafat.

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Professional Experience: Worked at Fort Smith Structural Steel Corporation in the summer of 1975 as a general trainee. Employeed as a junior engineer at William Brothers Engineering Company from June, 1976, to January, 1977, where he did his graduate internship. Employed by Oklahoma State University as a Graduate Assistant, Graduate-Associate, Instructor from 1977 to 1981. Employed by Wichita State University as Assistant Professor of Decision Science from August, 1981, to present.

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