DOMINANCE OF SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS

By

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PREFACE

This study is concerned with determining the dominance structure of the solution space of two term n-th order linear quasi-differential equations. This is a burgeoning field of inquiry, and in this study I have tried to give a comprehensive development of the current state of the theory. Special attention has been given to isolating and identifying those points at which the theory is weak and where further work is needed.

I would like to express my deep appreciation to my thesis advisor, Dr. Marvin Keener, for his unfailing aid and guidance in my studies. Without his help I might never have known the thrill of the hunt in Mathematics. I also wish to thank the other members of my committee, Dr. Hermann Burchard, Dr. John Wolfe, Dr. Thomas Karman, and Dr. Donald Fisher, for their assistance and suggestions in preparing this thesis. Special thanks are due Charlene Fries for her typing of the first draft, and to Barbara Newport for her expert and beautiful typing of the final copy.

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CHAPTER I

INTRODUCTION

The study of the solutions of a differential equation is frequently divided into two classes: quantitative behavior and qualitative behavior. A quantitative study provides solutions either by an explicit formula or by numerical techniques, and in one sense this type of description is best since questions of a numerical nature can be answered. However, for many applications involving differential equations, only the qualitative behavior of the solutions is of interest. It may be possible to give qualitative descriptions of solutions of differential equations for which particular solutions are unobtainable. Furthermore, particular solutions may not be useful for answering qualitative questions. Calculating one solution might be so difficult or time consuming that it would be impractical to find enough solutions to describe the solution space of a higher order problem. And if all the solutions could be found, information might be available only point by point, rendering any description in the large virtually impossible.

Much effort, therefore, has been devoted to the separate study of the qualitative behavior of solutions. Such study seeks to describe the behavior of the solutions in general terms apart from any of their specific numerical values. For the differential equations considered in this work, such efforts have been quite varied and disorganized in the past. Many different settings have been considered, and each has evolved its own notation. A major goal of the current study is the unification of these efforts within a consistent notational framework. Many of the questions discussed here are still being actively researched, and such a framework may make the remaining difficulties more clear.

To be specific, consider the n-th order linear homogeneous differential equation

$$Ly \equiv y^{(n)} + p_{n-1}(x)y^{(n-1)} + \ldots + p_1(x)y' + p_0(x)y = 0 \tag{1.1}$$

where the coefficients $p_i(x)$ are real-valued and continuous on the interval $I = [0,\infty)$ (other intervals may be considered, but primarily the interval $[0,\infty)$ is used here to simplify the discussion). One objective of this thesis is to study the behavior of solutions of (1.1) on the interval I. The conditions given in Equation (1.1) are sufficient to guarantee the existence and uniqueness of solutions of the initial value problem, and so the limiting behavior of a solution $y(x)$ is predetermined by the values $y(0)$, $y'(0), \ldots,$ $y^{(n-1)}(0)$. In practice, however, this behavior may be difficult to ascertain, and may be greatly affected by small changes in the initial conditions. One problem of current interest would be to determine a setting in which the limiting behavior of solutions can be effectively characterized in terms of their initial conditions.

The basic descriptive terminology and notation is developed in this introductory chapter, and a few examples are provided of the variety of behaviors possible for Equation (1.1). In Chapter II the notion of a disconjugate operator is explored and a canonical framework is established which is essential for the later chapters. The concept of dominance of solutions of differential equations is introduced in Chapter III, based on the work of Dolan and Klaasen. With the aid of several examples, the problems associated with trying to analyze Equation (1.1) in its full generality are also discussed in this chapter, suggesting that a more restricted equation should be considered. In Chapter IV, utilizing the recent work of Uri Elias, the equation $L_n y + p(x)y = 0$, where L_n is an n-th order disconjugate operator, is shown to be a very natural restriction of Equation (1.1) with sufficient conditions to allow a more detailed study. In Chapter V the dominance of solutions of the new equation is discussed, and in Chapter VI the focus is restricted still further to a consideration of the equation $y^{(n)} + p(x)y = 0$.

Definitions and Examples

In order to discuss the problem in more detail, some descriptive terminology must be established. A nontrivial solution $y(x)$ of Equation (1.1) is called <u>oscillatory</u> on the interval I provided it has an infinity of zeros in I, and <u>nonoscillatory</u> if the number of zeros in I is finite. Note that no nontrivial solution $y(x)$ can have an accumulation of zeros at some point $x = a$ in I, or else the continuity of y and its first $n-1$ derivatives in conjunction with Rolle's theorem would imply that $y(a) = y'(a) = \ldots = y^{(n-1)}(a) = 0$. This in turn would imply $y(x) \equiv 0$ by the uniqueness of solutions of the initial value problem, a contradiction. Thus the zeros of an oscillatory solution are isolated and unbounded in the interval I. The equation (1.1) is called nonoscillatory provided no nontrivial solution of (1.1) is oscillatory, and oscillatory otherwise. The equation (1.1) (and the operator L) is called <u>disconjugate</u> provided no nontrivial solution has more than $n-1$ zeros in I, counting multiplicities. That is, if $y(b) = y'(b) = \ldots = y^{(k-1)}(b) = 0$, $y^{(k)}(b) \neq 0$, then y is said to have a zero of multiplicity k (or order k) at $x = b$.

In what follows, the term "solution" means "nontrivial solution" unless otherwise noted.

In the second order case

$$y'' + p(x)y' + q(x)y = 0, \tag{1.2}$$

the classic 1836 work by Sturm [115] showed that either every solution of Equation (1.2) oscillates, or none do. This is illustrated by the equations

$$y'' + y = 0 \tag{1.3}$$

and

$$y'' - y = 0 \tag{1.4}$$

with solutions of the form $y = a\cos x + b\sin x$ and $y = ce^{x} + de^{-x}$, respectively, where a, b, c, and d are arbitrary constants. The second equation is disconjugate as well as nonoscillatory. When $n > 2$ in (1.1), however, the picture is more complicated. The equation

$$y^{(4)} - y = 0, \tag{1.5}$$

with solutions of the form $y = a\sin x + b\cos x + ce^{x} + de^{-x}$, has both oscillatory and nonoscillatory solutions. On the other hand, the equation

$$y^{(4)} - 5y'' + 4y = 0 \tag{1.6}$$

is nonoscillatory and disconjugate since its solutions are given by $y = ae^{2x} + be^{x} + ce^{-x} + de^{-2x}$.

A major question in the early study of disconjugacy was whether nonoscillation and disconjugacy were equivalent concepts for Equation (1.1) with $n > 2$. Leighton and Nehari [76] in 1958 answered the question in the affirmative for the fourth order equation

$$(r(x)y'')'' + p(x)y = 0 \tag{1.7}$$

where $r(x) > 0$ and either $p(x) > 0$ or $p(x) < 0$. Essentially, they showed that because of its special form, Equation (1.7) behaves quite similarly to the second order equation (1.2). In a like manner, other authors such as Keener [53, 56] have analyzed various forms of Equation (1.1) by identifying similarities with the second order problem. However, Gustafson [34] showed in 1970 that in general these concepts are not equivalent. For every $n > 2$, Gustafson gave an example of an n-th order equation for which there exist solutions with arbitrarily large numbers of zeros, but for which every solution is nonoscillatory. Thus for the general n-th order equation (1.1), disconjugacy is a stronger property than nonoscillation.

It is important to underscore the difficulty involved in trying to study Equation (1.1) in its most general form. As suggested before, Keener [54], Ridenhour [106], and others have used a variety of assumptions and techniques in approaching questions of nonoscillation and disconjugacy. A large part of the current study, beginning in Chapter IV, will be devoted to a type of equation which does guarantee the equivalence of these properties.

Several additional definitions will be useful in what follows. If Equation (1.1) is not disconjugate on the interval $[a,b)$, $b \leq \infty$, then there is a (nontrivial) solution $y(x)$ with at least n zeros in $[a,b)$. Define the first <u>conjugate point</u> $\eta_1(a)$ of a to be the infimum of those values c, $a < c < b$, for which a solution $y(x)$ exists with at least n zeros in $[a,c]$. If Equation (1.1) is disconjugate on $[a,\infty)$, set $\eta_1(a) = \infty$. In a like manner, the i-th conjugate point $\eta_i(a)$ of a is defined to be the infimum of those values c, $a < c$, for which there is a solution of Equation (1.1) with at least $n + i - 1$ zeros in $[a,c]$. If there is no such solution, take $\eta_i(a) = \infty$. The i-th focal point $\zeta_i(a)$ of a is the infimum of those values $c > a$ for which there is a solution with at least i zeros for each of $y(x)$, $y'(x)$, . . . , $y^{(n-1)}(x)$ in $[a,c]$.

This use of the terms conjugate point and focal point is rooted in the calculus of variations. Weierstrass introduced the concept of a conjugate point in 1879 for the second order Jacobi equation, where the absence of conjugate points is associated with the location of minimums of functionals. The term focal point (in German, Brennpunkte) was used first for the Jacobi equation by Kneser in about 1900 [104, p. 22]. Sherman [111], in 1965, extended the notion of conjugate point to the n-th order equation (1.1), and it is in this spirit that the term is used here. The extension of the focal point terminology to the n-th order equation is less well established. A definition based upon first order differential systems may be found in Reid [104]. The definition used here can be traced to Nehari [93].

Considering the solution $y = \sin x$ of Equation (1.3) reveals that $\eta_i(0) = i\pi$, for $i = 1,2,\ldots$, and that $\zeta_i(0) = \pi/2 + (i-1)\pi$, $i = 1,2,\ldots$. Equation (1.4), on the other hand, has $\eta_i(0) = \infty$ and $\zeta_i(0) = \infty$ for all i. Since $y = \sin x$ is also a solution of Equation (1.5), it is easy to see that $\eta_i(0) < i\pi + 2\pi$, but the actual minimum values which define $\eta_i(0)$, $i = 1,2,\ldots$, for Equation (1.5) are more difficult to determine. The problem of determining these values is discussed in Chapter IV. It is worth noting at this point, however, that it follows immediately from Rolle's theorem that $\zeta_i(a) \leq \eta_i(a)$, $i = 1,2,\ldots$, for every equation of the form (1.1).

A solution y(x) is said to have a (k,n-k) zero distribution on [a,b] if y has a zero of order not less than k at $x = a$ and not less than n-k at $x = b$. The interval [a,b] is called a (k,n-k) interval of oscillation for Equation (1.1) provided there is a solution which is positive on (a,b) and has a (k,n-k) zero distribution on [a,b]. Equation (1.1) is said to have (k,n-k) oscillation type provided there is a (k,n-k) interval of oscillation in every half-line $[M,\infty)$, $M > 0$. Finally, if no such interval exists in $[M,\infty)$ for some $M \geq 0$, then the equation is said to be (k,n-k) disconjugate there, and Equation (1.1) is called eventually (k,n-k) disconjugate. As will be made clear later, in certain situations there is a direct relationship between (k,n-k) disconjugacy and the existence of oscillatory solutions. Keener and Travis [57] have also given a definition for (k,n-k) disfocal, but this terminology will not be needed here.

For examples of these concepts, again consider Equation (1.3). The solution y(x) = sinx has a (1,1) zero distribution on $[0,\pi]$, and since $\sin x > 0$ for $x \in (0,\pi)$, this interval is a (1,1) interval of oscillation. It follows similarly that each interval $[k\pi,(k+1)\pi]$ is a (1,1) interval of oscillation, and so Equation (1.3) has (1,1) oscillation type. For Equation (1.4), no nontrivial solution ever has two zeros on any interval since the equation is disconjugate, so consequently it is also (1,1) disconjugate. Trivially, by the uniqueness of solutions, every equation of the form (1.1) is (n,0) and (0,n) disconjugate. In the previously mentioned paper of Leighton and Nehari [76], the fourth order equation (1.5) is shown to have (2,2) oscillation type, and to be (1,3) and (3,1) disconjugate on $[0,\infty)$. These results will follow from the more general work presented in Chapter IV.

A useful tool for analyzing the behavior of the solutions of Equation (1.1) is its adjoint. The differential operator

$$L^*z \equiv q_n(x)z^{(n)} + q_{n-1}(x)z^{(n-1)} + \ldots + q_0(x)z \tag{1.8}$$

is the <u>adjoint operator</u> of the operator L given by Equation (1.1) provided that for all $y,z \in C^n[0,\infty)$ we have

$$zLy - yL^*z = \frac{d}{dx} J(y,z), \tag{1.9}$$

where

$$J(y,z) = \sum_{0 \leq j+k<n} \alpha_{jk}(x)y^{(j)}z^{(k)}$$

is a bilinear form with coefficients $\alpha_{jk} \in C'$. As shown in Coppel [15], Equation (1.9) uniquely determines both the adjoint operator L^* and the bilinear form J. The equation

$$L^*y = 0 \tag{1.10}$$

is the <u>adjoint equation</u> of Equation (1.1). It is clear from Equation (1.9) that $L^{**} = L$. If $L^* = L$, then the operator L is said to be <u>self-adjoint</u>. Every second order equation of the form (1.2) can be written in the self-adjoint form

$$(r(x)y')' + s(x)y = 0 \tag{1.11}$$

by means of the integrating factor $\exp(\int p(x)dx)$. The fourth order equation (1.7), which was considered by Leighton and Nehari [76] as described earlier, is also self-adjoint. The similar form of the two equations

(1.7) and (1.11) provided Leighton and Nehari with the motivation for their work.

One of the properties associated with the adjoint operator which is of fundamental importance in the types of questions considered here is contained in the following theorem, which can be found in Coppel [15].

THEOREM 1.1: If Equation (1.1) has a solution y(x) with a (k,n-k) zero distribution on [a,b], then the adjoint equation (1.10) has a solution z(x) with an (n-k,k) zero distribution on [a,b].

Since Equations (1.1) and (1.10) have the same form, Theorem 1.1 can reduce the number of cases to be considered in the analysis of certain problems.

As will be seen in later chapters, disconjugate operators play a key role in establishing a setting in which the analysis of the limiting behavior of solutions is tractable. In the following chapter, the notion of a disconjugate operator is explored more fully.

CHAPTER II

DISCONJUGATE OPERATORS

Disconjugate operators have been studied, in effect, since the last century, but only recently has the study become formalized. The concept of disconjugacy is adapted from the calculus of variations, where disconjugate meant the absence of a conjugate point for the second order Jacobi equation. Wintner [131], in 1951, extended this notion and applied the term disconjugate to more general second order problems. Since then, as suggested by the definitions of the previous chapter, the terminology has spread to a wide variety of situations. A clear understanding of these situations first requires a more detailed analysis of the basic concept itself.

Disconjugacy and Interpolation

Recall that the operator L defined by the equation

$$Ly \equiv y^{(n)} + p_{n-1}(x)y^{(n-1)} + \ldots + p_0(x)y = 0, \tag{2.1}$$

with continuous coefficients $p_0(x), \ldots, p_{n-1}(x)$ on $I = [0,\infty)$ is said to be disconjugate on I provided no nontrivial solution of Equation (2.1) has more than n-1 zeros (counting multiplicaties) on I. In other words, the only solution with a total of n zeros on I is the identically zero solution. It is possible to draw a parallel between disconjugate operators as defined above and nonsingular matrices in linear algebra, where an $n \times n$ matrix A is said to be nonsingular provided the only vector solution $\vec{\alpha}$ of $A\vec{\alpha} = 0$ is the identically zero vector. One consequence of this property is that the equation $A\vec{\alpha} = \vec{\beta}$ always has a unique solution $\vec{\alpha}$ for each choice of $\vec{\beta}$. In the current setting the corresponding result is the following theorem, which may be found in Coppel [15], but which is proved here for completeness.

THEOREM 2.1: The operator L is disconjugate on I if and only if for every $m < n$ distinct points $x_1, \ldots, x_m$ in I, for arbitrary positive integers $r_1, \ldots, r_m$ with sum $r_1 + \ldots + r_m = n$, and for n arbitrary real numbers $\beta_{1,1}, \ldots, \beta_{1,r_1}, \beta_{2,1}, \ldots, \beta_{2,r_2}, \ldots, \beta_{m,r_m}$, there is a unique solution $y(x)$ of $Ly = 0$ such that

$$y^{(j-1)}(x_i) = \beta_{i,j}, \qquad 1 \leq j \leq r_i,\ 1 \leq i \leq m.$$

PROOF: Let $y_1(x), \ldots, y_n(x)$ be n linearly independent solutions of Equation (2.1) and let A be the matrix defined by

$$A = \begin{bmatrix} y_1(x_1) & \cdots & y_n(x_1) \\ y_1'(x_1) & \cdots & y_n'(x_1) \\ \vdots & & \vdots \\ y_1^{(r_1-1)}(x_1) & \cdots & y_n^{(r_1-1)}(x_1) \\ \vdots & & \vdots \\ y_1^{(r_m-1)}(x_m) & \cdots & y_n^{(r_m-1)}(x_m) \end{bmatrix}$$

Let $\vec{\nu}$ be a column vector with components $\nu_1, \ldots, \nu_n$, and suppose that

$A\vec{\nu} = 0$. Then the solution $\tilde{y}(x) = \nu_1 y_1(x) + \ldots + \nu_n y_n(x)$ has a zero of order r_i at each x_i in I, and so has at least n total zeros in I. If L is disconjugate, then $\tilde{y}(x) \equiv 0$, and so $\vec{\nu} = 0$. Therefore, A is a nonsingular matrix. If $\vec{\beta}$ is now chosen with components $\beta_{1,1}, \ldots, \beta_{m,r_m}$, then there is a unique column vector $\vec{\alpha}$ with components $\alpha_1, \ldots, \alpha_n$ such that $A\vec{\alpha} = \vec{\beta}$, so that $y(x) = \alpha_1 y_1(x) + \ldots + \alpha_n y_n(x)$ is the unique solution desired.

On the other hand, if L is not disconjugate, then $Ly = 0$ has a nontrivial solution $y(x)$ satisfying the conditions

$$y^{(j-1)}(x_i) = 0, \qquad 1 \leq j \leq r_i, \quad 1 \leq i \leq m,$$

for some choice of x_i and r_i, $1 \leq i \leq m$. Since the identically zero solution also satisfies these conditions, the solution y is not unique. This completes the proof of Theorem 2.1.

This theorem states that disconjugacy is equivalent with the ability to interpolate at up to n points (counting multiplicities) in I by means of solutions of the differential equation (2.1). It suggests a link between disconjugate operators and certain matrices which can be exploited to express L in a much more meaningful form than that given in Equation (2.1).

Pólya Factorization

In 1922, Pólya [102] revived an 1874 result due to Frobenius [30] and applied it to the operator L. If, as in the proof of Theorem 2.1, $\{y_1(x), \ldots, y_n(x)\}$ is a fundamental set of solutions for $Ly = 0$, then the Wronskians $W_k \equiv W(y_1, \ldots, y_k)$, $k = 1, \ldots, n$, may be defined as

$$W(y_1, \ldots, y_k) = \begin{vmatrix} y_1(x) & \cdots & y_k(x) \\ y_1'(x) & & y_k'(x) \\ \vdots & & \\ y_1^{(k-1)}(x) & \cdots & y_k^{(k-1)}(x) \end{vmatrix}$$

For notational convenience, define $W_0 \equiv 1$.

<u>THEOREM 2.2</u> (Frobenius): If the Wronskians W_i satisfy

$$W_i(x) > 0, \qquad i = 0, \ldots, n, \tag{2.2}$$

on the interval I, then the operator L can be expressed as

$$Ly = r_n(x)\frac{d}{dx}r_{n-1}(x)\frac{d}{dx} \cdots r_1(x)\frac{d}{dx}r_0(x)y \tag{2.3}$$

with $r_i(x) > 0$, $i = 0,1, \ldots, n$, where

$$r_0 = \frac{1}{y_1} = \frac{1}{W_1}, \tag{2.4a}$$

$$r_n = \frac{W_n}{W_{n-1}} \tag{2.4b}$$

and

$$r_i = \frac{W_i^2}{W_{i+1}W_{i-1}}, \qquad i = 1, \ldots, n-1. \tag{2.4c}$$

<u>THEOREM 2.3</u> (Pólya): The equation $Ly = 0$ has a fundamental set of solutions $\{y_1, \ldots, y_n\}$ such that the Wronskians W_i satisfy Equation (2.2) on I if and only if L is disconjugate on I. That is, L has the factorization (2.3) on I if and only if L is disconjugate there. (The factorization (2.3) is known as a Pólya factorization of L.)

Before the theorems are proved, it should be noted that if the Wronskians W_i are all nonzero on I, then the inequalities (2.2) can be achieved by replacing y_j by $-y_j$ as required in the fundamental set of solutions.

PROOF (Theorem 2.2): This proof is based on the fact that if two k-th order linear differential equations of the form (2.1) have the same fundamental set of solutions and the same leading coefficient, they must in fact be identical equations. For if not, then their difference would be a nontrivial equation of order less than k with k independent solutions, a contradiction.

If L_k is defined by

$$L_k y = \frac{W(y_1, \ldots, y_k, y)}{W_k}, \qquad k = 1, \ldots, n, \tag{2.5}$$

then $L_k y = 0$ is the k-th order differential equation with leading coefficient 1 and solutions $y_1, \ldots, y_k$. Consequently, $L_n y = Ly$. In addition,

$$\frac{d}{dx}\frac{W(y_1, \ldots, y_{n-1}, y)}{W_k} = 0$$

is a differential equation with leading coefficient $\frac{W_{k-1}}{W_k}$ and solutions $y_1, \ldots, y_k$. Thus it follows that

$$L_k y = \frac{W_k}{W_{k-1}} \frac{d}{dx} \frac{W(y_1, \ldots, y_{k-1}, y)}{W_k}$$

$$= \frac{W_k}{W_{k-1}} \frac{d}{dx}\left(\frac{W_{k-1}}{W_k} \cdot \frac{W(y_1, \ldots, y_{k-1}, y)}{W_{k-1}}\right),$$

or

$$L_k y = \frac{W_k}{W_{k-1}} \frac{d}{dx}\left(\frac{W_{k-1}}{W_k} L_{k-1}\right) . \tag{2.6}$$

Equation (2.6) holds for $1 < k \leq n$. If the definition of $L_k y$ is extended to include $L_0 y = y$, then Equation (2.6) holds for $k = 1$ as well. Applying this relation to the operator $L = L_n$ gives

$$L_n y = \frac{W_n}{W_{n-1}} \frac{d}{dx}\left(\frac{W_{n-1}}{W_n} L_{n-1} y\right) = \frac{W_n}{W_{n-1}} \frac{d}{dx}\left(\frac{W_{n-1}}{W_n} \cdot \frac{W_{n-1}}{W_{n-2}} \frac{d}{dx}\left(\frac{W_{n-2}}{W_{n-1}} L_{n-2}\, y\right)\right) .$$

Repeating in this fashion, applying Equation (2.6) a total of $n-1$ times, and identifying the coefficients as in Equation (2.4) yields the desired result. This completes the proof of the theorem.

PROOF (Theorem 2.3): Assume first that L is disconjugate on I, and let the fundamental set of solutions $\{y_1(x), \ldots, y_n(x)\}$ be defined by

$$y_i^{(n-j)}(0) = \delta_{ij},$$

where δ_{ij} denotes the Kronecker delta. If for some k the Wronskian W_k vanishes at a point $c > 0$, then some nontrivial linear combination of $y_1, \ldots, y_k$ would have k zeros at $x = c$ and $n-k$ zeros at $x = 0$. If L is disconjugate this cannot occur, and so every disconjugate operator has a factorization (2.3).

For the other direction, suppose L is given by the factorization (2.3). Following Willett [128] and Trench [125], a fundamental set of solutions $\{y_1, \ldots, y_n\}$ may be defined by fixing $a \in I$ and setting

$$y_1(x) = \frac{1}{r_0(x)},$$

$$y_2(x) = \frac{1}{r_0(x)} \int_a^x \frac{1}{r_1(s)}\, ds,$$

$$\vdots$$

$$y_n(x) = \frac{1}{r_0(x)} \int_a^x \frac{1}{r_1(s_1)} \int_a^{s_1} \frac{1}{r_2(s_2)} \cdots$$

$$\int_a^{s_{n-1}} \frac{1}{r_{n-1}(s_{n-1})}\, ds_{n-1} \ldots ds_2 ds_1. \tag{2.7}$$

Suppose now that some nontrivial linear combination $y(x) = \alpha_1 y_1(x) + \ldots + \alpha_n y_n(x)$ has at least n zeros on I. Then so does $r_0(x)y(x)$, and by Rolle's theorem it follows that $\frac{d}{dx} r_0(x)y(x)$ has at least $n-1$ zeros on I. Continuing in this manner, applying the factors of Equation (2.3) in succession, leads to the conclusion that $r_{n-1}(x)\frac{d}{dx} \ldots r_1(x)\frac{d}{dx} r_0(x)y$ has at least one zero on I. But from Equation (2.7) it is easy to see that for each $k \leq n-1$,

$$r_k(x)\frac{d}{dx} \ldots r_1(x)\frac{d}{dx} r_0(x) y_k = 0,$$

so that

$$r_{n-1}(x)\frac{d}{dx} \ldots r_1(x)\frac{d}{dx} r_0(x)y = r_{n-1}(x)\frac{d}{dx} \ldots r_1(x)\frac{d}{dx} r_0(x)y_n \equiv 1,$$

a contradiction. Thus the operator L given by Equation (2.3) is disconjugate. In fact, it can be shown that the coefficients $r_i(x)$, $i = 0,\ldots,n$, of the given factorization are the same as those determined by the solutions defined in Equation (2.7) by means of Equation (2.4). This is most easily seen by row reducing each W_i. This completes the proof of Theorem 2.3.

The proof of Theorem 2.3 can be used to gain information about the functions which locate conjugate points, as the following corollary shows.

COROLLARY 2.4: If $\eta_1(a) < \infty$ for Equation (2.1), then for some k, $1 \leq k \leq n-1$, there is a solution y(x) of Equation (2.1) with $n-k$ zeros at a and k zeros at $\eta_1(a)$.

PROOF: Let the fundamental set of solutions $y_1(x), \ldots, y_n(x)$ be defined by

$$y_i^{(n-j)}(a) = \delta_{ij}$$

where δ_{ij} denotes the Kronecker delta. Let these solutions define the Wronskians W_i, $i = 0, \ldots, n$. Since Equation (2.1) is disconjugate on $[a, \eta_1(a))$ but not on $[a, \eta_1(a) + \varepsilon)$ for $\varepsilon > 0$ arbitrarily small, then for some k, $1 \leq k \leq n-1$, the Wronskian W_k must vanish at $\eta_1(a)$. Then there is a nontrivial linear combination of $y_1, \ldots, y_k$ with k zeros at $\eta_1(a)$ and $n-k$ zeros at a, as desired. This completes the proof of Corollary 2.4.

A proof of this result based on Green's functions is due to Levin [78] [119, p. 169].

Clearly there is a strong connection between disconjugate operators and Čebyšev and Markov systems (and hence total positivity). From the definition above, a fundamental set of solutions $\{y_1, \ldots, y_n\}$ for Equation (2.1) forms a Čebyšev system on I if and only if L is disconjugate there. Theorem 2.3 shows that Equation (2.1) has a fundamental set of solutions forming a Markov system on I if and only if L is disconjugate there. While this terminology is merely noted in passing here, a development based on such systems can be found in Coppel [15].

Quasi-Derivatives

The Pólya factorization (2.3) of a disconjugate operator L leads very naturally to an extension of the idea of derivative. Let the operators L_k, $k = 0, \ldots, n$, be defined by

$$L_0 y = r_0(x)y, \quad L_k y = r_k(x)\frac{d}{dx}(L_{k-1}y), \qquad k = 1, \ldots, n, \tag{2.8}$$

where the positive functions $r_k(x)$, $k = 0, \ldots, n$, are those found in Equation (2.3). The function $L_k y$ is called the k-th <u>quasi-derivative</u> of y. Since $r_k(x) > 0$ for $k = 0, \ldots, n$, it follows that the quasi-derivatives of y exhibit many of the same characteristics as its derivatives, when both are defined. For example, $L_k y$ is increasing or decreasing as $L_{k+1}y$ is positive or negative. More importantly, y has a zero of exact order i at $x = \alpha$ if and only if $L_0 y(\alpha) = L_1 y(\alpha) = \ldots = L_{i-1} y(\alpha) = 0$, and $L_i y(\alpha) \neq 0$. For if $f(x) \in C^i$ is nonzero and if y has a zero of exact order i at $x = \alpha$, then the product differentiation formulas imply that $f(x)y(x)$ also has a zero of exact order i at $x = \alpha$. Thus $L_0 y(x) = r_0(x)y(x)$ has a zero of exact order i at $x = \alpha$, implying that $L_1 y(x) = r_1(x)\frac{d}{dx}L_0 y(x)$ has a zero of exact order $i - 1$ at $x = \alpha$. The claim then follows by an induction argument.

The similarity between derivatives and quasi-derivatives led Nehari [93], in 1967, to consider a generalization of Equation (2.1). Let the positive functions $r_i(x)$, $i = 0, \ldots, n$, be given, and let the quasi-derivatives $L_0 y, \ldots, L_n y$ be defined by Equation (2.8). Consider the operator L on I defined by

$$Ly \equiv L_n y + p_{n-1}(x)L_{n-1}y + \ldots + p_1(x)L_1 y + p_0(x)L_0 y, \tag{2.9}$$

with continuous coefficients $p_i(x)$ as for Equation (2.1). A function y is said to be a solution of $Ly = 0$ provided each quasi-derivative $L_0 y, \ldots, L_n y$ exists, and provided $Ly \equiv 0$ on I. In this new setting, Wronskians are defined in terms of quasi-derivatives, as opposed to ordinary derivatives. That is, the $i-j$ element of $W(y_1, \ldots, y_k)$ is taken to be $L_{i-1}y_j(x)$, for $1 \leq i,j \leq k$. Using these Wronskians, Nehari showed that the same analysis which applied to Equation (2.1) also applies to Equation (2.9): L is disconjugate on I if and only if it has a Pólya factorization there. However, note that the Pólya factorization for L is probably different from that for L_n for Equation (2.9), just as the factorization of L for Equation (2.1) is probably not d^n/dx^n.

Pólya factorization, and the canonical Trench factorization discussed later in this chapter, are extremely powerful tools for use in the study of disconjugate operators. For example, suppose L is a disconjugate operator given in the factored form (2.3), and suppose L^* is its adjoint as defined in Chapter I. Theorem 1.1 suggests that L^* is disconjugate as well, since it must be $(n-k,k)$ disconjugate for every choice of k. That this is in fact the case can now be shown, along with the remarkable result that L^* has a Pólya factorization in which the factors of L are reversed [15].

THEOREM 2.5: If L is disconjugate with factorization

$$Ly = r_n \frac{d}{dx} r_{n-1} \frac{d}{dx} \cdots r_1 \frac{d}{dx} r_0 y ,$$

then its adjoint L^* is the disconjugate operator $(-1)^n M$ given by

$$Mz = r_0 \frac{d}{dx} r_1 \frac{d}{dx} \cdots r_{n-1} \frac{d}{dx} r_n z. \tag{2.10}$$

PROOF: If it can be shown that the operator M with factorization (2.10) gives the adjoint of L, Theorem 2.3 will then imply the adjoint is disconjugate. Recall that the adjoint L^* of L is the unique operator defined by

$$zLy - yL^*z = \frac{d}{dx} J(y,z)$$

where J is a bilinear form in y and z.

Let $L_0y, \ldots, L_ny$ and $M_0z, \ldots, M_nz$ be the quasi-derivatives given by L and M, respectively, and let

$$J(y,z) = \sum_{k=1}^{n} (-1)^{k-1} (M_{k-1}z)(L_{n-k}y).$$

Then J(y,z) is a bilinear form, and

$$\begin{aligned} \frac{d}{dx} J(y,z) &= \sum_{k=1}^{n} (-1)^{k-1} \left[\left(\frac{d}{dx} M_{k-1}z\right)(L_{n-k}y) + (M_{k-1}z)\left(\frac{d}{dx} L_{n-k}y\right) \right] \\ &= \sum_{k=1}^{n} (-1)^{k-1} \left[\frac{1}{r_{n-k}} M_k z \cdot L_{n-k}y + M_{k-1}z \cdot \frac{1}{r_{n-k+1}} L_{n-k+1}y \right]. \end{aligned}$$

This last sum is a collapsing sum, and so

$$\frac{d}{dx} J(y,z) = \frac{1}{r_n} M_0 z \cdot L_n y + (-1)^{n-1} \frac{1}{r_0} M_n z \cdot L_0 y = zLy - (-1)^n yMz.$$

Thus $L^*z = (-1)^n Mz$. This completes the proof of Theorem 2.5.

Because of their importance in the study of differential equations, it is not surprising that factorization results have attracted the attention of many authors. For example, Frobenius and Pólya were not the only early contributors in this area. In 1833, Libri [83] arrived at an integral version of Theorem 2.2, while in 1931, Mammana [90] worked with a factorization of an operator L into linear factors of the form $(\frac{d}{dx} + p(x))$. More recently, Zettl [132-135] has produced a series of papers relating to factorizations and disconjugacy, and Kim [59] has obtained factorizations with higher order factors for operators which are not disconjugate.

In the next section the Polya factorization for a disconjugate operator is considered in more detail. The relationship between fundamental solutions, Wronskians, and factorization as presented in Theorems 2.2 and 2.3 is exploited to describe a canonical factorization.

Nonuniqueness and Canonical Factorizations

Although every disconjugate operator has a Pólya factorization as a result of Theorem 2.3, such factorizations are not unique. The Wronskians W_k which determine the coefficients depend upon the choice and order of the fundamental system $\{y_1, \ldots, y_n\}$ of solutions. For example, the second order operator L_2 with factors determined by the solutions $\{e^{-x}, e^{x}\}$ of $L_2 y = 0$ is

$$L_2 y = e^{x} \frac{d}{dx} e^{-2x} \frac{d}{dx} e^{x} y.$$

When the order of solutions is changed to $\{e^{x}, e^{-x}\}$, the corresponding Wronskians change to yield the alternate factorization

$$L_2 y = e^{-x} \frac{d}{dx} e^{2x} \frac{d}{dx} e^{-x} y.$$

This same operator has still a third factorization,

$$L_2 y = \frac{e^{x}}{e^{2x}+1} \frac{d}{dx} (e^{x} + e^{-x})^2 \frac{d}{dx} \frac{e^{x}}{e^{2x}+1} y,$$

derived from the solutions $\{e^x + e^{-x}, e^x\}$.

Because the factorization of a disconjugate operator is not unique, the definitions of the quasi-derivatives are also not unique. Therefore, certain questions involving quasi-derivatives are not always well-defined. Consider, for example, the problem of locating conjugate points and focal points for $Ly = 0$, where L is given in the generalized form (2.9). Multiple zeros of y counted with respect to $\frac{d}{dx}$ agree with those counted with respect to the quasi-derivatives, and so if L is re-expressed in terms of different quasi-derivatives, the existence and location of conjugate points is not altered. On the other hand, focal points can be affected by changes in the factorization when defined in terms of quasi-derivatives since the zeros of quasi-derivatives are not necessarily located at the immutable zeros of y.

Consider the example

$$y'' - y = 0$$

with solutions $y_1 = \sin x$ and $y_2 = \cos x$. In this form, it is clear that conjugate and focal points for $x = \pi$ are located by y_1, and so $\eta_1(\pi) = 2\pi$ and $\zeta_1(\pi) = 3\pi/2$. If the operator d^2/dx^2 used here is permitted to change, this equation may be written in the form

$$\frac{1}{x}\frac{d}{dx}x^2\frac{d}{dx}\frac{1}{x}y - y = 0$$

with solutions as before. The solution y_1 still locates the conjugate and focal points for $x = \pi$, and $\eta_1(\pi) = 2\pi$ is unchanged. With derivatives now given in terms of the quasi-derivatives of this new factorization, $\zeta_1(\pi)$ is a solution of $x^2\frac{d}{dx}\frac{\sin x}{x} = 0$, and so $\zeta_1(\pi) < \frac{3\pi}{2}$.

Changing the factorization for a disconjugate operator redistributes the weight of the coefficients among the quasi-derivatives. This in turn alters the growth relationships among the corresponding solutions. In the first example above, three ordered sets of solutions were considered. If the solutions in each of these are labeled in order as y_1 and y_2, then only the first set of solutions has the additional property

$$\lim_{x\to\infty} \frac{y_1(x)}{y_2(x)} = 0. \tag{2.11}$$

This property is desirable because it implies that the solution $y = ay_1 + by_2$ resembles y_2 for large values of x exactly when $b \neq 0$. Because of the condition (2.11), the solutions $y_1 = e^{-x}$ and $y_2 = e^{x}$ are said to form a <u>principal system</u> of solutions for $L_2 y = 0$.

For the more general equation (2.1), when the operator L is disconjugate, the solutions $y_1, \ldots, y_n$ form a principal system of solutions provided that these solutions are eventually positive and

$$\lim_{x\to\infty} \frac{y_i(x)}{y_j(x)} = 0, \qquad 1 \leq i < j \leq n. \tag{2.12}$$

It is known that Equation (2.1) has a principal system of solutions for every disconjugate operator L. These systems had been studied and discussed extensively by Coppel [15], Hartman [37, 38, 39], Levin [80], and Willett [128, 129], but it remained for Trench [125], in 1974, to establish a link between such systems and the factorization of L.

Recall that when L is given the factorization (2.3), then a fundamental system of solutions $\{y_1, \ldots, y_n\}$ of Equation (2.1) is determined by Equation (2.7). Suppose the factorization (2.3) has the additional property that

$$\int_0^\infty \frac{1}{r_i(x)}\,dx = \infty, \qquad 1 \le i \le n-1. \tag{2.13}$$

Then simple applications of l'Hopital's rule reveal that the system $\{y_1, \ldots, y_n\}$ is actually a principal system of solutions. The contribution made by Trench in 1974 was to show that every disconjugate operator L has a canonical factorization satisfying (2.13). His method is constructive, so that if an operator is given in terms of a Pólya factorization, then it gives a technique for determining the coefficients of a new factorization satisfying (2.13). Furthermore, Trench showed that the factorization of L satisfying (2.13) is unique up to positive multiplicative constants with product 1. The factorization of L satisfying (2.13) will be called the Trench factorization of L.

The proof of Trench's theorem is instructive both because of its constructive nature and because of the role played by the Trench factorization in what follows. By means of Lemmas 2.6 and 2.7, a device is constructed for placing one term of the factorization in the form (2.13) while altering the other terms as little as possible. This device is then employed in Theorem 2.8 for the existence portion of the result. Throughout these results, all coefficients in the factorizations of the various operators will be taken to be positive continuous functions on the interval I.

LEMMA 2.6: If

$$\int_0^\infty \frac{1}{p_1(x)}\,dx < \infty,$$

then the operator M given by

$$My = p_2 \frac{d}{dx} p_1 \frac{d}{dx} p_0 y$$

can be rewritten as

$$My = \tilde{p}_2 \frac{d}{dx} \tilde{p}_1 \frac{d}{dx} \tilde{p}_0 y,$$

with the property that

$$\int_0^\infty \frac{1}{\tilde{p}_1(x)} dx = \infty.$$

PROOF: Define the coefficients $\tilde{p}_0$, $\tilde{p}_1$, and $\tilde{p}_2$ according to

$$\tilde{p}_0(x) = p_0(x)\left(\int_x^\infty \frac{dt}{p_1(t)}\right)^{-1},$$

$$\tilde{p}_1(x) = p_1(x)\left(\int_x^\infty \frac{dt}{p_1(t)}\right)^{2},$$

and

$$\tilde{p}_2(x) = p_2(x)\left(\int_x^\infty \frac{dt}{p_1(t)}\right)^{-1}.$$

Then

$$\int_0^\infty \frac{1}{\tilde{p}_1(x)} dx = \int_0^\infty \left(\int_x^\infty \frac{dt}{p_1(t)}\right)^{-2} \frac{dx}{p_1(x)} = \lim_{b\to\infty} \left(\int_x^\infty \frac{dt}{p_1(t)}\right)^{-1}\Bigg|_0^b = \infty,$$

and so $\tilde{p}_1(x)$ satisfies (2.13), as desired. To verify that the operators are the same, observe that

$$\tilde{p}_2 \frac{d}{dx} \tilde{p}_1 \frac{d}{dx} \tilde{p}_0 y = \tilde{p}_2 \frac{d}{dx} \tilde{p}_1 \frac{d}{dx} \left[\left(\int_x^\infty \frac{dt}{p_1(t)}\right)^{-1} \cdot p_0(x) y\right]$$

$$= \tilde{p}_2 \frac{d}{dx} \tilde{p}_1 \left[\left(\int_x^\infty \frac{dt}{p_1(t)}\right)^{-1} \frac{d}{dx} p_0 y + p_0 y \left(\int_x^\infty \frac{dt}{p_1(t)}\right)^{-2} \cdot \frac{1}{p_1}\right]$$

$$= \tilde{p}_2 \frac{d}{dx} \left(\int_x^\infty \frac{dt}{p_1(t)} \cdot p_1 \frac{d}{dx} p_0 y + p_0 y\right)$$

$$= \tilde{p}_2 \cdot \int_x^\infty \frac{dt}{p_1(t)} \frac{d}{dx} p_1 \frac{d}{dx} p_0 y = p_2 \frac{d}{dx} p_1 \frac{d}{dx} p_0 y.$$

This completes the proof of the lemma.

It is clear from this lemma that the coefficient p_1 can be placed in the form (2.13) only at the cost of altering the coefficients to the left and right. These coefficients might now fail the condition (2.13) even if they met it before. For the factorization (2.3), only the functions $r_1(x), \ldots, r_{n-1}(x)$ need meet this condition, but even if only these are considered it is still reasonable to question whether the process of applying Lemma 2.6 will terminate. Lemma 2.7 indicates to what extent the changes to a coefficient such as $p_1(x)$ are stable.

<u>LEMMA 2.7</u>: Suppose the operator given by

$$My = p_3 \frac{d}{dx} p_2 \frac{d}{dx} p_1 \frac{d}{dx} p_0 y$$

satisfies

$$\int_0^\infty \frac{1}{p_2(x)}\,dx < \infty \quad \text{and} \quad \int_0^\infty \frac{1}{p_1(x)}\,dx = \infty. \tag{2.14}$$

Suppose further that Lemma 2.6 is applied to $p_2(x)$ so that

$$My = \tilde{p}_3 \frac{d}{dx} \tilde{p}_2 \frac{d}{dx} \tilde{p}_1 \frac{d}{dx} p_0 y$$

with

$$\int_0^\infty \frac{1}{\tilde{p}_2(x)}\,dx = \infty \quad \text{and} \quad \int_0^\infty \frac{1}{\tilde{p}_1(x)}\,dx < \infty.$$

If Lemma 2.6 is now applied to $\tilde{p}_1(x)$ yielding the factorization

$$My = q_3 \frac{d}{dx} q_2 \frac{d}{dx} q_1 \frac{d}{dx} q_0 y,$$

then both q_1 and q_2 satisfy the condition (2.13).

PROOF: After applying Lemma 2.6 to $p_2(x)$ the coefficients are

$$\tilde{p}_3(x) = p_3(x)\left(\int_x^\infty \frac{dt}{p_2(t)}\right)^{-1}, \quad \tilde{p}_2(x) = p_2(x)\left(\int_x^\infty \frac{dt}{p_2(t)}\right)^{2}, \tag{2.15}$$

and

$$\tilde{p}_1(x) = p_1(x)\left(\int_x^\infty \frac{dt}{p_2(t)}\right)^{-1}. \tag{2.16}$$

Therefore, Lemma 2.6 acting on $\tilde{p}_1(x)$ gives

$$q_3(x) = \tilde{p}_3(x),$$

$$q_2(x) = \tilde{p}_2(x)\left(\int_x^\infty \frac{dt}{\tilde{p}_1(t)}\right)^{-1},$$

$$q_1(x) = \tilde{p}_1(x)\left(\int_x^\infty \frac{dt}{\tilde{p}_1(t)}\right)^{2},$$

and

$$q_0(x) = \tilde{p}_0(x)\left(\int_x^\infty \frac{dt}{\tilde{p}_1(t)}\right)^{-1}.$$

Only the behavior of q_2 requires verification. Based on Equations (2.15) and (2.16), and integrating by parts, the appropriate computation is

$$\begin{aligned}
\int_0^b \frac{1}{q_2(x)}\,dx &= \int_0^b \frac{1}{\tilde{p}_2(x)} \int_x^\infty \frac{dt}{\tilde{p}_1(t)}\,dx \\
&= \int_0^b \frac{1}{p_2(x)}\left(\int_x^\infty \frac{dt}{p_2(t)}\right)^{-2} \cdot \int_x^\infty \frac{dt}{\tilde{p}_1(t)}\,dx \\
&= \left(\int_x^\infty \frac{dt}{p_2(t)}\right)^{-1} \cdot \int_x^\infty \frac{dt}{p_1(t)}\,\Bigg|_0^b + \int_0^b \left(\int_x^\infty \frac{dt}{p_2(t)}\right)^{-1} \frac{1}{\tilde{p}_1(x)}\,dx \\
&= \left(\int_x^\infty \frac{dt}{p_2(t)}\right)^{-1} \cdot \int_x^\infty \frac{dt}{\tilde{p}_1(t)}\,\Bigg|_0^b + \int_0^b \frac{1}{p_1(x)}\,dx.
\end{aligned}$$

The first term on the right is positive, while from Equation (2.14) the second diverges to $+\infty$ in the limit as b approaches ∞, and so q_2 satisfies Equation (2.13). This completes the proof of Lemma 2.7.

As suggested earlier, these lemmas provide the basis of an iterative technique for Trench factoring an arbitrary disconjugate operator.

THEOREM 2.8: Every disconjugate operator L on the interval I has a factorization of the form (2.3) with coefficients $r_i > 0$ satisfying (2.13) for $i = 1,2,\ldots,n-1$. This factorization is unique up to positive multiplicative constants with product 1.

PROOF: The existence portion of the proof involves a double induction argument. Lemmas 2.6 and 2.7 imply existence for operators of order $n=2$ and $n=3$. Suppose that existence has been established for operators of order $n-1$ where $n \geq 4$. Then the n-th order operator L may be written in the form

$$Ly = p_n \frac{d}{dx} p_{n-1} \frac{d}{dx} \cdots p_1 \frac{d}{dx} p_0 y$$

where the coefficients p_i, $i = 1,\ldots,n-2$, satisfy (2.13). If p_{n-1} satisfies Equation (2.13), there is nothing more to be shown, and taking $r_i = p_i$, $i = 0,1,\ldots,n$ gives the desired result. Otherwise, Lemma 2.6 may be applied to p_{n-1} yielding coefficients $\tilde{p}_n$, $\tilde{p}_{n-1}$, and $\tilde{p}_{n-2}$ where $\tilde{p}_{n-1}$ satisfies (2.13), but where $\tilde{p}_{n-2}$ may no longer meet this condition. If $\tilde{p}_{n-2}$ still satisfies (2.13), the required factorization is given by setting $r_i = p_i$, $i = 0,1,\ldots,n-3$, $r_i = \tilde{p}_i$, $i = n-2$, $n-1$, and n. If $\tilde{p}_{n-2}$ does not satisfy Equation (2.13), then Lemma 2.7 implies that Lemma 2.6 may be applied to $\tilde{p}_{n-2}$, giving coefficients q_{n-1}, q_{n-2}, and q_{n-3} where q_{n-1} and q_{n-2} satisfy Equation (2.13), but where q_{n-3} is suspect.

Suppose now that by repeated applications of Lemmas 2.6 and 2.7 the operator L has been transformed to

$$Ly = q_n \frac{d}{dx} q_{n-1} \frac{d}{dx} \cdots q_1 \frac{d}{dx} q_0 y$$

where the coefficients q_i, $1 \leq i < k$, $k < i \leq n-1$, satisfy (2.13), for some k, $1 \leq k \leq n-3$. If at this point q_k satisfies (2.13), the process terminates giving $r_i = q_i$, $i = 0,1, \ldots, n$. If not, then Lemmas 2.6 and 2.7 applied to q_k produce coefficients $\tilde{q}_{k+1}$, $\tilde{q}_k$, and $\tilde{q}_{k-1}$ with $\tilde{q}_{k+1}$ and $\tilde{q}_k$ satisfying (2.13), but with $\tilde{q}_{k-1}$ now suspect. By induction on k, this process may be continued until $k = 1$. If the procedure does not terminate as described above, then setting $r_i = \tilde{q}_i$, $i = 0,1,2$, and $r_i = q_i$, $i = 3, \ldots, n$ gives the Trench factorization for L. Finally, by induction on n, every disconjugate operator can be Trench factored.

To prove the essential uniqueness of the Trench factorization, suppose

$$Ly = r_n \frac{d}{dx} r_{n-1} \frac{d}{dx} \cdots r_1 \frac{d}{dx} r_0 y \tag{2.17}$$

and

$$Ly = p_n \frac{d}{dx} p_{n-1} \frac{d}{dx} \cdots p_1 \frac{d}{dx} p_0 y \tag{2.18}$$

are Trench factorizations for L. As discussed earlier, the fundamental set of solutions $\{y_1, \ldots, y_n\}$ for $Ly = 0$ described by Equation (2.7) using the coefficients of Equation (2.17) forms a principal system of solutions. Similarly, there is a principal system of solutions $\{z_1, \ldots, z_n\}$ for $Ly = 0$ as described by Equation (2.7) using the coefficients of Equation (2.18). Since the condition (2.12) must be met by both of these systems, it is clear that

$$y_i = \sum_{j=1}^{i} \alpha_{ij} z_j$$

where the coefficients α_{ij} are constants, with $\alpha_{ii} > 0$. Consequently for $k = 1, \ldots, n$,

$$W(y_1, \ldots, y_k) = C_k W(z_1, \ldots, z_k) \tag{2.19}$$

where $C_k = \prod_{i=1}^{k} \alpha_{ii}$. The proof of Theorem 2.3 revealed that the Wronskians of y_i and z_i give rise to the factorizations (2.17) and (2.18), respectively. Thus Equation (2.19) implies that the corresponding coefficients of the factorizations differ at most by positive constant multiples. Since these factorizations both define L, the product of these constants must be exactly 1. This completes the proof of Theorem 2.8.

The principal solutions of a disconjugate equation enjoy a natural dominance relationship as expressed by Equation (2.12). In order to be useful in the context of later chapters, this concept must be defined more explicitly and extended to settings involving oscillatory solutions as well. Once this has been done in Chapter III, the main questions to be considered here may be addressed.

CHAPTER III

DOMINANCE FOR THE GENERAL EQUATION

The concept of dominance of solutions of the general n-th order equation

$$Ly = y^{(n)} + p_{n-1}(x)y^{(n-1)} + \ldots + p_1(x)y' + p_0(x)y = 0, \tag{3.1}$$

with continuous coefficients, was studied by Dolan and Klaasen [16] in 1975 in an attempt to characterize the asymptotic behavior of its solutions. The major contribution provided by their work was that of defining the basic concepts used in discussing questions of the following kind: Given an oscillatory solution y and a nonoscillatory solution z of Equation (3.1), is the linear combination $y + \lambda z$ oscillatory or nonoscillatory? Although Dolan and Klaasen were primarily interested in identifying dominance between the class $\mathcal{N}$ of all nonoscillatory solutions and the class $\mathcal{O}$ of all oscillatory solutions of Equation (3.1), the terminology may be extended easily to a comparison between classes of nonoscillatory solutions alone. Dominance among pairs of oscillatory solutions is more difficult to define, and a suitable definition is still being sought. As examples considered here will show, under certain circumstances the dominance of solutions may be used to separate the solution space of Equation (3.1) into disjoint subsets from which the limiting behavior of all solutions is easily determined.

Dolan and Klaasen [16] identified two kinds of dominance. Again taking y to be oscillatory and z to be nonoscillatory, then y dominates z at zero provided $y + \lambda z$ oscillates for $|\lambda|$ sufficiently small. If $y + \lambda z$ oscillates for all real λ, then y dominates z at infinity (or simply, y dominates z). If, as suggested earlier, $\mathcal{O}$ and $\mathcal{N}$ are taken to be the classes of all oscillatory solutions and all nonoscillatory solutions of Equation (3.1), respectively, and if $P \subseteq \mathcal{N}$ and $Q \subseteq \mathcal{O}$, then Q is said to dominate P at zero provided y dominates z at zero for all $y \in Q$, $z \in P$. Similar definitions hold for Q dominates P at infinity, and for when the roles of y and z are reversed.

These definitions have equivalent expressions in terms of limits of quotients, the proofs of which are immediate from the definitions.

LEMMA 3.1: Let the solutions $y \in \mathcal{O}$ and $z \in \mathcal{N}$ be given. Then

(1) y dominates z at zero if and only if

$$\liminf_{x\to\infty} \frac{y(x)}{z(x)} < 0 < \limsup_{x\to\infty} \frac{y(x)}{z(x)}; \tag{3.2}$$

(2) z dominates y at zero if and only if

$$\limsup_{x\to\infty} \left|\frac{y(x)}{z(x)}\right| < \infty; \tag{3.3}$$

(3) y dominates z at infinity if and only if

$$\liminf_{x\to\infty} \frac{y(x)}{z(x)} = -\infty \text{ and } \limsup_{x\to\infty} \frac{y(x)}{z(x)} = \infty; \tag{3.4}$$

(4) z dominates y at infinity if and only if

$$\lim_{x\to\infty} \frac{y(x)}{z(x)} = 0. \tag{3.5}$$

Such equivalent formulations as the limits (3.2) through (3.5) provide for the concept of dominance can be immediately extended to pairs of nonoscillatory solutions. Specifically, if z and w are both nonoscillatory, then z is said to dominate w at zero provided $\limsup_{x\to\infty} \left|\frac{w(x)}{z(x)}\right| < \infty$, and z dominates w at infinity provided $\lim_{x\to\infty} \frac{w(x)}{z(x)} = 0$. When considering pairs of oscillatory solutions the problem of division by zero cannot be avoided, making comparable definitions difficult to find in that case. In Chapter V one definition is suggested, but the definition is only applicable to the equations considered in that chapter.

It is worth noting that if $y \in \mathcal{O}$ dominates $z \in \mathcal{N}$ at zero, then it is not necessarily the case that z dominates y at zero, since conceivably one might have $\limsup_{x\to\infty} \frac{y(x)}{z(x)} < \infty$ while $\liminf_{x\to\infty} \frac{y(x)}{z(x)} = -\infty$. However, if z also dominates y at zero, then necessarily $\limsup_{x\to\infty} \left|\frac{y(x)}{z(x)}\right| = c$, $0 < c < \infty$, and so their relative magnitude remains roughly the same as x approaches infinity.

The constant coefficient case of Equation (3.1) can provide several helpful examples. Consider the equation

$$y^{(4)} - y = 0. \tag{3.6}$$

A fundamental set of solutions for this equation is given by the functions $y_1 = e^x$, $y_2 = \sin x$, $y_3 = \cos x$, $y_4 = e^{-x}$. Clearly y_1 dominates y_2, y_3, and y_4 at infinity while y_2 and y_3 each dominate y_4 at infinity. In fact, y_1 dominates every other solution to Equation (3.6) at zero, since every solution can be written in the form $y = \sum_{i=1}^{4} c_i y_i$ for some choice of the constants c_i, i = 1,2,3,4, and hence $\limsup_{x\to\infty} \left|\frac{y}{y_1}\right| = c_1 < \infty$.

The same analysis applied to Equation (3.6) also applies to any constant coefficient equation of the form

$$y^{(n)} + ky = 0. \tag{3.7}$$

When $k < 0$, $-k$ always has a positive real n-th root λ_1, and $y_1 = e^{\lambda_1 x}$ dominates every other solution at zero. The solution y_1 dominates at infinity solutions given by the other roots of $-k$. When $(-1)^n k < 0$, $-k$ always has a negative real n-th root λ_2, and $y_2 = e^{\lambda_2 x}$ is dominated by every other solution at zero. The solution y_2 is in fact dominated at infinity by every solution linearly independent from y_2.

It is interesting that in the case of Equation (3.6), or in the case of Equation (3.7) when n is even and $k < 0$, there is no useful dominance relationship among the classes $\mathcal{O}$ and $\mathcal{N}$, and yet the dominance structure of the fundamental set of solutions given by the roots of $-k$ makes it possible to predict whether an arbitrary solution oscillates or does not oscillate based upon its initial conditions. Specifically, a particular solution $y = ae^x + b\cos x + c\sin x + de^{-x}$ of Equation (3.6) can be identified as belonging in $\mathcal{O}$ or $\mathcal{N}$ simply by observing which of the coefficients a, b, c, and d are not zero.

The ability to discern oscillatory or nonoscillatory behavior based upon initial conditions alone exists for every constant coefficient problem with roots (or conjugate pairs of roots) with distinct real parts. In the case of a repeated root λ, such as $\lambda = 1$ for

$$y'' - 2y' + y = 0,$$

solutions typically take the form $y_1 = e^{\lambda x}$ and $y_2 = xy_1$. Consequently, a useful dominance relation still exists for this case. On the other hand, the problem

$$y''' - 3y'' + 4y' - 2y = 0$$

has solutions e^x, $e^x\sin x$, and $e^x\cos x$, and no one solution dominates any other at infinity. Therefore, the process of determining whether a linear combination oscillates or is nonoscillatory will not be as simple as merely identifying which coefficients are nonzero.

Another example which will be considered in greater detail later is the Euler equation

$$x^n y^{(n)} - ky = 0 \tag{3.8}$$

on the interval $[1,\infty)$. Substituting x^α for y in Equation (3.8) reveals that $y = x^\alpha$ is a solution of Equation (3.8) whenever α is a solution of

$$\alpha(\alpha-1)(\ldots)(\alpha-n+1)-k = 0. \tag{3.9}$$

Depending on the parity of n and the sign and magnitude of k, Equation (3.9) may have anywhere from none up to n real roots. Real roots yield nonoscillatory solutions while complex conjugate roots yield pairs of oscillatory solutions. Regardless of these considerations, however, the roots $\alpha_1, \alpha_2, \ldots, \alpha_n$ of Equation (3.9) correspond to solutions $y_1, \ldots, y_n$ of Equation (3.8) which divide into three subsets: The set $\mathcal{O}$ of oscillatory solutions, the set $\mathcal{N}_1$ of nonoscillatory solutions which dominate $\mathcal{O}$, and the set $\mathcal{N}_2$ of nonoscillatory solutions which are dominated by $\mathcal{O}$. Using these solutions as a basis for the solution space of Equation (3.8), the oscillatory or nonoscillatory nature of an arbitrary linear combination again can be determined merely by observing which coefficients are not zero.

The notion of dominance has appeared, although not always by name, in the works of many authors, including Kim [64], Elias [25], Etgen and Taylor [27], Keener [53, 54], and Cheng [12]. One of the topics discussed in this regard is the uniqueness of the strongly decreasing solution of Equation (3.1). A solution $w(x)$ of Equation (3.1) is said to be <u>strongly decreasing</u> provided there exists a point $c \in I$ such that

$$(-1)^i w^{(i)}(x) > 0 \quad \text{for } x \geq c, \quad i = 0,1, \ldots, n-1. \tag{3.10}$$

By way of contrast, a solution $u(x)$ of Equation (3.1) is called <u>strongly increasing</u> provided there exists $c \in I$ such that

$$u^{(i)}(x) > 0 \quad \text{for } x \geq c, \quad i = 0,1, \ldots, n-1. \tag{3.11}$$

The strongly decreasing solution w is said to be <u>essentially unique</u> if the only solutions of Equation (3.1) satisfying (3.10) are constant multiples of w. Etgen and Taylor [27] have addressed the question of the essential uniqueness of the strongly decreasing solution for the equation $y^{(2m)} - py = 0$ with $p > 0$, and dominance plays an important role in their work. Using a systems approach, this same question is considered by Cheng [12]. In many respects, the types of results sought by these authors mirror the questions of dominance which are studied in the later chapters here.

Because Equation (3.1) analyzed by Dolan and Klaasen is so general, their results are necessarily limited in scope. Unless some less general equation is considered, such as the ones studied by Etgen and Taylor or Cheng, stronger results are not likely. In Chapter IV an equation is studied which seems to offer considerable hope of analysis along these lines.

CHAPTER IV

THE EQUATION $L_n y + py = 0$

Motivated by the examples of the previous chapters, it seems reasonable to consider the equation

$$L_n y + p(x)y = 0 \tag{4.1}$$

where $p(x)$ is continuous and nonzero on $I = [0,\infty)$, and where L_n is the factored disconjugate operator defined by

$$L_n y = \rho_n(x)\frac{d}{dx}\rho_{n-1}(x)\frac{d}{dx}\cdots\rho_1(x)\frac{d}{dx}\rho_0(x)y, \tag{4.2}$$

with $\rho_i(x)$ continuous and positive on I, $i = 0,1,\ldots,n$. Equation (4.1) is similar in form to the constant coefficient equation (3.7), and as has already been observed, the quasi-derivatives $L_i y$, $i = 0,1,\ldots,n$, defined implicitly by Equation (4.2) share many characteristics with ordinary derivatives. One might ask, then, whether Equation (4.1) shares any properties with Equation (3.7). The behavior of the solutions of Equation (3.7) is completely characterized by the nature of the n-th roots of -k. The main goal of the present chapter is to study the behavior of the solutions of Equation (4.1) and to describe a useful classification scheme for the solutions based on their behavior. Once this behavior is understood, questions of dominance such as those raised in Chapter III for the general equation may be addressed.

Classifying the solutions of a differential equation into oscillatory and nonoscillatory subsets, or on the basis of the signs of derivatives, is by no means new. Leighton and Nehari [76] used this technique in their study of the fourth order self-adjoint equation

$$[r(x)y'']'' + p(x)y = 0. \tag{4.3}$$

Some of their results were anticipated by Svec [117], who studied Equation (4.3) for $r(x) \equiv 1$, $p(x) \geq 0$. Svec also classified the solutions of the equation

$$y''' + p(x)y = 0 \tag{4.4}$$

for $p(x) \geq 0$, and this work on third order problems was continued in the major papers by Hanan [35] and Lazer [72].

Various n-th order equations related to Equation (4.1) have been studied by Kiguradze [58], Nehari [92-95], Johnson [47, 48], Kim [59-64], and Elias [18-26], as well as a host of others. Kiguradze devised a classification scheme for the nonoscillatory solutions of the equation

$$y^{(n)} + p(x)y = 0, \tag{4.5}$$

and this scheme has been used extensively by Kim [60-63]. Recently, Kim [64] extended the Kiguradze classification scheme to Equation (4.1), but the results obtained are not the best possible. Nehari [95], Peterson [99-101], and Elias [26] have attempted an analysis based on Green's functions, but this approach too does not appear to yield the type of results required here. Instead, the primary tool will be a counting technique first devised by Johnson [47] and later improved by Elias in the papers [18, 19, 24, 25].

Let $S(c_0, c_1, \ldots, c_n)$ denote the number of sign changes in the sequence $c_0, c_1, \ldots, c_n$ of nonzero real numbers. For a nontrivial solution $y(x)$ of Equation (4.1), and for $x > 0$, set

$$S(y,x-) = \lim_{\xi \to x-} S(L_0 y(\xi), L_1 y(\xi), \ldots, L_n y(\xi)), \tag{4.6}$$

$$S(y,x+) = \lim_{\xi \to x+} S(L_0 y(\xi), -L_1 y(\xi), \ldots, (-1)^n L_n y(\xi)). \tag{4.7}$$

Since $L_n y = -p(x)y$ from Equation (4.1), if any quasi-derivative had an accumulation of zeros in I, then by Rolle's theorem y would also have an accumulation of zeros, along with every other quasi-derivative. But then y would have a zero of multiplicity n at some point, contradicting the choice of y as a nontrivial solution. Consequently, every point $x > 0$ has a deleted neighborhood in which no quasi-derivative vanishes, and so the limits (4.6) and (4.7) exist. By virtue of the definitions, $S(y,x-)$ counts the number of sign changes in the sequence $L_0 y(x), \ldots, L_n y(x)$, while $S(y,x+)$ counts the number of sign agreements. Suppose $k = S(y,x+)$ for some point x at which no quasi-derivative vanishes. Then $S(y,x-) = n - k$, and there are $n - k$ changes in sign in the sequence of quasi-derivatives. Thus

$$\operatorname{sgn}[L_0 y(x)] = (-1)^{n-k} \operatorname{sgn}[L_n y(x)].$$

But $\rho_0(x) > 0$, so that $\operatorname{sgn}[y(x)] = \operatorname{sgn}[L_0 y(x)]$. Then, from Equation (4.1), the previous equation becomes

$$\operatorname{sgn}[y(x)] = (-1)^{n-k} \operatorname{sgn}[-p(x)y(x)],$$

or

$$\operatorname{sgn}[y(x)] = -\operatorname{sgn}[(-1)^{n-k} p(x)y(x)].$$

Therefore,

$$(-1)^{n-k}p(x) < 0. \tag{4.8}$$

This important restriction on k will appear again, and is known as the <u>parity condition</u> for Equation (4.1).

Zeros for a solution y of Equation (4.1) are counted according to the zeros of its quasi-derivatives, so that y has a zero of order three at $x = a$ if $L_0y(a) = L_1y(a) = L_2y(a) = 0$. In addition, by virtue of Equation (4.1), the quasi-derivatives $L_0y, \ldots, L_{n-1}y$ can be arranged in a cyclic order with L_0y following $L_{n-1}y$. If for example $n = 3$, and $L_0y(a) = 0 = L_2y(a)$, $L_1y(a) \neq 0$, then $x = a$ is a zero of order two for $L_2y(x)$. For the arbitrary interval $[a,b] \subseteq I$, let $a \leq x_1 \leq x_2 \leq \ldots \leq x_r \leq b$ be the zeros of $L_0y, \ldots, L_{n-1}y$ in the interval. In this enumeration of zeros, zeros of consecutive quasi-derivatives are counted as multiple zeros, while zeros of nonconsecutive quasi-derivatives are given separate subscripts. The exact multiplicity of the zero x_i is denoted by $n(x_i)$. For example, let $n = 5$ and suppose that $L_0y(a) = L_1y(a) = L_3y(a) = 0$, and $L_0y(b) = L_3y(b) = L_4y(b) = 0$. Then $x_1 = a$, $x_2 = a$, and $x_3 = b$ are the zeros, with $n(x_1) = 2$, $n(x_2) = 1$, and $n(x_3) = 3$. Finally, let $\langle m \rangle$ denote the greatest even integer not greater than m. Then an important relation between the sign changes in the list of quasi-derivatives and the zeros x_i in [a,b] is given by the following result of Elias [24], which generalizes Johnson [47].

<u>LEMMA 4.1</u>: Every solution y of Equation (4.1) satisfies

$$N(y) \equiv S(y,a+) + S(y,b-) + \sum_{a<x_i<b} \langle n(x_i) \rangle \leq n. \tag{4.9}$$

If $N(y) = n$, then $L_{j+1}y$ has exactly one sign change between two consecutive zeros of $L_j y$ in $[a,b]$. The quasi-derivative $L_{j+1}y$ changes sign before the first zero of $L_j y$ in $(a,b]$ if and only if $\operatorname{sgn}[L_{j+1}y(a+\varepsilon)] = \operatorname{sgn}[L_j y(a+\varepsilon)]$ for small $\varepsilon > 0$, and this sign change is unique. Similarly $L_{j+1}y$ changes sign after the last zero of $L_j y$ in $[a,b)$ if and only if $\operatorname{sgn}[L_{j+1}y(b-\varepsilon)] = -\operatorname{sgn}[L_j y(b-\varepsilon)]$ for small $\varepsilon > 0$, and this sign change is unique.

<u>PROOF</u>: Let j be an integer, $0 \leq j \leq n-1$. First note that if $L_j y(a) = 0$, then $\operatorname{sgn}[L_j y(a+\varepsilon)] = \operatorname{sgn}[L_{j+1}y(a+\varepsilon)]$ for $\varepsilon > 0$ sufficiently small, while if $L_j y(b) = 0$, then $\operatorname{sgn}[L_j y(b-\varepsilon)] = -\operatorname{sgn}[L_{j+1}y(b-\varepsilon)]$ for $\varepsilon > 0$ sufficiently small. Thus

$$S(y,a+) \geq \sum_{x_i=a} n(x_i), \qquad S(y,b-) \geq \sum_{x_i=b} n(x_i). \tag{4.10}$$

Let $\{x_{ij}\} \subseteq \{x_i\}$ be the zeros of $L_j y$ which are not zeros of $L_{j-1}y$. Hence $n(x_{ij})$ is the exact number of quasi-derivatives which vanish at x_{ij}, starting with $L_j y$.

Let γ_j be the total number of zeros of $L_j y$ in $[a,b]$, counting multiplicities. If the quasi-derivative $L_{j-1}y$ has zeros at the points $z_1 < \ldots < z_q$ in $[a,b]$ with multiplicities $m_1, \ldots, m_q$, then

$$\gamma_{j-1} = m_1 + \ldots + m_q.$$

At these same points, $L_j y$ has zeros of orders $m_1 - 1, \ldots, m_q - 1$, as well as new zeros at the points of $\{x_{ij}\}$. Thus,

$$\gamma_j = (m_1 - 1) + \ldots + (m_q - 1) + \Sigma n(x_{ij}) = \gamma_{j-1} - q + \Sigma n(x_{ij}).$$

Since $L_j y$ must change sign between each pair of zeros z_i, z_{i+1} of

$L_{j-1}y$, then L_jy has at least q-1 zeros of odd order on the interval (z_1,z_q). Hence, if for each zero of L_jy in (z_1,z_q), $n(x_{ij})$ is replaced by $\langle n(x_{ij})\rangle$, the greatest even integer in $n(x_{ij})$, this last equation becomes

$$\gamma_j \geq \gamma_{j-1} - 1 + \sum_{x_{ij}<z_1} n(x_{ij}) + \sum_{z_1<x_{ij}<z_q} \langle n(x_{ij})\rangle + \sum_{z_q<x_{ij}} n(x_{ij}).$$

Since z_1 is a zero of $L_{j-1}y$, then $\operatorname{sgn}[L_jy(z_1-\varepsilon)] = -\operatorname{sgn}[L_{j-1}y(z_1-\varepsilon)]$ for $\varepsilon > 0$ sufficiently small. So if $\operatorname{sgn}[L_jy(a+\varepsilon)] = \operatorname{sgn}[L_{j-1}y(a+\varepsilon)]$ for $\varepsilon > 0$ sufficiently small, then L_jy must change sign on (a,z_1). Employing a similar idea at $x=b$ leads to

$$\gamma_j \geq \gamma_{j-1} - 1 + S(L_{j-1}y(a+\varepsilon), L_jy(a+\varepsilon)) + \sum_{z_1<x_{ij}<z_q} \langle n(x_{ij})\rangle$$

$$+ S(L_{j-1}y(b-\varepsilon), L_jy(b-\varepsilon)). \tag{4.11}$$

Using the relation $L_ny = -py$, Equation (4.11) includes the case

$$\gamma_0 \geq \gamma_{n-1} - 1 + S(L_{n-1}y(a+\varepsilon), L_ny(a+\varepsilon)) + \sum_{z_1<x_{i,0}<z_q} \langle n(x_{i,0})\rangle$$

$$+ S(L_{n-1}y(b-\varepsilon), L_ny(b-\varepsilon)).$$

Note that $S(c_i,c_{i+1}) + S(c_{i+1},c_{i+2}) = S(c_i,c_{i+1},c_{i+2})$. Then adding Equation (4.11) for $j = 0,1, \ldots, n-1$ leads to

$$n \geq S(y,a+) + S(y,b-) + \sum_{a<x_i<b} \langle n(x_i)\rangle,$$

as desired.

If $N(y) = n$, then equality must hold in (4.11) for all values of j. This happens only when there is exactly one sign change between consecutive zeros of the quasi-derivatives, and when the zeros predicted by $S(y,a+)$ and $S(y,b-)$ are unique. This completes the lemma.

If no quasi-derivative for y vanishes at b, then clearly $S(y,b+) + S(y,b-) = n$. Taken together with (4.9), this implies

$$S(y,a+) \leq S(y,b+) \tag{4.12}$$

for all $b \geq a$ for which $L_i y(b) \neq 0$, $i = 0, \ldots, n-1$. Since every point b has a deleted neighborhood in which no quasi-derivative vanishes, Equation (4.7) extends (4.12) to all $b \geq a$. Thus $S(y,x+)$ is an increasing integer valued function of x. Since this function is bounded above by n, the following lemma of Elias [24] is immediate.

LEMMA 4.2: For every nontrivial solution y of Equation (4.1), there is an $x_0 > 0$ and an integer $k \geq 0$ such that $S(y,x+) \equiv k$ and $S(y,x-) \equiv n-k$ for every $x > x_0$. Furthermore, by virtue of (4.9), the functions $L_0 y, \ldots, L_{n-1} y$ may have only simple zeros on (x_0, ∞).

Recalling the parity condition (4.8), Lemma 4.2 implies that the solution space of Equation (4.1) may be divided into disjoint subsets or classes $S_k \equiv \{y \mid \lim_{x\to\infty} S(y,x+) = k\}$. There are two key questions to be answered in determining whether such a division is useful: Are the classes nonempty? Do the members of a class have any other common characteristics? The next theorem (Elias [24]) answers the first of these questions in the affirmative; a definitive answer to the second will require some further work.

THEOREM 4.3: For each k, $0 \leq k \leq n$, satisfying the parity condition (4.8), the class S_k is nonempty.

PROOF: The proof uses a limiting procedure to construct a solution in each class. To begin, let k satisfying $1 \leq k \leq n-1$ and $(-1)^{n-k}p(x) < 0$ be fixed, and let $y(x,s)$ be a nontrivial solution of Equation (4.1) satisfying the $n-1$ homogeneous boundary condition

$$L_i y(a) = 0, \qquad i = 0, \ldots, k-1, \tag{4.13a}$$

$$L_j y(s) = 0, \qquad j = 0, \ldots, n-k-2, \tag{4.13b}$$

where $0 \leq a < s$. The inequalities (4.10) reveal that $S(y(x,s),a+) \geq k$ and $S(y(x,s),s-) \geq n-k-1$. Since $S(y,a+) + S(y,s-) \leq n$ by Lemma 4.1, then $S(y(x,s),a+) \leq k+1$ and $S(y(x,s),s-) \leq n-k$. But since $(-1)^{n-k}p(x) < 0$, then $S(y(x,s),a+) = k+1$ would violate the parity condition (4.8), and so $S(y(x,s),a+) = k$. Similarly, $S(y(x,s),s-) = n-k-1$ would violate the parity condition, and so $S(y(x,s),s-) = n-k$. Since $S(y,x+)$ is increasing in x, then for any $x \in (a,s)$ it follows that

$$S(y(x,s),x+) \geq S(y(x,s),a+) = k,$$

while at the same time from (4.9),

$$S(y(x,s),x+) \leq n - S(y(x,s),s-) = k.$$

Therefore, $S(y(x,s),x+) = k$. Similar reasoning reveals that for all $x \in (a,s)$, $S(y(x,s),x-) = n-k$. That is,

$$S(y(x,s),x+) = k, \qquad a \leq x < s, \tag{4.14a}$$

$$S(y(x,s),x-) = n-k, \quad a < x \leq s. \tag{4.14b}$$

The solution $y(x,s)$ satisfying (4.13) is essentially unique, for if there were another independent solution for these conditions, then some linear combination $\tilde{y}$ of the two would have at least $k+1$ zeros at $x=a$ and at least $n-k-1$ zeros at $x=s$. Then the inequalities (4.10) and (4.9) would imply that $S(\tilde{y},a+)=k+1$ and $S(\tilde{y},s-)=n-k-1$, in violation of the parity condition.

For each $s>a$, normalize the solution $y(x,s)$ by setting $\sum_{j=0}^{n-1}(L_j y(a,s))^2=1$, and choose a monotone, increasing, unbounded sequence $\{s_i\}_{i=1}^{\infty}$ such that the functions $L_j y(x,s_i)$ converge for each j uniformly on compact subsets of $[0,\infty)$. Let $y(x)$ be the solution of Equation (4.1) such that $\lim_{i\to\infty} y(x,s_i)=y(x)$. Then by Lemma 4.2, there is an $x_0 \geq 0$ such that $S(y,x+)$ is constant for $x>x_0$. Take $x_1>x_0$ such that $L_j(x_1)\neq 0$ for $j=0,\ldots,n-1$. For i sufficiently large, then, $L_j y(x_1,s_i)\neq 0$ and $\operatorname{sgn}[L_j y(x_1)]=\operatorname{sgn}[L_j y(x_1,s_i)]$ for $j=0,\ldots,n-1$. Consequently, $S(y,x_1+)=S(y(x,s_i),x_1+)$ for i sufficiently large. If in addition i is chosen so large that $s_i>x_1$, then Equation (4.14) implies $S(y,x_1+)=k$, and hence by choice of x_1 the solution y is in the class S_k. Thus for $(-1)^{n-k}p(x)<0$, $1\leq k\leq n-1$, the class S_k is nonempty.

If $k=0$ is allowable by (4.8), let $y(x,s)$ satisfy the conditions

$$L_j y(s) = 0, \qquad j=0,\ldots,n-2,$$

normalized by $\sum_{j=0}^{n-1}(L_j y(a,s))^2=1$, and then argue as before. If $k=n$ satisfies (4.8), take $y(x)$ to be the unique solution of (4.1) satisfying

$$L_i y(a) = 0, \qquad i=0,\ldots,n-2,$$

$$L_{n-1}y(a) = 1.$$

Then $S(y,a+)=n$ by (4.10), and hence $S(y,x+)=n$ on $[a,\infty)$ since $S(y,x+)$ is increasing in x. This implies that no quasi-derivative can change sign in (a,∞), and so the solution $y(x)$ in fact satisfies $L_i y(x) > 0$, $i = 0, \ldots, n-1$, on (a,∞). Thus the class S_n is nonempty. This concludes the proof of Theorem 4.3.

The proof of Theorem 4.3 demonstrates not only that the class S_k is nonempty, but also that for every choice of $a \geq 0$, S_k contains a solution $y(x)$ such that $L_i y(a) = 0$, $i = 0, \ldots, k-1$, and $S(y, x+) = k$ for all $x \in [a,\infty)$. These solutions in fact generalize the "principal solutions" used by Leighton and Nehari [76] and the "fundamental solutions" employed by Hanan [35]. As will be seen shortly, a further generalization of these solutions is possible.

The evidence suggests that the classes S_k play a key role in the dominance of solutions of Equation (4.1). For the equation

$$y^{(4)} - y = 0,$$

for example, straight-forward computations show $e^{-x} \in S_0$, $e^{x} \in S_4$, and $\sin x, \cos x \in S_2$. In the sense defined in Chapter III, the solutions in class S_2 dominate those in S_0, and are themselves dominated by the solutions in class S_4. This role is supported further by the nature of the classes themselves.

While at first it might seem that there need not be any logical organization in the division of solutions of Equation (4.1) among the classes S_k, it will be shown in Theorem 4.14 that, for fixed k, the solutions in S_k are either all oscillatory or all nonoscillatory. Before this result can be proved, however, it will be necessary to develop the idea of extremal points for Equation (4.1).

Extremal Points and Extremal Solutions

The development in this section, leading eventually to a proof of the result described above, is based upon the recent paper [24] of Elias. The results in Elias' paper are a natural outgrowth of the work done on the fourth order equation (4.3) by Leighton and Nehari [76], and generalize many of their ideas. These results also involve an extension of the notion of conjugate point and focal point, and suggest several lines of future investigation as described in Chapter VII.

For the interval $[a,s] \subset [0,\infty)$, consider the boundary conditions

$$L_i y(a) = 0, \qquad i \in \{i_1, \ldots, i_k\}, \tag{4.15a}$$

$$L_j y(s) = 0, \qquad j \in \{j_1, \ldots, j_{n-k}\}, \tag{4.15b}$$

where $\{i_1, \ldots, i_k\}$ and $\{j_1, \ldots, j_{n-k}\}$ are two arbitrary (not necessarily distinct) sets of indices chosen from $\{0, \ldots, n-1\}$. If Equation (4.1) has a nontrivial solution y satisfying (4.15), then (4.10) implies $S(y,a+) \geq k$ and $S(y,s-) \geq n-k$. But since $S(y,a+) + S(y,s-) \leq n$ by Lemma 4.1, then $S(y,a+) = k$ and $S(y,s-) = n-k$. Therefore, Equation (4.1) has a nontrivial solution subject to (4.15) only if k satisfies the parity condition (4.8). In the remainder of this section, it will be assumed that $0 \leq k \leq n$ and $(-1)^{n-k} p(x) < 0$, unless explicitly stated otherwise.

Many classical results are more easily understood in light of this parity restriction on k for (4.15). If p is negative and $n = 2$, than a (1,1) zero distribution is prohibited, and hence the corresponding second order equation has no oscillatory solutions. For $n = 4$, this parity result reduces to the restrictions on possible zero distributions discovered by Leighton and Nehari [76]. The Class I and Class II solution conditions utilized by Hanan [35] for the general third order equation reduce to the parity conditions for $n = 3$ in Equation (4.1). General n-th order versions of the parity restrictions have appeared in the works of virtually every author considering higher order equations since the 1967 paper of Nehari [93]. The parity condition is primarily geometric in nature, and similar geometric considerations are at the heart of the disconjugacy conditions considered by Barrett [7], Peterson [97], Ridenhour and Sherman [108], and Keener [54] for the general fourth order problem.

As an extension of the notion of conjugate or focal point, define the i-th _extremal point_ $\theta_i(a)$ of a corresponding to (4.15) to be the i-th value of s in (a,∞) for which Equation (4.1) has a nontrivial solution satisfying (4.15). Such a nontrivial solution is called an _extremal solution_. These definitions presume that the points $s \in (a,\infty)$ for which extremal solutions exist have no accumulation point in $[a,\infty)$. Since no nontrivial solution may have n zeros at a, then a cannot be an accumulation point of the set of extremal points for (4.15). That there can be no accumulation point in (a,∞) will follow from Theorem 4.7, to be shown shortly. Solutions meeting these definitions were discussed by Leighton and Nehari [76] for the fourth order problem (4.3), while Johnson [47, 48, 49] and Elias [18, 19, 24, 25] have devoted much effort to the study of such solutions for Equation (4.1).

Since (4.15) has n conditions on y, nontrivial solutions may not always exist. If one of these conditions is removed, however, a solution always exists for the remaining set of $n - 1$ homogeneous conditions, and may be used to analyze (4.15).

LEMMA 4.4: If the $n-1$ boundary conditions

$$L_i y(a) = 0, \qquad i \in \{i_1, \ldots, i_k\}, \tag{4.16a}$$

$$L_j y(s) = 0, \qquad j \in \{j_1, \ldots, j_{n-k-1}\} \tag{4.16b}$$

are considered, then the following results hold:

(1) Equation (4.1) has an essentially unique solution $y(x,s)$ satisfying (4.16).

(2) At the point a, $S(y(x,s),a+) = k$, and $\operatorname{sgn}[L_{i+1}y(a+\varepsilon,s)] = \operatorname{sgn}[L_i y(a+\varepsilon,s)]$ if and only if $i \in \{i_1, \ldots, i_k\}$. In particular, no quasi-derivatives other than those specified in (4.16) can vanish at a.

(3) At the point s, $S(y(x,s),s-) = n-k$, and $n-k-1$ of the sign changes among consecutive quasi-derivatives are determined by $j_1, \ldots, j_{n-k-1}$. At most one additional quasi-derivative may vanish at s other than those specified in (4.16).

(4) $S(y(x,s),x+) \equiv k$ for $x \in [a,s)$ and $S(y(x,s),x-) \equiv n-k$ for $x \in (a,s]$.

(5) $L_t y(x,s)$, $t = 0, \ldots, n-1$, may have only simple zeros in (a,s). $L_{t+1}y(x,s)$ has exactly one simple zero between two consecutive zeros of $L_t y(x,s)$ in $[a,s]$.

(6) $L_t y(x,s)$, $t = 0, \ldots, n-1$, and its simple zeros are differentiable as functions of s.

PROOF: Note that the boundary conditions (4.16) are a generalized version of the boundary conditions (4.13) considered in the proof of Theorem 4.3. The inequalities (4.10) again show $S(y(x,s),a+) \geq k$ and $S(y(x,s),s-) \geq n-k-1$, so that parts (1) and (4) may be proved as in the case of the earlier theorem. Parts (2) and (3) then follow immediately from Lemma 4.1 and the parity condition (4.8).

The quasi-derivatives $L_t y(x,s)$, $t = 0, \ldots, n-1$, can have only simple zeros in (a,s) by virtue of (4.9) since $S(y(x,s),a+) + S(y(x,s),s-) = n$. Lemma 4.1 showed that $L_{t+1}y(x,s)$ has exactly one change of sign between the zeros of $L_t y(x,s)$ in $[a,s]$; since all the zeros of $L_{t+1}y(x,s)$ in (a,s) must now be simple zeros, $L_{t+1}y(x,s)$ has exactly one simple zero between the zeros of $L_t y(x,s)$ in $[a,s]$.

It remains only to prove (6). Let $\{u_1, \ldots, u_n\}$ be an independent set of solutions of Equation (4.1), and consider the determinant

$$z(x) \equiv \begin{vmatrix} L_{i_1}u_1(a), \ldots, L_{i_k}u_1(a), L_{j_1}u_1(s), \ldots, L_{j_{n-k-1}}u_1(s), u_1(x) \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ L_{i_1}u_n(a), \ldots, L_{i_k}u_n(a), L_{j_1}u_n(s), \ldots, L_{j_{n-k-1}}u_n(s), u_n(x) \end{vmatrix} \tag{4.17}$$

Clearly $z(x)$ is a solution of Equation (4.1) satisfying (4.16). Choose x_0 in (a,s) such that $y(x_0,s) \neq 0$. Then the determinant defining $z(x_0)$ must be nonzero since the solution $y(x,s)$ is essentially unique. Therefore, $z(x_0) \neq 0$, and thus $z(x)$ is a nontrivial solution of Equation (4.1). Hence $y(x,s) = z(x)$, and part (6) follows by the implicit function theorem applied to the determinant (4.17). This completes the proof of Lemma 4.4.

The proof of Lemma 4.4 rests on the fact that any solution y of Equation (4.1) satisfying the boundary conditions (4.16) must also satisfy $N(y) \geq n-1$, where $N(y)$ is given by (4.9). The proof can be extended to include any homogeneous set of k boundary conditions at a and $n-k-1$ conditions at s, or $k-1$ boundary conditions at a and $n-k$ conditions at s, which force $N(y) \geq n-1$. This is summarized for future reference in the following lemma.

LEMMA 4.5: The properties of Lemma 4.4 can be proved for the solutions of Equation (4.1) satisfying any set of $n-1$ homogeneous boundary conditions at a and s which give $N(y) \geq n-1$.

Comparing Lemma 4.1 and parts (2) and (3) of Lemma 4.4 reveals that $L_{i+1}y(x,s)$ has a simple zero before the first zero of $L_i y(x,s)$ in $(a,s]$ if and only if $i \in \{i_1, \ldots, i_k\}$, and that there is precisely one value of j other than those given by $\{j_1, \ldots, j_{n-k-1}\}$ for which $L_{j+1}y(x,s)$ has a simple zero after the last zero of $L_j y(x,s)$ in $[a,s)$. It is clear from the boundary conditions (4.15) and (4.16) that s is an extremal point for (4.15) if and only if $L_{j_{n-k}} y(x,s)\big|_{x=s} = 0$. Thus if s is an extremal point, then $y(x,s)$ is the essentially unique extremal solution. These remarks may be summarized as follows.

COROLLARY 4.6: Let $y = y(x,\theta_i)$ be the unique extremal solution corresponding to $\theta_i(a)$. Then $L_{t+1}y$ has exactly one simple zero between two consecutive zeros of $L_t y$ in $[a,\theta_i(a)]$, and these are its only zeros in $(a,\theta_i(a))$, for $t = 0, \ldots, n-1$.

The essential uniqueness of such solutions for the fourth order problem (4.3) was shown by Leighton and Nehari [76, Theorem 2.6]. For the case $n = 2$ the result is trivial since no solution may have a double zero. For $n = 3$ the essential uniqueness of these solutions can be derived from Hanan's Class I and Class II conditions. Johnson [49] also has several results of this type for Equation (4.1) with n even.

Equation (4.17) suggests a relationship between extremal points and certain Wronskian-like determinants. The next few results explore this relationship in order to establish a link between the extremal points of Equation (4.1) and its oscillatory solutions. Recall that, in addition, Theorem 4.7 will show that the set of extremal points of Equation (4.1) has no finite accumulation point.

By comparison with Equation (4.17), it is clear that the determinant $W(s) \equiv W(s,j_1, \ldots ,j_{n-k})$ defined by

$$W(s) = \begin{vmatrix} L_{i_1}u_1(a),\ldots,L_{i_k}u_1(a),L_{j_1}u_1(s),\ldots,L_{j_{n-k}}u_1(s) \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ L_{i_1}u_n(a),\ldots,L_{i_k}u_n(a),L_{j_1}u_n(s),\ldots,L_{j_{n-k}}u_n(s) \end{vmatrix} \tag{4.18}$$

satisfies $W(s) = L_{j_{n-k}}y(x,s)|_{x=s}$, and so $W(s)$ vanishes if and only if s is an extremal point for (4.15). A more precise relationship between the zeros of $W(s)$ and the extremal points of Equation (4.1) is expressed in the next theorem.

THEOREM 4.7: The extremal points for (4.15) are simple zeros of $W(s)$.

PROOF: Differentiating by columns in Equation (4.18) gives

$$\frac{d}{ds}W(s,j_1,\ldots,j_{n-k}) = \sum_{t=1}^{n-k} W(s,j_1,\ldots,j_{t-1},j_t+1,j_{t+1},\ldots,j_{n-k})/\rho_{j_t+1}(s). \tag{4.19}$$

The proof consists in showing that the derivative is nonzero for those values of s for which $W(s)$ vanishes. In particular, the proof will show that the right side of Equation (4.19) has at least one nonzero term, and that each nonzero term has the same sign. To that end, assume throughout the proof that s satisfies

$$W(s;j_1,\ldots,j_{n-k}) = 0. \tag{4.20}$$

Recall that $1 \leq k \leq n-1$ in the boundary conditions (4.15), so that there is at least one term on the right side of Equation (4.19) for which $j_t + 1 \neq j_{t+1}$. Suppose such a term vanished, so that

$$W(s; j_1, \ldots, j_t + 1, \ldots, j_{n-k}) = 0. \tag{4.21}$$

Equation (4.20) implies that $y_1(x) = y(x,s)$ is an extremal solution satisfying (4.15), while Equation (4.21) implies the existence of a nontrivial solution $y_2(x)$ satisfying the same boundary conditions with j_t replaced by $j_t + 1$. Since these solutions share $n-1$ boundary conditions, they are linearly dependent by Lemma 4.4. But then each has a total of $n-k+1$ zeros at s, contradicting part (3) of Lemma 4.4. Consequently, the determinant in Equation (4.21) cannot vanish, and the derivative in Equation (4.19) has at least one nonzero term. If it has exactly one nonzero term, the proof is complete.

Suppose now that the sum in Equation (4.19) has nonzero terms for $t = q$ and $t = r$, with $j_q < j_r$. In order to prove that these terms have the same sign, each will be shown to be a quasi-derivative of a constant multiple of the extremal solution for (4.15). By means of an auxiliary function, the signs of the constant multiples can be determined.

Let $y_1(x,s)$ and $y_2(x,s)$ be defined by the determinants in Table I. Then as in Lemma 4.4, $y_1(x,s)$ is the essentially unique solution of Equation (4.1) satisfying the boundary conditions

$$L_i y(a) = 0, \qquad i \in \{i_1, \ldots, i_k\},$$

$$L_j y(s) = 0, \qquad j \in \{j_1, \ldots, j_{n-k}\} \setminus \{j_q\},$$

while $y_2(x,s)$ is the essentially unique solution satisfying

TABLE I

MATRIX DEFINITIONS OF THE FUNCTIONS y_1, y_2, AND y_3

$$y_1(x,s) = \begin{vmatrix} L_{i_1}u_1(a), & \ldots, & L_{i_k}u_1(a), & L_{j_1}u_1(s), & \ldots, & L_{j_{q-1}}u_1(s), & L_{j_{q+1}}u_1(s), & \ldots, & L_{j_{n-k}}u_1(s), & u_1(x) \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ L_{i_1}u_n(a), & \ldots, & L_{i_k}u_n(a), & L_{j_1}u_n(s), & \ldots, & L_{j_{q-1}}u_n(s), & L_{j_{q+1}}u_n(s), & \ldots, & L_{j_{n-k}}u_n(s), & u_n(x) \end{vmatrix}$$

$$y_2(x,s) = \begin{vmatrix} L_{i_1}u_1(a), & \ldots, & L_{i_k}u_1(a), & L_{j_1}u_1(s), & \ldots, & L_{j_{r-1}}u_1(s), & L_{j_{r+1}}u_1(s), & \ldots, & L_{j_{n-k}}u_1(s), & u_1(x) \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ L_{i_1}u_n(a), & \ldots, & L_{i_k}u_n(a), & L_{j_1}u_n(s), & \ldots, & L_{j_{r-1}}u_n(s), & L_{j_{r+1}}u_n(s), & \ldots, & L_{j_{n-k}}u_n(s), & u_n(x) \end{vmatrix}$$

$$y_3(x,s) = \begin{vmatrix} L_{i_1}u_1(a), & \ldots, & L_{i_{k+1}}u_1(a), & L_{j_1}u_1(s), & \ldots, & L_{j_{q-1}}u_1(s), & L_{j_{q+1}}u_1(s), & \ldots, & L_{j_{r-1}}u_1(s), & L_{j_{r+1}}u_1(s), & \ldots, & L_{j_{n-k}}u_1(s), & u_1(x) \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ L_{i_1}u_n(a), & \ldots, & L_{i_{k+1}}u_n(a), & L_{j_1}u_n(s), & \ldots, & L_{j_{q-1}}u_n(s), & L_{j_{q+1}}u_n(s), & \ldots, & L_{j_{r-1}}u_n(s), & L_{j_{r+1}}u_n(s), & \ldots, & L_{j_{n-k}}u_n(s), & u_n(x) \end{vmatrix}$$

$$L_i y(a) = 0, \qquad i \in \{i_1, \ldots, i_k\},$$

$$L_j y(s) = 0, \qquad j \in \{j_1, \ldots, j_{n-k}\} \setminus \{j_r\}.$$

Since each of these sets of boundary conditions is a subset of those in (4.15), then both y_1 and y_2 must be extremal solutions, and so constant multiples of each other.

Differentiating and then exchanging columns in the definition of y_1 and y_2 gives

$$(-1)^{n-k-q} L_{j_q+1} y_1(x,s) = W(s; j_1, \ldots, j_q + 1, \ldots, j_{n-k}) \equiv W_q \tag{4.22}$$

and

$$(-1)^{n-k-r} L_{j_r+1} y_2(x,s) = W(s; j_1, \ldots, j_r + 1, \ldots, j_{n-k}) \equiv W_r. \tag{4.23}$$

The boundary conditions (4.13) imply that for $\varepsilon > 0$ sufficiently small, each of y_1 and y_2 satisfies

$$\operatorname{sgn}[L_j y(s-\varepsilon)] = -\operatorname{sgn}[L_{j+1} y(s-\varepsilon)] \tag{4.24}$$

if and only if $j \in \{j_1, \ldots, j_{n-k}\}$. Consequently,

$$\operatorname{sgn}[y_1(x,s)|_{s-\varepsilon}] = (-1)^q \operatorname{sgn}[L_{j_q+1} y_1(x,s)|_{s-\varepsilon}],$$

and this together with Equation (4.22) becomes

$$\operatorname{sgn}[W_q] = (-1)^{n-k} \operatorname{sgn}[y_1(x,s)|_{s-\varepsilon}].$$

Likewise, Equations (4.24) and (4.23) combine to give

$$\operatorname{sgn}[W_r] = (-1)^{n-k} \operatorname{sgn}[y_2(x,s)|_{s-\varepsilon}].$$

Since y_1 and y_2 are both constant multiples of the extremal solution

$y(x,s)$, their product has constant sign for all x, and so

$$\operatorname{sgn}[W_q W_r] = \operatorname{sgn}[y_1(x,s)|_{s-\varepsilon}\, y_2(x,s)|_{s-\varepsilon}] = \operatorname{sgn}[y_1(x,s) y_2(x,s)]. \tag{4.25}$$

Therefore, it suffices to show that y_1 and y_2 have the same sign. In order to do this, it is necessary to compare them at a point where it is known they do not vanish.

Choose i_{k+1} from $\{0, \ldots, n-1\} \setminus \{i_1, \ldots, i_k\}$, so that $L_{i_{k+1}} y(x,s)|_{x=a} \neq 0$, and define $y_3(x,s)$ by the determinant in Table I to be the unique solution of Equation (4.1) satisfying the boundary conditions

$$L_i y(a) = 0, \qquad i \in \{i_1, \ldots, i_k, i_{k+1}\}, \tag{4.26a}$$

$$L_j y(s) = 0, \qquad j \in \{j_1, \ldots, j_{n-k}\} \setminus \{j_q, j_r\}. \tag{4.26b}$$

Differentiating and exchanging rows as before reveals that

$$(-1)^{n-k-r} L_{j_r} y_3(x,s)|_{x=s} = (-1)^{n-k-1} L_{i_{k+1}} y_1(a,s) \tag{4.27}$$

and (recalling that the j_r column is missing from $y_3(x,s)$, with $j_r > j_q$)

$$(-1)^{n-k-q-1} L_{j_q} y_3(x,s)|_{x=s} = (-1)^{n-k-1} L_{i_{k+1}} y_2(a,s). \tag{4.28}$$

It is necessary to determine the signs of $L_t y_3(x,s)$, $t = 0, \ldots, n-1$, on a left neighborhood of s. From the boundary conditions (4.26) and the parity condition, the $n-k-2$ conditions $L_j y_3(x,s)|_{x=s} = 0$, $j \in \{j_1, \ldots, j_{n-k}\} \setminus \{j_q, j_r\}$, locate all the sign changes exactly. So, among the functions $L_{j_q} y_3, \ldots, L_{j_r} y_3$, there are exactly $r-q-1$ sign changes in a left neighborhood of s, or

$$\operatorname{sgn}[L_{j_r} y_3(x,s)|_{x=s}] = (-1)^{r-q-1} \operatorname{sgn}[L_{j_q} y_3(x,s)]. \tag{4.29}$$

Combining Equations (4.27), (4.28), and (4.29) shows

$$\operatorname{sgn}[L_{i_{k+1}} y_1(a,s)] = \operatorname{sgn}[L_{i_{k+1}} y_2(a,s)].$$

Consequently the products in Equation (4.25) are positive, and so the nonzero terms of the derivative (4.19) are of the same sign. This completes the proof of Theorem 4.7.

Suppose that there existed a sequence of points $\{s_i\}_{i=1}^{\infty}$ converging to some point s_0 in (a,∞) such that $W(s_i) = 0$, $i = 1, \ldots, \infty$. Since $W(s)$ is differentiable on (a,∞), then by Rolle's theorem there would be a sequence $s_i' \to s_0$ such that $\frac{d}{ds}W(s_i') = 0$, $i = 1, \ldots, \infty$. The continuity of W and $\frac{d}{ds}W$ would then imply $W(s_0) = \frac{d}{ds}W(s_0) = 0$, contradicting Theorem 4.7. Hence the set of extremal points has no finite accumulation point, and the points in the set may be numbered accordingly.

Each extremal point $\theta_i(a)$ depends upon the choice of a. For the general problem investigated in earlier chapters, it could happen that $\theta_i(a) = \theta_i(a')$ while $a \neq a'$. That this is no longer the case for Equation (4.1) is demonstrated next. In addition, the domain of θ_i is analyzed. This result is a generalization of earlier results due to Peterson [98], Johnson [47], Kim [59], and Nehari [95].

<u>THEOREM 4.8</u>: The function $\theta_i(a)$ is differentiable and strictly increasing, with domain of the form $[0,b)$, $0 \leq b \leq \infty$.

<u>PROOF</u>: To emphasize its dependence on a, denote the determinant (4.18) by $W(a,s)$. As Theorem 4.7 has shown, if $\theta_i(a)$ exists, then it is a simple zero of $W(a,s)$. Hence by the implicit function theorem $\theta_i(a)$ is defined and differentiable in a neighborhood of a, and

$$\frac{d\theta_i}{da} = \frac{\partial W}{\partial a} \Big/ \frac{\partial W}{\partial s}\Bigg|_{s=\theta_i(a)} .$$

Analyzing $\frac{\partial W}{\partial a}$ as in Theorem 4.7 shows it is nonvanishing, so that $\frac{d\theta_i}{da} \neq 0$ and θ_i is monotonic. Since θ_i can be continued as long as it is bounded, then it must be increasing. Otherwise, it could be continued for $b > a$ until $\theta_i(b) > b$ is contradicted.

If $\theta_i(a)$ exists, then θ_i is defined in some largest open interval A containing a. Let $a' = \inf A$. For every $b \in (a',a]$, $\theta_i(b)$ exists and $W(b,\theta_i(b)) = 0$. By the continuity of W and θ_i, $\theta_i(a')$ must also exist. If $a' > 0$, then θ_i is defined in a neighborhood of a', contradicting the definition of $a' = \inf A$. Thus $a' = 0$, and $\theta_i(a)$ is defined on some half-open subinterval of $[0,\infty)$. This completes the proof of Theorem 4.8.

From Lemma 4.4 it is known that the unique solution $y(x,s)$ of Equation (4.1) satisfying (4.16) has only simple zeros in (a,s), and at most one additional zero at s other than those specified by (4.16). The behavior of these zeros as functions of s is described next.

<u>THEOREM 4.9</u>: The number of simple zeros of $L_r y(x,s)$, $0 \leq r \leq n-1$, in (a,s) can vary, as s increases, only when a simple zero enters (a,s) through the variable endpoint s.

<u>PROOF</u>: Since $y(x,s)$ cannot have multiple zeros in (a,s), and since its simple zeros are continuous functions of s, their number can vary only when one simple zero enters or leaves the interval through the endpoints a or s. Lemma 4.4 forbids any additional zeros at a, and so the simple zeros may only enter or leave through s. The proof consists of showing that zeros may only enter through s.

Suppose $r \in \{j_1, \ldots, j_{n-k-1}\}$, so that $L_r y(x,s)|_{x=s} = 0$. If for certain values of s, a simple zero of $L_r y(x,s)$ does not enter or leave (a,s) at s, then $\operatorname{sgn}[L_r y(x,s)|_{x=s-\varepsilon}] = -\operatorname{sgn}[L_{r+1} y(x,s)|_{x=s-\varepsilon}]$ is constant, and so a simple zero of $L_{r+1} y(x,s)$ cannot enter (a,s) at s. On the other hand, if a zero of $L_r y(x,s)$ does enter or leave at s, then Rolle's theorem implies the same holds for $L_{r+1} y(x,s)$ as well. Therefore, it is sufficient to prove the theorem for those quasi-derivatives for which no condition is given at s in (4.16).

Assume for the moment that $r+1 \notin \{j_1, \ldots, j_{n-k-1}\}$ as well. If then $L_r y(x,s_0)$ has a zero at $x = s_0$, it must be a simple zero, and so by the implicit function theorem there exists a simple zero x(s) of $L_r y(x,s)$ such that $x(s_0) = s_0$ and

$$\left.\frac{dx}{ds}\right|_{s=s_0} = -\frac{\partial}{\partial s} L_r y(x,s) \Big/ \frac{\partial}{\partial x} L_r y(x,s) \Bigg|_{x=s_0,\, s=s_0} . \tag{4.30}$$

Recall from Equation (4.17) that $L_r y(x,s)$ has the determinant representation

$$L_r y(x,s) = \begin{vmatrix} L_{i_1} u_1(a), \ldots, L_{i_k} u_1(a), L_{j_1} u_1(s), \ldots, L_{j_{n-k-1}} u_1(s), L_r u_1(x) \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \qquad\qquad \vdots \\ L_{i_1} u_n(a), \ldots, L_{i_k} u_n(a), L_{j_1} u_n(s), \ldots, L_{j_{n-k-1}} u_n(s), L_r u_n(x) \end{vmatrix}$$

so that

$$\frac{\partial}{\partial x} L_r y(x,s)\Big|_{(s_0,s_0)} = W(s_0; j_1, \ldots, j_{n-k-1}, r+1)/\rho_{r+1}(s_0), \tag{4.31a}$$

$$\frac{\partial}{\partial s} L_r y(x,s)\Big|_{(s_0,s_0)} = \sum_{t=1}^{n-k-1} W(s_0; j_1, \ldots, j_t + 1, \ldots, j_{n-k-1}, r)/\rho_{j_t+1}(s_0). \tag{4.31b}$$

Since s_0 is an extremal point for the boundary conditions

$$L_i y(a) = 0, \qquad i \in \{i_1, \ldots, i_k\} \tag{4.32a}$$

$$L_j y(s) = 0, \qquad j \in \{j_1, \ldots, j_{n-k-1}, r\}, \tag{4.32b}$$

the proof of Theorem 4.7 shows that the nonzero derivatives in (4.31) have the same sign, and that $\frac{\partial}{\partial x} L_r y(x,s)\Big|_{(s_0,s_0)} \neq 0$. Thus, by Equation (4.30), $x'(s_0) \leq 0$. Let $d(s) \equiv x(s) - s$ measure the distance between the simple zero $x(s)$ and the variable endpoint s. Since $d(s_0) = 0$ and $d'(s_0) \leq -1$, the zero $x(s)$ enters the interval (a,s) at $s = s_0$.

Now consider the case where $r+1 \in \{j_1, \ldots, j_{n-k-1}\}$. The argument used above fails here since $\frac{\partial}{\partial x} L_r y(x,s)\Big|_{(s_0,s_0)} = 0$, and the implicit function theorem no longer applies. In order to analyze this case, it is necessary to locate an extremal point s_1 following s_0 for (4.32), which in fact may not exist. Since extremal points for (4.32) are simple zeros of $W(s;j_1, \ldots, j_{n-k-1}, r)$, for $\varepsilon > 0$ arbitrarily small, the zero $x(s)$ must either have entered $(a, s_0 + \varepsilon)$ at $s = s_0$ or left without entering. Therefore, choose c, $c > s_0$, and on $[c,\infty)$ redefine $p(x) \equiv p(c)$ and $\rho_i(x) \equiv \rho_i(c)$. Since the solution $y(x,s)$ is unaffected on $[a,c]$, the behavior of $x(s)$ is also unaffected. On the other hand, on $[c,\infty)$ Equation (4.1) now has constant coefficients, and it is well known that $\theta_i(c)$ exists for all i for this problem [21]. By Theorem 4.8 $\theta_i(a)$ exists for all i as well. Without loss of generality, therefore, assume the extremal point $s_1 > s_0$ exists. To prove that the zero $x(s)$ entered (a,s) at $s = s_0$, it will suffice to show that $L_r y(x,s_1)$ has more zeros in (a,s_1) than $L_r y(x,s_0)$ has in (a,s_0).

From the set of $n-k < n$ indices $\{j_1, \ldots, j_{n-k-1}, r\}$ choose an index, say j_q, for which $j_q + 1$ is not also in the set. Let $y_1(x,s)$ be the essentially unique solution satisfying the conditions

$$L_i y(a) = 0, \qquad i \in \{i_1, \ldots, i_k\}, \tag{4.33a}$$

$$L_j y(s) = 0, \qquad j \in \{j_1, \ldots, j_{n-k-1}, r\} \setminus \{j_q\}. \tag{4.33b}$$

Clearly $y(x,s_0) = y_1(x,s_0)$ and $y(x,s_1) = y_1(x,s_1)$, up to constant multiples, since the extremals for (4.32) also satisfy (4.33). Since $j_q + 1 \notin \{j_1, \ldots, j_{n-k-1}, r\}$, the first part of the proof implies $L_{j_q} y_1(x,s_1)$ has one more zero in (a,s_1) than has $L_{j_q} y_1(x,s_0)$ in (a,s_0). By Corollary 4.7, it follows that $L_r y_1(x,s_1)$ has exactly one more zero in (a,s_1) than does $L_r y(x,s_0)$ in (a,s_0). This completes the proof of Theorem 4.9.

For s sufficiently close to a, the solution y(x,s) and its quasi-derivatives can have no zeros in (a,s) other than those predicted by Rolle's theorem from the boundary conditions (4.16). If there were an additional zero for values of s arbitrarily close to a, then Rolle's theorem would guarantee at least one zero for each quasi-derivative $L_t y(x,s)$. Choosing an appropriate sequence $s_i \to a^+$ and passing to the limit would then give a nontrivial solution with n zeros at $x = a$, a contradiction to the uniqueness of solutions for Equation (4.1).

As s increases, Theorem 4.9 has shown that new zeros of the quasi-derivatives enter (a,s) through the enpoint s. Since the zeros in (a,s) are subject to Rolle's theorem and the constraints of Lemmas 4.1 and 4.8, the exact order in which these zeros enter depends solely on the boundary conditions (4.16). Also, as s increases, the quasi-derivatives $L_r y(x,s)$, $r \in \{0, \ldots, n-1\} \setminus \{j_1, \ldots, j_{n-k-1}\}$, necessarily vanish in decreasing order at $x = s$. An example will help clarify exactly how these orders are determined by (4.15).

Suppose $n = 5$ and $p(x) > 0$, and consider the boundary conditions

$$L_i y(a) = 0, \qquad i = 0,1,$$

$$L_j y(s) = 0, \qquad j = 0,2.$$

By Rolle's theorem, $L_r y(x,s)$ vanishes in (a,s) for $r = 1,2,3$, regardless of how close s is to a. Assume s starts so near a that these are the only zeros in (a,s). As s increases, the first new zero at s must occur for $L_4 y(x,s)$, since any other location would force $L_4 y(x,s)$ to vanish in (a,s), implying that the zero has already entered there. Allowing s to increase still further so that $L_4 y(x,s)$ now vanishes in (a,s), the next zero at s must occur for $L_3 y(x,s)$ so as to predict the zero of $L_4 y(x,s)$ by Rolle's theorem. If this zero entered by way of any other quasi-derivative, it would force $L_3 y(x,s)$ to have an extra zero in (a,s), implying a zero had entered previously. The new zero of $L_3 y(x,s)$ at s raises the multiplicity of the zero of $L_2 y(x,s)$ at s. Therefore, in order to maintain $S(y(x,s),s-) = 3$, as s increases beyond this point both $L_2 y(x,s)$ and $L_3 y(x,s)$ gain a new simple zero in (a,s). That is, the multiple zero of $L_2 y(x,s)$ at s breaks up into two simple zeros, and the new simple zero of $L_3 y(x,s)$ is then forced by Rolle's theorem. Finally, $L_1 y(x,s)$ would vanish at s, causing new simple zeros to enter (a,s) for both $L_0 y(x,s)$ and $L_1 y(x,s)$.

Once a pattern such as this has been established, it cannot be varied but must instead be repeated as s increases (provided new zeros continue to appear), by the same argument with Rolle's theorem. Therefore, Theorem 4.9 has the following corollary:

COROLLARY 4.10: Let ℓ_t be the number of zeros predicted by Rolle's theorem for $L_t y(x,s)$ in (a,s) for (4.16) with s near a. If $0 \le t \le j_{n-k}$, then the quasi-derivative $L_t y(x,\theta_i(a))$ of the extremal solution $y(x,\theta_i(a))$ has exactly $i + \ell_t - 1$ simple zeros in $(a,\theta_i(a))$, $i = 1,2,\ldots$. If $t > j_{n-k}$, then $L_t y(x,\theta_i(a))$ has $i + \ell_t$ simple zeros in $(a,\theta_i(a))$.

Theorem 4.9 and Corollary 4.10 extend a result of Leighton and Nehari [76, Theorem 3.6] describing extremal solutions for the conjugate points of Equation (4.3) with $p(x) < 0$.

The boundary conditions (4.16) were obtained from (4.15) by deleting a condition at s, and the results which followed described the behavior of the solution $y(x,s)$ as s varied. Using precisely the same methods of proof, analogous theorems describe the behavior of solutions which satisfy boundary conditions obtained from (4.15) by deleting a condition at a, and then allowing s to vary.

Of particular interest in the proof of Theorem 4.14 is the situation where the sets $\{i_1, \ldots, i_k\}$ and $\{j_1, \ldots, j_{n-k}\}$ of (4.15) are disjoint. The following lemma is slightly more general than the version used by Elias [24], but has the advantage of permitting a proof of Theorem 4.17 which does not require any reference to the Trench factorization of the operator L_n in Equation (4.1).

LEMMA 4.11: Consider the boundary conditions

$$L_i y(a) = 0, \qquad i \in \{i_1, \ldots, i_k\}, \tag{4.34a}$$

$$L_j y(s) = 0, \qquad j \in \{0,1,\ldots,n-1\} \setminus \{i_1,\ldots,i_k\}. \tag{4.34b}$$

If there is a solution $y(x)$ of Equation (4.1) which satisfies

$$L_t y(x) \neq 0, \qquad t = 0,1,\ldots,n-1, \tag{4.35a}$$

$$\operatorname{sgn}[L_i y(x)] = \operatorname{sgn}[L_{i+1} y(x)], \qquad i \in \{i_1,\ldots,i_k\}, \tag{4.35b}$$

$$\operatorname{sgn}[L_j y(x)] = -\operatorname{sgn}[L_{j+1} y(x)], \quad j \in \{0,\ldots,n-1\} \setminus \{i_1,\ldots,i_k\}, \tag{4.35c}$$

on (a,b), then $\theta_1(a) \geq b$ for (4.34). If $i_k \neq n-1$ and if $\theta_1(a) \geq b$, then there is a solution $y(x)$ satisfying (4.35) on (a,b).

PROOF: In this setting, with the assumption that $i_k \neq n-1$, the boundary conditions (4.16) derived from (4.34) give $\ell_t = 0$ for $t = 0,\ldots,n-1$. If $\theta_1(a) \geq b$, then $y(x,b)$ satisfies (4.35) by Corollary 4.10 and Lemma 4.4. This proves the second part of the lemma.

To prove the first part, assume $\theta_1(a) < b$, and suppose $y_1(x)$ satisfies (4.35). By the continuity of θ_1, there is an $a' > a$ such that $\theta_1(a') < b$. Let $y_2(x)$ be the extremal solution for $\theta_1(a')$ on $[a',\theta_1(a')]$, and assume, without loss of generality, that y_1 and y_2 have the same sign in $(a',\theta_1(a'))$. Set $w_\lambda = y_1 - \lambda y_2$, and note that $L_t w_0(x) \neq 0$ on $[a',\theta_1(a')]$ by (4.35) for $t = 0,\ldots,n-1$. Define λ_0 to be the smallest positive value of λ such that $L_t w_\lambda$ vanishes in $[a',\theta_1(a')]$ for some t. Continuity with respect to λ implies no quasi-derivative of w_{λ_0} changes sign in $[a',\theta_1(a'))$. Consequently, w_{λ_0} must vanish at either a' or $\theta_1(a')$. Suppose $L_t w_{\lambda_0}(a') = 0$ for some t. In light of the boundary conditions on y_2, necessarily $t \in \{0,\ldots,n-1\} \setminus \{i_j,\ldots,i_k\}$. Since $\operatorname{sgn}[L_t w_{\lambda_0}] = \operatorname{sgn}[L_t y_1]$ by definition of λ_0, then $\operatorname{sgn}[L_t w_{\lambda_0}] = -\operatorname{sgn}[L_{t+1} w_{\lambda_0}]$ by (4.35), contradicting $L_t w_{\lambda_0}(a') = 0$. A similar argument at $\theta_1(a')$ shows the solutions y_1 and y_2 are incompatible. This completes the proof of Lemma 4.11.

The incompatibility of the conditions (4.34) and (4.35) is indeed the heart of the proof of Theorem 4.14. The basic idea of the proof is this: if the class S_k contains a nonoscillatory solution, then for some $a > 0$, the solution satisfies conditions of the form (4.35) on $[a,\infty)$. On the other hand, the existence of an oscillatory solution will lead to the existence of an extremal point $\theta_1(a')$ for some $a' > a$ and for conditions of the form (4.34). The difficulty lies in that the set of indices $\{i_1, \ldots, i_k\}$ in (4.34) and (4.35) as determined by these solutions might be different. This requires, then, showing that the existence of an extremal point for one set of conditions of the form (4.15) implies the existence of extremal points for every other set of conditions of the form (4.15). Theorems 4.12 and 4.13 accomplish this task, allowing Theorem 4.14 to be proved.

<u>THEOREM 4.12</u>: If all the extremal points $\theta_i(a)$, $i = 1,2, \ldots,$ exist for one system of boundary conditions of type (4.15), then all the extremal points exist for every system of boundary conditions of type (4.15).

<u>PROOF</u>: It is sufficient to show that all the extremal points $\theta_i(a)$, $i = 1,2, \ldots,$ of (4.15) exist if and only if the corresponding extremal points exist when a left or right boundary condition is increased by one. Let $\tilde{\theta}_i(a)$, $i = 1,2, \ldots,$ represent the extremal points for the boundary conditions

$$L_i y(a) = 0, \qquad i \in \{i_1, \ldots, i_k\}, \tag{4.36a}$$

$$L_j y(s) = 0, \qquad j \in \{j_1, \ldots, j_q + 1, \ldots, j_{n-k}\}, \tag{4.36b}$$

where $j_q + 1 \notin \{j_1, \ldots, j_q, \ldots, j_{n-k}\}$. The proof shows that the points

$\{\theta_i(a)\}$ and $\{\tilde{\theta}_i(a)\}$ must separate each other.

Let $y(x,s)$ be the solution satisfying the $n-1$ conditions

$$L_i y(a) = 0, \qquad i \in \{i_1, \ldots, i_k\}, \tag{4.37a}$$

$$L_j y(s) = 0, \qquad j \in \{j_1, \ldots, j_{n-k}\} \setminus \{j_q\}. \tag{4.37b}$$

For s sufficiently close to a, all the zeros of $y(x,s)$ and its quasi-derivatives on (a,s) are given by Rolle's theorem applied to (4.37). As the argument before Corollary 4.10 shows, when s increases the quasi-derivative $L_{j_q+1}y(x,s)$ gains a zero at $x=s$ before $L_{j_q}y(x,s)$. As s increases still more, $L_{j_q}y(x,s)$ must vanish at s before $L_{j_q+1}y(x,s)$ can gain its second zero. Likewise, allowing s to increase again, $L_{j_q+1}y(x,s)$ must achieve its second zero at $x=s$ before $L_{j_q}y(x,s)$ does. Applying this argument to the points $\theta_i(a)$ and $\tilde{\theta}_i(a)$ reveals $\tilde{\theta}_i(a) < \theta_i(a) < \tilde{\theta}_{i+1}(a) < \theta_{i+1}(a)$. Hence (4.15) has infinitely many extremal points if and only if (4.36) does.

A similar argument applies when a system of boundary conditions differs from (4.15) by one condition at $x=a$. In that case, $y(x,s)$ is defined by deleting a condition at a instead of s. Theorems such as those proved earlier would show that, as s increases, zeros could enter (a,s) only by way of the endpoint a, allowing the proof to be completed. Actually the earlier results make new results about the endpoint a easy to prove, since at $s=\theta_i(a)$, the solution $y(x,s)$ defined by deleting a condition at a must be the unique extremal solution. This completes the proof of Theorem 4.12.

The point $x_0 \geq 0$ at which a nonoscillatory solution y finally achieves its constant value of $S(y,x+)$ can be arbitrarily large in general. In order to arrive at a contradiction in the proof of Theorem 4.14, it is necessary to generate extremal solutions for $\theta_1(a)$ for arbitrarily large values of a. That is one function of the next theorem.

THEOREM 4.13: Let $y(x)$ be an oscillatory solution of Equation (4.1) which satisfies $S(y,x+) \equiv k$ for x sufficiently large. Then for every system of type (4.15), and for every $a \geq 0$, all the extremal points $\theta_i(a)$, $i = 1, 2, \ldots,$ exist.

PROOF: From Lemma 4.2 there is an $x_0 \geq 0$ such that $S(y,x+) = k$ for $x > x_0$. Choose the point $a \geq x_0$ and the integer $m > n^2$ arbitrarily large, and let t_1 and s_1 be chosen so that $y(x)$ has at least m simple zeros in $(t_1,s_1) \subset (a,\infty)$, with no quasi-derivatives vanishing at either t_1 or s_1.

Since $S(y,t_1+) = k$ and $S(y,s_1-) = n-k$, there are sets of indices $\{i_1, \ldots, i_k\}$ and $\{j_1, \ldots, j_{n-k}\}$ such that

$$c_i \equiv L_{i+1}y(t_1)/L_iy(t_1) > 0, \qquad i \in \{i_1, \ldots, i_k\},$$

$$-d_j \equiv L_{j+1}y(s_1)/L_jy(s_1) < 0, \qquad j \in \{j_1, \ldots, j_{n-k}\}.$$

For $x_0 < t_1 < s$, let $u(x,s)$ be the solution of Equation (4.1) satisfying the boundary conditions

$$L_{i+1}u(t_1) - c_iL_iu(t_1) = 0, \qquad i \in \{i_1, \ldots, i_k\}, \tag{4.38a}$$

$$L_{j+1}u(s) + d_jL_ju(s) = 0, \qquad j \in \{j_1, \ldots, j_{n-k-1}\}. \tag{4.38a}$$

Every solution u of (4.38) satisfies

$$N(u) \geq S(u,t_1+) + S(u,s-) \geq k + (n-k-1) = n-1,$$

and so by Lemma 4.5 the solution $u(x,s)$ has the properties given by Lemma 4.4. In particular, $u(x,s)$ is essentially unique; since $y(x)$ also meets (4.38) for $s = s_1$, then $u(x,s_1) \equiv y(x)$. Thus $u(x,s_1)$ has at least m simple zeros in (t_1,s_1).

By Lemma 4.4, simple zeros of $u(x,s)$ cannot coincide or meet t_1 as s decreases. In addition, when s is sufficiently close to t_1, at least one quasi-derivative must be nonzero, so that as s decreases toward t_1, zeros of the quasi-derivatives must leave (t_1,s) through s.

Suppose that, as s decreases from s_1 toward t_1, the first zero of $L_{j_{n-k}}u(x,s)$ leaves (t_1,s) for $s = s_2$, $t_1 < s_2 < s_1$, so that

$$L_{j_{n-k}}u(x,s_2)\Big|_{x=s_2} = 0.$$

As the number of zeros of $L_{j_{n-k}}u(x,s)$ in (t_1,s) decreases by one, Rolle's theorem implies that $u(x,s)$ loses at most $j_{n-k} < n$ zeros from (t_1,s). Thus $u(x,s_2)$ still has at least $m - n > n^2 - n$ zeros in (t_1,s_2).

Let $v(x,s)$ be the solution which satisfies

$$L_{i+1}v(t_1) - c_iL_iv(t_1) = 0, \qquad i \in \{i_1, \ldots, i_k\},$$

$$L_{j+1}v(s) + d_jL_jv(s) = 0, \qquad j \in \{j_1, \ldots, j_{n-k-2}\},$$

$$L_{j_{n-k}}v(s) = 0.$$

By Lemma 4.5, $v(x,s)$ also has the properties specified in Lemma 4.4, and hence it follows that $v(x,s_2) \equiv u(x,s_2)$ and has at least $m - n$ zeros in (t_1,s_2). Now let s decrease until $L_{j_{n-k-1}}v(x,s)$ has a zero at $s = s_3$, $t_1 < s_3 < s_2$. Repeating in this fashion merges the zeros of $y(x)$ at the endpoint $\bar{s} = s_{n-k}$. In a similar manner, by deleting conditions one by one at t from

$$L_{i+1}w(t) - c_iL_iw(t) = 0, \qquad i \in \{i_1, \ldots, i_k\},$$

$$L_jw(\bar{s}) = 0, \qquad j \in \{j_1, \ldots, j_{n-k}\},$$

and allowing t to increase toward $\overline{s}$ merges the zeros at the endpoint $\overline{t} = t_k$. Thus, there is a solution $\overline{y}(x)$ which satisfies

$$L_i\overline{y}(\overline{t}) = 0, \qquad i \in \{i_1, \ldots, i_k\}, \tag{4.39a}$$

$$L_j\overline{y}(\overline{s}) = 0, \qquad j \in \{j_1, \ldots, j_{n-k}\}, \tag{4.39b}$$

Furthermore, $\overline{y}$ has at least $m - n^2$ simple zeros in $(\overline{t},\overline{s}) \subset (t_1,s_1)$.

The point $\overline{s}$ is an extremal point for $\overline{t}$; that is, $\overline{s} = \theta_i(\overline{t})$ for some integer $i \geq 1$. Theorem 4.8 implies $\theta_i(a) \leq \overline{s}$, and so is defined. Since $\overline{y}$ has at least $m - n^2$ simple zeros in $(\overline{t},\overline{s})$, then Corollary 4.10 implies $i \geq m - n^2 - \ell_0$, where the constant ℓ_0 is determined by the boundary conditions (4.39). Because m may be chosen arbitrarily large, it follows that $\theta_i(a)$ exists for $i = 1, 2, \ldots$. Even though the conditions (4.39) may change with m, some set of conditions must be repeated infinitely often, and so $\theta_i(a)$ exists for every set of conditions of the form (4.15) by Theorem 4.12. Finally, since a itself was chosen arbitrarily large, $\theta_i(a)$ is defined for all $a \geq 0$. This completes the proof of Theorem 4.13.

Theorem 4.14 may now be proved easily. Theorems 4.3, 4.13, and 4.14 demonstrate the utility of the classes S_k, $0 \leq k \leq n$, $(-1)^{n-k}p(x) < 0$.

<u>THEOREM 4.14</u>: For fixed k, the solutions in S_k are either all oscillatory or all nonoscillatory.

<u>PROOF</u>: Recall that, for $0 \leq k \leq n$, $(-1)^{n-k}p(x) \leq 0$, the class S_k is the set of solutions $\{y \mid \lim_{x\to\infty} S(y,x+) = k\}$. Suppose S_k contains both a nonoscillatory solution u(x) and an oscillatory solution v(x). Let $a > 0$ be chosen so large that no quasi-derivative of u(x) vanishes on $[a,\infty)$. Since $u \in S_k$, then $S(u,x+) \equiv k$ for $x \geq a$. Consequently, there is a set of

indices $\{i_1, \ldots, i_k\}$ such that

$$\operatorname{sgn}[L_i u(x)] = \operatorname{sgn}[L_{i+1} u(x)], \qquad i \in \{i_1, \ldots, i_k\},$$

$$\operatorname{sgn}[L_j u(x)] = \operatorname{sgn}[L_{j+1} u(x)], \qquad j \in \{0, \ldots, n-1\} \setminus \{i_1, \ldots, i_k\},$$

on $[a,\infty)$. By Lemma 4.11, $\theta_1(a) = \infty$ for the boundary conditions of the form (4.34) given by the indices $\{i_1, \ldots, i_k\}$. But since $v(x)$ is an oscillatory solution in class S_k, $\theta_1(a)$ must exist for every set of boundary conditions and for all $a > 0$ by Theorem 4.13, a contradiction. This completes the proof of Theorem 4.14.

At this point, the classes S_k, $0 \leq k \leq n$ satisfying (4.8), have been shown to be nonempty disjoint classes whose union is the entire solution space of Equation (4.1) (minus the trivial solution) with the additional property that each class consists solely of either oscillatory or nonoscillatory solutions. It is reasonable to suspect that the solution space of Equation (4.1) has a basis associated with the classes, and this in fact is the case. For convenience of notation in the next result, add the trivial solution to each class. This result, due to Elias [25], will play a fundamental role in Chapter V.

THEOREM 4.15: Let $a \geq 0$ be a fixed point. There exists a basis $\{u_0, \ldots, u_{n-1}\}$ of the solution space of Equation (4.1) such that

(1) u_i, $i = 0, \ldots, n-1$, has a zero of multiplicity i at a.

(2) For $1 \leq k \leq n-1$, $(-1)^{n-k} p < 0$, S_k contains span $\{u_{k-1}, u_k\}$, the two-dimensional subspace spanned by u_{k-1} and u_k. If $(-1)^n p < 0$ then $u_0 \in S_0$, and if $p < 0$, then $u_{n-1} \in S_n$.

(3) If the set S_k, $1 \leq k \leq n-1$, consists of nonoscillatory solutions, then span $\{u_{k-1}, u_k\}$ contains two solutions v,w such that $L_t v / L_t w \to \infty$ monotonically as $x \to \infty$, $t = 0, \ldots, n-1$. If S_k consists of oscillatory solutions, then the zeros of every two linearly independent solutions in span $\{u_{k-1}, u_k\}$ separate each other in (a, ∞).

PROOF: For $1 \leq k \leq n-1$, $(-1)^{n-k} p < 0$, let $y_1(x,s)$ be the solution of Equation (4.1) satisfying the $n-1$ conditions

$$L_i y(a) = 0, \qquad i = 0, \ldots, k-1, \tag{4.40a}$$

$$L_j y(s) = 0, \qquad j = 0, \ldots, n-k-2, \tag{4.40b}$$

and let $y_2(x,s)$ be the solution satisfying the $n-1$ conditions

$$L_i y(a) = 0, \qquad i = 0, \ldots, k-2, k, \tag{4.41a}$$

$$L_j y(s) = 0, \qquad j = 0, \ldots, n-k-2. \tag{4.41b}$$

Normalize these solutions so that $\sum_{t=0}^{n-1} (L_t y(a,s))^2 = 1$, and define $u_k(x) = \lim y_1(x,s)$, $u_{k-1}(x) = \lim y_2(x,s)$ as a tends to infinity through an appropriately chosen sequence. By using properly chosen subsequences, it may be assumed that both y_1 and y_2 converge along the same sequence of s values. As in Theorem 4.3, both u_k and u_{k-1} must belong in S_k. If $k = 0$ $(k = n)$ is allowable by the parity condition, then take u_0 (u_n) to be the solution found by Theorem 4.3 to be in class S_0 (S_n). From these definitions, part (1) is immediate.

Since y_1 and y_2 converge along the same sequence, then clearly $c_1 u_k(x) + c_2 u_{k-1}(x) = \lim_{s\to\infty} (c_1 y_1(x,s) + c_2 y_2(x,s))$. The conditions (4.40) and (4.41) imply that the combination $y(x,s) \equiv c_1 y_1(x,s) + c_2 y_2(x,s)$ must satisfy the $n-2$ boundary conditions

$$L_i y(a) = 0, \qquad i = 0, \ldots, k-2,$$

$$L_j y(s) = 0, \qquad j = 0, \ldots, n-k-2.$$

Therefore, $S(y(x,s),a+) \geq k-1$, $S(y(x,s),s-) \geq n-k-1$. But the parity condition (4.8) then forces $S(y(x,s),a+) = k$, $S(y(x,s),s-) = n-k$. Now proceeding exactly as in Theorem 4.3, the limit function $\lim_{s\to\infty} y(x,s) = c_1u_1(x) + c_2u_2(x)$ must belong in S_k. Thus S_k contains span $\{u_{k-1},u_k\}$, and part (2) is shown.

Suppose that the class S_k, $1 \leq k \leq n-1$, consists of nonoscillatory solutions. Then there exists $c \geq 0$ sufficiently large so that $L_t u_k \neq 0$ on $[c,\infty)$, $t = 0, \ldots, n-1$. If, for some point $x_0 > c$, it happened that $(L_r u_{k-1}/L_r u_k)'(x_0) = 0$, then $(L_{r+1}u_{k-1}L_r u_k - L_r u_{k-1}L_{r+1}u_k)(x_0) = 0$. Consequently, there would be constants c_1 and c_2 so that the solution $y(x) = c_1u_k(x) + c_2u_{k-1}(x)$ satisfied $L_r y(x_0) = L_{r+1}y(x_0) = 0$. Now, the combination of u_k and u_{k-1} must have at least $k-1$ initial zeros, so that $S(y,a+) = k$ by the parity condition. But from Lemma 4.1, for $b > x_0$, $S(y,a+) + \langle n(x_0)\rangle + S(y,b-) \leq n$. Hence $S(y,b-) \leq n-k-2$, or $S(y,b+) \geq k+2$, contradicting the fact that $y \in S_k$. Therefore $(L_r u_{k-1}/L_r u_k)' \neq 0$, and the quotient is monotone and hence convergent in $R \cup \{\infty\}$. If the limit is ∞, then clearly the limit of $L_{r-1}u_{k-1}/L_{r-1}u_k$ cannot be finite, and so $L_t u_{k-1}/L_t u_k \to \infty$ for all t, $t = 0,1, \ldots, n-1$, by the cyclic nature of the quasi-derivatives. On the other hand, if the limit is λ, $|\lambda| < \infty$, then setting $v = u_k$ and $w = u_{k-1} - \lambda u_k$ gives $\lim_{x\to\infty} (L_r v/L_r w) = \infty$, and the result follows as above.

Suppose finally that S_i consists of oscillatory solutions. If there are two linearly independent solutions, $v,w \in \operatorname{span}\{u_{k-1},u_k\}$ whose zeros do not separate each other on (a,∞), then there is a linear combination of the form $y = c_1 v + c_2 w$ such that, for some index r and for some point $x_0 > a$, $L_r y(x_0) = L_{r+1} y(x_0) = 0$. As before, this implies $S(y, x_0+) \geq k+2$, a contradiction since $y \in \text{span}\{u_{k-1}, u_k\}$. This completes the proof of Theorem 4.15.

The study of oscillatory solutions and extremal points, used here to produce a proof of Theorem 4.14, is an area worthy of greater consideration than can be afforded in this study. There remains, however, one additional result from this study which cannot justly be omitted. As discussed in Chapter I, nonoscillation is not equivalent to eventual disconjugacy for general n-th order differential equations. That these properties are equivalent for Equation (4.1) has been shown by Elias [20, 24]. The original proof [20] involved many of the same manipulations with boundary conditions required for the broader paper [24], and undoubtedly inspired much of the later work.

In order to produce a proof which does not depend upon the Trench factorization, a slightly more general lemma is needed here than used by Elias [24, Lemma 4]. Otherwise, the proof which follows for Theorem 4.17 is based on that in Reference [24].

<u>LEMMA 4.16</u>: Let $\tilde{\theta}_i(a)$ represent the extremal points for the boundary conditions

$$L_i y(a) = 0, \qquad i = 0,1, \ldots, k-1, \tag{4.42a}$$

$$L_j y(s) = 0, \qquad j = 0,1, \ldots, n-k-1, \tag{4.42b}$$

and let $\theta_i(a)$ represent the extremal points for (4.15). If $\tilde{\theta}_1(a)$ exists, then $\theta_1(a) \leq \tilde{\theta}_1(a)$, with equality only when (4.15) is equivalent to (4.42).

PROOF: Exactly as in the proof of Theorem 4.12, it can be shown that the extremal point decreases when either a left or right boundary condition is increased by one. That is, the first extremal point for (4.15) must occur before the first extremal point for the conditions

$$L_i y(a) = 0, \qquad i \in \{i_1, \ldots, i_k\},$$

$$L_j y(s) = 0, \qquad j \in \{j_1, \ldots, j_q - 1, \ldots, j_{n-k}\}.$$

This suffices to prove the lemma.

THEOREM 4.17: Equation (4.1) is eventually disconjugate if and only if there does not exist an oscillatory solution.

PROOF: One direction is trivial, since the existence of an oscillatory solution implies the existence of $\eta_1(a)$ for arbitrarily large values of a.

For the other direction, note from Corollary 2.4 that the conjugate point $\eta_1(a)$ coincides with the extremal point $\tilde{\theta}_1(a)$ for the boundary conditions (4.42) for some k, $1 \leq k \leq n-1$. If Equation (4.1) is not eventually disconjugate, then $\eta_1(a)$ exists for arbitrarily large values of a, and hence for some fixed k, the extremal point $\tilde{\theta}_1(a)$ exists for (4.42) for arbitrarily large values of a. The theorem will follow if it can be shown that this implies the existence of an oscillatory solution for Equation (4.1).

Consider the solution $u_k \in S_k$ of Equation (4.1) as given by Theorem 4.15 which has a zero of multiplicity k at $x = 0$. If u_k is oscillatory, then the theorem is proved. If u_k is nonoscillatory, then there is a point $a > 0$ such that no quasi-derivative of u_k vanishes on $[a,\infty)$. Thus, for some set of indices $\{i_1, \ldots, i_k\}$, u_k satisfies the conditions (4.35) of Lemma 4.11 on the interval $[a,\infty)$. Consequently, $\theta_1(a) = \infty$ for the boundary conditions (4.34). But since $\tilde{\theta}_1(a)$ exists, then Lemma 4.16 implies $\theta_1(a) \leq \tilde{\theta}_1(a) < \infty$, a contradiction. This contradiction shows that u_k cannot be oscillatory. This completes the proof of Theorem 4.17.

Using the basis $\{u_0, \ldots, u_{n-1}\}$ and the classes S_k, the dominance of the solutions of Equation (4.1) may now be considered in Chapter V.

CHAPTER V

DOMINANCE FOR $L_n y + py = 0$

In the study of a second order differential equation, the classic work of Sturm [110] showed that either all solutions oscillate or none do. Because of this simplicity in the structure of the solution space, dominance results such as those described in Chapter III would be of little value. As the order of the equation increases, however, the solution space becomes more complex and varied, and there is a greater need to discriminate between the behaviors of the various solutions.

For the fourth order problem (4.3) studied by Leighton and Nehari [76], the work of Keener [53, Theorems 3.4 and 4.7] effectively characterizes the dominance relationships among oscillatory and nonoscillatory solutions. Lazer [72] has also characterized dominance relationships for the solutions of the third order equation $y''' + ry' + py = 0$. On the other hand, as discussed in Chapter III, there are third order equations for which such results do not exist, and so any reasonable analysis of dominance must consider an equation less general than Equation (3.1).

As in Chapter IV, consider the equation

$$L_n y + p(x)y = 0, \qquad (5.1)$$

where $p(x)$ is continuous and nonzero on $I = [0,\infty)$, and where L_n is the factored disconjugate operator defined by

$$L_n y = \rho_n(s)\frac{d}{dx}\rho_{n-1}(x)\frac{d}{dx}\cdots\rho_1(s)\frac{d}{dx}\rho_0(x)y, \tag{5.2}$$

with $\rho_i(x)$ continuous and positive on I for $i = 0,1,\ldots,n$. For each solution y of Equation (5.1) set $S(y) = \lim_{x\to\infty} S(y,x+)$, and define $S_k \equiv \{y \mid S(y) = k\}$. Whenever mention is made of a class S_k of solutions in this chapter, it will be assumed that $0 \leq k \leq n$ and $(-1)^{n-k}p(x) < 0$. For emphasis, these values of k will be called <u>admissible</u> values. It will be convenient at times to refer to the classes S_k as the <u>Elias classes</u> for Equation (5.1).

The main question to be discussed here is the following: Under what conditions does the class S_k dominate the class S_j for every $j < k$? At this point, the question is not even well defined. If two distinct classes S_k and S_j both consist of oscillatory solutions, then the definitions of dominance given in Chapter III do not apply. Therefore, in addition to the definitions found in that chapter, several new definitions are needed. If y and z are both solutions of Equation (5.1), then y <u>dominates</u> z <u>by class at zero</u> provided there is an $\varepsilon > 0$ such that $S(y + \lambda z) = S(y)$ whenever $|\lambda| < \varepsilon$. The solution y <u>dominates</u> z <u>by class at infinity</u> (or simply, <u>by class</u>) provided $S(y + \lambda z) = S(y)$ for all real values of λ. If P and Q are subsets of the solution space of Equation (5.1), then P dominates Q by class at zero provided y dominates z by class at zero for every choice of $y \in P$, $z \in Q$. The set P dominates Q by class at infinity provided y dominates z by class at infinity for every $y \in P$, $z \in Q$.

Let $\mathcal{O}$ be the set of oscillatory solutions of Equation (5.1), and let $\mathcal{N}$ be the set of nonoscillatory solutions. If $y \in \mathcal{O}$ and $z \in \mathcal{N}$, then domination by class implies domination in the sense of Chapter III, since

$k = S(y)$ and $j = S(z)$ imply $S_k \subseteq \mathcal{O}$ and $S_j \subseteq \mathcal{N}$. Whether the converse is true remains an open question at this time. As part (3) of Theorem 4.15 shows, it is possible to find two solutions v and w in a nonoscillatory class such that w dominates v by class at infinity and yet w does not dominate v even at zero in the sense of Chapter III.

While its proof is almost trivial, the following lemma of Elias [25] provides a key starting point for discussing dominance.

<u>LEMMA 5.1</u>: If $y_i \to y$ as $i \to \infty$ and if $S(y_i) = k$, $i = 1,2, \ldots$, then $S(y) \leq k$.

<u>PROOF</u>: Suppose $S(y) = j > k$. Then there is a point $x_0 \geq 0$ such that $S(y,x+) = j$ for all $x > x_0$. Choose $x_1 > x_0$ so that no quasi-derivative of y vanishes at x_1. Since $y_i \to y$, then there is an integer i_0 such that $\operatorname{sgn}(L_t y(x_1)) = \operatorname{sgn}(L_t y_i(x_1))$ for $0 < t < n$ whenever $i < i_0$. Consequently, $S(y_i, x_1+) = j$, and since $S(y_i, x+)$ is increasing in x, this contradicts $S(y_i) = k$. This contradiction completes the proof of Lemma 5.1.

As an example, consider the solutions $y_1 = \sin x$ and $y_2 = e^{-x}$ of the equation $y^{(4)} - y = 0$, and note that $S(y_1) = 2$ while $S(y_2) = 0$. In addition, $S(ay_1 + by_2) = 2$ for every linear combination with $a \neq 0$, since eventually such a combination must oscillate, and S_2 is the only oscillatory class for this equation. Let the sequence of functions $\{z_i\}$ be defined by $z_i = y_1 + y_2/i$, $i = 1,2, \ldots$. Then $z_i \to y_1$ as $i \to \infty$, and $S(z_i) = S(y_1)$ for each i. On the other hand, if the sequence $\{w_i\}$ is defined by $w_i = y_1/i + y_2$, $i = 1,2, \ldots$, then $w_i \to y_2$ as $i \to \infty$ while $S(w_i) > S(y_2)$. Thus both possibilities indicated by Lemma 5.1 may actually occur.

Suppose y dominates z by class at infinity, and set $z_i = y/i + z =$

$(y + iz)/i$. By the definition of dominance by class at infinity, it follows that $S(z_i) = S(y)$ for every i, while $z_i \to z$ as $i \to \infty$. Thus Lemma 5.1 has the following immediate consequence:

COROLLARY 5.2: If y dominates z by class at infinity, then $S(y) \geq S(z)$.

The major concern of what follows is to determine to what extent the converse of Corollary 5.2 is true. Lemma 5.1 comes tantalizingly close to proving that if $S(y) \geq S(z)$, then y must dominate z by class at zero. Instead, the strongest result available at this point from Lemma 5.1 is more a description of dominance in the sense of Chapter III:

THEOREM 5.3 (Elias [25]): For each q, $\bigcup_{k \geq q} S_k$ dominates the entire solution space of Equation (5.1) at zero.

PROOF: The proof is virtually immediate from Lemma 5.1. If $y_1 \in \bigcup_{k \geq q} S_k$ and if y_2 is any solution of (5.1), then $y_1 + \lambda y_2 \to y_1$ as $\lambda \to 0$. If there existed a sequence $\lambda_i \to 0$ as $i \to \infty$ such that $S(y_1 + \lambda_i y_2) < q$ for all i, then the lemma would imply $S(y_1) < q$, a contradiction. Consequently, there is an $\varepsilon > 0$ such that $S(y_1 + \lambda y_2) \geq q$ whenever $|\lambda| < \varepsilon$. This completes the proof.

This theorem does not provide much information about comparisons between individual pairs of solutions or pairs of classes. Such information is more difficult to extract from Lemma 5.1, as the next theorem, again due to Elias [25], demonstrates.

THEOREM 5.4: For every pair of solutions y_1, y_2 such that $S(y_1) > S(y_2)$, there exists a positive constant c so that (in the extended real numbers)

$$\max_{0 \leq t \leq n-1} |L_t y_1(x)/L_t y_2(x)| \geq c > 0, \qquad 0 \leq x < \infty. \tag{5.3}$$

If in addition $y_1 \in \mathcal{O}$ and $y_2 \in \mathcal{N}$, then

$$\limsup_{x\to\infty} |L_r y_1(x)/L_r y_2(x)| > 0, \qquad r = 0, \ldots, n-1. \tag{5.4}$$

PROOF: Assume that (5.3) fails. Then there is an infinite sequence $\{x_i\}$ such that for every $\varepsilon > 0$ there is an integer i_ε with the property that

$$\max_{0 \leq t \leq n-1} |L_t y_1(x_i)/L_t y_2(x_i)| < \varepsilon, \qquad i > i_\varepsilon.$$

Thus

$$\varepsilon|L_t y_2(x_i)| > |L_t y_1(x_i)|, \qquad t = 0, \ldots, n-1, \quad i > i_\varepsilon,$$

and therefore,

$$\operatorname{sgn}[\varepsilon L_t y_2(x_i)] = \operatorname{sgn}[L_t(\varepsilon y_2 + y_1)(x_i)], \qquad t = 0, \ldots, n-1, \quad i > i_\varepsilon.$$

In other words,

$$S(\varepsilon y_2, x_i+) = S(\varepsilon y_2 + y_1, x_i+), \qquad i > i_\varepsilon.$$

If the sequence $\{x_i\}$ were unbounded, this would imply

$$S(y_2) = S(\varepsilon y_2 + y_1).$$

As $\varepsilon \to 0$, $\varepsilon y_2 + y_1 \to y_1$, so that by Lemma 5.1 this last equation would imply

$$S(y_1) \leq S(\varepsilon y_2 + y_1) = S(y_2).$$

Since $S(y_2) < S(y_1)$ by assumption, this is impossible, so that the sequence $\{x_i\}$ must be bounded, and must have a finite accumulation point x_0.

No nontrivial solution of Equation (5.1) has a zero of multiplicity n at x_0, so that for some t_0, $0 \leq t_0 \leq n-1$, the quasi-derivative $L_{t_0}y_1$ does not vanish at x_0. Since $L_{t_0}y_1$ is continuous, there exists $\delta > 0$ such that $L_{t_0}y_1(x) \neq 0$, $x \in [x_0 - \delta, x_0 + \delta]$. Thus $L_{t_0}y_1$ is bounded away from zero on this compact interval. Since $|L_{t_0}y_2|$ must also be bounded above here, the quotient

$$|L_{t_0}y_1(x)/L_{t_0}y_2(x)|$$

is bounded away from zero on $[x_0 - \delta, x_0 + \delta]$. This contradicts the choice of x_0 as an accumulation point of the sequence $\{x_i\}$, so that Equation (5.3) must hold

Suppose now y_2 is nonoscillatory, and that for some r, $0 \leq r \leq n-1$, (5.4) fails. Then $\lim_{x\to\infty}|L_r y_1(x)/L_r y_2(x)| = 0$, and for each $\varepsilon > 0$ there is a point $x_\varepsilon \geq 0$ such that

$$|L_r y_1(x)/L_r y_2(x)| < \varepsilon, \qquad x > x_\varepsilon.$$

If $|\lambda| \geq \varepsilon$, then $L_r y_1(x)/L_r y_2(x) - \lambda \neq 0$ on (x_ε, ∞), and since $L_r y_2(x) \neq 0$ for x sufficiently large, then $L_r(y_1 - \lambda y_2) \neq 0$ on (x_ε, ∞). Consequently, $y_1 - \lambda y_2$ is a nonoscillatory solution, so for some $\tilde{x}_\lambda \geq x_\varepsilon$, $L_t(y_1 - \lambda y_2) \neq 0$, $t = 0, \ldots, n-1$, on $(\tilde{x}_\lambda, \infty)$. The solution $y_1 - \varepsilon y_2$ is itself nonoscillatory, and since $y_1 \in \mathcal{O}$, $y_2 \in \mathcal{N}$, then $\tilde{x}_\lambda$ may be chosen so that $\tilde{x}_\lambda \leq \tilde{x}_\varepsilon$. Thus for every λ satisfying $|\lambda| \geq \varepsilon$, $L_t(y_1 - \lambda y_2) \neq 0$, $t = 0, \ldots, n-1$, on $(\tilde{x}_\varepsilon, \infty)$, and so $|L_t y_1/L_t y_2| < \varepsilon$ on $(\tilde{x}_\varepsilon, \infty)$. Therefore, $L_t y_1/L_t y_2 \to 0$ as $x \to \infty$ for every t, $t = 0, \ldots, n-1$, contradicting (5.3). This completes the proof of Theorem 5.4.

REMARK: In the statement of this theorem as it appears in Reference [25], Elias omits the condition $y_1 \in \mathcal{O}$ from the hypotheses for (5.4). Without this additional assumption, the conclusion is false. The proof fails because it is no longer possible to choose $\tilde{x}_\lambda \leq \tilde{x}_\varepsilon$, and so the inequality $L_t(y_1 - \lambda y_2) \neq 0$ could not be forced to hold over a fixed interval for all values of λ. More significantly, an easy counterexample exists. Consider, for the interval $[1,\infty)$, the equation

$$x^2(x^2(x^2(x^2y')')')' - y = 0. \tag{5.5}$$

This equation has solutions $y_1 = e^{1/x}$, $y_2 = e^{-1/x}$, $y_3 = \cos \frac{1}{x}$, and $y_4 = \sin \frac{1}{x}$, all of which are nonoscillatory. Direct computation reveals that $y_1 \in S_0$, $y_2 \in S_4$, and $y_3, y_4 \in S_2$. In particular, note that $\lim_{x\to\infty} y_1 = 1$ while $\lim_{x\to\infty} y_4 = 0$, which contradicts (5.4) for $r = 0$.

The problem with Theorem 5.4 hints at a much more significant difficulty in the task of analyzing dominance for Equation (5.1). The solutions y_1, y_2, y_3 of Equation (5.5) all have limit 1 as $x \to \infty$, so that no one solution of the three dominates another at infinity. A closer examination of the quasi-derivatives for these solutions shows that $y_3 \in S_2$ is dominated by class at zero by both $y_1 \in S_0$ and $y_2 \in S_4$, and yet y_3 is not dominated by either of the others by class at infinity. For some reason, the sign structure of the quasi-derivatives as identified by the classes S_0, S_2, and S_4 does not reflect the actual growth of the solutions as $x \to \infty$.

Recall that the operator L_n as defined in Equation (5.2) is Trench factored provided

$$\int_0^\infty \frac{1}{\rho_i(x)}\,dx = \infty, \qquad i = 1, \ldots, n-1. \tag{5.6}$$

Taking a cue from the principal systems discussed in Chapter II, when the operator in Equation (5.5) is Trench factored the equation becomes

$$x^5 (x^3 y)^{(4)} - y = 0. \tag{5.7}$$

In this new factorization all three of the solutions y_1, y_2, and y_3 are in the class S_4, which explains the similarity in the limiting behavior. The solutions $y_4 = \sin \frac{1}{x}$ and $y_5 = \sinh \frac{1}{x}$ are in S_2, while $y_6 = \sin \frac{1}{x} - \sinh \frac{1}{x}$ is in class S_0. The class S_4 dominates S_2 by class at infinity, and S_2 in turn dominates S_0 by class at infinity.

Since Equations (5.5) and (5.7) have only nonoscillatory solutions, it is tempting to think that the preceding work is a direct result of the existence of principal systems for Trench factored disconjugate equations of the form (2.1). It is important, therefore, to recognize that in Equation (5.7) only the operator L_n has been refactored into Trench form. Theorem 4.17 in the previous chapter revealed that nonoscillation is equivalent to eventual disconjugacy for Equation (5.1), and so Equation (5.7) does have a Trench factorization for its left-hand side. But in general such a factorization would have little value, and would be difficult to find. Instead, the operator L_n of Equation (5.1) might be Trench factored in an attempt to redistribute the weight of its coefficients and obtain more growth information from the quasi-derivatives.

The next two results lend further support to the idea that the Trench factorization of L_n is the natural choice for discussing questions of dominance [25, 64].

LEMMA 5.5: Let y be a nonoscillatory solution of Equation (5.1) such that $y(x) > 0$ on some interval $[a,\infty)$. If (5.6) holds, then there is an integer k, $0 \leq k \leq n$, such that

$$L_i y(x) > 0, \qquad i = 0, \ldots, k, \tag{5.8a}$$

$$(-1)^{j-k} L_j y(x) > 0, \qquad j = k, \ldots, n, \tag{5.8b}$$

on some interval $[b,\infty)$, $b \geq a$. Furthermore,

$$\lim_{x\to\infty} L_i y(x) = \infty, \qquad i = 0, \ldots, k-2, \tag{5.9a}$$

$$\lim_{x\to\infty} L_j y(x) = 0, \qquad j = k+1, \ldots, n-1. \tag{5.9b}$$

PROOF: As in the previous chapter, since y is nonoscillatory there is a point $b \geq a$ such that all the quasi-derivatives of y are nonzero on $[b,\infty)$. If y does not satisfy (5.8) on this interval for $k = 0$, then there is an integer t, $0 \leq t \leq n-1$, such that

$$\operatorname{sgn}[L_t y(x)] = \operatorname{sgn}[L_{t+1} y(x)], \qquad x \geq b. \tag{5.10}$$

From the definition of the quasi-derivatives $L_t y$ and $L_{t-1} y$ as given implicitly by Equation (5.2), it follows that

$$L_{t-1} y(x) = L_{t-1} y(b) + \int_b^x \frac{1}{\rho_t(s)} L_t y(s)\, ds. \tag{5.11}$$

If $L_t y(x)$ and $L_{t+1} y(x)$ are negative on $[b,\infty)$, then $L_t y(x)$ is a negative decreasing function bounded above by $L_t(b) < 0$. Hence Equation (5.11) leads to the inequality

$$L_{t-1} y(x) \leq L_{t-1} y(b) + L_t y(b) \int_b^x \frac{1}{\rho_t(s)}\, ds\,. \tag{5.12}$$

Since $L_t y(b) < 0$, then (5.6) implies $L_{t-1} y(x)$ must eventually be negative, and thus $L_{t-1}y(x) < 0$ on $[b,\infty)$ because it is nonzero there. Continuing in this fashion for $L_{t-2}y(x), \ldots, L_0y(x)$ eventually leads to the contradiction $L_0y(x) < 0$ on $[b,\infty)$. Consequently, $L_ty(x)$ and $L_{t+1}y(x)$ must both be positive on $[b,\infty)$, so that $L_ty(x)$ is positive and increasing on $[b,\infty)$, bounded below by $L_ty(b) > 0$.

Equation (5.11) now leads to the inequality

$$L_{t-1}y(x) \geq L_{t-1}y(b) + L_ty(b)\int_b^x \frac{1}{\rho_t(s)}\,ds\,. \tag{5.13}$$

By virtue of (5.6) again, together with $L_ty(b) > 0$, (5.13) implies $L_{t-1}y(x)$ is eventually positive, and so $L_{t-1}y(x) > 0$ on $[b,\infty)$. Continuing as before, it follows that

$$L_iy(x) > 0, \qquad i = 0,1, \ldots, t+1, \quad x \geq b. \tag{5.14}$$

Let k be the largest integer for which

$$L_iy(x) > 0, \qquad i = 0,1, \ldots, k, \quad x \geq b. \tag{5.15}$$

Equation (5.10) led to (5.14), so that the maximality of k implies $k \geq t+1$ for every t satisfying Equation (5.10). Therefore, the quasi-derivatives must alternate in sign beyond L_ky, so that

$$(-1)^{j-k}L_jy(x) > 0, \qquad j = k, \ldots, n, \quad x \geq b. \tag{5.16}$$

This completes the verification of Equation (5.8).

For the second part of the lemma, observe that $L_ty(x)$ is positive and increasing on $[b,\infty)$, bounded below by $L_ty(b) > 0$, for every integer t, $0 \leq t \leq k-1$. Since the integral on the right-hand side of (5.13) is positive and unbounded as $x \to \infty$ for $1 \leq t \leq n-1$, taking $i = t-1$ it follows that

$$\lim_{x\to\infty} L_i y(x) = \infty, \qquad i = 1, \ldots, k-2,$$

which is the first portion of (5.9).

Let the ingeger j now be chosen from $k+1, \ldots, n-1$. Without loss of generality it may be assumed that $L_j y(x) > 0$ for $x > b$; otherwise y may be replaced by -y without affecting this portion of the proof. From (5.8), $L_{j+1}y(x) < 0$ on $[b,\infty)$, so that $L_j y(x)$ is a positive decreasing function on that interval. Consequently, $\lim_{x\to\infty} L_j y(x) = c \geq 0$ must exist. Suppose $c \neq 0$. Then Equation (5.11) becomes

$$L_{j-1}y(x) = L_{j-1}y(b) + \int_b^x \frac{1}{\rho_j(s)} L_j y(s)\, ds$$

$$\geq L_{j-1}y(b) + c\int_b^x \frac{1}{\rho_j(x)} ds ,$$

from which (5.6) implies $L_{j-1}y(x)$ is eventually positive. Thus

$$\operatorname{sgn}[L_{j-1}y(x)] = \operatorname{sgn}[L_j y(x)], \qquad x \geq b,$$

contradicting (5.8). This contradiction shows $c = 0$. This completes the proof of the lemma.

The strict restrictions (5.8) and (5.9) on the sign and growth of the quasi-derivatives of (eventually positive) nonoscillatory solutions under the assumption (5.6) gives the solution space of Equation (5.1) sufficient structure to permit a proof of a dominance result for non-oscillatory solutions.

<u>THEOREM 5.6</u>: With the assumption (5.6), if y_1 and y_2 are two non-oscillatory solutions of Equation (5.1) and if $S(y_1) > S(y_2)$, then

$$\lim_{x\to\infty} |L_t y_1(x)/L_t y_2(x)| = \infty, \qquad t = 0, \ldots, n-1. \tag{5.17}$$

Consequently, y_1 dominates y_2 at infinity and by class at infinity.

PROOF: Without loss of generality, assume y_1 and y_2 are eventually positive; otherwise, y_i may be replaced by $-y_i$ as required without affecting the limits in (5.17). Let $k_1 = S(y_1)$, $k_2 = S(y_2)$, and observe that the parity condition forces $k_2 \leq k_1 - 2$.

Since $L_{k_2+1}y_2(x)$ is eventually negative by (5.8), the eventually positive function $L_{k_2}y_2(x)$ must have a finite limit. But $L_{k_2}y_1(x)$ is unbounded by (5.9), and so (5.17) holds for $t = k_2$. Similarly, $L_{k_1-1}y_1(x)$ must have a nonzero limit since $L_{k_1}y_1(x) > 0$ for x sufficiently large. Since (5.9) shows $\lim_{x\to\infty} L_{k_1-1}y_2(x) = 0$, (5.17) holds for $t = k_1 - 1$ as well. In fact, the limits (5.9) show easily that (5.17) holds for all t, $k_2 \leq t \leq k_1 - 1$.

If $\lim_{x\to\infty} L_{k_2-1}y_2(x) < \infty$, then (5.17) follows for $t = k_2 - 1$ as for the case $t = k_2$. Otherwise, the limit in (5.17) is indeterminate, and l'Hôpital's rule may be employed. Furthermore, l'Hôpital's rule may be applied to all the indeterminate limits (5.17) for $0 \leq t \leq k_2 - 2$, $k_1 + 1 \leq t \leq n - 1$. In the case of $t = n - 1$, observe that (provided the limit is indeterminate)

$$\begin{aligned}\lim_{x\to\infty} |L_{n-1}y_1(x)/L_{n-1}y_2(x)| &= \lim_{x\to\infty} |L_1(x)/L_n y_2(x)| \\ &= \lim_{x\to\infty} |(-p(x)y_1(x)/(-p(x)y_2(x)| \\ &= \lim_{x\to\infty} |L_0y_1(x)/L_0y_2(x)|.\end{aligned}$$

Thus, after an appropriate number of application of l'Hôpital's rule, (5.17) holds for $0 \leq t \leq k_2 - 2$ or $k_1 + 1 \leq t \leq n - 1$ by comparison with $t = k_2 - 1$. Finally, if $\lim_{x\to\infty} L_{k_1}y_1(x)$ is nonzero, (5.17) follows immediately for $t = k_1$ since $\lim_{x\to\infty} L_{k_1} y_2(x) = 0$. Otherwise, the limit is indeterminate and l'Hôpital's rule may be used.

From (5.17), y_1 dominates y_2 at infinity by definition. Also, (5.17) shows that for any real constant λ,

$$\operatorname{sgn}[L_t y_1(x)] = \operatorname{sgn}[L_t(y_1 + \lambda y_2)(x)]$$

for x sufficiently large. Thus $S(y_1 + \lambda y_2) = S(y_1)$, and so y_1 dominates y_2 by class at infinity. This completes the proof of Theorem 5.6.

The strength of the Trench factorization and Lemma 5.5 can be demonstrated further by an easy oscillation criterion for Equation (5.1). Recall that the classes S_0 and S_n, when admissible by the parity condition, must be nonoscillatory. In the case n even and $p(x) < 0$, both S_0 and S_n are admissible, and Equation (5.1) is said to have property (H) provided all other classes are oscillatory. This property has figured prominently in the efforts of Etgen and Taylor [27, 29] to show that the class S_0 has at most one linearly independent solution.

Consider the second order equation

$$y'' + p(x)y = 0,$$

with $p(x)$ continuous on $[0,\infty)$. Wintner [126] and Leighton [74] have shown that this equation is oscillatory if $\int_0^\infty p(x)dx = \infty$. The following theorem is in the same vein:

<u>THEOREM 5.7</u>: If Equation (5.1) is Trench factored, and if

$$\int_0^\infty \frac{|p(x)|}{\rho_0(x)\rho_n(x)} dx = \infty, \tag{5.18}$$

then for all k, $1 \leq k \leq n-1$, the class S_k is oscillatory. In particular, if n is even and $p(x) < 0$, then Equation (5.1) has property (H).

<u>PROOF</u>: Let k, $1 \leq k \leq n-1$, $(-1)^{n-k} p < 0$, be given. Equation (5.1) can be rewritten as

$$\rho_n \frac{d}{dx} \rho_{n-1} \cdots \frac{d}{dx} \rho_0 y + (-1)^{n-k-1} |p| y = 0,$$

or, dividing by $|p|$,

$$\frac{\rho_n}{|p|} \frac{d}{dx} \rho_{n-1} \cdots \frac{d}{dx} \rho_0 y + (-1)^{n-k-1} y = 0 . \tag{5.19}$$

Applying the operator L_k to both sides of Equation (5.19) gives

$$L_k \frac{\rho_n}{|p|} \frac{d}{dx} \rho_{n-1} \cdots \frac{d}{dx} \rho_k \cdots \frac{d}{dx} \rho_0 y + (-1)^{n-k+1} L_k y = 0,$$

which is the same as

$$\rho_k \frac{d}{dx} \rho_{k-1} \cdots \frac{d}{dx} \frac{\rho_n \rho_0}{|p|} \frac{d}{dx} \rho_{n-1} \cdots$$

$$\rho_{k+1} \frac{d}{dx} L_k y + (-1)^{n-k+1} L_k y = 0. \tag{5.20}$$

Equation (5.20) is an n-th order equation in the function $L_k y$, and the operator

$$\tilde{L}_n \equiv \rho_k \frac{d}{dx} \cdots \rho_1 \frac{d}{dx} \frac{\rho_n \rho_0}{|p|} \frac{d}{dx} \cdots \rho_{k+1} \frac{d}{dx} \tag{5.21}$$

implicitly defines quasi-derivatives $\tilde{L}_i$, $0 \leq i \leq n$. The condition (5.18) is merely the condition that $\tilde{L}_n$ is Trench factored, since the coefficients ρ_i, $1 \leq i \leq n-1$, already satisfy (5.6). Since $L_n y = -py = (-1)^{n-k} |p| y$, then Equation (5.21) shows

$$S(\tilde{L}_0(L_k y),\ldots,\tilde{L}_n(L_k y)) = S(L_k y,\ldots,L_{n-1}y,(-1)^{n-k}L_0 y,\ldots,(-1)^{n-k}L_k y)$$
$$= S(L_k y,\ldots,L_{n-1}y,L_n y) + S(L_0 y,\ldots,L_k y)$$
$$= S(L_0 y,\ldots,L_n y).$$

Thus $S(y) = S(L_k y)$. If $y \in S_k$ is nonoscillatory, then by Lemma 5.5, $\operatorname{sgn}(L_k y) = -\operatorname{sgn}(L_{k+1}y)$, while $\operatorname{sgn}(\tilde{L}_0(L_k y)) = \operatorname{sgn}(\tilde{L}_1(L_k y))$. Since $\tilde{L}_0(L_k y) = L_k y$ and $\tilde{L}_1(L_k y) = L_{k+1}y$, this is a contradiction, so that $y \in S_k$ could not have been nonoscillatory. This completes the proof of Theorem 5.7.

Theorem 5.7 is not the most complete result known of this type (see [64, Theorem 5] or [32, Theorem 4], for example), but it lends support to the use of the Trench factored form of L_n in analyzing Equation (5.1).

What art eht problems associated with changing the factorization? The most obvious concern is that the analysis of Equation (5.1) for one factorization might not carry over to another factorization. As Equations (5.5) and (5.7) have already shown, the Elias classes S_k are not invariant for changes in the factorization, and domination results that hold for one form may break down in another. Certain properties are preserved, however, and they and their consequences are discussed in Theorem 5.8 and Corollary 5.9.

<u>THEOREM 5.8</u>: Let L_n and $\tilde{L}_n$ be two different Pólya factorizations of the same disconjugate operator. For fixed $a \geq 0$, let $\{u_0, \ldots, u_{n-1}\}$ and $\{\tilde{u}_0, \ldots, \tilde{u}_{n-1}\}$ be the fundamental sets of solutions given by Theorem 4.15 for the equations $L_n y + py = 0$ and $\tilde{L}_n y + py = 0$, respectively, and let S_k and $\tilde{S}_k$, $0 \leq k \leq n$, $(-1)^{n-k}p < 0$, be the corresponding Elias classes.

Then

(1) $\mathrm{span}\{u_{k-1},u_k\} = \mathrm{span}\{\tilde{u}_{k-1},\tilde{u}_k\}$ for $1 \le k \le n-1$, $(-1)^{n-k}p < 0$.

(2) If $(-1)^n p < 0$, then u_0 and $\tilde{u}_0$ are linearly dependent. If $p < 0$, then u_{n-1} and $\tilde{u}_{n-1}$ are linearly dependent.

(3) The class S_k is oscillatory if and only if the class $\tilde{S}_k$ is oscillatory.

<u>PROOF</u>: For $1 \le k \le n-1$, $(-1)^{n-k}p < 0$, it is necessary to consider four sets of boundary conditions. Specifically, let $y_1(x,s)$ be the solution of Equation (5.1) satisfying the $n-1$ conditions

$$L_i y(a) = 0, \qquad i = 0,\ldots,k-1, \tag{5.22a}$$

$$L_j y(s) = 0, \qquad j = 0,\ldots,n-k-2, \tag{5.22b}$$

and let $y_2(x,s)$ be the solution satisfying

$$L_i y(a) = 0, \qquad i = 0,\ldots,k-2,k, \tag{5.23a}$$

$$L_j y(s) = 0, \qquad j = 0,\ldots,n-k-2. \tag{5.23b}$$

In addition, let $\tilde{y}_1(x,s)$ be the solution satisfying the conditions

$$\tilde{L}_i y(a) = 0, \qquad i = 0,\ldots,k-1, \tag{5.24a}$$

$$\tilde{L}_j y(s) = 0, \qquad j = 0,\ldots,n-k-2, \tag{5.24b}$$

and let $\tilde{y}_2(x,s)$ be the solution satisfying

$$\tilde{L}_i y(a) = 0, \qquad i = 0,\ldots,k-2,k, \tag{5.25a}$$

$$\tilde{L}_j y(s) = 0, \qquad j = 0,\ldots,n-k-2. \tag{5.25b}$$

The solutions u_k, u_{k-1}, $\tilde{u}_k$, and $\tilde{u}_{k-1}$ are defined in terms of limits over appropriate sequences in s of the solutions $y_1(x,s)$, $y_2(x,s)$, $\tilde{y}_1(x,s)$, and $\tilde{y}_2(x,s)$, respectively. Suppose the sequence $\{s_i\}$, $s_i \to \infty$, is chosen so that $y_1(x,s_i)$ converges uniformly on compact subsets of $[0,\infty)$. Then there is a subsequence $\{s_i'\} \subseteq \{s_i\}$ such that $y_2(x,s_i')$ converges uniformly on compact subsets of $[0,\infty)$. Continuing in this fashion, there is a sequence $\{\sigma_i\} \subseteq \{s_i\}$ so that all four of the functions $y_1(x,\sigma_i)$, $y_2(x,\sigma_i)$, $\tilde{y}_1(x,\sigma_i)$, and $\tilde{y}_2(x,\sigma_i)$ converge uniformly on compact subsets of $[0,\infty)$, as $\sigma_i \to \infty$. Without affecting the results of Theorem 4.15, it may be assumed then that u_k, u_{k-1}, $\tilde{u}_k$, and $\tilde{u}_{k-1}$ are defined as limits over the same sequence $\{\sigma_i\}$, $\sigma_i \to \infty$.

As discussed in Chapter II, multiple zeros of y counted with respect to $\frac{d}{dx}$ agree in number with those counted with respect to quasi-derivatives. This principle applied first to the zeros of (5.22) and then to the zeros of y with respect to $\frac{d}{dx}$ shows that $y_1(x,s)$ must also satisfy (5.24). Since solutions satisfying these conditions are essentially unique by Lemma 4.5, then $y_1(x,s) = \tilde{y}_1(x,s)$ when these solutions are properly normalized. Consequently, $\tilde{u}_k = u_k$.

The proof of Theorem 4.15 showed that $S(y_2(x,s),a+) = k$. Therefore, $L_{k-1}y_2(a,s) \neq 0$, for then $S(y_2(x,s),a+) \geq k+1$. Applying the principle described above, $\tilde{L}_{k-1}y_2(a,s) \neq 0$, and $y_2(x,s)$ must satisfy the $n-2$ boundary conditions

$$\tilde{L}_i y(a) = 0, \qquad i = 0, \ldots, k-2,$$

$$\tilde{L}_j y(s) = 0, \qquad j = 0, \ldots, n-k-2.$$

In addition, since $S(y_1(x,s),a+) = k$, then $\tilde{L}_k y_1(a,s) \neq 0$, so that there exists a constant $\lambda_s = \tilde{L}_k y_2(a,s)/\tilde{L}_k y_1(a,s)$ for which $y_3(x,s) = y_2(x,s) - \lambda_s y_1(x,s)$ must satisfy the boundary conditions (5.25). By the essential uniqueness of these solutions, it follows that

$$\tilde{y}_2(x,s) = y_2(x,s) - \lambda_s y_1(x,s) \tag{5.26}$$

when these solutions are properly normalized.

Notice that the sequence $\tilde{L}_k y_1(a,\sigma_i) = \tilde{L}_k \tilde{y}_1(a,\sigma_i)$ converges to $\tilde{L}_k \tilde{u}_k(a)$, which is not zero by Theorem 4.15, since $\tilde{u}_k \in \tilde{S}_k$, and already has k zeros at $x = a$. On the other hand $\tilde{L}_k y_2(a,\sigma_i)$ converges to $\tilde{L}_k u_{k-1}(a)$, which is finite. Thus the sequence $\{\lambda_{\sigma_i}\}$ is bounded, and so there is a subsequence $\{\sigma_i'\} \subset \{\sigma_i\}$ and a constant λ such that $\lambda_{\sigma_i'} \to \lambda$ as $\sigma_i' \to \infty$. Consequently, passing to the limit along this subsequence, (5.26) becomes

$$\tilde{u}_{k-1} = u_{k-1} - \lambda u_k. \tag{5.27}$$

Since it is already known that $\tilde{u}_k = u_k$, then $\text{span}\{\tilde{u}_{k-1}, \tilde{u}_k\} \subseteq \text{span}\{u_{k-1}, u_k\}$. Reversing the roles of u_{k-1} and $\tilde{u}_{k-1}$ obtains the reverse containment, and completes the proof of part (1).

The solution u_{n-1} is defined to be the unique solution satisfying the initial conditions

$$L_i y(a) = 0, \qquad i = 0, \ldots, n-2,$$

$$L_{n-1} y(a) = 1,$$

and as a result of the principle enunciated earlier, it follows that u_{n-1} and $\tilde{u}_{n-1}$ are linearly dependent. If $(-1)^n p < 0$, then u_0 is defined as u_k from (5.22) with $k = 0$, so that as in the earlier case, u_0 and $\tilde{u}_o$ are linearly dependent. This completes part (2).

Part (3) is the immediate consequence of (1), (2), and Theorem

4.14. This completes the proof of Theorem 5.8.

The following corollary shows that Theorem 5.8 justifies a change in factorization in studying the dominance characteristics of the solutions of Equation (5.1).

COROLLARY 5.9: Suppose, in addition to the hypotheses of Theorem 5.8, it is known that $\tilde{S}_k$ dominates $\tilde{S}_j$ by class at infinity whenever $k > j$. If $y = \sum_{i=0}^{n-1} c_i u_i$ is a nontrivial solution of Equation (5.1), and if t is the largest index such that the constant c_t is nonzero, then y is oscillatory or nonoscillatory as u_t is oscillatory or nonoscillatory.

PROOF: Suppose $t = k$, $(-1)^{n-k} p < 0$, so that $u_t = u_k \in S_k$. Theorem 5.8 shows that $c_k u_k + c_{k-1} u_k \in \tilde{S}_k$, and also that $c_j u_j$ and $c_{j-1} u_{j-1} \in \tilde{S}_j$ for all $j < k$. Since $\tilde{S}_k$ dominates $\tilde{S}_j$ by class at infinity, it follows that $y = \sum_{i=0}^{n-1} c_i u_i \in \tilde{S}_k$. Consequently, part (3) of Theorem 5.8 shows that the behavior of y is determined by the behavior of u_k.

If $t = 0$, n, or $k - 1$, the argument is similar. This completes the proof of Corollary 5.9.

All the theorems and examples so far seem to indicate that the best setting from which to analyze the dominance structure of the solution space of Equation (5.1) is to Trench factor L_n. From this point on, then, it will be assumed that the operator L_n is in its Trench factored form, with coefficients given in Equation (5.2) satisfying (5.6). In Theorem 5.6 it was seen that, with this assumption, the nonoscillatory Elias classes of Equation (5.1) satisfy a natural dominance relationship. In fact, with (5.6), all of the Elias classes have satisfied this natural dominance relationship in every example studied to date. This suggests the following conjecture, which remains an open question:

<u>CONJECTURE 5.10</u>: If Equation (5.1) is Trench factored, then S_k dominates S_j by class at infinity for every $j < k$, for all admissible values of j and k.

The main problem that has arisen in trying to demonstrate this conjecture has been a basic lack of understanding of the growth of oscillatory solutions of Equation (5.1). Because nonoscillatory solutions are eventually single-signed, the Trench factorization can restrict their growth, as in Lemma 5.5.

Perhaps Conjecture 5.10 is too strong--it may not be possible to obtain the necessary information only from the Trench factorization. It is still reasonable to suggest that the control on the nonoscillatory solutions is great enough to permit a weakened version of Conjecture 5.10 to be proved:

<u>CONJECTURE 5.11</u>: If Equation (5.1) is Trench factored, and if S_k is nonoscillatory, then S_k dominates S_j by class at infinity for every $j < k$, for all admissible values of j and k.

While this, too, remains an open question, a little more information can be extracted from the recent work of Kim [64]. If A is a set of solutions of Equation (5.1), define q(A) to be the maximum number of linearly independent solutions in A such that every nontrivial linear combination of the solutions is again in A. Theorem 4.15 may be interpreted as saying that for $1 \leq k \leq n-1$, $q(S_k) \geq 2$, while $q(S_0) \geq 1$ and $q(S_n) \geq 1$ when these classes are admissible. As will be seen below, for $1 \leq k \leq n-1$, then $q(S_k) = 2$, while $q(S_0) = 1$ and $q(S_n) = 1$ whenever these classes are admissible. But first, an intermediate result:

THEOREM 5.12 (Kim [35]): For $1 \leq k \leq n-1$, if the class S_k is nonoscillatory, and if $y_1, y_2 \in S_k$ are eventually positive solutions such that

$$\lim_{x\to\infty} \frac{y_2(x)}{y_1(x)} = \infty, \tag{5.28}$$

then either

$$\limsup_{x\to\infty} \frac{w(x)}{y_1(x)} \neq \infty, \tag{5.29a}$$

or

$$\limsup_{x\to\infty} \frac{w(x)}{y_2(x)} \neq 0 \tag{5.29b}$$

for every solution w of Equation (5.1).

If the class S_n is admissible, and if $y_1 \in S_n$ is eventually positive, then

$$\limsup_{x\to\infty} \frac{w(x)}{y_1(x)} \neq \infty \tag{5.30}$$

for every solution w of Equation (5.1).

PROOF: For the first part of the theorem, let k, $1 \leq k \leq n-1$, be given, and let w be an arbitrary solution of Equation (5.1). Suppose that w is oscillatory. If $S(w) > k$, then $\limsup_{x\to\infty} \frac{w(x)}{y_2(x)} > 0$ by Theorem 5.4. Suppose, on the other hand, that $S(w) < k$ and $\limsup_{x\to\infty} \frac{w(x)}{y_2(x)} = 0$ while $\limsup_{x\to\infty} \frac{w(x)}{y_1(x)} = \infty$. Since $y_1(x)$ is eventually positive, this second limit implies $w(x) - \lambda y_1(x)$ is an oscillatory solution for every $\lambda \in [0,\infty)$. Thus the solution $\frac{1}{\lambda}w(x) - y_1(x)$ oscillates for every positive λ. Since $\lim_{x\to\infty} (\frac{1}{\lambda}w(x) - y_1(x)) = -y_1(x) \in S_k$, Lemma 5.1 implies that there is a constant $\lambda_0 > 0$ such that $S(\frac{1}{\lambda_0} w(x) - y_1(x)) \geq k$. Since S_k is nonoscillatory it follows that $S(\frac{1}{\lambda_0} w(x) - y_1(x)) > k$.

Applying Theorem 5.4 to $w(x) - \lambda_0 y_1(x)$ and $y_2(x)$ gives

$$\limsup_{x\to\infty} \frac{w(x) - \lambda_0 y_1(x)}{y_2(x)} > 0.$$

However, since $\lim_{x\to\infty} \frac{y_1(x)}{y_2(x)} = 0$ by Equation (5.28), then

$$\limsup_{x\to\infty} \frac{w(x) - \lambda_0 y_1(x)}{y_2(x)} = \limsup_{x\to\infty} \frac{w(x)}{y_2(x)} = 0.$$

This contradiction verifies Equation (5.29) when w is oscillatory.

If w is nonoscillatory and $S(w) \neq k$, then Equation (5.29) follows immediately from Theorem 5.6. Thus it remains only to argue the case where $w \in S_k$. Suppose $\limsup_{x\to\infty} \frac{w(x)}{y_2(x)} = 0$. Since both solutions are positive, then $\lim_{x\to\infty} \frac{w(x)}{y_2(x)} = 0$. Suppose in addition that $\limsup_{x\to\infty} \frac{w(x)}{y_1(x)} = \infty$. If $\lim_{x\to\infty} \frac{w(x)}{y_1(x)}$ does not exist, then for some constant λ the solution $w(x) - \lambda y_1(x)$ is oscillatory. But clearly

$$\limsup_{x\to\infty} \frac{w(x) - \lambda y_1(x)}{y_1(x)} = \infty$$

and

$$\limsup_{x\to\infty} \frac{w(x) - \lambda y_1(x)}{y_2(x)} = 0,$$

contradicting the first portion of the proof. Hence $\lim_{x\to\infty} \frac{w(x)}{y_1(x)} = \infty$. From these limits it follows that $c_1 y_1 + c_2 y_2 + c_3 w \in S_k$ for every choice of the constants c_1, c_2, c_3.

Let $\{x_i\}$ be a positive, increasing, unbounded sequence of real numbers. Since the solutions y_1, y_2 and w are necessarily linearly independent, there exist nontrivial constants $c_{i,1}$, $c_{i,2}$, and $c_{i,3}$, normalized so that $\sum_{j=1}^{3} c_{ij}^2 = 1$, with the property that the solution $z_i \equiv c_{i,1}y_1 + c_{i,2}y_2 + c_{i,3}w$ has a double zero at the point x_i. Since this solution is in S_k, then $S(z_i,x_i+) \leq k$; consequently, by Lemma 4.1, $S(z_i,x+) < k$ for $x < x_i$, $i = 1,2,\ldots$. Using subsequences as necessary, let $c_j = \lim_{i\to\infty} c_{i,j}$, $j = 1,2,3$, and set $z = c_1y_1 + c_2y_2 + c_3w$. The solution z must be in S_k by the previous paragraph, so that for some x_0 sufficiently large, $S(z,x+) = k$ for $x > x_0$. Let $x > x_0$ be chosen so that no quasi-derivative of z vanishes at x, and choose $x_i > x$ so large that $\operatorname{sgn}(L_t z_i(x)) = \operatorname{sgn}(L_t z(x))$, $t = 0,\ldots,n$. Then $S(z_i,x+) = k$, contradicting the choice of $x_i > x$. This contradiction shows that $w \in S_k$ cannot be chosen so that Equation (5.29) is violated.

For the second part of the theorem, observe that (5.30) is immediate from Theorem 5.6 if w is nonoscillatory and $S(w) \neq n$. If w is oscillatory and (5.30) is violated, then $w - \lambda y_1$ is oscillatory for every positive λ. But $\frac{1}{\lambda}w - y_1$ converges to $-y_1 \in S_n$ as $\lambda \to \infty$, and so for some λ_0, Lemma 5.1 implies $w - \lambda_0 y_1 \in S_n$. But then $w - \lambda_0 y_1$ is nonoscillatory, a contradiction. Thus, as before, it remains only to consider the case $w \in S_n$ nonoscillatory.

If $\limsup_{x\to\infty} \frac{w(x)}{y_1(x)} = \infty$, then as before it may be argued that $\lim_{x\to\infty} \frac{w(x)}{y_1(x)}$ must exist, leading to the conclusion that every nontrivial combination $d_1y_1 + d_2w_2 \in S_n$. If y satisfies $S(y,x+) = n$, then every quasi-derivative of y must agree in sign. Thus, if x_0 is even a simple zero of y, it follows that $S(y,x+)<n$ for $x<x_0$. Hence, choosing constants $d_{i,1}$ and $d_{i,2}$ so that $d_{i,1}y_1 + d_{i,2}w$ has a simple zero at x_i, $x_i \to 0$, the proof may be com-

pleted as in the previous part. This completes the proof of Theorem 5.12.

If it could be shown that, for $0 \le k \le n-1$, $\bigcup_{j<k} S_j$ is a linear subspace of dimension exactly $k+1$, this would imply the truth of Conjecture 5.10. Thus the next two results of Kim [64] provide partial support for the conjecture above.

<u>THEOREM 5.13</u>: For $1 \le k \le n-1$, $q(S_k) = 2$. In addition $q(S_0) = 1$ and $q(S_n) = 1$ whenever these classes are admissible.

<u>PROOF</u>: For $1 \le k \le n-1$, if class S_k contains three solutions y_1, y_2, and y_3 such that every nontrivial linear combination is again in S_k, then exactly as in the proof of Theorem 5.12 it is possible to construct a combination $c_1y_1 + c_2y_2 + c_3y_3$ with $S(c_1y_1 + c_2y_2 + c_3y_3) < k$, a contradiction. Alternatively, if S_k is nonoscillatory, then as in Theorem 4.15 part (3), it is possible to construct solutions u, v, w $\in$ span$\{y_1, y_2, y_3\}$ such that

$$\lim_{x\to\infty} \frac{u(x)}{v(x)} = \infty$$

and

$$\lim_{x\to\infty} \frac{v(x)}{w(x)} = \infty,$$

contradicting the conclusion of Theorem 5.12. Thus $q(S_k) = 2$.

For $k = n$, if class S_n contains two solutions y_1, y_2 such that every nontrivial linear combination is in S_n, then again as in the proof of Theorem 5.12 this leads to a contradiction, and hence $q(S_n) \le 1$. By Theorem 4.15, $q(S_n) \ge 1$ when the class is admissible, and so $q(S_n) = 1$.

Finally, for $k = 0$, if $y \in S_0$, then y cannot vanish anywhere in $[0,\infty)$. For if $y(x_0) = 0$, then $\mathrm{sgn}[L_0y(x_0+\varepsilon)] = \mathrm{sgn}[L_1y(x_0+\varepsilon)]$ for $\varepsilon > 0$ sufficiently small, and so $S(y,x_0+) \geq 1$. Since $S(y,x+)$ is an increasing function, this contradicts $y \in S_0$. Consequently, S_0 cannot contain two linearly independent solutions for which every nontrivial linear combination is back in S_0. Thus it follows that $q(S_0) = 1$. This completes the proof of Theorem 5.13.

Note that Theorem 5.13 says $q(S_0) = 1$, not $\dim(S_0) = 1$ as a subspace. Thus this theorem does not show the strongly decreasing solution is essentially unique. In fact, Theorem 5.13 does not require the Trench factorization of L_n for its proof, and so is applicable to the example of Equation (5.5). Both the solutions $y_1 = e^{1/x}$ and $y_6 = \sin\frac{1}{x} - \sinh\frac{1}{x}$ were seen to be in the class S_0, and yet these solutions are linearly independent. Theorem 5.13 is not violated, however, because the combination $y_1 + 2y_6$ is in class S_2 and not S_0.

The next theorem comes closer to describing the dimension of $\bigcup_{j \leq k} S_j$, and does require the Trench factorization [64].

THEOREM 5.14: Suppose the class S_k is nonoscillatory for some k $0 < k \leq n$, and set

$$y(x) = \begin{cases} y_1 \in S_k & \text{for } k = n \\ y_1 & \text{if } y_1, y_2 \in S_k \text{ and } \lim\limits_{x\to\infty} \dfrac{y_2(x)}{y_1(x)} = \infty. \end{cases} \tag{5.31}$$

Then there are at most k linearly independent solutions $v_i(x)$, $i = 1, \ldots, k$, such that

$$\limsup_{x\to\infty} \frac{|v_i(x)|}{y(x)} \neq \infty. \tag{5.32}$$

PROOF: The result is trivial for $k = n$ since the dimension of the entire solution space is n.

Suppose that for some k, $0 < k < n$, there are $k+1$ linearly independent solutions $v_1(x), \ldots, v_{k+1}(x)$ satisfying (5.32). Without loss of generality, assume y_2 is eventually positive, and choose $b \geq 0$ so that $L_i y_1(x) \neq 0$ on $[b,\infty)$, $0 \leq i \leq n$. Let $v = \sum_{i=1}^{k+1} c_i v_i$ be a nontrivial linear combination satisfying

$$L_i v(b) = 0, \qquad i = 0, \ldots, k-1, \tag{5.33}$$

and such that $v(x)$ is positive somewhere in (b,∞). Since $\lim_{x\to\infty} \frac{y_2(x)}{y_1(x)} = \infty$, then (5.32) implies

$$\lim_{x\to\infty} \frac{v(x)}{y_2(x)} = 0, \tag{5.34}$$

and hence

$$\lim_{x\to\infty} \frac{L_j v(x)}{L_j y_2(x)} = 0, \qquad j = 0,1, \ldots, n. \tag{5.35}$$

For (5.34) implies that $y_2 + \lambda v$ is nonoscillatory for every λ, so that the limits in (5.35) must exist. Since the limit with $j = 0$ is zero by (5.34), the others must also be zero by virtue of (5.6).

Let H be the set of positive numbers λ such that

$$L_j y_2 - \lambda L_j v \geq 0 \text{ on } [b,\infty), \qquad j = 0,1, \ldots, k-1.$$

The limits (5.35) show H is nonempty, and since $v(x) > 0$ somewhere in $[b,\infty)$, H must be bounded above. Set $\lambda_0 = \sup H$, and let

$$z(x) = y_2(x) - \lambda_0 v(x).$$

If $L_i z(x) > 0$ on $[b,\infty)$ for $i = 0, \ldots, k-1$, then given $\varepsilon_1 > 0$, by (5.35) there would exist $c > b$ such that

$$L_i y_2(x) > (\lambda_0 + \varepsilon_1) L_i v(x) \quad \text{on } [c,\infty), \qquad i = 0, \ldots, k-1.$$

In addition, since $L_i z(x) > 0$ on $[b,c]$, there would exist ε_2, $0 < \varepsilon_2 < \varepsilon_1$, such that

$$L_i y_2(x) > (\lambda_0 + \varepsilon_2) L_i v(x) \quad \text{on } [b,c], \qquad i = 0, \ldots, k-1,$$

and hence

$$L_i y_2(x) > (\lambda_0 + \varepsilon_2) L_i v(x) \quad \text{on } [b,\infty), \qquad i = 0, \ldots, k-1,$$

contradicting the choice of λ_0. Therefore, $L_i z(x_0) = 0$ for some i, $0 \leq i \leq k-1$, and for some point $x_0 \in [b,\infty)$. By the choice of b and from (5.33), it is clear that $x_0 \neq b$. From the definition of H it follows that $L_i z \geq 0$ on (b,∞). Since $L_i z(x_0) = 0$, then $L_{i+1} z(x_0) = 0$ and changes sign at x_0. This will contradict the definition of H and the choice of λ_0 unless $i + 1 = k$. Thus $L_k z(x_0) = 0$.

The limits (5.35) imply $z \in S_k$, and therefore $S(z,x_0+) \leq k$. Since $L_i z(x) = L_{k-1} z(x) \geq 0$ on (b,∞), then $L_k z(x_0 + \varepsilon) > 0$ for $\varepsilon > 0$ sufficiently small. Since $L_k z$ changes sign at x_0, then $L_{k+1} z(x_0 + \varepsilon) > 0$ as well for $\varepsilon > 0$ sufficiently small. In addition, by definition of H and the choice of λ_0, for $0 \leq j \leq k-1$, $L_j z(x) \geq 0$ on (b,∞), so that $S(z,x_0+) \geq k+1$. This contradiction shows that $k+1$ linearly independent solutions $v_i(x)$, $i = 1, \ldots, k+1$, cannot be found satisfying (5.32). This completes the proof of Theorem 5.14.

Theorem 5.14 places an upper bound on the number of solutions which $y(x)$, chosen as in Equation (5.31), can dominate at zero. In fact, this maximum number is actually achieved for each nonoscillatory class S_k, $0 < k \leq n-1$, for a certain choice of the function $y(x)$, as the following theorem shows.

THEOREM 5.15: Suppose the class S_k is nonoscillatory for some k, $0 < k \leq n$. If $y(x) \in S_n$, then there are exactly n linearly independent solutions v_i, $i = 0, \ldots, n-1$, such that

$$\limsup_{x\to\infty} \frac{|v_i(x)|}{y(x)} \neq \infty. \tag{5.36}$$

If $0 < k < n$, then there exist solutions v, w S_k with $\lim_{x\to\infty} \frac{v(x)}{w(x)} = \infty$ such that there are exactly k linearly independent solutions v_i, $i = 0, \ldots, k-1$, such that (5.36) is satisfied with $y = w$.

PROOF: The case $k = n$ follows immediately from Theorem 5.12, which states in part that (5.36) is satisfied for any choice of $y \in S_n$. Thus, taking $\{v_0, \ldots, v_{n-1}\}$ to be any basis for the solution space suffices.

For $0 < k < n$, consider the $k+1$ sets of $n-1$ boundary conditions described by

$$L_i y(a) = 0, \qquad i \in \{0, \ldots, k\} \setminus \{t\}, \tag{5.37a}$$

$$L_j y(s) = 0, \qquad j = 0, \ldots, n-k-2, \tag{5.37b}$$

$0 \leq t \leq k$, and suppose $y_t(x,s)$ satisfies (5.37) for the given value of t. There exists a sequence $\{s_i\}$, $s_i \to \infty$, such that when appropriately normalized the functions $y_t(x,s_i) \to y_t(x)$ uniformly on compact sets, $0 \leq t \leq k$. As in the proof of Theorem 4.15, $y_t \in S_k$ for each t, and $\lim_{x\to\infty} \frac{y_t(x)}{y_r(x)}$ exists, finite or infinite, whenever $0 \leq t, r \leq k$. Note that $y_k = u_k$, $y_{k-1} = u_{k-1}$, where $\{u_0, \ldots, u_{n-1}\}$ is the basis described by Theorem 4.15. Let

$v, w \in S_k$ be as in Theorem 4.15 so that in particular $\lim_{x\to\infty} \frac{v(x)}{w(x)} = \infty$. The claim is that there exist k independent solutions satisfying (5.36) with $y(x) = w(x)$.

Recall from the proof of Theorem 4.15 that $v(x)$ was chosen to be either $y_k(x)$ or $y_{k-1}(x)$. For example, if $\lim_{x\to\infty} \frac{y_k(x)}{y_{k-1}(x)} = c < \infty$, then $v(x) = y_{k-1}(x)$, $w(x) = y_k(x) - cy_{k-1}(x)$. Hence, for $0 \le t \le k$,

$$\lim_{x\to\infty} \frac{y_t(x)}{v(x)} = c_t < \infty \tag{5.38}$$

exists in the extended real numbers. Suppose for some t, say t_0, this limit is infinite. Then clearly

$$\lim_{x\to\infty} \frac{y_{t_0}(x)}{w(x)} = \infty, \tag{5.39}$$

and $v(x)$ violates Theorem 5.12 for the pair $w(x)$, $y_{t_0}(x)$. Hence the limit in (5.38) is finite for every t. Without loss of generality, assume $v(x) = y_k(x)$; otherwise, reverse the role of $y_k(x)$ and $y_{k-1}(x)$ in what follows. Define the set of functions $v_i(x)$, $i = 0, \ldots, k-1$, by

$$v_i(x) = y_i(x) - c_i v(x) = y_i(x) - c_i y_k(x).$$

Then, for $0 \le i \le k-1$,

$$\lim_{x\to\infty} \frac{v_i(x)}{v(x)} = 0,$$

and so (5.36) follows from Theorem 5.12. This completes the proof of Theorem 5.15.

Consider the Euler equation

$$x^4 y^{(4)} - \varepsilon y = 0 \tag{5.40}$$

on the interval $[1,\infty)$. This equation has solutions of the form x^{α}, where α is a root of the equation

$$\alpha(\alpha-1)(\alpha-2)(\alpha-3) - \varepsilon = 0. \tag{5.41}$$

Choose $\varepsilon > 0$ so small that this equation has four real roots, $\alpha_0 < 0$, $1 < \alpha_1 < \alpha_2 < 2$, and $3 < \alpha_3$. In this example $x^{\alpha_0} \in S_0$, $\{x^{\alpha_1}, x^{\alpha_2}\} \subseteq S_2$, $x^{\alpha_3} \in S_4$. Set $y_i = x^{\alpha_i}$, $0 \le i \le 3$.

Observe that y_1 dominates both itself and y_0 at zero, achieving the maximum allowed by Theorem 5.14 without benefit of the argument used in Theorem 5.15. Note also that y_2 dominates both y_0 and y_1 by class at infinity. Strictly speaking, y_2 cannot dominate itself by class at infinity since $y_2 - \lambda y_2$ fails to be in S_2 for $\lambda = 1$. However, y_2 does dominate the solution $y_1 + y_2$ by class at infinity, and this solution is independent of both y_0 and y_1. This example suggests the following corollaries to Theorems 5.14 and 5.15.

<u>COROLLARY 5.16</u>: Suppose the class S_k is nonoscillatory for some k, $0 < k < n$, and suppose $y_1, y_2 \in S_k$ satisfy $\lim\limits_{x\to\infty} \dfrac{y_2(x)}{y_1(x)} = \infty$. Then y_2 dominates at most $k+1$ linearly independent solutions by class at infinity.

<u>PROOF</u>: Suppose there are $k+2$ linearly independent solutions v_i, $i = 1, \ldots, k+2$, such that

$$y_2 + \lambda v_i \in S_2, \qquad i = 1, \ldots, k+2 \tag{5.42}$$

for all real values of λ.

If $\lim\limits_{x\to\infty} \dfrac{v_i(x)}{y_2(x)}$ does not exist in the extended real numbers for some i, $1 \le i \le k+2$, then there is a constant λ_0 such that the solution

$y_2 + \lambda_0 v_i$ is oscillatory, contradicting (5.42). Thus

$$\lim_{x\to\infty} \frac{v_i(x)}{y_2(x)} = c_i \leq \infty, \qquad i = 1, \ldots, k+2 \tag{5.43}$$

exists in the extended real numbers. If any limit in (5.43) is infinite for some i, then $y_2(x)$ will violate Theorem 5.12 for the pair $v_i(x)$, $y_1(x)$. Consequently, $c_i < \infty$, and so

$$\lim_{x\to\infty} \frac{v_i(x) - c_i y_2(x)}{y_2(x)} = 0, \qquad i = 1, \ldots, k+2.$$

By Theorem 5.12, then

$$\lim_{x\to\infty} \frac{v_i(x) - c_i y_2(x)}{y_1(x)} \neq \infty, \qquad i = 1, \ldots, k+2$$

Since the set $\{v_i - c_i y_2\}_{i=1}^{k+2}$ contains at least $k+1$ linearly independent solutions, this contradicts Theorem 5.14. This completes the proof of Corollary 5.16.

COROLLARY 5.17: Suppose the class S_k is nonoscillatory for some k, $0 < k < n$. Then there exist solutions $v, w \in S_k$ with $\lim_{x\to\infty} \frac{v(x)}{w(x)} = \infty$, such that there are exactly $k+1$ linearly independent solutions dominated by v by class at infinity.

PROOF: This is virtually immediate from the proof of Theorem 5.15. Let $y_i(x)$, $i = 0, \ldots, k$ be defined as in that proof, and again assume without loss of generality that $v(x) = y_k(x)$. Then the proof of Theorem 5.15 showed that $v(x)$ dominates the $k+1$ solutions $y_0, \ldots, y_{k-1}, y_k + y_{k-1}$ by class at infinity. This completes the proof of Corollary 5.17.

While these results have provided some information about the dominance of solutions of Equation (5.1), they are too weak to lead to results such as Conjecture 5.10 or 5.11. The reason in part has to do with the imprecise nature of the limits (5.32) and (5.36). If these limits could be strengthened to $\lim_{x\to\infty} \frac{v_i(x)}{y(x)} = 0$, then Conjecture 5.11 would follow almost immediately. Even so, it would not follow that an oscillatory class dominated all lower order nonoscillatory classes.

Considerations such as these suggest that the major problem is located at the interface between groups of oscillatory classes and nonoscillatory classes. That is, if the classes S_i, $k_1 \leq i \leq k_2$, are all nonoscillatory, if $y_i, y_{i-1} \in S_i$ are chosen so that $\lim_{x\to\infty} \frac{y_i(x)}{y_{i-1}(x)} = \infty$, $k_1 \leq i \leq k_2$, and if w is an oscillatory solution, then either $\limsup_{x\to\infty} \frac{w(x)}{y_{k_1-1}(x)} \neq \infty$ or $\limsup_{x\to\infty} \frac{w(x)}{y_{k_2}(x)} > 0$. Thus w must dominate, or be dominated by, every solution $y_{k_1}, \ldots, y_{k_2-1}$. However, the relationship between w(x) and y_{k_2} or y_{k_1-1} remains unclear at this time.

The study of the dominance properties of solutions of Equation (5.1) is still quite active. Even if the interface problem outlined above could be solved, many questions would still remain about the interaction between pairs of oscillatory Elias classes.

Another major question not even touched on here is whether there are restrictions, other than those due to parity, as to which classes may be oscillatory or nonoscillatory. Even though this is not directly a question of dominance, it does deal with a description of the possible locations of the interfaces mentioned earlier, and so affects the dominance problem. Chapter VI takes up this question for a slightly less general version of Equation (5.1).

CHAPTER VI

THE EQUATION $y^{(n)} + py = 0$

Chapters IV and V considered the equation

$$L_n y + p(x)y = 0, \tag{6.1}$$

where $p(x)$ is a nonvanishing continuous function on $I = [0,\infty)$, and where L_n is the factored disconjugate operator defined by

$$L_n y = \rho_n(x)\frac{d}{dx}\rho_{n-1}(x)\frac{d}{dx}\cdots\rho_1(x)\frac{d}{dx}\rho_0(x)y, \tag{6.2}$$

with $\rho_i(x)$ continuous and positive on I. The equation considered in this chapter is

$$y^{(n)} + p(x)y = 0, \tag{6.3}$$

which is of the form (6.1) with coefficients $\rho_i(x) \equiv 1$, $i = 0, \ldots, n$. Note in particular that the operator $L_n = d^n/dx^n$ used in Equation (6.3) is Trench factored. As a consequence, the results discussed in Chapters IV and V for Equation (6.1) apply to Equation (6.3) as well. In addition, because of the simplicity of the operator d^n/dx^n, many results can be obtained for Equation (6.3) that are unavailable in the general case.

Historically, interest in Equation (6.3) was the natural outgrowth of the surge of activity in lower order problems which followed the 1958 paper of Leighton and Nehari [76], even though it was studied at least as early as 1955 by Mikusiński [91]. After the 1967 Nehari paper [93] which laid the groundwork for the use of quasi-derivatives, and the 1974 paper by Trench [125] which established a canonical factorization for disconjugate operators, Equation (6.3) served at least in part as a motivating factor for the study of Equation (6.1). For example, the papers [62, 63] of Kim yielded techniques for Equation (6.3) he was able to apply later to Equation (6.1) in Reference [64]. Throughout this time, however, Equation (6.3) also maintained a separate identity, and is still of tremendous current interest apart from Equation (6.1).

The primary distinguishing feature of Equation (6.3) is that integrals derived from the operator d^n/dx^n do not involve the unknown coefficients of the general theory. As a result, such integrals frequently can be simplified in terms of elementary functions. The following lemma, which has its origins in the work of Kiguradze [58], is an example.

<u>LEMMA 6.1</u>: For $1 \leq k \leq n-1$, $(-1)^{n-k}p(x) < 0$, if the Elias class S_k of Equation (6.3) is nonoscillatory, then there are positive constants A and B such that

$$Ax^k \geq y(x) \geq Bx^{k-1} \tag{6.4}$$

for x sufficiently large, and for every $y \in S_k$ which is eventually positive. If $y \in S_0$ is eventually positive, then

$$A \geq y(x) > 0, \tag{6.5}$$

and if $y \in S_n$ is eventually positive, then

$$y(x) \geq Bx^{n-1} \tag{6.6}$$

for some choice of the positive constants A and B, and for x sufficiently large.

PROOF: Let k, $1 \le k \le n-1$, $(-1)^{n-k} p(x) < 0$, be chosen so that S_k is nonoscillatory. If $y \in S_k$ is eventually positive, then by Lemma 5.5 y must satisfy the conditions

$$y^{(i)}(x) > 0, \qquad i = 0, \ldots, k, \tag{6.7a}$$

$$(-1)^{j-k} y^{(j)}(x) > 0, \qquad j = k, \ldots, n, \tag{6.7b}$$

on some interval $[b,\infty)$. As a result, $y^{(k)}(x)$ is a positive decreasing function on $[b,\infty)$. Take $M \equiv \max\{r!y^{(k-r)}(b)/b^r \mid 0 \le r \le k\}$ and $N \equiv \min\{(r-1)!y^{(k-r)}(b)/b^{r-1} \mid 1 \le r \le k\}$. In particular, then $M \ge y^{(k)}(x)$ for $x \in [b,\infty)$ so that integrating from b to x gives

$$Mx - Mb \ge y^{(k-1)}(x) - y^{(k-1)}(b), \qquad x \in [b,\infty).$$

By choice of M, $Mb \ge y^{(k-1)}(b)$, so that in fact

$$Mx \ge y^{(k-1)}(x), \qquad x \in [b,\infty).$$

Now $y^{(k-1)}(x)$ is a positive increasing function, and thus by choice of N,

$$Mx \ge y^{(k-1)}(x) \ge N, \qquad x \in [b,\infty).$$

Continuing to integrate in this fashion, employing the definition of M and N to remove the constants of integration, eventually yields

$$\frac{Mx^k}{k!} \ge y(x) \ge \frac{Nx^{k-1}}{(k-1)!}, \qquad x \in [b,\infty).$$

Identifying $A = \frac{M}{k!}$ and $B = \frac{N}{(k-1)!}$ produces Equation (6.4).

The proofs for Equations (6.5) and (6.6) are handled similarly. This completes the proof of Lemma 6.1.

It is clear that the technique used in this lemma can be duplicated for Equation (6.1) when the operator L_n is Trench factored, but in that case the inequality (6.4) becomes ([25, p. 31])

$$A\rho_0^{-1}(x)\int_0^x \rho_1^{-1}(s_1)\int_b^{s_1} \ldots \int_b^{s_{k-1}} \rho_k^{-1}(s_k)ds_k \ldots ds_1 \geq y(x)$$

$$\geq B\rho_0^{-1}(x)\int_0^x \rho_1^{-1}(s_1)\int_b^{s_1} \ldots \int_b^{s_{k-2}} \rho_{k-1}^{-1}(s_{k-1})ds_{k-1} \ldots ds_1, \quad (6.8)$$

on some interval $[b,\infty)$.

The proof of Lemma 6.1 is particularly easy because it combines the strength of the general results known for the Elias classes of Equation (6.1) with the simplicity due to the form of Equation (6.3). In his 1978 paper [61], Kim did not have the benefit of knowledge of the Elias classes, and so was forced to consider more possible cases than actually existed. In light of what is now known about Equation (6.1), much of which Kim discovered independently (see References [62, 63, 64]), the 1978 paper can be seen to contain little more than a weakened version of Lemma 6.1.

As was observed in Chapter V, the oscillatory solutions of equations such as (6.1) and (6.3) do not seem to be governed by the same rules as control the nonoscillatory solutions. Using Lemma 6.1, this can actually be used to advantage to test for the existence of oscillatory solutions, as described in the next result.

<u>COROLLARY 6.2</u>: Let y be a solution of Equation (6.3). If for some k, $1 \leq k \leq n-1$, $(-1)^{n-k}p(x) < 0$, and for every choice of positive constants A and B there is an increasing unbounded sequence $\{x_i\}$ of real numbers such that

$$Ax_i^{k+1} \geq y(x_i) \geq Bx_i^k \quad (6.9)$$

for all i sufficiently large, then y is oscillatory. If $(-1)^n p(x) > 0$ and

$$\lim_{i\to\infty} y(x_i) = 0, \tag{6.10}$$

or if $p(x) > 0$ and for every choice of $B > 0$,

$$y(x_i) \geq Bx_i^n, \tag{6.11}$$

then y is oscillatory.

Corollary 6.2 is an immediate consequence of Lemma 6.1. As an example, consider the constant coefficient problem

$$y''' - y = 0, \tag{6.12}$$

with solutions $y_1 = e^x$, $y_2 = \sin(\frac{\sqrt{3}}{2}x)e^{-\frac{1}{2}x}$, and $y_3 = \cos(\frac{\sqrt{3}}{2}x)e^{-\frac{1}{2}x}$. Note that $y_1 \in S_3$ and $y_2, y_3 \in S_1$. Since $\lim_{x\to\infty} y_2(x) = 0$, it follows from Equation (6.10) that y_2 oscillates, which is clearly the case for this example. On the other hand, while the conditions (6.9) through (6.11) are sufficient, they are not necessary to identify oscillation. The solution y_2 of Equation (6.12) is also a solution of the equation

$$y^{(6)} - y = 0. \tag{6.13}$$

In the case of Equation (6.13), the condition (6.10) no longer applies since $(-1)^6(-1) < 0$. Thus the limit $y_2(x) \to 0$ as $x \to \infty$ is acceptable behavior for a nonoscillatory solution, and so the oscillatory behavior of y_2 is not detectable by Corollary 6.2.

Lemma 6.1 shows that, for $1 \leq k \leq n-1$, nonoscillatory solutions in the class S_k exhibit polynomial growth. Read [103] has obtained results which describe the growth of solutions in the classes S_0 and S_n more accurately than Lemma 6.1, for the case n even. Read's results predict the exponential growth and decay of the nonoscillatory solutions of the equation $y^{(2n)} - \lambda y = 0$ where $\lambda > 0$ is a constant.

While such results as these provide additional information about the nature of the solutions of Equation (6.3), they fail to shed any new light on the problem of the dominance of the solutions. Since the nonoscillatory solutions satisfy a natural dominance relationship as described by Theorem 5.6, and since this dominance can be disrupted only at the interface between oscillatory and nonoscillatory classes, what is needed are results describing the location of the oscillatory classes among the list of Elias classes for Equation (6.3).

In 1976, Nehari [95] claimed to have shown that, for n even, if some class S_k, $1 \leq k \leq n-1$, was oscillatory, then all the classes S_j, $1 \leq j \leq n-1$, were oscillatory. If this result had been true, the question of dominance would have been virtually answered by Theorem 5.6 alone for Equation (6.3) with n even. However, Jones [50] in 1980 was able to produce counterexamples to Nehari's result and other related claims. Such an example will be discussed here shortly. Nonetheless, the failure of Nehari's result left unanswered the question of which classes could contain oscillatory solutions, and when.

A sufficient test for the existence of oscillatory solutions for Equation (6.1) was devised by Kim [64]. For notational purposes, it is necessary to define several iterated integrals. For $1 \leq \ell \leq n-1$, and for $a \geq 0$, set

$$\psi_{\ell\ell}(t) = \int_a^t \rho_\ell^{-1}(s)\,ds, \qquad \psi_{k\ell}(t) \equiv 1 \text{ if } k < \ell,$$

$$\psi_{k\ell}(t) = \int_a^t \rho_k^{-1}(s)\psi_{k-1,\ell}(s)\,ds, \qquad \ell+1 \leq k \leq n-1,$$

$$\phi_{\ell\ell}(t) = \int_a^t \rho_\ell^{-1}(s)\,ds, \qquad \phi_{k\ell}(t) \equiv 1 \text{ if } k > \ell,$$

$$\phi_{k\ell}(t) = \int_a^t \rho_k^{-1}(s)\,\phi_{k+1,\ell}(s)\,ds, \qquad 1 \le k \le \ell - 1,$$

and for $1 \le k \le n-1$ define

$$H_k \equiv \int_a^\infty \psi_{n-1,k+1}(t)\,\phi_{1,k-1}(t)\,\rho_n^{-1}(t)\,\rho_0^{-1}(t)\,|p(t)|\,dt.$$

Kim [64] proved that if $H_k = \infty$, then the class S_k is oscillatory. Unfortunately, the condition $H_k = \infty$ can be difficult to verify, and the complexity of the integral in the general case for Equation (6.1) makes it unclear just how strong a result this is. As an example of the application of the test, consider the equation

$$x(x(x(xy')')')' - y = 0,$$

for $x \in [1,\infty)$. The admissible classes for this equation are S_0, S_2, and S_4. For the fourth order equation the classes S_0 and S_4 are nonoscillatory, and so only the class S_2 has the potential to be oscillatory. In order to test the condition $H_2 = \infty$, the functions $\psi_{3,3}(t)$ and $\phi_{1,1}(t)$ must be computed. Applying the definitions above with $a = 1$ gives $\psi_{3,3}(t) = \phi_{1,1}(t) = \ell n t$, $t > 1$, and so

$$H_2 = \int_0^\infty \ell n^2 t/t\,dt = \lim_{b\to\infty} (\ell n b)^3/3 = \infty.$$

Hence the class S_2 is oscillatory; in fact $\sin \ell n x \in S_2$, which does oscillate as predicted.

When expressed in terms of Equation (6.3), the conditions $H_k = \infty$, $k = 1, \ldots, n-1$, are all equivalent to the condition

$$\int_a^\infty t^{n-2}|p(t)|\,dt = \infty. \tag{6.14}$$

Thus for the problem (6.3) this result is not sensitive enough to detect the oscillation of one class S_k when another class S_j is nonoscillatory, $1 \le j,k \le n-1$. When the test (6.14) is successful, however, it does imply that only S_0 and S_n, if these are admissible, are nonoscillatory. (Compare this result to Theorem 5.7.) When $n=2$, Equation (6.14) reduces to the classic test of Wintner [131] and Leighton [74] cited earlier, while for $n=4$ the corresponding test was given by Leighton and Nehari [76]. For arbitrary n, condition (6.14) was known to Anan'eva and Balaganskiǐ [5] and Kondrat'ev [66], while a slightly more general test was given by Kiguradze [58].

Consider the Euler equation

$$y^{(8)} - \lambda x^{-8} y = 0, \tag{6.15}$$

for $x \in [1,\infty)$, with solutions of the form $y = x^{\alpha}$ where α is a root of the equation

$$\alpha(\alpha-1)\ldots(\alpha-7) - \lambda = 0. \tag{6.16}$$

The constant $\lambda > 0$ can be chosen so that Equation (6.16) has six real roots $\alpha_1 < 0$, $1 < \alpha_2 < \alpha_3 < 2$, $4 < \alpha_4 < \alpha_5 < 5$, $6 < \alpha_6$, and two complex conjugate roots with real part 7/2. It follows then that the classes S_0, S_2, S_6, and S_8 are nonoscillatory, while S_4 is oscillatory, so that the test (6.14) fails for Equation (6.15). This example is also the counterexample for the earlier cited claim of Nehari [95].

If now λ is taken so large that Equation (6.16) has only two real roots, then S_k is oscillatory for all admissible k, $1 \le k \le n-1$, and yet Equation (6.14) still fails since the integrand has been changed only by a constant multiple. Thus, even though the test for oscillation devised by Kim is one of the best results of its kind for Equation (6.1), its merit is limited in dealing with Equation (6.3).

The first positive efforts in analyzing the relative location of the oscillatory classes in the list of Elias classes for Equation (6.3) are due to Jones [51], published in 1981. Some of the intermediate results used by Jones were also shown independently by Elias [23], but Jones' conclusions based on these results were quite novel. In essence, he showed that in the ordered list of Elias classes, the nonoscillatory classes are gathered at the beginning and end, with the oscillatory classes collected in the middle. Thus, in discussing the dominance of Equation (6.3), there can be at most two interfaces between oscillatory and nonoscillatory classes.

Jones' result is based on a series of comparison theorems which in turn rely on the following lemma, due in part to Kiguradze [58] and Lazer [72], and also found in Elias [23].

LEMMA 6.3: Suppose for some integer $k \geq 2$, $y \in C^{k+1}[a,b)$ satisfies the conditions

$$y^{(i)}(a) = 0, \quad i = 0, \ldots, k-1, \qquad y^{(k)}(a) = 1, \tag{6.17a}$$

$$y^{(i)}(x) \geq 0, \quad i = 0, \ldots, k, \qquad y^{(k+1)}(x) < 0, \quad x \in (a,b). \tag{6.17b}$$

Then, for $j = 1, 2, \ldots, k-1$,

$$y(x)/y^{(j)}(x) \geq (k-j)!(x-a)^j/k!, \qquad x \in (a,b). \tag{6.18}$$

PROOF: The proof given here is different from that of Reference [51]. Applying Taylor's theorem about the point a to both $y(x)$ and $y^{(j)}(x)$ gives

$$y(x) = y(a) + y'(a)(x-a) + \ldots + y^{(k)}(a)\,\frac{(x-a)^k}{k!}$$

$$+ \int_a^x \frac{(x-t)^k}{k!}\, y^{(k+1)}(t)\,dt,$$

$$y^{(j)}(x) = y^{(j)}(a) + \ldots + y^{(k)}(a)\,\frac{(x-a)^{k-j}}{(k-j)!}$$

$$+ \int_a^x \frac{(x-t)^{k-j}}{(k-j)!}\, y^{(k+1)}(t)\,dt.$$

The conditions at a in (6.17) reduce these to

$$y(x) = \frac{(x-a)^k}{k!} + \int_a^x \frac{(x-t)^k}{k!}\, y^{(k+1)}(t)\,dt,$$

$$y^{(j)}(x) = \frac{(x-a)^{k-j}}{(k-j)!} + \int_a^x \frac{(x-t)^{k-j}}{(k-j)!}\, y^{(k+1)}(t)\,dt.$$

Consequently, the result will follow if it can be shown that

$$\frac{(x-a)^k + \int_a^x (x-t)^k y^{(k+1)}(t)\,dt}{(x-a)^{k-j} + \int_a^x (x-t)^{k-j} y^{(k+1)}(t)\,dt} \geq (x-a)^j\,. \tag{6.19}$$

The sign conditions in (6.17) imply both the numerator and denominator are positive, and so (6.19) is equivalent to

$$\int_a^x (x-t)^k y^{(k+1)}(t)\,dt \geq (x-a)^j \int_a^x (x-t)^{k-j} y^{(k+1)}(t)\,dt. \tag{6.20}$$

For $t \in (a,x)$, then $(x-t)^k \leq (x-a)^j (x-t)^{k-j}$. Since $y^{(k+1)}(t) \leq 0$ on the interval (a,x), then $(x-t)^k y^{(k+1)}(t) \geq (x-a)^j (x-t)^{k-j} y^{(k+1)}(t)$, and thus (6.20) must hold. This in turn proves (6.19), from which (6.18) follows. This completes the proof of Lemma 6.3.

A series of five comparison theorems are required for the proof of Jones' main result. The first, Theorem 6.4, can be deduced from Theorems 3.3 and 3.5 in the paper [95] by Nehari. It will be convenient, however, to prove this result in a different manner here, and thereby introduce a technique used again in Theorem 6.5. After Theorem 6.4, the remaining four comparison theorems establish relationships between classes of nonoscillatory solutions for equations of different orders. By starting with Equation (6.3), stepping down to a lower order equation, and then back up to Equation (6.3), it will be possible to establish relationships between different Elias classes for Equation (6.3). Theorem 6.4 will play an important role in the proof of Theorem 6.6, as well as the major result.

THEOREM 6.4: If Equation (6.3) is eventually $(k, n-k)$ disconjugate, and if $\tilde{p}(x)$ is a continuous nonvanishing function on $[0,\infty)$ such that $\mathrm{sgn}[\tilde{p}(x)] = \mathrm{sgn}[p(x)]$ and $|p(x)| > |\tilde{p}(x)|$, then the equation

$$y^{(n)} + \tilde{p}(x)y = 0 \tag{6.21}$$

is also eventually $(k, n-k)$ disconjugate.

PROOF: Observe that any k, $0 \le k \le n$, satisfying the parity condition for Equation (6.21) also satisfies the parity condition for Equation (6.3). The proof is trivial for $k=0$ or $k=n$ since no nontrivial solution ever has n zeros at one point. Assume, then, that $1 \le k \le n-1$. Note that the hypotheses imply S_k is nonoscillatory by Theorem 4.13.

Let $y \in S_k$ for Equation (6.21). Since y is nonoscillatory, there is an interval $[b,\infty)$ on which y satisfies the conditions

$$y^{(i)}(x) > 0, \qquad i = 0, \ldots, k, \tag{6.22a}$$

$$(-1)^{j-k} y^{(j)}(x) > 0, \qquad j = k, \ldots, n. \tag{6.22b}$$

If $y^{(k)}(x)$ is expanded about some point $a > b$ by Taylor's formula, then

$$y^{(k)}(x) = y^{(k)}(a) + y^{(k+1)}(a)(x-a) + \ldots + \frac{y^{(n-1)}(a)}{(n-k-1)!}(x-a)^{n-k-1}$$

$$+ \int_a^x \frac{(x-t)^{n-k-1}}{(n-k-1)!} y^{(n)}(t)\,dt. \tag{6.23}$$

For $x < a$, by virtue of (6.22), the polynomial terms on the right side of Equation (6.23) are all positive, and so

$$y^{(k)}(x) \geq \int_a^x \frac{(x-t)^{n-k-1}}{(n-k-1)!} y^{(n)}(t)\,dt$$

$$= (-1)^{n-k} \int_x^a \frac{(t-x)^{n-k-1}}{(n-k-1)!} y^{(n)}(t)\,dt. \tag{6.24}$$

In the limit as $a \to \infty$, (6.24) becomes

$$y^{(k)}(x) \geq (-1)^{n-k} \int_x^\infty \frac{(t-x)^{n-k-1}}{(n-k-1)!} y^{(n)}(t)\,dt, \tag{6.25}$$

for $x \in [0,\infty)$. From Equation (6.21) and the parity condition, however,

$$y^{(n)}(t) = (-1)^{n-k}|p(t)|y(t).$$

Inserting this expression in (6.25) yields

$$y^{(k)}(x) \geq \int_x^\infty \frac{(t-x)^{n-k-1}}{(n-k-1)!} |p(t)|y(t)\,dt. \tag{6.26}$$

Now expand $y(x)$ about the point b to obtain

$$y(x) = y(b) + y'(b)(x-b) + \ldots + \frac{y^{(k-1)}(b)}{(k-1)!}(x-b)^{k-1}$$

$$+ \int_b^x \frac{(x-t)^{k-1}}{(k-1)!} y^{(k)}(t)\,dt$$

$$\geq y(b) + \ldots + \frac{y^{(k-1)}(b)}{(k-1)!}(x-b)^{k-1}$$

$$+ \int_b^x \frac{(x-t)^{k-1}}{(k-1)!} \int_t^\infty \frac{(s-t)^{n-k-1}}{(n-k-1)!} |p(s)|y(s)\,ds\,dt. \tag{6.27}$$

Recalling that $|p(x)| \geq |\tilde{p}(x)|$, then finally from (6.27) comes

$$y(x) > y(b) + \ldots + \frac{y^{(k-1)}(b)}{(k-1)!}(x-b)^{k-1} + \int_b^x \frac{(x-t)^{k-1}}{(k-1)!} \int_t^\infty \frac{(s-t)^{n-k-1}}{(n-k-1)!} |\tilde{p}(s)| y(s) \, ds dt. \tag{6.28}$$

Define the sequence of functions $\{y_i(x)\}$ by setting $y_0(x) = y(x)$, and taking

$$y_i(x) = y(b) + y'(b)(x-b) + \ldots + \frac{y^{(k-1)}(b)}{(k-1)!}(x-b)^{k-1} + \int_b^x \frac{(x-t)^{k-1}}{(k-1)!} \int_t^\infty \frac{(s-t)^{n-k-1}}{(n-k-1)!} |\tilde{p}(s)| y_{i-1}(s) \, ds dt.$$

All of the terms on the right side of Equation (6.29) which do not depend on i are nonnegative for $x \geq b$. Since $y(x) \geq y(b) > 0$ on $[b,\infty)$, an induction argument shows $y_i(x) \geq y(b)$ on $[b,\infty)$ as well. Beginning with (6.28), another induction argument reveals $y_i(x) \leq y_{i-1}(x)$, $i = 1,2,\ldots$. Finally, differentiating in Equation (6.29) shows that $y_i^{(j)}(b) = y^{(j)}(b)$, $0 \leq j \leq k-1$, $i \geq 1$. Consequently, the sequence $\{y_i(x)\}$ has a positive limit function $z(x)$ satisfying $z^{(j)}(b) = y^{(j)}(b)$, $0 \leq j \leq k-1$. By the monotone convergence theorem applied to Equation (6.29), $z(x)$ also satisfies the equation

$$z(x) = z(b) + z'(b)(x-b) + \ldots + \frac{z^{(k-1)}(b)}{(k-1)!}(x-b)^{k-1} + \int_b^x \frac{(x-t)^{k-1}}{(k-1)!} \int_t^\infty \frac{(s-t)^{n-k-1}}{(n-k-1)!} |\tilde{p}(s)| z(s) \, ds dt. \tag{6.30}$$

Differentiating in Equation (6.30) and recalling that $z(s) > 0$ for $s \geq b$ on the right gives that

$$z^{(i)}(x) > 0, \qquad i = 0, \ldots, k,$$

$$(-1)^{j-k} z^{(j)}(x) > 0, \qquad j = k, \ldots, n,$$

on $[b,\infty)$, and also that $z(x)$ satisfies Equation (6.21). Therefore, $z \in S_k$ for Equation (6.21), and so Equation (6.21) is eventually $(k,n-k)$ disconjugate. This completes the proof of Theorem 6.4.

Theorem 6.4 provides a very nice generalization of the classical Sturm comparison theorem. If the class S_k is oscillatory for Equation (6.21), this theorem shows that S_k must oscillate for Equation (6.3) as well. More important for the moment is the technique of producing an appropriately structured nonoscillatory solution for one equation from that of another. This technique is used again in Theorem 6.5, which provides the "step back up" to Equation (6.3) described earlier.

<u>THEOREM 6.5</u>: If the class S_k is nonoscillatory for the n-2 order equation

$$y^{(n-2)} + \frac{x^2 p(x)}{(n-k-1)(n-k-2)} y = 0, \tag{6.31}$$

then the class S_k is also nonoscillatory for Equation (6.3), $0 \le k \le n-3$.

<u>PROOF</u>: Since the class S_0 is always nonoscillatory, the proof is trivial for $k = 0$, and so assume $1 \le k \le n-3$. Let y be a solution of Equation (6.31) and assume $y \in S_k$. Since y is nonoscillatory, Theorem 5.5 implies there is an interval $[b,\infty)$ on which y satisfies the conditions

$$y^{(i)}(x) > 0, \qquad i = 0, \ldots, k \tag{6.32a}$$

$$(-1)^{j-k} y^{(j)}(x) > 0, \qquad j = k, \ldots, n-2. \tag{6.32b}$$

Expanding $y^{(k)}(x)$ about the point $a > b$ as before leads to the inequality

$$y^{(k)}(x) \geq (-1)^{n-k-2} \int_x^\infty \frac{(t-x)^{n-k-3}}{(n-k-3)!} y^{(n-2)}(t)\,dt. \tag{6.33}$$

From Equation (6.31) and the parity condition, however,

$$y^{(n-2)}(t) = (-1)^{n-k-2} \frac{t^2|p(t)|}{(n-k-1)(n-k-2)} y(t).$$

Inserting this expression in (6.33) yields

$$y^{(k)}(x) \geq \int_x^\infty \frac{(t-x)^{n-k-3}t^2|p(t)|}{(n-k-1)!} y(t)\,dt$$

$$\geq \int_x^\infty \frac{(t-x)^{n-k-1}|p(t)|}{(n-k-1)!} y(t)\,dt. \tag{6.34}$$

Now $y(x)$ may be expanded about the point b as in Theorem 6.4 to obtain

$$y(x) \geq y(b) + \ldots + \frac{y^{(k-1)}(b)}{(k-1)!}(x-b)^{k-1}$$

$$+ \int_b^x \frac{(x-t)^{k-1}}{(k-1)!} \int_t^\infty \frac{(s-t)^{n-k-1}}{(n-k-1)!} |p(s)|y(s)\,ds\,dt. \tag{6.35}$$

Constructing a sequence and passing to the limit as in Theorem 6.4 produces a positive function $z(x)$ which satisfies the conditions

$$z^{(i)}(x) > 0, \qquad i = 0, \ldots, k,$$

$$(-1)^{j-k} z^{(j)}(x) > 0, \qquad j = k, \ldots, n.$$

on $[b,\infty)$, and which also satisfies Equation (6.3). Therefore, $z \in S_k$ for Equation (6.3) so that S_k is nonoscillatory. This completes the proof of Theorem 6.5.

Notice that the "step back up" from (6.31) to the original equation (6.3) takes place without any change in the k. Jones' theorem describes a relationship between nonoscillatory classes and adjacent classes. To accomplish this, a shift in the k must occur. This shift in fact takes place in the "step down" from Equation (6.3) as described by Theorem 6.6.

THEOREM 6.6: If, for some k, $2 \le k \le n$, Equation (6.3) is $(k, n-k)$ disconjugate on $[0,\infty)$, then the $n-2$ order equation

$$y^{(n-2)} + \frac{x^2 p(x)}{(k)(k-1)} y = 0 \tag{6.36}$$

is eventually $(k-2, n-k)$ disconjugate. In particular, the class S_{k-2} for Equation (6.36) is nonoscillatory.

PROOF: Since Equation (6.3) is $(k, n-k)$ disconjugate on $[0,\infty)$, then there is a solution $y(x) \in S_k$ satisfying the boundary conditions

$$y^{(i)}(0) = 0, \quad i = 0, \ldots, k-1, \qquad y^{(k)}(0) = 1, \tag{6.37}$$

as well as satisfying the inequalities

$$y^{(i)}(x) > 0, \qquad i = 0, \ldots, k, \tag{6.38a}$$

$$(-1)^{j-k} y^{(j)}(x) > 0, \qquad j = k, \ldots, n, \tag{6.38b}$$

on the interval $(0,\infty)$. In fact, the solution $u_k(x)$ defined in Chapter IV satisfies this description. Since

$$y^{(n)} + py = (y'')^{(n-2)} + (py/y'')y'',$$

then the equation

$$z^{(n-2)} + (py/y'')z = 0 \tag{6.39}$$

has a solution $z = y''$ such that

$$z^{(i)}(x) > 0, \qquad i = 0, \ldots, k-2,$$

$$(-1)^{j-k+2} z^{(j)}(x) > 0, \qquad j = k-2, \ldots, n-2,$$

for $x \in (0,\infty)$. Thus Equation (6.39) must be eventually $(k-2, n-k)$ disconjugate.

The function $y(x)$ satisfying Equations (6.37) and (6.38) meets the hypothesis of Lemma 6.3), so that $y(x)/y''(x) \geq x^2/((k)(k-1))$ on $(0,\infty)$. That is,

$$|p(x)y(x)/y''(x)| \geq \left|\frac{x^2 p(x)}{(k)(k-1)}\right|,$$

and then Theorem 6.4 implies that Equation (6.36) is eventually $(k-2, n-k)$ disconjugate. This completes the proof of Theorem 6.6.

When n is even, Equation (6.3) is self-adjoint, and so by Theorem 1.1 if Equation (6.3) is $(k, n-k)$ disconjugate then it is also $(n-k, k)$ disconjugate. Thus a discussion which deals with $(k, n-k)$ disconjugacy for $k \leq n/2$ also applies to $k \geq n/2$. The same is not true for the case where n is odd. In this case the adjoint for Equation (6.3) is the equation

$$y^{(n)} - py = 0. \tag{6.40}$$

It is necessary, then, to develop theorems like the previous two which handle this particular situation. The proofs of Theorems 6.7 and 6.8 are similar to those just completed for Theorem 6.5 and Theorem 6.6, respectively, and so are omitted.

<u>THEOREM 6.7</u>: If the class S_k is nonoscillatory for the $n-1$ order equation

$$y^{(n-1)} + \frac{xp(x)}{n-k-1} y = 0, \tag{6.41}$$

then the class S_k is also nonoscillatory for the adjoint equation (6.40), $0 < k < n-2$.

THEOREM 6.8: If, for some k, $1 \leq k \leq n$, Equation (6.3) is $(k, n-k)$ disconjugate on $[0,\infty)$, then the $n-1$ order equation

$$y^{(n-1)} + \frac{xp(x)}{k} y = 0 \tag{6.42}$$

is eventually $(k-1, n-k)$ disconjugate.

The main theorem of Jones [24] may now be given.

THEOREM 6.9: If the class S_k is nonoscillatory for Equation (6.3), and if $k \leq n/2$, then the class S_j is nonoscillatory for all admissible values $j \leq k$ and for all $j \geq n-k$. If $k \geq n/2$, then the class S_j is nonoscillatory for all $j \geq k$ and for all $j \leq n-k$.

PROOF: It is enough to prove the first half of the theorem, since then the second half would follow by considering the adjoint of Equation (6.3). Suppose $k \leq n/2$ is given, and S_k is nonoscillatory. Choose $y \in S_k$, and let $b \geq 0$ be such that y satisfies the inequalities (6.22) on $[b,\infty)$. Then Equation (6.3) is $(k, n-k)$ disconjugate on $[b,\infty)$. Let $t = x - b$, and set $\tilde{p}(t) = p(t+b)$, so that the equation

$$y^{(n)} + \tilde{p}(t) y = 0 \tag{6.43}$$

is $(k, n-k)$ disconjugate on $[0,\infty)$.

By Theorem 6.6, the equation

$$y^{(n-2)} + \frac{t^2 \tilde{p}(t)}{(k)(k-1)} y = 0 \tag{6.44}$$

is eventually $(k-2, n-k)$ disconjugate. But $k \leq n/2$ implies $k \leq n-k+1$

and $k-1 \leq n-k$, so that by Theorem 6.4 the equation

$$y^{(n-2)} + \frac{t^2\tilde{p}(t)}{(n-k+1)(n-k)} y = 0 \tag{6.45}$$

is eventually $(k-2, n-k)$ disconjugate. Theorem 6.5 then shows that the class S_{k-2} is nonoscillatory for Equation (6.43), and hence for Equation (6.3) as well. Since $j < k$ must satisfy the parity condition in order for S_j to be defined, it follows that S_j is nonoscillatory for all admissible values of j, $j \leq k$.

If n is even, then Equation (6.3) is self-adjoint, and so $(k, n-k)$ disconjugacy is equivalent to $(n-k, k)$ disconjugacy. Thus the class S_k is nonoscillatory exactly when S_{n-k} is nonoscillatory, and therefore if S_j is nonoscillatory for $j \leq k$, then S_j is nonoscillatory for $j \geq n-k$.

If n is odd and S_k is nonoscillatory for $k \leq n/2$, then Equation (6.43) is $(k, n-k)$ disconjugate on $[0,\infty)$. From Theorem 6.8 then, the equation

$$y^{(n-1)} + \frac{t\tilde{p}(t)}{k} y = 0$$

is eventually $(k-1, n-k)$ disconjugate. Since $k \leq n/2$, then $k \leq n-k$, and then by Theorem 6.4 the equation

$$y^{(n-1)} + \frac{t\tilde{p}(t)}{n-k} y = 0$$

is also eventually $(k-1, n-k)$ disconjugate. Now applying Theorem 6.7 shows that the class S_{k-1} is nonoscillatory for the adjoint equation (6.40). From the first part of the proof, the classes S_j for Equation (6.40) are nonoscillatory for all $j \leq k-1$. Since k does not satisfy the parity condition for Equation (6.40), this is equivalent to saying that S_j is nonoscillatory for all admissible values $j \leq k$. Finally, since Equation (6.40) is the adjoint for Equation (6.3) when n is odd, this implies S_j is nonoscillatory for Equation (6.3) for all admissible values of j, $j \geq n-k$. This completes the proof of Theorem 6.9.

The two Euler equations

$$y^{(10)} - \frac{\lambda}{x^{10}} y = 0, \tag{6.46}$$

$$y^{(7)} - \frac{\lambda}{x^{7}} y = 0, \tag{6.47}$$

provide excellent illustrations of the principles given in Theorem 6.9.

Recall that x^α is a solution of Equation (6.46) if α is a root of the characteristic equation

$$f(\alpha) - \lambda = 0 \tag{6.48}$$

where $f(\alpha) = \alpha(\alpha-1) \ldots (\alpha-9)$. Let the points $\beta_1 < \beta_2 < \ldots < \beta_9$ denote the zeros of $f'(\alpha)$. Symmetry and elementary claculus show that $f(\beta_i) = f(\beta_{10-i})$, $i = 1,2, \ldots, 9$, and $|f(\beta_i)| > |f(\beta_{i+1})|$, $i = 1,2,3,4$. For $\lambda > f(\beta_2)$, Equation (6.48) has two real roots and eight complex roots, from which it follows that the classes S_0 and S_{10} are nonoscillatory while S_2, S_4, S_6. and S_8 are oscillatory.

If these classes are considered as set-valued functions of λ, then as λ decreases until $f(\beta_2) > \lambda > f(\beta_4)$, the classes S_2 and S_8 become non-oscillatory, while S_4 and S_6 remain oscillatory. Allowing λ to decrease still further until $f(\beta_4) > \lambda > 0$ causes these two remaining classes to become nonoscillatory. At each step, the division of these classes into oscillatory and nonoscillatory groups satisfies the restrictions of Theorem 6.9. In particular, note that since Equation (6.46) is self-adjoint, the classes change from oscillatory to nonoscillatory in k, n - k pairs. Also note that as λ decreases toward zero, the number of nonoscillatory classes does not decrease, in accordance with Theorem 6.4.

For Equation (6.47), the associated characteristic equation is

$$g(\alpha) - \lambda = 0 \tag{6.49}$$

where $g(\alpha) = \alpha(\alpha - 1) \ldots (\alpha - 6)$. Again let $\beta_1 < \beta_2 < \ldots < \beta_6$ denote the zeros of $g'(\alpha)$. In this case, $g(\beta_i) = -g(\beta_{7-i})$, and $|g(\beta_i)| > |g(\beta_{i+1})|$, $i = 1,2$. For $\lambda > g(\beta_6)$, the class S_0 is nonoscillatory while S_2, S_4, and and S_6 are all oscillatory. As λ decreases successively through $g(\beta_6)$, $g(\beta_2)$, and $g(\beta_4)$, the classes S_6, S_2, and S_4 become nonoscillatory in turn. Again, this is the grouping predicted by Theorem 6.9. Any remaining oscillatory classes are always grouped and centered in the ordered list of Elias classes.

The behavior of the adjoint of Equation (6.47) as discussed above can be observed by allowing λ to pass through negative values. Not surprisingly, S_7 is always nonoscillatory, while S_1, S_5, and S_3 each changes from oscillatory to nonoscillatory as λ approaches zero through negative values.

Perhaps more intriguing is to observe the behavior of Equation (6.46) as λ passes through negative values. Consider the equation

$$y^{(10)} + \frac{\lambda}{x^{10}} y = 0 \tag{6.50}$$

for positive values of λ. If the classes S_1, S_3, . . . , S_9 which arise from Equation (6.50) are listed in order with the classes S_0, S_2, . . . , S_{10} coming from Equation (6.46), then the resulting list S_0, S_1, S_2, S_3, . . . , S_9, S_{10} still satisfies the conclusions of Theorem 6.9 as a result of the ordering of the magnitudes of the extrema of $f(\alpha)$. That this remarkable result holds in general for Equation (6.3) has been shown in a recent paper by Etgen and Taylor [29] for the case n even. In fact, the conclusion follows immediately from the proof of Theorem 6.9 for all values of n.

To be more explicit, suppose the sign of $p(x)$ is chosen so that $(-1)^n p < 0$. Then the class S_k corresponds to Equation (6.3) for all even admissible values of k, while S_k corresponds to Equation (6.40) for all odd admissible values of k. Let these classes form a single listing, $S_0, S_1, \ldots, S_n$.

<u>THEOREM 6.10</u>: If S_k is nonoscillatory for some $k \leq n/2$, then the class S_j is nonoscillatory for all $j \leq k$ and all $j \geq n-k$.

<u>PROOF</u>: The proof of Theorem 6.9 showed that if Equation (6.3) is eventually $(k, n-k)$ disconjugate, then Equation (6.40) is $(k-1, n-k-1)$ disconjugate. Thus, if S_k is nonoscillatory then so is S_{k-1}. It then follows that S_j is nonoscillatory for all $j \leq k$. By considering the adjoints of Equations (6.3) and (6.40), Theorem 1.1 then shows that S_j is nonoscillatory for all $j \geq n-k$.

Results such as Theorem 6.9 and Theorem 6.10 greatly simplify the analysis of the dominance of solutions for Equation (6.3). Theorem 6.9 can be restated as follows:

<u>COROLLARY 6.11</u>: Let $\{u_0, \ldots, u_{n-1}\}$ be the basis of solutions for Equation (6.3) given by Theorem 4.15. If the solution u_k is nonoscillatory for some $k \leq n/2$, the u_j is nonoscillatory for all $j \leq k$. If u_k is nonoscillatory for some $k \geq n/2$, then u_j is nonoscillatory for all $j \geq k$.

Combining this result with the results of Chapter V quickly leads to the strongest dominance result known so far for Equation (6.3):

<u>COROLLARY 6.12</u>: Let $\{u_0, \ldots, u_{n-1}\}$ be the basis of solutions for Equation (6.3) given by Theorem 4.15. For $0 < j < n$, if the class S_j is nonoscillatory, assume without loss of generality that $\lim\limits_{x\to\infty} \frac{u_j(x)}{u_{j-1}(x)} = \infty$. Set $k_0 = \max\{k \leq n/2 \mid u_k$ is nonoscillatory$\}$, and set $k_1 = \min\{k \geq n/2 \mid u_k$ is nonoscillatory$\}$. If $y(x)$ is any oscillatory solution of Equation (6.3), then y dominates u_k at infinity for all $k \leq k_0 - 1$, and y is dominated at infinity and by class at infinity by u_k for all $k \geq k_1 + 1$.

As an example of the use of this corollary, consider Equation (6.47) once again. Choose λ such that $g(\beta_2) > \lambda > g(\beta_4) > 0$, so that the classes S_0, S_2, and S_6 are all nonoscillatory, while S_4 is oscillatory. That is, the solutions u_0, u_1, u_2, u_5, and u_6 are nonoscillatory while u_3 and u_4 both oscillate. In this case, $k_0 = 2$ and $k_1 = 5$, and so Corollary 6.12 claims correctly that u_3 and u_4 dominate u_0 and u_1, while they themselves are dominated by u_6. The corollary does not detect the dominance of u_2 by u_3 and u_4, or the dominance of u_3 and u_4 by u_5. The solutions u_2 and u_5 are the nonoscillatory solutions at the interface between oscillatory and nonoscillatory classes for Equation (6.47). It is for precisely these solutions, that is u_{k_0} and u_{k_1}, that the current theory fails to provide any answers.

As a final example, it is fitting to consider a problem which calls on many of the resources developed so far. For specificity, consider the equation

$$y^{(11)} - p(x)y = 0. \tag{6.51}$$

If it is known that class S_7 of the adjoint for Equation (6.51) is nonoscillatory, then the classes S_8 and S_{10} for Equation (6.51) must also be nonoscillatory. If any solution $y(x)$ satisfies boundary conditions at $x = a$ such that $S(y,a+) \geq 8$, it must then belong in either S_8 or S_{10}, and so be nonoscillatory. If $S(y,a+) = 10$, then $y(x)$ must dominate by class at infinity every solution in classes $S_0, \ldots, S_8$.

CHAPTER VII

SUMMARY

The primary purpose of this thesis has been to develop the theory of the dominance of solutions for the equation

$$L_n y + py = 0 \tag{7.1}$$

within a consistent and unified framework, and to demonstrate how such a theory can be used to determine the character of individual solutions. This development is intended to be self-contained so that a newcomer to the field might move comfortably from this study to the current research literature. At the same time, the intention has been to present the major results together in context so that even those well versed in the subject might find insights here.

The main results are those presented in Chapters IV, V, and VI. In these chapters, it was observed that three major tools simplify the analysis of the solution space of Equation (7.1). First, the parity condition $(-1)^{n-k}p(x) < 0$ brings together the cases n even or odd and p(x) positive or negative that have been considered separately by most authors in the past. Second, the Elias classes S_k form a useful subdivision of the solution space of Equation (7.1) that is most noteworthy for the similarity in behavior of the solutions within each class. Finally, the canonical Trench factorization serves to bring out the differences in the behavior of solutions in distinct Elias classes. Whether these differences are strong enough to allow for the proof of a dominance result such as Conjecture 5.10 remains to be seen.

The three tools of parity condition, Elias classes, and Trench factorization form the framework that was sought for the study of the solutions of Equation (7.1). These tools were applied to the less general equation $y^{(n)} + py = 0$ in Chapter VI, and with the aid of a comparison theorem due to Nehari [95], several stronger results were obtained than had been available for Equation (7.1). The most striking weakness in the dominance theory developed in Chapters V and VI is its inability to deal directly with oscillatory solutions.

Throughout this study, the coefficient $p(x)$ in Equation (7.1) has been taken to be nonvanishing on $[0,\infty)$. Virtually every result given here holds as written if the condition on p is relaxed to permit isolated zeros which are not sign changes. Most of the proofs involved counting arguments, and accounting for the zeros of p would have unnecessarily complicated these proofs.

Notably absent from these pages are the many comparison theorems and integral tests for oscillation for Equation (7.1) and its related equations. In the approximately twenty-five years of its modern era, oscillation theory has grown from the basic Sturm theorems into a field far too broad to be adequately sampled here. A certain single-mindedness of purpose has had to be maintained in order to preserve the intended unity and clarity of presentation. Thus, even though some of the results contained here are major theorems in their own right, their significance has been downplayed to keep the other results in a proper perspective. For example, the comparison theorem, Theorem 6.4, is well known and easily generalized to include all equations of the form (7.1). But such a generalization would have been out of place here in the absence of the other tools required to produce a Jones-type theorem for Equation (7.1). The classic books by Swanson [119], Coppel [15], and Kreith [69] each present their own broad view of the subject, particularly for lower order problems. The more recent book by Reid [104] is an excellent treatment of the Sturmian theory for self-adjoint differential systems. To the best of the author's knowledge, there are no recent compilations which include quasi-derivatives and higher order problems of the kind addressed in this study.

Through the use of the framework of analysis described here, it is now possible to specify some of the questions or problems still to be resolved, or areas which warrant further investigation for Equation (7.1).

<u>PROBLEM 1</u>: If L_n in Equation (7.1) is a Trench factored differential operator defined by Equations (5.2) and (5.6), does the class S_k dominate the class S_j by class at infinity for all admissible values $j < k$?

Recall that the strongly decreasing solution $w(x)$ of Equation (7.1) satisfies $(-1)^j L_j w(x) > 0$, $j = 0, \ldots, n-1$, so that $w(x) \in S_0$. A positive result to Problem 1 would also give an affirmative answer to the next problem.

<u>PROBLEM 2</u>: If L_n in Equation (7.1) is a Trench factored differential operator defined by Equations (5.2) and (5.6), and if $(-1)^n p(x) < 0$, is the strongly decreasing solution $w(x) \in S_0$ essentially unique? That is, is $S_0 \cup \{0\}$ a linear subspace of dimension one?

In the absence of as strong a result as suggested by Problem 1, the relationships among the oscillatory solutions remain a mystery, suggesting a number of questions.

PROBLEM 3: Does there exist an analogue to Theorem 6.9 for Equation (7.1) which isolates the oscillatory classes into a contiguous collection of Elias classes?

PROBLEM 4: Must a linear combination of two oscillatory solutions chosen from distinct Elias classes always oscillate?

Problem 4 is equivalent to asking if, given two solutions such that one is oscillatory and one is nonoscillatory, must one always dominate the other at infinity? An affirmative answer here would permit the construction of a basis of solutions such that the oscillatory or nonoscillatory behavior of linear combinations from the basis depends solely on which coefficients are nonzero, as in Corollary 5.9.

PROBLEM 5: Do the signs of the quasi-derivatives of oscillatory solutions have any structure analogous to Equation (5.8) of Lemma 5.5? Do oscillatory solutions have any growth estimates comparable to Equations (5.9) or (6.8)?

Answers to questions such as these would provide valuable clues as to how to proceed in resolving questions such as Problem 1.

Finally, even though conjugate points and extremal points were used primarily as tools to gain information about the Elias classes, they constitute a subject worthy of separate consideration. The work of Chapter IV suggests the following problems:

PROBLEM 6: Does the conjugate point $\eta_i(a)$ always correspond to some extremal point $\theta_i(a)$?

PROBLEM 7: Does the focal point $\zeta_i(a)$ always correspond to some extremal point $\theta_i(a)$?

PROBLEM 8: Let i be fixed. Among all the possible boundary conditions, for which does $\theta_i(a)$ appear first? In particular, what is the effect on $\theta_i(a)$ as the number of boundary conditions at each endpoint is changed?

Since it is known that extremal points occur at the simple zeros of certain Wronskian determinants, one goal of these problems is to determine whether it is possible to effectively locate conjugate points and focal points using these Wronskians, and if so, how.

One of the continuing goals of the research which has culminated in this study has been to identify the many sources that deal with Equation (7.1) and its variations. As the problem has grown, these sources have come to include disconjugacy and factorization results, Green's functions, nonlinear and retarded equations, and so forth, in addition to dominance and oscillation results. Furthermore, much of the work on equations such as (7.1) has been motivated by lower order examples. The bibliography given here, then, has three purposes beyond providing references to those materials on which this study depends. First and foremost, it is intended to provide a comprehensive guide to the literature on the classification and dominance of the solutions of Equation (7.1), as an aid to those who might wish to work in this area. Second, it includes papers of historical significance which have helped motivate the more modern efforts. Finally, by providing references to a few select works which are only indirectly related to the main ideas of this study, it is intended to serve as a guide for anyone interested in investigating some of the related topics of oscillation theory.

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