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# Entanglement by linear $SU(2)$ transformations: generation and evolution of quantum vortex states

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**Abstract**

We consider the evolution of a two-mode system of bosons under the action of a Hamiltonian that generates linear $SU(2)$ transformations. The Hamiltonian is generic in that it represents a host of entanglement mechanisms, which can thus be treated in a unified way. We start by solving the quantum dynamics analytically when the system is initially in a Fock state. We show how the two modes get entangled by evolution to produce a coherent superposition of vortex states in general, and a single vortex state under certain conditions. The degree of entanglement between the modes is measured by finding the explicit analytical dependence of the Von Neumann entropy on the system parameters. The reduced state of each mode is analysed by means of its correlation function and spatial coherence function. Remarkably, our analysis is shown to be equally as valid for a variety of initial states that can be prepared from a two-mode Fock state via a unitary transformation and for which the results can be obtained by mere inspection of the corresponding results for an initial Fock state. As an example, we consider a quantum vortex as the initial state and also find conditions for its revival and charge conjugation. While studying the evolution of the initial vortex state, we have encountered and explained an interesting situation in which the entropy of the system does not evolve whereas its wavefunction does. Although the modal concept has been used throughout the paper, it is important to note that the theory is equally applicable for a two-particle system in which each particle is represented by its bosonic creation and annihilation operators.

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(Some figures in this article are in colour only in the electronic version)

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## 1. Introduction

Non-classical properties of quantum states are actively being studied for their relevance in quantum computation. It is known that quantum entanglement is the key to performing communication and information processing tasks that cannot be realized classically. For this reason, there has been a surge of activity towards preparing, identifying and quantifying entangled systems [1].

An important source of quantum entanglement has been the polarization-entangled two-photon states generated from type-II phase-matched parametric down conversion [2]. A variety of other entangled states can be produced by using various polarizing components. More recently, the subject of quantum information processing has been given a new direction with the realization that a number of quantum logic operations can be performed using single photons and methods of linear optics [3]. Even a method for quantum teleportation was proposed and implemented [4]. Clearly one needs to examine, in full generality, the question of transformation of an arbitrary input state by a device which can mix different states.

We note that a number of special cases for the generation of entanglement using linear optical devices have been investigated. Huang and Agarwal [5] considered multimode systems described by a Hamiltonian that is quadratic in the mode operators. They derived conditions for the generation of an entangled state when the input state was represented by a Gaussian density matrix. Their treatment covered a large class of states including squeezed coherent states and even states with thermal noise. However, they did not consider the case of input fields in Fock states. More recently and more specifically, Kim *et al* [6] examined the question of the generation of entangled states by a beam splitter using Fock states as input fields.

In this paper we specialize to intensity- or number-preserving linear transformations belonging to the $SU(2)$ group. For two-mode states characterized by the annihilation operators $a$ and $b$, such transformations can be generated by evolution under a Hamiltonian of the form

$$H = g(a^\dagger b\,\mathrm{e}^{\mathrm{i}\phi} + \mathrm{h.c}) + \Omega(a^\dagger a - b^\dagger b) \tag{1}$$

where $g$ and $\Omega$ are real constants. Introducing the generators of the $SU(2)$ group as

$$J_1 = (a^\dagger b + ab^\dagger)/2, \qquad J_2 = (a^\dagger b - ab^\dagger)/2\mathrm{i}, \qquad J_3 = (a^\dagger a - b^\dagger b)/2, \tag{2}$$

the Hamiltonian (1) can be rewritten in the form

$$H = v_1 J_1 + v_2 J_2 + v_3 J_3 \tag{3}$$

with

$$v_1 = 2g\cos\phi, \qquad v_2 = -2g\sin\phi, \qquad v_3 = 2\Omega. \tag{4}$$

Motivation for the present work comes from the realization that a Hamiltonian of the form (1) can represent a host of entanglement mechanisms which can thus be treated in a unified way. Several examples are given as follows.

The beam splitter, used by many authors as an entangler [7], can be described by (1) for $\Omega = 0$ if one defines its amplitude reflection and transmission coefficients by $\cos g$ and $\sin g$ respectively, while $\phi$ denotes the phase difference between the reflected and transmitted fields.

The parametric frequency conversion by a strong pump field (of frequency $\omega$) in $\chi^{(2)}$ material can also be represented by an interaction Hamiltonian of the form (1) [8] with $\Omega = 0$. Here $a$ and $b$ are the annihilation operators for the signal (of frequency $\omega_a$) and the idler (of frequency $\omega_b$) respectively, $g$ is a coupling constant that depends on the amplitude of the pump mode and $\phi = \Delta\omega t$ where $\Delta\omega = \omega + \omega_b - \omega_a$. We should note, however, that this Hamiltonian does not support parametric down conversion.

Polarizing elements such as half- and quarter wave plates also can act as entangling devices. Quantum mechanically, polarized light is represented by a pair of orthogonal polarization modes (described by boson mode operators $a, b$), or as points on the Poincaré sphere. The effect of a polarizing element on the field is a $SU(2)$ transformation of the mode operators which corresponds to rotations on the Poincaré sphere. The transformations are generated by Hamiltonians of the form (1).

Finally, following the work of Wineland *et al* [9], we consider a single laser cooled ion confined in a two-dimensional harmonic trap. The internal and motional degrees of freedom of the ion can be coupled by applying two classical laser beams. If $a$ and $b$ represent the two oscillatory modes of the ion's quantized motion and $\phi$ denotes the difference in phase between the two applied fields, then, under certain conditions [10], the Hamiltonian for the ion's motion will be of the form (1) in the interaction picture.

The present work is also relevant in the context of parallel developments in the field of optical vortices. An optical vortex of order $l$ centred at the origin ($r = 0$) has a field distribution of the form $F(r)\exp(\mathrm{i}l\phi)$. The distribution is such that the field intensity tends to zero as $r \to 0$ whereas the phase shift in one cycle around the origin is $2\pi l$, where $l$ is an integer. The azimuthal mode index $l$ has a physical meaning in that the vortex carries an orbital angular momentum of $l\hbar$ per photon [11]. This angular momentum can be imparted to microscopic particles in order to manipulate them optically [12, 13]. In recent years, this understanding has led to considerable interest in the generation and study of optical vortices both in free space [14] and in guided media [15, 16].

A physically realizable field distribution that contains optical vortices is a higher-order Laguerre–Gaussian (LG) beam whose waist-plane field amplitude is given by [17]

$$u_{mn}^{\mathrm{LG}}(x, y, \omega) = \sqrt{\frac{2}{\pi\omega^2}}\frac{(-1)^p p!}{\sqrt{m!n!}}\,\mathrm{e}^{-\mathrm{i}\theta(m-n)}(r\sqrt{2}/\omega)^{|m-n|}L_p^{|m-n|}(2r^2/\omega^2)\,\mathrm{e}^{-r^2/\omega^2} \tag{5}$$

where $r^2 = x^2 + y^2, \theta = \arctan(y/x), \omega$ is the beam waist, $p = \min(n, m)$ and $L_p^l(x)$ is a generalized Laguerre polynomial. LG beams can be produced directly from a laser [18]. In fact, in a hydrodynamic formulation of laser beam dynamics in terms of LG modes, vortices were found to occur in transverse laser patterns [19, 20]. Usually, however, LG beams are produced by the conversion or combination of Hermite–Gaussian (HG) beams that are emitted by most laser cavities. This is made possible because of the fact that any LG mode can be expressed in terms of HG modes. The waist-plane amplitude of the HG modes has the form

$$u_{n,m}^{\mathrm{HG}}(x, y, \omega) = \Phi_n(x, \omega)\Phi_m(y, \omega), \tag{6}$$

where

$$\Phi_n(x, \omega) = \left(\frac{\sqrt{2}}{\sqrt{\pi}2^n wn!}\right)^{1/2} H_n(\sqrt{2}x/w)\exp(-x^2/w^2) \tag{7}$$

and $H_n(x)$ is a Hermite polynomial. The decomposition of a LG mode in terms of HG modes is given as [14]

$$u_{n,m}^{\mathrm{LG}}(x, y, \omega) = \sum_{k=0}^{m+n} i^k b(n, m, k)u_{m+n-k,k}^{\mathrm{HG}}(x, y, \omega) \tag{8a}$$

$$b(n, m, k) = \sqrt{\frac{(n+m)!k!}{2^{n+m}n!m!}}\frac{1}{k!}\frac{\mathrm{d}^k}{\mathrm{d}t^k}[(1-t)^n(1+t)^m]|_{t=0}. \tag{8b}$$

The vortices as discussed above appear on the transverse amplitude profile of *classical* wave fields. Vortices can also occur in the configuration space representation of quantum systems of matter or radiation. Since the HG modes are also the energy eigenfunctions of a quantum oscillator, quantum vortices should arise in the study of wave packets of a quantum system that could be a two-dimensional harmonic oscillator like an ion in a two-dimensional trap. For a two-mode radiation field characterized by the annihilation operators $a, b$, and represented by a state vector $|\psi\rangle$, the quantum vortex will appear in the quadrature distribution $|\langle x, y|\psi\rangle|^2$ where $|x, y\rangle$ is the eigenvector of $(a + a^\dagger)/\sqrt{2}$ and $(b + b^\dagger)/\sqrt{2}$. Quadrature distributions can be measured by a homodyne method [21]. Vortices of matter will appear in the configuration space probability distribution. Recently it has been shown that the HG and LG modes are unitarily related [22] and the Poincaré sphere [23] representing LG beams has an underlying $SU(2)$ structure [24]. The Hamiltonian (1) is therefore ideally suited to explore the possibility of generating quantum vortices.

The objective and the plan of the paper are as follows. In section 2 we obtain the state vector and the wavefunction of a two-mode system which is initially in a Fock state and is acted upon by the Hamiltonian (1). We show how the two modes get entangled by evolution and under certain conditions evolve into a vortex state. The degree of entanglement between the modes is measured by finding the dependence of the von Neumann entropy on the system parameters. In section 3 the above analysis is carried out when the two-mode system is initially in a state that can be obtained from a Fock state via a unitary transformation. As an example, a quantum vortex is used as the initial state. We also find conditions for the revival and the charge conjugation of the vortex. In section 4 we consider the structure of the reduced state of each mode. The paper ends with concluding remarks in section 5.

## 2. Generation of quantum entanglement and creation of a quantum vortex using an initial two-mode Fock state

### 2.1. Evolution of the state vector

Let us consider the evolution of a two-mode Fock state $|N - j, j\rangle$ when the Hamiltonian is given by (1) and the total number ($N$) of photons in the two modes is constant. The resulting state $|\psi_{Nj}(t)\rangle = U(t)|N - j, j\rangle$ can be obtained by the use of the disentangling theorem. In what follows, we use a different method. We write $|N - j, j\rangle$ as

$$|N - j, j\rangle = \frac{(\hat{a}^\dagger)^{N-j}(\hat{b}^\dagger)^j}{\sqrt{N - j!j!}}|0, 0\rangle \tag{9}$$

and define a new pair of operators

$$\begin{pmatrix}\hat{a}(t)\\ \hat{b}(t)\end{pmatrix} = U^\dagger(t)\begin{pmatrix}\hat{a}\\ \hat{b}\end{pmatrix}U(t), \tag{10}$$

where $U(t) = \exp(-\mathrm{i}Ht)$ is the time evolution operator. Then $|\psi_{Nj}(t)\rangle$ can be written in a compact form as

$$|\psi_{Nj}(t)\rangle = \frac{[\hat{a}^\dagger(-t)]^{N-j}[\hat{b}^\dagger(-t)]^j}{\sqrt{N - j!j!}}|0, 0\rangle. \tag{11}$$

Note that the state at time $t$ is obtained by using the operators evaluated at time $-t$. The explicit expressions for $\hat{a}(t)$ and $\hat{b}(t)$ can be obtained by solving the Heisenberg equations for the operators. We get

$$\begin{pmatrix}\hat{a}(t)\\ \hat{b}(t)\end{pmatrix} = \mathbf{V}\begin{pmatrix}\hat{a}(0)\\ \hat{b}(0)\end{pmatrix} = \mathbf{V}\begin{pmatrix}\hat{a}\\ \hat{b}\end{pmatrix} \tag{12}$$

where $\mathbf{V} = \{v_{ij}\}$ is a $2 \times 2$ unitary matrix. Setting $\sigma = \sqrt{\Omega^2 + g^2}$ and $\Omega = \sigma \cos\Theta$, the matrix elements are written as

$$\begin{aligned} v_{11} &= \cos\sigma t - \mathrm{i}\cos\Theta\sin\sigma t, & v_{12} &= -\mathrm{i}\,\mathrm{e}^{\mathrm{i}\phi}\sin\Theta\sin\sigma t, \\ v_{21} &= -\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\phi}\sin\Theta\sin\sigma t, & v_{22} &= \cos\sigma t + \mathrm{i}\cos\Theta\sin\sigma t. \end{aligned} \tag{13}$$

Note that

$$v_{11} = v_{22}^*, \qquad v_{12} = -v_{21}^* \qquad \text{and} \qquad |v_{21}|^2 + |v_{22}|^2 = 1. \tag{14}$$

Substitution in (11) followed by binomial expansion and the use of (9) yields

$$|\psi_{Nj}(t)\rangle = \sum_{m=0}^{N-j}\sum_{n=0}^{j} b_{mn}|N-(m+n), m+n\rangle \tag{15}$$

where

$$b_{mn} = \binom{N-j}{m}\binom{j}{n}\binom{N}{N-j}^{1/2}\binom{N}{m+n}^{-1/2}(v_{11})^{N-j-m}(v_{21})^m(v_{12})^{j-n}(v_{22})^n. \tag{16}$$

*The two modes in the state $|\psi_{Nj}\rangle$ are entangled in the sense that the above double sum cannot be reduced to the product of two single-mode summations.*

It is instructive to briefly mention the case when the two modes are initially in a Glauber coherent state $|\alpha, \beta\rangle$. Since the Hamiltonian (1) conserves photon numbers, the state at time $t$ will also be a coherent state:

$$U(t)|\alpha, \beta\rangle = |\alpha(t), \beta(t)\rangle. \tag{17}$$

Applying (12) on $|\alpha, \beta\rangle$, we immediately obtain

$$\begin{pmatrix}\alpha(t)\\ \beta(t)\end{pmatrix} = \mathbf{V}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}. \tag{18}$$

Furthermore, the unitarity of $\mathbf{V}$ ensures that

$$|\alpha(t)|^2 + |\beta(t)|^2 = |\alpha|^2 + |\beta|^2. \tag{19}$$

Thus *no entanglement occurs if each input mode is in a coherent state*.

In what follows, we exploit coherent states as generating functions of number states to reduce the double sum in (15) to a single sum. Expanding both sides of (17) in number states and recalling that $|\alpha(t)|^2 + |\beta(t)|^2 = |\alpha|^2 + |\beta|^2$, we get the relation

$$\sum_m\sum_n \frac{\alpha^m\beta^n}{\sqrt{m!n!}}U(t)|m, n\rangle = \sum_p\sum_q \frac{\alpha^{p+q}}{\sqrt{p!q!}}\xi_{pq}(\tau)|p, q\rangle \tag{20}$$

where $\tau = \beta/\alpha$ and

$$\xi_{pq}(\tau) = (v_{11} + v_{12}\tau)^p(v_{21} + v_{22}\tau)^q = \sum_{k=0}^{p+q}\frac{\tau^k}{k!}\partial_\tau^{(k)}\xi_{pq}(\tau)\rfloor_{\tau\to 0}. \tag{21}$$

Substituting in (20) and equating the coefficient of $\alpha^{N-j}\beta^j$, one gets

$$|\psi_{Nj}(t)\rangle = U(t)|N-j, j\rangle = \sum_{q=0}^{N} C_{Nj}^{(q)}|N-q, q\rangle \tag{22}$$

where

$$C_{Nj}^{(q)} = \frac{1}{j!}\left[\frac{(N-j)!j!}{(N-q)!q!}\right]^{1/2}\partial_\tau^{(j)}\xi_{N-q,q}(\tau)\rfloor_{\tau\to 0}. \tag{23}$$

Some useful properties of $\left|C_{Nj}^{(q)}\right|^2$ are derived in appendix A.

### 2.2. The wavefunction—a coherent superposition of vortex states

The corresponding wavefunction in configuration space is obtained as follows. Using the relation

$$\langle y|q\rangle = \frac{\mathrm{e}^{-y^2/2} H_q(y)}{\sqrt{2^q q! \sqrt{\pi}}}, \qquad H_q(y) = (-1)^q \, \mathrm{e}^{y^2} \partial_y^{(q)} \, \mathrm{e}^{-y^2}, \tag{24}$$

and the corresponding expression for $\langle x|N-q\rangle$, we obtain

$$\begin{aligned}\psi_{Nj}(x, y, t) &= \langle x, y|U(t)|N-j, j\rangle \\ &= \frac{\mathrm{e}^{-(x^2+y^2)/2}}{\sqrt{\pi 2^N}} \sum_{q=0}^{N} C_{Nj}^{(q)} \frac{H_{N-q}(x) H_q(y)}{\sqrt{(N-q)!q!}} \\ &= \frac{(-1)^N \, \mathrm{e}^{(x^2+y^2)/2}}{\sqrt{\pi 2^N}} \sum_{q=0}^{N} C_{Nj}^{(q)} \frac{\partial_x^{N-q} \partial_y^q \, \mathrm{e}^{-(x^2+y^2)}}{\sqrt{(N-q)!q!}}.\end{aligned} \tag{25}$$

The wavefunction has a more appealing form in polar coordinates as shown below. Writing $x = r\cos\theta$, $y = r\sin\theta$, and defining

$$\gamma_{\pm}(\tau) = v_{11} + v_{12}\tau \pm i(v_{21} + v_{22}\tau), \tag{26}$$

we get (see appendix B)

$$\psi_{Nj}(x, y, t) = \sum_{n=0}^{N} b_{Nj}^{(n)} u_{N-n,n}(r, \theta) \tag{27}$$

where

$$\begin{aligned} b_{Nj}^{(n)} &= \frac{1}{j!} \sqrt{\frac{(N-j)!j!}{(N-n)!n!2^N}} \zeta_{Nn}^{(j)}(0) \qquad && \zeta_{Nn}(\tau) = \gamma_+(\tau)^{N-n} \gamma_-(\tau)^n \\ \zeta_{Nn}^{(j)}(0) &= \partial_\tau^j \zeta_{Nn}(\tau)\rfloor_{\tau\to 0} && u_{mn}(r, \theta) = u_{mn}^{\mathrm{LG}}(x, y, \sqrt{2}). \end{aligned} \tag{28}$$

Recall that for $m \neq n$, $u_{mn}(r, \theta)$ represents a vortex of order $|m-n|$ and charge $m-n$ embedded in a Gaussian host beam of waist $\omega = \sqrt{2}$. Thus *for odd values of N, the wavefunction $\psi_{Nj}(x, y, t)$ becomes a coherent superposition of vortex states, whereas for even values of N, the superposition will also contain a state (corresponding to $n = N/2$) that does not have a vortex character* [25].

### 2.3. Creation of a single quantum vortex

In this section we will derive conditions for the creation of a single quantum vortex. We reiterate that for light fields, the vortex will appear in the quadrature distribution whereas for other systems it will be in the probability distribution in configuration space.

The initial two-mode Fock state can evolve into a single vortex state when the summation in (27) collapses into a single term. This happens whenever $\gamma_+(0)$ or $\gamma_-(0)$ is zero. It is easy to show that $|v_{21}|^2 = 1/2$ for both these cases.

If $\gamma_+(0) = 0$, then $v_{11} + \mathrm{i}v_{21} = 0$ and taking the complex conjugate of this equation, $v_{22} + \mathrm{i}v_{12} = 0$. Then $\gamma_+(\tau) = 2v_{12}\tau$ and $\gamma_-(\tau) = -2\mathrm{i}v_{21}$ so that $\zeta_{Nn}^{(j)}(0) = (2v_{12})^{N-n}(-2\mathrm{i}v_{21})^n j!\delta_{N-j,n}$ and, finally,

$$\psi_{Nj}(x, y, t)\rfloor_{\gamma_+(0)=0} = 2^{N/2} i^{j-N} v_{21}^{N-j} v_{12}^{j} u_{j,N-j}(r, \theta). \tag{29}$$

The condition $\gamma_+(0) = 0$ implies that $\Omega = -g\sin\phi$ and $\sigma\cos\sigma t = -g\cos\phi\sin\sigma t$. We give two examples for which these conditions are satisfied.

(i) Setting $\Omega = 0$, $\phi = \pi$ and $\sigma t = \pi/4$, we get

$$\psi_{Nj}(x, y, t) = i^j u_{j,N-j}(r, \theta). \tag{30}$$

From (22), one obtains the corresponding state vector

$$U_0|N-j, j\rangle = \sum_{q=0}^{N} D_{Nj}^{(q)}|N-q, q\rangle \tag{31}$$

where

$$U_0 = \exp\left[\frac{\mathrm{i}\pi}{4}(a^\dagger b + ab^\dagger)\right] \tag{32}$$

and

$$\begin{aligned} D_{Nj}^{(q)} &= C_{Nj}^{(q)}\rfloor_{\Omega=0,\phi=\pi,\sigma t=\pi/4} \\ &= \sqrt{\frac{(N-j)!j!}{2^N(N-q)!q!}}\frac{i^q}{j!}\left[\partial_\tau^j(1+\mathrm{i}\tau)^{N-q}(1-\mathrm{i}\tau)^q\right]_{\tau\to 0}. \end{aligned} \tag{33}$$

(ii) Setting $\Omega = g$, $\phi = -\pi/2$ and $\sigma t = \pi/2$, we get

$$\psi_{Nj}(x, y, t) = (-i)^{j+N} u_{j,N-j}(r, \theta). \tag{34}$$

The operator form of the corresponding state vector is given by[4]

$$\exp\left[\frac{\mathrm{i}\pi}{2\sqrt{2}}\{\mathrm{i}(a^\dagger b - ab^\dagger) - (a^\dagger a - b^\dagger b)\}\right]|N-j, j\rangle. \tag{35}$$

Following a similar analysis for $\gamma_-(0) = 0$, one obtains

$$\psi_{Nj}(x, y, t)\rfloor_{\gamma_-(0)=0} = 2^{N/2} i^{N-j} v_{21}^{N-j} v_{12}^{j} u_{N-j,j}(r, \theta). \tag{36}$$

The condition $\gamma_-(0) = 0$ yields $\Omega = g\sin\phi$ and $\sigma\cos\sigma t = g\cos\phi\sin\sigma t$. These two conditions are satisfied , for example, when $\Omega = \phi = 0$ and $\sigma t = \pi/4$. The corresponding wavefunction is the complex conjugate of (30).

We end this section by noting that the above conditions can be physically realized for a given entangling device. We give an example in the context of a frequency converter. Suppose the signal (of frequency $\omega_a$) and the idler (of frequency $\omega_b$) are initially in Fock states and the converter is pumped at the difference frequency $\omega_a - \omega_b$. Replacing $t$ by $L/c$, where $L$ is the length of the non-linear medium and $c$ is the speed of light, one can adjust the pump amplitude such that $gL/c = \pi/4$. This set-up corresponds to $\Omega = \phi = 0$ and $gt = \pi/4$. In this case the quadrature distribution of the output state will be a single quantum vortex as mentioned above.

### *2.4. Entanglement of the two modes*

Initially the two modes are not entangled as the state vector $|N-j, j\rangle$ is the direct product of the state vectors for each mode. In configuration space, this would imply that $\psi_{Nj}(x, y, 0)$ is separable in $x$ and $y$ as indeed it is. Furthermore, as the time dependence arises solely in $v_{ij}$ which vary as $\cos\sigma t$ or $\sin\sigma t$, the initial state is revived whenever $\sigma t = k\pi$, where $k$ is an integer. For even values of $k$, the revival is exact whereas for odd values of $k$, it is within an overall factor of $(-)^N$. At other times, the two modes are entangled as is evident in expression ((15) or (22)) for the state vector and expression ((25) or (27)) for the corresponding wavefunction.

[4] Although relation (31) has been derived earlier in a different context [22], relation (35) has not been reported in the literature to the best of our knowledge.

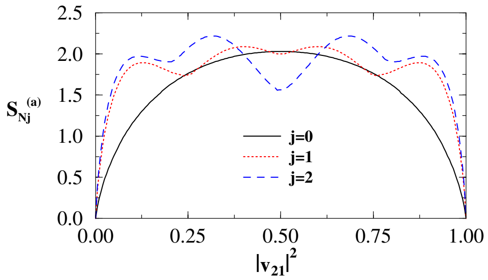


**Figure 1.** Plot of $S_{Nj}^{(a)}$ as a function of $|v_{21}|^2$ for $N = 4$ and $j = 0, 1, 2$.

### *2.5. Degree of entanglement*

Note that the two-mode system $|\psi_{Nj}(t)\rangle$ is in a pure state whereas the reduced state of each mode, determined by a partial trace operation, will be a mixed state. The reduced density operators of modes '$a$' and '$b$' are given respectively by

$$\rho_{Nj}^{(a)} = \mathrm{Tr}_b|\psi_{Nj}\rangle\langle\psi_{Nj}| = \sum_{q=0}^{N}\left|C_{Nj}^{(q)}\right|^2|q\rangle\langle q| \tag{37a}$$

$$\rho_{Nj}^{(b)} = \mathrm{Tr}_a|\psi_{Nj}\rangle\langle\psi_{Nj}| = \sum_{q=0}^{N}\left|C_{Nj}^{(N-q)}\right|^2|q\rangle\langle q|. \tag{37b}$$

The corresponding von Neumann entropies $S_{Nj}^{(a)}$ and $S_{Nj}^{(b)}$ provide a measure of the degree of entanglement between the two modes:

$$S_{Nj}^{(a)} = -\sum_{q=0}^{N}\left|C_{Nj}^{(q)}\right|^2\log\left|C_{Nj}^{(q)}\right|^2 \tag{38a}$$

$$S_{Nj}^{(b)} = -\sum_{q=0}^{N}\left|C_{Nj}^{(N-q)}\right|^2\log\left|C_{Nj}^{(N-q)}\right|^2. \tag{38b}$$

By virtue of relations (A.4), we get

$$S_{Nj}^{(a)}\rfloor_{|v_{21}|^2\to 1-R} = S_{Nj}^{(a)}\rfloor_{|v_{21}|^2\to R} = S_{N,N-j}^{(a)}\rfloor_{|v_{21}|^2\to R}. \tag{39}$$

Changing the summation index from $q$ to $N-q$ in the expression for $S_{Nj}^{(b)}$, one obtains $S_{Nj}^{(a)} = S_{Nj}^{(b)}$. Thus the symmetry relations (39) hold good for $S_{Nj}^{(b)}$ as well. These observations hold for any bipartite system in a pure state.

It is remarkable that for a given value of $N$, $j$ and $q$, the dynamics of $\left|C_{Nj}^{(q)}\right|^2$ depends on $|v_{21}|^2 = \sin^2\Theta\sin^2\sigma t$ only (see appendix A). This important observation implies that *(a) the entropy $S_{Nj}^{(a)}$ and the reduced density operator $\rho_{Nj}^{(a)}$ are independent of $\phi$ and (b) are symmetric with respect to the interchange of $\Theta$ and $\sigma t$*. In figure 1, we plot $S_{Nj}^{(a)}$ as a function of $|v_{21}|^2$ for $N = 4$ and $j = 0, 1, 2$.

Trivially, for $|v_{21}|^2 = 0$, the initial pure state $|N-j, j\rangle$ either does not evolve or is fully revived and the entropy of the reduced state is zero. For $|v_{21}|^2 = 1$, the initial state swaps the photon numbers in the two modes and becomes $|j, N-j\rangle$ which is also a pure state. For all other values of $|v_{21}|^2$, the initially pure state becomes a mixed state and the entropy of the

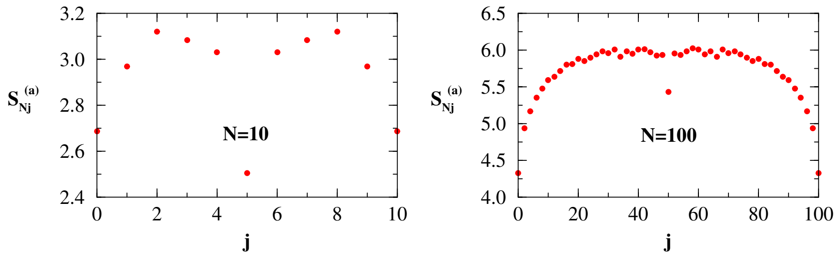


**Figure 2.** Plot of $S_{Nj}^{(a)}$ as a function of $j$ for $|v_{21}|^2 = 1/2$ and a given total number of photons $N$.

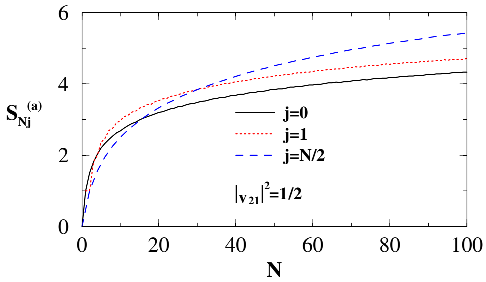


**Figure 3.** Plot of $S_{Nj}^{(a)}$ as a function of $N$ for $|v_{21}|^2 = 1/2$ and $j = 0, 1, N/2$.

reduced state becomes non-zero. Recall that for $|v_{21}|^2 = 1/2$ and $N - j \neq j$, the quantum state becomes a vortex. Thus *a quantum vortex is indeed an entangled state*. To quantify the degree of entanglement for a vortex state, we plot $S_{Nj}^{(a)}$ as a function of $j$ for $|v_{21}|^2 = 1/2$ and a given total number of photons $N$ (see figure 2). It is clear that the entropy of the state without a vortex ($j = N/2$) is less than the entropy of the neighbouring ($j \sim N/2$) vortex states ($N - j \neq j$). This reduction in entropy can be attributed to the symmetry of the $j = N/2$ state and traced to the highly oscillatory nature of the Jacobi polynomial appearing in equation (A.5). For a given value of $N$, the minimum in the entropy of a *vortex state* occurs for $j = 0, N$ in which case $\left|C_{Nj}^{(q)}\right|^2$ is a binomial distribution (see appendix A). Interestingly, for $j = 0, N$, the vortex state will have the maximum allowed order ($N$). Thus *the vortex state of maximum order will have minimum entropy* which is counter-intuitive. One would have expected that the more twists the phase of the state has, more energetic and more entropic it would be. Note that the symmetry of $S_{Nj}^{(a)}$ about $|v_{21}|^2 = 1/2$ in figure 1 and about $j = N/2$ in figure 2 is contained in the relations (39). Note also that the vorticity or non-vorticity of the state of lowest entropy depends on the value of $N$ (see figure 3).

We end this section by comparing the entropy values in figures 1–3 with $\log_2(N + 1)$, the maximum entropy possible for a given $N$ with an entirely mixed state. For $N = 4$, 10 and 100, $\log_2(N + 1)$ has the values 2.321 93, 3.459 43 and 6.658 21, respectively.

## 3. Evolution of an initial vortex state

### *3.1. Evolution of the state vector*

Let us assume that the two modes are initially in a quantum vortex state as in (31). Then the state vector at time $t$ will be given by

$$|\tilde{\psi}_{Nj}(t)\rangle = U(t)U_0|N - j, j\rangle. \tag{40}$$

Proceeding as in section 2.1, we define

$$\begin{pmatrix}\hat{a}(t)\\ \hat{b}(t)\end{pmatrix} = [U(t)U_0]^\dagger \begin{pmatrix}\hat{a}\\ \hat{b}\end{pmatrix}[U(t)U_0] \tag{41}$$

and obtain

$$\begin{pmatrix}\hat{a}(t)\\ \hat{b}(t)\end{pmatrix} = \mathbf{V}\begin{pmatrix}\hat{a}(0)\\ \hat{b}(0)\end{pmatrix}. \tag{42}$$

Note, however, that in this case,

$$\begin{pmatrix}\hat{a}(0)\\ \hat{b}(0)\end{pmatrix} = U_0^\dagger \begin{pmatrix}\hat{a}\\ \hat{b}\end{pmatrix} U_0 = \mathbf{W}\begin{pmatrix}\hat{a}\\ \hat{b}\end{pmatrix}, \qquad \mathbf{W} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & \mathrm{i}\\ \mathrm{i} & 1\end{pmatrix}. \tag{43}$$

Thus

$$\begin{pmatrix}\hat{a}(t)\\ \hat{b}(t)\end{pmatrix} = \tilde{\mathbf{V}}\begin{pmatrix}\hat{a}\\ \hat{b}\end{pmatrix}, \qquad \tilde{\mathbf{V}} = \mathbf{V}\mathbf{W}. \tag{44}$$

The action and the effect of the unitary operator $U_0$ are now clear. $U_0$ transforms the two-mode Fock state into a different initial state *before* its time evolution begins and thus $U_0$ can be regarded as the operator for *initial state preparation*. The effect of $U_0$ is contained in the unitary matrix **W**. As a result, the overall unitary evolution matrix changes from **V** to $\tilde{\mathbf{V}} = \mathbf{V}\mathbf{W}$.

In the present case, $U_0$, as given by (32), prepares a quantum vortex state as the initial state and the corresponding expression for **W** is given as in (43). A different expression for $U_0$ will generate an initial state that is different from a quantum vortex. Yet for all these initial states the dynamics is essentially solved once the corresponding dynamics for a two-mode Fock state is worked out as in section 2.1. In each case, one need only calculate the matrix **W** and replace **V** by $\tilde{\mathbf{V}}$. *In this sense, our theory not only provides a unified approach to entanglement through a generic Hamiltonian but also promises wide applicability to a variety of initial states.*

In the present case, the matrix elements of $\tilde{\mathbf{V}} = \{\tilde{v}_{ij}\}$ are obtained easily as

$$\begin{aligned}\tilde{v}_{11} &= (v_{11} + \mathrm{i}v_{12})/\sqrt{2}, \qquad & \tilde{v}_{12} &= (v_{12} + \mathrm{i}v_{11})/\sqrt{2},\\ \tilde{v}_{21} &= (v_{21} + \mathrm{i}v_{22})/\sqrt{2}, \qquad & \tilde{v}_{22} &= (v_{22} + \mathrm{i}v_{21})/\sqrt{2}.\end{aligned} \tag{45}$$

Using (14), one can also show that

$$\tilde{v}_{11} = \tilde{v}_{22}^*, \qquad \tilde{v}_{12} = -\tilde{v}_{21}^* \qquad \text{and} \qquad |\tilde{v}_{21}|^2 + |\tilde{v}_{22}|^2 = 1. \tag{46}$$

It is now trivial to obtain the wave vector and the wavefunction by borrowing the corresponding results from the previous section. We simply replace $v_{ij}$ by $\tilde{v}_{ij}$ for $i, j = 1, 2$ and, for the sake of clarity and comparison, use the same nomenclature for the new expressions except for a ˜ (tilde) over them. Thus

$$|\tilde{\psi}_{Nj}(t)\rangle = \sum_{q=0}^{N} \tilde{C}_{Nj}^{(q)}|N-q, q\rangle \tag{47}$$

where

$$\tilde{C}_{Nj}^{(q)} = \frac{1}{j!}\left[\frac{(N-j)!j!}{(N-q)!q!}\right]^{1/2} \partial_\tau^{(j)}\tilde{\xi}_{N-q,q}(\tau)\rfloor_{\tau\to 0} \tag{48a}$$

$$\tilde{\xi}_{pq}(\tau) = (\tilde{v}_{11} + \tilde{v}_{12}\tau)^p(\tilde{v}_{21} + \tilde{v}_{22}\tau)^q. \tag{48b}$$

Furthermore, the results of appendix A can be used to write

$$\left|\tilde{C}_{Nj}^{(q)}\right|^2 = (N-j)!(N-q)!q!(j!)^{-1}(1-|\tilde{v}_{21}|^2)^N \left(\frac{|\tilde{v}_{21}|^2}{1-|\tilde{v}_{21}|^2}\right)^{q-j} \left|f_{Nj}^{(q)}(|\tilde{v}_{21}|^2)\right|^2 \tag{49}$$

$$= \begin{cases} \delta_{q,j}, & |\tilde{v}_{21}|^2 \to 0, \\ \delta_{q,N-j}, & |\tilde{v}_{21}|^2 \to 1, \end{cases} \tag{50}$$

and

$$\left|\tilde{C}_{Nj}^{(q)}\right|^2\Big\rfloor_{|\tilde{v}_{21}|^2\to 1-R} = \left|\tilde{C}_{Nj}^{(N-q)}\right|^2\Big\rfloor_{|\tilde{v}_{21}|^2\to R} = \left|\tilde{C}_{N,N-j}^{(q)}\right|^2\Big\rfloor_{|\tilde{v}_{21}|^2\to R}. \tag{51}$$

### 3.2. The wavefunction

The corresponding wavefunction in configuration space can be read off from equation (27). We get

$$\tilde{\psi}_{Nj}(x,y,t) = \sum_{n=0}^{N} \tilde{b}_{Nj}^{(n)} u_{N-n,n}(r,\theta), \tag{52}$$

where

$$\tilde{b}_{Nj}^{(n)} = \frac{1}{j!}\sqrt{\frac{(N-j)!j!}{(N-n)!n!2^N}}\,\tilde{\zeta}_{Nn}^{(j)}(0) \tag{53a}$$

$$\tilde{\zeta}_{Nn}(\tau) = \tilde{\gamma}_+(\tau)^{N-n}\tilde{\gamma}_-(\tau)^n \tag{53b}$$

$$\tilde{\gamma}_\pm(\tau) = \tilde{v}_{11} + \tilde{v}_{12}\tau \pm \mathrm{i}(\tilde{v}_{21} + \tilde{v}_{22}\tau). \tag{53c}$$

Thus *a quantum vortex state evolves into a superposition of vortex states* under the action of the Hamiltonian (1).

### 3.3. Revival and charge conjugation

It can be shown that if $\tilde{\gamma}_+(0)$ or $\tilde{\gamma}_-(0)$ is zero, then the summation in (52) reduces to a single term. Specifically, if $\tilde{\gamma}_+(0) = 0$, then $\operatorname{Im} v_{21} = \operatorname{Im} v_{22} = 0$ and

$$\tilde{\psi}_{Nj}(x,y,t) = (\mathrm{i}v)^j v^{*N-j} u_{j,N-j}(r,\theta) \tag{54}$$

with $v = \operatorname{Re} v_{22} + \mathrm{i}\operatorname{Re} v_{21}$. The above conditions are satisfied for the following cases: (a) $\sin\sigma t = 0$ for arbitrary values of $\Theta$ and $\phi$. This includes the initial state ($t = 0$) and the state upon revival ($\sigma t = \pi$). (b) $\sin\Theta = \sin\phi = 1$ for arbitrary time. In this case the initial vortex state becomes an eigenstate of the corresponding Hamiltonian.

On the other hand, if $\tilde{\gamma}_-(0) = 0$, then $\operatorname{Re} v_{21} = \operatorname{Re} v_{22} = 0$ and

$$\tilde{\psi}_{Nj}(x,y,t) = (v)^j(-iv^*)^{N-j} u_{N-j,j}(r,\theta) \tag{55}$$

with $v = \operatorname{Im} v_{22} + \mathrm{i}\operatorname{Im} v_{21}$. Note that $u_{N-j,j}(r,\theta) = u_{j,N-j}^*(r,\theta)$ and thus $\tilde{\gamma}_-(0) = 0$ is the condition for *'charge conjugation'* or *'helicity reversal'* of the initial vortex state. This condition is fulfilled whenever $\sin\Theta\sin\phi = 0$ and $\cos\sigma t = 0$.

*3.4. Degree of Entanglement in the superposition state (52)*

The reduced density operator of mode '$a$' and the corresponding von Neumann entropy are given respectively by

$$\tilde{\rho}_{Nj}^{(a)} = \mathrm{Tr}_b|\tilde{\psi}_{Nj}\rangle\langle\tilde{\psi}_{Nj}| = \sum_{q=0}^{N}\left|\tilde{C}_{Nj}^{(q)}\right|^2|q\rangle\langle q| \tag{56a}$$

$$\tilde{S}_{Nj}^{(a)} = -\sum_{q=0}^{N}\left|\tilde{C}_{Nj}^{(q)}\right|^2\log\left|\tilde{C}_{Nj}^{(q)}\right|^2. \tag{56b}$$

It is clear that $\tilde{S}_{Nj}^{(a)}$ will depend on $|\tilde{v}_{21}|^2$ in exactly the same way as $S_{Nj}^{(a)}$ does on $|v_{21}|^2$ except that the form of $|\tilde{v}_{21}|^2$ as a function of $\Theta, \phi$ and $\sigma t$ is quite different from $|v_{21}|^2$. Notably, $|\tilde{v}_{21}|^2$ depends on $\phi$ while $|v_{21}|^2$ does not. Explicitly,

$$|\tilde{v}_{21}|^2 = \tfrac{1}{2} - \sin\Theta\sin\sigma t(\cos\phi\cos\sigma t - \sin\phi\cos\Theta\sin\sigma t). \tag{57}$$

*3.5. A case of constant entropy*

Note that if $\Theta = \phi = \pi/2$ or $\Theta = 0$, then $|\tilde{v}_{21}|^2 = 1/2$ so that $\left|\tilde{C}_{Nj}^{(q)}\right|^2$ and the entropy $\tilde{S}_{Nj}^{(a)}$ will not evolve with time. The underlying reason is as follows.

Using expressions (2) for the $SU(2)$ generators, we obtain

$$J_3|N-q,q\rangle = \frac{N-2q}{2}|N-q,q\rangle. \tag{58}$$

Noting that $U_0$, as given by (32), can be written as $U_0 = \exp(\mathrm{i}\pi J_1/2)$ and using the relation $J_2 = U_0 J_3 U_0^{\dagger}$, we also get

$$J_2 U_0|N-j,j\rangle = \frac{N-2j}{2}U_0|N-j,j\rangle \tag{59}$$

For $\Theta = \phi = \pi/2$, the Hamiltonian reduces to $H = -2gJ_2$ for which $U_0|N-j,j\rangle$ becomes an eigenstate by virtue of (59).

The condition $\Theta = 0$ corresponds to $g = 0$ and the Hamiltonian reduces to $2\Omega J_3$. From (31), (40) and (47), one then immediately obtains $\tilde{C}_{Nj}^{(q)} = \exp(-\mathrm{i}\Omega t[N-2q])D_{Nj}^{(q)}$ so that $\left|\tilde{C}_{Nj}^{(q)}\right|^2$, and consequently, the entropy $\tilde{S}_{Nj}^{(a)}$ become independent of time. The corresponding wavefunction is given by (52) where the coefficients $\tilde{b}_{Nj}^{(n)}$ have the value

$$\tilde{b}_{Nj}^{(n)} = \frac{N!}{j!}\binom{N}{j}^{-1/2}\binom{N}{n}^{-1/2}(-i)^{N-n}(\sin\Omega t)^{N-n+j}(\cos\Omega t)^{n-j}f_{Nj}^{(n)}(\cos^2\Omega t). \tag{60}$$

It is interesting that although $\left|\tilde{C}_{Nj}^{(q)}\right|^2$ and the entropy $\tilde{S}_{Nj}^{(a)}$ remain constant for $\Theta = 0$, the initial vortex state will continue to evolve with time, as shown in figure 4 [27]. For $N = 4$ and $j = 0$, the initial state at $t = 0$ (figure 4($a$)) corresponding to $\cos^2\Omega t = 1.0$ is a vortex of order 4 and charge $-4$ as given by equations (30) and (31). Recall that $\Omega = \sigma\cos\Theta$. Thus $\Theta = 0$ corresponds to $\Omega = \sigma$. Furthermore, if $\cos^2\Omega t = 0.0$, then, with $\Theta = 0$, the conditions for charge conjugation as given below equation (55) are satisfied and we obtain the complex conjugate of the initial vortex. For $\cos^2\Omega t = 0.5$, it is more convenient to use Cartesian co-ordinates. Using the expression for $\tilde{C}_{Nj}^{(q)}$ as given above and the configuration space representation of number states as given by (24), one can use the summation theorem for Hermite polynomials [26] to obtain

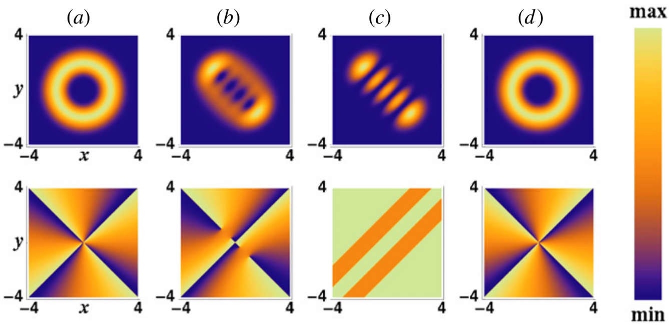


**Figure 4.** Time evolution of an initial vortex state for $\Theta = 0$ even though the entropy $\tilde{S}_{Nj}^{(a)}$ remains constant. Shown here are the contour plots of the absolute square (top row) and the phase (bottom row) of $\tilde{\psi}_{Nj}(x, y, t)$ as functions of $x$ and $y$ at different times with $b_{Nj}^{(n)}$ as given in (60). Here, $N = 4$, $j = 0$ and $\cos^2 \Omega t$ has the values (*a*) 1.0, (*b*) 0.9, (*c*) 0.5 and (*d*) 0.0. The horizontal and the vertical axes refer to the $x$ and $y$ coordinates, respectively. For the phase plots, we have used the convention that the phase ranges from $-\pi$ to $\pi$. Note that for $\cos^2 \Omega t = 0$ the state becomes the complex conjugate of the initial state as the direction of phase change is reversed.

$$\tilde{\psi}_{40}(x, y, t)\rfloor_{\Omega t=\frac{\pi}{4}} = -\frac{\mathrm{e}^{-(x^2+y^2)/2}}{\sqrt{\pi 2^4 4!}} H_4\left(\frac{x-y}{\sqrt{2}}\right) \tag{61a}$$

$$= \frac{\mathrm{e}^{-(x^2+y^2)/2}}{\sqrt{24\pi}}[-(x-y)^4 + 6(x-y)^2 - 3]. \tag{61b}$$

Thus, the wavefunction is a Gaussian modulated by a Hermite polynomial. Clearly, its value is real and, therefore, its phase is either zero or $\pi$ depending respectively on whether the wavefunction is $\geqslant 0$ or negative. It is easy to show that the wavefunction vanishes whenever $(x - y)^2 = 3 \pm \sqrt{6}$. Finally, it may be of some interest to realize that the wavefunction corresponds to a $SU(2)$ coherent state $-|\tau, N\rangle$ in the Schwinger representation [27] with $\tau = -1$ and $N = 4$.

## 4. Structure of the reduced state

In order to determine the structure of the reduced state for each mode, we first consider the correlation function in the $x$-space of mode '$a$' given by $\langle x|\rho_{Nj}^{(a)}|y\rangle$. A classical analogue of this function is the mutual coherence function of a partially coherent source [28]. Thus the process of reduction of a pure two-mode state into a mixed state by a partial trace operation over one mode amounts to loss of coherence and information. We can also define the spatial coherence function $\gamma_{Nj}^{(a)}(l)$ for the reduced state by

$$\gamma_{Nj}^{(a)}(l) = \int \langle x|\rho_{Nj}^{(a)}|x + l\rangle \,\mathrm{d}x. \tag{62}$$

When the system is initially in a two-mode Fock state, one obtains

$$\langle x|\rho_{Nj}^{(a)}|y\rangle = \sum_{q=0}^{N} \frac{\left|C_{Nj}^{(q)}\right|^2}{2^q q! \sqrt{\pi}} \mathrm{e}^{-(x^2+y^2)/2} H_q(x) H_q(y). \tag{63}$$

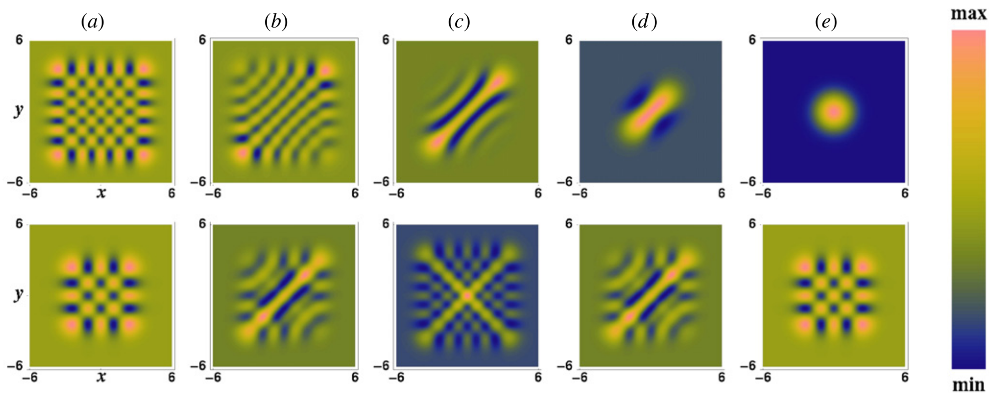


**Figure 5.** Contour plots of the correlation function as a function of $x$ and $y$ for different values of $|v_{21}|^2$. The parameters are as follows: $N = 8$; $j = 0$ (top row), $j = 4$ (bottom row); $|v_{21}|^2$ has the values (*a*) 1.0, (*b*) 0.9, (*c*) 0.5, (*d*) 0.1 and (*e*) 0.0. The horizontal and the vertical axes refer to the $x$ and $y$ coordinates, respectively.

The corresponding expression for $\gamma_{Nj}^{(a)}(l)$ is obtained by evaluating the standard integral [26] in (62). We get

$$\gamma_{Nj}^{(a)}(l) = \sum_{q=0}^{N} \left|C_{Nj}^{(q)}\right|^2 \mathrm{e}^{-l^2/4} L_q(l^2/2) \tag{64}$$

where $L_q(x) = L_q^0(x)$ is a Laguerre polynomial. Note that only one term survives in the summations over $q$ by virtue of (A.3) whenever $|v_{21}|^2 = 0$ or 1. Thus

$$\langle x|\rho_{Nj}^{(a)}|y\rangle = \frac{\mathrm{e}^{-(x^2+y^2)/2}}{\sqrt{\pi}} \begin{cases} \dfrac{H_j(x)H_j(y)}{2^j j!}, & |v_{21}|^2 \to 0, \\ \dfrac{H_{N-j}(x)H_{N-j}(y)}{2^{N-j}(N-j)!}, & |v_{21}|^2 \to 1, \end{cases} \tag{65}$$

and

$$\gamma_{Nj}^{(a)}(l) = \mathrm{e}^{-l^2/4} \begin{cases} L_j(l^2/2), & |v_{21}|^2 \to 0, \\ L_{N-j}(l^2/2), & |v_{21}|^2 \to 1. \end{cases} \tag{66}$$

Furthermore, one can use equation (A.4) and the definition (37*a*) to get

$$\langle x|\rho_{Nj}^{(a)}|y\rangle\rfloor_{|v_{21}|^2\to 1-R} = \langle x|\rho_{N,N-j}^{(a)}|y\rangle\rfloor_{|v_{21}|^2\to R} \tag{67a}$$

$$\gamma_{Nj}^{(a)}(l)\big\rfloor_{|v_{21}|^2\to 1-R} = \gamma_{N,N-j}^{(a)}\big\rfloor_{|v_{21}|^2\to R}. \tag{67b}$$

When the system is initially in a vortex state, equations (63)–(67*b*) are still valid provided that $|v_{21}|^2$ is replaced by $|\tilde{v}_{21}|^2$. In figure 5 we present contour plots of the correlation function as a function of $x$ and $y$ for a set of values of $|v_{21}|^2$ when $N = 8$ and $j = 0$ (top row), $j = 4$ (bottom row). The intricate patterns for $|v_{21}|^2 = 0$ and 1 can be explained by using equation (65). Furthermore, the identical nature of patterns for $N = 8$, $j = 4$ on either side of $|v_{21}|^2 = 1/2$ can be attributed to the property (67*a*). Finally in figure 6 we plot $\gamma_{Nj}^{(a)}(l)$ as a function of $l$ and $|v_{21}|^2$ when $N = 4$ and $j = 0, 2$. The patterns for $|v_{21}|^2 = 0$ and 1 follow from equation (66) and the symmetry of the plot for $j = 2$ about $|v_{21}|^2 = 1/2$ follows from (67*b*).

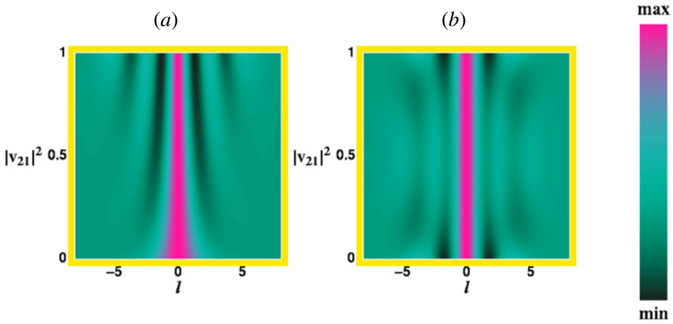


**Figure 6.** Contour plot of $\gamma_{Nj}^{(a)}(l)$ as a function of $l$ and $|v_{21}|^2$ when $N = 4$ and $j = 0$ (left), $j = 2$ (right).

## 5. Conclusion

In conclusion, we have studied, in a general way, entanglement produced in a two-mode bosonic system by linear $SU(2)$ transformations leading to the generation and evolution of quantum vortex states. The linear $SU(2)$ transformations are generated by evolving the system under the action of a generic Hamiltonian that mimics a variety of entanglement mechanisms. We have demonstrated that these transformations produce a coherent superposition of quantum vortices in general, and a single quantum vortex under certain conditions. Furthermore, as one would expect, a vortex state is found to be an entangled state. When the system is a light field, the vortex will appear in the quadrature distribution that can be measured by a homodyne method [21]. Explicit analytical results were obtained when the system was initially either in a Fock state or in a quantum vortex state. In the latter case, we have also found conditions for its revival and charge conjugation. A simple recipe was provided to accommodate all other cases for which the initial state can be reached from a Fock state by a unitary transformation. Thus we not only provide a unified approach to entanglement through a generic Hamiltonian but also predict the wide applicability of our results to a variety of initial states.

The ideas developed in this paper can be applied not only to light fields but also to matter waves such as the Bose Einstein condensates (BEC). In recent years, the BEC has proved to be an excellent laboratory for studying (both bipartite and many-particle) entanglement [29–33]. The entanglement of the modes as well as the entanglement of the atoms in a BEC have been considered. We mention parenthetically that our work is relevant in the former case. It is also well known that several mechanisms exist for the generation of vortices in a BEC [34–37]. Additionally, Whyte *et al* [38] have used the similarity between BECs and laser light to propose a method for generating Hermite–Gaussian type modes in a so-called *light pulse resonator*. Thus it should indeed be possible to generate vortices in a two-component BEC by entangling the two modes of the BEC by linear $SU(2)$ transformations via an entangling device such as a beam splitter.

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## Appendix A. Some useful properties of $\left|C_{Nj}^{(q)}\right|^2$

Using Leibniz's rule for the $j$th derivative of a product and the relations (14), one obtains

$$\left|C_{Nj}^{(q)}\right|^2 = (N-j)!(N-q)!q!(j!)^{-1}(1-|v_{21}|^2)^N\left(\frac{|v_{21}|^2}{1-|v_{21}|^2}\right)^{q-j}\left|f_{Nj}^{(q)}(|v_{21}|^2)\right|^2 \tag{A.1}$$

with

$$f_{Nj}^{(q)}(|v_{21}|^2) = \sum_{k=0}^{j}\frac{(-1)^k\binom{j}{k}}{(N-q-k)!(q-j+k)!}\left(\frac{|v_{21}|^2}{1-|v_{21}|^2}\right)^k. \tag{A.2}$$

It is easy to show that

$$\left|C_{Nj}^{(q)}\right|^2 = \begin{cases}\delta_{q,j}, & |v_{21}|^2\to 0,\\ \delta_{q,N-j}, & |v_{21}|^2\to 1.\end{cases} \tag{A.3}$$

Next we derive some important symmetry properties of $\left|C_{Nj}^{(q)}\right|^2$. First we show that

$$f_{Nj}^{(q)}(1-R) = (-1)^j\left(\frac{1-R}{R}\right)^j f_{Nj}^{(N-q)}(R)$$

$$f_{N,N-j}^{(q)}(R) = (-1)^{N-q}\frac{(N-j)!}{j!}\left(\frac{R}{1-R}\right)^{N-q} f_{Nj}^{(q)}(1-R)$$

where $0\leqslant R\leqslant 1$. The first relation is obtained from (A.2) by changing the summation index from $k$ to $j-k$ and the second relation is proved by exploiting the non-negativity of the factorials in (A.2) and changing the summation range accordingly. Using these two relations we immediately get

$$\left|C_{Nj}^{(q)}\right|^2\Big]_{|v_{21}|^2\to 1-R} = \left|C_{Nj}^{(N-q)}\right|^2\Big]_{|v_{21}|^2\to R} = \left|C_{N,N-j}^{(q)}\right|^2\Big]_{|v_{21}|^2\to R}. \tag{A.4}$$

Note that if $|v_{21}|^2 = 1/2$, then $|v_{11}|^2 = |v_{22}|^2 = |v_{12}|^2 = 1/2$ as well. Additionally, if $j=0$ or $N$, then $\left|C_{Nj}^{(q)}\right|^2 = 2^{-N}\binom{N}{q}$ is a binomial distribution whereas if $j=N/2$, then

$$\left|C_{N/2,N/2}^{(q)}\right|^2 = (N!)^{-1}\left[(N/2)!P_{N/2}^{(N/2-q,q-N/2)}(0)\right]^2\binom{N}{q} \tag{A.5}$$

where $P_n^{(\alpha,\beta)}(x)$ is a Jacobi polynomial.

## Appendix B. Derivation of equation (27)

Substituting expression (23) for $C_{Nj}^{(q)}$ in (25) and performing the summation over $q$ *before* differentiation with respect to $\tau$, we get

$$\psi_{Nj}(x,y,t) = \frac{(-1)^N}{N!}\sqrt{\frac{(N-j)!}{j!2^N\pi}}\,\mathrm{e}^{(x^2+y^2)/2}\left[\partial_\tau^{(j)}\hat{A}^N(\tau)\,\mathrm{e}^{-(x^2+y^2)}\right]_{\tau\to 0} \tag{B.1}$$

where

$$\hat{A}(\tau) = (v_{11}+v_{12}\tau)\partial_x + (v_{21}+v_{22}\tau)\partial_y. \tag{B.2}$$

We introduce $z = x+\mathrm{i}y$ so that $x^2+y^2 = zz^*$ and $\hat{A}(\tau) = \gamma_+(\tau)\partial_z + \gamma_-(\tau)\partial_{z^*}$ with $\gamma_\pm$ given by (26). Next we expand $\hat{A}(\tau)^N$ binomially and then use the relation

$$\partial_z^m\partial_{z^*}^n\,\mathrm{e}^{-zz^*} = \begin{cases}(-1)^n m!\,\mathrm{e}^{-zz^*}z^{n-m}L_m^{n-m}(zz^*), & m\leqslant n,\\ (-1)^m n!\,\mathrm{e}^{-zz^*}z^{*m-n}L_n^{m-n}(zz^*), & n\leqslant m,\end{cases} \tag{B.3}$$

to evaluate $\hat{A}(\tau)^N\,\mathrm{e}^{-(x^2+y^2)}$. Collecting all the terms we finally obtain (27).

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