SELECTING THE BEST TREATMENT

THROUGH LIKELIHOODS

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in partial fulfillment of the requirements
for the Degree of
DOCTOR OF PHILOSOPHY
July, 1982

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ACKNOWLEDGMENTS

I wish to express my thanks and appreciation to my adviser, Dr. Nitis Mukhopadhyay, for suggesting this topic, and for his valuable advice and encouragement. I also wish to thank my committee chairman, Dr. J. Leroy Folks, and my committee members, Dr. Ronald M. McNew and Dr. Kenneth E. Case, for their helpful suggestions and assistance. I would also like to thank my many friends who helped very much to make my stay in O.S.U. pleasant, enjoyable and useful.

I want to acknowledge the financial support I have received during the past two years from the Department of Statistics, and Ming-Chi Institute of Technology.

Special gratitude is given to my mother, Wei-Lain Chou, my wife, Yu-Tzu Lin, and my two children, Chin-Yu Chou and Chin-Tin Chou for their gracious acceptance of my long absence.

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CHAPTER I

INTRODUCTION

Everyone is constantly faced with the problem of choosing one out of several alternatives. The choice is a decision about which alternative is the "best" (in some well-defined sense). Ranking and selection procedures are statistical techniques suitable for comparing k populations. We assume at the outset that the populations are not all the same and can be ordered in some meaningful way, from worst to best. These selection procedures are designed specifically to identify the best single population, or the best subset of populations, or some subset of populations that includes the best population, or the like.

In the framework of testing of hypotheses, the classical procedure attempts to determine whether all the k parameters $\theta_1, \ldots, \theta_k$ have a common value. Each parameter represents the same type of description, attribute, or response for all populations, but the populations may differ. The classical procedure permits us to decide about the following null hypothesis, sometimes called the "homogeneity hypothesis".

$$H_0: \quad \theta_1=\theta_2=\ldots=\theta_k.$$

The alternative hypothesis, which may be implicit or explicit is that the parameters do not all have the same θ values.

If a test of homogeneity is the primary and final goal of an investigation or experiment, alternative methods of statistical analysis are not needed. However, there are many practical situations where other kinds of information or other goals are of interest. For example, suppose that the null hypothesis of homogeneity is rejected. The investigator is seldom satisfied with terminating with this decision. In particular, he may want (a) to determine which populations differ from which others, and in what direction, (b) to see which populations can be considered best in some well-defined sense of the term "best". In case (a), techniques of multiple comparisons or simultaneous inferences are frequently appropriate. The method of multiple comparisons may also provide information that is relevant for case (b). But, there is no explicit guarantee that the probability that "the alternative selected is the best alternative" is suitably large. Ranking and selection procedures are designed to accomplish this goal.

When the goal is to select the one best population out of k populations, a test of homogeneity of all k populations is really inadequate. The test of homogeneity can only tell us whether or not the parameters are equivalent; this test is not set up to resolve the problem of choosing the single best. Although some modifications and extensions of the test of homogeneity have been formulated to provide further information, no modification can be appropriate if we assume at the outset that for any two different treatments, differences in parameters must surely exist. Moreover, if we must make a choice among the k populations, the conclusion corresponding to the null hypothesis H_0, namely that all k populations have the same parameter value, is neither realistic nor useful. The ranking and selection procedures have been designed specifically to resolve such practical problems.

Procedures for selection and ranking started to develop through the pioneering work of R. E. Bechhofer (1954) (assuming normality and known variances). During the next 28 years such procedures have been developed for more complex and more realistic settings. These studies can be grouped into one of the two fundamental approaches, namely, (1) Indifference Zone Approach of Bechhofer (1954), (2) Subset Selection Approach of Gupta (1956). These areas are both vast and rich for pursuing research work. Four published books authored by Bechhofer et al. (1968), Gibbons et al. (1977), Gupta and Panchapakesan (1979), Gupta and Huang (1981) will undoubtedly prove our claim. The recently published categorized bibliography by Dudewicz and Koo (1981) will be immensely useful. There are very useful discussions in Dudewicz (1976, 1980), Dudewicz and Dalal (1975), Mukhopadhyay (1979, 1980a, 1981a, 1981b).

The area of the usual analysis of variance is very much dependent on the assumption of normality of the parent populations. We will follow this same old path, and assume that we wish to select the "best" population from a set of k (≥ 2) normal populations. The "best" population is defined to be as the one having (i) the smallest variance or (ii) the largest mean. More specifically, in this study we discuss two separate problems: one involving the selection of the smallest normal variance, the other involving the selection of the largest normal mean. In general, the "best" population can, however, be defined in any reasonable way pertinent to the problem. One is referred to Bechhofer et al. (1968) for discussions on these aspects. For the two problems mentioned earlier, we adopt the indifference zone approach with a target value P^* of the probability of correct selection (CS). We show that the proposed sequential procedures for both the problems result in a substantial "saving" in the average sample sizes compared with the corresponding well-known fixed sample size procedures (see Gibbons et al. (1977)). We present simulation results in detail for the cases of two populations as well as of three populations for both the problems. We also study various asymptotic behavior (as $P^* \to 1$) of stopping times involved in our statistical methods in both the problems.

The organization of this thesis is as follows: The relevant literature is reviewed in Chapter II. Chapter III deals with the selection of the smallest normal variance with procedures developed through comparisons of ratios of likelihoods. Chapter IV deals with the selection of the largest mean through procedures developed along the lines of Hall's (1962, 1980) sequential tests and Mukhopadhyay's (1979) modified rules. Chapter V contains a brief summary of the thesis. To make the thesis easy to read, we put some important useful theorems in Appendix A, and the tedius proofs of the main theorems (3.1, 4.1, 4.3 and 4.4) have been deferred to Appendix B.

CHAPTER II

REVIEW OF LITERATURE

There is a considerable amount of literature on the subject of selecting the "best" treatment. For a complete bibliography, one is referred to Dudewicz and Koo (1981). As pointed out in Chapter I, the selection procedures could primarily be classified under one of the two formulations, namely: (1) Indifference Zone Approach and (2) Subset Selection Approach.

2.1. Indifference Zone Approach

Theoretical statistics concerned itself too little with problems in which the basic observations come from several sources or populations until the 1950's. Bechhofer (1954, 1958), Bechhofer and Sobel (1954), and Bechhofer et al. (1954) brought a change in thinking through their pioneering work in ranking and selection. Bechhofer brought this subject to full light of day with a context other than the type described by saying (as in classical ANOVA), "We have k populations, but would like to test the hypothesis that we really only have one." The essential formulation of Bechhofer given in 1954 is as follows:

There exist populations (sources of observations) $\Pi_1, \ldots, \Pi_k$ $(k \geq 2)$ with respective unknown means $\mu_1, \ldots, \mu_k$ for their observations, and a common known variance σ^2; a goal of selecting the population associated with $\mu_{(k)} = \max(\mu_1, \ldots, \mu_k)$, having a probability requirement that Prob (CS) $\geq P^*$, $(1/k < P^* < 1)$ if $\mu_{(k)} - \mu_{(k-1)} \geq \delta^*$ $(\delta^* > 0)$; and a procedure of selecting the population yielding $\bar{X}_{max} = \max(\bar{X}_1, \ldots, \bar{X}_k)$, where $\bar{X}_i$ is the sample mean from the population Π_i, i=1, . . ., k.

Since it does not explicitly seek to control the Prob (CS) at parameter points satisfying $\mu_{(k)} - \mu_{(k-1)} < \delta^*$, this has thus obtained the name of a "zone" where one is "indifferent" to select the best population. Srivastava (1966) applied Chow and Robbins' (1965) sequential theory to various selection and slippage problems and gave a class of "asymptotically efficient" sequential procedures for such problems. Robbins et al. (1968) proposed a sequential procedure for selecting the largest of k normal means with common unknown variance. They had established that the sequential procedure was "asymptotically consistent" and "efficient" (in the sense of Chow and Robbins (1965)) and that the cost of ignorance of σ^2 was of little importance when the sequential procedure was used, for all $0 < \sigma^2 < \infty$ and $\delta^* > 0$. Sobel (1977) gave new results on selecting the best population where "consistency" is measured by smallness of the inter (α, β)-range. Bishop and Gibbons (revision of Bishop (1978)) showed how to apply complete ranking theory (to six New England states), and indicated how the results would be of considerable interest in the insurance industry. Ranking in terms of variability is also covered. Mukhopadhyay (1980) developed a sequential procedure through likelihoods, rather than just deciding through the largest sample mean alone and the procedure was shown to have substantial asymptotic saving in the average sample sizes compared to the known procedures now being used in practice (see Gibbons et al. (1977), section 2.3).

2.2. Subset Selection Approach

The area of subset selection procedures (which is equivalent to the idea of elimination) originated from the basic ideas of Gupta (1956, 1965). No indifference zone is usually brought to bear and the orientation is toward working with data already collected, rather than toward determining a sample size for designing the experiment. Since the methods have different goals, different input, and so on, it is very difficult to make any meaningful comparisons. In Paulson's (1964) paper, sequential procedures are given for selecting the normal population with the largest mean when (a) the k populations have a common known variance or (b) the k populations have a common but unknown variance, so that in each case the probability of making the correct selection exceeds a specified value when the largest mean exceeds all other means by at least a specified amount. Desu and Sobel (1968) had obtained some theorems and tables for the problem of selecting a fixed-size subset of normal populations with a common known variance. Sobel (1969) considered the problem of selecting s populations and asserting that they contain at least one of the t best populations. The original problem of Gupta (1956) has been abstracted and generalized by Deverman and Gupta (1969), and Gupta and Panchapakesan (1972). A procedure that controls the probability of eliminating those populations which are distinctly inferior is treated by Desu (1970), and a similar type of result is also available in Carroll et al. (1975). Two stage procedures for the subset selection problem in the case of normal distributions with unknown (not necessarily equal) variances were given in Dudewicz and Dalal (1975). Lee, in the revised version of Lee (1977), gives a very clear elaboration of the approaches discussed in McDonald (1979) and Gupta and Hsu (1977) for subset selection as well as indifference zone. By modifying the Dudewicz and Dalal's (1975) procedure for problems of selecting the population having the largest mean from k normal populations with unknown variances, Rinott (1978) derived some inequalities and used them to obtain a lower-bound of the probability of correct selection. Those bounds were applied for the determination of the second-stage sample size which was required in order to achieve a prescribed probability of correct selection. Mukhopadhyay (1979) had shown that the procedures of Rinott (1978) was "asymptotically inefficient" in the sense of Chow and Robbins (1965) for all $k \geq 2$. Some two-stage procedures having all the properties of Rinott's procedures, together with the highly desirable property of "asymptotic efficiency" were also proposed in Mukhopadhyay (1979).

It turns out that these two fundamental formulations dominated the whole area of selection and ranking theory. As we pointed out earlier, we will follow Bechhofer's (1954) formulation through the "indifference zone" approach.

CHAPTER III

SELECTING THE SMALLEST NORMAL VARIANCE

3.1. Defining the Problem

Suppose there are ($k \geq 2$) independent normal populations $\Pi_1, \ldots, \Pi_k$ where Π_i is assumed to have the mean μ_i and unknown variance σ_i^2 with $0 < \sigma_i^2 < \infty$, $i=1, \ldots, k$. We follow the usual notation of ordering and write the ordered variances as $\sigma_{(1)}^2 \leq \ldots \leq \sigma_{(k)}^2$. Our goal is to select the population having the variance $\sigma_{(1)}^2$, that is variance. We will refer to such a population as the "best" population. For practical applications where one faces this type of selection problems, one is referred to Chapter 5 of Gibbons et al. (1977) and sections 6.5 and 6.6 of Gupta and Panchapakesan (1979). Once we develop our procedures in the following sections, just by looking at our decision rules it will be clear that these are different from the ones discussed in Hoel (1971) for this particular problem except for R1(2).

We will restrict our attention to the "indifference zone approach" only and follow the formulation originated in Bechhofer (1954). Following the standard notation, we assume that we are given two numbers δ^* and P*, $0 < \delta^* < 1$ and $k^{-1} < P^* < 1$. Let $\Omega(\delta^*) = \{(\sigma_1^2, \ldots, \sigma_k^2, \mu_1, \ldots, \mu_k): \delta^{*-1}\sigma_{(1)}^2 \leq \sigma_{(2)}^2\}$. Naturally, we are assuming $\mu_1, \ldots, \mu_k$ to be unknown. If any of the μ's are known, we will drop them from the parameter vector in $\Omega(\delta^*)$. We wish to propose sequential procedures for selection of the smallest variance $\sigma_{(1)}^2$ such that $P(CS) \geq P^*$ if $(\sigma_1^2,$

$\ldots, \sigma_k^2, \mu_1, \ldots, \mu_k) \varepsilon \Omega(\delta^*)$, where "CS" stands for the correct selection. The configuration $\delta^{*-1}\sigma_{(1)}^2 = \sigma_{(2)}^2 = \ldots = \sigma_{(k)}^2$ is referred to as the least favorable configuration (LFC) or a slippage configuration in this context.

We plan to develop sequential procedures to select the "best" population through likelihoods under the LFC, as developed in Mukhopadhyay (1980a) for a different problem. We also present detail comparisons of our procedure with the existing fixed-sample procedures as discussed in Gibbons et al. (1977) for some values of k. For numerical comparisons we also consider σ^2-values in $\Omega(\delta^*)$ but not in the LFC.

We always take one sample at a time from each population and thus take the same number of samples from each population. We denote $\{X_{i1}, \ldots, X_{in}\}$ as iid random variables from the population Π_i, i=1, . . ., k and $n \geq 2$. Having recorded n observations from each population, we let $\bar{X}_{in} = n^{-1}\sum_{j=1}^{n} X_{ij}$ and $S_{in}^2 = (n-1)^{-1}\sum_{j=1}^{n}(X_{ij} - X_{in})^2$, i=1, . . ., k. Unless otherwise specified we will use this notations throughout and assume all the means $\mu_1, \ldots, \mu_k$ to be completely unknown. The cases where some or all the μ's are known will be addressed separately and the notation will be modified accordingly.

3.2. Likelihood Procedures

As mentioned earlier we will propose sequential procedures in the case of LFC to select the "best" population with $P(CS) \geq P^*$. This problem can be viewed as a multiple hypothesis testing problem of deciding among the k hypotheses, namely, $H_i: \sigma_i^2 = \sigma_{(1)}^2$, i=1, . . ., k. This kind of an approach was also adopted in Mukhopadhyay (1980). Let us define a statistic $\underset{\sim}{T}_n = (T_{1n}, \ldots, T_{k-1n})$ where $T_{i-1n} = S_{in}^2/S_{1n}^2$, i=2, . . .,

k. Also, let $\underset{\sim}{\theta} = (\theta_1, \ldots, \theta_{k-1})$ where $\theta_i = \sigma_1{}^2/\sigma_{i+1}{}^2$, i=1, . . ., k-1. This choice of $\underset{\sim}{T}_n$ seems natural to us because we wish to have ratios of variances, namely, $\theta_1, \ldots, \theta_{k-1}$, as the distance measures. Using the results of Hall et al. (1965), T_{in} can be easily seen to be invariantly sufficient for θ_i, i=1, . . ., k-1, with respect to the group of (non-zero) scale transformations. It is easy to see that $H_1, \ldots, H_k$ can be equivalently rephrased in terms of θ-values (under the LFC) in the following way:

$$H_1:(\theta_i = \delta^* \text{ for all } i=1, \ldots, k-1),$$

$$H_j:(\theta_{j-1} = \delta^{*-1} \text{ and } \theta_i = 1 \text{ for all } i \neq j-1)$$

where j=2, 3, . . .,k. Using the multivariate F-distribution (see page 240 of Johnson and Kotz (1972), we obtain the probability density function (pdf) of $\underset{\sim}{T}_n$ as

$$f(\underset{\sim}{T}_n|\theta) = C(n,k)\prod_{i=1}^{k-1} T_{in}{}^{\frac{1}{2}(n-3)}\prod_{i=1}^{k-1} \theta_i{}^{\frac{1}{2}(n-1)}/$$

$$(1 + \sum_{i=1}^{k-1}\theta_i T_{in})^{\frac{1}{2}k(n-1)} \qquad \ldots (3.1)$$

where $C(n,k) = \{\Gamma\frac{1}{2}(n-1)\}^{-k}\{\Gamma\frac{1}{2}k(n-1)\}$. We will follow the convention that $f(\underset{\sim}{T}_n|\theta)$ is of a specified form whenever $T_{in} > 0$ for all i=1, . . ., k-1, while $f(\underset{\sim}{T}_n|\theta)$ is zero otherwise. We will maintain this throughout with the understanding that the likelihood ratios are computed when T_{1n}, . . ., T_{k-1n} are all positive. Writing f_{jn} as the likelihood of $\underset{\sim}{T}_n$ under H_j, we obtain from (3.1) the following expression for $f_{j\,n}$ for j=1, . . ., k:

$$f_{jn} = C(n,k)\delta^{*\frac{1}{2}(k-1)(n-1)}\prod_{i=1}^{k-1}T_{in}^{\frac{1}{2}(n-3)}/(1 + \delta^{*}\sum_{i=1}^{k-1}T_{in})^{\frac{1}{2}k(n-1)} \text{ for } j=1,$$

$$= C(n,k)\delta^{*-\frac{1}{2}(n-1)}\prod_{i=1}^{k-1} T_{in}^{\frac{1}{2}(n-3)}/\ (1 + \sum_{i=1,\neq j-1}^{k-1}T_{1n} + \delta^{*-1}T_{j-1n})^{\frac{1}{2}k(n-1)} \text{ for}$$

$j = 2, \ldots, k$. Letting the constants $C_{pq} = 1$ if $p=q$, $C_{pq} = \delta^*$ if $p \neq q$, for $p, q = 1, \ldots, k$. It is easy to see that $f_{jn}/f_{in} = Y_{ijn}^{\frac{1}{2}k(n-1)}$, where $Y_{ijn} = (C_{j1} + C_{j2}T_{1n} + \ldots + C_{jk}T_{k-1n}) \cdot (C_{i1} + C_{i2}T_{1n} + \ldots + C_{ik}T_{k-1n})^{-1}$ for all $i, j = 1, \ldots k$.

Being motivated by Khan's (1973) results, we choose a doubly indexed sequence of constants $a_{ij} = (k-1)(1-P^*)^{-1}$ for all $i \neq j = 1, \ldots, k$. Following the sequential rules of Mukhopadhyay (1980a), we now define the stopping rule in the present context as follows:

$$R1(k): N = \inf\{n \geq 2: \sup_{j\neq i}(a_{ij}f_{jn}/f_{in}) \leq 1 \text{ for some } i\},$$
$$= \infty \text{ if no such } n.$$

When N stops with i, we decide for the hypothesis H_i, that is, we declare that Π_i has the smallest variance, $i=1, \ldots, k$.

One major valid question is whether R1(k) is a bonafide stopping rule, that is whether $P(N < \infty|H_i) = 1$ for all $i = 1, \ldots, k$. From equation (4.2) of Khan (1973) it is obvious that $P(N < \infty|H_i) = 1$ if

$$P\{\lim_{n\to\infty} \inf \sup_{j\neq i} (f_{jn}/f_{in}) = 0|H_i\} = 1, \qquad \ldots (3.3)$$

for all $i = 1, \ldots, k$. Now $P\{\lim_{n\to\infty} \inf \sup_{j\neq i} (f_{jn}/f_{in}) = 0|H_i\}$

$$= P\{\lim_{n\to\infty} \inf(n-1)\sup_{j\neq i}\ln(Y_{ijn}) = -\infty|H_i\}. \qquad \ldots (3.4)$$

Using the strong law of large numbers, as $n\to\infty$, $\sup_{j\neq i} \ln(Y_{ijn})$ converges almost surely (a.s.) to $\ln\{k\delta^*\{k\delta^* + (1-\delta^*)^2\}^{-1}\}$ under H_i, $i = 1, \ldots, k$. Notice that this limiting value is negative and thus the probability in (3.4) turns out to be one. This verifies the sufficient condition (3.3). So, indeed the stopping variable N of the rule R1(k) is finite with probability one under any of the hypothesis H_i, i=1, . . ., k.

Remark 3.1. Suppose we are also interested in examining the termination property in the case $\sigma_{(1)}^2 < \sigma_{(2)}^2$. To be specific, without any loss of generality, let us assume that $\sigma_{(i)}^2 = \sigma_i^2$, i=1, . . ., k. It can be easily verified that for $\chi=1+\delta^*(\sigma_2^2 + \sigma_3^2 + \ldots + \sigma_k^2)\sigma_1^{-2}$,

$$\lim_{n\to\infty} \ln(Y_{ijn}) = \ln\{\chi[\chi + (1 - \delta^*)(\sigma_2^2\sigma_1^{-2} - 1)]^{-1}\} \quad \ldots (3.5)$$

almost surely. Since the limit in (3.5) is negative, the sufficient condition (3.3) still holds.

Following Khan (1973), it is straight-forward to see that $P(CS|H_i) \geq P^*$ for i=1, . . ., k, since $\sum_{i\neq j} a_{ij}^{-1} = 1 - P^*$ for every j=1, . . ., k. We may stress that this is an exact result.

Although N is finite with probability one under any H_i, i=1, . . ., k, it may be necessary to truncate the rule R1(k) at some stage for practical purposes. We propose the following truncated version:

R1*(k): We take one sample at a time from each population (after we start with two samples from each) and continue checking with the rule R1(k) if we can stop. When we reach the stage n=m we terminate sampling regardless of R1(k). We decide for the population Π_ℓ as being the "best", where $\sup_{j\neq\ell}\{a_{\ell j}f_{jm}/f_{\ell m}\} = \min_i \sup_{j\neq i}\{a_{ij}f_{jm}/f_{im}\}$.

This seems to be the natural way of truncation of R1(k) along the lines of Wald's (1947) procedures.

Remark 3.2: When the rule R1(k) tells to stop, the rule indeed selects that population which has the smallest sample variance at the stopping stage. To justify this remark, suppose i=1, and we have for

$j=2$, $a_{12}f_{2n}/f_{1n} = a_{12}\{(C_{11} + C_{12}T_{1n} + \cdot\cdot\cdot + C_{1k}T_{(k-1)n})(C_{21} + C_{22}T_{1n} + \cdot\cdot\cdot + C_{2k}T_{(k-1)n})^{-1}\}^{\frac{1}{2}k(n-1)}$,

$j=3$, $a_{13}f_{3n}/f_{1n} = a_{13}\{(C_{11} + C_{12}T_{1n} + \cdot\cdot\cdot + C_{1k}T_{(k-1)n})(C_{31} + C_{32}T_{1n} + \cdot\cdot\cdot + C_{3k}T_{(k-1)n})^{-1}\}^{\frac{1}{2}k(n-1)}$,

$\vdots$

$j=k$, $a_{1k}f_{kn}/f_{1n} = a_{1k}\{(C_{11} + C_{12}T_{1n} + \cdot\cdot\cdot + C_{1k}T_{(k-1)n})(C_{k1} + C_{k2}T_{1n} + \cdot\cdot\cdot + C_{kk}T_{(k-1)n})^{-1}\}^{\frac{1}{2}k(n-1)}$.

Suppose we accept H_1 and thus let $a_{12}f_{2n}/f_{1n}$ be the $\sup_{j\neq 1}(a_{ij}f_{jn}/f_{1n})$. Then $(k-1)(1-p^*)^{-1}f_{2n}f_{1n}^{-1} \leq 1$ and $0 < P^* < 1$ imply

$$1 + \delta^*T_{1n} + \cdot\cdot\cdot + \delta^*T_{(k-1)n} \leq \delta^* + T_{1n} + \delta^*T_{2n} + \cdot\cdot\cdot + \delta^*T_{(k-1)n},$$

which further implies $(1 - \delta^*) \leq (1 - \delta^*)T_{1n}$. Since $0 < \delta^* < 1$, we then obtain $T_{1n} \geq 1$.

By the property of supremum, namely, $a_{12}f_{2n}/f_{1n}$, we have

$T_{jn} \geq T_{1n}$, $j=3, \ldots, k$, which implies that

S_{1n}^2 is the smallest variance.

The other cases (i.e., $i=2, 3, \ldots, k$) can be verified similarly.∇

<u>Remark 3.3</u>. In the case when all the μ's are known, we will redefine $S_{in}^2 = n^{-1}\sum_{j=1}^{n}(X_{ij} - \mu_i)^2$, $i=1, \ldots, k$, and take $T_{i-1\ n} = S_{in}^2/S_{1n}^2$, $i=2, \ldots, k$. Then, the likelihood ratio in (3.2) will have the same form with exponent $-kn/2$. The rules R1(k) and R1*(k) will change very little, while all their properties will carry over in this situation.

3.3. The Special Case of Two Populations

In this special case, the rule R1(k) takes the following form:

$$\text{R1(2)}: \quad N = \inf\{n\geq 2: \{(\delta^* + T_{1n})/(1 + \delta^* T_{1n})\}^{-n+1} \notin I(P^*)\}, \qquad \ldots (3.6)$$

$$= \infty \text{ if no such } n,$$

where $I(P^*)$ is the interval $\left((1-P^*), (1-P^*)^{-1}\right)$. At stage N, we accept H_1 or H_2 according as the lower or the upper boundary is crossed.

Notice that the form in (3.6) can also be stated equivalently as

$$N(P^*) = \inf\{n\geq 2: n-1 \geq -\ln(1-P^*)\left|\ln\left((1+\delta^* T_{1n})/(\delta^*+T_{1n})\right)\right|^{-1}\} \qquad \ldots (3.7)$$

Now, we have the following theorem summarizing the asymptotic properties of the rule in (3.7). The numbers C and D are defined as follows:

$$C = -\{\ln(1-P^*)\}/\ln\{(1 + \delta^{*2})/2\delta^*\}, \qquad \ldots (3.8)$$

$$D^2 = \tfrac{1}{2}\{-\ln(1-P^*)\}\{\ln\left((1 + \delta^{*2})/2\delta^*\right)\}^{-3}. \qquad \ldots (3.9)$$

<u>Theorem 3.1</u>. <u>For fixed</u> μ_1, μ_2 <u>in</u> $(-\infty,\infty)$ <u>and</u> σ_1, σ_2 <u>in</u> $(0,\infty)$, <u>for either hypothesis</u> H_1 or H_2 <u>we have for the rule in</u>(3.6):

(i) N is a non-decreasing function of P^*, $N\to\infty$ a.s. as $P^*\to 1$,

$$N/C \to 1 \text{ a.s. as } P^* \to 1. \qquad \ldots (3.10)$$

(ii) $$(N - C)/D \xrightarrow{\mathcal{L}} N(0,1) \text{ as } P^* \to 1. \qquad \ldots (3.11)$$

Proof: see Appendix B (p. 90).

<u>Remark 3.4</u>. At this stage we could not prove (or disprove) that NC^{-1} is uniformly integrable. So, although $NC^{-1} \to 1$ a.s. and $(N - C)/D$

$\xrightarrow{\mathcal{L}}$ N(0, 1) as $P^* \to 1$, we are unable to conclude that $E(NC^{-1})$ converges (or does not converge) to 1 as $P^* \to 1$. However, we conjecture that E(N) is finite for all fixed $P^* \varepsilon(\frac{1}{2}, 1)$.

Remark 3.5. Even if Π_1 and Π_2 are not normal, theorem 3.1 still holds with $D = 4^{-1}(\beta_4 - 1)\{-\ell n(1-P^*)\}\{\ell n((1+\delta^{*2})/2\delta^*)\}^{-3}$, where $\beta_4 = \{\sigma_2^{-4} \cdot E(X_{21} - \mu_2)^4\}$, and it is assumed that $1 < \beta_4 < \infty$. This modification can easily be verified along the lines of Ghosh and Mukhopadhyay (1975). Such comments are also valid for theorem 3.2 in subsection 3.3.2.

Remark 3.6. It is obvious that Hoel's (1971) procedure when restricted to the case k = 2 coincides with our procedure R1(2). However, Hoel's (1971) procedure and our procedure R1(k) do not match for $k \geq 3$. One main reason is that Hoel (1971) developed his procedure through elimination of "inferior" populations, while in our procedure R1(k) we do not capitalize on "elimination" at all. Another difference is that we look at $(T_{1n}, \ldots, T_{k-1n})$ all together through $f(\underset{\sim}{T}_n | \underset{\sim}{\theta})$, while in Hoel (1971) the comparisons are made in pairs. It seems that our procedure R1(k) together with some kind of improvised "elimination" as in Hoel (1971), would have considerably improved performances over Hoel's (1971) procedure. This point is, however, presently under further study.

3.3.1. Moderate Sample Size Behavior of R1(2) and Comparison With Fixed Sample Procedures

We are going to use the rule R1(2) and compare with the fixed sample rule (FSR) as given in Gibbons et al. (1977), Chapter 5. We look at Table G.1 of the same book. For each Δ^* and P^*, we compute $\delta^* = \Delta^{*2}$ (where Δ^* comes from the Table G.1) and generate two populations Π_1 and

Π_2 in an IBM 370/168 computer system for simulation purposes.

We used Subroutine RANDU to generate uniform variates in (0, 1), e.g. look at p. 77 of the IBM application program (1970). We then used SLAM random sampling procedures discussed on pp. 565-566 of Pritsker and Pegden (1979) to obtain samples from a standard normal distribution. We generate Π_1 as N(0, 1) and Π_2 as $N(0, \delta^{*-1})$ so that the hypothesis H_1 is deliberately made to be true. For each pair of values (Δ*, P*) we repeat the experiment using the rule R1*(2), while truncation point m is taken to be n* = n* (Δ*, P*) which is the sample size needed by the FSR. Notice that n* = ν + 1 where ν is the quantity coming from Table G.1 of Gibbons et al. (1977).

For each entry of (Δ*, P*), we repeat the experiment 200 times using the rule R1*(2). In Table I, under the "untruncated part" we compute the average sample size $\bar{N}$, its standard error $S(\bar{N})$ and P, the proportion of correctly deciding for H_1 for all the repetitions (out of 200) which did not have to be truncated; under the heading "truncated" we report T, the number of truncations and P', the proportion (out of T truncations) of correctly deciding for H_1; under the "over all" category we report $\bar{N}$, $S(\bar{N})$, and P" computed from all the 200 repetitions; under the "asymptotic" category we provide with C and $D(200)^{-\frac{1}{2}} = D'$, say.

We compute the "overall saving η" in the following way. η = (n*p" - $\bar{N}$P*)/n*P", where n* is the sample size needed by the FSR and $\bar{N}$ is the "over all" average sample size. We should stress that all the entries in columns four and beyond are estimated from simulated experiments.

In Table II, all the notations remain the same as explained above. However, for <u>each</u> (Δ*, P*) we report two rows - the first row is the result when we use R1*(2) with m = n*, and the second row is the same

TABLE I

SIMULATION RESULT FOR THE RULE R1(2), BOTH THE MEANS UNKNOWN AND TRUNCATION AT n*

			Untruncated part			Truncated		Over all				Asymptotic	
Δ*	P*	n*	$\bar{N}$	$S(\bar{N})$	P	T	P'	P''	$\bar{N}$	$S(\bar{N})$	η	C	D'
0.50	0.95	8	5.01	0.09	0.953	28	0.786	0.930	5.43	0.11	0.31	3.97	0.13
	0.99	14	7.55	0.17	0.968	13	0.692	0.950	7.97	0.20	0.41	6.11	0.16
0.60	0.95	13	7.19	0.17	0.910	23	0.739	0.890	7.86	0.20	0.35	6.65	0.29
	0.99	24	11.83	0.33	0.995	13	1.000	0.995	12.62	0.37	0.48	10.23	0.36
0.70	0.95	24	12.43	0.39	0.944	23	0.783	0.925	13.76	0.43	0.41	12.73	0.76
	0.99	45	20.55	0.66	0.984	15	0.867	0.975	22.38	0.76	0.50	19.56	0.94
0.75	0.95	35	16.94	0.58	0.959	30	0.733	0.925	19.65	0.67	0.42	19.07	1.39
	0.99	67	28.30	0.92	0.983	19	0.789	0.965	31.98	1.16	0.51	29.31	1.72
0.80	0.95	56	25.35	0.88	0.966	26	0.731	0.935	29.34	1.06	0.47	31.06	2.89
	0.99	110	45.36	1.60	0.989	12	0.833	0.980	49.34	1.85	0.55	47.75	3.58
0.85	0.95	104	48.31	1.80	0.907	28	0.750	0.885	56.11	2.07	0.42	57.70	7.32
	0.99	206	85.29	3.07	0.984	14	0.929	0.980	93.74	3.59	0.54	88.70	9.07
0.90	0.95	245	107.46	4.08	0.940	17	0.824	0.930	119.15	4.62	0.50	135.93	26.45
	0.99	489	190.67	6.87	1.000	7	0.571	0.985	201.11	7.68	0.58	208.96	32.80
0.95	0.95	1030	433.75	16.43	0.967	18	0.833	0.955	487.41	19.23	0.53	570.36	227.35
	0.99	2058	797.18	31.67	0.979	4	1.000	0.980	822.40	33.46	0.60	876.78	281.88

TABLE II

SIMULATION RESULT FOR THE RULE R1(2), BOTH THE MEANS UNKNOWN AND TRUNCATION AT n* AND 2n*

			Untruncated part			Truncated		Over all				Asymptotic	
Δ*	P*	n*	$\bar{N}$	$S(\bar{N})$	P	T	P'	P"	$\bar{N}$	$S(\bar{N})$	n	C	D'
0.50	0.75	3	2.99	0.01	0.725	69	0.565	0.670	3.00	0.01	-0.12	1.84	0.09
		6	3.51	0.06	0.771	8	0.625	0.765	3.49	0.06	-0.14		
	0.90	6	3.95	0.07	0.810	21	0.762	0.805	4.17	0.08	0.22	3.06	0.12
		12	4.42	0.12	0.829	1	1.000	0.830	4.43	0.12	0.20		
0.60	0.75	4	3.25	0.03	0.710	38	0.526	0.675	3.40	0.04	0.06	3.08	0.20
		8	3.65	0.07	0.713	5	0.800	0.715	3.66	0.07	0.04		
	0.90	9	5.49	0.12	0.890	37	0.649	0.845	6.14	0.14	0.27	5.11	0.25
		18	6.60	0.22	0.900	5	1.000	0.900	6.66	0.21	0.26		
0.70	0.75	6	4.24	0.09	0.787	64	0.531	0.705	4.81	0.08	0.15	5.89	0.52
		12	5.46	0.17	0.782	12	0.583	0.770	5.49	0.16	0.11		
	0.90	15	8.76	0.23	0.903	46	0.739	0.865	10.20	0.26	0.29	9.78	0.66
		30	10.92	0.40	0.905	10	0.800	0.900	11.12	0.38	0.26		
0.75	0.75	8	5.40	0.12	0.721	64	0.531	0.660	6.24	0.12	0.11	8.82	0.95
		16	7.05	0.23	0.741	15	0.533	0.725	7.12	0.22	0.08		
	0.90	22	11.56	0.36	0.906	40	0.750	0.875	13.65	0.41	0.36	14.66	1.22
		44	14.96	0.62	0.913	5	0.800	0.910	15.14	0.61	0.30		
0.80	0.75	11	7.32	0.17	0.798	76	0.539	0.700	8.72	0.16	0.15	14.37	1.97
		22	9.43	0.33	0.796	33	0.515	0.750	9.69	0.28	0.12		
	0.90	35	17.33	0.58	0.910	34	0.647	0.865	20.34	0.67	0.40	23.88	2.53
		70	22.41	0.98	0.915	1	1.000	0.915	22.47	0.98	0.37		
0.85	0.75	18	10.98	0.33	0.765	81	0.605	0.700	13.82	0.31	0.22	26.70	4.98
		36	16.11	0.63	0.771	21	0.619	0.755	16.31	0.56	0.15		
	0.90	64	28.85	1.06	0.923	45	0.778	0.890	36.76	1.33	0.42	44.35	6.41
		128	39.99	1.83	0.923	4	0.500	0.915	40.47	1.81	0.38		
0.90	0.75	43	23.88	0.91	0.777	79	0.645	0.725	31.43	0.86	0.24	62.90	17.99
		86	33.37	1.42	0.804	32	0.500	0.755	34.91	1.22	0.19		
	0.90	149	71.05	2.89	0.938	39	0.794	0.910	86.25	3.19	0.43	104.48	23.19
		298	91.44	4.29	0.943	8	0.750	0.935	93.74	4.20	0.39		
0.95	0.75	174	97.06	3.71	0.754	74	0.662	0.720	125.53	3.52	0.25	263.94	154.66
		348	146.74	6.23	0.790	14	0.857	0.795	148.64	5.82	0.19		
	0.90	626	259.84	11.33	0.898	34	0.617	0.850	322.08	13.55	0.46	438.39	199.32
		1252	354.66	18.58	0.889	2	0.500	0.885	357.38	18.50	0.42		

thing with $m = 2n^*$, where it is particularly important to note that the second row is obtained from all those sequences in the first row which did not terminate by itself within n^* samples but had to go somewhere between $n^* + 1$ and $2n^*$ (including truncation) for stopping. Unless otherwise specified, for each (Δ^*, P^*) we repeat the experiment 200 times. Table II is needed particularly because for $P^* = .75$ or $.90$, the results do not look very impressive when we truncate at $m = n^*$, while the performance improves considerably when we choose $m = 2n^*$.

In Table III, we present results for each (Δ^*, P^*) without truncation, while the repetitions for each entry have been 1000. Thus, $\bar{N}$, $S(\bar{N})$ and P'' are computed from all the repetitions for each entry of (Δ^*, P^*).

In Table IV, we present results for the case when the rule $R1^*(2)$ is used with $m = n^*$, but actually σ_1^2, σ_2^2 are not in the LFC. We generate $\Pi_1 \sim N(0,1)$ and $\Pi_2 \sim N(0, r/\delta^*)$ where $r = 1.1, 1.3, 1.5, 2.0$, and 2.5; however, the rule $R1^*(2)$ is used without any change at all.

Remark 3.7. In Tables I and II we see that on the average the percentage of saving increases when Δ^* or P^* increases. For instance, when $\Delta^* = .6$, $P^* = .95$, the saving $\eta = .35$, while for $\Delta^* = .6$, $P^* = .99$ saving $\eta = .48$; or for $\Delta^* = .75$, $P^* = .95$ the saving $\eta = .42$. The proportion of overall correct decision, namely P'', on the average is lower than the proportion P of untruncated part - this is due to the low proportions of truncation (on the average) at $m = n^*$. In Table II, one may note that when we increased m to $2n^*$, we get increments of P'' at the expense of losing some saving. This feature can also be seen by comparing Tables I and II with Table III.

TABLE III

SIMULATION RESULT FOR THE RULE R1(2), BOTH THE MEANS UNKNOWN AND WITHOUT TRUNCATION WITH ONE THOUSAND REPETITIONS

Δ*	P*	n*	$\bar{N}$	$S(\bar{N})$	P"	η	C	D'
0.50	0.75	3	3.59	0.04	0.766	-0.17	1.84	0.04
	0.90	6	4.42	0.06	0.838	0.21	3.06	0.05
	0.95	8	6.13	0.09	0.933	0.22	3.97	0.06
	0.99	14	8.37	0.11	0.985	0.40	6.11	0.07
0.60	0.75	4	4.30	0.07	0.770	-0.05	3.08	0.09
	0.90	9	7.04	0.13	0.881	0.20	5.11	0.11
	0.95	13	8.65	0.14	0.937	0.33	6.65	0.13
	0.99	24	13.07	0.21	0.984	0.45	10.23	0.16
0.70	0.75	6	6.34	0.13	0.768	-0.03	5.89	0.23
	0.90	15	11.28	0.24	0.891	0.24	9.78	0.30
	0.95	24	14.66	0.28	0.949	0.39	12.73	0.34
	0.99	45	22.90	0.40	0.990	0.49	19.56	0.42
0.75	0.75	8	8.40	0.19	0.777	-0.01	8.82	0.42
	0.90	22	15.83	0.35	0.904	0.28	14.66	0.54
	0.95	35	20.59	0.45	0.935	0.40	19.07	0.62
	0.99	67	33.05	0.61	0.979	0.50	29.31	0.77
0.80	0.75	11	12.33	0.27	0.794	-0.06	14.37	0.88
	0.90	35	23.49	0.50	0.895	0.33	23.88	1.13
	0.95	56	32.07	0.67	0.943	0.42	31.06	1.29
	0.99	110	53.00	1.10	0.986	0.52	47.75	1.60
0.85	0.75	18	20.34	0.48	0.781	-0.03	26.70	2.23
	0.90	64	43.66	1.05	0.900	0.32	44.35	2.87
	0.95	104	56.86	1.24	0.950	0.45	57.70	3.27
	0.99	206	91.80	1.88	0.990	0.55	88.70	4.06
0.90	0.75	43	43.22	1.06	0.795	0.05	62.90	8.05
	0.90	149	93.04	2.15	0.902	0.38	104.48	10.37
	0.95	245	129.65	3.09	0.936	0.46	139.93	11.83
	0.99	489	208.71	4.23	0.980	0.57	208.96	14.67
0.95	0.75	174	165.75	4.08	0.786	0.09	263.94	69.16
	0.90	626	363.25	8.56	0.898	0.42	438.39	89.14
	0.95	1030	499.12	11.49	0.948	0.51	570.36	101.67
	0.99	2058	902.08	18.96	0.983	0.56	876.78	126.06

TABLE IV

SIMULATION RESULT FOR THE RULE R1(2), BOTH THE MEANS UNKNOWN AND TRUNCATION AT n*: THE PARAMETERS BETTER THAN LFC

				Untruncated part			Truncated		Over all			
Δ*	P*	n*	r	$\bar{N}$	$S(\bar{N})$	P	T	P'	P"	$\bar{N}$	$S(\bar{N})$	η
0.50	0.75	3	1.1	3.00	0.00	0.791	66	0.530	0.705	3.00	0.00	-0.06
			1.3	3.00	0.00	0.800	60	0.567	0.730	3.00	0.00	-0.03
			1.5	3.00	0.00	0.817	58	0.638	0.765	3.00	0.00	0.02
			2.0	3.00	0.00	0.848	55	0.673	0.800	3.00	0.00	0.06
			2.5	3.00	0.00	0.878	53	0.642	0.815	3.00	0.00	0.08
	0.90	6	1.1	3.75	0.07	0.891	26	0.846	0.885	4.04	0.08	0.32
			1.3	3.72	0.07	0.920	24	0.792	0.905	4.00	0.08	0.34
			1.5	3.74	0.07	0.913	16	0.813	0.905	3.92	0.08	0.35
			2.0	3.72	0.06	0.938	6	0.500	0.925	3.79	0.07	0.39
			2.5	3.62	0.06	0.938	6	0.833	0.935	3.70	0.06	0.41
	0.95	8	1.1	5.04	0.09	0.955	23	0.870	0.945	5.38	0.10	0.32
			1.3	5.04	0.09	0.957	16	0.813	0.945	5.28	0.10	0.34
			1.5	4.90	0.08	0.968	12	0.917	0.965	5.09	0.09	0.37
			2.0	4.78	0.07	0.985	4	1.000	0.985	4.85	0.08	0.42
			2.5	4.67	0.07	0.985	2	1.000	0.985	4.71	0.07	0.43
	0.99	14	1.1	7.48	0.18	0.973	14	0.786	0.960	7.94	0.21	0.42
			1.3	7.30	0.17	0.984	9	1.000	0.985	7.61	0.19	0.45
			1.5	7.05	0.15	0.995	5	1.000	0.995	7.23	0.17	0.49
			2.0	6.70	0.14	0.995	1	1.000	0.995	6.74	0.14	0.52
			2.5	6.27	0.11	0.995	0	-----	0.995	6.27	0.11	0.55
0.60	0.75	4	1.1	3.27	0.04	0.718	58	0.655	0.700	3.48	0.04	0.07
			1.3	3.31	0.04	0.760	50	0.680	0.740	3.48	0.04	0.12
			1.5	3.31	0.04	0.769	44	0.773	0.770	3.46	0.04	0.16
			2.0	3.32	0.04	0.811	31	0.677	0.790	3.43	0.04	0.19
			2.5	3.31	0.04	0.828	26	0.654	0.805	3.40	0.04	0.21
	0.90	9	1.1	5.56	0.12	0.921	35	0.629	0.870	6.17	0.14	0.29
			1.3	5.49	0.12	0.932	23	0.478	0.880	5.90	0.13	0.33
			1.5	5.26	0.11	0.943	26	0.654	0.905	5.75	0.13	0.37
			2.0	5.09	0.10	0.957	13	0.923	0.955	5.34	0.11	0.44
			2.5	4.87	0.09	0.963	9	0.889	0.960	5.06	0.10	0.47

TABLE IV (Continued)

				Untruncated part			Truncated		Over all			
Δ*	P*	n*	r	$\bar{N}$	S($\bar{N}$)	P	T	P'	P"	$\bar{N}$	S($\bar{N}$)	η
	0.95	13	1.1	7.01	0.16	0.954	26	0.500	0.895	7.79	0.20	0.36
			1.3	6.53	0.15	0.955	24	0.792	0.935	7.31	0.20	0.43
			1.5	6.70	0.15	0.962	15	0.867	0.955	7.18	0.18	0.45
			2.0	6.68	0.16	0.974	6	0.833	0.970	6.87	0.17	0.48
			2.5	6.08	0.13	0.979	5	0.600	0.970	6.25	0.15	0.53
	0.99	24	1.1	11.56	0.30	0.985	6	1.000	0.985	11.93	0.32	0.50
			1.3	10.42	0.25	0.990	6	0.833	0.985	10.82	0.30	0.55
			1.5	9.98	0.23	0.990	2	1.000	0.990	10.12	0.25	0.58
			2.0	8.98	0.19	0.995	1	1.000	0.995	9.06	0.20	0.62
			2.5	8.41	0.17	1.000	0	-----	1.000	8.41	0.17	0.65
0.70	0.75	6	1.1	4.34	0.08	0.806	61	0.656	0.760	4.85	0.08	0.20
			1.3	4.37	0.08	0.841	55	0.709	0.805	4.82	0.08	0.25
			1.5	4.38	0.08	0.849	48	0.708	0.815	4.77	0.08	0.27
			2.0	4.29	0.08	0.879	35	0.771	0.860	4.59	0.08	0.33
			2.5	4.21	0.07	0.905	32	0.750	0.880	4.50	0.08	0.36
	0.90	15	1.1	8.37	0.22	0.904	23	0.783	0.890	9.14	0.25	0.38
			1.3	7.85	0.21	0.972	24	0.750	0.945	8.71	0.25	0.45
			1.5	7.71	0.20	0.968	15	0.600	0.940	8.26	0.23	0.47
			2.0	7.22	0.16	0.984	8	1.000	0.985	7.54	0.19	0.54
			2.5	6.77	0.14	0.990	4	1.000	0.990	6.94	0.16	0.58
	0.95	24	1.1	12.10	0.35	0.956	17	0.941	0.955	13.12	0.40	0.46
			1.3	10.91	0.33	0.978	14	0.857	0.970	11.83	0.39	0.52
			1.5	10.86	0.31	0.995	5	0.400	0.980	11.36	0.34	0.54
			2.0	9.42	0.24	1.000	2	0.500	0.995	9.57	0.26	0.62
			2.5	8.80	0.20	1.000	1	0.000	0.995	8.88	0.21	0.65
	0.99	45	1.1	18.56	0.51	0.979	7	1.000	0.980	19.49	0.60	0.56
			1.3	16.86	0.48	0.995	3	1.000	0.995	17.29	0.54	0.62
			1.5	14.59	0.38	1.000	3	1.000	1.000	15.05	0.46	0.67
			2.0	12.98	0.30	1.000	1	1.000	1.000	13.14	0.34	0.71
			2.5	12.08	0.28	1.000	0	-----	1.000	12.08	0.28	0.74

TABLE IV (Continued)

				Untruncated part			Truncated		Over all			
Δ*	P*	n*	r	$\bar{N}$	$S(\bar{N})$	P	T	P'	P"	$\bar{N}$	$S(\bar{N})$	η
0.75	0.75	8	1.1	5.53	0.12	0.784	61	0.672	0.750	6.29	0.11	0.21
			1.3	5.45	0.11	0.821	49	0.735	0.800	6.08	0.11	0.29
			1.5	5.40	0.11	0.883	55	0.782	0.855	6.12	0.11	0.33
			2.0	5.24	0.10	0.899	31	0.839	0.890	5.67	0.11	0.40
			2.5	5.26	0.10	0.931	25	0.840	0.920	5.60	0.11	0.43
	0.90	22	1.1	11.03	0.35	0.929	45	0.822	0.905	13.51	0.43	0.39
			1.3	11.44	0.32	0.950	20	0.800	0.935	12.50	0.36	0.45
			1.5	10.42	0.29	0.959	7	1.000	0.960	10.83	0.32	0.54
			2.0	9.77	0.27	1.000	4	0.750	0.995	10.02	0.29	0.59
			2.5	8.77	0.21	1.000	2	1.000	1.000	8.90	0.23	0.64
	0.95	35	1.1	15.89	0.53	0.961	20	0.750	0.940	17.81	0.63	0.49
			1.3	14.78	0.52	0.963	9	0.889	0.960	15.69	0.58	0.55
			1.5	13.73	0.44	0.969	4	0.750	0.965	14.16	0.48	0.60
			2.0	11.86	0.32	0.995	1	1.000	0.995	11.98	0.34	0.67
			2.5	10.72	0.26	0.995	1	1.000	0.995	10.85	0.29	0.70
	0.99	67	1.1	26.30	0.89	1.000	6	1.000	1.000	27.52	0.99	0.59
			1.3	23.66	0.71	1.000	2	1.000	1.000	24.09	0.77	0.64
			1.5	20.58	0.59	1.000	0	-----	1.000	20.58	0.59	0.70
			2.0	17.15	0.40	1.000	0	-----	1.000	17.15	0.40	0.75
			2.5	15.15	0.29	1.000	0	-----	1.000	15.15	0.29	0.78
0.80	0.75	11	1.1	7.39	0.19	0.813	60	0.600	0.749	8.48	0.18	0.23
			1.3	6.94	0.17	0.867	56	0.679	0.814	8.09	0.18	0.32
			1.5	6.80	0.14	0.898	42	0.786	0.874	7.69	0.17	0.40
			2.0	6.85	0.15	0.947	28	0.821	0.930	7.43	0.16	0.46
			2.5	6.72	0.14	0.989	20	0.700	0.960	7.15	0.16	0.49
	0.90	35	1.1	17.71	0.60	0.905	32	0.750	0.880	20.48	0.68	0.40
			1.3	16.32	0.54	0.952	14	0.857	0.945	17.63	0.60	0.52
			1.5	14.66	0.46	0.984	8	0.875	0.980	15.48	0.52	0.59
			2.0	12.97		0.985	1	1.000	0.985	13.08	0.39	0.66
			2.5	11.29	0.29	1.000	0	-----	1.000	11.29	0.29	0.71

TABLE IV (Continued)

				Untruncated part			Truncated		Over all			
Δ*	P*	n*	r	$\bar{N}$	$S(\bar{N})$	P	T	P'	P''	$\bar{N}$	$S(\bar{N})$	η
	0.95	56	1.1	24.85	0.92	0.935	15	0.600	0.910	27.19	1.03	0.49
			1.3	21.94	0.72	0.990	8	0.875	0.985	23.30	0.84	0.60
			1.5	20.04	0.64	0.990	2	1.000	0.990	20.40	0.69	0.65
			2.0	16.46	0.47	0.995	0	-----	0.997	16.46	0.47	0.72
			2.5	14.08	0.34	1.000	0	-----	1.000	14.08	0.34	0.76
	0.99	110	1.1	41.84	1.43	0.990	1	1.000	0.990	42.18	1.47	0.62
			1.3	34.41	1.10	1.000	0	-----	1.000	34.41	1.10	0.69
			1.5	29.92	0.79	1.000	0	-----	1.000	29.92	0.79	0.73
			2.0	23.29	0.55	1.000	0	-----	1.000	23.29	0.55	0.79
			2.5	20.09	0.37	1.000	0	-----	1.000	20.09	0.37	0.82
0.85	0.75	18	1.1	11.19	0.35	0.838	70	0.671	0.780	13.57	0.32	0.31
			1.3	10.61	0.31	0.914	61	0.705	0.850	12.87	0.32	0.40
			1.5	10.49	0.29	0.968	42	0.643	0.900	12.07	0.31	0.47
			2.0	9.87	0.25	0.983	27	0.741	0.950	10.97	0.29	0.54
			2.5	9.42	0.22	0.989	11	0.818	0.980	9.90	0.25	0.60
	0.90	64	1.1	27.75	0.94	0.936	27	0.444	0.870	32.65	1.20	0.47
			1.3	25.78	1.01	0.974	10	0.900	0.970	27.69	1.12	0.60
			1.5	23.40	0.83	0.995	5	1.000	0.995	24.41	0.92	0.66
			2.0	18.72	0.61	0.995	0	-----	0.995	18.72	0.61	0.74
			2.5	15.79	0.43	1.000	0	-----	1.000	15.79	0.43	0.78
	0.95	104	1.1	40.76	1.48	0.990	5	0.800	0.985	42.34	1.60	0.61
			1.3	33.29	1.21	1.000	2	1.000	1.000	34.00	1.30	0.69
			1.5	28.61	0.91	1.000	1	1.000	1.000	28.99	0.98	0.74
			2.0	22.54	0.67	1.000	0	-----	1.000	22.54	0.67	0.79
			2.5	19.46	0.44	1.000	0	-----	1.000	19.47	0.44	0.82
	0.99	206	1.1	70.36	2.51	0.995	3	1.000	0.995	72.39	2.73	0.65
			1.3	54.10	1.70	1.000	0	-----	1.000	54.10	1.70	0.74
			1.5	45.54	1.29	1.000	0	-----	1.000	45.54	1.29	0.78
			2.0	33.31	0.71	1.000	0	-----	1.000	33.31	0.71	0.84
			2.5	28.97	0.52	1.000	0	-----	1.000	28.97	0.52	0.86

TABLE IV (Continued)

Δ*	P*	n*	r	Untruncated part N̄	S(N̄)	P	Truncated T	P'	Over all P"	N̄	S(N̄)	η
0.90	0.75	43	1.1	24.36	0.83	0.888	66	0.636	0.805	30.51	0.83	0.34
			1.3	22.99	0.72	0.945	37	0.757	0.910	26.70	0.80	0.49
			1.5	21.09	0.67	0.977	26	0.769	0.950	23.94	0.78	0.56
			2.0	17.96	0.54	0.995	5	1.000	0.995	18.59	0.60	0.67
			2.5	15.67	0.43	1.000	2	0.500	0.995	15.95	0.47	0.72
	0.90	149	1.1	64.42	2.38	0.967	16	0.875	0.960	71.19	2.73	0.55
			1.3	51.62	2.02	0.995	1	0.000	0.990	52.11	2.07	0.68
			1.5	39.48	1.34	0.995	0	-----	0.995	39.48	1.34	0.76
			2.0	29.42	0.77	1.000	0	-----	1.000	29.42	0.77	0.82
			2.5	24.30	0.55	1.000	0	-----	1.000	24.30	0.55	0.85
	0.95	245	1.1	93.48	3.79	0.990	5	0.600	0.980	97.27	4.06	0.62
			1.3	67.56	2.21	0.995	1	1.000	0.995	68.45	2.37	0.73
			1.5	51.97	1.57	1.000	0	-----	1.000	51.97	1.57	0.80
			2.0	36.58	0.77	1.000	0	-----	1.000	36.58	0.77	0.86
			2.5	31.18	0.60	1.000	0	-----	1.000	31.18	0.60	0.88
	0.99	489	1.1	147.93	5.55	1.000	0	-----	1.000	147.93	5.55	0.70
			1.3	98.50	2.71	1.000	0	-----	1.000	98.50	2.71	0.80
			1.5	76.82	2.03	1.000	0	-----	1.000	76.82	2.03	0.84
			2.0	53.55	1.00	1.000	0	-----	1.000	53.55	1.000	0.89
			2.5	44.98	0.67	1.000	0	-----	1.000	44.98	0.67	0.91
0.95	0.75	174	1.1	89.17	3.36	0.948	47	0.766	0.905	109.11	3.62	0.48
			1.3	71.33	2.43	0.995	11	1.000	0.995	76.98	2.83	0.67
			1.5	57.45	1.81	1.000	1	1.000	1.000	58.03	1.89	0.75
			2.0	38.45	1.01	1.000	0	-----	1.000	38.45	1.01	0.83
			2.5	31.77	0.70	1.000	0	-----	1.000	31.77	0.70	0.86
	0.90	626	1.1	217.63	8.79	0.984	7	0.714	0.975	231.93	10.01	0.66
			1.3	134.56	4.95	1.000	0	-----	1.000	134.56	4.95	0.81
			1.5	98.19	2.97	1.000	0	-----	1.000	98.19	2.97	0.86
			2.0	64.37	1.34	1.000	0	-----	1.000	64.37	1.34	0.91
			2.5	51.99	0.91	1.000	0	-----	1.000	51.99	0.91	0.93

TABLE IV (Continued)

				Untruncated part			Truncated		Over all			
Δ*	P*	n*	r	$\bar{N}$	S($\bar{N}$)	P	T	P'	P"	$\bar{N}$	S($\bar{N}$)	η
	0.95	1030	1.1	277.23	11.14	1.000	0	------	1.000	277.23	11.14	0.74
			1.3	151.60	4.57	1.000	0	------	1.000	151.60	4.57	0.86
			1.5	112.06	2.69	1.000	0	------	1.000	112.06	2.69	0.90
			2.0	75.13	1.31	1.000	0	------	1.000	75.13	1.31	0.93
			2.5	62.45	0.87	1.000	0	------	1.000	62.45	0.87	0.94
	0.99	2058	1.1	482.37	17.77	1.000	0	------	1.000	482.37	17.77	0.77
			1.3	260.78	6.51	1.000	0	------	1.000	260.78	6.51	0.87
			1.5	187.16	4.07	1.000	0	------	1.000	187.16	4.07	0.91
			2.0	125.50	1.95	1.000	0	------	1.000	125.50	1.95	0.94
			2.5	100.08	1.24	1.000	0	------	1.000	100.08	1.24	0.95

Remark 3.8. In Table IV, that is, the cases not in the LFC, one may note that P" as well as the over all saving η increases quite considerably as the value of r increases.

Remark 3.9. While computing "saving", there is an alternative way to define it. Using interpolations or extrapolations in Table G.1 of Gibbons et al. (1977), we first compute $n^{**} = n(\Delta^*, P'')$, the sample size required by the FSR to achieve minimum protection P" (under the LFC). The "over all saving ρ" is now computed as $\left(1 - \bar{N}(n^{**})^{-1}\right)$, where $\bar{N}$ is the "over all" average sample size. In Table V we present some values of ρ for the case of both the means being unknown and having truncation at n*.

TABLE V

SOME VALUES OF ρ FOR BOTH THE MEANS UNKNOWN AND TRUNCATION AT n*

Δ* \ P*	0.75	0.90	0.95	0.99
0.50	0.00	-0.04	0.22	0.00
0.60	0.15	-0.02	0.13	0.64
0.70	0.20	0.07	0.24	0.28
0.75	0.11	0.20	0.27	0.20
0.80	0.13	0.08	0.38	0.38
0.85	0.19	0.32	-0.10	0.38
0.90	0.21	0.46	0.37	0.50
0.99	0.20	-0.00	0.54	0.45

3.3.2. Use of Wald's Boundaries and Moderate Sample Performance

We are still going to decide for H_1 or H_2 where $H_1:\sigma_1{}^2 = \delta^*\sigma_2{}^2$, and $H_2:\sigma_2{}^2 = \delta^*\sigma_1{}^2$. We let type I and type II errors be equal, that is, $\alpha = \beta = \frac{1}{2}(1-P^*)$. We now borrow some notations from the proof of theorem 3.1 (p. 90-91), and let U , V , C, D mean the same things as there then Wald's (1947) sequential probability ratio test (SPRT) will look like this:

$$\text{R2(2):}\quad N'(P^*) \equiv N' = \inf\{n\geq 2: n-1 \geq V_n\ \ell n[(1+P^*)/(1-P^*)]\},$$
$$= \infty \text{ if no such n.}$$

We accept H_1 or H_2 if $U_n{}^{n-1} \leq (1-P^*)/(1+P^*)$ or $U_n{}^{n-1} \geq (1+P^*)/(1-P^*)$, respectively.

The truncated version of R2(2), at stage m is defined as follows: R2*(2): If the procedure R2(2) reaches the mth stage, but would need more samples to stop, we truncate the sequence and decide for H_1 or H_2 according as $U_m{}^{m-1} \leq 1$ or $U_m{}^{m-1} > 1$, respectively

It is very easy to check that, with probability one, the random sample size required for the rule R1(2) to stop is at most as large as that required by the rule R2(2).

We can prove the following theorem in the same way we proved our theorem 3.1 on pp. 90-91. The proof is omitted.

Theorem 3.2. *For fixed μ_1, μ_2 in $(-\infty, \infty)$ and σ_1, σ_2 in $(0, \infty)$, for either hypothesis H_1 or H_2 we have for the rule* R2(2):

$$(N - C^*)/D^* \xrightarrow{L} N(0, 1), \text{ as } P^* \to 1, \text{ where}$$

$$C^* = -\{\ell n[(1-P^*)(1+P^*)^{-1}]\}\{\ell n[(1 + \delta^{*2})/2\delta^*]\}^{-1},$$

$$D^{*2} = \tfrac{1}{2}\{-\ln[(1-P^*)(1+P^*)^{-1}]\}\{\ln[(1+\delta^{*2})/2\ \delta^*]\}^{-3}.$$

The Table VI is much like Table I except that we now use the rule R2*(2) for simulation. All the entries are self explanatory as earlier. Under the heading "asymptotic" we provide the values of C* and D" = D* $(200)^{-\frac{1}{2}}$. One may note that the performance of R1*(2) is better than that of R2*(2). This is quite expected in view of our remarks made in the paragraph just before theorem 3.2.

3.3.3. Only One Population Mean Known

Without any loss of generality we may assume that μ_1 is known while μ_2 is unknown. The basic structure of notations will remain the same as in subsections 3.3.1 and 3.3.2; however, we redefine $S_{1n}^2 = n^{-1}\sum_{j=1}^{n}(X_{1j} - \mu_1)^2$, $S_{2n}^2 = (n-1)^{-1}\sum_{j=1}^{n}(X_{2j} - \bar{X}_{2n})^2$, where $\bar{X}_{2n} = n^{-1}\sum_{j=1}^{n}X_{2j}$, for $n \geq 2$ and let $T_{1n} = S_{2n}^2/S_{1n}^2$, $\theta_1 = \sigma_1^2/\sigma_2^2$. Then $f(T_{1n}|\theta_1) = a(n)T_{1n}^{\frac{1}{2}(n-3)}\theta_1^{\frac{1}{2}(n-1)}$ $/\{n+(n-1)\theta_1 T_{1n}\}^{\frac{1}{2}(2n-1)}$, . . . (3.12)

where $a(n) = \{\Gamma\tfrac{1}{2}(2n-1)\}\{\Gamma\tfrac{1}{2}n\,\Gamma\tfrac{1}{2}(n-1)\}^{-1}n^{\frac{1}{2}n}(n-1)^{\frac{1}{2}(n-1)}$.

From (3.12) it follows that

$$f_{1n}/f_{2n} = \{\delta^*+n^{-1}(n-1)T_{1n}\}^{\frac{1}{2}(2n-1)}\{1+n^{-1}(n-1)\ \delta^* T_{1n}\}^{\frac{1}{2}(-2n+1)}\ \delta^{*-\frac{1}{2}}. \quad . . . (3.13)$$

As in the rule R1(2), we define the following rule:

$$\text{R3(2)}: N = \inf\{n \geq 2 \text{ such that } f_{1n}/f_{2n} \notin I(P^*)\}, \quad . . . (3.14)$$
$$= \infty \text{ if no such } n.$$

At stage N, we accept H_1 or H_2 according as the lower or upper boundary is crossed.

The form in (3.14) can be equivalently stated as:

TABLE VI

SIMULATION RESULT FOR WALD'S RULE R2*(2),
BOTH THE MEANS UNKNOWN

			Untruncated part			Truncated		Over all				Asymptotic	
Δ*	P*	n*	$\bar{N}$	S($\bar{N}$)	P	T	P'	P''	$\bar{N}$	S($\bar{N}$)	η	C*	D''
0.50	0.75	3	3.00	0.00	0.769	92	0.674	0.725	3.00	0.00	-0.03	2.58	0.11
	0.90	6	4.54	0.06	0.914	49	0.816	0.890	4.90	0.06	0.17	3.91	0.13
	0.95	8	5.78	0.11	0.954	47	0.851	0.930	6.30	0.11	0.20	4.86	0.15
	0.99	14	8.29	0.17	0.989	14	0.929	0.985	8.69	0.19	0.38	7.02	0.17
0.60	0.75	4	3.64	0.05	0.733	110	0.700	0.715	3.84	0.03	-0.01	4.32	0.23
	0.90	9	6.32	0.12	0.935	61	0.656	0.850	7.14	0.12	0.16	6.54	0.28
	0.95	13	8.64	0.21	0.968	45	0.778	0.925	9.62	0.21	0.24	8.14	0.32
	0.99	24	13.01	0.32	1.000	14	0.786	0.985	13.78	0.36	0.42	11.75	0.38
0.70	0.75	6	4.99	0.08	0.802	104	0.702	0.750	5.52	0.05	0.08	8.27	0.61
	0.90	15	9.99	0.24	0.977	71	0.789	0.910	11.77	0.23	0.22	12.51	0.75
	0.95	24	13.18	0.37	0.957	37	0.757	0.920	15.19	0.42	0.35	15.56	0.84
	0.99	45	23.05	0.69	0.989	18	0.833	0.975	25.03	0.77	0.44	22.49	1.01
0.75	0.75	8	6.52	0.11	0.845	116	0.664	0.740	7.38	0.08	0.07	12.39	1.12
	0.90	22	13.59	0.39	0.911	65	0.738	0.855	16.33	0.38	0.22	18.74	1.38
	0.95	35	17.95	0.49	0.955	44	0.773	0.915	21.70	0.63	0.36	23.32	1.54
	0.99	67	32.70	1.02	0.984	16	0.750	0.965	35.44	1.15	0.46	33.69	1.85
0.80	0.75	11	8.74	0.20	0.820	139	0.727	0.755	10.31	0.10	0.07	20.18	2.33
	0.90	35	20.60	0.62	0.955	66	0.697	0.870	25.35	0.63	0.25	30.53	2.87
	0.95	56	31.24	0.96	0.968	45	0.867	0.945	36.81	1.04	0.34	37.99	3.20
	0.99	110	50.92	1.74	0.994	21	0.905	0.985	57.12	2.02	0.48	54.88	3.84
0.85	0.75	18	12.73	0.38	0.884	131	0.649	0.730	16.18	0.22	0.13	37.48	5.90
	0.90	64	37.88	1.25	0.928	62	0.758	0.875	45.98	1.22	0.26	56.71	7.25
	0.95	104	53.72	1.76	0.975	39	0.846	0.950	63.53	2.00	0.39	70.56	8.09
	0.99	206	95.04	3.00	0.995	11	0.909	0.990	101.15	3.35	0.51	101.95	9.72
0.90	0.75	43	29.11	0.96	0.863	127	0.685	0.750	37.93	0.59	0.12	88.29	21.32
	0.90	149	84.10	2.54	0.929	59	0.797	0.890	103.25	2.76	0.30	133.60	26.22
	0.95	245	115.11	4.40	0.987	43	0.791	0.945	143.04	5.12	0.41	166.23	29.25
	0.99	489	220.93	7.40	1.000	11	1.000	1.000	235.68	8.22	0.52	240.18	35.16
0.95	0.75	174	114.80	3.73	0.850	120	0.650	0.730	150.32	2.54	0.11	370.48	183.23
	0.90	626	343.82	12.33	0.971	61	0.688	0.885	429.89	12.57	0.30	560.59	225.39
	0.95	1030	507.15	17.92	0.983	24	0.750	0.955	569.89	19.84	0.45	697.51	251.41
	0.99	2058	902.33	33.74	0.994	22	0.636	0.955	1029.46	39.47	0.48	1007.79	302.20

$$N = \inf\{n \geq 2 \text{ such that } n-\tfrac{1}{2} \geq -W_n \ln(1-P^*)\}, \text{ where}$$

$$W_n^{-1} = \left|\ln\{\left[\delta^* + n^{-1}(n-1)T_{1n}\right]/\left[1+n^{-1}(n-1)\delta^* T_{1n}\right]\} - (2n-1)^{-1}\ln\delta^*\right|.$$

Remark 3.10. The asymptotic (as $P^* \to 1$) distribution $N^{\frac{1}{2}}(W_N - a)/b$ and $N^{\frac{1}{2}}(W_{N-1} - a)/b$ are both standard normal, where a and b are the same as in the proof of theorem 3.1. The asymptotic distribution (as $P^* \to 1$) of $(N-C)/D$ is again $N(0,1)$, where the numbers C and D are defined in (3.8) and (3.9), respectively, N being given by (3.14). This can be justified along the lines of the proof of theorem 3.1 given on page 90 of Appendix B. The truncated version of R3(2), namely R3*(2), is exactly the same as R1*(2) except that we use f_{1n}/f_{2n} from (3.12) in the rule. The Tables VII and VIII should be read just like the Tables I and II.

Remark 3.11. One can see, however, that the average over-all sample sizes $\bar{N}$ in Tables VII and VIII are mostly smaller than the corresponding entries in Tables I and II. This is naturally expected to happen because one known mean adds some additional information in some sense, which is reflected in our ability to decide for H_1 or H_2 somewhat earlier. But, there is no rigorous mathematical justification known to us at this stage for this to be so.

3.4. The Special Case of Three Populations

In the case of all the μ's being unknown, we use the rules R1(k) and R1*(k) specialized for k = 3. In this situation, one can prove the following theorem without much difficulty. We omit its proof.

Theorem 3.3: For fixed μ_i in $(-\infty,\infty)$ and σ_i^2 in $(0, \infty)$, for each hypothesis H_i, i=1, 2, 3, we have for the rule R1(3):

<u>N is a non-decreasing function of P*</u>, $N \to \infty$ <u>a.s. as</u> $P^* \to 1$, <u>and</u> $N/C^{**} \to 1$ <u>a.s. as</u> $P^* \to 1$, <u>where</u>

$$C^{**} = \{-2\ln \tfrac{1}{2}(1-P^*)\}\{3\ln\{(1+ \delta^* + \delta^{*2})/3\ \delta^*\}\}^{-1}.$$

In Table IX we present simulation results for the rule R1(3) truncated at $m=n^*=\upsilon+1$ where υ comes from the Table G.1 of Gibbons et al. (1977) for each pair (Δ^*, P^*) and we let $\delta^* = \Delta^{*2}$. We generate normal populations in the same way we explained at the beginning of subsection 3.3.1. We generate Π_1 as $N(0, 1)$ and both Π_2 and Π_3 are generated as $N(0, \delta^{*-1})$, so that the hypothesis H_1 is deliberately made to be true. We estimate $\bar{N}$, $S(\bar{N})$ for the "untruncated part" and "overall" as in Table I. Under each of these headings, when we report "proportion" we subdivide it into three parts--a part is labeled as proportion of times we decided for H_i, i=1, 2, 3, with that particular category of heading. For each pair of (Δ^*, P^*) we estimate the quantities from 200 repetitions in columns four and beyond. The amount of "saving η" is computed in the same way as in Table I.

<u>Remark 3.12</u>: Comments like those in remark 3.7 are still valid for Table IX for the overall proportion of times we decide for H_1.

<u>3.4.1. Only One Mean Known</u>

Without any loss of generality we assume that μ_1 is known, while μ_2, μ_3 are both unknown. We let $S_{1n}^2 = n^{-1}\sum_{j=1}^{n}(X_{1j} - \mu_1)^2$, S_{2n}^2 and S_{3n}^2 be the same as in section 3.2 for $n \geq 2$. We define, as earlier, $\underset{\sim}{T}_n = (T_{1n}, T_{2n})$ where $T_{1n} = S_{2n}^2/S_{1n}^2$ and $T_{2n} = S_{3n}^2/S_{1n}^2$. Using the notations of section 3.3, we get

$$f(\underset{\sim}{T}_n|\underset{\sim}{\theta}) = b(n)(T_{1n}T_{2n})^{\frac{1}{2}(n-3)}(\theta_1\theta_2)^{\frac{1}{2}(n-1)}\{1 + n^{-1}(n-1)T_{1n}\theta_1 +$$

$$n^{-1}(n-1)T_{2n}\theta_2\}^{-\frac{1}{2}(3n-2)},$$

where $b(n) = \{n^{-1}(n-1)\}^{n-1}\Gamma\tfrac{1}{2}(3n-2)/\{\Gamma\tfrac{1}{2}n\{\Gamma\tfrac{1}{2}(n-1)\}^2\}$.

As written earlier $f_{jn} = f(\underset{\sim}{T}_n|\underset{\sim}{\theta})$ under H_j. We can compute f_{in}/f_{jn} for all $i \neq j = 1, 2, 3$.

Now, the sequential procedure for this case will be just like R1(k), with k=3, making sure that we work with this new f_{in}/f_{jn}. We also use the rule R1*(k) with m=n*, n* coming from the appropriate Table in G.1 of Gibbons et al. (1977), for a pair (Δ*, P*). In Table X we present results on simulating this procedure for several pairs of (Δ*, P*). These entries should be interpreted in the same way as in Table IX.

3.4.2. Exactly Two Means Known

Without any loss of generality we assume that μ_1 and μ_2 are known, while μ_3 is unknown. For $n \geq 2$, we let S_{1n}^2 and S_{3n}^2 be the same as in subsection 3.4.1, however, we define $S_{2n}^2 = n^{-1}\sum_{j=1}^{n}(X_{2j} - \mu_2)^2$. Writing $T_{1n} = S_{2n}^2/S_{1n}^2$, $T_{2n} = S_{3n}^2/S_{1n}^2$, we obtain

$$f(\underset{\sim}{T}_n|\underset{\sim}{\theta}) = d(n)(T_{1n}T_{2n})^{\frac{1}{2}(n-3)}(\theta_1\theta_2)^{\frac{1}{2}(n-1)}/\{1+T_{1n}\theta_1+n^{-1}(n-1)T_{2n}\theta_2\}^{\frac{1}{2}(3n-1)}$$

where $d(n) = \{n^{-1}(n-1)\}^{\frac{1}{2}(n-1)}\{\Gamma\tfrac{1}{2}(3n-1)\,(\Gamma\tfrac{1}{2}(n-1)\,(\Gamma\tfrac{1}{2}n)^2)\}^{-1}$.

As earlier, we write $f_{jn} = f(\underset{\sim}{T}_n|\underset{\sim}{\theta})$ under H_j and we can easily obtain f_{in}/f_{jn} for all $i \neq j = 1, 2, 3$.

Again, the sequential procedure for this case will look just like R1(k) with k=3, making sure that we substitute these new f_{in}/f_{jn} ratios in the rule. We can easily define a truncated version as in subsection 3.4.2. We report some simulated results on these rules in Table XI, and the entries mean the same things as in Tables IX and X.

TABLE VII

SIMULATION RESULT FOR THE RULE R3(2), ONE OF THE TWO MEANS KNOWN AND TRUNCATION AT n*

			Untruncated part			Truncated		Over all				Asymptotic	
Δ*	P*	n*	$\bar{N}$	S($\bar{N}$)	P	T	P'	P''	$\bar{N}$	S($\bar{N}$)	η	C	D'
0.50	0.95	8	4.56	0.10	0.933	22	0.682	0.905	4.94	0.12	0.35	3.97	0.13
	0.99	14	6.95	0.19	0.978	19	0.842	0.965	7.62	0.23	0.44	6.11	0.16
0.60	0.95	13	6.57	0.20	0.919	39	0.564	0.850	7.83	0.24	0.33	6.65	0.29
	0.99	24	11.38	0.33	0.979	12	1.000	0.980	12.14	0.38	0.49	10.23	0.36
0.70	0.95	24	12.01	0.38	0.944	23	0.913	0.940	13.39	0.43	0.44	12.73	0.76
	0.99	45	19.76	0.59	0.979	9	0.667	0.965	20.90	0.68	0.52	19.56	0.94
0.75	0.95	35	15.73	0.56	0.927	22	0.727	0.905	17.85	0.66	0.46	19.07	1.39
	0.99	67	27.56	0.96	0.984	10	0.800	0.975	29.54	1.09	0.55	29.31	1.72
0.80	0.95	56	24.41	0.91	0.907	28	0.607	0.865	28.84	1.10	0.43	31.06	2.89
	0.99	110	46.65	1.73	0.990	9	1.000	0.990	49.51	1.90	0.55	47.75	3.58
0.85	0.95	104	45.00	1.66	0.973	14	0.857	0.965	49.13	1.88	0.54	57.70	7.32
	0.99	206	79.78	2.76	0.995	11	0.636	0.975	86.72	3.31	0.57	88.70	9.07
0.90	0.95	245	106.96	4.05	0.950	20	0.850	0.940	120.76	4.68	0.50	135.93	26.45
	0.99	489	193.07	7.08	1.000	7	0.857	0.995	203.43	7.84	0.59	208.96	32.80
0.95	0.95	1030	423.05	17.05	0.962	17	0.824	0.950	474.64	19.68	0.54	570.36	227.35
	0.99	2058	772.23	30.10	0.995	8	0.875	0.990	823.56	33.97	0.60	876.78	281.88

TABLE VIII

SIMULATION RESULT FOR THE RULE R3(2), ONE OF THE TWO MEANS KNOWN AND TRUNCATION AT n* AND 2n*

			Untruncated part			Truncated		Over all				Asymptotic	
Δ*	P*	n*	$\bar{N}$	S($\bar{N}$)	P	T	P'	P"	$\bar{N}$	S($\bar{N}$)	η	C	D'
0.50	0.75	3	2.45	0.04	0.631	59	0.525	0.600	2.62	0.03	-0.09	1.84	0.09
		6	3.06	0.08	0.687	5	0.800	0.690	3.06	0.08	-0.11		
	0.90	6	3.61	0.09	0.837	28	0.536	0.795	3.94	0.10	0.26	3.06	0.12
		12	4.12	0.13	0.856	5	0.800	0.855	4.17	0.13	0.27		
0.60	0.75	4	2.97	0.06	0.677	45	0.622	0.665	3.20	0.06	0.10	3.08	0.20
		8	3.41	0.10	0.679	16	0.688	0.680	3.46	0.09	0.05		
	0.90	9	4.93	0.13	0.872	36	0.639	0.830	5.66	0.16	0.32	5.11	0.25
		18	6.08	0.24	0.881	7	0.714	0.875	6.18	0.23	0.29		
0.70	0.75	6	4.07	0.10	0.709	59	0.542	0.660	4.64	0.09	0.12	5.89	0.52
		12	5.32	0.19	0.747	14	0.786	0.750	5.37	0.18	0.11		
	0.90	15	7.90	0.23	0.864	23	0.696	0.845	8.72	0.26	0.38	9.78	0.66
		30	9.20	0.35	0.864	2	1.000	0.865	9.26	0.35	0.36		
0.75	0.75	8	5.14	0.14	0.735	68	0.588	0.685	6.11	0.13	0.16	8.82	0.95
		16	6.83	0.25	0.756	20	0.550	0.735	6.95	0.22	0.11		
	0.90	22	11.96	0.40	0.909	35	0.771	0.885	13.72	0.42	0.37	14.66	1.22
		44	14.72	0.59	0.913	5	1.000	0.915	14.91	0.58	0.33		
0.80	0.75	11	6.88	0.19	0.780	68	0.559	0.705	8.28	0.19	0.20	14.37	1.97
		22	8.94	0.31	0.788	21	0.619	0.770	9.16	0.28	0.19		
	0.90	35	17.91	0.63	0.839	32	0.781	0.830	20.65	0.69	0.36	23.88	2.53
		70	22.28	0.98	0.857	4	0.750	0.855	22.54	0.97	0.32		
0.85	0.75	18	10.52	0.35	0.746	70	0.571	0.685	13.14	0.34	0.24	26.70	4.98
		36	15.08	0.63	0.757	19	0.632	0.745	15.36	0.57	0.19		
	0.90	64	30.64	1.12	0.920	38	0.579	0.855	36.98	1.29	0.39	44.35	6.41
		126	41.46	1.97	0.929	3	0.333	0.920	41.80	1.95	0.36		
0.90	0.75	43	24.37	0.84	0.779	69	0.609	0.720	30.80	0.84	0.25	62.90	17.99
		86	33.30	1.39	0.778	24	0.542	0.750	34.46	1.24	0.20		
	0.90	149	68.70	2.56	0.950	39	0.718	0.905	84.36	3.05	0.44	104.48	23.19
		298	91.76	4.44	0.938	6	0.833	0.935	93.48	4.36	0.40		
0.95	0.75	174	97.69	3.73	0.764	73	0.712	0.745	125.54	3.52	0.27	263.94	154.66
		348	147.25	6.32	0.805	15	0.600	0.790	149.26	5.86	0.19		
	0.90	626	273.47	11.43	0.894	30	0.600	0.850	326.35	13.19	0.45	438.39	199.32
		1252	363.59	18.67	0.890	0	———	0.890	363.59	18.67	0.41		

TABLE IX

SIMULATION RESULT FOR THE RULE R1(3), ALL THE THREE MEANS UNKNOWN AND TRUNCATION AT n*

			Untruncated part					Truncated				Over all						
					proportion				proportion			proportion						
Δ*	P*	n*	$\bar{N}$	$S(\bar{N})$	H_1	H_2	H_3	T	H_1	H_2	H_3	H_1	H_2	H_3	$\bar{N}$	$S(\bar{N})$	η	C**
0.50	0.75	5	4.13	0.07	0.800	0.082	0.118	90	0.600	0.256	0.144	0.710	0.160	0.130	4.52	0.05	0.05	3.48
	0.90	8	5.80	0.12	0.911	0.022	0.067	66	0.697	0.091	0.212	0.840	0.045	0.115	6.53	0.11	0.13	4.57
	0.95	10	7.11	0.14	0.979	0.014	0.007	59	0.780	0.135	0.085	0.920	0.050	0.030	7.96	0.14	0.18	5.40
	0.99	17	10.06	0.22	0.983	0.017	0.000	25	0.800	0.120	0.080	0.960	0.030	0.010	10.93	0.25	0.34	7.31
0.60	0.75	6	5.12	0.09	0.756	0.180	0.064	122	0.574	0.189	0.237	0.645	0.185	0.170	5.66	0.05	-0.10	5.31
	0.90	12	8.68	0.18	0.962	0.038	0.000	67	0.776	0.119	0.105	0.900	0.065	0.035	9.79	0.17	0.18	7.21
	0.95	17	10.85	0.24	0.954	0.020	0.026	48	0.771	0.104	0.125	0.910	0.040	0.050	12.33	0.26	0.24	8.65
	0.99	28	16.91	0.38	1.000	0.000	0.000	17	0.941	0.059	0.000	0.995	0.005	0.000	17.86	0.41	0.37	11.99
0.70	0.75	11	8.38	0.18	0.839	0.097	0.064	107	0.636	0.178	0.186	0.730	0.140	0.130	9.78	0.13	0.09	9.51
	0.90	22	14.66	0.35	0.892	0.065	0.043	61	0.738	0.098	0.164	0.845	0.075	0.080	16.90	0.34	0.18	13.26
	0.95	32	19.09	0.52	0.956	0.025	0.019	40	0.675	0.125	0.200	0.900	0.045	0.055	21.67	0.55	0.29	16.10
	0.99	54	29.22	0.80	0.989	0.006	0.005	24	0.750	0.083	0.167	0.960	0.015	0.025	32.19	0.90	0.39	22.68
0.75	0.75	15	10.99	0.24	0.770	0.110	0.120	100	0.650	0.210	0.140	0.710	0.160	0.130	13.00	0.19	0.08	13.90
	0.90	33	20.94	0.58	0.930	0.021	0.049	58	0.724	0.172	0.104	0.870	0.065	0.065	24.44	0.57	0.23	19.59
	0.95	48	28.14	0.72	0.982	0.000	0.018	33	0.727	0.121	0.152	0.940	0.020	0.040	31.42	0.80	0.34	23.89
	0.99	82	44.68	1.17	0.995	0.005	0.000	14	0.786	0.143	0.071	0.980	0.015	0.005	47.29	1.28	0.42	33.88

TABLE IX (Continued)

			Untruncated part						Truncated			Over all						
					proportion				proportion			proportion						
Δ*	P*	n*	$\bar{N}$	$S(\bar{N})$	H_1	H_2	H_3	T	H_1	H_2	H_3	H_1	H_2	H_3	$\bar{N}$	$S(\bar{N})$	η	C**
0.80	0.75	24	16.40	0.44	0.857	0.044	0.099	109	0.670	0.147	0.183	0.755	0.100	0.145	20.54	0.33	0.15	22.22
	0.90	53	32.86	0.87	0.910	0.055	0.034	55	0.782	0.145	0.073	0.875	0.080	0.045	38.40	0.90	0.26	31.58
	0.95	77	42.00	1.21	0.964	0.030	0.006	34	0.823	0.059	0.118	0.940	0.035	0.025	47.95	1.37	0.37	38.65
	0.99	134	69.74	2.00	0.989	0.011	0.000	19	0.842	0.158	0.000	0.975	0.025	0.000	75.85	2.25	0.43	55.08
0.85	0.75	42	27.76	0.79	0.913	0.049	0.038	97	0.506	0.247	0.247	0.715	0.145	0.140	34.67	0.65	0.13	40.71
	0.90	97	55.34	1.63	0.930	0.051	0.019	43	0.698	0.093	0.209	0.880	0.060	0.060	64.30	1.76	0.32	58.21
	0.95	142	75.83	2.24	0.941	0.024	0.035	30	0.867	0.133	0.000	0.930	0.040	0.030	85.76	2.54	0.38	71.44
	0.99	251	124.41	3.35	0.995	0.000	0.005	10	1.000	0.000	0.000	0.995	0.000	0.005	130.74	3.73	0.48	102.18
0.90	0.75	95	60.13	1.90	0.772	0.132	0.096	86	0.628	0.209	0.163	0.710	0.165	0.125	75.13	1.63	0.16	95.01
	0.90	227	127.31	4.02	0.892	0.054	0.054	51	0.627	0.157	0.216	0.825	0.080	0.095	152.73	4.29	0.27	136.44
	0.95	334	173.34	5.72	0.958	0.012	0.030	34	0.912	0.029	0.059	0.950	0.015	0.035	200.66	6.39	0.40	167.77
	0.99	592	299.14	9.76	0.978	0.005	0.017	18	0.722	0.056	0.222	0.955	0.010	0.035	325.50	10.68	0.43	240.54
0.95	0.75	393	243.86	8.31	0.864	0.100	0.036	90	0.633	0.245	0.122	0.760	0.165	0.075	310.97	6.96	0.22	396.59
	0.90	948	504.22	16.83	0.904	0.057	0.038	43	0.605	0.163	0.232	0.840	0.080	0.080	599.63	18.48	0.32	570.91
	0.95	1399	728.72	22.67	0.937	0.023	0.040	27	0.704	0.185	0.111	0.905	0.045	0.050	819.21	25.45	0.39	702.78
	0.99	2490	1159.78	35.01	0.995	0.000	0.005	15	0.933	0.067	0.000	0.990	0.005	0.005	1259.55	40.80	0.49	1008.96

TABLE X

SIMULATION RESULT FOR THE RULE R1(3), ONE OF THE THREE MEANS KNOWN, OTHER TWO UNKNOWN AND TRUNCATION AT n*

			Untruncated part					Truncated				Over all						
					proportion				proportion			proportion						
Λ^*	P^*	n^*	$\bar{N}$	$S(\bar{N})$	H_1	H_2	H_3	T	H_1	H_2	H_3	H_1	H_2	H_3	$\bar{N}$	$S(\bar{N})$	η	C**
0.50	0.75	5	3.91	0.08	0.698	0.129	0.173	84	0.619	0.226	0.155	0.665	0.170	0.165	4.37	0.06	0.02	3.48
	0.90	8	5.50	0.12	0.903	0.035	0.063	56	0.714	0.089	0.197	0.850	0.050	0.100	6.20	0.12	0.18	4.57
	0.95	10	6.65	0.14	0.939	0.054	0.007	52	0.904	0.019	0.077	0.930	0.045	0.025	7.52	0.15	0.23	5.40
	0.99	17	9.56	0.22	0.994	0.006	0.000	22	0.773	0.091	0.136	0.970	0.015	0.015	10.38	0.26	0.38	7.31
0.60	0.75	6	4.83	0.10	0.699	0.194	0.108	107	0.608	0.168	0.224	0.650	0.180	0.170	5.46	0.06	-0.05	5.31
	0.90	12	8.38	0.19	0.940	0.045	0.015	67	0.821	0.075	0.104	0.900	0.055	0.045	9.60	0.18	0.20	7.21
	0.95	17	10.40	0.25	0.955	0.013	0.032	46	0.783	0.109	0.109	0.915	0.035	0.050	11.92	0.28	0.27	8.65
	0.99	28	16.15	0.37	0.995	0.000	0.005	14	0.857	0.143	0.000	0.985	0.010	0.005	16.98	0.41	0.39	11.99
0.70	0.75	11	7.88	0.21	0.809	0.085	0.106	106	0.642	0.123	0.236	0.720	0.105	0.175	9.54	0.15	0.10	9.51
	0.90	22	14.71	0.38	0.921	0.043	0.036	61	0.738	0.147	0.115	0.865	0.075	0.060	16.94	0.36	0.20	13.26
	0.95	32	18.34	0.53	0.957	0.031	0.012	37	0.622	0.189	0.189	0.895	0.060	0.045	20.87	0.57	0.31	16.10
	0.99	54	27.71	0.82	0.983	0.006	0.012	27	0.815	0.074	0.111	0.960	0.015	0.025	31.26	0.95	0.40	22.68
0.75	0.75	15	10.46	0.27	0.804	0.120	0.076	108	0.667	0.185	0.148	0.730	0.155	0.115	12.91	0.20	0.12	13.90
	0.90	33	20.65	0.59	0.919	0.027	0.054	52	0.654	0.192	0.154	0.850	0.070	0.080	23.86	0.58	0.23	19.59
	0.95	48	26.88	0.75	0.970	0.012	0.018	32	0.875	0.063	0.062	0.955	0.020	0.025	30.26	0.84	0.37	23.89
	0.99	82	43.87	1.18	0.984	0.016	0.000	18	0.944	0.000	0.056	0.980	0.015	0.005	47.31	1.33	0.42	33.88

TABLE X (Continued)

			Untruncated part					Truncated				Over all						
					proportion				proportion			proportion						
*	P*	n*	$\bar{N}$	$S(\bar{N})$	H_1	H_2	H_3	T	H_1	H_2	H_3	H_1	H_2	H_3	$\bar{N}$	$S(\bar{N})$		C**
0.80	0.75	24	16.40	0.48	0.872	0.053	0.075	106	0.717	0.132	0.151	0.790	0.095	0.115	20.43	0.35	0.20	22.22
	0.90	53	32.61	0.91	0.912	0.054	0.034	53	0.660	0.227	0.113	0.845	0.100	0.055	38.02	0.92	0.24	31.58
	0.95	77	41.50	1.28	0.953	0.035	0.012	29	0.793	0.103	0.104	0.930	0.045	0.025	46.65	1.41	0.38	38.65
	0.99	134	68.04	2.00	0.984	0.005	0.011	16	0.813	0.125	0.062	0.970	0.015	0.015	73.32	2.24	0.44	55.08
0.85	0.75	42	28.95	0.85	0.898	0.061	0.041	102	0.598	0.216	0.186	0.745	0.140	0.115	35.61	0.62	0.15	40.71
	0.90	97	54.69	1.64	0.937	0.044	0.019	42	0.595	0.143	0.262	0.865	0.065	0.070	63.58	1.78	0.32	58.21
	0.95	142	75.84	2.30	0.964	0.024	0.012	34	0.794	0.118	0.088	0.935	0.030	0.025	87.09	2.60	0.38	71.44
	0.99	251	122.38	3.33	0.995	0.000	0.005	12	1.000	0.000	0.000	0.995	0.000	0.005	130.10	3.81	0.48	102.18
0.90	0.75	95	59.23	1.86	0.811	0.108	0.081	89	0.551	0.292	0.157	0.695	0.190	0.115	75.15	1.63	0.15	95.01
	0.90	227	125.01	3.84	0.862	0.079	0.059	48	0.688	0.166	0.146	0.820	0.100	0.080	149.49	4.25	0.28	136.44
	0.95	334	170.15	5.54	0.959	0.012	0.296	31	0.871	0.065	0.064	0.945	0.020	0.035	195.55	6.29	0.41	167.77
	0.99	592	294.09	9.89	0.978	0.006	0.016	19	0.842	0.053	0.105	0.965	0.010	0.025	322.39	10.88	0.44	240.54
0.95	0.75	393	244.04	8.03	0.843	0.087	0.070	85	0.635	0.259	0.106	0.755	0.160	0.085	307.35	6.96	0.22	396.59
	0.90	948	478.59	17.41	0.902	0.065	0.033	47	0.681	0.149	0.170	0.850	0.085	0.065	588.90	19.40	0.34	570.91
	0.95	1399	702.54	23.26	0.935	0.030	0.035	30	0.700	0.267	0.033	0.900	0.065	0.035	807.01	26.49	0.39	702.78
	0.99	2490	1140.51	34.08	0.995	0.000	0.005	17	0.941	0.059	0.000	0.990	0.005	0.005	1255.22	41.03	0.50	1008.96

TABLE XI

SIMULATION RESULT FOR THE RULE R1(3), TWO OF THE THREE MEANS KNOWN, ONE UNKNOWN AND TRUNCATION AT n*

			Untruncated part					Truncated				Over all						
					proportion				proportion			proportion						
Δ*	P*	n*	$\bar{N}$	S($\bar{N}$)	H_1	H_2	H_3	T	H_1	H_2	H_3	H_1	H_2	H_3	$\bar{N}$	S($\bar{N}$)	η	C**
0.50	0.75	5	3.33	0.10	0.928	0.007	0.065	62	0.742	0.065	0.193	0.870	0.025	0.105	3.85	0.09	0.34	3.48
	0.90	8	4.89	0.12	0.944	0.013	0.043	39	0.744	0.000	0.256	0.905	0.010	0.085	5.50	0.13	0.32	4.57
	0.95	10	5.78	0.16	0.988	0.006	0.005	30	0.833	0.033	0.134	0.965	0.010	0.025	6.41	0.18	0.37	5.40
	0.99	17	8.66	0.22	0.995	0.000	0.005	12	0.667	0.083	0.250	0.975	0.005	0.020	9.16	0.25	0.45	7.31
0.60	0.75	6	4.25	0.11	0.948	0.009	0.043	85	0.671	0.129	0.200	0.830	0.060	0.110	5.00	0.09	0.25	5.31
	0.90	12	7.41	0.20	0.968	0.019	0.013	45	0.800	0.089	0.111	0.930	0.035	0.035	8.44	0.21	0.32	7.21
	0.95	17	9.35	0.26	0.988	0.000	0.012	36	0.667	0.111	0.222	0.930	0.020	0.050	10.73	0.30	0.36	8.65
	0.99	28	14.42	0.40	0.989	0.005	0.006	10	0.800	0.200	0.000	0.980	0.015	0.005	15.10	0.43	0.46	11.99
0.70	0.75	11	7.18	0.19	0.908	0.017	0.075	80	0.713	0.112	0.175	0.830	0.055	0.115	8.71	0.17	0.28	9.51
	0.90	22	12.93	0.39	0.949	0.013	0.038	44	0.705	0.182	0.113	0.895	0.050	0.055	14.93	0.40	0.32	13.26
	0.95	32	16.98	0.51	0.965	0.018	0.017	29	0.862	0.035	0.103	0.950	0.020	0.030	19.16	0.57	0.40	16.10
	0.99	54	26.79	0.81	0.989	0.006	0.005	19	0.684	0.105	0.211	0.960	0.015	0.025	29.38	0.92	0.44	22.68
0.75	0.75	15	9.51	0.29	0.876	0.035	0.089	87	0.724	0.092	0.184	0.810	0.060	0.130	11.90	0.25	0.27	13.90
	0.90	33	19.14	0.57	0.918	0.027	0.055	54	0.759	0.111	0.130	0.875	0.050	0.075	22.88	0.60	0.29	19.59
	0.95	48	25.10	0.76	0.972	0.006	0.022	23	0.870	0.043	0.087	0.960	0.010	0.030	27.73	0.85	0.43	23.89
	0.99	82	41.61	1.20	0.995	0.005	0.000	12	0.833	0.000	0.167	0.985	0.005	0.010	44.04	1.32	0.46	33.88

TABLE XI (Continued)

			Untruncated part					Truncated				Over all						
					proportion				proportion			proportion						
Λ*	P*	n*	$\bar{N}$	S($\bar{N}$)	H_1	H_2	H_3	T	H_1	H_2	H_3	H_1	H_2	H_3	$\bar{N}$	S($\bar{N}$)	η	C**
0.80	0.75	24	14.65	0.47	0.894	0.027	0.080	87	0.747	0.080	0.172	0.830	0.050	0.120	18.72	0.42	0.30	22.22
	0.90	53	31.23	0.90	0.929	0.032	0.039	45	0.778	0.133	0.089	0.895	0.055	0.050	36.13	0.95	0.31	31.58
	0.95	77	40.99	1.30	0.966	0.017	0.017	23	0.913	0.000	0.087	0.960	0.015	0.025	45.14	1.41	0.42	38.65
	0.99	134	67.05	2.08	0.984	0.005	0.011	14	0.786	0.143	0.071	0.970	0.015	0.015	71.74	2.28	0.45	55.08
0.85	0.75	42	25.61	0.83	0.917	0.033	0.050	80	0.550	0.213	0.237	0.770	0.105	0.125	32.17	0.75	0.25	40.71
	0.90	97	52.06	1.67	0.950	0.037	0.013	39	0.821	0.102	0.077	0.925	0.050	0.025	60.83	1.84	0.39	58.21
	0.95	142	72.65	2.30	0.959	0.030	0.012	31	0.839	0.032	0.129	0.940	0.030	0.030	83.40	2.64	0.41	71.44
	0.99	251	119.14	3.57	0.990	0.005	0.005	8	0.875	0.000	0.125	0.985	0.005	0.010	124.41	3.89	0.50	102.18
0.90	0.75	95	58.94	1.90	0.867	0.080	0.053	87	0.586	0.264	0.150	0.745	0.160	0.095	74.63	1.66	0.21	95.01
	0.90	227	121.57	4.16	0.918	0.034	0.048	53	0.660	0.151	0.189	0.850	0.065	0.085	149.51	4.50	0.30	136.44
	0.95	334	166.37	5.69	0.982	0.006	0.012	33	0.879	0.030	0.091	0.965	0.010	0.025	194.03	6.48	0.43	167.77
	0.99	592	285.76	9.90	0.989	0.000	0.011	22	0.727	0.091	0.182	0.960	0.010	0.030	319.45	11.12	0.44	240.54
0.95	0.75	393	232.40	7.69	0.866	0.109	0.025	81	0.617	0.222	0.161	0.765	0.155	0.080	297.44	7.22	0.26	396.59
	0.90	948	478.68	17.66	0.916	0.045	0.039	46	0.543	0.283	0.174	0.830	0.100	0.070	586.63	19.51	0.33	570.91
	0.95	1399	708.55	23.29	0.933	0.034	0.033	22	0.682	0.182	0.136	0.905	0.050	0.045	784.50	25.76	0.41	702.78
	0.99	2490	1148.83	34.72	0.995	0.005	0.000	15	1.000	0.000	0.000	0.995	0.005	0.000	1249.42	40.72	0.50	1008.96

Remark 3.13: In Tables X and XI, one can see that on the average all average sample sizes $\bar{N}$ in Tables X and XI are mostly smaller than the corresponding entries in Table IX.

CHAPTER IV

SELECTING THE LARGEST NORMAL MEAN

4.1. Defining the Problem

Suppose there are $k(\geq 2)$ independent normal populations $\Pi_1, \ldots, \Pi_k$, where Π_i is assumed to have the mean μ_i and common variance σ^2 $(0 < \sigma^2 < \infty)$, $i=1, 2, \ldots, k$. Let $\mu_{(1)} \leq \ldots \leq \mu_{(k)}$ be the ordered μ-values. The problem is to select the population having the mean $\mu_{(k)}$, which is also referred to as the "best" population. For practical applications where one faces this type of selection problems, the reader is referred to Chapter 2 of Gibbons et al. (1977) and section 6.2 of Gupta and Panchapakesan (1979).

We will restrict our attention to the "indifference zone approach" only and follow the formulation originated in Bechhofer (1954). Following the standard notations, we assume that we are given two numbers δ^* and P^*, $0 < \delta^* < \infty$, $k^{-1} < P^* < 1$. Let $\psi = (\mu_1, \ldots, \mu_k, \sigma^2)$, and $\Omega(\delta^*) = \{\psi: \mu_{(k)} - \mu_{(k-1)} \geq \delta^*\}$. We wish to propose sequential procedures for selecting the largest mean $\mu_{(k)}$ such that $P(CS) \geq P^*$ if $\psi \in \Omega(\delta^*)$. The configuration $\mu_{(1)} = \ldots = \mu_{(k-1)} = \mu_{(k)} - \delta^*$ is referred to as the least favorable configuration (LFC) or a slippage configuration in this context.

We plan to develop sequential procedures to select the "best" population under the LFC when σ^2 is unknown, by appealing to the rules developed in Mukhopadhyay (1980a) for known σ^2. We compare our procedures in detail with existing fixed-sample procedures as discussed in Gibbons et al. (1977) for some values of k. For numerical comparisons, we also present some modified rules along the lines of Baker (1950), Hall (1962), and Mukhopadhyay (1979, 1980b).

4.2. The Common Variance is Known

In motivating our developments for unknown σ^2, we first refer briefly to the case when σ^2 is known. This means that ψ is now defined by $\psi = (\mu_1, \ldots, \mu_k)$ only. We wish to decide among the k hypotheses $(k \geq 2)$, H_i: $\mu_i = \mu_{(k)}$, $i=1, \ldots, k$. Having recorded $\{X_{i1}, \ldots, X_{in}\}$ from Π_i, $i=1, \ldots, k$, and utilizing the maximal invariant (with respect to the group of location shifts by the same amount), we let the sequence $\underset{\sim}{U}_j = (X_{2j} - X_{1j}, \ldots, X_{kj} - X_{1j})' = (U_{1j}, \ldots, U_{k-1\ j})'$ say, for $j=1, \ldots, n$. Now, $\underset{\sim}{U}_1, \ldots \underset{\sim}{U}_n$, are iid $N_{k-1}(\underset{\sim}{\theta}, \sigma^2\Sigma)$, where $\underset{\sim}{\theta} = (\mu_2 - \mu_1, \ldots, \mu_k - \mu_1)' = (\theta_1, \ldots, \theta_{k-1})'$ say, and $\Sigma = (\sigma_{ij})$ where $\sigma_{ii} = 2$, $\sigma_{ij} = 1$ for all $1 \leq i \neq j \leq k-1$. Then Σ^{-1} can be written as (σ^{ij}) where $\sigma^{ii} = (k-1)/k$, $\sigma^{ij} = -1/k$ for all $1 \leq i \neq j \leq k-1$. The previous k hypotheses can be equivalently stated as follows:

$$H_1: \ (\theta_i = -\delta^* \text{ for all } i=1, \ldots, k-1)$$
$$H_j: \ (\theta_{j-1} = \delta^* \text{ and } \theta_i = 0 \text{ for all } i \neq j-1), \text{ where}$$

$j=2, \ldots, k$. Mukhopadhyay (1980a) proposed the following stopping rule for deciding among $H_1, H_2, \ldots, H_k$.

$$P1(k): \ N = \inf\{n \geq 1: \ \delta^*\sigma^{-2} \sup_{j \neq 1}\Big[\sum_{\ell \neq 1}^{n}(X_{j\ell} - X_{i\ell})\Big] \leq \ln\big[(1-P^*)(k-1)^{-1}\big] \text{ for some } i\},$$
$$= \infty \text{ if no such } n, \qquad \ldots (4.1)$$

and when N stops with i, we decide for the hypothesis H_i, that is, we declare that Π_i has the largest mean, i=1, . . ., k.

Although N is finite with probability one under H_i, i=1, . . ., k, it may be necessary in some practical situations to truncate the rule P1(k) at some stage. The truncated version is proposed as follows:

P1*(k): We take one sample at a time from each population and continue checking with the rule P1(k) to see if we can stop. When we reach the stage n=m, and P1(k) does not stop by itself, but we wish to terminate sampling, we decide for Π_ℓ as being the "best" population, where

$$\sup_{j\neq\ell}\{\exp\left[\delta^*\sigma^{-2}\sum_{h=1}^{m}(X_{jh}-X_{\ell h})\right]\} = \min_{i}\ \sup_{j\neq i}\{\exp\left[\delta^*\sigma^{-2}\sum_{h=1}^{m}(X_{jh}-X_{ih})\right]\}.$$

4.3. The Common Variance Is Unknown

In this section we propose several procedures dealing with the same selection problem under the assumption that σ^2 is unknown. Mukhopadhyay's (1980a) procedure P1(k) as discussed in section 4.2 serves as a foundation for the following procedures.

4.3.1. Procedure P2(k)

Let $S_n^{\,2} = (kn-k)^{-1}\sum_{i=1}^{k}\left[\sum_{j=1}^{n}X_{ij}^{\,2} - n^{-1}(\sum_{j=1}^{n}X_{ij})^2\right]$, $Z_{ij\ell} = X_{j\ell} - X_{i\ell}$, and $\bar{Z}_{ijn} = n^{-1}\sum_{\ell=1}^{n}Z_{ij\ell}$, where i≠j=1, . . ., k. Being motivated by the developments of Mukhopadhyay (1979, 1980a), we now propose the following stopping rule:

P2(k): Suppose $n^* = \max\{\left[(-\delta^{*-2}\ln\{(1-P^*)(k-1)^{-1}\})^{\frac{1}{1+\gamma}}\right]^+, 2\}$ where $\gamma>0$, and $[y]^+$ denotes the largest integer $\leq y$.

We define $N = \inf\{n\geq n^*:\ \delta^* S_n^{\,-2}\sup_{j\neq i} n\bar{Z}_{ijn} \leq \ln\left[(1-)^*)(k-1)^{-1}\right]$ for some i},

$= \infty$ if no such n.

When N stops with i, we decide for the hypothesis H_i, that is, we declare that Π_i has the largest mean, i=1, . . ., k.

Remark 4.1: The particular choice of n* in the rule P2(k) is motivated from the rule P2(2)' of section 4.4.2. In fact, the rule P2(k) can be proposed with any starting sample size (≥ 2). Moreover, when N stops for the rule P2(k), we can prove that P2(k) indeed selects that population which has the largest sample mean at the stopping stage.

Proof: Considering $-\delta^* S_n^2 \sum_{\ell=1}^{n}(X_{i\ell} - X_{j\ell})$, for i=1, and taking j=2, . . ., k. We have respectively the expressions

$$\delta^* S_n^2 \sum_{\ell=1}^{n}(X_{2\ell} - X_{1\ell}),\ \delta^* S_n^2 \sum_{\ell=1}^{n}(X_{3\ell} - X_{1\ell}),\ \ldots,\ \delta^* S_n^2 \sum_{\ell=1}^{n}(X_{k\ell} - X_{1\ell}).$$

Suppose the decision is made in favor of H_1 so that $\exp\left[-\delta^* S_n^2 \sum_{\ell=1}^{n}(X_{1\ell} - X_{2\ell})\right]$ is the $\sup_{j\neq 1} \exp\left[-\delta^* S_n^2 \sum_{\ell=1}^{n}(X_{1\ell} - X_{j\ell})\right]$ and $\delta^* S_n^2 \sum_{\ell=1}^{n}(X_{2\ell} - X_{1\ell}) < \ln\{(1-p^*)(k-1)^{-1}\} < 0$, which implies $\sum_{\ell=1}^{n} X_{2\ell} < \sum_{\ell=1}^{n} X_{1\ell}$, or $\bar{X}_{2n} < \bar{X}_{1n}$.
For j=3, . . ., k, we have

$$\delta^* S_n^2 \sum_{\ell=1}^{n}(X_{j\ell} - X_{1\ell}) < \delta^* S_n^2 \sum_{\ell=1}^{n}(X_{2\ell} - X_{1\ell}),$$

which implies $\sum_{\ell=1}^{n} X_{j\ell} < \sum_{\ell=1}^{n} X_{2\ell}$, or $\bar{X}_{jn} < \bar{X}_{2n}$, j=3, . . ., k. Hence, $\bar{X}_{1n}$ is the largest sample mean.

Similarly, we can verify our comment for i=2, . . ., k. ∇

4.3.2. Procedure P3(k)

Let S_m^2 be computed as S_n^2 for a fixed n=m, and let $r_n(S_m) = \delta^* \cdot \delta^* \sum_{\ell=1} z_{ij\ell}/S_m^2$, $a_m = \frac{1}{2}\nu(\alpha^{-2/\nu} - 1) \simeq (-\ln\alpha)\{1-(\ln\alpha)/\nu\}$, where $\alpha = (1-p^*)/(k-1)$, and $\nu = k(m-1)$. Now, utilizing the test procedures of Baker (1950) and Hall (1962), we propose the following stopping rule.

P3(k): Observe $\{X_{i1}, \ldots, X_{im}; i=1, \ldots, k\}$ and then $Z_{ij(m+1)}$, $Z_{ij(m+2)}, \ldots$, successively. For each $n \geq m$ after observing Z_{ijn}, we stop sampling and accept H_i, i=1, . . ., k, if $\sup_{j \neq i} r_n(S_m) \leq -a_m$ for some i, where m is taken to be the n* which is defined in our procedure P2(k) of section 4.3.2.

4.3.3. Procedure P4(k)

Let S_n^2, $r_n(S_n)$, and a_n be the same as in the section 4.3.2 with m=n. We propose the following stopping rule.

P4(k): Observe $\{X_{i1}, X_{i2}, \ldots; i=1, \ldots, k\}$ and thus obtain Z_{ij1}, $Z_{ij2}, \ldots$, successively. For each $n \geq 2$, after observing Z_{ijn}, we stop sampling and accept H_i if $\sup_{j \neq i} r_n(S_n) \leq -a_n$ for some i.

4.4. The Special Case of Two Populations

We will discuss this special case separately under both the situations when the common variance is known or unknown. We will also investigate the moderate sample size behaviors of our proposed rules in separate subsections.

4.4.1. The Common Variance Is Known

In this case, the rule P1(k) defined in (4.1) takes the following form:

$$\text{P1(2)}: \quad N = \inf \{n \geq 1 \text{ such that } \exp\left[\delta^* \sigma^{-2} \sum_{i=1}^{n} Z_{12i}\right] \notin I(P^*)\},$$
$$= \infty \text{ if no such } n,$$

where $I(P^*)$ is the interval $[(1-P^*), (1-P^*)^{-1}]$, $Z_{12i} = X_{2i} - X_{1i}$, i=1, . . ., n. At stage N, we accept H_1 or H_2 according as the lower or the upper boundary is crossed.

Note that this stopping variable can also be stated equivalently as

$$N(P^*) \equiv N = \inf\{n \geq 1: |\delta^* \sigma^{-2} \sum_{i=1}^{n} z_{12i}| \geq -\ln(1-P^*)\}. \quad \ldots (4.2)$$

The corresponding truncated rule is proposed as follows:

P1*(2): When we reach the stage n=m, but P1(2) does not stop by itself, we may wish to terminate sampling and we accept $H_1(H_2)$ if $\exp\left[\delta^*\sigma^{-2}\sum_{i=1}^{m} z_{12i}\right] \geq (<) 1$.

For the rest of this subsection, let us write $z_i = z_{12i}$, i=1, 2, ... From (4.2), we have $\delta^*\sigma^{-2}E|\sum_{i=1}^{N} z_i| \geq -\ln(1-P^*)$, which implies $\delta^*\sigma^{-2}E\sum_{i=1}^{N}|z_i| \geq -\ln(1-P^*)$, and we then obtain $E(N) \geq -\sigma^2[\ln(1-P^*)]/\delta^*E(|z_1|)$. Since $z_i \sim N(\theta, 2\sigma^2)$, applying truncated normal distribution (see Johnson and Kotz (1970), p. 81), we have

$$E(|z_1|) = \theta[2\Phi(2^{-\frac{1}{2}}\theta/\sigma) - 1] + 2\pi^{-\frac{1}{2}}\sigma \exp(-\theta^2/4\sigma^2),$$

and under H_1 or H_2, we obtain

$$E(N) \geq \sigma^2[-\ln(1-P^*)]/\delta^*\{\delta^*[2\Phi(2^{-\frac{1}{2}}\delta^*/\sigma) - 1] + 2\pi^{-\frac{1}{2}}\sigma \exp(-\delta^{*2}/4\sigma^2)\}. \quad \ldots (4.3)$$

Applying Jensen's inequality and Wald's (1947) first equation (Appendix A.5 and A.10), we also have

$$E(N) \leq \sigma^2\delta^{*-2}[-\ln(1-P^*)] + 1, \text{ under } H_1 \text{ or } H_2. \quad \ldots (4.4)$$

In proving (4.3) and (4.4) we tacitly assumed that E(N) is finite under H_1 or H_2. However, this assumption can easily be relaxed by using the monotone convergence theorem and a truncation argument as in Chow and Robbins (1965).

Now, we have the following theorems summarizing the asymptotic properties of the stopping rule P1(2). The quantities C and D are defined in (4.7).

Theorem 4.1: For fixed known σ in $(0, \infty)$ and μ_1, μ_2 in $(-\infty, \infty)$, for either hypothesis H_1 or H_2, for the rule P1(2), we have the following:

(i) N is a non-decreasing function of P^*, $N \to \infty$ a.s. as $P^* \to 1$, and $N/C \to 1$ a.s. as $P^* \to 1$. . . . (4.5)

(ii) $(N - C)/D \xrightarrow{\mathcal{L}} N(0, 1)$ as $P^* \to 1$ . . . (4.6)

where $C = -\sigma^2 \delta^{*-2} \ln(1-P^*)$ and $D = 2^{\frac{1}{2}}\sigma^2\{-\ln(1-P^*)\}^{\frac{1}{2}}/\delta^{*2}$. . . . (4.7)

Proof: See Appendix B (p. 91).

Theorem 4.2: For the rule P1(2), $\lim_{P^* \to 1} E(N/C) = 1$.

Proof: From (4.4), we get $E(N/C) \leq 1 + C^{-1}$, which implies $\limsup_{P^* \to 1} E(N/C) \leq 1$.

Applying Fatou's Lemma (Appendix A.3), we see that

$$E\{\liminf_{P^* \to 1}(N/C)\} \leq \liminf_{P^* \to 1} E(N/C).$$

Since $N/C \to 1$ a.s. as $P^* \to 1$, $E\{\liminf_{P^* \to 1} (N/C)\} = 1$, and thus $\liminf_{P^* \to 1} E(N/C) \geq 1$. Combining these facts about the lim sup and lim inf, we obtain theorem 4.2. ∇

4.4.2. The Common Variance Is Unknown

In this subsection, we will discuss several procedures as in section 4.3 for k=2. We still prefer to write $Z_{12i} = Z_i$, i=1, 2, ...

Procedure P2(2)

Let S_n^2 be the same as in the rule P2(k) with k=2, while z_i is the same as in subsection 4.4.1. The rule P2(2) takes the following form:

$$P2(2):\quad N = \inf\{n \geq 2 \text{ such that } \exp(\delta^* S_n^{-2} \sum_{i=1}^{n} z_i) \notin I(P^*)\},$$
$$= \infty \text{ if no such } n,$$

where I(P*) is as in rule P1(2). At stage N, we accept H_1 or H_2 according as the lower or upper boundary is crossed.

This stopping variable can equivalently be stated as

$$N(P^*) \equiv N = \inf\{n \geq 2:\ n\delta^* \geq -S_n^2 |\bar{z}_n|^{-1} \ln(1-P^*)\}, \qquad \ldots (4.8)$$
$$= \infty \text{ if no such } n.$$

By using Helmert's orthogonal transformation, we can write S_n^2 as:

$$S_n^2 = (2n-2)^{-1} \sum_{i=1}^{2(n-1)} Y_i^2 \qquad \ldots (4.9)$$

where Y_i's are iid $N(0, \sigma^2)$, i=1, . . ., 2(n-1).

Following the lines of theorem 4.1, it will be easy to prove the following theorem. Its proof is deferred to Appendix B (p. 93).

Theorem 4.3: For fixed μ_1, μ_2 in $(-\infty, \infty)$ and σ in $(0, \infty)$ for either hypothesis H_1 or H_2, for the rule P2(2), we have the following:

(i) N is a non-decreasing function of P*, $N \to \infty$ a.s. as $P^* \to 1$,

$$N/C \to 1 \text{ a.s. as } P^* \to 1, \qquad \ldots (4.10)$$

(ii) $(N-C)/D \xrightarrow{\mathcal{L}} N(0, 1)$ as $P^* \to 1$, where $C = -\sigma^2 \delta^{*-2} \ln(1-P^*)$, and

$$D = 2^{\frac{1}{2}} \sigma^2 \delta^{*-2} \{-\ln(1-P^*)\}^{\frac{1}{2}}. \qquad \ldots (4.11)$$

Theorem 4.4: For the rule P2(2), we have $\lim_{P^*\to 1} E(N/C) = 1$.

proof: see Appendix B (p. 94).

Being motivated by the results of Mukhopadhyay (1980b), a higher proportion of correct selection is achieved by proposing the following modification of the rule P2(2), namely P2(2)'.

P2(2)': Let $n^* = \max\{[\{-\delta^{*-2}\ell n(1-P^*)\}^{\frac{1}{1+\gamma}}]^+, 2\}$, where $\gamma > 0$, and $[y]^+$ is the largest integer $\leq$ y.

$$N = \inf\{n \geq n^*, \text{ such that } n\delta^* \geq -\{S_n^2|\bar{Z}_n|^{-1} + (\delta^* n^\gamma)^{-1}\}\{\ell n(1-P^*)\},$$
$$= \infty \text{ if no such } n.$$

At stage N, we accept $H_1(H_2)$, if $\sum_{i=1}^{N} X_{2i} - \sum_{i=1}^{N} X_{1i} \leq (>) 0$.

This choice of n* is quite natural, because at the stopping stage, we have

$$\delta^* N \geq -\ell n(1-P^*)\left[S_N^2|\bar{Z}_N|^{-1} + (\delta^* N^\gamma)^{-1}\right]$$
$$\geq -(\delta^* N^\gamma)^{-1}\ell n(1-P^*),$$

which implies $N \geq \{-\delta^{*-2}\ell n(1-P^*)\}^{\frac{1}{1+\gamma}}$.

It is easy to see that P2(2)' has the same asymptotic properties (as $P^*\to 1$) as those of P2(2), namely, theorems 4.3 and 4.4. We will present numerical results about the rules P2(2) and P2(2)' in subsection 4.4.3. The property obtained in theorem 4.2 and theorem 4.4 is referred as "asymptotic efficiency".

Procedure P3(2)

Let S_m^2, a_m, Z_{ijm}, and $r_n(S_m)$ be defined as in subsection 4.3.2, and let $b_m = -\frac{1}{2}\nu(\beta^{-2/\nu}-1) \simeq (\ell n\,\beta)[1-(\ell n\,\beta)/\nu]$, with $\alpha = \beta = 1-P^*$, and m=n* as in procedure P2(k) of subsection 4.3.1. The stopping rule is proposed as follows:

P3(2): Observe $\{X_{i1}, \ldots, X_{im};\ i=1, 2\}$ and then $Z_{ij(m+1)}$, $Z_{ij(m+2)}, \ldots$

successively. For $n \geq m$, after observing Z_{ijn}, we stop sampling and accept H_1 if $r_n(S_m) \leq b_m$, accept H_2 if $r_n(S_m) \geq a_m$, and we continue sampling if $b_m < r_n(S_m) < a_m$.

Procedure P4(2)

Let S_n^2, a_n, b_n, Z_{ijn}, and $r_n(S_n)$ be defined as in the procedure P3(2) with m=n. The stopping rule in this case is proposed as follows:

P4(2): Observe $\{X_{i1}, X_{i2}, \ldots;\ i=1, 2\}$ and thus obtain Z_{ij1}, Z_{ij2}, . . . successively. For each $n \geq 2$, after observing Z_{ijn}, we stop sampling and accept H_1 if $r_n(S_n) \leq b_n$, accept H_2 if $r_n(S_n) \geq a_n$, and we continue sampling if $b_n < r_n(S_n) < a_n$.

We will present some numerical results regarding the rules P3(2) and P4(2) in the following subsection 4.4.3.

4.4.3. Moderate Sample Size Behavior of Our Rules and Comparisons With Fixed Sample Size Procedures

We are going to use our proposed rules (with k=2), compare them one by one with the fixed sample rule (FSR) (as given in Gibbons et al. (1977), Chapter 2). We look at Table A.1 from the same book. For each n' and P*, we compute $\delta^* = \zeta_t \sigma(n')^{-\frac{1}{2}}$ (where ζ_t comes from Table A.1), and we generate two populations Π_1 and Π_2 in an IBM 370/168 Computer for simulation purposes.

We used subroutine RANDU to generate Uniform (0, 1) variates and subroutine GAUSS to obtain samples from normal variates (see p. 77 of

IBM application program, 1970). We generate Π_1 as $N(\delta^*, 1)$ and Π_2 as $N(0, 1)$ so that the hypothesis H_1 is deliberately made to be true. For each pair of values of (δ^*, P^*), we repeat the experiment 200 times using rules P1*(2), P2(2), P2(2)', P3(2), and P4(2) for Tables XII, XIII, XIV, XV, and XVI, respectively. When we use P2(2), sometimes we fall short of P*. To remedy this, we suggest taking some extra samples of fixed size, say R, once the rule P2(2) stops by itself. The Table XVII suggests that on the average this (fixed) extra sample size is possibly five. Our Table XVII presents results for R=3, 5, and 10. In Table XII, under the "untruncated part", we estimate the average sample size $\bar{N}$, its standard error $S(\bar{N})$ and P, the relative frequency of correct decision (in favor of H_1) for all the repetitions (out of 200) which did not have to be truncated; under the heading "truncated", we report T, the number of truncations and P', the relative frequency of correct decision (in favor of H_1) out of T truncations; under the "over all" category we report $\bar{N}$, $S(\bar{N})$, and P" computed from all the 200 repetitions; under the "asymptotic" category we provide with values of C and $D(200)^{-\frac{1}{2}} = D'$ in Tables XII, XIII, XIV, XV, XVI, and XVII, where C and D are given in 4.7. We estimate the "overall saving η" {Mukhopadhyay and Chou (1981)} in the same way as on page 17, namely $\eta = (n'p'' - \bar{N}P^*)/n'P''$, where n' is the sample size needed by the FSR, $\bar{N}$ is the "over all" average sample size. We should stress that all the entries in columns four and beyond are estimated from simulated data. In Tables XIV and XV, n* is the starting sample size, γ is taken to be 1/2, 1/3, and 1/4. In Table XVII, R is the number of extra samples taken, after the rule P2(2) stops by itself.

<u>Remark 4.2</u>: From Table XII, we notice that, on the average, the

TABLE XII

SIMULATION RESULT FOR THE RULE P1*(2), VARIANCE KNOWN

			Untruncated part			Truncated		Over all				Asymptotic	
n'	P*	δ*	$\bar{N}$	S($\bar{N}$)	P	T	P'	P''	$\bar{N}$	S($\bar{N}$)	η	C	D'
2	0.90	1.282	1.485	0.043	0.940	66	0.864	0.915	1.655	0.034	0.186	1.402	0.092
	0.95	1.645	1.430	0.039	0.988	35	0.943	0.980	1.530	0.035	0.258	1.107	0.064
4	0.90	0.906	2.390	0.086	0.966	54	0.815	0.925	2.825	0.080	0.313	2.804	0.185
	0.95	1.163	2.425	0.078	0.976	33	0.848	0.955	2.685	0.078	0.332	2.214	0.128
6	0.90	0.740	3.221	0.112	0.961	46	0.739	0.910	3.860	0.119	0.364	4.206	0.277
	0.95	0.950	3.226	0.109	0.981	41	0.829	0.950	3.795	0.117	0.368	3.322	0.192
10	0.90	0.573	5.317	0.183	0.957	39	0.795	0.925	6.230	0.198	0.394	7.010	0.462
	0.95	0.736	4.911	0.169	0.994	31	0.839	0.970	5.700	0.193	0.442	5.536	0.320
16	0.90	0.453	7.969	0.284	0.938	38	0.658	0.885	9.495	0.321	0.397	11.216	0.739
	0.95	0.582	7.476	0.293	0.965	30	0.833	0.945	8.755	0.329	0.450	8.858	0.512
30	0.90	0.331	14.987	0.554	0.956	42	0.786	0.920	18.140	0.616	0.408	21.029	1.386
	0.95	0.425	12.640	0.491	0.966	22	0.864	0.955	14.550	0.582	0.518	16.604	0.960
60	0.90	0.234	28.238	1.060	0.919	28	0.714	0.890	32.685	1.201	0.449	42.059	2.772
	0.95	0.300	26.788	0.977	0.962	16	0.750	0.945	29.445	1.102	0.507	33.217	1.919
120	0.90	0.165	60.137	2.195	0.926	25	0.720	0.900	67.620	2.378	0.437	84.118	5.544
	0.95	0.212	48.732	1.976	0.978	21	0.857	0.965	56.215	2.350	0.539	66.434	3.838

TABLE XIII

SIMULATION RESULT FOR THE RULE P2(2), VARIANCE UNKNOWN

n'	P*	δ *	$\bar{N}$	$S(\bar{N})$	P	η	C	D'
2	0.90	1.282	2.575	0.086	0.945	-0.226	1.402	0.092
	0.95	1.645	2.295	0.045	0.990	-0.101	1.107	0.064
4	0.90	0.906	3.380	0.155	0.915	0.169	2.804	0.185
	0.95	1.163	3.120	0.117	0.950	0.220	2.214	0.128
6	0.90	0.740	4.395	0.232	0.915	0.280	4.206	0.277
	0.95	0.950	4.040	0.190	0.925	0.380	3.322	0.192
10	0.90	0.573	5.905	0.355	0.885	0.399	7.010	0.462
	0.95	0.736	4.930	0.289	0.940	0.502	5.536	0.320
16	0.90	0.453	8.480	0.568	0.880	0.458	11.216	0.739
	0.95	0.582	6.630	0.440	0.900	0.563	8.858	0.512
30	0.90	0.331	15.875	1.135	0.865	0.449	21.029	1.386
	0.95	0.425	12.050	0.768	0.900	0.576	16.608	0.960
60	0.90	0.234	32.360	2.088	0.885	0.452	42.059	2.772
	0.95	0.300	25.635	1.619	0.925	0.561	33.217	1.919
120	0.90	0.165	60.375	4.191	0.855	0.470	84.118	5.544
	0.95	0.212	48.715	3.036	0.910	0.576	66.434	3.838

TABLE XIV

SIMULATION RESULT FOR THE RULE P2(2)',
VARIANCE UNKNOWN

γ	n'	P*	δ *	n*	$\bar{N}$	$S(\bar{N})$	P	η	C	D'
1/2	10	0.90	0.573	4	9.725	0.312	0.945	0.074	7.010	0.462
		0.95	0.736	4	8.905	0.384	0.990	0.145	5.536	0.320
	16	0.90	0.453	6	15.595	0.605	0.985	0.109	11.216	0.739
		0.95	0.582	5	12.440	0.434	0.980	0.246	8.858	0.512
	30	0.90	0.331	8	26.655	1.072	0.970	0.176	21.029	1.386
		0.95	0.425	7	21.140	0.715	0.975	0.313	16.608	0.960
	60	0.90	0.234	13	47.520	1.696	0.950	0.250	42.059	2.772
		0.95	0.300	11	39.390	1.460	0.980	0.364	33.217	1.919
	120	0.90	0.165	20	96.220	4.376	0.950	0.240	84.118	5.544
		0.95	0.212	17	74.245	3.036	0.985	0.403	66.434	3.838
1/3	10	0.90	0.573	5	10.970	0.340	0.970	-0.018	7.010	0.462
		0.95	0.736	4	9.630	0.352	0.985	0.071	5.536	0.320
	16	0.90	0.453	7	17.025	0.574	0.985	0.028	11.216	0.739
		0.95	0.582	6	13.450	0.422	0.985	0.189	8.858	0.512
	30	0.90	0.331	10	30.105	1.070	0.970	0.069	21.029	1.386
		0.95	0.425	9	24.895	0.886	0.995	0.208	16.608	0.960
	60	0.90	0.234	17	54.905	1.783	0.970	0.151	42.059	2.772
		0.95	0.300	14	44.015	1.365	0.975	0.285	33.217	1.919
	120	0.90	0.165	28	111.215	4.611	0.960	0.131	84.118	5.544
		0.95	0.212	24	83.515	3.062	0.980	0.325	66.434	3.838

TABLE XIV (Continued)

γ	n'	P*	δ *	n*	$\bar{N}$	S($\bar{N}$)	P	η	C	D'
1/4	10	0.90	0.573	5	11.945	0.368	0.985	-0.091	7.010	0.462
		0.95	0.736	4	10.180	0.354	0.985	0.018	5.536	0.320
	16	0.90	0.453	7	18.130	0.574	0.985	-0.035	11.216	0.739
		0.95	0.582	6	14.665	0.440	0.990	0.120	8.858	0.512
	30	0.90	0.331	12	33.055	1.193	0.985	-0.007	21.029	1.386
		0.95	0.425	10	26.510	0.873	0.990	0.152	16.608	0.960
	60	0.90	0.234	20	59.775	1.736	0.980	0.085	42.059	2.772
		0.95	0.300	17	48.920	1.503	0.985	0.214	33.217	1.919
	120	0.90	0.165	35	123.055	4.497	0.965	0.044	84.118	5.544
		0.95	0.212	29	92.070	3.008	0.985	0.260	66.434	3.838

TABLE XV

SIMULATION RESULT FOR THE RULE P3(2), VARIANCE UNKNOWN

γ	n'	P*	δ *	n*	$\bar{N}$	$S(\bar{N})$	P	η	C	D'
1/2	10	0.90	0.573	4	9.505	0.478	0.940	0.090	7.010	0.462
		0.95	0.736	4	9.910	0.538	0.980	0.039	5.536	0.320
	16	0.90	0.453	6	14.790	0.819	0.960	0.133	11.216	0.739
		0.95	0.582	5	12.345	0.600	0.965	0.240	8.858	0.512
	30	0.90	0.331	8	23.865	1.319	0.960	0.254	21.029	1.386
		0.95	0.425	7	20.910	1.159	0.980	0.324	16.608	0.960
	60	0.90	0.234	13	41.510	2.229	0.940	0.338	42.059	2.772
		0.95	0.300	11	37.880	1.824	0.970	0.382	33.217	1.919
	120	0.90	0.165	20	81.855	4.146	0.940	0.347	84.118	5.544
		0.95	0.212	17	68.530	3.436	0.965	0.438	66.434	3.838
1/3	10	0.90	0.573	5	9.715	0.463	0.945	0.075	7.010	0.462
		0.95	0.736	4	9.910	0.538	0.980	0.039	5.536	0.320
	16	0.90	0.453	7	14.395	0.715	0.960	0.157	11.216	0.739
		0.95	0.582	6	12.210	0.515	0.960	0.245	8.858	0.512
	30	0.90	0.331	10	24.695	1.303	0.970	0.236	21.029	1.386
		0.95	0.425	9	20.800	1.014	0.980	0.328	16.608	0.960
	60	0.90	0.234	17	42.170	2.010	0.930	0.320	42.059	2.772
		0.95	0.300	14	37.770	1.808	0.985	0.393	32.217	1.919
	120	0.90	0.165	28	79.075	4.140	0.935	0.366	84.118	5.544
		0.95	0.212	24	67.845	3.435	0.980	0.452	66.434	3.838

TABLE XV (Continued)

γ	n'	P*	$\hat{\delta}$ *	n*	$\bar{N}$	S($\bar{N}$)	P	η	C	D'
1/4	10	0.90	0.573	5	9.715	0.463	0.945	0.075	7.010	0.462
		0.95	0.736	4	9.910	0.538	0.980	0.039	5.536	0.320
	16	0.90	0.453	7	14.395	0.715	0.960	0.157	11.216	0.739
		0.95	0.582	6	12.210	0.515	0.960	0.245	8.858	0.512
	30	0.90	0.331	12	25.155	1.285	0.950	0.206	21.029	1.386
		0.95	0.425	10	21.610	1.051	0.990	0.309	16.608	0.960
	60	0.90	0.234	20	43.730	1.901	0.945	0.306	42.059	2.772
		0.95	0.300	17	36.940	1.576	0.970	0.397	33.217	1.919
	120	0.90	0.165	35	80.775	3.986	0.940	0.356	84.118	5.544
		0.95	0.212	29	67.750	3.210	0.970	0.447	66.434	3.838

TABLE XVI

SIMULATION RESULT FOR THE RULE P4(2), VARIANCE UNKNOWN

n'	P*	δ *	$\bar{N}$	S($\bar{N}$)	P	η	C	D'
2	0.90	1.282	3.200	0.180	0.975	-0.477	1.402	0.092
	0.95	1.645	2.770	0.064	0.995	-0.322	1.107	0.064
4	0.90	0.906	4.060	0.172	0.915	0.002	2.804	0.185
	0.95	1.163	3.955	0.143	0.965	0.027	2.214	0.128
6	0.90	0.740	5.535	0.272	0.940	0.117	4.206	0.277
	0.95	0.950	5.025	0.214	0.930	0.144	3.322	0.192
10	0.90	0.573	7.150	0.362	0.935	0.312	7.010	0.462
	0.95	0.736	6.320	0.321	0.930	0.354	5.536	0.320
16	0.90	0.453	10.465	0.597	0.900	0.346	11.216	0.739
	0.95	0.582	9.360	0.518	0.925	0.399	8.858	0.512
30	0.90	0.331	19.530	1.207	0.945	0.380	21.029	1.386
	0.95	0.425	16.025	0.864	0.950	0.466	16.608	0.960
60	0.90	0.234	36.355	1.971	0.940	0.420	42.059	2.772
	0.95	0.300	29.980	1.541	0.945	0.498	33.217	1.919
120	0.90	0.165	69.715	4.131	0.915	0.429	84.118	5.544
	0.95	0.212	56.370	3.012	0.950	0.530	66.434	3.838

TABLE XVII

SIMULATION RESULT FOR THE RULE P2(2) WITH R MORE ADDITIONAL SAMPLES WHEN THE SAMPLING TERMINATES, VARIANCE UNKNOWN

R	n'	P*	δ*	$\bar{N}$	S($\bar{N}$)	P	η	C	D'
3	6	0.90	0.740	7.010	0.188	0.935	-0.125	4.206	0.277
		0.95	0.950	6.600	0.170	0.940	-0.112	3.322	0.192
	10	0.90	0.573	8.295	0.319	0.890	0.161	7.010	0.462
		0.95	0.736	8.390	0.334	0.955	0.165	5.536	0.320
	16	0.90	0.453	10.810	0.546	0.885	0.313	11.216	0.739
		0.95	0.582	9.695	0.397	0.900	0.360	8.858	0.512
	30	0.90	0.331	18.685	1.110	0.910	0.384	21.029	1.386
		0.95	0.425	16.110	0.843	0.925	0.448	16.608	0.960
	60	0.90	0.234	32.880	1.902	0.890	0.446	42.059	2.772
		0.95	0.300	28.890	1.482	0.930	0.508	33.217	1.919
	120	0.90	0.165	60.420	3.675	0.855	0.470	84.118	5.544
		0.95	0.212	51.555	3.340	0.895	0.544	66.434	3.838
5	6	0.90	0.740	8.970	0.209	0.965	-0.394	4.206	0.277
		0.95	0.950	8.650	0.179	0.965	-0.419	3.322	0.192
	10	0.90	0.573	10.385	0.337	0.935	0.000	7.010	0.462
		0.95	0.736	10.290	0.335	0.950	-0.029	5.536	0.320
	16	0.90	0.453	13.365	0.533	0.905	0.169	11.216	0.739
		0.95	0.582	12.095	0.452	0.945	0.240	8.858	0.512

TABLE XVII (CONTINUED)

R	n'	P*	δ *	$\bar{N}$	S($\bar{N}$)	P	η	C	D'
	30	0.90	0.331	20.400	1.187	0.910	0.327	21.029	1.386
		0.95	0.425	18.625	0.887	0.955	0.382	16.608	0.960
	60	0.90	0.234	35.635	1.914	0.905	0.409	42.059	2.772
		0.95	0.300	30.025	1.476	0.920	0.483	33.217	1.919
	120	0.90	0.165	67.025	4.183	0.890	0.435	84.118	5.544
		0.95	0.212	55.895	2.907	0.935	0.527	66.434	3.838
10	6	0.90	0.740	13.865	0.187	0.970	-1.144	4.206	0.277
		0.95	0.950	13.805	0.185	0.995	-1.197	3.322	0.192
	10	0.90	0.573	15.480	0.313	0.975	-0.429	7.010	0.462
		0.95	0.736	15.315	0.311	0.970	-0.500	5.536	0.320
	16	0.90	0.453	18.745	0.574	0.940	-0.122	11.216	0.739
		0.95	0.582	17.035	0.428	0.960	-0.054	8.858	0.512
	30	0.90	0.331	24.950	1.024	0.935	0.199	21.029	1.386
		0.95	0.425	24.010	0.882	0.935	0.187	16.608	0.960
	60	0.90	0.234	40.025	1.862	0.920	0.347	42.059	2.772
		0.95	0.300	32.495	1.488	0.895	0.425	33.217	1.919
	120	0.90	0.165	71.910	4.084	0.875	0.384	84.118	5.544
		0.95	0.212	63.310	3.271	0.950	0.472	66.434	3.838

percentage of saving increases when n' (or P*) increases. For instance, when n' = 10, P* = 0.90, we have η = 0.394, while for n' = 10, P* = 0.95, we have η = 0.442; or for n' = 60, P* = 0.90, we have η = 0.449. The over all relative frequency of correct decision in favor of H_1, namely P", on the average, is lower than P for the untruncated part. We believe that this is due to having relatively low frequencies of correct decision under truncation (on the average) at m=n'.

Remark 4.3: Comparing Tables XIII and XIV, we notice that the overall relative frequencies of correct selection in Table XIII are not too impressive, in general. By sacrificing some saving, that is, taking some more samples in Table XIV we obtain considerably increased amount of overall relative frequencies of correct selection. Also P increases while γ and η decrease.

Remark 4.4: Table XV presents the numerical results for the procedure P3(2) with the starting sample size n* as defined in procedure P2(k). It shows an impressive amount of saving with respect to FSR, and our results for the relative frequencies of correct selection are also very encouraging.

Remark 4.5: In Table XVI, we notice that, by sacrificing some saving, we obtain considerably increased amounts of overall relative frequencies of correct selection for moderate values of n'. In general, we have quite an impressive amount of saving with respect to FSR.

Remark 4.6: One alternative way to increase the relative frequencies of correct selection for the rule P2(2) is to take some more extra samples of fixed size, say R, when the original rule P2(2) stops. In Table XVII, we see that R=5 seems to be a good guess for this extra fixed sample size.

4.5. The Special Case of Three Populations

We will discuss this problem under the situations when the common variance is known or unknown in separate subsections.

4.5.1. The Common Variance Is Known

In this case, the rule P1(k) takes the following form.

P1(3): $N = \inf\{n \geq 1: \delta^*\sigma^{-2}\sup_{j\neq i}\{\sum_{\ell=1}(X_{j\ell}-X_{i\ell})\} \leq \ln\{\frac{1}{2}(1-P^*)\}$ for some $i\}$,
$= \infty$ if no such n.

At stage N, if we stop with i, we accept the hypothesis H_i.

P1*(3): When we reach the stage n=m, and P1(3) does not stop by itself, we may wish to terminate sampling. In this case, we accept H_ℓ when

$$\delta^*\sigma^{-2}\sup_{j\neq\ell}\left[\sum_{h=1}^{m}(X_{jh} - X_{\ell h})\right] = \min_i \sup_{j\neq i} \delta^*\sigma^{-2}\left[\sum_{h=1}^{m}(X_{jh} - X_{ih})\right].$$

One can prove the following theorem without much difficulty. We omit its proof.

<u>Theorem 4.5</u>: <u>For fixed</u> σ <u>in</u> $(0, \infty)$ <u>and</u> μ_i <u>in</u> $(-\infty, \infty)$, <u>for each hypothesis</u> H_i, i=1, 2, 3, <u>we have the following for the rule</u> P1(3): <u>N is a non-decreasing function of</u> P*, $N\to\infty$ <u>a.s. as</u> $P^*\to 1$, <u>and</u> $N/C'\to 1$ <u>a.s. as</u> $P^*\to 1$, <u>where</u> $C' = -\sigma^2\delta^{*-2}\ln\left[\frac{1}{2}(1-P^*)\right]$.

In Table XVIII we present numerical results for the rule P1*(3) truncated at m=n'. Given P* and n', we obtain ζ_t from table A.1 of Gibbons et al. (1977), and then compute $\delta^* = \zeta_t\sigma(n')^{-\frac{1}{2}}$ as in section 4.4.3. Using the same program routines explained in section 4.4.3, we generate Π_1 as $N(\delta^*, 1)$, and both Π_2, Π_3 as $N(0, 1)$, so that the hypothesis H_1 is deliberately made to be true. We estimate $\bar{N}$, $S(\bar{N})$ for the "untruncated part" and "over all" part as in Table XII. Under each of these headings, when we report "proportion", we subdivide it into three parts--a part is labelled as proportion of frequencies we decided in favor of H_i, i=1, 2, 3, with that particular category of heading. For each pair of (n', P*) (or (P*, δ *)), we estimate the quantities from 200 repetitions in columns four and beyond. The amount of "saving n" is computed in the same way as in Table XII.

Remark 4.7: Comments like those in remark 4.2 are still valid for Table XVIII for the overall relative frequencies of correct selection in favor of H_1.

4.5.2. The Common Variance Is Unknown

In this subsection, we will discuss several procedures as in Section 4.3 for k=3.

Procedure P2(3)

Let $S_n{}^2$, $Z_{ij\ell}$, $\bar{Z}_{ijn}$ and n* as defined in P2(k) with k=3. Then our rule takes the following form.

P2(3): $N = \inf\{n \geq n^*, \delta^* S_n{}^{-2} \sup_{j \neq i} n\bar{Z}_{ijn} \leq \ln\left[\tfrac{1}{2}(1-P^*)\right]$ for some i},

$= \infty$ if no such n,

and when N stops with i, we decide for the hypothesis H_i, that is to declare that Π_i has the largest mean, i=1, . . ., k.

It is fairly simple to prove the following theorem.

Theorem 4.6: For fixed unknown σ in $(0, \infty)$, and μ_i in $(-\infty, \infty)$, for each hypothesis H_i, i=1, 2, 3, we have the following for the rule P2(3): N is a non-decreasing function of P*, $N \to \infty$ a.s. as $P^* \to 1$, and $N/C' \to 1$ a.s. as $P^* \to 1$, where $\delta^{*2} C' = -\sigma^2 \ln\left[\tfrac{1}{2}(1-P^*)\right]$.

In Table XIX, we present numerical results on simulating this procedure for several pairs of (P*, δ *). These entries should be interpreted in the same way as explained in Table XVIII under the "over all" category.

Remark 4.8: In Table XIX, for $P^* = 0.75$ most of the sample sizes were overly estimated, consequently we have considerably more in terms of extra proportion of frequencies we decide for H_1; while for $P^*>.90$ the sample sizes are under estimated, and consequently we lose in terms of having smaller proportion of correct decision. In general, we obtain considerably increased amounts of overall relative frequencies of correct selection in favor of H_1. Also the relative frequency under H_1 increases while γ and η decrease.

Procedure P3(3)

Let S_m^2, a_m, Z_{ijm}, and $r_n(S_m)$ be as defined in procedure P3(k) with k=3. In this case, the stopping rule takes the following form.

P3(3): Observe $\{X_{i1}, \ldots, X_{im};\ i=1, 2, 3\}$ and then $Z_{ij(m+1)}$, $Z_{ij(m+2)}$, . . . successively. For each $n \geq m$ after observing Z_{ijn}, we stop sampling and accept H_i, i=1, 2, 3, if $\sup_{j \neq i} r_n(S_m) \leq -a_m$ for some i, where m is taken to be n* which is defined in procedure P2(k).

In Table XX, we present results on simulating this procedure P3(3) for several pairs (P^*, δ^*). These entries should be interpreted in the same way as in Table XIX.

Remark 4.9: In Table XX we present the numerical results procedure P3(3) with starting sample size n* as defined in procedure P2(k). We notice that by sacrificing some saving, we obtain considerably increased amount of overall relative frequencies of correct selection, for small sample sizes n'. In general, we have an impressive amount of saving with respect to FSR.

TABLE XVIII

SIMULATION RESULT FOR THE RULE P1*(3), VARIANCE KNOWN

			Untruncated part					Truncated				Over all						
					proportion				proportion			proportion						
n'	P*	δ *	$\bar{N}$	$S(\bar{N})$	H_1	H_2	H_3	T	H_1	H_2	H_3	H_1	H_2	H_3	$\bar{N}$	$S(\bar{N})$	η	C'
2	0.75	1.01	1.769	0.048	0.872	0.064	0.064	122	0.680	0.180	0.139	0.755	0.135	0.110	1.910	0.020	0.051	2.023
	0.90	1.58	1.607	0.045	1.000	0.000	0.000	83	0.771	0.145	0.084	0.905	0.060	0.035	1.770	0.030	0.120	1.204
	0.95	1.92	1.539	0.042	0.986	0.000	0.014	59	0.898	0.017	0.085	0.960	0.005	0.035	1.675	0.033	0.171	1.005
	0.99	2.56	1.461	0.037	1.000	0.000	0.000	20	0.900	0.100	0.000	0.990	0.010	0.000	1.515	0.035	0.243	0.810
4	0.75	0.72	3.108	0.087	0.940	0.012	0.048	117	0.547	0.231	0.222	0.710	0.140	0.150	3.630	0.048	0.041	4.046
	0.90	1.12	2.730	0.070	0.980	0.007	0.013	48	0.771	0.167	0.063	0.930	0.045	0.025	3.035	0.066	0.266	2.409
	0.95	1.36	2.688	0.075	0.981	0.006	0.013	40	0.750	0.200	0.050	0.935	0.045	0.020	2.950	0.071	0.251	2.009
	0.99	1.81	2.528	0.066	1.000	0.000	0.000	20	0.909	0.045	0.046	0.990	0.005	0.005	2.690	0.067	0.328	1.620
6	0.75	0.59	4.208	0.148	0.945	0.027	0.028	128	0.672	0.195	0.133	0.770	0.135	0.095	5.355	0.081	0.131	6.069
	0.90	0.91	3.837	0.122	0.992	0.008	0.000	77	0.779	0.078	0.143	0.910	0.035	0.055	4.670	0.106	0.230	3.613
	0.95	1.11	3.736	0.108	0.982	0.018	0.000	37	0.838	0.054	0.108	0.955	0.025	0.020	4.155	0.108	0.311	3.014
	0.99	1.48	3.658	0.095	0.995	0.000	0.005	16	0.750	0.063	0.187	0.975	0.005	0.020	3.845	0.099	0.349	2.430
10	0.75	0.45	6.517	0.234	0.885	0.057	0.058	113	0.681	0.142	0.177	0.770	0.105	0.125	8.485	0.159	0.174	10.115
	0.90	0.71	6.175	0.191	0.964	0.029	0.007	63	0.730	0.143	0.127	0.890	0.065	0.045	7.380	0.182	0.254	6.022
	0.95	0.86	5.929	0.160	0.981	0.019	0.000	44	0.886	0.091	0.023	0.960	0.035	0.005	6.825	0.173	0.325	5.023
	0.99	1.14	5.387	0.152	0.995	0.000	0.005	14	0.929	0.000	0.071	0.990	0.000	0.010	5.710	0.164	0.429	4.049

TABLE XVIII (Continued)

			Untruncated part					Truncated				Over all						
					proportion				proportion			proportion						
n'	P*	δ*	$\bar{N}$	$S(\bar{N})$	H_1	H_2	H_3	T	H_1	H_2	H_3	H_1	H_2	H_3	$\bar{N}$	$S(\bar{N})$	η	C'
16	0.75	0.36	10.929	0.348	0.828	0.071	0.101	101	0.713	0.139	0.149	0.770	0.105	0.125	13.490	0.249	0.179	16.184
	0.90	0.56	9.567	0.300	0.973	0.007	0.020	50	0.760	0.140	0.100	0.920	0.040	0.040	11.175	0.299	0.317	9.636
	0.95	0.68	8.719	0.284	0.963	0.025	0.012	40	0.750	0.125	0.125	0.920	0.045	0.035	10.175	0.307	0.343	8.036
	0.99	0.90	8.299	0.249	0.995	0.000	0.005	13	1.000	0.000	0.000	0.995	0.000	0.005	8.800	0.269	0.453	6.479
30	0.75	0.26	18.648	0.643	0.857	0.067	0.076	95	0.674	0.200	0.126	0.770	0.130	0.100	24.040	0.524	0.219	30.345
	0.90	0.41	17.599	0.563	0.966	0.014	0.020	53	0.774	0.132	0.094	0.915	0.045	0.040	20.885	0.567	0.315	18.067
	0.95	0.49	16.812	0.510	0.952	0.042	0.006	35	0.686	0.200	0.114	0.905	0.070	0.025	19.120	0.551	0.331	15.068
	0.99	0.66	13.962	0.431	1.000	0.000	0.000	15	0.667	0.133	0.200	0.975	0.010	0.015	15.165	0.499	0.487	12.148
60	0.75	0.19	37.510	1.354	0.854	0.063	0.083	104	0.702	0.173	0.125	0.775	0.120	0.105	49.205	1.027	0.206	60.691
	0.90	0.29	33.731	1.078	0.945	0.028	0.027	55	0.855	0.091	0.055	0.920	0.045	0.035	40.955	1.141	0.332	36.135
	0.95	0.35	32.048	1.009	0.958	0.024	0.018	33	0.818	0.182	0.000	0.935	0.050	0.015	36.660	1.118	0.379	30.135
	0.99	0.47	28.521	0.865	1.000	0.000	0.000	12	0.833	0.083	0.084	0.990	0.005	0.005	30.410	0.971	0.493	24.295
120	0.75	0.13	72.971	2.690	0.794	0.118	0.088	98	0.602	0.153	0.245	0.700	0.135	0.165	96.015	2.157	0.143	121.381
	0.90	0.20	62.409	1.970	0.940	0.020	0.040	51	0.863	0.078	0.059	0.920	0.035	0.045	77.095	2.306	0.372	72.270
	0.95	0.25	65.440	2.124	0.958	0.024	0.018	34	0.824	0.088	0.088	0.935	0.035	0.030	74.715	2.284	0.367	60.271
	0.99	0.33	58.746	1.747	0.995	0.000	0.005	11	0.727	0.182	0.091	0.980	0.010	0.010	62.115	1.925	0.477	48.590

TABLE XIX

SIMULATION RESULT FOR THE RULE P2(3), VARIANCE UNKNOWN

γ	n'	P*	δ*	n*	$\bar{N}$	$S(\bar{N})$	H_1	H_2	H_3	η
1/2	6	0.75	0.59	4	9.300	0.278	0.865	0.060	0.075	-0.344
		0.90	0.91	3	6.350	0.171	0.925	0.050	0.025	-0.030
		0.95	1.11	3	5.300	0.132	0.960	0.015	0.025	0.126
		0.99	1.48	2	4.650	0.118	0.980	0.015	0.005	0.217
	10	0.75	0.45	5	13.065	0.340	0.865	0.060	0.075	-0.133
		0.90	0.71	4	9.240	0.254	0.905	0.065	0.030	0.081
		0.95	0.86	3	8.170	0.226	0.955	0.020	0.025	0.187
		0.99	1.14	3	6.820	0.160	0.960	0.000	0.040	0.297
	16	0.75	0.36	7	20.250	0.610	0.830	0.105	0.065	-0.144
		0.90	0.56	5	13.680	0.398	0.890	0.045	0.065	0.135
		0.95	0.68	5	12.050	0.318	0.925	0.035	0.040	0.227
		0.99	0.90	4	10.195	0.239	0.980	0.010	0.010	0.356
	30	0.75	0.26	10	33.545	1.076	0.845	0.080	0.075	0.008
		0.90	0.41	7	24.175	0.763	0.915	0.050	0.035	0.207
		0.95	0.49	7	20.835	0.575	0.900	0.060	0.040	0.267
		0.99	0.66	6	17.375	0.439	0.965	0.015	0.020	0.406
	60	0.75	0.19	16	66.260	2.016	0.810	0.100	0.090	-0.023
		0.90	0.29	11	44.935	1.461	0.875	0.060	0.065	0.230
		0.95	0.35	10	38.780	1.140	0.910	0.050	0.040	0.325
		0.99	0.47	9	31.905	0.847	0.955	0.020	0.025	0.449
	120	0.75	0.13	25	121.385	4.685	0.730	0.115	0.155	-0.039
		0.90	0.20	18	80.430	2.432	0.890	0.060	0.050	0.322
		0.95	0.25	16	73.380	2.196	0.905	0.035	0.060	0.358
		0.99	0.33	14	62.430	1.717	0.960	0.025	0.015	0.463

TABLE XIX (Continued)

γ	n'	P*	δ*	n*	$\bar{N}$	$S(\bar{N})$	H_1	H_2	H_3	η
1/3	6	0.75	0.59	4	10.300	0.256	0.900	0.055	0.045	-0.431
		0.90	0.91	3	6.910	0.169	0.945	0.025	0.030	-0.097
		0.95	1.11	3	5.600	0.132	0.960	0.025	0.015	0.076
		0.99	1.48	2	4.890	0.111	0.980	0.005	0.015	0.177
	10	0.75	0.45	6	15.040	0.399	0.840	0.085	0.075	-0.343
		0.90	0.71	4	10.175	0.242	0.945	0.030	0.025	0.031
		0.95	0.86	4	8.895	0.233	0.970	0.020	0.010	0.129
		0.99	1.14	3	7.630	0.173	0.975	0.000	0.025	0.225
	16	0.75	0.36	9	22.430	0.570	0.825	0.095	0.080	-0.274
		0.90	0.56	6	15.035	0.359	0.920	0.045	0.035	0.081
		0.95	0.68	5	13.535	0.343	0.950	0.025	0.025	0.154
		0.99	0.90	5	11.165	0.238	0.990	0.005	0.005	0.302
	30	0.75	0.26	13	38.810	1.042	0.825	0.085	0.090	-0.176
		0.90	0.41	9	27.110	0.822	0.935	0.020	0.045	0.130
		0.95	0.49	8	23.120	0.537	0.900	0.055	0.045	0.187
		0.99	0.66	7	19.415	0.443	0.980	0.010	0.010	0.346
	60	0.75	0.19	22	75.420	2.459	0.770	0.100	0.130	-0.224
		0.90	0.29	15	51.210	1.489	0.890	0.060	0.050	0.137
		0.95	0.35	13	43.315	1.112	0.925	0.045	0.030	0.259
		0.99	0.47	11	35.420	0.799	0.985	0.010	0.005	0.407
	120	0.75	0.13	37	140.495	4.387	0.785	0.095	0.120	-0.119
		0.90	0.20	25	92.625	2.749	0.885	0.055	0.060	0.215
		0.95	0.25	22	86.429	2.374	0.930	0.045	0.025	0.264
		0.99	0.33	19	70.325	1.673	0.965	0.020	0.015	0.399

TABLE XIX (Continued)

γ	n'	P*	δ*	n*	$\bar{N}$	S($\bar{N}$)	H_1	H_2	H_3	η
1/4	6	0.75	0.59	5	10.585	0.262	0.915	0.045	0.040	-0.446
		0.90	0.91	3	7.235	0.185	0.980	0.010	0.010	-0.107
		0.95	1.11	3	5.870	0.132	0.985	0.010	0.005	0.056
		0.99	1.48	3	5.035	0.107	0.990	0.005	0.005	0.161
	10	0.75	0.45	7	16.415	0.422	0.880	0.070	0.050	-0.399
		0.90	0.71	5	10.855	0.245	0.930	0.035	0.035	-0.050
		0.95	0.86	4	9.295	0.209	0.945	0.020	0.035	0.066
		0.99	1.14	4	7.850	0.168	0.975	0.000	0.025	0.203
	16	0.75	0.36	10	24.685	0.612	0.845	0.090	0.065	-0.369
		0.90	0.56	7	16.340	0.395	0.940	0.030	0.030	0.022
		0.95	0.68	6	13.995	0.339	0.945	0.020	0.035	0.121
		0.99	0.90	5	12.020	0.251	0.995	0.000	0.005	0.253
	30	0.75	0.26	16	42.550	1.072	0.835	0.090	0.075	-0.274
		0.90	0.41	11	29.935	0.896	0.940	0.030	0.030	0.045
		0.95	0.49	9	24.945	0.546	0.930	0.035	0.035	0.151
		0.99	0.66	8	20.455	0.420	0.975	0.010	0.015	0.308
	60	0.75	0.19	27	82.045	2.225	0.820	0.090	0.090	-0.251
		0.90	0.29	18	55.600	1.456	0.885	0.060	0.055	0.058
		0.95	0.35	16	46.800	1.170	0.945	0.035	0.020	0.216
		0.99	0.47	13	37.560	0.705	0.965	0.015	0.020	0.358
	120	0.75	0.13	47	154.595	4.589	0.810	0.080	0.110	-0.193
		0.90	0.20	31	102.075	2.838	0.935	0.025	0.040	0.181
		0.95	0.25	27	91.615	2.268	0.945	0.020	0.035	0.233
		0.99	0.33	23	76.645	1.770	0.970	0.025	0.005	0.348

TABLE XX

SIMULATION RESULT FOR THE RULE P3(3), VARIANCE UNKNOWN

γ	n'	P*	δ*	n*	$\bar{N}$	S($\bar{N}$)	H_1	H_2	H_3	η
1/2	6	0.75	0.59	4	10.950	0.537	0.935	0.035	0.030	-0.464
		0.90	0.91	3	8.395	0.353	0.970	0.015	0.015	-0.298
		0.95	1.11	3	6.955	0.270	0.970	0.020	0.010	-0.135
		0.99	1.48	2	8.575	0.412	1.000	0.000	0.000	-0.415
	10	0.75	0.45	5	14.620	0.808	0.900	0.065	0.035	-0.218
		0.90	0.71	4	12.295	0.591	0.995	0.000	0.005	-0.112
		0.95	0.86	3	10.450	0.512	0.975	0.010	0.010	-0.018
		0.99	1.14	3	10.050	0.389	1.00	0.000	0.000	0.005
	16	0.75	0.36	7	23.050	1.222	0.890	0.050	0.060	-0.214
		0.90	0.56	5	16.050	0.689	0.955	0.015	0.030	0.055
		0.95	0.68	5	14.720	0.673	0.980	0.010	0.010	0.108
		0.99	0.90	4	13.655	0.528	1.000	0.000	0.000	0.155
	30	0.75	0.26	10	38.445	2.062	0.885	0.080	0.035	-0.086
		0.90	0.41	7	28.140	1.348	0.965	0.015	0.020	0.125
		0.95	0.49	7	25.135	1.225	0.960	0.035	0.005	0.171
		0.99	0.66	6	21.215	0.827	1.000	0.000	0.000	0.300
	60	0.75	0.19	16	80.795	3.981	0.875	0.060	0.065	-0.154
		0.90	0.29	11	53.755	2.549	0.955	0.015	0.030	0.156
		0.95	0.35	10	45.120	1.945	0.960	0.015	0.025	0.256
		0.99	0.47	9	35.155	1.330	0.995	0.000	0.005	0.417
	120	0.75	0.13	25	146.325	7.683	0.855	0.080	0.065	-0.070
		0.90	0.20	18	95.820	4.435	0.940	0.030	0.030	0.235
		0.95	0.25	16	87.180	3.400	0.970	0.015	0.015	0.288
		0.99	0.33	14	72.490	2.599	0.995	0.000	0.005	0.399

TABLE XX (Continued)

γ	n'	P*	δ*	n*	$\bar{N}$	$S(\bar{N})$	H_1	H_2	H_3	η
1/3	6	0.75	0.59	4	10.950	0.537	0.935	0.035	0.030	-0.464
		0.90	0.91	3	8.395	0.353	0.970	0.015	0.015	-0.298
		0.95	1.11	3	6.955	0.270	0.970	0.020	0.010	-0.135
		0.99	1.48	2	8.575	0.412	1.000	0.000	0.000	-0.415
	10	0.75	0.45	6	14.690	0.714	0.915	0.050	0.035	-0.204
		0.90	0.71	4	12.295	0.591	0.995	0.000	0.005	-0.112
		0.95	0.86	4	9.770	0.455	0.995	0.000	0.005	0.067
		0.99	1.14	3	10.050	0.389	1.000	0.000	0.000	0.005
	16	0.75	0.36	9	25.295	1.316	0.930	0.030	0.040	-0.275
		0.90	0.56	6	16.210	0.694	0.980	0.010	0.010	0.070
		0.95	0.68	5	14.720	0.673	0.980	0.010	0.010	0.108
		0.99	0.90	5	12.605	0.456	0.995	0.005	0.000	0.216
	30	0.75	0.26	13	40.635	1.981	0.905	0.050	0.045	-0.123
		0.90	0.41	9	26.185	1.223	0.945	0.030	0.025	0.169
		0.95	0.49	8	24.865	1.128	0.965	0.020	0.015	0.184
		0.99	0.66	7	20.755	0.792	1.000	0.000	0.000	0.315
	60	0.75	0.19	22	79.755	3.913	0.880	0.060	0.060	-0.133
		0.90	0.29	15	52.810	2.401	0.970	0.010	0.020	0.183
		0.95	0.35	13	42.920	1.841	0.960	0.020	0.020	0.292
		0.99	0.47	11	35.300	1.276	1.000	0.000	0.000	0.418
	120	0.75	0.13	37	146.325	7.473	0.860	0.080	0.060	-0.063
		0.90	0.20	25	94.110	4.418	0.940	0.025	0.035	0.249
		0.95	0.25	22	85.745	3.211	0.955	0.020	0.025	0.289
		0.99	0.33	19	70.495	2.718	1.000	0.000	0.000	0.418

TABLE XX (Continued)

γ	n'	P*	δ*	n*	$\bar{N}$	S($\bar{N}$)	H_1	H_2	H_3	η
1/4	6	0.75	0.59	5	10.605	0.448	0.920	0.055	0.025	-0.441
		0.90	0.91	3	8.395	0.353	0.970	0.015	0.015	-0.298
		0.95	1.11	3	6.955	0.270	0.970	0.020	0.010	-0.135
		0.99	1.48	3	6.905	0.267	0.995	0.000	0.005	-0.145
	10	0.75	0.45	7	15.555	0.759	0.935	0.040	0.025	-0.247
		0.90	0.71	5	11.130	0.490	0.985	0.005	0.010	-0.017
		0.95	0.86	4	9.770	0.455	0.995	0.000	0.005	0.067
		0.99	1.14	4	9.120	0.325	1.000	0.000	0.000	0.097
	16	0.75	0.36	10	23.755	1.213	0.905	0.040	0.055	-0.230
		0.90	0.56	7	16.210	0.636	0.970	0.010	0.020	0.060
		0.95	0.68	6	14.535	0.609	0.995	0.005	0.000	0.133
		0.99	0.90	5	12.605	0.456	0.995	0.005	0.000	0.216
	30	0.75	0.26	16	38.200	1.745	0.900	0.060	0.040	-0.061
		0.90	0.41	11	27.385	1.255	0.975	0.010	0.015	0.157
		0.95	0.49	9	24.365	1.043	0.965	0.030	0.005	0.200
		0.99	0.66	8	20.195	0.763	1.000	0.000	0.000	0.334
	60	0.75	0.19	27	80.580	4.067	0.875	0.055	0.070	-0.151
		0.90	0.29	18	53.155	2.354	0.960	0.020	0.020	0.169
		0.95	0.35	16	44.330	1.748	0.970	0.015	0.015	0.276
		0.99	0.47	13	34.185	1.202	0.995	0.000	0.005	0.433
	120	0.75	0.13	47	142.555	7.087	0.855	0.080	0.065	-0.042
		0.90	0.20	31	95.855	4.646	0.955	0.015	0.030	0.247
		0.95	0.25	27	87.135	3.220	0.960	0.025	0.015	0.281
		0.99	0.33	23	67.635	2.620	0.995	0.005	0.000	0.439

Procedure P4(3)

Let S_n^2, $r_n(S_n)$, and a_n be as defined in procedure P4(k) with k=3. In this case, the stopping rule takes the following form.

P4(3): Observe $\{X_{i1}, X_{i2}, \ldots; i=1, 2, 3\}$ and thus obtain Z_{ij1}, Z_{ij2}, . . ., successively. For each $n \geq 2$, after observing Z_{ijn}, we stop sampling and accept H_i, if $\sup_{j \neq i} r_n(S_n) \leq -a_n$ for some i.

In Table XXI, we present results on simulating this procedure for several pairs (P^*, δ^*). These entries mean the same things as in Table XIX.

Remark 4.10: Comments like those in remark 4.5 are still valid for Table XXI for overall relative frequencies of correct selection in favor of H_1.

TABLE XXI

SIMULATION RESULT FOR THE RULE P4(3), VARIANCE UNKNOWN

n'	P*	δ*	$\bar{N}$	$S(\bar{N})$	H_1	H_2	H_3	η
2	0.75	1.01	4.255	0.157	0.940	0.035	0.025	-0.697
	0.90	1.58	3.335	0.091	1.000	0.000	0.000	-0.501
	0.95	1.92	2.975	0.074	0.980	0.000	0.020	-0.442
	0.99	2.56	2.715	0.056	1.000	0.000	0.000	-0.344
4	0.75	0.72	5.770	0.242	0.865	0.070	0.065	-0.251
	0.90	1.12	4.530	0.152	0.975	0.010	0.015	-0.045
	0.95	1.36	4.200	0.126	1.000	0.000	0.000	0.003
	0.99	1.81	3.800	0.111	0.990	0.000	0.010	0.050
6	0.75	0.59	8.565	0.411	0.890	0.055	0.055	-0.203
	0.90	0.91	6.315	0.270	0.945	0.030	0.025	-0.002
	0.95	1.11	5.320	0.178	0.980	0.015	0.005	0.140
	0.99	1.48	4.955	0.156	0.990	0.000	0.010	0.174
10	0.75	0.45	12.420	0.659	0.915	0.065	0.020	-0.018
	0.90	0.71	9.450	0.426	0.970	0.010	0.020	0.123
	0.95	0.86	7.910	0.330	0.965	0.025	0.010	0.221
	0.99	1.14	6.945	0.232	0.995	0.000	0.005	0.309
16	0.75	0.36	19.155	1.138	0.865	0.055	0.080	-0.038
	0.90	0.56	13.455	0.581	0.950	0.015	0.035	0.203
	0.95	0.68	12.310	0.506	0.980	0.015	0.005	0.254
	0.99	0.90	10.015	0.347	0.995	0.000	0.005	0.377
30	0.75	0.26	32.365	1.695	0.855	0.080	0.065	0.054
	0.90	0.41	23.515	1.178	0.960	0.010	0.030	0.265
	0.95	0.49	21.130	0.949	0.945	0.035	0.020	0.292
	0.99	0.66	16.700	0.671	0.995	0.000	0.005	0.446
60	0.75	0.19	69.480	3.238	0.845	0.065	0.090	-0.028
	0.90	0.29	45.580	2.154	0.920	0.020	0.060	0.257
	0.95	0.35	38.475	1.721	0.940	0.030	0.030	0.352
	0.99	0.47	30.675	1.122	0.985	0.005	0.010	0.486
120	0.75	0.13	135.570	6.584	0.845	0.090	0.065	-0.003
	0.90	0.20	83.230	3.549	0.900	0.050	0.050	0.306
	0.95	0.25	78.120	3.067	0.940	0.035	0.025	0.342
	0.99	0.33	64.210	2.549	0.990	0.000	0.010	0.465

CHAPTER V

SUMMARY

The objective of this thesis is to develop procedures to solve the following two problems: (a) the selection of the smallest normal variance, (b) the selection of the largest normal mean. We adopt the indifference zone approach with a target value P* of the probability of correct selection. For the first problem, we develop sequential procedures through comparisons of likelihoods. For the second problem, by appealing to the rules developed in Mukhopadhyay (1980a) for the case of common variance being known, we develop some sequential procedures when the common variance is unknown. For numerical comparisons, we also present some modified rules along the lines of Baker (1950), Hall (1962), and Mukhopadhyay (1979, 1980b).

The proposed sequential procedures for both the problems result in a substantial "saving" in the average sample sizes compared to the corresponding well known (Chapter 2, Gibbons et al. (1977)) fixed sample size procedures. We suggest, however, two separate methods of the "saving" and work primarily with one of these notions.

For the first problem, we consider some special cases for some or all of the population means being known. In the cases k=2 and k=3, we have presented extensive numerical results through simulations suggesting the merits (in almost all the simulations) of our proposed procedures. For the case k=2, we have studied various asymptotic behavior (as $P^*\to 1$) of the stopping time involved in our statistical methods, and these are summarized in theorems 3.1 and 3.2. For the case k=3, we present some partial asymptotic results (as $P^*\to 1$) in theorem 3.3.

For the second problem, our major findings are in the case when the common variance is unknown. In the cases k=2 and k=3, we have also presented extensive numerical results through simulations suggesting the merits (in almost all the simulations) of our proposed procedures. For the case k=2, we have also studied various asymptotic behavior (as $P^*\to 1$) of the stopping time involved in our statistical methods, and these are summarized in theorems 4.1, 4.2, 4.3 and 4.4, we further obtain the property of "asymptotic efficiency". For the case k=3, we present some partial asymptotic results (as $P^*\to 1$) summarized in our theorems 4.5 and 4.6.

We have discussed the situations where there is only one population with the smallest variance for the first problem, and the situations where there is only one population with the largest mean. The situations where there are more than one "best" population may be solved by modifying our present solutions. The solutions for these types of practical problems are yet to be designed and studied along the lines of our suggestions in this dissertation.

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APPENDIXES

APPENDIX A

STATEMENTS OF SOME IMPORTANT RESULTS

A.1. Anscombe's (1952) Results

Let $\{Y_n\}$, n=1, 2, . . ., be an infinite sequence of random variables (r.v.'s). Suppose there exists a real number θ, a sequence of positive numbers $\{w_n\}$, and a distribution function $F(X)$, such that the following conditions are satisfied:

(C1) Convergence of $\{Y_n\}$: For any X such that F(X) is continuous {a "continuity point" of F(X)}, Prob $(Y_n - \theta \leq Xw_n) \to F(X)$ as $n\to\infty$.

(C2) Uniform continuity in probability of $\{Y_n\}$: Given any small positive ε and η, there is a large ν and a small positive c such that, for every $n>\nu$, Prob$\{|Y_{n'} - Y_n| < \varepsilon w_n$ simultaneously for all integers n' such that $|n' - n| < cn\} > 1-\eta$.

Let $\{X_n\}$, n=1, 2, . . ., denote an infinite sequence of r.v.'s, not necessarily independent. For each r, let Y_n and Z_n be functions of X_1, . . ., X_n, Suppose that $\{Y_n\}$ satisfies conditions C1 and C2 above. Let $\{a_r\}$, r=1, 2, . . ., be a decreasing sequence of positive numbers converging to zero. Let $\{N_r\}$ be a sequence of r.v.'s defined by the condition: N_r is the least positive integer n such that $Z_n \leq a_r$; and let $\{n_r\}$ be the sequence of positive integers defined by the conditions: n_r is the least n such that $w_n \leq a_r$.

(C3) Convergence of $\{w_n\}$: $\{w_n\}$ is decreasing, and it tends to zero such that $w_n/w_{n+1} \to 1$ as $n\to\infty$.

(C4) Convergence of $\{N_r\}$: N_r is a proper r.v. for all r, and $N_r/n_r \to 1$ in probability as $r\to\infty$.

Theorem:

(1) Under conditions C1 - C4, $\text{Prob}\{Y_{N_r} - \theta \le Xa_r\} \to F(X)$ as $r\to\infty$, at all continuity points X of F(X).

(2) If $X_1, X_2, \ldots,$ are independently and identically distributed, $Y_n = n^{-1}\sum_{i=1}^{n} X_i$, C1 and C3 hold, and F(X) is proper and continuous which imply that condition C2 also holds.

A.2. Dominated Ergodic Theorem (Wiener, 1939)

Theorem:

Let S be a measurable set of points of finite measure. Let T be a transformation of S into itself, which transforms every measurable subset of S into a set of equal measure, and whose inverse has the same property. Let f(P) be a function defined, over S and of Lebesque class L. Let $f(P) \ge 0$ on S and let $f^*(P) = \sup_{0<A<\infty} \frac{1}{A+1}\sum_{n=0}^{A} f(T^n P)$. Then if f(p) belongs to L^p (p>1), so does f*(p); while if $\int_S f(p)\log f(p)\, dV_p < \infty$, then f*(p) belong to L.

A.3. Fatou's Lemma

Theorem:

Let $g_n \ge f$(integrable) be a sequence of integrable functions. Then $\liminf_{n\to\infty} g_n$ is integrable and

$$\int \liminf_{n\to\infty} g_n \, d\mu \le \liminf_{n\to\infty} \int g_n \, d\mu$$

A.4. Ghosh and Mukhopadhyay's (1975) Result

Let $\{N_\nu, \nu \geq 1\}$ be a sequence of positive integer valued r.v.'s defined as follows: N_ν is the smallest positive integer $n(\geq n_0)$ for which $n \geq \psi_\nu T_n$, where n_0 is the starting sample size, $\{\psi_\nu\}$ is a sequence of positive constants, $\to\infty$ as $\nu\to\infty$, and $T_n(n\geq n_0)$ are statistics such that $P\{T_n \leq 0\} = 0$ for all $n \geq n_0$.

Theorem:

For the sequence of stopping times defined above, if $N_\nu^{\frac{1}{2}}(T_{N_\nu} - a)/b \xrightarrow{\mathcal{L}} N(0, 1)$ as $\nu\to\infty$, and

$N_\nu^{\frac{1}{2}}(T_{N_\nu - 1} - a)/b \xrightarrow{\mathcal{L}} N(0, 1)$ as $\nu\to\infty$, where $a(>0)$ and $b(>0)$ are constants, then $a^{\frac{1}{2}}(N_\nu - a\psi_\nu)/b\psi_\nu^{\frac{1}{2}} \xrightarrow{\mathcal{L}} N(0, 1)$ as $\nu\to\infty$.

A.5. Jensen's Inequality

Theorem:

Let υ be a real valued convex function, and X and $\upsilon(X)$ be integrable r.v.'s, then for each Borel subfield g:

$$\upsilon\{E(X|g)\} \leq E\{\nu(X)|g\}.$$

A.6. Mann and Wald's Theorem (Rao, 1973, p. 385)

Theorem:

Let $\{T_n\}$, $n=1, 2, \ldots$, be a sequence of statistics such that $n^{\frac{1}{2}}(T_n - \theta) \xrightarrow{\mathcal{L}} N\{0, \sigma^2(\theta)\}$. Let g be a function of a single variable admitting the first derivative g'. Then

$$n^{\frac{1}{2}}\{g(T_n) - g(\theta)\} \xrightarrow{\mathcal{L}} N\left(0, \{g'(\theta)\sigma(\theta)\}^2\right), \text{ if } g'(\theta) \neq 0.$$

Further let g' be continuous, then

$$n^{\frac{1}{2}}\{g(T_n) - g(\theta)\}/g'(T_n) \xrightarrow{\mathcal{L}} N\{0, \sigma^2(\theta)\},$$ and if $\sigma(\theta)$ is also continuous, then $n^{\frac{1}{2}}\{g(T_n) - g(\theta)\}/g'(T_n)\sigma(T_n) \xrightarrow{\mathcal{L}} N(0, 1)$.

A.7. Monotone Convergence Theorem

Theorem:

Let g_n be a sequence of non-negative, non-decreasing, and measurable functions, and let $\lim_{n\to\infty} g_n = g$, a.e. ($\mu$). Then g is measurable and $\lim_{n\to\infty}\int g_n d\mu = \int g d\mu$.

A.8. Starr and Woodroofe's (1968) Result

Let $X_1, X_2, \ldots,$ be a sequence of iid r.v.'s with finite expectation EX_1. Let $\bar{X}_n = n^{-1}\sum_{i=1}^{n} X_i$, $n \geq 1$, and let $\{C_n\}$ be any sequence of constants and m any positive integer. Suppose a stopping time based on the sequence $X_1, X_2, \ldots,$ is defined by

$$N = \{\text{smallest } n \geq m \text{ such that } \bar{X}_n \leq C_n\},$$
$$= \infty \text{ if no such n exists.}$$

Assume that $P(N<\infty) = 1$, so that $\bar{X}_N$ is well-defined.

Theorem:

If $E\bar{X}_N$ exists, then $E\bar{X}_N \leq EX_1$.

A.9. Strong Law of Large Numbers

Theorem:

Let $X_1, X_2, \ldots,$ be a sequence of iid r.v.'s. Then a necessary

<u>and sufficient condition that</u> $\bar{X}_n \to \mu$ <u>a.s. is that</u> $E(X_1)$ <u>exists and is equal to</u> μ.

A.10. Wald's First Equation

<u>Theorem</u>:

<u>Let</u> $X_1, X_2, \ldots,$ <u>be iid real-valued r.v.'s and</u> N <u>be a stopping time such that</u>

(i) $E\{|X_1|\} < \infty$,

(ii) <u>the event</u> $\{N \geq j\}$ <u>depends only on</u> $X_1, X_2, \ldots, X_{j-1}$,

(iii) $E(N) < \infty$,

<u>Then</u> $E(\sum_{i=1}^{N} X_i) = E(N)E(X_1)$.

APPENDIX B

PROOFS OF THEOREMS

B.1. Theorem 3.1

Let us work under the hypothesis H_1. Under H_2, a similar proof can easily be constructed. The first two parts of (3.10) are obvious. For the other part, we write the basic inequality:

$$-V_N \ln(1-P^*) \leq N \leq 1 - V_{N-1} \ln(1-P^*) \qquad \ldots (B.1)$$

where $V_n^{-1} = |\ln\{(1 + \delta^* T_{1n})/(\delta^* + T_{1n})\}| = |\ln(U_n)|$, say. Using the strong law of large numbers (Appendix A.9) and the fact that $N \to \infty$ a.s. as $P^* \to 1$, we conclude from (B1) that $N/C \to 1$ a.s. as $P^* \to 1$, where

$$C = -\{\ln(1-P^*)\}/\ln\{(1+\delta^{*2})/2\delta^*\}. \qquad \ldots (B.2)$$

We now proceed for a proof of (3.11) under H_1. Note that $n^{\frac{1}{2}}(T_{1n} - \delta^{*-1})/2^{\frac{1}{2}}\delta^{*-1} \overset{\mathcal{L}}{\to} N(0, 1)$ as $n \to \infty$, which implies $n^{\frac{1}{2}}\{U_n - 2\delta^*(1+\delta^{*2})^{-1}\}/2^{\frac{1}{2}}\delta^*(1+\delta^{*2})^{-1} \overset{\mathcal{L}}{\to} N(0, 1)$ as $n \to \infty$, and this further implies

$$(2n)^{\frac{1}{2}}\{\ln(U_n) - \ln\left(2\delta^*(1+\delta^{*2})^{-1}\right)\} \overset{\mathcal{L}}{\to} N(0, 1), \qquad \ldots (B.3)$$

as $n \to \infty$, by using Mann and Wald's theorem (Appendix A.6).

Using Mann and Wald's theorem all over again, we can write from (B.3) that

$$n^{\frac{1}{2}}\{V_n - (\ell n\{(1+\delta^{*2})/2\delta^*\})^{-1}\}/2^{-\frac{1}{2}}\{\ell n((1+\delta^{*2})/2\delta^*)\}^{-2} \xrightarrow{\mathcal{L}} N(0,\ 1), \quad \ldots \text{(B.4)}$$

as $n\to\infty$. One may note that the same result will also hold for V_{n-1}.

Since the sampling is carried out from normal distributions, we can use Helmert's orthogonal transformation on the X-variables and write $S_{in}{}^2 = (n-1)^{-1}\sum_{j=2}^{n} Y_{ij}{}^2$ where $Y_{i2}, \ldots, Y_{in}$ are iid $N(0,\ \sigma_i{}^2)$, $i=1,\ 2$. Then, we can apply Anscombe's (1952) results (Appendix A.1) to write

$$N^{\frac{1}{2}}(T_{1N} - \delta^{*-1})/2^{\frac{1}{2}}\delta^{*-1} \xrightarrow{\mathcal{L}} N(0,\ 1) \text{ as } P^*\to 1.$$

Now, retracing all the previous steps, we obtain from (B.4) that

$$N^{\frac{1}{2}}(V_N - a)/b \xrightarrow{\mathcal{L}} N(0,\ 1), \text{ as } P^*\to 1,$$

where $a^{-1} = \ell n((1+\delta^{*2})/2\delta^*)$, and $b = 2^{-\frac{1}{2}}\{\ell n((1+\delta^{*2})/2\delta^*)\}^{-2}$. . . . (B.5)

It may be remarked that the same result as in (B.5) also holds for V_{N-1}. Then we can apply a theorem of Ghosh and Mukhopadhyay (1975) (Appendix A.4) with $\psi_\nu = -\ell n(1-P^*)$. We obtain

$$a^{\frac{1}{2}}(N - a\psi_\nu)/b\psi_\nu{}^{\frac{1}{2}} \xrightarrow{\mathcal{L}} N(0,\ 1), \text{ as } P^*\to 1. \quad \ldots \text{(B.6)}$$

Now equation (B.6) can be rewritten as $(N - C)/D \xrightarrow{\mathcal{L}} N(0,\ 1)$ as $P^*\to 1$, where C is given (B.2) and D is given by

$$D^2 = b^2(\psi_\nu/a) = -\tfrac{1}{2}(\ell n(1-P^*))\{\ell n((1+\delta^{*2})/2\delta^*)\}^{-3}. \quad \ldots \text{(B.7)}$$

This completes the proof of theorem 3.1. ∇

B.2. Theorem 4.1

We are going to work under the hypothesis H_2, while a similar proof can easily be constructed under H_1. The first two parts of (4.5) are obvious. For the other part, we write the basic inequality,

$$-V_N \sigma^2 \delta^{*-1} \ln(1-P^*) \leq N \leq 1 - V_{N-1} \sigma^2 \delta^{*-1} \ln(1-P^*) \quad \ldots (B.8)$$

where $V_n^{-1} = |\bar{Z}_n|$. Using the strong law of large numbers (Appendix A.9) and the fact that $N \to \infty$ a.s. as $P^* \to 1$, we conclude from (B.8) that $N/C \to 1$ a.s. as $P^* \to 1$, where $C = -\sigma^2 \delta^{*-2} \ln(1-P^*)$. . . . (B.9)

We now proceed for a proof of (4.5) under H_2. Note that for fixed values of n, $n^{\frac{1}{2}}(\bar{Z}_n - \delta^*)/2^{\frac{1}{2}}\sigma \xrightarrow{L} N(0, 1)$ as $n \to \infty$.

Using Mann and Wald's theorem (Appendix A.6), we obtain

$$n^{\frac{1}{2}}(V_n - |\delta^*|^{-1})/2^{\frac{1}{2}}\sigma|\delta^*|^{-2} \xrightarrow{L} N(0, 1) \text{ as } n \to \infty. \quad \ldots (B.10)$$

One may note that the same result also holds for V_{n-1}.

Since the sampling is carried out from normal distributions, $\bar{Z}_n = n^{-1} \Sigma Z_i$, where $Z_1, \ldots, Z_n$ are iid $N\theta, 2\sigma^2)$. Applying Anscombe's (1952) results (Appendix A.1), we have

$$N^{\frac{1}{2}}(\bar{Z}_N - \delta^*)/2^{\frac{1}{2}}\sigma \xrightarrow{L} N(0, 1) \text{ as } P^* \to 1. \quad \ldots (B.11)$$

Now, retracing all the previous steps we obtain from (B.10) that

$$N^{\frac{1}{2}}(V_N - a)/b \xrightarrow{L} N(0, 1) \text{ as } P^* \to 1, \quad \ldots (B.12)$$

where $a = \delta^{*-1}$ and $b = 2^{\frac{1}{2}} \delta^{*-2}$. It may be remarked that the same result as in (B.12) also holds for V_{N-1}. Utilizing a theorem of Ghosh and Mukhopadhyay (1975) (Appendix A.4) with $\psi_\nu = -\sigma^2 \ln(1-P^*)/\delta^*$, we obtain

$$a^{\frac{1}{2}}(N - a\psi_\nu)/b\psi_\nu^{\frac{1}{2}} \xrightarrow{L} N(0, 1) \text{ as } P^* \to 1. \quad \ldots (B.13)$$

Now, equation (B.13) can be written as

$$(N - C)/D \xrightarrow{L} N(0, 1) \text{ as } P^* \to 1, \text{ where C is given in (B.9)}$$

and D is given by $D = b\ (\psi_\nu/a)^{\frac{1}{2}}$,

$$= 2^{\frac{1}{2}}\left(-\ell n(1-P^*)\right)^{\frac{1}{2}}\sigma^2/\,\delta^* \, . \qquad . \; . \; . \; (B.14)$$

This completes the proof of theorem 4.1. ∇

B.3. Theorem 4.3

Let us work under the hypothesis H_2. Under H_1, a similar proof can easily be constructed. The first two parts of (4.10) is obvious. For the other part of (4.10), we write the basic inequality:

$$-V_N\,\delta^{*-1}\ell n(1-P^*) \leq N \leq 1 - V_{N-1}\,\delta^{*-1}\ell n(1-P^*) \qquad . \; . \; . \; (B.15)$$

where $V_n = S_n^{\,2}|\bar{Z}_n|^{-1}$. Using the strong law of large numbers (Appendix A.9) and the fact that $N\to\infty$ a.s. as $P^*\to 1$, we conclude from (B.15) that $N/C\to 1$ a.s. as $P^*\to 1$, where $C = -\sigma^2\,\delta^{*-2}\ell n(1-P^*)$. . . . (B.16)

We now proceed for a proof of (4.11) under H_2. Note that for fixed values of n, $n^{\frac{1}{2}}(\bar{Z}_n - \delta^*)/2^{\frac{1}{2}}\sigma \xrightarrow{\mathcal{L}} N(0,\ 1)$ as $n\to\infty$, which implies

$$n^{\frac{1}{2}}(\bar{Z}_n S_n^{-2} - \delta^*\sigma^{-2})/2^{\frac{1}{2}}\sigma^{-1} \xrightarrow{\mathcal{L}} N(0,\ 1) \text{ as } n\to\infty.$$

Using Mann and Wald's theorem (Appendix A.6), we have

$$n^{\frac{1}{2}}(S_n^{\,2}|\bar{Z}_n|^{-1} - \sigma^2\,\delta^{*-1})/2^{\frac{1}{2}}\sigma^3\,\delta^{*-2} \xrightarrow{\mathcal{L}} N(0,\ 1) \text{ as } n\to\infty. \qquad . \; . \; . \; (B.17)$$

One may note that the same result also holds for V_{n-1}.

Since the sampling is carried out from normal distributions, we can use Helmert's orthogonal transformation on X-variables and write $S_n^{\,2} = (2n-2)^{-1}\sum_{i=1}^{2(n-1)} Y_i^{\,2}$, where Y_i's are iid $N(0,\sigma^2)$. Then we can apply Anscombe's (1952) results (Appendix A.4) to write

$$N^{\frac{1}{2}}(\bar{Z}_N - \delta^*)/2^{\frac{1}{2}}\sigma \xrightarrow{\mathcal{L}} N(0,\ 1) \text{ as } P^*\to 1. \qquad . \; . \; . \; (B.18)$$

Retracing all the previous steps, we obtain from (B.17)

$$N^{\frac{1}{2}}(V_N - a)/b \xrightarrow{\mathcal{L}} N(0,\ 1) \text{ as } P^*\to 1, \qquad \ldots \text{ (B.19)}$$

where $a = \sigma^2\delta^{*-1}$, and $b = 2^{\frac{1}{2}}\sigma^3\delta^{*-2}$. It may be remarked that the same result as in (B.19) also holds for V_{N-1}. Then applying a theorem of Ghosh and Mukhopadhyay (1975) (Appendix A.4), with $\psi_\nu = -\delta^{*-1}\ell n(1-P^*)$, we obtain $a^{\frac{1}{2}}(N - a\psi_\nu)/b\psi_\nu^{\frac{1}{2}} \xrightarrow{\mathcal{L}} N(0,\ 1)$ as $P^*\to 1$. . . . (B.20)

Equation (B.20) can be rewritten as $(N - C)/D^* \xrightarrow{\mathcal{L}} N(0,\ 1)$ as $P^*\to 1$, where C is given in (B.16) and D* is given as

$$D^* = b(\psi_\nu/a)^{\frac{1}{2}} = 2^{\frac{1}{2}}\sigma^2\delta^{*-2}\{-\ell n(1-P^*)\}^{\frac{1}{2}}.$$

This completes the proof of theorem 4.3. ∇

B.4. Theorem 4.4

Let $S_n{}^2 = \frac{1}{2}(S_{1n}{}^2 + S_{2n}{}^2)$, where $S_{jn}{}^2 = (n-1)^{-1}\sum_{i=1}^{n}(X_{ji} - \bar{X}_{jn})^2$, j=1,2.

Let $U = \sup_{n\geq 2}\{S_{1n}\}$, $V = \sup_{n\geq 2}\{S_{2n}\}$. . . . (B.21)

Then $U^2 \leq 2\ \sup_{n\geq 2}\{\sum_{i=1}^{n}(X_{1i} - \mu_1)^2/n\}$, . . . (B.22)

and $V^2 \leq 2\ \sup_{n\geq 2}\{\sum_{i=1}^{n}(X_{2i} - \mu_2)^2/n\}$. . . . (B.23)

Since the forth moment of X is finite, by Wiener's (1939) dominated ergodic theorem (see Appendix A.2), the right hand sides of (B.22) and (B.23) are integrable. Thus $EU^2 < \infty$ and $EV^2 < \infty$. Notice that

$$S^2_{N-1} = \tfrac{1}{2}(U^2 + V^2), \qquad \ldots \text{ (B.24)}$$

which implies that S^2_{N-1} is integrable, that is $E(S^2_{N-1}) < \infty$.

From (4.8), it follows that $S^2_{N-1} \leq -\ \delta^*|\sum_{i=1}^{N-1} z_i|/\ell n(1-P^*)$. Utilizing the orthogonal transformation (as shown in (4.9)), we let $T_n = \sum_{i=1}^{n} z_i$.

Given $\underset{\sim}{T} = (T_1, T_2, \ldots)$, we write

$$C_n = -\delta^* |T_n| / \ell n(1-P^*). \qquad \ldots \text{ (B.25)}$$

From (B.24), (B.25) and theorem 1 of Starr and Woodroof (1968), we obtain $E(S^2_{N-1}|\underset{\sim}{T}) \leq E(Y_1{}^2|\underset{\sim}{T}) = E(Y_1{}^2) = \sigma^2$, and then

$$E(S^2_{N-1}) = E\big(E(S^2_{N-1}|\underset{\sim}{T})\big)\ \sigma^2 \qquad \ldots \text{ (B.26)}$$

By looking at stage N-1, we have $|\delta^* \sum_{i=1}^{N-1} Z_i / S^2_{N-1}| \leq -\ell n(1-P^*)$, which implies $|\sum_{i=1}^{N-1} Z_i| \leq -S^2_{N-1} \delta^{*-1} \ell n(1-P^*)$, and we conclude that

$$E|\sum_{i=1}^{N-1} Z_i| \leq -\delta^{*-1} E(S^2_{N-1}) \ell n(1-P^*).$$

Now, applying Jensen's inequality (Appendix A.5) and Wald's (1947) first equation (Appendix A.10), we have

$$E(N-1)|E(Z_1)| = |E \sum_{i=1}^{N-1} Z_i| \leq E|\sum_{i=1}^{N-1} Z_i| \leq -\delta^{*-1} E(S^2_{N-1}) \ell n(1-P^*).$$

From (B.26), under H_1 or H_2, we thus obtain $E(N-1) \leq -\delta^{*-2}\sigma^2 \ell n(1-P^*)$. Hence, $E(N) \leq -\delta^{*-2}\sigma^2 \ell n(1-P^*) + 1$, which implies

$$E(N/C) \leq 1 + C^{-1}.$$

Thus,
$$\limsup_{P^* \to 1} E(N/C) \leq 1.$$

Applying Fatou's Lemma (Appendix A.3), we also have

$$E\big(\liminf_{P^* \to 1} (N/C)\big) \leq \liminf_{P^* \to 1} E(N/C).$$

Since $N/C \to 1$ a.s. as $P^* \to 1$, we obtain $\lim_{P^* \to 1} E(N/C) = 1$.
This completes the proof of theorem 4.4. ∇

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