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UNSYMMETRIC FREE VIBRATIONS OF

SANDWICH SHELLS OF REVOLUTION

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UNSYMMETRIC FREE VIBRATIONS OF

SANDWICH SHELLS OF REVOLUTION

CHAPTER 1

SURVEY OF DEVELOPMENTS IN THE ANALYSIS OF VIBRATIONAL CHARACTERISTICS OF THIN ELASTIC SHELLS

1.1 Introduction

During the past decade the field of aircraft and missile structures has undergone continuous and rapid changes. The emphasis is continually on strong, but light, structures in an effort to meet the extremely stringent requirements for low structural weight.

In the construction of space-vehicle structures, as well as in atmospheric vehicles, thin-shell structures are of great importance. For example, ballistic missiles and spacecraft are constructed basically of cylindrical and conical sections. Missile bodies and rocket-motor cases are formed by cylindrical shells. Conical shells find applications in rocket-motor nozzles, interstage fairings, skirts and nose sections. Nose cones are generally spherical, paraboloidal, ogival, or conical sections, or combinations of these. Other shell applications include pressure vessels, loudspeaker cones, submarine hulls and radomes.

In addition to static strength requirements, dynamic problems are very real. Acoustic instability in rocket motors, for instance, can excite vibrations in various missile components which can threaten their structural integrity. Unsteady aerodynamic loads or aerothermoelastic coupling can also produce critical loadings. Therefore, a thorough knowledge of the resonant frequencies and modal shapes of the structural components is a necessity.

In order to attain high-strength, low-weight structures, sandwich structures came into wide use. A sandwich structure is identified as having two thin, stiff facings (which can be dissimilar) separated by a relatively thick, low-density core which is highly flexible in shear.

It is the purpose of this dissertation to investigate free unsymmetric vibrations of sandwich shells of revolution.

1.2 General Thin-Shell Theory

The basis for the present-day shell theory was developed in the nineteenth century. Aron (1)* developed a theory of shells based on the hypotheses of Kirchhoff**. Later Love (2) corrected some inaccuracies in Aron's theory including those found in the formulas for change in curvature. However, more recent investigators such as Mushtari and Sachenkov (3) have determined that Aron's approximations were perhaps more accurate than those of Love. Love's shell theory also appears in his book (4).

In addition to the hypotheses of Kirchhoff, the so-called classical

---

*Underlined numbers in parentheses refer to references at the end of the dissertation.

**a) A normal to the middle surface before deformation remains normal after deformation.

b) The normal stresses normal to the middle surface are negligible compared to other stresses.

s..ll theory further assumes that: a) the ratio of the thickness to the radius of curvature is small as compared to unity; and b) the strains are small and the nonlinear terms in the strain-displacement expressions can be neglected.

The early investigations took the form primarily of mathematical theory. In the last few decades many useful techniques have been developed for the study of static and dynamic shell problems. Many books have been written concerning shell theory. Among them are those by Vlasov (5), Timoshenko and Woinowsky-Krieger (6), Flügge (7,8) and Novozhilov (9).

Hildebrand, Reissner and Thomas (10) discussed various systems of equations relating to shell theory. They then considered small axisymmetric deformations of orthotropic shells.

Vlasov (11) used the strain tensor component expressions for general curvilinear orthogonal coordinates (attributed by Love to Lamé) to derive the basic differential equations of motion for thin elastic shells.

Reissner (12) derived the stress-strain relationships for axisymmetric deformations of thin shells. Naghdi (13), also, formulated the stress-strain relationships, to include the effect of boundary conditions, in terms of general orthogonal curvilinear coordinates for thin, elastic, isotropic, uniform-thickness shells.

Later, Naghdi (14) included the effect of transverse shear deformation on the bending of shells of revolution. The differential equations developed included the effect of a varying thickness. A general solution for the differential equations was obtained using an extension of asymptotic integration.

Budiansky and Radkowski (15) used a finite-difference numerical-analysis technique for finding stresses and deflections due to unsymmetric bending of shells of revolution.

Cohen (16), also, used a finite-difference method for a computer analysis of unsymmetrical deformations of orthotropic shells of revolution.

1.3 Vibrations of Homogeneous Shells of Revolution

a. General and Miscellaneous Shells of Revolution

According to Epstein (17), the classical theory of vibrations of thin elastic plates was developed by Poisson (18) and Kirchhoff (19). Later, their theory was extended by Love (20) to include thin shells. In the dynamic analysis of thin shells, these early investigators retained the assumptions of Kirchhoff. That is, transverse shear deformations were ignored.

The Rayleigh inextensional theory* for thin shells was applied by Saunders and Paslay (21) to a sphere-cone shell combination to solve for frequencies of vibration. The calculated values agreed with experimental results within 5%.

Lin and Lee (22) represented a sphere-cone shell combination by a paraboloidal shell to remove the discontinuities at the junction of the sphere and cone encountered by Saunders and Paslay. Only inextensional vibrations were considered by Lin and Lee.

Garnet, Goldberg and Salerno (23) derived the general equations for torsional vibrations of a shell of revolution. Then a particular example of a conical frustum was solved for the first five modes of torsional vibration.

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*It is assumed that the middle surface deforms but without extension.

Shiraishi and DiMaggio (24) used a perturbation solution to solve for axisymmetric, non-torsional, extensional vibrations of prolate spheroidal shells.

Cohen (25) used a finite-difference method for an analysis of free vibrations of orthotropic shells of revolution.

b. Circular Cylindrical Shells

Among the early investigations of vibrations of cylindrical shells are those of Strutt (26) and Van Urk and Hut (27).

Epstein (17) considered flexural and torsional vibrations of cylindrical shells.

Baron and Bleich (28) used a one-term Rayleigh procedure to calculate frequencies of free vibrations of long thin cylindrical shells. Data were tabulated for various length-to-radius ratios.

Mirsky and Herrmann (29) generalized a Timoshenko-type theory (the inclusion of the effect of transverse shear strain) to calculate unsymmetric vibratory motions of cylindrical shells. Their theory included rotatory inertia as well as membrane and bending effects.

Yu (30) used the Donnell-type* equations to analyze vibrations of thin cylindrical shells.

c. Conical Shells

Goldberg (31) used a power series for an approximate solution to the equations of motion to analyze axisymmetric vibrations of conical shells.

Herrmann and Mirsky (32) applied a one-term Rayleigh procedure to determine the axisymmetric frequencies of vibration of a conical

*Donnell (35) simplified some of the early classical shell theory in his analysis of thin-walled tubes under torsion.

frustum. They ignored transverse shear deformation and rotatory inertia in their analysis.

Shulman (33) used stress functions and displacement functions to study the free vibrations of conical shells. Shulman also treated the problem of flutter of cylindrical and conical shells.

Saunders, Wisniewski and Paslay (34) included the bending and membrane effects in a Rayleigh-Ritz analysis of a conical frustum. The boundary conditions which they considered were that the cone was built-in on the small end and either simply supported or free on the large end. They found that the boundary conditions were not as influential on the higher frequencies as on the lower frequencies.

Goldberg, Bogdanoff and Marcus (36) solved the differential equations of motion for the vibrational frequencies of conical shells. A specific example of a loudspeaker cone was considered in detail. Their results showed good agreement with a power series approach previously used.

Later, Goldberg and Bogdanoff (37) developed a method for calculating frequencies for both axisymmetric and unsymmetric vibrational modes of a conical frustum by means of numerical integration. Their theory accounts for variations along the generators of thickness and material properties as well as axisymmetric temperature variations.

A Runge-Kutta process for numerical integration was used by Goldberg, Bogdanoff and Alspaugh (38) to solve a system of eight first-order differential equations for frequencies of vibration and the associated mode shapes for conical shells.

Watkins and Clary (39) conducted an experimental investigation of the vibrational characteristics of thin-walled conical frustum shells.

Four stainless-steel conical frustum shells were tested using an electromagnetic shaker as the source of excitation. Higher natural frequencies produced a greater number of circumferential waves at the large end than at the small end. This effect was observed to increase with conicity (increasing taper). In nearly all cases the observed frequencies were lower than those predicted theoretically.

Garnet and Kempner (40) used a two-term Rayleigh-Ritz procedure to study axisymmetric free vibrations of conical frustum shells. Their notable contribution was that of including transverse shear flexibility and rotatory inertia in the analysis. Comparing their results with those obtained from the classical shell theory, they found that for the lowest modes of vibration inclusion of the rotatory inertia had a negligible effect. However, the effect of transverse shear was more pronounced, particularly for short cones. It should be noted that rotatory inertia will have a greater effect at higher frequencies of vibration.

Hu (41) applied the Galerkin procedure to determine free axisymmetric and unsymmetric vibrations of truncated conical shells. Five-term Fourier series were used for modal functions. His analysis included the effects of transverse shear and rotatory inertia. He considered the boundary conditions: a) free-free, b) clamped-clamped, and c) simple-simple.

1.4 Vibrations of Sandwich Shells

Yu (42) investigated the vibrational properties of sandwich cylindrical shells. He considered only axisymmetric modes of vibration including strictly torsional modes. In all of Yu's work on sandwich plates and shells, transverse shear effects were included in the facings as well as in the core. Therefore, he had to allow for different shear rotations in the facings and core.

Chu (43), also, considered different shear rotations in the facings and core of a honeycomb sandwich shell. Chu used a one-term Rayleigh procedure to determine the free vibrations and the propagation of waves in the axial direction in a sandwich cylindrical shell.

Azar (44) extended the existing theory to cover axisymmetric free vibrations of sandwich shells of revolution, including transverse shear deformation of the core and rotatory inertia and orthotropy of the facings and core. The differential equations of motion were derived for a sandwich shell of revolution, but their solution was found to be untractable. The shell of revolution was then approximated by a series of conical frustums which were studied by means of a three-term Rayleigh-Ritz procedure. Numerical results were presented for the first two modes of vibration.

Habip (45) has presented an extensive survey and list of references concerning developments in the analysis of sandwich structures.

1.5 Closure

Very little has been published concerning the dynamic behavior of sandwich structures. Specifically, the only one to attack the vibrational problem of sandwich shells other than cylindrical was Azar. Since his study was limited to axisymmetric vibrations, a natural extension of the existing "state-of-the-art" is the determination of unsymmetric modes of vibration. That is the avenue pursued in the present dissertation.

CHAPTER 2

FORMULATION OF THE STRAIN AND KINETIC ENERGY EXPRESSIONS FOR A CIRCULAR CONICAL SHELL

2.1 Coordinate System

The conical-coordinate system shown in Figure 2-1 is used throughout this dissertation. The x, θ and z axes form an orthogonal coordinate system. The x axis coincides with the meridional direction on the middle surface of the shell, the θ axis with the circumferential direction on the middle surface, and the z axis is normal to the middle surface. The displacements of the middle surface are u, v and w which are in the x, θ and z directions, respectively.

2.2 Derivation of Strain Equations

Love (46) gave expressions for the six components of strain (neglecting nonlinear terms) for a general curvilinear orthogonal coordinate system. These strain components are

$$\left.\begin{aligned}
e_{\alpha\alpha} &= h_1 u_{\alpha,\alpha} + h_1 h_2 u_\beta (1/h_1)_{,\beta} + h_3 h_1 u_\gamma (1/h_1)_{,\gamma} \\
e_{\beta\beta} &= h_2 u_{\beta,\beta} + h_2 h_3 u_\gamma (1/h_2)_{,\gamma} + h_1 h_2 u_\alpha (1/h_2)_{,\alpha} \\
e_{\gamma\gamma} &= h_3 u_{\gamma,\gamma} + h_3 h_1 u_\alpha (1/h_3)_{,\alpha} + h_2 h_3 u_\beta (1/h_3)_{,\beta} \\
e_{\beta\gamma} &= (h_2/h_3)(h_3 u_\gamma)_{,\beta} + (h_3/h_2)(h_2 u_\beta)_{,\gamma} \\
e_{\gamma\alpha} &= (h_3/h_1)(h_1 u_\alpha)_{,\gamma} + (h_1/h_3)(h_3 u_\gamma)_{,\alpha} \\
e_{\alpha\beta} &= (h_1/h_2)(h_2 u_\beta)_{,\alpha} + (h_2/h_1)(h_1 u_\alpha)_{,\beta}
\end{aligned}\right\} \quad (2\text{-}1)$$

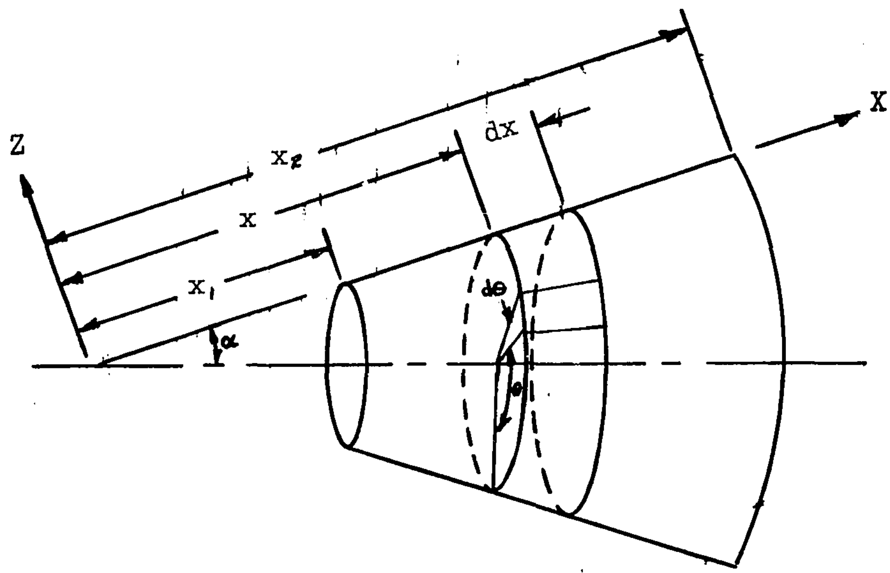

Figure 2-1. Conical coordinate system of the middle surface of the shell.

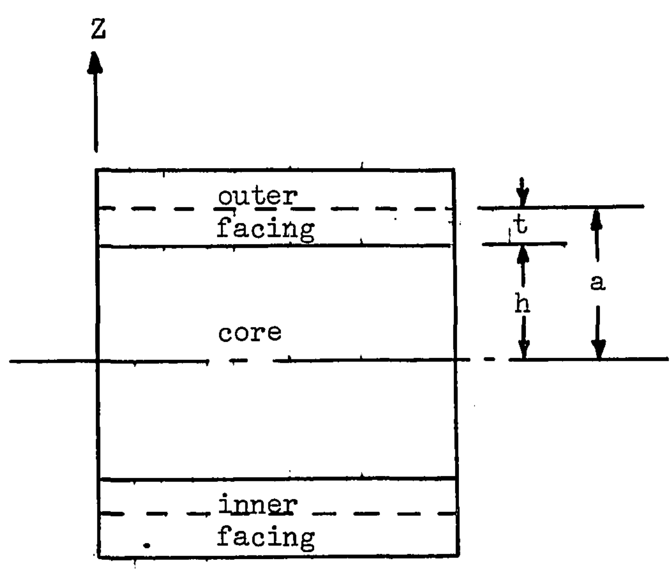

Figure 2-2. Dimensioning of a symmetrical sandwich shell (all dimensions typical).

where α and β coincide with the lines of curvature of the middle surface, γ is measured along the outward-drawn normal, and the h's are defined below.*

The infinitesimal distance, ds, between the points (α, β, γ) and $(\alpha + d\alpha, \beta + d\beta, \gamma + d\gamma)$ is given by

$$(ds)^2 = (d\alpha / h_1)^2 + (d\beta / h_2)^2 + (d\gamma / h_3)^2 \tag{2-2}$$

Equation (2-2) defines the h's which appear in Equation (2-1).

Now, the conical coordinate system shown in Figure 2-3 is considered. Here, the middle surface is represented by $O_o P_o$, point P is a point in the shell but not on the middle surface, and θ is measured from an arbitrary meridional plane.

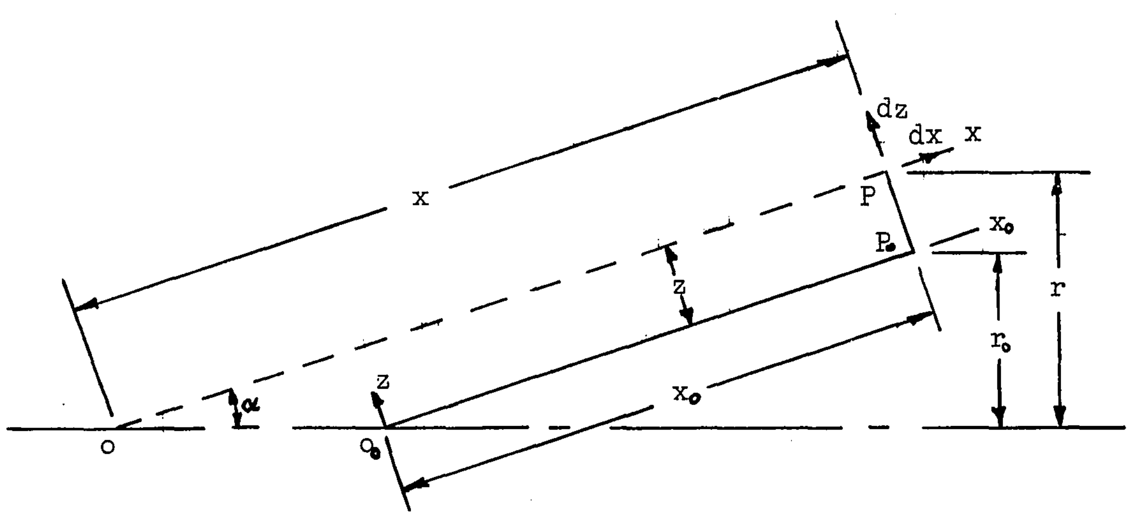

Figure 2-3. Conical Coordinate System

*The notation $(\ )_{,\alpha}$ indicates a partial derivative of the enclosed variable with respect to the coordinate following the comma. This notation is used throughout this dissertation.

Under the conical coordinate system the infinitesimal distance, ds, is given by

$$(ds)^2 = (dx)^2 + (rd\theta)^2 + (dz)^2 \qquad (2\text{-}3)$$

where

$$r = r_o + z\cos\alpha = x_o \sin\alpha + z\cos\alpha \qquad (2\text{-}4)$$

Comparison of Equations (2-2) and (2-3) reveals that in the conical coordinate system

$$\left.\begin{aligned} 1/h_1 &= 1 \\ 1/h_2 &= r \\ 1/h_3 &= 1 \end{aligned}\right\} \qquad (2\text{-}5)$$

where the following coordinate replacements are made:

$$\begin{aligned} \alpha &\rightarrow x \\ \beta &\rightarrow \theta \\ \gamma &\rightarrow z \end{aligned}$$

Substitution of Equations (2-4,5) into (2-1), remembering that $(\ )_{,x} \equiv (\ )_{,x_o}$ gives

$$\left.\begin{aligned} e_{xx} &= u_{x,x} \\ e_{\theta\theta} &= (1/r)(u_{\theta,\theta} + u_x \sin\alpha + u_z \cos\alpha) \\ e_{zz} &= u_{z,z} \\ e_{\theta z} &= (1/r)(u_{z,\theta} - u_\theta \cos\alpha) + u_{\theta,z} \\ e_{zx} &= u_{z,x} + u_{x,z} \\ e_{x\theta} &= (1/r)(u_{x,\theta} - u_\theta \sin\alpha) + u_{\theta,x} \end{aligned}\right\} \qquad (2\text{-}6)$$

2.3 Hypotheses

Before proceeding further, it is in order to present a brief listing of the hypotheses on which the analysis is based. The following assumptions are those generally made in the study of sandwich shells:

1. The core is capable of resisting transverse shearing stresses only, i. e. it has no resistance to bending or extension.
2. The core is linearly elastic and can be orthotropic.
3. The facings are linearly elastic and can be orthotropic.
4. The facings resist membrane (extensional), bending and transverse shearing strains.
5. The facings are identical, i.e. the sandwich construction is symmetrical (see Figure 2-2).
6. Since all deflections are assumed to be small, the strain-displacement relationships can be linearized.
7. All material damping effects are neglected.
8. All thermal and initial-stress effects are neglected.
9. The shell thickness is small compared to the smallest radius of curvature of the shell.
10. The facings and the core furnish both translational and rotatory inertia effects.
11. Lines which are straight and normal to the middle surface before deformation, remain straight during deformation, but not necessarily normal to the middle surface. (This is in contrast to the Kirchhoff hypothesis which precludes transverse-shear deformation. It states that lines which are straight and normal to the middle surface

before deformation remain so during deformation.)

12. All interactions with the surrounding fluid (air or water) are neglected; i.e. the shell is assumed to be in a vacuum.

2.4 Core of the Shell

In Equation (2-6) it is assumed that the displacements u_x and u_θ are dependent on the z-coordinate as well as on the displacements of the middle surface. The displacements of the core u_x, u_θ and u_z can be assumed in the following form:

$$\left.\begin{aligned} u_x &= u(x,\theta,t) + z\,\psi_x(x,\theta,t) \\ u_\theta &= v(x,\theta,t) + z\,\psi_\theta(x,\theta,t) \\ u_z &= w(x,\theta,t) \end{aligned}\right\} \quad (2\text{-}7)$$

where u, v and w are displacements of the middle surface and ψ_x and ψ_θ represent the angles of rotation of normals to the middle surface as a result of deformation in the meridional and circumferential directions respectively. The first two equations imply that lines originally straight and normal to the middle surface remain straight during deformation. The third equation implies that the shell is incompressible in the thickness direction. Dar (47) has studied the effect of thickness-direction motion on the free vibration of sandwich plates and found the effect to be negligible for practical sandwich geometries. Thus, the assumption made here appears to be valid.

Substitution of Equations (2-7) into Equations (2-6) gives

$$\begin{aligned} e_{xx} &= u_{,x} + z\,\psi_{x,x} \\ e_{\theta\theta} &= (x\sin\alpha)^{-1} v_{,\theta} + x^{-1}u + (x\tan\alpha)^{-1}w + z\left[(x\sin\alpha)^{-1}\psi_{\theta,\theta} + x^{-1}\psi_x\right] \end{aligned}$$

$$\left.\begin{aligned}
e_{zz} &= 0 \\
e_{\theta z} &= (x \sin\alpha)^{-1} w_{,\theta} + \psi_\theta - (x \tan\alpha)^{-1} v \\
e_{zx} &= w_{,x} + \psi_x \\
e_{x\theta} &= (x \sin\alpha)^{-1} u_{,\theta} - x^{-1} v + v_{,x} \\
&\quad + z\left[(x \sin\alpha)^{-1} \psi_{x,\theta} - x^{-1}\psi_\theta + \psi_{\theta,x}\right]
\end{aligned}\right\} \quad (2\text{-}8)$$

where $r = x \sin\alpha$. Equations (2-8) represent the state of strain in the core.

The usual engineering assumption, when investigating sandwich shell structures, is that the core is incapable of resisting extensional, bending and in-plane shearing strains. These assumptions are particularly applicable to a typical honeycomb core. Thus, all strain energy in the core arises from the transverse shearing strains $e_{\theta z}$ and e_{zx}. Therefore, in the core

$$\left.\begin{aligned}
\sigma_{xx} &= \sigma_{\theta\theta} = \sigma_{x\theta} = 0 \\
\sigma_{zx} &= G_{zx}\, e_{zx} \\
\sigma_{\theta z} &= G_{\theta z}\, e_{\theta z}
\end{aligned}\right\} \quad (2\text{-}9)$$

Thus, the ~~only~~ nonzero stress resultants in the core are given by

$$\begin{Bmatrix} Q_x \\ \\ Q_\theta \end{Bmatrix} = \int_{-h}^{h} \begin{Bmatrix} \sigma_{zx} K_x \\ \\ \sigma_{\theta z} K_\theta \end{Bmatrix} dz \qquad (2\text{-}10)$$

where Q_x and Q_θ are transverse shearing forces (lb/in), K_x and K_θ are the dynamic shear coefficients, and h is the half-thickness of the core. Figure 2-2 shows the cross-sectional dimensioning of a sandwich structure.

It is noted that since the core cannot resist any bending, the transverse shear stresses are uniform through the thickness of the core. Then integration of Equations (2-10), after substitutions from Equations (2-8,9), gives

$$\left.\begin{aligned} Q_x &= 2hK_x G_{zx}\, e_{zx} \\ Q_\theta &= 2hK_\theta G_{\theta z}\, e_{\theta z} \end{aligned}\right\} \quad (2\text{-}11)$$

2.5 Facings of the Shell

The identical facings also experience transverse shear, but to a different degree than the core. These rotation angles in the facings are distinguished from the corresponding core angles by a prime, ψ_x' and ψ_θ'. Thus, for the outer facing the displacements are assumed in the following form:

$$\left.\begin{aligned} u_x &= u(x,\theta,t) + h\psi_x(x,\theta,t) + (z-h)\psi_x'(x,\theta,t) \\ u_\theta &= v(x,\theta,t) + h\psi_\theta(x,\theta,t) + (z-h)\psi_\theta'(x,\theta,t) \\ u_z &= w(x,\theta,t) \end{aligned}\right\} \quad (2\text{-}12)$$

For the inner facing the displacements are assumed to be

$$\left.\begin{aligned} u_x &= u(x,\theta,t) - h\psi_x(x,\theta,t) + (z+h)\psi_x'(x,\theta,t) \\ u_\theta &= v(x,\theta,t) - h\psi_\theta(x,\theta,t) + (z+h)\psi_\theta'(x,\theta,t) \\ u_z &= w(x,\theta,t) \end{aligned}\right\} \quad (2\text{-}13)$$

Substitution of Equations (2-12,13) into Equations (2-6) gives

$$\left\{\begin{matrix} e^o_{xx} \\ e^i_{xx} \end{matrix}\right\} = u_{,x} \pm h(\psi_{x,x} - \psi'_{x,x}) + z\,\psi'_{x,x}$$

$$
\left.
\begin{aligned}
\begin{Bmatrix} e^{o}_{\theta\theta} \\ e^{i}_{\theta\theta} \end{Bmatrix} &= (x \sin\alpha)^{-1} \left[v_{,\theta} \pm h(\psi_{\theta,\theta} - \psi'_{\theta,\theta})\right] \\
&\quad + x^{-1}\left[u \pm h(\psi_{x} - \psi'_{x}) \right] + (x \tan\alpha)^{-1} w \\
&\quad + z \left[(x \sin\alpha)^{-1}\psi'_{\theta,\theta} + x^{-1}\psi'_{x} \right] \\
e^{o}_{zz} &= e^{i}_{zz} = 0 \\
\begin{Bmatrix} e^{o}_{\theta z} \\ e^{i}_{\theta z} \end{Bmatrix} &= (x \sin\alpha)^{-1} w_{,\theta} - (x \tan\alpha)^{-1} (v \pm h\psi_{\theta}) + \psi'_{\theta} \\
e^{o}_{zx} &= e^{i}_{zx} = w_{,x} + \psi'_{x} \\
\begin{Bmatrix} e^{o}_{x\theta} \\ e^{i}_{x\theta} \end{Bmatrix} &= (x \sin\alpha)^{-1}\left[u_{,\theta} \pm h(\psi_{x,\theta} - \psi'_{x,\theta})\right] \\
&\quad -x^{-1}\left[v \pm h(\psi_{\theta} - \psi'_{\theta})\right] + \left[v_{,x} \pm h(\psi_{\theta} - \psi'_{\theta,x})\right] \\
&\quad +z \left[(x \sin\alpha)^{-1}\psi'_{x,\theta} - x^{-1}\psi'_{\theta} + \psi'_{\theta,x} \right]
\end{aligned}
\right\} \quad (2\text{-}14)
$$

The superscripts o and i refer to the outer and inner facings, respectively. The e_{xx}, $e_{\theta\theta}$ and $e_{x\theta}$ strains can be written

$$
\left.
\begin{aligned}
\begin{Bmatrix} e^{o}_{xx} \\ e^{i}_{xx} \end{Bmatrix} &= \begin{Bmatrix} \epsilon^{o}_{x} \\ \epsilon^{i}_{x} \end{Bmatrix} + z\chi_{x} \\
\begin{Bmatrix} e^{o}_{\theta\theta} \\ e^{i}_{\theta\theta} \end{Bmatrix} &= \begin{Bmatrix} \epsilon^{o}_{\theta} \\ \epsilon^{i}_{\theta} \end{Bmatrix} + z\chi_{\theta} \\
\begin{Bmatrix} e^{o}_{x\theta} \\ e^{i}_{x\theta} \end{Bmatrix} &= \begin{Bmatrix} \epsilon^{o}_{x\theta} \\ \epsilon^{i}_{x\theta} \end{Bmatrix} + z\chi_{x\theta}
\end{aligned}
\right\} \quad (2\text{-}15)
$$

Thus, Equations (2-14) represent the state of strain in the facings.

In addition to membrane (extensional and in-plane shearing) and bending strains, it is also considered that the facings resist the transverse shearing strains $e_{\theta z}$ and e_{zx}. This lends consistency to the inclusion of the rotatory inertia of the facings in a subsequent section.

For the outer facing the following stresses can be calculated:

$$
\left.
\begin{aligned}
\sigma_{xx}^{Mo} &= \bar{E}_x'(\epsilon_x^o + \mu_x'\epsilon_\theta^o) \\
\sigma_{\theta\theta}^{Mo} &= \bar{E}_\theta'(\epsilon_\theta^o + \mu_\theta'\epsilon_x^o) \\
\sigma_{x\theta}^{Mo} &= G_{x\theta}'\epsilon_{x\theta}^o \\
\sigma_{xx}^{Bo} &= z\bar{E}_x'(\chi_x + \mu_x'\chi_\theta) \\
\sigma_{\theta\theta}^{Bo} &= z\bar{E}_\theta'(\chi_\theta + \mu_\theta'\chi_x) \\
\sigma_{\theta z}^o &= G_{\theta z}'e_{\theta z}^o \\
\sigma_{zx}^o &= G_{zx}'e_{zx}^o
\end{aligned}
\right\} \quad (2\text{-}16)
$$

where the M and B superscripts refer to membrane and bending effects. The primes indicate facing properties. Also, the identities

$$
\begin{aligned}
\bar{E}_x' &\equiv E_x'/\ (1-\mu_x'\mu_\theta') \\
\bar{E}_\theta' &\equiv E_\theta'/\ (1-\mu_x'\mu_\theta')
\end{aligned}
$$

are used.

The stress resultants for the outer facing are given by

$$
\begin{Bmatrix} F_x^o \\ F_\theta^o \\ F_{x\theta}^o \\ Q_x^o \\ Q_\theta^o \\ M_x^o \\ M_\theta^o \\ M_{x\theta}^o \end{Bmatrix} = \int_h^{h+t} \begin{Bmatrix} \sigma_{xx}^{Mo} \\ \sigma_{\theta\theta}^{Mo} \\ \sigma_{x\theta}^{Mo} \\ k_x'\sigma_{zx}^o \\ k_\theta'\sigma_{\theta z}^o \\ z\,\sigma_{xx}^{Bo} \\ z\,\sigma_{\theta\theta}^{Bo} \\ z\,\sigma_{x\theta}^{Bo} \end{Bmatrix} dz \qquad (2\text{-}17)
$$

Equations similar to Equations (2-16,17) can be written for the inner facing with the superscript o changed to i. Also, the integration limits in Equation (2-17) are changed to -h for the upper limit and -h-2t for the lower limit.

The complete listing of stress resultants for the inner and outer facings is:

$$
\left.
\begin{aligned}
F_x^o &= 2t\bar{E}'_x(\epsilon_x^o + \mu'_x\epsilon_\theta^o) \\
F_\theta^o &= 2t\bar{E}'_\theta(\epsilon_\theta^o + \mu'_\theta\epsilon_x^o) \\
F_{x\theta}^o &= 2tG'_{x\theta}\epsilon_{x\theta}^o \\
Q_x^o &= 2tK'_xG'_{zx}e_{zx}^o \\
Q_\theta^o &= 2tK'_\theta G'_{\theta z}e_{\theta z}^o \\
M_x^o &= \tfrac{2}{3}t^3\bar{E}'_x(\chi_x + \mu'_x\chi_\theta) \\
M_\theta^o &= \tfrac{2}{3}t^3\bar{E}'_\theta(\chi_\theta + \mu'_\theta\chi_x) \\
M_{x\theta}^o &= \tfrac{2}{3}t^3G'_{x\theta}\chi_{x\theta} \\
F_x^i &= 2t\bar{E}'_x(\epsilon_x^i + \mu'_x\epsilon_\theta^i) \\
F_\theta^i &= 2t\bar{E}'_\theta(\epsilon_\theta^i + \mu'_\theta\epsilon_x^i) \\
F_{x\theta}^i &= 2tG'_{x\theta}\epsilon_{x\theta}^i \\
Q_x^i &= 2tK'_xG'_{zx}e_{zx}^i \\
Q_\theta^i &= 2tK'_\theta G'_{\theta z}e_{\theta z}^i \\
M_x^i &= \tfrac{2}{3}t^3\bar{E}'_x(\chi_x + \mu'_x\chi_\theta) \\
M_\theta^i &= \tfrac{2}{3}t^3\bar{E}'_\theta(\chi_\theta + \mu'_\theta\chi_x) \\
M_{x\theta}^i &= \tfrac{2}{3}t^3G'_{x\theta}\chi_{x\theta}
\end{aligned}
\right\} \quad (2\text{-}18)
$$

2.6 Total Strain Energy

The total strain energy of the composite shell is comprised of the strain energy of the core plus the strain energy of the facings; or

$$V = V^c + V^F \qquad (2\text{-}19)$$

where V is the total strain energy of the composite shell and V^c and V^F are the strain energies of the core and facings, respectively.

The strain energy of the core is expressed by

$$V^c = \frac{1}{2} \int_x \int_\theta \left(Q_x e_{zx} + Q_\theta e_{\theta z} \right) x \sin\alpha \, d\theta dx \tag{2-20}$$

Substituting from Equations (2-11) into Equation (2-20) gives

$$V^c = h \int_x \int_\theta \left(K_x G_{zx} e^2_{zx} + K_\theta G_{\theta z} e^2_{\theta z} \right) x \sin\alpha \, d\theta dx \tag{2-21}$$

The strain energy of the facing is

$$\begin{aligned} V^F = \frac{1}{2} \int_x \int_\theta \Big\{ & F^o_x \epsilon^o_x + F^o_\theta \epsilon^o_\theta + F^o_{x\theta} \epsilon^o_{x\theta} + Q^o_x e^o_{zx} + Q^o_\theta e^o_{\theta z} \\ & + \left[M^o_x + M^i_x + a\left(F^o_x - F^i_x \right) \right] \chi_x + \left[M^o_\theta + M^i_\theta + a\left(F^o_\theta - F^i_\theta \right) \right] \chi_\theta \\ & + \left[M^o_{x\theta} + M^i_{x\theta} + a\left(F^o_{x\theta} - F^i_{x\theta} \right) \right] \chi_{x\theta} + F^i_x \epsilon^i_x + F^i_\theta \epsilon^i_\theta \\ & + F^i_{x\theta} \epsilon^i_{x\theta} + Q^i_x e^i_{zx} + Q^i_\theta e^i_{\theta z} \Big\} x \sin\alpha \, d\theta dx \end{aligned} \tag{2-22}$$

Substituting from Equations (2-18) into Equation (2-22) gives

$$\begin{aligned} V^F = \int_x \int_\theta \Big\{ & t\bar{E}'_x \left[(\epsilon^o_x)^2 + (\epsilon^i_x)^2 \right] + t\bar{E}'_\theta \left[(\epsilon^o_\theta)^2 + (\epsilon^i_\theta)^2 \right] + \left[t\bar{E}'_x \mu'_x \right. \\ & \left. + t\bar{E}'_\theta \mu'_\theta \right] \epsilon^o_x \epsilon^o_\theta + \left[t\bar{E}'_x \mu'_x + t\bar{E}'_\theta \mu'_\theta \right] \epsilon^i_x \epsilon^i_\theta + 2tK'_x G'_{zx} (e^o_{zx})^2 \\ & + tK'_\theta G'_{\theta z} \left[(e^o_{\theta z})^2 + (e^i_{\theta z})^2 \right] + \frac{2}{3} t^3 \bar{E}'_x \chi^2_x + \frac{2}{3} t^3 \bar{E}'_\theta \chi^2_\theta \\ & + \frac{2}{3} t^3 \left(\bar{E}'_x \mu'_x + \bar{E}'_\theta \mu'_\theta \right) \chi_x \chi_\theta + at\bar{E}'_x \left[(\epsilon^o_x - \epsilon^i_x) + \mu'_x (\epsilon^o_\theta - \epsilon^i_\theta) \right] \chi_x \\ & + at\bar{E}'_\theta \left[(\epsilon^o_\theta - \epsilon^i_\theta) + \mu'_\theta (\epsilon^o_x - \epsilon^i_x) \right] + \frac{2}{3} t^3 G'_{x\theta} \chi^2_{x\theta} \\ & + tG'_{x\theta} \left[(\epsilon^o_{x\theta})^2 + (\epsilon^i_{x\theta})^2 \right] \\ & + atG'_{x\theta} \left(\epsilon^o_{x\theta} - \epsilon^i_{x\theta} \right) \chi_{x\theta} \Big\} x \sin\alpha \, d\theta dx \end{aligned} \tag{2-23}$$

The following identities are introduced for convenience:

$$\eta_1 \equiv t\bar{E}'_x \qquad \eta_3 \equiv t\left(\mu'_x \bar{E}'_x + \mu'_\theta \bar{E}'_\theta \right)$$

$$\eta_2 \equiv t\bar{E}'_\theta \qquad \eta_4 \equiv 2tK'_x G'_{zx}$$

$$\left.\begin{aligned}
\eta_5 &\equiv t K'_\theta G'_{\theta z} & \eta_{12} &\equiv a t \mu'_\theta \bar{E}'_\theta \\
\eta_6 &\equiv \tfrac{2}{3} t^3 \bar{E}'_x & \eta_{13} &\equiv \tfrac{2}{3} t^3 G'_{x\theta} \\
\eta_7 &\equiv \tfrac{2}{3} t^3 \bar{E}'_\theta & \eta_{14} &\equiv t G'_{x\theta} \\
\eta_8 &\equiv \tfrac{2}{3} t^3 (\mu'_x \bar{E}'_x + \mu'_\theta \bar{E}'_\theta) & \eta_{15} &\equiv a t G'_{x\theta} \\
\eta_9 &\equiv a t \bar{E}'_x & \eta_{16} &\equiv h K_x G_{zx} \\
\eta_{10} &\equiv a t \bar{E}'_x \mu'_x & \eta_{17} &\equiv h K_\theta G_{\theta z} \\
\eta_{11} &\equiv a t \bar{E}'_\theta & &
\end{aligned}\right\} \quad (2\text{-}24)$$

Also, the three displacements of the middle surface and the four rotation angles of normals to the middle surface are assumed to have the following form:

$$\left.\begin{aligned}
u(x,\theta,t) &= U(x) \cos n\theta \sin\omega t \\
v(x,\theta,t) &= V(x) \sin n\theta \sin\omega t \\
w(x,\theta,t) &= W(x) \cos n\theta \sin\omega t \\
\psi_x(x,\theta,t) &= \Phi_x(x) \cos n\theta \sin\omega t \\
\psi'_x(x,\theta,t) &= \Phi'_x(x) \cos n\theta \sin\omega t \\
\psi_\theta(x,\theta,t) &= \Phi_\theta(x) \sin n\theta \sin\omega t \\
\psi'_\theta(x,\theta,t) &= \Phi'_\theta(x) \sin n\theta \sin\omega t
\end{aligned}\right\} \quad (2\text{-}25)$$

where n is the circumferential wave number and ω is the circular frequency of free vibration. The normal modal functions (which are functions of x only) must be chosen to meet the boundary conditions. The maximum value of strain energy then occurs when $\sin\omega t = 1$.

Substituting from Equations (2-8) and (2-14) into Equations

(2-21) and (2-23), respectively and using the identities of Equations (2-24), after adding, gives

$$
\begin{aligned}
V = \iint_{x\,\theta} \Big\{ & 2\eta_1 \left(u_{,x}^2 + h^2\psi_{x,x}^2 - 2h^2\psi_{x,x}\psi'_{x,x} + h^2\psi'^2_{x,x}\right) + 2\eta_2\big[(x\sin\alpha)^{-2}\left(v_{,\theta}^2 + h^2\psi_{\theta,\theta}^2\right. \\
& \left. - 2h^2\psi_{\theta,\theta}\psi'_{\theta,\theta} + h^2\psi'^2_{\theta,\theta}\right) + 2x^{-2}\csc\alpha\left(u v_{,\theta} + h^2\psi_x\psi_{\theta,\theta} - h^2\psi'_x\psi_{\theta,\theta} - h^2\psi_x\psi'_{\theta,\theta}\right. \\
& \left. + h^2\psi'_x\psi'_{\theta,\theta}\right) + 2x^{-2}\csc^2\alpha\cos\alpha\, v_{,\theta}w + x^{-2}\left(u^2 + h^2\psi_x^2 - 2h^2\psi_x\psi'_x + h^2\psi'^2_x\right) \\
& + 2x^{-2}\cot\alpha\, uw + (x\tan\alpha)^{-2}w^2\big] + 2\eta_3\big[(x\sin\alpha)^{-1}\left(u_{,x}v_{,\theta} + x^{-1}uu_{,x} + (x\tan\alpha)^{-1}u_{,x}w\right. \\
& + h^2(x\sin\alpha)^{-1}\left(\psi_{x,x}\psi_{\theta,\theta} - \psi_{x,x}\psi'_{\theta,\theta}\right) + h^2x^{-1}\left(\psi_x\psi_{x,x} - \psi_{x,x}\psi'_x\right) + h^2(x\sin\alpha)^{-1}\left(\psi'_{x,x}\psi'_{\theta,\theta}\right. \\
& \left. - \psi'_{x,x}\psi_{\theta,\theta}\right) + h^2x^{-1}\left(\psi'_{x,x}\psi'_x - \psi'_{x,x}\psi_x\right)\big] + \eta_4\left(w_{,x}^2 + 2w_{,x}\psi'_x + \psi'^2_x\right) + 2\eta_5\big[(x\sin\alpha)^{-2}w_{,\theta}^2 \\
& - 2x^{-2}\csc^2\alpha\cos\alpha\, v w_{,\theta} + 2(x\sin\alpha)^{-1}w_{,\theta}\psi'_\theta + (x\tan\alpha)^{-2}\left(v^2 + h^2\psi_\theta^2\right) + \psi'^2_\theta \\
& - 2(x\tan\alpha)^{-1}v\psi'_\theta\big] + \eta_6\psi'^2_{x,x} + \eta_7\left[(x\sin\alpha)^{-2}\psi'^2_{\theta,\theta} + 2x^{-2}\csc\alpha\,\psi'_{\theta,\theta}\psi'_x + x^{-2}\psi'^2_x\right] \\
& + \eta_8\left[(x\sin\alpha)^{-1}\psi'_{x,x}\psi'_{\theta,\theta} + x^{-1}\psi'_x\psi'_{x,x}\right] + 2h\eta_9\left(\psi_{x,x}\psi'_{x,x} - \psi'^2_{x,x}\right) + 2h\eta_{10}\big[(x\sin\alpha)^{-1}\left(\psi'_{x,x}\psi_{\theta,\theta}\right. \\
& \left. - \psi'_{x,x}\psi'_{\theta,\theta}\right) + x^{-1}\left(\psi_x\psi'_{x,x} - \psi'_x\psi'_{x,x}\right)\big] + 2h\eta_{11}\big[(x\sin\alpha)^{-2}\left(\psi_{\theta,\theta}\psi'_{\theta,\theta} - \psi'^2_{\theta,\theta}\right) \\
& + x^{-2}\csc\alpha\left(\psi_x\psi'_{\theta,\theta} - 2\psi'_x\psi'_{\theta,\theta} + \psi'_x\psi_{\theta,\theta}\right) + x^{-2}\left(\psi_x\psi'_x - \psi'^2_x\right)\big] + 2h\eta_{12}\big[(x\sin\alpha)^{-1}\left(\psi_{x,x}\psi'_{\theta,\theta}\right. \\
& \left. - \psi'_{x,x}\psi'_{\theta,\theta}\right) + x^{-1}\left(\psi_{x,x}\psi'_x - \psi'_x\psi'_{x,x}\right)\big] + \eta_{13}\big[(x\sin\alpha)^{-2}\psi'^2_{x,\theta} - 2x^{-2}\csc\alpha\,\psi'_{x,\theta}\psi'_\theta \\
& + 2(x\sin\alpha)^{-1}\psi'_{x,\theta}\psi'_{\theta,x} + x^{-2}\psi'^2_\theta - 2x^{-1}\psi'_\theta\psi'_{\theta,x} + \psi'^2_{\theta,x}\big] + 2\eta_{14}\big[(x\sin\alpha)^{-2}\left(u_{,\theta}^2\right. \\
& \left. + h^2\psi_{x,\theta}^2 - 2h^2\psi_{x,\theta}\psi'_{x,\theta} + h^2\psi'^2_{x,\theta}\right) - 2x^{-2}\csc\alpha\left(u_{,\theta}v + h^2\psi_{x,\theta} - h^2\psi_{x,\theta}\psi'_\theta\right. \\
& \left. - h^2\psi'_{x,\theta}\psi_\theta + h^2\psi'_{x,\theta}\psi'_\theta\right) + 2(x\sin\alpha)^{-1}\left(u_{,\theta}v_{,x} + h^2\psi_{x,\theta}\psi_{\theta,x} - h^2\psi_{x,\theta}\psi'_{\theta,x}\right. \\
& \left. - h^2\psi'_{x,\theta}\psi_{\theta,x} + h^2\psi'_{x,\theta}\psi'_{\theta,x}\right) - 2x^{-1}\left(v v_{,x} + h^2\psi_\theta\psi_{\theta,x} - h^2\psi_\theta\psi'_{\theta,x} - h^2\psi'_\theta\psi_{\theta,x}\right. \\
& \left. + h^2\psi'_\theta\psi'_{\theta,x}\right) + x^{-2}\left(v^2 + h^2\psi_\theta^2 - 2h^2\psi_\theta\psi'_\theta + h^2\psi'^2_\theta\right) + h^2\psi_{\theta,x}^2 - 2h^2\psi_{\theta,x}\psi'_{\theta,x} \\
& + h^2\psi'^2_{\theta,x}\big] + 2h\eta_{15}\big[(x\sin\alpha)^{-2}\left(\psi_{x,\theta}\psi'_{x,\theta} - \psi'^2_{x,\theta}\right) - x^{-2}\csc\alpha\left(\psi'_{x,\theta}\psi_\theta\right. \\
& \left. - 2\psi'_{x,\theta}\psi'_\theta + \psi_{x,\theta}\psi'_\theta\right) + (x\sin\alpha)^{-1}\left(\psi'_{x,\theta}\psi_{\theta,x} - 2\psi'_{x,\theta}\psi'_{\theta,x} + \psi_{x,\theta}\psi'_{\theta,x}\right) \\
& + x^{-2}\left(\psi_\theta\psi'_\theta - \psi'^2_\theta\right) - x^{-1}\left(\psi_{\theta,x}\psi'_\theta - 2\psi'_\theta\psi'_{\theta,x} + \psi_\theta\psi'_{\theta,x}\right) + \psi_{\theta,x}\psi'_{\theta,x} - \psi'^2_{\theta,x}\big]
\end{aligned}
$$

(Continued)

$$+\eta_{16}(w_{,x}^2 + 2w_{,x}\psi_x + \psi_x^2) + \eta_{17}[(x\sin\alpha)^{-2}w_{,\theta}^2 + 2(x\sin\alpha)^{-1}w_{,\theta}\psi_\theta$$
$$-2x^{-2}\csc^2\alpha\cos\alpha\, w_{,\theta}v + \psi_\theta^2 - 2(x\tan\alpha)^{-1}v\psi_\theta$$
$$+(x\tan\alpha)^{-2}v^2]\}\, x\sin\alpha\, d\theta\, dx \qquad (2\text{-}26)$$

After substitution of Equations (2-25) into Equation (2-26) and indicating the maximum value of strain energy with V_M and after integrating over θ, the following equation is obtained:

$$\begin{aligned}
V_M = \pi \int_x \{ & 2\eta_1(U_{,x}^2 + h^2\Phi_{x,x}^2 - 2h^2\Phi_{x,x}\Phi'_{x,x} + h^2\Phi'^2_{x,x}) + 2\eta_2[n^2(x\sin\alpha)^{-2}(V^2 + h^2\Phi_\theta^2 \\
& - 2h^2\Phi_\theta\Phi'_\theta + h^2\Phi'^2_\theta) + 2nx^{-2}\csc\alpha(UV + h^2\Phi_x\Phi_\theta - h^2\Phi'_x\Phi_\theta - h^2\Phi_x\Phi'_\theta + h^2\Phi'_x\Phi'_\theta) \\
& + 2nx^{-2}\csc^2\alpha\cos\alpha\, VW + x^{-2}(U^2 + h^2\Phi_x^2 - 2h^2\Phi_x\Phi'_x + h^2\Phi'^2_x) + 2x^{-2}\cot\alpha\, UW \\
& + (x\tan\alpha)^{-2}W^2] + 2\eta_3[n(x\sin\alpha)^{-1}U_{,x}V + x^{-1}UU_{,x} + (x\tan\alpha)^{-1}U_{,x}W \\
& + nh^2(x\sin\alpha)^{-1}(\Phi_{x,x}\Phi_\theta - \Phi_{x,x}\Phi'_\theta) + h^2x^{-1}(\Phi_x\Phi_{x,x} - \Phi_{x,x}\Phi'_x) + nh^2(x\sin\alpha)^{-1}(\Phi'_{x,x}\Phi'_\theta \\
& - \Phi'_{x,x}\Phi_\theta) + h^2x^{-1}(\Phi'_{x,x}\Phi'_x - \Phi_x\Phi'_{x,x})] + \eta_4(W_{,x}^2 + 2W_{,x}\Phi'_x + \Phi'^2_x) + 2\eta_5[n^2(x\sin\alpha)^{-2}W^2 \\
& + 2nx^{-2}\csc^2\alpha\cos\alpha\, VW - 2n(x\sin\alpha)^{-1}W\Phi'_\theta + (x\tan\alpha)^{-2}(V^2 + h^2\Phi_\theta^2) + \Phi'^2_\theta \\
& - 2(x\tan\alpha)^{-1}V\Phi'_\theta] + \eta_6\Phi'^2_{x,x} + \eta_7[n^2(x\sin\alpha)^{-2}\Phi'^2_\theta + 2nx^{-2}\csc\alpha\,\Phi'_x\Phi_\theta \\
& + x^{-2}\Phi'^2_x] + \eta_8[n(x\sin\alpha)^{-1}\Phi'_{x,x}\Phi'_\theta + x^{-1}\Phi'_x\Phi'_{x,x}] + 2h\eta_9(\Phi_{x,x}\Phi'_{x,x} - \Phi'^2_{x,x}) \\
& + 2h\eta_{10}[n(x\sin\alpha)^{-1}(\Phi'_{x,x}\Phi_\theta - \Phi'_\theta\Phi'_{x,x}) + x^{-1}(\Phi_x\Phi'_{x,x} - \Phi'_x\Phi'_{x,x})] + 2h\eta_{11}[n^2(x\sin\alpha)^{-2}(\Phi_\theta\Phi'_\theta \\
& - \Phi'^2_\theta) + nx^{-2}\csc\alpha(\Phi_x\Phi'_\theta - 2\Phi'_x\Phi'_\theta + \Phi'_x\Phi_\theta) + x^{-2}(\Phi_x\Phi'_x - \Phi'^2_x)] + 2h\eta_{12}[n(x\sin\alpha)^{-1}\Phi_{x,x}\Phi'_\theta \\
& - \Phi'_{x,x}\Phi'_\theta) + x^{-1}(\Phi_{x,x}\Phi'_x - \Phi'_x\Phi'_{x,x})] + \eta_{13}[n^2(x\sin\alpha)^{-2}\Phi'^2_x + 2nx^{-2}\csc\alpha\,\Phi'_x\Phi'_\theta \\
& - 2n(x\sin\alpha)^{-1}\Phi'_x\Phi'_{\theta,x} + x^{-2}\Phi'^2_\theta - 2x^{-1}\Phi'_\theta\Phi'_{\theta,x} + \Phi'^2_{\theta,x}] + 2\eta_{14}[n^2(x\sin\alpha)^{-2}(U^2 + h^2\Phi_x^2 \\
& - 2h^2\Phi_x\Phi'_x + h^2\Phi'^2_x) - 2nx^{-2}\csc\alpha(-UV - h^2\Phi_x\Phi_\theta + h^2\Phi_x\Phi'_\theta + h^2\Phi'_x\Phi_\theta \\
& - h^2\Phi'_x\Phi'_\theta) + 2n(x\sin\alpha)^{-1}(-UV_{,x} - h^2\Phi_x\Phi_{\theta,x} + h^2\Phi_x\Phi'_{\theta,x} + h^2\Phi'_x\Phi_{\theta,x} \\
& - h^2\Phi'_x\Phi'_{\theta,x}) - 2x^{-1}(VV_{,x} + h^2\Phi_\theta\Phi_{\theta,x} - h^2\Phi_\theta\Phi'_{\theta,x} - h^2\Phi'_\theta\Phi_{\theta,x}
\end{aligned}$$

(Continued)

$$
\begin{aligned}
&+h^2\Phi'_\theta\Phi'_{\theta,x}) + x^{-2}(V^2 + h^2\Phi_\theta^2 - 2h^2\Phi_\theta\Phi'_\theta + h^2\Phi'^2_\theta) + V_{,x}^2 + h^2\Phi_{\theta,x}^2 \\
&-2h^2\Phi_{\theta,x}\Phi'_{\theta,x} + h^2\Phi'^2_{\theta,x}] + 2h\eta_{15}[n^2(x\sin\alpha)^{-2}(\Phi_x\Phi'_x - \Phi'^2_x) \\
&-nx^{-2}\csc\alpha(-\Phi'_x\Phi_\theta + 2\Phi'_x\Phi'_\theta - \Phi_x\Phi'_\theta) + n(x\sin\alpha)^{-1}(-\Phi'_x\Phi_{\theta,x} + 2\Phi'_x\Phi'_{\theta,x} \\
&-\Phi_x\Phi'_{\theta,x}) + x^{-2}(\Phi_\theta\Phi'_\theta - \Phi'^2_\theta) - x^{-1}(\Phi_{\theta,x}\Phi'_\theta - 2\Phi'_\theta\Phi'_{\theta,x} + \Phi_\theta\Phi'_{\theta,x}) + \Phi_{\theta,x}\Phi'_{\theta,x} \\
&-\Phi'^2_{\theta,x}] + \eta_{16}(W_{,x}^2 + 2W_{,x}\Phi_x + \Phi_x^2) + \eta_{17}[n^2(x\sin\alpha)^{-2}W^2 \\
&-2n(x\sin\alpha)^{-1}W\Phi_\theta + 2nx^{-2}\csc^2\alpha\cos\alpha\, VW + \Phi_\theta^2 - 2(x\tan\alpha)^{-1}V\Phi_\theta \\
&\qquad + (x\tan\alpha)^{-2}V^2]\} \, x\sin\alpha \, dx
\end{aligned}
\tag{2-27}
$$

Equation (2-27) gives the maximum strain energy for the composite shell.

2.7 Total Kinetic Energy

The total kinetic energy for the composite shell consists of the sum of the rotatory inertias of the core and facings in addition to the translational inertias of the core and facings. This can be expressed as

$$
\begin{aligned}
T = \frac{1}{2}\iint_{x\,\theta} &[\bar{M}(\dot{u}^2 + \dot{v}^2 + \dot{w}^2) + \bar{I}(\dot{\psi}_x^2 + \dot{\psi}_\theta^2) \\
&+ 2\bar{I}'(\dot{\psi}'^2_x + \dot{\psi}'^2_\theta)] \, x\sin\alpha \, dx \, d\theta
\end{aligned}
\tag{2-28}
$$

where

$\bar{M} \equiv m + 2m'$ = total mass per unit area

$m \equiv$ core mass per unit area

$m' \equiv$ mass of one facing per unit area

$\bar{I} \equiv$ mass moment of inertia of core

$\bar{I}' \equiv$ mass moment of inertia of one facing

Substituting Equations (2-25) into Equation (2-28) and integrating over θ gives

$$T_M = \frac{\pi}{2}\omega^2 \int_x [\bar{M}(U^2 + V^2 + W^2) + \bar{I}(\varphi_x^2 + \varphi_\theta^2) + 2\bar{I}'(\varphi_x'^2 + \varphi_\theta'^2)]\, x \sin\alpha \, dx \qquad (2\text{-}29)$$

Equation (2-29) gives the maximum kinetic energy for the composite shell which occurs when $\cos\omega t = 1$.

CHAPTER 3

APPLICATION OF RAYLEIGH-RITZ METHOD

3.1 Rayleigh-Ritz Method

Rayleigh's method for determining the frequency of a vibrating conservative system consists of assuming a deflection modal shape, and then calculating and equating the maximum kinetic and potential energies. The Rayleigh method leads to higher frequencies than "exact" solutions yield due to the restraint added to the system by assuming a modal shape.

Ritz's contribution to the Rayleigh method was to assume a deflection mode as a series of several terms rather than a single term as Rayleigh had done.* This deflection mode can be written

$$q_i = A_{ij}\phi_j \tag{3-1}$$

where q_i is a generalized coordinate, A_{ij} are the undetermined parameters, and ϕ_j is the set of functions chosen to meet the boundary conditions.

As in the Rayleigh method the expressions (3-1) are substituted into the expression

$$T_M = V_M \tag{3-2}$$

*This is known as Ritz's method when applied to equilibrium problems and the Rayleigh-Ritz method when applied to eigenvalue problems.

where the subscript M indicates the maximum kinetic and potential energies. Because of computational necessity the infinite series (3-1) is truncated at a finite number of terms, n, prior to substitution into Equation (3-2). The undetermined parameters A_{ij} are then chosen so as to minimize the expression $V_M - T_M$ by means of the following system of equations:

$$\frac{\partial}{\partial A_{ij}}(V_M - T_M) = 0 \tag{3-3}$$

A system of n linear homogeneous algebraic equations containing n + 1 unknowns then exists. Writing Equation (3-3) in matrix form gives

$$[A]\{\delta\} - \lambda [B]\{\delta\} = 0 \tag{3-4}$$

or

$$\left([A] - \lambda [B]\right)\{\delta\} = 0 \tag{3-5}$$

where

$[A]$ = n x n stiffness matrix

λ = eigenvalue

$[B]$ = n x n inertia matrix

$\{\delta\}$ = n x 1 displacement matrix (eigenvector)

For a non-trivial solution, the determinant formed by

$$\left|[A] - \lambda [B]\right|$$

must vanish. The values of λ (eigenvalues) which make this determinant vanish are the quantities sought.

Mathematically, λ represents the roots of the system of equations in the matrix expression (3-5). Thus, if there are n equations in expression (3-5) there will be n values of λ. In this particular application, since

$$\lambda = \omega^2 x_1^2 \rho' / \bar{E}'_x$$

the square root of λ can be thought of as a non-dimensional frequency parameter where ω is the circular frequency of vibration.

An eigenvector can be calculated for each λ found. The eigenvector is composed of the undetermined parameters of the assumed modal functions. Thus, after the eigenvector is determined the modal shapes for each coordinate can be determined.

3.2 Coordinate Transformation

In applying the Rayleigh-Ritz procedure, a series of trigonometric functions is assumed for each of the three displacements, U, V, W and the four rotations, φ_x, φ_θ, φ'_x, φ'_θ (although a series of polynomials could be used*). However, the integration of Equation (2-27) would lead to sine and cosine integrals of the form

$$\int (1/t) \sin t \, dt \quad \text{and} \quad \int (1/t) \cos t \, dt \tag{3-6}$$

which lead to infinite series. In order to reduce the computational labor, a change of variables appears attractive. The most commonly used substitution is of the form

$$x = x_1 e^y \quad \text{or} \quad y = \ln(x/x_1) \tag{3-7}$$

*It would be more unwieldy, computationally, to set these up to fit the boundary conditions.

where y is the new variable.

In using the change of variables in (3-7), the case of the complete cone where x = 0 at the apex cannot be considered. However, the assumption, already made, that the total thickness of the composite shell is small as compared to the smallest radius of curvature is not valid at the apex of a cone. Therefore, the change of variables does not limit the investigation beyond the existing assumptions.

From the change of variables (3-7)

$$\left.\begin{aligned}
dy &= \frac{1}{x}\,dx \\
\frac{dU}{dx} &= \frac{dU}{dy}\frac{dy}{dx} = \frac{1}{x}\frac{dU}{dy} = (x_1 e^{y})^{-1}\frac{dU}{dy} \\
\frac{dV}{dx} &= (x_1 e^{y})^{-1}\frac{dV}{dy} \\
&\text{etc.}
\end{aligned}\right\} \qquad (3\text{-}8)$$

where the functions U, V, etc., are functions of y.

Substituting relationships (3-8) into Equation (2-22) to make the change of variables gives

$$\begin{aligned}
V_M = \pi \int_y \Big\{ & (\bar\eta_1 + n^2\bar\eta_2)U^2 + \bar\eta_3 UU_{,y} + \bar\eta_4 U_{,y}^2 + n\bar\eta_5 UV + \bar\eta_6 UW + n\bar\eta_7 U_{,y}V + \bar\eta_8 U_{,y}W \\
& - n\bar\eta_9 UV_{,y} + (n^2\bar\eta_{10} + \bar\eta_{11})V^2 + n\bar\eta_{12}VW - \bar\eta_{13}e^{y}V\Phi_\theta - \bar\eta_{14}e^{y}V\Phi_\theta' - \bar\eta_{15}VV_{,y} + \bar\eta_{16}V_{,y}^2 \\
& + (\bar\eta_{17} + n^2\bar\eta_{18})W^2 - n\bar\eta_{19}e^{y}W\Phi_\theta - n\bar\eta_{20}e^{y}W\Phi_\theta' + \bar\eta_{21}W_{,y}^2 + \bar\eta_{22}e^{y}W_{,y}\Phi_x' \\
& + \bar\eta_{23}e^{y}W_{,y}\Phi_x + (\bar\eta_{24} + n^2\bar\eta_{25} + \bar\eta_{26}e^{2y})\Phi_x^2 + \bar\eta_{27}\Phi_{x,y}^2 + \bar\eta_{28}\Phi_x\Phi_{x,y} + (\bar\eta_{29} \\
& + n^2\bar\eta_{30})\Phi_x\Phi_x' + \bar\eta_{31}\Phi_x\Phi_{x,y}' + \bar\eta_{32}\Phi_{x,y}\Phi_x' + \bar\eta_{33}\Phi_{x,y}\Phi_{x,y}' + (\bar\eta_{34} + n^2\bar\eta_{35} + \bar\eta_{36}e^{2y})\Phi_x'^2 \\
& + \bar\eta_{37}\Phi_x'\Phi_{x,y}' + n\bar\eta_{38}\Phi_x\Phi_\theta + n\bar\eta_{39}\Phi_x\Phi_\theta' + n\bar\eta_{40}\Phi_{x,y}\Phi_\theta + n\bar\eta_{41}\Phi_{x,y}\Phi_\theta' \\
& + n\bar\eta_{39}\Phi_x'\Phi_\theta + n\bar\eta_{42}\Phi_x'\Phi_\theta' + n\bar\eta_{43}\Phi_x'\Phi_{\theta,y} + n\bar\eta_{44}\Phi_x'\Phi_{\theta,y}' - n\bar\eta_{45}\Phi_x\Phi_{\theta,y} \\
& + n\bar\eta_{43}\Phi_x\Phi_{\theta,y}' + n\bar\eta_{46}\Phi_{x,y}'\Phi_\theta + n\bar\eta_{47}\Phi_{x,y}'\Phi_\theta' + \bar\eta_{48}\Phi_{\theta,y}^2 + (\bar\eta_{49} + n^2\bar\eta_{50}
\end{aligned}$$

(Continued)

$$+\bar{\eta}_{51}e^{2y})\Phi_\theta^2 + (\bar{\eta}_{52} + n^2\bar{\eta}_{53})\Phi_\theta\Phi_\theta' - \bar{\eta}_{52}\Phi_\theta\Phi_{\theta,y}' - \bar{\eta}_{52}\Phi_{\theta,y}\Phi_\theta'$$

$$+\bar{\eta}_{52}\Phi_{\theta,y}\Phi_{\theta,y}' + (\bar{\eta}_{54} + n^2\bar{\eta}_{55} + \bar{\eta}_{56}e^{2y})\Phi_\theta'^2 + \cancel{\bar{\eta}_{54}\Phi_{\theta,y}'^2 - \bar{\eta}_{48}\Phi_\theta\Phi_{\theta,y}}$$

$$+\bar{\eta}_{57}\Phi_\theta'\Phi_{\theta,y}' + \bar{\eta}_{58}\Phi_{x,y}'^2 \Big\}\, dy \qquad (3\text{-}9)$$

where the coefficients $\bar{\eta}_i$ are defined as

$\bar{\eta}_1 \equiv 2\eta_2 \sin\alpha$

$\bar{\eta}_2 \equiv 2\eta_{14} \csc\alpha$

$\bar{\eta}_3 \equiv 2\eta_3 \sin\alpha$

$\bar{\eta}_4 \equiv 2\eta_1 \sin\alpha$

$\bar{\eta}_5 \equiv 4(\eta_2 + \eta_{14})$

$\bar{\eta}_6 \equiv 4\eta_2 \cos\alpha$

$\bar{\eta}_7 \equiv 2\eta_3$

$\bar{\eta}_8 \equiv 2\eta_3 \cos\alpha$

$\bar{\eta}_9 \equiv 4\eta_{14}$

$\bar{\eta}_{10} \equiv 2\eta_2 \csc\alpha$

$\bar{\eta}_{11} \equiv (2\eta_5 + \eta_{17})\csc\alpha\cos^2\alpha + 2\eta_{14}$

$\bar{\eta}_{12} \equiv 2(2\eta_2 + 2\eta_5 + \eta_{17})\csc\alpha\cos\alpha$

$\bar{\eta}_{13} \equiv 2x_1\eta_{17}\cos\alpha$

$\bar{\eta}_{14} \equiv 4x_1\eta_5\cos\alpha$

$\bar{\eta}_{15} \equiv 4\eta_{14}\sin\alpha$

$\bar{\eta}_{16} \equiv 2\eta_{14}\sin\alpha$

$\bar{\eta}_{17} \equiv 2\eta_2\csc\alpha\cos^2\alpha$

$\bar{\eta}_{18} \equiv (2\eta_5 + \eta_{17})\csc\alpha$

$\bar{\eta}_{19} \equiv 2x_1\eta_{17}$

$\bar{\eta}_{20} \equiv 4x_1\eta_5$

$\bar{\eta}_{21} \equiv (\eta_4 + \eta_{16})\sin\alpha$

$\bar{\eta}_{22} \equiv 2x_1\eta_4\sin\alpha$

$\bar{\eta}_{23} \equiv 2x_1\eta_{16}\sin\alpha$

$\bar{\eta}_{24} \equiv 2h^2\eta_2\sin\alpha$

$\bar{\eta}_{25} \equiv 2h^2\eta_{14}\csc\alpha$

$\bar{\eta}_{26} \equiv x_1^2\eta_{16}\sin\alpha$

$\bar{\eta}_{27} \equiv 2h^2\eta_1\sin\alpha$

$\bar{\eta}_{28} \equiv 2h^2\eta_3\sin\alpha$

$\bar{\eta}_{29} \equiv 2h(\eta_{11} - 2h\eta_2)\sin\alpha$

$\bar{\eta}_{30} \equiv 2h(\eta_{15} - 2h\eta_{14})\csc\alpha$

$\bar{\eta}_{31} \equiv 2h(\eta_{10} - h\eta_3)\sin\alpha$

$\bar{\eta}_{32} \equiv 2h(\eta_{12} - h\eta_3)\sin\alpha$

$\bar{\eta}_{33} \equiv 2h(\eta_9 - 2h\eta_1)\sin\alpha$

$\bar{\eta}_{34} \equiv (2h^2\eta_2 + \eta_7 - 2h\eta_{11})\sin\alpha$

$\bar{\eta}_{35} \equiv (\eta_{13} + 2h^2\eta_{14} - 2h\eta_{15})\csc\alpha$

$\bar{\eta}_{36} \equiv x_1^2\eta_4\sin\alpha$

$\bar{\eta}_{37} \equiv (2h^2\eta_3 + \eta_8 - 2h\eta_{10} - 2h\eta_{12})\sin\alpha$

$\bar{\eta}_{38} \equiv 4h^2(\eta_2 + \eta_{14})$

$\bar{\eta}_{39} \equiv 2h(\eta_{11} - 2h\eta_2 - 2h\eta_{14} + \eta_{15})$

$\bar{\eta}_{40} \equiv 2h^2\eta_3$

$\bar{\eta}_{41} \equiv 2h(\eta_{12} - h\eta_3)$

$\bar{\eta}_{42} \equiv 2(2h^2\eta_2 + \eta_7 - 2h\eta_{11} + \eta_{13}$
$\quad + 2h^2\eta_{14} - 2h\eta_{15})$

$\bar{\eta}_{43} \equiv 2h(2h\eta_{14} - \eta_{15})$

$\bar{\eta}_{44} \equiv 2(2h\eta_{15} - \eta_{13} - 2h^2\eta_{14})$

$\bar{\eta}_{45} \equiv 4h^2\eta_{14}$

$\bar{\eta}_{46} \equiv 2h(\eta_{10} - h\eta_5)$

$\bar{\eta}_{47} \equiv 2h^2\eta_5 + \eta_8 - 2h\eta_{10} - 2h\eta_{12}$

$\bar{\eta}_{48} \equiv 2h^2\eta_{14}\sin\alpha$

$\bar{\eta}_{49} \equiv 2h^2(\eta_5\csc\alpha\cos^2\alpha + \eta_{14}\sin\alpha)$

$\bar{\eta}_{50} \equiv 2h^2\eta_2\csc\alpha$

$\bar{\eta}_{51} \equiv x_1^2\eta_{17}\sin\alpha$

$\bar{\eta}_{52} \equiv 2h(\eta_{15} - 2h\eta_{14})\sin\alpha$

$\bar{\eta}_{53} \equiv 2h(\eta_{11} - 2h\eta_2)\csc\alpha$

$\bar{\eta}_{54} \equiv (\eta_{13} + 2h^2\eta_{14} - 2h\eta_{15})\sin\alpha$

$\bar{\eta}_{55} \equiv (2h^2\eta_2 + \eta_7 - 2h\eta_{11})\csc\alpha$

$\bar{\eta}_{56} \equiv 2x_1^2\eta_5\sin\alpha$

$\bar{\eta}_{57} \equiv -2\bar{\eta}_{54}$

$\bar{\eta}_{58} \equiv (2h^2\eta_1 + \eta_6 - 2h\eta_9)\sin\alpha$

Similarly, substituting relationships (3-8) into Equation (2-29) to make the change of variables gives

$$T_M = \frac{\pi}{2}\omega^2 x_1^3 \rho' \int_y e^{2y}\sin\alpha \left[M(U^2 + V^2 + W^2) + x_1^2 I(\Phi_x^2 + \Phi_\theta^2) + 2x_1^2 I'(\Phi_x'^2 + \Phi_\theta'^2)\right] dy \qquad (3\text{-}10)$$

where

$$M \equiv 2\left[(\rho/\rho')(h/x_1) + 2(t/x_1)\right]$$
$$I \equiv (2/3)(\rho/\rho')(h/x_1)^3$$
$$I' \equiv (2/3)(t/x_1)\left[3(h/x_1)^2 + 6(h/x_1)(t/x_1) + 4(t/x_1)^2\right]$$

3.3 Approximation of a Shell of Revolution

Since the purpose of this dissertation is to treat shells of revolution in addition to conical shells, a scheme used by Azar (44) is used here. A shell of revolution can be approximated by a series of conical shells as shown in Figure 3-1. The first subscript on the x-dimensions refers to the segment number and the second subscript refers to the small or large ends, 1 and 2, respectively.

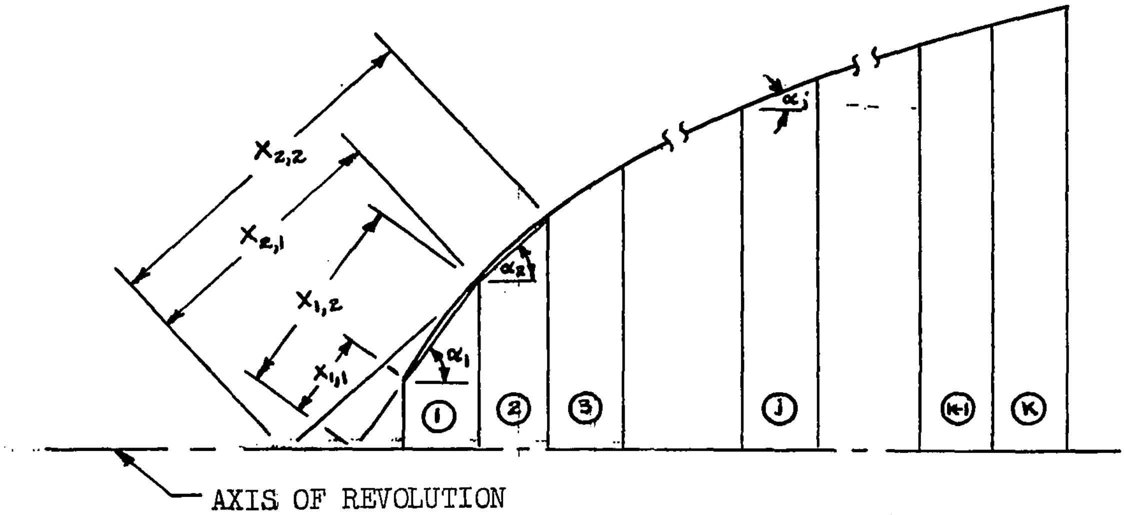

Figure 3-1 Shell of revolution approximated by K conical segments.

The total strain and kinetic energies are found by summing over K number of conical segments as follows:

$$V = \sum_{j=1}^{K} V_j \qquad T = \sum_{j=1}^{K} T_j \tag{3-11}$$

Thus, Equations (3-9,10) must be modified to accommodate the summing of energies of the K conical segments. Boundary conditions are imposed at the edges $x_{1,1}$ and $x_{K,2}$ only.

3.4 Boundary Conditions

Since there are three displacements and four angles of rotation of the normal to the middle surface involved in the state of deformation of the composite shell, the specification of boundary conditions could assume many forms. The boundary condition considered here is that of simply-supported ends in the sense that the normal and circumferential displacements are zero at each end of the shell. In addition the meridional stress resultant and moment are zero at the ends, so that

$$v = w = F_x = M_x = 0$$

on the edges $y = 0$, $\ln(x_{K,2}/x_{1,1})$ $(x = x_{1,1}, x_{K,2})$. (These are the same boundary conditions used by Cohen (25) and Hu (41) in their studies of unsymmetric free vibrations of conical shells.) By specifying that $v = w = 0$, the shell is actually "fixed" in the circumferential direction, so that $\varphi_\theta = \varphi_\theta' = 0$ on the edges. Thus the complete set of boundary conditions is

$$v = w = \varphi_\theta = \varphi_\theta' = F_x = M_x = 0 \tag{3-12}$$

on the edges $y = 0$, $\ln(x_{K,2}/x_{1,1})$.

Modal functions which satisfy the first four boundary conditions are given by the following sine series:

$$V = x_{1,1} \sum_m B_m \sin \beta_m y$$
$$W = x_{1,1} \sum_m C_m \sin \beta_m y$$
$$\varphi_\theta = \sum_m D_m \sin \beta_m y$$
$$\varphi_\theta' = \sum_m E_m \sin \beta_m y$$

where

$$\sum_m = \sum_{m=1}^{\infty}$$

$$\beta_m = m\pi / \ln(x_{K,2}/x_{1,1})$$

$$y = \ln(x/x_{1,1})$$

B_m, C_m, D_m, E_m = undetermined non-dimensional parameters

The fifth boundary condition is that the meridional constraint must vanish at the edges. Now F_x is calculated by

$$F_x = F_x^o + F_x^i \tag{3-12}$$

Substituting from Equations (2-18) into (3-12) gives

$$F_x = 2t\bar{E}_x'[\epsilon_x^o + \epsilon_x^i + \mu_x'(\epsilon_\theta^o + \epsilon_\theta^i)] \tag{3-13}$$

and substituting from Equations (2-14) and (2-15) into Equation (3-13) gives

$$\begin{aligned} F_x = \; & 2t\bar{E}_x'\Big((u_{,x} + h\psi_{x,x} - h\psi_{x,x}') + (u_{,x} - h\psi_{x,x} + h\psi_{x,x}') \\ & + \mu_x'\big\{ [\bar{x}^{-1}\csc\alpha(v_{,\theta} + h\psi_{\theta,\theta} - h\psi_{\theta,\theta}') + \bar{x}^{-1}(u + h\psi_x - h\psi_x') + \bar{x}^{-1}\cot\alpha\, w] \\ & + [\bar{x}^{-1}\csc\alpha(v_{,\theta} - h\psi_{\theta,\theta} + h\psi_{\theta,\theta}') + \bar{x}^{-1}(u - h\psi_x + h\psi_x') + \bar{x}^{-1}\cot\alpha\, w]\big\}\Big) \end{aligned}$$

or

$$F_x = 4t\bar{E}_x'\,[u_{,x} + \mu_x' x^{-1}(\csc\alpha\, v_{,\theta} + u + \cot\alpha\, w)]$$

Since it is specified that v and w vanish on the edges, the requirement that $F_x = 0$ on the edges, after making the transformation $y = \ln(x/x_{1,1})$, can be replaced by

$$U_{,y} + \mu_x' U = 0 \tag{3-14}$$

on the edges $y = 0$, $\ln(x_{K,2}/x_{1,1})$.

If U is assumed to be of the form

$$U = x_{1,1}\, e^{-\mu_x' y} \sum A_m \cos \beta_m y \tag{3-15}$$

Equation (3-14) is satisfied.

The final boundary condition is $M_x = 0$ on the edges. M_x can be calculated by

$$M_x = M_x^o + M_x^i + a(F_x^o - F_x^i) \tag{3-16}$$

Substituting from Equations (2-18) into Equation (3-16) gives

$$M_x = \tfrac{4}{3} t^3 \bar{E}_x' [\chi_x + \mu_x' \chi_\theta] + 2at\bar{E}_x' [\epsilon_x^o - \epsilon_x^i + \mu_x' (\epsilon_x^o - \epsilon_x^i)] \tag{3-17}$$

By Equations (2-14) and (2-15), Equation (3-17) becomes

$$\begin{aligned} M_x = {} & \tfrac{4}{3} t^3 \bar{E}_x' [\psi'_{x,x} + \mu_x' \bar{x}^{-1} (\csc\alpha\, \psi'_{\theta,\theta} + \psi'_x)] + 2at\bar{E}_x' \Big((u_{,x} + h\psi_{x,x} \\ & - h\psi'_{x,x}) - (u_{,x} - h\psi_{x,x} + h\psi'_{x,x}) + \mu_x' \bar{x}^{-1} \{ [\csc\alpha(v + h\psi_{\theta,\theta} - h\psi'_{\theta,\theta}) \\ & + (u + h\psi_x - h\psi'_x) + \cot\alpha\, w] - [\csc\alpha(v - h\psi_{\theta,\theta} + h\psi'_{\theta,\theta}) \\ & + (u - h\psi_x + h\psi'_x) + \cot\alpha\, w] \} \Big) \end{aligned}$$

or

$$\begin{aligned} M_x = {} & \tfrac{4}{3} t^3 \bar{E}_x' [\psi'_{x,x} + \mu_x' \bar{x}^{-1} (\csc\alpha\, \psi'_{\theta,\theta} + \psi'_x)] \\ & + 4hat\bar{E}_x' \{ \psi_{x,x} - \psi'_{x,x} + \mu_x' \bar{x}^{-1} [\csc\alpha(\psi_{\theta,\theta} - \psi'_{\theta,\theta}) + \psi_x - \psi'_x] \} \end{aligned}$$

Since ψ_θ and ψ'_θ vanish on the edges, the requirement that M_x, also, vanish on the edges can be replaced by

$$\psi'_{x,x} + \mu'_x \bar{x}^{1} \psi'_x + (3ha/t^2)[\psi_{x,x} - \psi'_{x,x} + \mu'_x \bar{x}^{1}(\psi_x - \psi'_x)] = 0 \qquad (3\text{-}18)$$

on the edges. After making the transformation $y = \ln(x/x_{1,1})$, Equation (3-18) can be written

$$(1 - 3ha/t^2)(\Psi'_{x,y} + \mu'_x \Psi'_x) + (3ha/t^2)(\Psi_{x,y} + \mu'_x \Psi_x) = 0 \qquad (3\text{-}19)$$

Equation (3-19) must be satisfied on the edges for any values of the coefficients $1-(3ha/t^2)$ and $(3ha/t^2)$. Therefore, the conditions

$$\left.\begin{aligned} \Psi'_{x,y} + \mu'_x \Psi'_x &= 0 \\ \Psi_{x,y} + \mu'_x \Psi_x &= 0 \end{aligned}\right\} \qquad (3\text{-}20)$$

must both be met on the edges. The following modal functions will satisfy Equations (3-20) on the boundaries:

$$\left.\begin{aligned} \Psi_x &= e^{-\mu'_x y} \sum_m F_m \cos \beta_m y \\ \Psi'_x &= e^{-\mu'_x y} \sum_m G_m \cos \beta_m y \end{aligned}\right\} \qquad (3\text{-}21)$$

In summary, the complete set of modal functions is

$$\left.\begin{aligned} U &= x_{1,1} e^{-\mu'_x y} \sum_m A_m \cos \beta_m y \\ V &= x_{1,1} \sum_m B_m \sin \beta_m y \\ W &= x_{1,1} \sum_m C_m \sin \beta_m y \\ \Psi_\theta &= \sum_m D_m \sin \beta_m y \\ \Psi'_\theta &= \sum_m E_m \sin \beta_m y \\ \Psi_x &= e^{-\mu'_x y} \sum_m F_m \cos \beta_m y \\ \Psi'_x &= e^{-\mu'_x y} \sum_m G_m \cos \beta_m y \end{aligned}\right\} \qquad (3\text{-}22)$$

which satisfies the boundary conditions

$$v = w = \psi_\theta = \psi_\theta' = F_x = M_x = 0$$

on the edges $y = 0$, $\ln(x_{K,2}/x_{1,1})$.

3.5 Inertia Coefficients

Each of the modal functions, Equations (3-22), is an infinite series which must be truncated at a finite number of terms for computational considerations. Garnet and Kempner (40) used a two-term series for each modal function while Azar (44) used three-term series. For this study each of the modal functions will be truncated at three terms.

Using the following identies

$$\underline{C}_1 = \cos\beta_1 y \qquad \underline{S}_1 = \sin\beta_1 y$$

$$\underline{C}_2 = \cos\beta_2 y \qquad \underline{S}_2 = \sin\beta_2 y$$

$$\underline{C}_3 = \cos\beta_3 y \qquad \underline{S}_3 = \sin\beta_3 y$$

When Equations (3-22) are substituted into Equation (3-10) the following expression for the total kinetic energy is obtained, after collecting terms:

$$T_M = \frac{\pi}{2}\omega^2 x_{1,1}^5 \rho' \Sigma \Big(\sin\alpha_j \int_{\phi_j}^{\tau_j} \Big\{ e^{2(1-\mu_x')y} \big[(MA_1^2$$

$$+ IF_1^2 + 2I'G_1^2)\underline{C}_1^2 + (MA_2^2 + IF_2^2 + 2I'G_2^2)\underline{C}_2^2$$

$$+ (MA_3^2 + IF_3^2 + 2I'G_3^2)\underline{C}_3^2 + 2(MA_1A_2 + IF_1F_2$$

$$+ 2I'G_1G_2)\underline{C}_1\underline{C}_2 + 2(MA_1A_3 + IF_1F_3 + 2I'G_1G_3)\underline{C}_1\underline{C}_3$$

$$+ 2(MA_2A_3 + IF_2F_3 + 2I'G_2G_3)\underline{C}_2\underline{C}_3\big] + e^{2y}\big[(MB_1^2$$

(Continued)

$$\left.\begin{aligned}
\frac{\partial T_M}{\partial A_1} &= \bar{\lambda}\,\Sigma\,\sin\alpha_j\, M\,(A_1\sigma_{1,j} + A_2\sigma_{4,j} + A_3\sigma_{5,j})\\
\frac{\partial T_M}{\partial A_2} &= \bar{\lambda}\,\Sigma\,\sin\alpha_j\, M\,(A_1\sigma_{4,j} + A_2\sigma_{2,j} + A_3\sigma_{6,j})\\
\frac{\partial T_M}{\partial A_3} &= \bar{\lambda}\,\Sigma\,\sin\alpha_j\, M\,(A_1\sigma_{5,j} + A_2\sigma_{6,j} + A_3\sigma_{3,j})\\
\frac{\partial T_M}{\partial B_1} &= \bar{\lambda}\,\Sigma\,\sin\alpha_j\, M\,(B_1\sigma_{7,j} + B_2\sigma_{10,j} + B_3\sigma_{11,j})\\
\frac{\partial T_M}{\partial B_2} &= \bar{\lambda}\,\Sigma\,\sin\alpha_j\, M\,(B_1\sigma_{10,j} + B_2\sigma_{8,j} + B_3\sigma_{12,j})\\
\frac{\partial T_M}{\partial B_3} &= \bar{\lambda}\,\Sigma\,\sin\alpha_j\, M\,(B_1\sigma_{11,j} + B_2\sigma_{12,j} + B_3\sigma_{9,j})\\
\frac{\partial T_M}{\partial C_1} &= \bar{\lambda}\,\Sigma\,\sin\alpha_j\, M\,(C_1\sigma_{7,j} + C_2\sigma_{10,j} + C_3\sigma_{11,j})\\
\frac{\partial T_M}{\partial C_2} &= \bar{\lambda}\,\Sigma\,\sin\alpha_j\, M\,(C_1\sigma_{10,j} + C_2\sigma_{8,j} + C_3\sigma_{12,j})\\
\frac{\partial T_M}{\partial C_3} &= \bar{\lambda}\,\Sigma\,\sin\alpha_j\, M\,(C_1\sigma_{11,j} + C_2\sigma_{12,j} + C_3\sigma_{9,j})\\
\frac{\partial T_M}{\partial D_1} &= \bar{\lambda}\,\Sigma\,\sin\alpha_j\, I\,(D_1\sigma_{7,j} + D_2\sigma_{10,j} + D_3\sigma_{11,j})\\
\frac{\partial T_M}{\partial D_2} &= \bar{\lambda}\,\Sigma\,\sin\alpha_j\, I\,(D_1\sigma_{10,j} + D_2\sigma_{8,j} + D_3\sigma_{12,j})\\
\frac{\partial T_M}{\partial D_3} &= \bar{\lambda}\,\Sigma\,\sin\alpha_j\, I\,(D_1\sigma_{11,j} + D_2\sigma_{12,j} + D_3\sigma_{9,j})\\
\frac{\partial T_M}{\partial E_1} &= \bar{\lambda}\,\Sigma\,\sin\alpha_j\, 2I'\,(E_1\sigma_{7,j} + E_2\sigma_{10,j} + E_3\sigma_{11,j})\\
\frac{\partial T_M}{\partial E_2} &= \bar{\lambda}\,\Sigma\,\sin\alpha_j\, 2I'\,(E_1\sigma_{10,j} + E_2\sigma_{8,j} + E_3\sigma_{12,j})\\
\frac{\partial T_M}{\partial E_3} &= \bar{\lambda}\,\Sigma\,\sin\alpha_j\, 2I'\,(E_1\sigma_{11,j} + E_2\sigma_{12,j} + E_3\sigma_{9,j})\\
\frac{\partial T_M}{\partial F_1} &= \bar{\lambda}\,\Sigma\,\sin\alpha_j\, I\,(F_1\sigma_{1,j} + F_2\sigma_{4,j} + F_3\sigma_{5,j})\\
\frac{\partial T_M}{\partial F_2} &= \bar{\lambda}\,\Sigma\,\sin\alpha_j\, I\,(F_1\sigma_{4,j} + F_2\sigma_{2,j} + F_3\sigma_{6,j})\\
\frac{\partial T_M}{\partial F_3} &= \bar{\lambda}\,\Sigma\,\sin\alpha_j\, I\,(F_1\sigma_{5,j} + F_2\sigma_{6,j} + F_3\sigma_{3,j})\\
\frac{\partial T_M}{\partial G_1} &= \bar{\lambda}\,\Sigma\,\sin\alpha_j\, 2I'\,(G_1\sigma_{1,j} + G_2\sigma_{4,j} + G_3\sigma_{5,j})\\
\frac{\partial T_M}{\partial G_2} &= \bar{\lambda}\,\Sigma\,\sin\alpha_j\, 2I'\,(G_1\sigma_{4,j} + G_2\sigma_{2,j} + G_3\sigma_{6,j})\\
\frac{\partial T_M}{\partial G_3} &= \bar{\lambda}\,\Sigma\,\sin\alpha_j\, 2I'\,(G_1\sigma_{5,j} + G_2\sigma_{6,j} + G_3\sigma_{3,j})
\end{aligned}\right\}\qquad (3\text{-}25)$$

Thus the right-hand portions of the twenty one equations, Equations (3-25), using the coefficients of (3-26), can be expressed in the matrix form

$$\bar{\lambda} \begin{bmatrix} & & \\ & b_{ij} & \\ & & \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ B_1 \\ \cdot \\ \cdot \\ \cdot \\ G_3 \end{bmatrix} \tag{3-27}$$

or

$$\bar{\lambda} \, [B] \{\delta\} \tag{3-28}$$

The matrix, $[B]$, has diagonal symmetry ($b_{ij} = b_{ji}$).

3.6 Stiffness Coefficients

Substituting the modal functions from Equations (3-22), truncating each series at three terms, into Equation (3-9) gives the following expression after collecting terms:

$$
\begin{aligned}
V_M = \pi \sum \Big(\int_{\phi_j}^{\tau_j} \Big\{ & \beta_1^2 (X_1^2 \bar{\eta}_{16} B_1^2 + X_1^2 \bar{\eta}_{21} C_1^2 + \bar{\eta}_{48} D_1^2 + \bar{\eta}_{52} D_1 E_1 + \bar{\eta}_{54} E_1^2)\, \underline{C}_1^2 \\
& + \beta_2^2 (X_1^2 \bar{\eta}_{16} B_2^2 + X_1^2 \bar{\eta}_{21} C_2^2 + \bar{\eta}_{48} D_2^2 + \bar{\eta}_{52} D_2 E_2 + \bar{\eta}_{54} E_2^2)\, \underline{C}_2^2 + \beta_3^2 (X_1^2 \bar{\eta}_{16} B_3^2 + X_1^2 \bar{\eta}_{21} C_3^2 \\
& + \bar{\eta}_{48} D_3^2 + \bar{\eta}_{52} D_3 E_3 + \bar{\eta}_{54} E_3^2)\, \underline{C}_3^2 + \beta_1 \beta_2 [2X_1^2 \bar{\eta}_{16} B_1 B_2 + 2X_1^2 \bar{\eta}_{21} C_1 C_2 + 2\bar{\eta}_{48} D_1 D_2 \\
& + \bar{\eta}_{52} (D_1 E_2 + D_2 E_1) + 2\bar{\eta}_{54} E_1 E_2]\, \underline{C}_1 \underline{C}_2 + \beta_1 \beta_3 [2X_1^2 \bar{\eta}_{16} B_1 B_3 + 2X_1^2 \bar{\eta}_{21} C_1 C_3 \\
& + 2\bar{\eta}_{48} D_1 D_3 + \bar{\eta}_{52} (D_1 E_3 + D_3 E_1) + 2\bar{\eta}_{54} E_1 E_3]\, \underline{C}_1 \underline{C}_3 + \beta_2 \beta_3 [2X_1^2 \bar{\eta}_{16} B_2 B_3 \\
& + 2X_1^2 \bar{\eta}_{21} C_2 C_3 + 2\bar{\eta}_{48} D_2 D_3 + \bar{\eta}_{52} (D_2 E_3 + D_3 E_2) + 2\bar{\eta}_{54} E_2 E_3]\, \underline{C}_2 \underline{C}_3 \\
& + [X_1^2 (n^2 \bar{\eta}_{10} + \bar{\eta}_{11}) B_1^2 + n \bar{\eta}_{12} X_1^2 B_1 C_1 + X_1^2 (\bar{\eta}_{17} + n^2 \bar{\eta}_{18}) C_1^2 + (\bar{\eta}_{49} + n^2 \bar{\eta}_{50}) D_1^2 \\
& + (\bar{\eta}_{52} + n^2 \bar{\eta}_{53}) D_1 E_1 + (\bar{\eta}_{54} + n^2 \bar{\eta}_{55}) E_1^2]\, \underline{S}_1^2 + [X_1^2 (n^2 \bar{\eta}_{10} + \bar{\eta}_{11}) B_2^2 + n X_1^2 \bar{\eta}_{12} B_2 C_2 \\
& + X_1^2 (\bar{\eta}_{17} + n^2 \bar{\eta}_{18}) C_2^2 + (\bar{\eta}_{49} + n^2 \bar{\eta}_{50}) D_2^2 + (\bar{\eta}_{52} + n^2 \bar{\eta}_{53}) D_2 E_2 + (\bar{\eta}_{54} + n^2 \bar{\eta}_{55}) E_2^2]\, \underline{S}_2^2 \\
& + [X_1^2 (n^2 \bar{\eta}_{10} + \bar{\eta}_{11}) B_3^2 + n X_1^2 \bar{\eta}_{12} B_3 C_3 + X_1^2 (\bar{\eta}_{17} + n^2 \bar{\eta}_{18}) C_3^2 + (\bar{\eta}_{49} + n^2 \bar{\eta}_{50}) D_3^2 \\
& + (\bar{\eta}_{52} + n^2 \bar{\eta}_{53}) D_3 E_3 + (\bar{\eta}_{54} + n^2 \bar{\eta}_{55}) E_3^2]\, \underline{S}_3^2 + [2X_1^2 (n^2 \bar{\eta}_{10} + \bar{\eta}_{11}) B_1 B_2 \\
& + n X_1^2 \bar{\eta}_{12} (B_2 C_1 + B_1 C_2) + 2X_1^2 (\bar{\eta}_{17} + n^2 \bar{\eta}_{18}) C_1 C_2 + 2(\bar{\eta}_{49} + n^2 \bar{\eta}_{50}) D_1 D_2 \\
& + (\bar{\eta}_{52} + n^2 \bar{\eta}_{53})(D_1 E_2 + D_2 E_1) + 2(\bar{\eta}_{54} + n^2 \bar{\eta}_{55}) E_1 E_2]\, \underline{S}_1 \underline{S}_2 + [2X_1^2 (n^2 \bar{\eta}_{10} \\
& + \bar{\eta}_{11}) B_1 B_3 + n X_1^2 \bar{\eta}_{12} (B_3 C_1 + B_1 C_3) + 2X_1^2 (\bar{\eta}_{17} + n^2 \bar{\eta}_{18}) C_1 C_3 + 2(\bar{\eta}_{49} + n^2 \bar{\eta}_{50}) D_1 D_3 \\
& + (\bar{\eta}_{52} + n^2 \bar{\eta}_{53})(D_1 E_3 + D_3 E_1) + 2(\bar{\eta}_{54} + n^2 \bar{\eta}_{55}) E_1 E_3]\, \underline{S}_1 \underline{S}_3 + [2X_1^2 (n^2 \bar{\eta}_{10} \\
& + \bar{\eta}_{11}) B_2 B_3 + n X_1^2 \bar{\eta}_{12} (B_3 C_2 + B_2 C_3) + 2X_1^2 (\bar{\eta}_{17} + n^2 \bar{\eta}_{18}) C_2 C_3 + 2(\bar{\eta}_{49} + n^2 \bar{\eta}_{50}) D_2 D_3 \\
& + (\bar{\eta}_{52} + n^2 \bar{\eta}_{53})(D_2 E_3 + D_3 E_2) + 2(\bar{\eta}_{54} + n^2 \bar{\eta}_{55}) E_2 E_3]\, \underline{S}_2 \underline{S}_3 + \beta_1 (-X_1^2 \bar{\eta}_{15} B_1^2 \\
& - 2\bar{\eta}_{52} D_1 E_1 - \bar{\eta}_{48} D_1^2 + \bar{\eta}_{57} E_1^2)\, \underline{S}_1 \underline{C}_1 + \beta_2 (-X_1^2 \bar{\eta}_{15} B_1 B_2 - \bar{\eta}_{52} D_1 E_2 - \bar{\eta}_{52} D_2 E_1 \\
& - \bar{\eta}_{48} D_1 D_2 + \bar{\eta}_{57} E_1 E_2)\, \underline{S}_1 \underline{C}_2 + \beta_3 (-X_1^2 \bar{\eta}_{15} B_1 B_3 - \bar{\eta}_{52} D_1 E_3 - \bar{\eta}_{52} D_3 E_1 - \bar{\eta}_{48} D_1 D_3 \\
& + \bar{\eta}_{57} E_1 E_3)\, \underline{S}_1 \underline{C}_3 + \beta_1 (-X_1^2 \bar{\eta}_{15} B_1 B_2 - \bar{\eta}_{52} D_2 E_1 - \bar{\eta}_{52} D_1 E_2 - \bar{\eta}_{48} D_1 D_2 + \bar{\eta}_{57} E_1 E_2)\, \underline{C}_1 \underline{S}_2 \\
& + \beta_2 (-X_1^2 \bar{\eta}_{15} B_2^2 - 2\bar{\eta}_{52} D_2 E_2 - \bar{\eta}_{48} D_2^2 + \bar{\eta}_{57} E_2^2)\, \underline{S}_2 \underline{C}_2 + \beta_3 (-X_1^2 \bar{\eta}_{15} B_2 B_3 \\
& - \bar{\eta}_{52} D_2 E_3 - \bar{\eta}_{52} D_3 E_2 - \bar{\eta}_{48} D_2 D_3 + \bar{\eta}_{57} E_2 E_3)\, \underline{S}_2 \underline{C}_3 + \beta_1 (-X_1^2 \bar{\eta}_{15} B_1 B_3 - \bar{\eta}_{52} D_3 E_1
\end{aligned}
$$

(Continued)

$$-\bar{\eta}_{52}D_1E_3 - \bar{\eta}_{48}D_1D_3 + \bar{\eta}_{57}E_1E_3)\underline{c}_1\underline{s}_3 + \beta_2(-x_1^2\bar{\eta}_{15}B_2B_3 - \bar{\eta}_{52}D_3E_2 - \bar{\eta}_{52}D_2E_3$$

$$-\bar{\eta}_{48}D_2D_3 + \bar{\eta}_{57}E_2E_3)\underline{c}_2\underline{s}_3 + \beta_3(-x_1^2\bar{\eta}_{15}B_3^2 - 2\bar{\eta}_{52}D_3E_3 - \bar{\eta}_{48}D_3^2 + \bar{\eta}_{57}E_3^2)\underline{s}_3\underline{c}_3$$

$$-(\bar{\eta}_{13}D_1B_1 + \bar{\eta}_{14}B_1E_1 + n\bar{\eta}_{19}C_1D_1 + n\bar{\eta}_{20}C_1E_1)x_1e^{y}\underline{s}_1^2 - (\bar{\eta}_{13}B_2D_2 + \bar{\eta}_{14}B_2E_2$$

$$+n\bar{\eta}_{19}C_2D_2 + n\bar{\eta}_{20}C_2E_2)x_1e^{y}\underline{s}_2^2 - (\bar{\eta}_{13}B_3D_3 + \bar{\eta}_{14}B_3E_3 + n\bar{\eta}_{19}C_3D_3 + n\bar{\eta}_{20}C_3E_3)x_1e^{y}\underline{s}_3^2$$

$$-[\bar{\eta}_{13}(B_1D_2+B_2D_1) + \bar{\eta}_{14}(B_1E_2+B_2E_1) + n\bar{\eta}_{19}(C_1D_2+C_2D_1) + n\bar{\eta}_{20}(C_1E_2+C_2E_1)]x_1e^{y}\underline{s}_1\underline{s}_2$$

$$-[\bar{\eta}_{13}(B_1D_3+B_3D_1) + \bar{\eta}_{14}(B_1E_3+B_3E_1) + n\bar{\eta}_{19}(C_1D_3+C_3D_1) + n\bar{\eta}_{20}(C_1E_3+C_3E_1)]xe^{y}\underline{s}_1\underline{s}_3$$

$$-[\bar{\eta}_{13}(B_2D_3+B_3D_2) + \bar{\eta}_{14}(B_2E_3+B_3E_2) + n\bar{\eta}_{19}(C_2D_3+C_3D_2) + n\bar{\eta}_{20}(C_2E_3+C_3E_2)]xe^{y}\underline{s}_2\underline{s}_3$$

$$+(\bar{\eta}_{51}D_1^2+\bar{\eta}_{56}E_1^2)e^{2y}\underline{s}_1^2 + (\bar{\eta}_{51}D_2^2+\bar{\eta}_{56}E_2^2)e^{2y}\underline{s}_2^2 + (\bar{\eta}_{51}D_3^2+\bar{\eta}_{56}E_3^2)e^{2y}\underline{s}_3^2$$

$$+2(\bar{\eta}_{51}D_1D_2+\bar{\eta}_{56}E_1E_2)e^{2y}\underline{s}_1\underline{s}_2 + 2(\bar{\eta}_{51}D_1D_3+\bar{\eta}_{56}E_1E_3)e^{2y}\underline{s}_1\underline{s}_3$$

$$+2(\bar{\eta}_{51}D_2D_3+\bar{\eta}_{56}E_2E_3)e^{2y}\underline{s}_2\underline{s}_3 + \beta_1(\bar{\eta}_{22}C_1G_1+\bar{\eta}_{23}C_1F_1)x_1e^{(1-\mu)y}\underline{c}_1^2$$

$$+\beta_2(\bar{\eta}_{22}C_2G_2+\bar{\eta}_{23}C_2F_2)x_1e^{(1-\mu)y}\underline{c}_2^2 + \beta_3(\bar{\eta}_{22}C_3G_3+\bar{\eta}_{23}C_3F_3)x_1e^{(1-\mu)y}\underline{c}_3^2$$

$$+[\bar{\eta}_{22}(\beta_1C_1G_2+\beta_2C_2G_1) + \bar{\eta}_{23}(\beta_1C_1F_2+\beta_2C_2F_1)]x_1e^{(1-\mu)y}\underline{c}_1\underline{c}_2 + [\bar{\eta}_{22}(\beta_1C_1G_3$$

$$+\beta_3C_3G_1) + \bar{\eta}_{23}(\beta_1C_1F_3+\beta_3C_3F_1)]x_1e^{(1-\mu)y}\underline{c}_1\underline{c}_3 + [\bar{\eta}_{22}(\beta_2C_2G_3+\beta_3C_3G_2)$$

$$+\bar{\eta}_{23}(\beta_2C_2F_3+\beta_3C_3F_2)]x_1e^{(1-\mu)y}\underline{c}_2\underline{c}_3 + (\bar{\eta}_{26}F_1^2+\bar{\eta}_{36}G_1^2)e^{2(1-\mu)y}\underline{c}_1^2$$

$$+(\bar{\eta}_{26}F_2^2+\bar{\eta}_{36}G_2^2)e^{2(1-\mu)y}\underline{c}_2^2 + (\bar{\eta}_{26}F_3^2+\bar{\eta}_{36}G_3^2)e^{2(1-\mu)y}\underline{c}_3^2 + 2(\bar{\eta}_{26}F_1F_2$$

$$+\bar{\eta}_{36}G_1G_2)e^{2(1-\mu)y}\underline{c}_1\underline{c}_2 + 2(\bar{\eta}_{26}F_1F_3+\bar{\eta}_{36}G_1G_3)e^{2(1-\mu)y}\underline{c}_1\underline{c}_3 + 2(\bar{\eta}_{26}F_2F_3$$

$$+\bar{\eta}_{36}G_2G_3)e^{2(1-\mu)y}\underline{c}_2\underline{c}_3 + n\beta_1(-x_1^2\bar{\eta}_9A_1B_1 + \bar{\eta}_{43}D_1G_1 + \bar{\eta}_{44}E_1G_1 - \bar{\eta}_{45}D_1F_1$$

$$+\bar{\eta}_{43}E_1F_1)e^{-\mu y}\underline{c}_1^2 + n\beta_2(-x_1^2\bar{\eta}_9A_2B_2 + \bar{\eta}_{43}D_2G_2 + \bar{\eta}_{44}E_2G_2 - \bar{\eta}_{45}D_2F_2 + \bar{\eta}_{43}E_2F_2)e^{-\mu y}\underline{c}_2^2$$

$$+n\beta_3(-x_1^2\bar{\eta}_9A_3B_3 + \bar{\eta}_{43}D_3G_3 + \bar{\eta}_{44}E_3G_3 - \bar{\eta}_{45}D_3F_3 + \bar{\eta}_{43}E_3F_3)e^{-\mu y}\underline{c}_3^2$$

$$+n[-x_1^2\bar{\eta}_9(\beta_1A_2B_1+\beta_2A_1B_2) + \bar{\eta}_{43}(\beta_1D_1G_2+\beta_2D_2G_1) + \bar{\eta}_{44}(\beta_1E_1G_2+\beta_2E_2G_1)$$

$$-\bar{\eta}_{45}(\beta_1D_1F_2+\beta_2D_2F_1) + \bar{\eta}_{43}(\beta_1E_1F_2+\beta_2E_2F_1)]e^{-\mu y}\underline{c}_1\underline{c}_2 + n[-x_1^2\bar{\eta}_9(\beta_1A_3B_1$$

$$+\beta_3A_1B_3) + \bar{\eta}_{43}(\beta_1D_1G_3+\beta_3D_3G_1) + \bar{\eta}_{44}(\beta_1E_1G_3+\beta_3E_3G_1) - \bar{\eta}_{45}(\beta_1D_1F_3$$

(Continued)

$$+\beta_3 D_3 F_1) + \bar{\eta}_{43}(\beta_1 E_1 F_3 + \beta_3 E_3 F_1)]\, e^{-\mu y}\, \underline{S}_1 \underline{C}_3 \;+ n\,[-x_1^2\,\bar{\eta}_9(\beta_2 A_3 B_2 + \beta_3 A_2 B_3)$$

$$+\bar{\eta}_{43}(\beta_2 D_2 G_3 + \beta_3 D_3 G_2) + \bar{\eta}_{44}(\beta_2 E_2 G_3 + \beta_3 E_3 G_2) - \bar{\eta}_{45}(\beta_2 D_2 F_3 + \beta_3 D_3 F_2)$$

$$+\bar{\eta}_{43}(\beta_2 E_2 F_3 + \beta_3 E_3 F_2)]\, e^{-\mu y}\, \underline{C}_2 \underline{C}_3 \;+\; (n x_1^2 \bar{\eta}_5 A_1 B_1 + \bar{\eta}_6 x_1^2 A_1 C_1 - \mu n x_1^2 \bar{\eta}_7 A_1 B_1$$

$$-\mu x_1^2 \bar{\eta}_8 A_1 C_1 + n\bar{\eta}_{38} D_1 F_1 + n\bar{\eta}_{39} E_1 F_1 - \mu n \bar{\eta}_{40} D_1 F_1 - \mu n \bar{\eta}_{41} E_1 F_1 + n\bar{\eta}_{42} E_1 G_1$$

$$+ n\bar{\eta}_{39} D_1 G_1 - \mu n \bar{\eta}_{46} D_1 G_1 - \mu n \bar{\eta}_{47} E_1 G_1)\, e^{-\mu y}\, \underline{S}_1 \underline{C}_1 \;+\; (n x_1^2 \bar{\eta}_5 A_2 B_1 + \bar{\eta}_6 x_1^2 A_2 C_1$$

$$-\mu n x_1^2 \bar{\eta}_7 A_2 B_1 - \mu x_1^2 \bar{\eta}_8 A_2 C_1 + n\bar{\eta}_{38} D_1 F_2 + n\bar{\eta}_{39} E_1 F_2 - \mu n \bar{\eta}_{40} D_1 F_2 - \mu n \bar{\eta}_{41} E_1 F_2$$

$$+ n\bar{\eta}_{42} E_1 G_2 + n\bar{\eta}_{39} D_1 G_2 - \mu n \bar{\eta}_{46} D_1 G_2 - \mu n \bar{\eta}_{47} E_1 G_2)\, e^{-\mu y}\, \underline{S}_1 \underline{C}_2 + (n\bar{\eta}_5 x_1^2 A_3 B_1$$

$$+ x_1^2 \bar{\eta}_6 A_3 C_1 - \mu n x_1^2 \bar{\eta}_7 A_3 B_1 - \mu x_1^2 \bar{\eta}_8 A_3 C_1 + n\bar{\eta}_{38} D_1 F_3 + n\bar{\eta}_{39} E_1 F_3 - \mu n \bar{\eta}_{40} D_1 F_3$$

$$-\mu n \bar{\eta}_{41} E_1 F_3 + n\bar{\eta}_{42} E_1 G_3 + n\bar{\eta}_{39} D_1 G_3 - \mu n \bar{\eta}_{46} D_1 G_3 - \mu n \bar{\eta}_{47} E_1 G_3)\, e^{-\mu y}\, \underline{S}_1 \underline{C}_3$$

$$+ (n x_1^2 \bar{\eta}_5 A_1 B_2 + x_1^2 \bar{\eta}_6 A_1 C_2 - \mu n x_1^2 \bar{\eta}_7 A_1 B_2 - \mu x_1^2 \bar{\eta}_8 A_1 C_2 + n\bar{\eta}_{38} D_2 F_1$$

$$+ n\bar{\eta}_{39} E_2 F_1 - \mu n \bar{\eta}_{40} D_2 F_1 - \mu n \bar{\eta}_{41} E_2 F_1 + n\bar{\eta}_{42} E_2 G_1 + n\bar{\eta}_{39} D_2 G_1 - \mu n \bar{\eta}_{46} D_2 G_1$$

$$-\mu n \bar{\eta}_{47} E_2 G_1)\, e^{-\mu y}\, \underline{C}_1 \underline{S}_2 \;+\; (n x_1^2 \bar{\eta}_5 A_2 B_2 + x_1^2 \bar{\eta}_6 A_2 C_2 + n\bar{\eta}_{38} D_2 F_2$$

$$-\mu n x_1^2 \bar{\eta}_7 A_2 B_2 - \mu x_1^2 \bar{\eta}_8 A_2 C_2 + n\bar{\eta}_{39} E_2 F_2 - \mu n \bar{\eta}_{40} D_2 F_2 - \mu n \bar{\eta}_{41} E_2 F_2$$

$$+ n\bar{\eta}_{42} E_2 G_2 + n\bar{\eta}_{39} D_2 G_2 - \mu n \bar{\eta}_{46} D_2 G_2 - \mu n \bar{\eta}_{47} E_2 G_2)\, e^{-\mu y}\, \underline{S}_2 \underline{C}_2$$

$$+ (n x_1^2 \bar{\eta}_5 A_3 B_2 + x_1^2 \bar{\eta}_6 A_3 C_2 - \mu n x_1^2 \bar{\eta}_7 A_3 B_2 - \mu x_1^2 \bar{\eta}_8 A_3 C_2 + n\bar{\eta}_{38} D_2 F_3 + n\bar{\eta}_{39} E_2 F_3$$

$$-\mu n \bar{\eta}_{40} D_2 F_3 - \mu n \bar{\eta}_{41} E_2 F_3 + n\bar{\eta}_{42} E_2 G_3 + n\bar{\eta}_{39} D_2 G_3 - \mu n \bar{\eta}_{46} D_2 G_3$$

$$-\mu n \bar{\eta}_{47} E_2 G_3)\, e^{-\mu y}\, \underline{S}_2 \underline{C}_3 \;+\; (n x_1^2 \bar{\eta}_5 A_1 B_3 + x_1^2 \bar{\eta}_6 A_1 C_3 - \mu n x_1^2 \bar{\eta}_7 A_1 B_3$$

$$-\mu x_1^2 \bar{\eta}_8 A_1 C_3 + n\bar{\eta}_{38} D_3 F_1 + n\bar{\eta}_{39} E_3 F_1 - \mu n \bar{\eta}_{40} D_3 F_1 - \mu n \bar{\eta}_{41} E_3 F_1 + n\bar{\eta}_{42} E_3 G_1$$

$$+ n\bar{\eta}_{39} D_3 G_1 - \mu n \bar{\eta}_{46} D_3 G_1 - \mu n \bar{\eta}_{47} E_3 G_1)\, e^{-\mu y}\, \underline{C}_1 \underline{S}_3 \;+\; (n x_1^2 \bar{\eta}_5 A_2 B_3$$

$$+ x_1^2 \bar{\eta}_6 A_2 C_3 - \mu n x_1^2 \bar{\eta}_7 A_2 B_3 - \mu x_1^2 \bar{\eta}_8 A_2 C_3 + n\bar{\eta}_{38} D_3 F_2 + n\bar{\eta}_{39} E_3 F_2$$

$$-\mu n \bar{\eta}_{40} D_3 F_2 - \mu n \bar{\eta}_{41} E_3 F_2 + n\bar{\eta}_{42} E_3 G_2 + n\bar{\eta}_{39} D_3 G_2 - \mu n \bar{\eta}_{46} D_3 G_2$$

$$-\mu n \bar{\eta}_{47} E_3 G_2)\, e^{-\mu y}\, \underline{C}_2 \underline{S}_3 \;+\; (n x_1^2 \bar{\eta}_5 A_3 B_3 + x_1^2 \bar{\eta}_6 A_3 C_3 - \mu n x_1^2 \bar{\eta}_7 A_3 B_3$$

(Continued)

$$-\mu\bar{\eta}_8 x_1^2 A_3C_3 + n\bar{\eta}_{38} D_3F_3 + n\bar{\eta}_{39} E_3F_3 - \mu n\bar{\eta}_{40} D_3F_3 - \mu n\bar{\eta}_{41} E_3F_3$$

$$+ n\bar{\eta}_{42} E_3G_3 + n\bar{\eta}_{39} D_3G_3 - \mu n\bar{\eta}_{46} D_3G_3 - \mu n\bar{\eta}_{47} E_3G_3)\, e^{-\mu y}\, \underline{s}_3\underline{c}_3$$

$$-\beta_1(n x_1^2\bar{\eta}_7 A_1B_1 + x_1^2\bar{\eta}_8 A_1C_1 + n\bar{\eta}_{40} D_1F_1 + n\bar{\eta}_{41} E_1F_1 + n\bar{\eta}_{46} D_1G_1 + n\bar{\eta}_{47} E_1G_1)\, e^{-\mu y}\, \underline{s}_1^2$$

$$-\beta_2(n x_1^2\bar{\eta}_7 A_2B_2 + x_1^2\bar{\eta}_8 A_2C_2 + n\bar{\eta}_{40} D_2F_2 + n\bar{\eta}_{41} E_2F_2 + n\bar{\eta}_{46} D_2G_2 + n\bar{\eta}_{47} E_2G_2)\, e^{-\mu y}\, \underline{s}_2^2$$

$$-\beta_3(n x_1^2\bar{\eta}_7 A_3B_3 + x_1^2\bar{\eta}_8 A_3C_3 + n\bar{\eta}_{40} D_3F_3 + n\bar{\eta}_{41} E_3F_3 + n\bar{\eta}_{46} D_3G_3 + n\bar{\eta}_{47} E_3G_3)\, e^{-\mu y}\, \underline{s}_3^2$$

$$-[n x_1^2\bar{\eta}_7(\beta_2A_2B_1 + \beta_1A_1B_2) + x_1^2\bar{\eta}_8(\beta_2A_2C_1 + \beta_1A_1C_2) + n\bar{\eta}_{40}(\beta_1D_2F_1 + \beta_2D_1F_2)$$

$$+ n\bar{\eta}_{41}(\beta_1E_2F_1 + \beta_2E_1F_2) + n\bar{\eta}_{46}(\beta_1D_2G_1 + \beta_2D_1G_2) + n\bar{\eta}_{47}(\beta_1E_2G_1 + \beta_2E_1G_2)]\, e^{-\mu y}\, \underline{s}_1\underline{s}_2$$

$$-[n x_1^2\bar{\eta}_7(\beta_3A_3B_1 + \beta_1A_1B_3) + x_1^2\bar{\eta}_8(\beta_3A_3C_1 + \beta_1A_1C_3) + n\bar{\eta}_{40}(\beta_1D_3F_1 + \beta_3D_1F_3)$$

$$+ n\bar{\eta}_{41}(\beta_1E_3F_1 + \beta_3E_1F_3) + n\bar{\eta}_{46}(\beta_1D_3G_1 + \beta_3D_1G_3) + n\bar{\eta}_{47}(\beta_1E_3G_1 + \beta_3E_1G_3)]\, e^{-\mu y}\, \underline{s}_1\underline{s}_3$$

$$-[n x_1^2\bar{\eta}_7(\beta_2A_2B_3 + \beta_3A_3B_2) + x_1^2\bar{\eta}_8(\beta_3A_3C_2 + \beta_2A_2C_3) + n\bar{\eta}_{40}(\beta_2D_3F_2 + \beta_3D_2F_3)$$

$$+ n\bar{\eta}_{41}(\beta_2E_3F_2 + \beta_3E_2F_3) + n\bar{\eta}_{46}(\beta_2D_3G_2 + \beta_3D_2G_3) + n\bar{\eta}_{47}(\beta_2E_3G_2 + \beta_3E_2G_3)]\, e^{-\mu y}\, \underline{s}_2\underline{s}_3$$

$$+[(\bar{\eta}_1 + n^2\bar{\eta}_2 - \mu\bar{\eta}_3 + \mu^2\bar{\eta}_4) x_1^2 A_1^2 + (\bar{\eta}_{24} + n^2\bar{\eta}_{25} + \mu^2\bar{\eta}_{27} - \mu\bar{\eta}_{28}) F_1^2 + (\bar{\eta}_{29}$$

$$+ n^2\bar{\eta}_{30} - \mu\bar{\eta}_{31} - \mu\bar{\eta}_{32} + \mu^2\bar{\eta}_{33}) F_1G_1 + (\bar{\eta}_{34} + n^2\bar{\eta}_{35} - \mu\bar{\eta}_{37} + \mu^2\bar{\eta}_{58}) G_1^2]\, e^{-2\mu y}\, \underline{c}_1^2$$

$$+[(\bar{\eta}_1 + n^2\bar{\eta}_2 - \mu\bar{\eta}_3 + \mu^2\bar{\eta}_4) x_1^2 A_2^2 + (\bar{\eta}_{24} + n^2\bar{\eta}_{25} + \mu^2\bar{\eta}_{27} - \mu\bar{\eta}_{28}) F_2^2 + (\bar{\eta}_{29}$$

$$+ n^2\bar{\eta}_{30} - \mu\bar{\eta}_{31} - \mu\bar{\eta}_{32} + \mu^2\bar{\eta}_{33}) F_2G_2 + (\bar{\eta}_{34} + n^2\bar{\eta}_{35} - \mu\bar{\eta}_{37} + \mu^2\bar{\eta}_{58}) G_2^2]\, e^{-2\mu y}\, \underline{c}_2^2$$

$$+[(\bar{\eta}_1 + n^2\bar{\eta}_2 - \mu\bar{\eta}_3 + \mu^2\bar{\eta}_4) x_1^2 A_3^2 + (\bar{\eta}_{24} + n^2\bar{\eta}_{25} + \mu^2\bar{\eta}_{27} - \mu\bar{\eta}_{28}) F_3^2 + (\bar{\eta}_{29}$$

$$+ n^2\bar{\eta}_{30} - \mu\bar{\eta}_{31} - \mu\bar{\eta}_{32} + \mu^2\bar{\eta}_{33}) F_3G_3 + (\bar{\eta}_{34} + n^2\bar{\eta}_{35} - \mu\bar{\eta}_{37} + \mu^2\bar{\eta}_{58}) G_3^2]\, e^{-2\mu y}\, \underline{c}_3^2$$

$$+[2(\bar{\eta}_1 + n^2\bar{\eta}_2 - \mu\bar{\eta}_3 + \mu^2\bar{\eta}_4) x_1^2 A_1A_2 + 2(\bar{\eta}_{24} + n^2\bar{\eta}_{25} + \mu^2\bar{\eta}_{27} - \mu\bar{\eta}_{28}) F_1F_2 + (\bar{\eta}_{29}$$

$$+ n^2\bar{\eta}_{30} - \mu\bar{\eta}_{31} - \mu\bar{\eta}_{32} + \mu^2\bar{\eta}_{33})(F_1G_2 + F_2G_1) + 2(\bar{\eta}_{34} + n^2\bar{\eta}_{35} - \mu\bar{\eta}_{37}$$

$$+ \mu^2\bar{\eta}_{58}) G_1G_2]\, e^{-2\mu y}\, \underline{c}_1\underline{c}_2 + [2(\bar{\eta}_1 + n^2\bar{\eta}_2 - \mu\bar{\eta}_3 + \mu^2\bar{\eta}_4) x_1^2 A_1A_3 + 2(\bar{\eta}_{24} + n^2\bar{\eta}_{25}$$

$$+ \mu^2\bar{\eta}_{27} - \mu\bar{\eta}_{28}) F_1F_3 + (\bar{\eta}_{29} + n^2\bar{\eta}_{30} - \mu\bar{\eta}_{31} - \mu\bar{\eta}_{32} + \mu^2\bar{\eta}_{33})(F_1G_3 + F_3G_1)$$

$$+ 2(\bar{\eta}_{34} + n^2\bar{\eta}_{35} - \mu\bar{\eta}_{37} + \mu^2\bar{\eta}_{58}) G_1G_3]\, e^{-2\mu y}\, \underline{c}_1\underline{c}_3 + [2(\bar{\eta}_1 + n^2\bar{\eta}_2 - \mu\bar{\eta}_3$$

(Continued)

$$+\mu^2\bar\eta_4)X_1^2 A_2A_3 + 2(\bar\eta_{24}+n^2\bar\eta_{25}+\mu^2\bar\eta_{27}-\mu\bar\eta_{28})F_2F_3 + (\bar\eta_{29}+n^2\bar\eta_{30}-\mu\bar\eta_{31}$$

$$-\mu\bar\eta_{32}+\mu^2\bar\eta_{33})(F_2G_3+F_3G_2) + 2(\bar\eta_{34}+n^2\bar\eta_{35}-\mu\bar\eta_{37}+\mu^2\bar\eta_{58})G_2G_3]e^{-2\mu y}\underline{c}_2\underline{c}_3$$

$$+\beta_1[X_1^2(-\bar\eta_3+2\mu\bar\eta_4)A_1^2 + (2\mu\bar\eta_{27}-\bar\eta_{28})F_1^2 + (-\bar\eta_{31}-\bar\eta_{32}+2\mu\bar\eta_{33})F_1G_1$$

$$+(-\bar\eta_{37}+2\mu\bar\eta_{58})G_1^2]e^{-2\mu y}\underline{s}_1\underline{c}_1 + \beta_1[X_1^2(-\bar\eta_3+2\mu\bar\eta_4)A_1A_2 + (2\mu\bar\eta_{27}-\bar\eta_{28})F_1F_2$$

$$+(-\bar\eta_{31}+\mu\bar\eta_{33})F_2G_1 + (-\bar\eta_{32}+\mu\bar\eta_{33})F_1G_2 + (-\bar\eta_{37}+2\mu\bar\eta_{58})G_1G_2]e^{-2\mu y}\underline{s}_1\underline{c}_2$$

$$+\beta_1[X_1^2(-\bar\eta_3+2\mu\bar\eta_4)A_1A_3 + (2\mu\bar\eta_{27}-\bar\eta_{28})F_1F_3 + (-\bar\eta_{31}+\mu\bar\eta_{33})F_3G_1 + (-\bar\eta_{32}$$

$$+\mu\bar\eta_{33})F_1G_3 + (-\bar\eta_{37}+2\mu\bar\eta_{58})G_1G_3]e^{-2\mu y}\underline{s}_1\underline{c}_3 + \beta_2[X_1^2(-\bar\eta_3+2\mu\bar\eta_4)A_1A_2$$

$$+(2\mu\bar\eta_{27}-\bar\eta_{28})F_1F_2 + (-\bar\eta_{31}+\mu\bar\eta_{33})F_1G_2 + (-\bar\eta_{32}+\mu\bar\eta_{33})F_2G_1 + (-\bar\eta_{37}$$

$$+2\mu\bar\eta_{58})G_1G_2]e^{-2\mu y}\underline{c}_1\underline{s}_2 + \beta_2[X_1^2(-\bar\eta_3+2\mu\bar\eta_4)A_2^2 + (2\mu\bar\eta_{27}-\bar\eta_{28})F_2^2$$

$$+(-\bar\eta_{31}-\bar\eta_{32}+2\mu\bar\eta_{33})F_2G_2 + (-\bar\eta_{37}+2\mu\bar\eta_{58})G_2^2]e^{-2\mu y}\underline{s}_2\underline{c}_2 + \beta_2[X_1^2(-\bar\eta_3$$

$$+2\mu\bar\eta_4)A_2A_3 + (2\mu\bar\eta_{27}-\bar\eta_{28})F_2F_3 + (-\bar\eta_{31}+\mu\bar\eta_{33})F_3G_2 + (-\bar\eta_{32}+\mu\bar\eta_{33})F_2G_3$$

$$+(-\bar\eta_{37}+2\mu\bar\eta_{58})G_2G_3]e^{-2\mu y}\underline{s}_2\underline{c}_3 + \beta_3[X_1^2(-\bar\eta_3+2\mu\bar\eta_4)A_1A_3 + (2\mu\bar\eta_{27}$$

$$-\bar\eta_{28})F_1F_3 + (-\bar\eta_{31}+\mu\bar\eta_{33})F_1G_3 + (-\bar\eta_{32}+\mu\bar\eta_{33})F_3G_1 + (-\bar\eta_{37}$$

$$+2\mu\bar\eta_{58})G_1G_3]e^{-2\mu y}\underline{c}_1\underline{s}_3 + \beta_3[X_1^2(-\bar\eta_3+2\mu\bar\eta_4)A_2A_3 + (2\mu\bar\eta_{27}-\bar\eta_{28})F_2F_3$$

$$+(-\bar\eta_{31}+\mu\bar\eta_{33})F_2G_3 + (-\bar\eta_{32}+\mu\bar\eta_{33})F_3G_2 + (-\bar\eta_{37}+2\mu\bar\eta_{58})G_2G_3]e^{-2\mu y}\underline{c}_2\underline{s}_3$$

$$+\beta_3[X_1^2(-\bar\eta_3+2\mu\bar\eta_4)A_3^2 + (2\mu\bar\eta_{27}-\bar\eta_{28})F_3^2 + (-\bar\eta_{31}-\bar\eta_{32}+2\mu\bar\eta_{33})F_3G_3$$

$$+(-\bar\eta_{37}+2\mu\bar\eta_{58})G_3^2]e^{-2\mu y}\underline{s}_3\underline{c}_3 + \beta_1^2(X_1^2\bar\eta_4A_1^2 + \bar\eta_{27}F_1^2 + \bar\eta_{33}F_1G_1$$

$$+\bar\eta_{58}G_1^2)e^{-2\mu y}\underline{s}_1^2 + \beta_2^2(X_1^2\bar\eta_4A_2^2 + \bar\eta_{27}F_2^2 + \bar\eta_{33}F_2G_2 + \bar\eta_{58}G_2^2)e^{-2\mu y}\underline{s}_2^2$$

$$+\beta_3^2(X_1^2\bar\eta_4A_3^2 + \bar\eta_{27}F_3^2 + \bar\eta_{33}F_3G_3 + \bar\eta_{58}G_3^2)e^{-2\mu y}\underline{s}_3^2 + \beta_1\beta_2[2X_1^2\bar\eta_4A_1A_2$$

$$+2\bar\eta_{27}F_1F_2 + \bar\eta_{33}(F_1G_2+F_2G_1) + 2\bar\eta_{58}G_1G_2]e^{-2\mu y}\underline{s}_1\underline{s}_2 + \beta_1\beta_3[2X_1^2\bar\eta_4A_1A_3$$

$$+2\bar\eta_{27}F_1F_3 + \bar\eta_{33}(F_1G_3+F_3G_1) + 2\bar\eta_{58}G_1G_3]e^{-2\mu y}\underline{s}_1\underline{s}_3 + \beta_2\beta_3[2X_1^2\bar\eta_4A_2A_3$$

$$+2\bar\eta_{27}F_2F_3 + \bar\eta_{33}(F_2G_3+F_3G_2) + 2\bar\eta_{58}G_2G_3]e^{-2\mu y}\underline{s}_2\underline{s}_3\}\,dy\Big) \qquad (3\text{-}29)$$

In Equation (3-29), for convenience of writing, $x_1 \equiv x_{1,1}$, $\mu \equiv \mu'_x$.

Integrating Equation (3-29), using the integration coefficients from Appendix A, gives

$$
\begin{aligned}
V_M = \pi \sum \Big\{ & \sigma_{1,j}(\bar\eta_{26} F_1^2 + \bar\eta_{36} G_1^2) + \sigma_{2,j}(\bar\eta_{26} F_2^2 + \bar\eta_{36} G_2^2) + \sigma_{3,j}(\bar\eta_{26} F_3^2 + \bar\eta_{36} G_3^2) \\
& + \sigma_{4,j}(2\bar\eta_{26} F_1F_2 + 2\bar\eta_{36} G_1G_2) + 2\sigma_{5,j}(\bar\eta_{26} F_1F_3 + \bar\eta_{36} G_1G_3) + 2\sigma_{6,j}(\bar\eta_{26} F_2F_3 \\
& + \bar\eta_{36} G_2G_3) + \sigma_{7,j}(\bar\eta_{51} D_1^2 + \bar\eta_{56} E_1^2) + \sigma_{8,j}(\bar\eta_{51} D_2^2 + \bar\eta_{56} E_2^2) + \sigma_{9,j}(\bar\eta_{51} D_3^2 \\
& + \bar\eta_{56} E_3^2) + 2\sigma_{10,j}(\bar\eta_{51} D_1D_2 + \bar\eta_{56} E_1E_2) + 2\sigma_{11,j}(\bar\eta_{51} D_1D_3 + \bar\eta_{56} E_1E_3) \\
& + 2\sigma_{12,j}(\bar\eta_{51} D_2D_3 + \bar\eta_{56} E_2E_3) + \beta_1^2 \sigma_{13,j}(x_1^2 \bar\eta_{16} B_1^2 + x_1^2 \bar\eta_{21} C_1^2 + \bar\eta_{48} D_1^2 \\
& + \bar\eta_{52} D_1E_1 + \bar\eta_{54} E_1^2) + \beta_2^2 \sigma_{14,j}(x_1^2 \bar\eta_{16} B_2^2 + x_1^2 \bar\eta_{21} C_2^2 + \bar\eta_{48} D_2^2 + \bar\eta_{52} D_2E_2 \\
& + \bar\eta_{54} E_2^2) + \beta_3^2 \sigma_{15,j}(x_1^2 \bar\eta_{16} B_3^2 + x_1^2 \bar\eta_{21} C_3^2 + \bar\eta_{48} D_3^2 + \bar\eta_{52} D_3E_3 + \bar\eta_{54} E_3^2) \\
& + \beta_1\beta_2 \sigma_{16,j}(2x_1^2 \bar\eta_{16} B_1B_2 + 2x_1^2 \bar\eta_{21} C_1C_2 + 2\bar\eta_{48} D_1D_2 + \bar\eta_{52} D_1E_2 + \bar\eta_{52} D_2E_1 \\
& + 2\bar\eta_{54} E_1E_2) + \beta_1\beta_3 \sigma_{17,j}(2x_1^2 \bar\eta_{16} B_1B_3 + 2x_1^2 \bar\eta_{21} C_1C_3 + 2\bar\eta_{48} D_1D_3 + \bar\eta_{52} D_1E_3 \\
& + \bar\eta_{52} D_3E_1 + 2\bar\eta_{54} E_1E_3) + \beta_2\beta_3 \sigma_{18,j}(2x_1^2 \bar\eta_{16} B_2B_3 + 2x_1^2 \bar\eta_{21} C_2C_3 \\
& + 2\bar\eta_{48} D_2D_3 + \bar\eta_{52} D_2E_3 + \bar\eta_{52} D_3E_2 + 2\bar\eta_{54} E_2E_3) + \sigma_{19,j}\big[x_1^2(n^2\bar\eta_{10} + \bar\eta_{11}) B_1^2 \\
& + n x_1^2 \bar\eta_{12} B_1C_1 + x_1^2(\bar\eta_{17} + n^2\bar\eta_{18}) C_1^2 + (\bar\eta_{49} + n^2\bar\eta_{50}) D_1^2 + (\bar\eta_{52} + n^2\bar\eta_{53}) D_1E_1 \\
& + (\bar\eta_{54} + n^2\bar\eta_{55}) E_1^2\big] + \sigma_{20,j}\big[x_1^2(n^2\bar\eta_{10} + \bar\eta_{11}) B_2^2 + n x_1^2 \bar\eta_{12} B_2C_2 + x_1^2(\bar\eta_{17} \\
& + n^2\bar\eta_{18}) C_2^2 + (\bar\eta_{49} + n^2\bar\eta_{50}) D_2^2 + (\bar\eta_{52} + n^2\bar\eta_{53}) D_2E_2 + (\bar\eta_{54} + n^2\bar\eta_{55}) E_2^2\big] \\
& + \sigma_{21,j}\big[x_1^2(n^2\bar\eta_{10} + \bar\eta_{11}) B_3^2 + x_1^2 n \bar\eta_{12} B_3C_3 + x_1^2(\bar\eta_{17} + n^2\bar\eta_{18}) C_3^2 + (\bar\eta_{49} \\
& + n^2\bar\eta_{50}) D_3^2 + (\bar\eta_{52} + n^2\bar\eta_{53}) D_3E_3 + (\bar\eta_{54} + n^2\bar\eta_{55}) E_3^2\big] + \sigma_{22,j}\big[2x_1^2(n^2\bar\eta_{10} \\
& + \bar\eta_{11}) B_1B_2 + n x_1^2 \bar\eta_{12} B_2C_1 + x_1^2 n \bar\eta_{12} B_1C_2 + 2x_1^2(\bar\eta_{17} + n^2\bar\eta_{18}) C_1C_2 + 2(\bar\eta_{49} \\
& + n^2\bar\eta_{50}) D_1D_2 + (\bar\eta_{52} + n^2\bar\eta_{53})(D_1E_2 + D_2E_1) + 2(\bar\eta_{54} + n^2\bar\eta_{55}) E_1E_2\big] \\
& + \sigma_{23,j}\big[2x_1^2(n^2\bar\eta_{10} + \bar\eta_{11}) B_1B_3 + n x_1^2 \bar\eta_{12} B_3C_1 + n x_1^2 \bar\eta_{12} B_1C_3 + 2x_1^2(\bar\eta_{17}
\end{aligned}
$$

(Continued)

$$+n^2\bar{\eta}_{18})C_1C_3 + 2(\bar{\eta}_{49} + n^2\bar{\eta}_{50})D_1D_3 + (\bar{\eta}_{52} + n^2\bar{\eta}_{53})(D_1E_3 + D_3E_1) + 2(\bar{\eta}_{54}$$

$$+n^2\bar{\eta}_{55})E_1E_3] + \sigma_{24,j}[2X_1^2(n^2\bar{\eta}_{10} + \bar{\eta}_{11})B_2B_3 + nX_1^2\bar{\eta}_{12}(B_3C_2 + B_2C_3) + 2X_1^2(\bar{\eta}_{17}$$

$$+n^2\bar{\eta}_{18})C_2C_3 + 2(\bar{\eta}_{49} + n^2\bar{\eta}_{50})D_2D_3 + (\bar{\eta}_{52} + n^2\bar{\eta}_{53})(D_2E_3 + D_3E_2) + 2(\bar{\eta}_{54}$$

$$+n^2\bar{\eta}_{55})E_2E_3] + \sigma_{25,j}\ \beta_1(-X_1^2\bar{\eta}_{15}B_1^2 - 2\bar{\eta}_{52}D_1E_1 - \bar{\eta}_{48}D_1^2 + \bar{\eta}_{57}E_1^2)$$

$$+\beta_2\sigma_{26,j}(-X_1^2\bar{\eta}_{15}B_1B_2 - \bar{\eta}_{52}D_1E_2 - \bar{\eta}_{52}D_2E_1 - \bar{\eta}_{48}D_1D_2 + \bar{\eta}_{57}E_1E_2) + \beta_3\sigma_{27,j}[-X_1^2\bar{\eta}_{15}B_1B_3$$

$$-\bar{\eta}_{52}(D_1E_3 + D_3E_1) - \bar{\eta}_{48}D_1D_3 + \bar{\eta}_{57}E_1E_3] + \beta_1\sigma_{28,j}[-X_1^2\bar{\eta}_{15}B_1B_2 - \bar{\eta}_{52}(D_2E_1 + D_1E_2)$$

$$-\bar{\eta}_{48}D_1D_2 + \bar{\eta}_{57}E_1E_2] + \beta_2\sigma_{29,j}[-X_1^2\bar{\eta}_{15}B_2^2 - 2\bar{\eta}_{52}D_2E_2 - \bar{\eta}_{48}D_2^2 + \bar{\eta}_{57}E_2^2]$$

$$+\beta_3\sigma_{30,j}[-X_1^2\bar{\eta}_{15}B_2B_3 - \bar{\eta}_{52}(D_2E_3 + D_3E_2) - \bar{\eta}_{48}D_2D_3 + \bar{\eta}_{57}E_2E_3]$$

$$+\beta_1\sigma_{31,j}[-X_1^2\bar{\eta}_{15}B_1B_3 - \bar{\eta}_{52}(D_3E_1 + D_1E_3) - \bar{\eta}_{48}D_1D_3 + \bar{\eta}_{57}E_1E_3]$$

$$+\beta_2\sigma_{32,j}[-X_1^2\bar{\eta}_{15}B_2B_3 - \bar{\eta}_{52}(D_3E_2 + D_2E_3) - \bar{\eta}_{48}D_2D_3 + \bar{\eta}_{57}E_2E_3]$$

$$+\beta_3\sigma_{33,j}[-X_1^2\bar{\eta}_{15}B_3^2 - 2\bar{\eta}_{52}D_3E_3 - \bar{\eta}_{48}D_3^2 + \bar{\eta}_{57}E_3^2] + X_1\sigma_{34,j}(-\bar{\eta}_{13}B_1D_1$$

$$-\bar{\eta}_{14}B_1E_1 - n\bar{\eta}_{19}C_1D_1 - n\bar{\eta}_{20}C_1E_1) + X_1\sigma_{35,j}(-\bar{\eta}_{13}B_2D_2 - \bar{\eta}_{14}B_2E_2 - n\bar{\eta}_{19}C_2D_2$$

$$-n\bar{\eta}_{20}C_2E_2) + X_1\sigma_{36,j}(-\bar{\eta}_{13}B_3D_3 - \bar{\eta}_{14}B_3E_3 - n\bar{\eta}_{19}C_3D_3 - n\bar{\eta}_{20}C_3E_3)$$

$$+X_1\sigma_{37,j}[-\bar{\eta}_{13}(B_1D_2 + B_2D_1) - \bar{\eta}_{14}(B_1E_2 + B_2E_1) - n\bar{\eta}_{19}(C_1D_2 + C_2D_1) - n\bar{\eta}_{20}(C_1E_2$$

$$+C_2E_1)] + X_1\sigma_{38,j}[-\bar{\eta}_{13}(B_1D_3 + B_3D_1) - \bar{\eta}_{14}(B_1E_3 + B_3E_1) - n\bar{\eta}_{19}(C_1D_3 + C_3D_1)$$

$$-n\bar{\eta}_{20}(C_1E_3 + C_3E_1)] + X_1\sigma_{39,j}[-\bar{\eta}_{13}(B_2D_3 + B_3D_2) - \bar{\eta}_{14}(B_2E_3 + B_3E_2)$$

$$-n\bar{\eta}_{19}(C_2D_3 + C_3D_2) - n\bar{\eta}_{20}(C_2E_3 + C_3E_2)] + \beta_1X_1\sigma_{40,j}(\bar{\eta}_{22}C_1G_1 + \bar{\eta}_{23}C_1F_1)$$

$$+\beta_2X_1\sigma_{41,j}(\bar{\eta}_{22}C_2G_2 + \bar{\eta}_{23}C_2F_2) + \beta_3X_1\sigma_{42,j}(\bar{\eta}_{22}C_3G_3 + \bar{\eta}_{23}C_3F_3)$$

$$+X_1\sigma_{43,j}[\bar{\eta}_{22}(\beta_1C_1G_2 + \beta_2C_2G_1) + \bar{\eta}_{23}(\beta_1C_1F_2 + \beta_2C_2F_1)] + X_1\sigma_{44,j}[\bar{\eta}_{22}(\beta_1C_1G_3$$

$$+\beta_3C_3G_1) + \bar{\eta}_{23}(\beta_1C_1F_3 + \beta_3C_3F_1)] + X_1\sigma_{45,j}[\bar{\eta}_{22}(\beta_2C_2G_3 + \beta_3C_3G_2)$$

$$+\bar{\eta}_{23}(\beta_2C_2F_3 + \beta_3C_3F_2)] + n\beta_1\sigma_{46,j}(-X_1^2\bar{\eta}_9A_1B_1 + \bar{\eta}_{43}D_1G_1 + \bar{\eta}_{44}E_1G_1$$

$$-\bar{\eta}_{45}D_1F_1 + \bar{\eta}_{43}E_1F_1) + n\beta_2\sigma_{47,j}(-X_1^2\bar{\eta}_9A_2B_2 + \bar{\eta}_{43}D_2G_2 + \bar{\eta}_{44}E_2G_2 - \bar{\eta}_{45}D_2F_2$$

(Continued)

$$+\bar{\eta}_{43}E_2F_2) + n\beta_3\sigma_{48,j}(-X_i^2\bar{\eta}_9A_3B_3 + \bar{\eta}_{43}D_3G_3 + \bar{\eta}_{44}E_3G_3 - \bar{\eta}_{45}D_3F_3 + \bar{\eta}_{43}E_3F_3)$$

$$+n\sigma_{49,j}[-X_i^2\bar{\eta}_9(\beta_1A_2B_1 + \beta_2A_1B_2) + \bar{\eta}_{43}(\beta_1D_1G_2 + \beta_2D_2G_1) + \bar{\eta}_{44}(\beta_1E_1G_2$$

$$+\beta_2E_2G_1) - \bar{\eta}_{45}(\beta_1D_1F_2 + \beta_2D_2F_1) + \bar{\eta}_{43}(\beta_1E_1F_2 + \beta_2E_2F_1)] + n\sigma_{50,j}[-X_i^2\bar{\eta}_9(\beta_1A_3B_1$$

$$+\beta_3A_1B_3) + \bar{\eta}_{43}(\beta_1D_1G_3 + \beta_3D_3G_1) + \bar{\eta}_{44}(\beta_1E_1G_3 + \beta_3E_3G_1) - \bar{\eta}_{45}(\beta_1D_1F_3$$

$$+\beta_3D_3F_1) + \bar{\eta}_{43}(\beta_1E_1F_3 + \beta_3E_3F_1)] + n\sigma_{51,j}[-X_i^2\bar{\eta}_9(\beta_2A_3B_2 + \beta_3A_2B_3)$$

$$+\bar{\eta}_{43}(\beta_2D_2G_3 + \beta_3D_3G_2) + \bar{\eta}_{44}(\beta_2E_2G_3 + \beta_3E_3G_2) - \bar{\eta}_{45}(\beta_2D_2F_3 + \beta_3D_3F_2)$$

$$+\bar{\eta}_{43}(\beta_2E_2F_3 + \beta_3E_3F_2)] + \beta_1\sigma_{52,j}(-X_i^2n\bar{\eta}_7A_1B_1 - X_i^2\bar{\eta}_8A_1C_1 - n\bar{\eta}_{40}D_1F_1$$

$$-n\bar{\eta}_{41}E_1F_1 - n\bar{\eta}_{46}D_1G_1 - n\bar{\eta}_{47}E_1G_1) + \beta_2\sigma_{53,j}(-X_i^2n\bar{\eta}_7A_2B_2 - X_i^2\bar{\eta}_8A_2C_2 - n\bar{\eta}_{40}D_2F_2$$

$$-n\bar{\eta}_{41}E_2F_2 - n\bar{\eta}_{46}D_2G_2 - n\bar{\eta}_{47}E_2G_2) + \beta_3\sigma_{54,j}(-X_i^2n\bar{\eta}_7A_3B_3 - X_i^2\bar{\eta}_8A_3C_3 - n\bar{\eta}_{40}D_3F_3$$

$$-n\bar{\eta}_{41}E_3F_3 - n\bar{\eta}_{46}D_3G_3 - n\bar{\eta}_{47}E_3G_3) + \sigma_{55,j}[-X_i^2n\bar{\eta}_7(\beta_2A_2B_1 + \beta_1A_1B_2)$$

$$-X_i^2\bar{\eta}_8(\beta_2A_2C_1 + \beta_1A_1C_2) - n\bar{\eta}_{40}(\beta_1D_2F_1 + \beta_2D_1F_2) - n\bar{\eta}_{41}(\beta_1E_2F_1 + \beta_2E_1F_2)$$

$$-n\bar{\eta}_{46}(\beta_1D_2G_1 + \beta_2D_1G_2) - n\bar{\eta}_{47}(\beta_1E_2G_1 + \beta_2E_1G_2)] + \sigma_{56,j}[-X_i^2n\bar{\eta}_7(\beta_3A_3B_1$$

$$+\beta_1A_1B_3) - X_i^2\bar{\eta}_8(\beta_3A_3C_1 + \beta_1A_1C_3) - n\bar{\eta}_{40}(\beta_1D_3F_1 + \beta_3D_1F_3) - n\bar{\eta}_{41}(\beta_1E_3F_1$$

$$+\beta_3E_1F_3) - n\bar{\eta}_{46}(\beta_1D_3G_1 + \beta_3D_1G_3) - n\bar{\eta}_{47}(\beta_1E_3G_1 + \beta_3E_1G_3)]$$

$$+\sigma_{57,j}[-X_i^2n\bar{\eta}_7(\beta_2A_2B_3 + \beta_3A_3B_2) - X_i^2\bar{\eta}_8(\beta_2A_2C_3 + \beta_3A_3C_2) - n\bar{\eta}_{40}(\beta_2D_3F_2$$

$$+\beta_3D_2F_3) - n\bar{\eta}_{41}(\beta_2E_3F_2 + \beta_3E_2F_3) - n\bar{\eta}_{46}(\beta_2D_3G_2 + \beta_3D_2G_3) - n\bar{\eta}_{47}(\beta_2E_3G_2$$

$$+\beta_3E_2G_3)] + \sigma_{58,j}[X_i^2n(\bar{\eta}_5 - \mu\bar{\eta}_7)A_1B_1 + X_i^2(\bar{\eta}_6 - \mu\bar{\eta}_8)A_1C_1 + n(\bar{\eta}_{38} - \mu\bar{\eta}_{40})D_1F_1$$

$$+n(\bar{\eta}_{39} - \mu\bar{\eta}_{41})E_1F_1 + n(\bar{\eta}_{42} - \mu\bar{\eta}_{47})E_1G_1 + n(\bar{\eta}_{39} - \mu\bar{\eta}_{46})D_1G_1]$$

$$+\sigma_{59,j}[X_i^2n(\bar{\eta}_5 - \mu\bar{\eta}_7)A_2B_1 + X_i^2(\bar{\eta}_6 - \mu\bar{\eta}_8)A_2C_1 + n(\bar{\eta}_{38} - \mu\bar{\eta}_{40})D_1F_2 + n(\bar{\eta}_{39}$$

$$-\mu\bar{\eta}_{41})E_1F_2 + n(\bar{\eta}_{42} - \mu\bar{\eta}_{47})E_1G_2 + n(\bar{\eta}_{39} - \mu\bar{\eta}_{46})D_1G_2] + \sigma_{60,j}[X_i^2n(\bar{\eta}_5$$

$$-\mu\bar{\eta}_7)A_3B_1 + X_i^2(\bar{\eta}_6 - \mu\bar{\eta}_8)A_3C_1 + n(\bar{\eta}_{38} - \mu\bar{\eta}_{40})D_1F_3 + n(\bar{\eta}_{39} - \mu\bar{\eta}_{41})E_1F_3$$

$$+n(\bar{\eta}_{42} - \mu\bar{\eta}_{47})E_1G_3 + n(\bar{\eta}_{39} - \mu\bar{\eta}_{46})D_1G_3] + \sigma_{61,j}[X_i^2n(\bar{\eta}_5 - \mu\bar{\eta}_7)A_1B_2$$

(Continued)

$$+X_i^2(\bar\eta_6-\mu\bar\eta_8)A_1C_2+n(\bar\eta_{38}-\mu\bar\eta_{40})D_2F_1+n(\bar\eta_{39}-\mu\bar\eta_{41})E_2F_1+n(\bar\eta_{42}$$

$$-\mu\bar\eta_{47})E_2G_1+n(\bar\eta_{39}-\mu\bar\eta_{46})D_2G_1]+\sigma_{62,j}[X_i^2n(\bar\eta_5-\mu\bar\eta_7)A_2B_2+X_i^2(\bar\eta_6$$

$$-\mu\bar\eta_8)A_2C_2+n(\bar\eta_{38}-\mu\bar\eta_{40})D_2F_2+n(\bar\eta_{39}-\mu\bar\eta_{41})E_2F_2+n(\bar\eta_{42}-\mu\bar\eta_{47})E_2G_2$$

$$+n(\bar\eta_{39}-\mu\bar\eta_{40})D_2G_2]+\sigma_{63,j}[X_i^2n(\bar\eta_5-\mu\bar\eta_7)A_3B_2+X_i^2(\bar\eta_6-\mu\bar\eta_8)A_3C_2$$

$$+n(\bar\eta_{38}-\mu\bar\eta_{40})D_2F_3+n(\bar\eta_{39}-\mu\bar\eta_{41})E_2F_3+n(\bar\eta_{42}-\mu\bar\eta_{47})E_2G_3+n(\bar\eta_{39}$$

$$-\mu\bar\eta_{46})D_2G_3]+\sigma_{64,j}[X_i^2n(\bar\eta_5-\mu\bar\eta_7)A_1B_3+X_i^2(\bar\eta_6-\mu\bar\eta_8)A_1C_3+n(\bar\eta_{38}$$

$$-\mu\bar\eta_{40})D_3F_1+n(\bar\eta_{39}-\mu\bar\eta_{41})E_3F_1+n(\bar\eta_{42}-\mu\bar\eta_{47})E_3G_1+n(\bar\eta_{39}-\mu\bar\eta_{46})D_3G_1]$$

$$+\sigma_{65,j}[X_i^2n(\bar\eta_5-\mu\bar\eta_7)A_2B_3+X_i^2(\bar\eta_6-\mu\bar\eta_8)A_2C_3+n(\bar\eta_{38}-\mu\bar\eta_{40})D_3F_2$$

$$+n(\bar\eta_{39}-\mu\bar\eta_{41})E_3F_2+n(\bar\eta_{42}-\mu\bar\eta_{47})E_3G_2+n(\bar\eta_{39}-\mu\bar\eta_{46})D_3G_2]+\sigma_{66,j}[X_i^2n(\bar\eta_5$$

$$-\mu\bar\eta_7)A_3B_3+X_i^2(\bar\eta_6-\mu\bar\eta_8)A_3C_3+n(\bar\eta_{38}-\mu\bar\eta_{40})D_3F_3+n(\bar\eta_{39}-\mu\bar\eta_{41})E_3F_3$$

$$+n(\bar\eta_{42}-\mu\bar\eta_{47})E_3G_3+n(\bar\eta_{39}-\mu\bar\eta_{46})D_3G_3]+\sigma_{67,j}[X_i^2(\bar\eta_1+n^2\bar\eta_2-\mu\bar\eta_3$$

$$+\mu^2\bar\eta_4)A_1^2+(\bar\eta_{24}+n^2\bar\eta_{25}+\mu^2\bar\eta_{27}-\mu\bar\eta_{28})F_1^2+(\bar\eta_{29}+n^2\bar\eta_{30}-\mu\bar\eta_{31}-\mu\bar\eta_{32}$$

$$+\mu^2\bar\eta_{33})F_1G_1+(\bar\eta_{34}+n^2\bar\eta_{35}-\mu\bar\eta_{37}+\mu^2\bar\eta_{58})G_1^2]+\sigma_{68,j}[X_i^2(\bar\eta_1+n^2\bar\eta_2$$

$$-\mu\bar\eta_3+\mu^2\bar\eta_4)A_2^2+(\bar\eta_{24}+n^2\bar\eta_{25}+\mu^2\bar\eta_{27}-\mu\bar\eta_{28})F_2^2+(\bar\eta_{29}+n^2\bar\eta_{30}-\mu\bar\eta_{31}$$

$$-\mu\bar\eta_{32}+\mu^2\bar\eta_{33})F_2G_2+(\bar\eta_{34}+n^2\bar\eta_{35}-\mu\bar\eta_{37}+\mu^2\bar\eta_{58})G_2^2]+\sigma_{69,j}[X_i^2(\bar\eta_1$$

$$+n^2\bar\eta_2-\mu\bar\eta_3+\mu^2\bar\eta_4)A_3^2+(\bar\eta_{24}+n^2\bar\eta_{25}+\mu^2\bar\eta_{27}-\mu\bar\eta_{28})F_3^2+(\bar\eta_{29}+n^2\bar\eta_{30}$$

$$-\mu\bar\eta_{31}-\mu\bar\eta_{32}+\mu^2\bar\eta_{33})F_3G_3+(\bar\eta_{34}+n^2\bar\eta_{35}-\mu\bar\eta_{37}+\mu^2\bar\eta_{58})G_3^2]$$

$$+\sigma_{70,j}[2X_i^2(\bar\eta_1+n^2\bar\eta_2-\mu\bar\eta_3+\mu^2\bar\eta_4)A_1A_2+2(\bar\eta_{24}+n^2\bar\eta_{25}+\mu^2\bar\eta_{27}-\mu\bar\eta_{28})F_1F_2$$

$$+(\bar\eta_{29}+n^2\bar\eta_{30}-\mu\bar\eta_{31}-\mu\bar\eta_{32}+\mu^2\bar\eta_{33})(F_1G_2+F_2G_1)+2(\bar\eta_{34}+n^2\bar\eta_{35}-\mu\bar\eta_{37}$$

$$+\mu^2\bar\eta_{58})G_1G_2]+\sigma_{71,j}[2X_i^2(\bar\eta_1+n^2\bar\eta_2-\mu\bar\eta_3+\mu^2\bar\eta_4)A_1A_3+2(\bar\eta_{24}+n^2\bar\eta_{25}+\mu^2\bar\eta_{27}$$

$$-\mu\bar\eta_{28})F_1F_3+(\bar\eta_{29}+n^2\bar\eta_{30}-\mu\bar\eta_{31}-\mu\bar\eta_{32}+\mu^2\bar\eta_{33})(F_1G_3+F_3G_1)+2(\bar\eta_{34}$$

$$+n^2\bar\eta_{35}-\mu\bar\eta_{37}+\mu^2\bar\eta_{58})G_1G_3]+\sigma_{72,j}[2X_i^2(\bar\eta_1+n^2\bar\eta_2-\mu\bar\eta_3+\mu^2\bar\eta_4)A_2A_3$$

(Continued)

$$
\begin{aligned}
&+2(\bar\eta_{24}+n^2\bar\eta_{25}+\mu^2\bar\eta_{27}-\mu\bar\eta_{28})F_2F_3+(\bar\eta_{29}+n^2\bar\eta_{30}-\mu\bar\eta_{31}-\mu\bar\eta_{32}\\
&+\mu^2\bar\eta_{33})(F_2G_3+F_3G_2)+2(\bar\eta_{34}+n^2\bar\eta_{35}-\mu\bar\eta_{37}+\mu^2\bar\eta_{58})G_2G_3]+\beta_1\sigma_{73,j}[X_1^2(-\bar\eta_3\\
&+2\mu\bar\eta_4)A_1^2+(2\mu\bar\eta_{27}-\bar\eta_{28})F_1^2+(-\bar\eta_{31}-\bar\eta_{32}+2\mu\bar\eta_{33})F_1G_1+(-\bar\eta_{37}+2\mu\bar\eta_{58})G_1^2]\\
&+\beta_1\sigma_{74,j}[X_1^2(-\bar\eta_3+2\mu\bar\eta_4)A_1A_2+(2\mu\bar\eta_{27}-\bar\eta_{28})F_1F_2+(-\bar\eta_{31}+\mu\bar\eta_{33})F_2G_1\\
&+(-\bar\eta_{32}+\mu\bar\eta_{33})F_1G_2+(-\bar\eta_{37}+2\mu\bar\eta_{58})G_1G_2]+\beta_1\sigma_{75,j}[X_1^2(-\bar\eta_3+2\mu\bar\eta_4)A_1A_3\\
&+(2\mu\bar\eta_{27}-\bar\eta_{28})F_1F_3+(-\bar\eta_{31}+\mu\bar\eta_{33})F_3G_1+(-\bar\eta_{32}+\mu\bar\eta_{33})F_1G_3+(-\bar\eta_{37}\\
&+2\mu\bar\eta_{58})G_1G_3]+\beta_2\sigma_{76,j}[X_1^2(-\bar\eta_3+2\mu\bar\eta_4)A_1A_2+(2\mu\bar\eta_{27}-\bar\eta_{28})F_1F_2\\
&+(-\bar\eta_{31}+\mu\bar\eta_{33})F_1G_2+(-\bar\eta_{32}+\mu\bar\eta_{33})F_2G_1+(-\bar\eta_{37}+2\mu\bar\eta_{58})G_1G_2]\\
&+\beta_2\sigma_{77,j}[X_1^2(-\bar\eta_3+2\mu\bar\eta_4)A_2^2+(2\mu\bar\eta_{27}-\bar\eta_{28})F_2^2+(-\bar\eta_{31}-\bar\eta_{32}+2\mu\bar\eta_{33})F_2G_2\\
&+(-\bar\eta_{37}+2\mu\bar\eta_{58})G_2^2]+\beta_2\sigma_{78,j}[X_1^2(-\bar\eta_3+2\mu\bar\eta_4)A_2A_3+(2\mu\bar\eta_{27}-\bar\eta_{28})F_2F_3\\
&+(-\bar\eta_{31}+\mu\bar\eta_{33})F_3G_2+(-\bar\eta_{32}+\mu\bar\eta_{33})F_2G_3+(-\bar\eta_{37}+2\mu\bar\eta_{58})G_2G_3]\\
&+\beta_3\sigma_{79,j}[X_1^2(-\bar\eta_3+2\mu\bar\eta_4)A_1A_3+(2\mu\bar\eta_{27}-\bar\eta_{28})F_1F_3+(-\bar\eta_{31}+\mu\bar\eta_{33})F_1G_3\\
&+(-\bar\eta_{32}+\mu\bar\eta_{33})F_3G_1+(-\bar\eta_{37}+2\mu\bar\eta_{58})G_1G_3]+\beta_3\sigma_{80,j}[X_1^2(-\bar\eta_3+2\mu\bar\eta_4)A_2A_3\\
&+(2\mu\bar\eta_{27}-\bar\eta_{28})F_2F_3+(-\bar\eta_{31}+\mu\bar\eta_{33})F_2G_3+(-\bar\eta_{32}+\mu\bar\eta_{33})F_3G_2+(-\bar\eta_{37}\\
&+2\mu\bar\eta_{58})G_2G_3]+\beta_3\sigma_{81,j}[X_1^2(-\bar\eta_3+2\mu\bar\eta_4)A_3^2+(2\mu\bar\eta_{27}-\bar\eta_{28})F_3^2\\
&+(-\bar\eta_{31}-\bar\eta_{32}+2\mu\bar\eta_{33})F_3G_3+(-\bar\eta_{37}+2\mu\bar\eta_{58})G_3^2]+\beta_1^2\sigma_{82,j}(X_1^2\bar\eta_4A_1^2\\
&+\bar\eta_{27}F_1^2+\bar\eta_{33}F_1G_1+\bar\eta_{58}G_1^2)+\beta_2^2\sigma_{83,j}(X_1^2\bar\eta_4A_2^2+\bar\eta_{27}F_2^2+\bar\eta_{33}F_2G_2\\
&+\bar\eta_{58}G_2^2)+\beta_3^2\sigma_{84,j}(X_1^2\bar\eta_4A_3^2+\bar\eta_{27}F_3^2+\bar\eta_{33}F_3G_3+\bar\eta_{58}G_3^2\\
&+\beta_1\beta_2\sigma_{85,j}[2X_1^2\bar\eta_4A_1A_2+2\bar\eta_{27}F_1F_2+\bar\eta_{33}(F_1G_2+F_2G_1)+2\bar\eta_{58}G_1G_2]\\
&+\beta_1\beta_3\sigma_{86,j}[2X_1^2\bar\eta_4A_1A_3+2\bar\eta_{27}F_1F_3+\bar\eta_{33}(F_1G_3+F_3G_1)+2\bar\eta_{58}G_1G_3]\\
&+\beta_2\beta_3\sigma_{87,j}[2X_1^2\bar\eta_4A_2A_3+2\bar\eta_{27}F_2F_3+\bar\eta_{33}(F_2G_3+F_3G_2)+2\bar\eta_{58}G_2G_3]\}
\end{aligned}
\tag{3-30}
$$

Taking partial derivatives of the strain energy expression, Equation (3-30), with respect to each of the undetermined parameters yields the following set of twenty one equations:

$$
\begin{aligned}
\frac{\partial V_M}{\partial A_1} = \pi x_i^2 \sum \Big(& A_1\big[2\sigma_{67,j}(\bar{\eta}_1 + n^2\bar{\eta}_2 - \mu\bar{\eta}_3 + \mu^2\bar{\eta}_4) + 2\beta_1\sigma_{73,j}(-\bar{\eta}_3 + 2\mu\bar{\eta}_4) \\
& + 2\beta_2^2\sigma_{82,j}\bar{\eta}_4\big] + A_2\big[2\sigma_{70,j}(\bar{\eta}_1 + n^2\bar{\eta}_2 - \mu\bar{\eta}_3 + \mu^2\bar{\eta}_4) + (\beta_1\sigma_{74,j} + \beta_2\sigma_{76,j})(-\bar{\eta}_3 \\
& + 2\mu\bar{\eta}_4) + 2\beta_1\beta_2\sigma_{85,j}\bar{\eta}_4\big] + A_3\big[2\sigma_{71,j}(\bar{\eta}_1 + n^2\bar{\eta}_2 - \mu\bar{\eta}_3 + \mu^2\bar{\eta}_4) + (\beta_1\sigma_{75,j} \\
& + \beta_3\sigma_{79,j})(-\bar{\eta}_3 + 2\mu\bar{\eta}_4) + 2\beta_1\beta_3\sigma_{86,j}\bar{\eta}_4\big] + B_1\big\{n\big[-\beta_1\sigma_{46,j}\bar{\eta}_9 - \beta_1\sigma_{52,j}\bar{\eta}_7 \\
& + \sigma_{58,j}(\bar{\eta}_5 - \mu\bar{\eta}_7)\big]\big\} + B_2\big\{n\big[-\beta_2\sigma_{49,j}\bar{\eta}_9 - \beta_1\sigma_{55,j}\bar{\eta}_7 + \sigma_{61,j}(\bar{\eta}_5 - \mu\bar{\eta}_7)\big]\big\} \\
& + B_3\big\{n\big[-\beta_3\sigma_{50,j}\bar{\eta}_9 - \beta_1\sigma_{56,j}\bar{\eta}_7 + \sigma_{64,j}(\bar{\eta}_5 - \mu\bar{\eta}_7)\big]\big\} + C_1\big[-\beta_1\sigma_{52,j}\bar{\eta}_8 \\
& + \sigma_{58,j}(\bar{\eta}_6 - \mu\bar{\eta}_8)\big] + C_2\big[-\beta_1\sigma_{55,j}\bar{\eta}_8 + \sigma_{61,j}(\bar{\eta}_6 - \mu\bar{\eta}_8)\big] + C_3\big[-\beta_1\sigma_{56,j}\bar{\eta}_8 \\
& + \sigma_{64,j}(\bar{\eta}_6 - \mu\bar{\eta}_8)\big]\Big)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial V_M}{\partial A_2} = \pi x_i^2 \sum \Big\{ & A_1\big[2\sigma_{70,j}(\bar{\eta}_1 + n^2\bar{\eta}_2 - \mu\bar{\eta}_3 + \mu^2\bar{\eta}_4) + (\beta_1\sigma_{79,j} + \beta_2\sigma_{76,j})(-\bar{\eta}_3 + 2\mu\bar{\eta}_4) \\
& + 2\beta_1\beta_2\sigma_{85,j}\bar{\eta}_4\big] + A_2\big[2\sigma_{68,j}(\bar{\eta}_1 + n^2\bar{\eta}_2 - \mu\bar{\eta}_3 + \mu^2\bar{\eta}_4) + 2\beta_2\sigma_{77,j}(-\bar{\eta}_3 + 2\mu\bar{\eta}_4) \\
& + 2\beta_2^2\sigma_{83,j}\bar{\eta}_4\big] + A_3\big[2\sigma_{72,j}(\bar{\eta}_1 + n^2\bar{\eta}_2 - \mu\bar{\eta}_3 + \mu^2\bar{\eta}_4) + (\beta_2\sigma_{78,j} + \beta_3\sigma_{80,j})(-\bar{\eta}_3 \\
& + 2\mu\bar{\eta}_4) + 2\beta_2\beta_3\sigma_{87,j}\bar{\eta}_4\big] + B_1 n\big[-\beta_1\sigma_{49,j}\bar{\eta}_9 - \beta_2\sigma_{55,j}\bar{\eta}_7 + \sigma_{59,j}(\bar{\eta}_5 - \mu\bar{\eta}_7)\big] \\
& + B_2 n\big[-\beta_2\sigma_{47,j}\bar{\eta}_9 - \beta_2\sigma_{53,j}\bar{\eta}_7 + \sigma_{62,j}(\bar{\eta}_5 - \mu\bar{\eta}_7)\big] + B_3 n\big[-\beta_3\sigma_{51,j}\bar{\eta}_9 \\
& - \beta_2\sigma_{57,j}\bar{\eta}_7 + \sigma_{65,j}(\bar{\eta}_5 - \mu\bar{\eta}_7)\big] + C_1\big[-\beta_2\sigma_{55,j}\bar{\eta}_8 + \sigma_{59,j}(\bar{\eta}_6 - \mu\bar{\eta}_8)\big] \\
& + C_2\big[-\beta_2\sigma_{53,j}\bar{\eta}_8 + \sigma_{62,j}(\bar{\eta}_6 - \mu\bar{\eta}_8)\big] + C_3\big[-\beta_2\sigma_{57,j}\bar{\eta}_8 + \sigma_{65,j}(\bar{\eta}_6 - \mu\bar{\eta}_8)\big]\Big\}
\end{aligned}
$$

$$
\frac{\partial V_M}{\partial A_3} = \pi x_i^2 \sum \Big\{ A_1\big[2\sigma_{71,j}(\bar{\eta}_1 + n^2\bar{\eta}_2 - \mu\bar{\eta}_3 + \mu^2\bar{\eta}_4) + (\beta_1\sigma_{75,j} + \beta_3\sigma_{79,j})(-\bar{\eta}_3 + 2\mu\bar{\eta}_4)
$$

(Continued)

$$
\begin{aligned}
&+2\beta_1\beta_3\sigma_{86,j}\bar{\eta}_4] + A_2[2\sigma_{72,j}(\bar{\eta}_1 + n^2\bar{\eta}_2 - \mu\bar{\eta}_3 + \mu^2\bar{\eta}_4) + (\beta_2\sigma_{78,j} + \beta_3\sigma_{80,j})(-\bar{\eta}_3 \\
&+2\mu\bar{\eta}_4) + 2\beta_2\beta_3\sigma_{87,j}\bar{\eta}_4] + A_3[2\sigma_{69,j}(\bar{\eta}_1 + n^2\bar{\eta}_2 - \mu\bar{\eta}_3 + \mu^2\bar{\eta}_4) + 2\beta_3\sigma_{81,j}(-\bar{\eta}_3 \\
&+2\mu\bar{\eta}_4) + 2\beta_3^2\sigma_{84,j}\bar{\eta}_4] + B_1 n[-\beta_1\sigma_{50,j}\bar{\eta}_9 - \beta_3\sigma_{56,j}\bar{\eta}_7 + \sigma_{60,j}(\bar{\eta}_5 - \mu\bar{\eta}_7)] \\
&+B_2 n[-\beta_2\sigma_{51,j}\bar{\eta}_9 - \beta_3\sigma_{57,j}\bar{\eta}_7 + \sigma_{63,j}(\bar{\eta}_5 - \mu\bar{\eta}_7)] + B_3 n[-\beta_3\sigma_{48,j}\bar{\eta}_9 \\
&-\beta_3\sigma_{54,j}\bar{\eta}_7 + \sigma_{66,j}(\bar{\eta}_5 - \mu\bar{\eta}_7)] + C_1[-\beta_3\sigma_{56,j}\bar{\eta}_8 + \sigma_{60,j}(\bar{\eta}_6 - \mu\bar{\eta}_8)] \\
&+C_2[-\beta_3\sigma_{57,j}\bar{\eta}_8 + \sigma_{63,j}(\bar{\eta}_6 - \mu\bar{\eta}_8)] + C_3[-\beta_3\sigma_{54,j}\bar{\eta}_8 + \sigma_{66,j}(\bar{\eta}_6 - \mu\bar{\eta}_8)]\Big\}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial V_M}{\partial B_1} = \pi\sum\Big\{ & A_1 n x_i^2[-\beta_1\sigma_{46,j}\bar{\eta}_9 - \beta_1\sigma_{52,j}\bar{\eta}_7 + \sigma_{58,j}(\bar{\eta}_5 - \mu\bar{\eta}_7)] + A_2 n x_i^2[-\beta_1\sigma_{49,j}\bar{\eta}_9 \\
&-\beta_2\sigma_{55,j}\bar{\eta}_7 + \sigma_{59,j}(\bar{\eta}_5 - \mu\bar{\eta}_7)] + A_3 n x_i^2[-\beta_1\sigma_{50,j}\bar{\eta}_9 - \beta_3\sigma_{56,j}\bar{\eta}_7 + \sigma_{60,j}(\bar{\eta}_5 - \mu\bar{\eta}_7)] \\
&+2B_1 x_i^2[\beta_1^2\sigma_{13,j}\bar{\eta}_{16} + \sigma_{19,j}(n^2\bar{\eta}_{10} + \bar{\eta}_{11}) - \beta_1\sigma_{25,j}\bar{\eta}_{15}] + B_2 x_i^2[2\beta_1\beta_2\sigma_{16,j}\bar{\eta}_{16} \\
&+2\sigma_{22,j}(n^2\bar{\eta}_{10} + \bar{\eta}_{11}) - \beta_2\sigma_{26,j}\bar{\eta}_{15} - \beta_1\sigma_{28,j}\bar{\eta}_{15}] + B_3 x_i^2[2\beta_1\beta_3\sigma_{17,j}\bar{\eta}_{16} + 2\sigma_{23,j}(n^2\bar{\eta}_{10} \\
&+\bar{\eta}_{11}) - \beta_3\sigma_{27,j}\bar{\eta}_{15} - \beta_1\sigma_{31,j}\bar{\eta}_{15}] + C_1 x_i^2 n\sigma_{19,j}\bar{\eta}_{12} + C_2 x_i^2 n\sigma_{22,j}\bar{\eta}_{12} + C_3 x_i^2 n\sigma_{23,j}\bar{\eta}_{12} \\
&-D_1 x_1\sigma_{34,j}\bar{\eta}_{13} - D_2 x_1\sigma_{37,j}\bar{\eta}_{13} - D_3 x_1\sigma_{38,j}\bar{\eta}_{13} - E_1 x_1\sigma_{34,j}\bar{\eta}_{14} - E_2 x_1\sigma_{37,j}\bar{\eta}_{14} \\
&-E_3 x_1\sigma_{38,j}\bar{\eta}_{14}\Big\}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial V_M}{\partial B_2} = \pi\sum\Big\{ & A_1 n x_i^2[-\beta_2\sigma_{49,j}\bar{\eta}_9 - \beta_1\sigma_{55,j}\bar{\eta}_7 + \sigma_{61,j}(\bar{\eta}_5 - \mu\bar{\eta}_7)] + A_2 n x_i^2[-\beta_2\sigma_{47,j}\bar{\eta}_9 \\
&-\beta_2\sigma_{53,j}\bar{\eta}_7 + \sigma_{62,j}(\bar{\eta}_5 - \mu\bar{\eta}_7)] + A_3 n x_i^2[-\beta_2\sigma_{51,j}\bar{\eta}_9 - \beta_3\sigma_{57,j}\bar{\eta}_7 + \sigma_{63,j}(\bar{\eta}_5 - \mu\bar{\eta}_7)] \\
&+B_1 x_i^2[2\beta_1\beta_2\sigma_{16,j}\bar{\eta}_{16} + 2\sigma_{22,j}(n^2\bar{\eta}_{10} + \bar{\eta}_{11}) - \beta_2\sigma_{26,j}\bar{\eta}_{15} - \beta_1\sigma_{28,j}\bar{\eta}_{15}] \\
&+2B_2 x_i^2[\beta_2^2\sigma_{14,j}\bar{\eta}_{16} + \sigma_{20,j}(n^2\bar{\eta}_{10} + \bar{\eta}_{11}) - \beta_2\sigma_{29,j}\bar{\eta}_{15}] + B_3 x_i^2[2\beta_2\beta_3\sigma_{18,j}\bar{\eta}_{16} \\
&+2\sigma_{24,j}(n^2\bar{\eta}_{10} + \bar{\eta}_{11}) - \beta_3\sigma_{30,j}\bar{\eta}_{15} - \beta_2\sigma_{32,j}\bar{\eta}_{15}] + C_1 x_i^2 n\sigma_{22,j}\bar{\eta}_{12} + C_2 x_i^2 n\sigma_{20,j}\bar{\eta}_{12} \\
&+C_3 x_i^2 n\sigma_{24,j}\bar{\eta}_{12} - D_1 x_1\sigma_{37,j}\bar{\eta}_{13} - D_2 x_1\sigma_{35,j}\bar{\eta}_{13} - D_3 x_1\sigma_{39,j}\bar{\eta}_{13} - E_1 x_1\sigma_{37,j}\bar{\eta}_{14} \\
&-E_2 x_1\sigma_{35,j}\bar{\eta}_{14} - E_3 x_1\sigma_{39,j}\bar{\eta}_{14}\Big\}
\end{aligned}
$$

$$\begin{aligned}\frac{\partial V_M}{\partial B_3} = \pi\sum\Big\{ & A_1 n X_i^2\left[-\beta_3\sigma_{50,j}\bar{\eta}_9 - \beta_1\sigma_{56,j}\bar{\eta}_7 + \sigma_{64,j}(\bar{\eta}_5-\mu\bar{\eta}_7)\right] + A_2 n X_i^2\left[-\beta_3\sigma_{51,j}\bar{\eta}_9\right. \\ & \left.-\beta_2\sigma_{57,j}\bar{\eta}_7 + \sigma_{65,j}(\bar{\eta}_5-\mu\bar{\eta}_7)\right] + A_3 n X_i^2\left[-\beta_3\sigma_{48,j}\bar{\eta}_9 - \beta_3\sigma_{54,j}\bar{\eta}_7 + \sigma_{66,j}(\bar{\eta}_5-\mu\bar{\eta}_7)\right] \\ & + B_1X_i^2\left[2\beta_1\beta_3\sigma_{17,j}\bar{\eta}_{16} + 2\sigma_{23,j}(n^2\bar{\eta}_{10}+\bar{\eta}_{11}) - \beta_3\sigma_{27,j}\bar{\eta}_{15} - \beta_1\sigma_{31,j}\bar{\eta}_{15}\right] \\ & + B_2X_i^2\left[2\beta_2\beta_3\sigma_{18,j}\bar{\eta}_{16} + 2\sigma_{24,j}(n^2\bar{\eta}_{10}+\bar{\eta}_{11}) - \beta_3\sigma_{30,j}\bar{\eta}_{15} - \beta_2\sigma_{32,j}\bar{\eta}_{15}\right] \\ & + 2B_3X_i^2\left[\beta_3^2\sigma_{15,j}\bar{\eta}_{16} + \sigma_{21,j}(n^2\bar{\eta}_{10}+\bar{\eta}_{11}) - \beta_3\sigma_{33,j}\bar{\eta}_{15}\right] + C_1X_i^2 n\,\sigma_{23,j}\bar{\eta}_{12} \\ & + C_2X_i^2 n\,\sigma_{24,j}\bar{\eta}_{12} + C_3X_i^2 n\,\sigma_{21,j}\bar{\eta}_{12} - D_1X_1\sigma_{38,j}\bar{\eta}_{13} - D_2X_1\sigma_{39,j}\bar{\eta}_{13} - D_3X_1\sigma_{36,j}\bar{\eta}_{13} \\ & - E_1X_1\sigma_{38,j}\bar{\eta}_{16} - E_2X_1\sigma_{39,j}\bar{\eta}_{16} - E_3X_1\sigma_{36,j}\bar{\eta}_{16}\Big\}\end{aligned}$$

$$\begin{aligned}\frac{\partial V_M}{\partial C_1} = \pi\sum\Big\{ & A_1X_i^2\left[-\beta_1\sigma_{52,j}\bar{\eta}_8 + \sigma_{58,j}(\bar{\eta}_6-\mu\bar{\eta}_8)\right] + A_2X_i^2\left[-\beta_2\sigma_{55,j}\bar{\eta}_8 + \sigma_{59,j}(\bar{\eta}_6\right. \\ & \left.-\mu\bar{\eta}_8)\right] + A_3X_i^2\left[-\beta_3\sigma_{56,j}\bar{\eta}_8 + \sigma_{60,j}(\bar{\eta}_6-\mu\bar{\eta}_8)\right] + B_1X_i^2 n\,\sigma_{19,j}\bar{\eta}_{12} + B_2X_i^2 n\,\sigma_{22,j}\bar{\eta}_{12} \\ & + B_3X_i^2 n\,\sigma_{23,j}\bar{\eta}_{12} + 2C_1X_i^2\left[\sigma_{13,j}\beta_1^2\bar{\eta}_{21} + \sigma_{19,j}(\bar{\eta}_{17}+n^2\bar{\eta}_{18})\right] + 2C_2X_i^2\left[\beta_1\beta_2\sigma_{16,j}\bar{\eta}_{21}\right. \\ & \left.+\sigma_{22,j}(\bar{\eta}_{17}+n^2\bar{\eta}_{18})\right] + 2C_3X_i^2\left[\beta_1\beta_3\sigma_{17,j}\bar{\eta}_{21} + \sigma_{23,j}(\bar{\eta}_{17}+n^2\bar{\eta}_{18})\right] - D_1X_1 n\,\sigma_{34,j}\bar{\eta}_{19} \\ & - D_2X_1 n\,\sigma_{37,j}\bar{\eta}_{19} - D_3X_1 n\,\sigma_{38,j}\bar{\eta}_{19} - E_1X_1 n\,\sigma_{34,j}\bar{\eta}_{20} - E_2X_1 n\,\sigma_{37,j}\bar{\eta}_{20} - E_3X_1 n\,\sigma_{38,j}\bar{\eta}_{20} \\ & + F_1X_1\beta_1\sigma_{40,j}\bar{\eta}_{23} + F_2X_1\beta_1\sigma_{43,j}\bar{\eta}_{23} + F_3X_1\beta_1\sigma_{44,j}\bar{\eta}_{23} + G_1X_1\beta_1\sigma_{40,j}\bar{\eta}_{22} \\ & + G_2X_1\beta_1\sigma_{43,j}\bar{\eta}_{22} + G_3X_1\beta_1\sigma_{44,j}\bar{\eta}_{22}\Big\}\end{aligned}$$

$$\begin{aligned}\frac{\partial V_M}{\partial C_2} = \pi\sum\Big\{ & A_1X_i^2\left[-\beta_1\sigma_{55,j}\bar{\eta}_8 + \sigma_{61,j}(\bar{\eta}_6-\mu\bar{\eta}_8)\right] + A_2X_i^2\left[-\beta_2\sigma_{53,j}\bar{\eta}_8 + \sigma_{62,j}(\bar{\eta}_6\right. \\ & \left.-\mu\bar{\eta}_8)\right] + A_3X_i^2\left[-\beta_3\sigma_{57,j}\bar{\eta}_8 + \sigma_{63,j}(\bar{\eta}_6-\mu\bar{\eta}_8)\right] + B_1X_i^2 n\,\sigma_{22,j}\bar{\eta}_{12} + B_2X_i^2 n\,\sigma_{20,j}\bar{\eta}_{12} \\ & + B_3X_i^2 n\,\sigma_{24,j}\bar{\eta}_{12} + 2C_1X_i^2\left[\beta_1\beta_2\sigma_{16,j}\bar{\eta}_{21} + \sigma_{22,j}(\bar{\eta}_{17}+n^2\bar{\eta}_{18})\right] + 2C_2X_i^2\left[\beta_2^2\sigma_{14,j}\bar{\eta}_{21}\right. \\ & \left.+\sigma_{20,j}(\bar{\eta}_{17}+n^2\bar{\eta}_{18})\right] + 2C_3X_i^2\left[\beta_2\beta_3\sigma_{18,j}\bar{\eta}_{21} + \sigma_{24,j}(\bar{\eta}_{17}+n^2\bar{\eta}_{18})\right] - D_1X_1 n\,\sigma_{37,j}\bar{\eta}_{19} \\ & - D_2X_1 n\,\sigma_{35,j}\bar{\eta}_{19} - D_3X_1 n\,\sigma_{39,j}\bar{\eta}_{19} - E_1X_1 n\,\sigma_{37,j}\bar{\eta}_{20} - E_2X_1 n\,\sigma_{35,j}\bar{\eta}_{20} - E_3X_1 n\,\sigma_{39,j}\bar{\eta}_{20} \\ & + F_1X_1\beta_2\sigma_{43,j}\bar{\eta}_{23} + F_2X_1\beta_2\sigma_{41,j}\bar{\eta}_{23} + F_3X_1\beta_2\sigma_{45,j}\bar{\eta}_{23} + G_1X_1\beta_2\sigma_{43,j}\bar{\eta}_{22}\end{aligned}$$

(Continued)

$$+ G_2 x_1 \beta_2 \sigma_{41,j} \bar{\eta}_{22} + G_3 x_1 \beta_2 \sigma_{45,j} \bar{\eta}_{22} \Big\}$$

$$\frac{\partial V_M}{\partial C_3} = \pi \Sigma \Big\{ A_1 x_i^2 [-\beta_1 \sigma_{56,j} \bar{\eta}_8 + \sigma_{64,j} (\bar{\eta}_6 - \mu \bar{\eta}_8)] + A_2 x_i^2 [-\beta_2 \sigma_{57,j} \bar{\eta}_8 + \sigma_{65,j} (\bar{\eta}_6 - \mu \bar{\eta}_8)] + A_3 x_i^2 [-\beta_3 \sigma_{54,j} \bar{\eta}_8 + \sigma_{66,j} (\bar{\eta}_6 - \mu \bar{\eta}_8)] + B_1 x_i^2 n \sigma_{23,j} \bar{\eta}_{12} + B_2 x_i^2 n \sigma_{24,j} \bar{\eta}_{12} + B_3 x_i^2 n \sigma_{21,j} \bar{\eta}_{12} + 2C_1 x_i^2 [\beta_1 \beta_3 \sigma_{17,j} \bar{\eta}_{21} + \sigma_{23,j} (\bar{\eta}_{17} + n^2 \bar{\eta}_{18})] + 2C_2 x_i^2 [\beta_2 \beta_3 \sigma_{18,j} \bar{\eta}_{21} + \sigma_{24,j} (\bar{\eta}_{17} + n^2 \bar{\eta}_{18})] + 2C_3 x_i^2 [\beta_3^2 \sigma_{15,j} \bar{\eta}_{21} + \sigma_{21,j} (\bar{\eta}_{17} + n^2 \bar{\eta}_{18})] - D_1 x_1 n \sigma_{38,j} \bar{\eta}_{19} - D_2 x_1 n \sigma_{39,j} \bar{\eta}_{19} - D_3 x_1 n \sigma_{36,j} \bar{\eta}_{19} - E_1 x_1 n \sigma_{38,j} \bar{\eta}_{20} - E_2 x_1 n \sigma_{39,j} \bar{\eta}_{20} - E_3 x_1 n \sigma_{36,j} \bar{\eta}_{20} + F_1 x_1 \beta_3 \sigma_{44,j} \bar{\eta}_{23} + F_2 x_1 \beta_3 \sigma_{45,j} \bar{\eta}_{23} + F_3 x_1 \beta_3 \sigma_{42,j} \bar{\eta}_{23} + G_1 x_1 \beta_3 \sigma_{44,j} \bar{\eta}_{22} + G_2 x_1 \beta_3 \sigma_{45,j} \bar{\eta}_{22} + G_3 x_1 \beta_3 \sigma_{42,j} \bar{\eta}_{22} \Big\}$$

$$\frac{\partial V_M}{\partial D_1} = \pi \Sigma \Big\{ -B_1 x_1 \sigma_{34,j} \bar{\eta}_{13} - B_2 x_1 \sigma_{37,j} \bar{\eta}_{13} - B_3 x_1 \sigma_{38,j} \bar{\eta}_{13} - C_1 x_1 n \sigma_{34,j} \bar{\eta}_{19} - C_2 x_1 n \sigma_{37,j} \bar{\eta}_{19} - C_3 x_1 n \sigma_{38,j} \bar{\eta}_{19} + 2D_1 [\sigma_{7,j} \bar{\eta}_{57} + \beta_1^2 \sigma_{13,j} \bar{\eta}_{48} + \sigma_{19,j} (\bar{\eta}_{49} + n^2 \bar{\eta}_{50}) - \sigma_{25,j} \beta_1 \bar{\eta}_{48}] + D_2 [2\sigma_{10,j} \bar{\eta}_{57} + 2\beta_1 \beta_2 \sigma_{16,j} \bar{\eta}_{48} + 2\sigma_{22,j} (\bar{\eta}_{49} + n^2 \bar{\eta}_{50}) - \beta_2 \sigma_{26,j} \bar{\eta}_{48} - \beta_1 \sigma_{28,j} \bar{\eta}_{48}] + D_3 [2\sigma_{11,j} \bar{\eta}_{57} + 2\beta_1 \beta_3 \sigma_{17,j} \bar{\eta}_{48} + 2\sigma_{23,j} (\bar{\eta}_{49} + n^2 \bar{\eta}_{50}) - \beta_3 \sigma_{27,j} \bar{\eta}_{48} - \beta_1 \sigma_{31,j} \bar{\eta}_{48}] + E_1 [\beta_1^2 \sigma_{13,j} \bar{\eta}_{52} + \sigma_{19,j} (\bar{\eta}_{52} + n^2 \bar{\eta}_{53}) - 2\beta_1 \sigma_{25,j} \bar{\eta}_{52}] + E_2 [\beta_1 \beta_2 \sigma_{16,j} \bar{\eta}_{52} + \sigma_{22,j} (\bar{\eta}_{52} + n^2 \bar{\eta}_{53}) - \bar{\eta}_{52} (\beta_2 \sigma_{26,j} + \beta_1 \sigma_{28,j})] + E_3 [\beta_1 \beta_3 \sigma_{17,j} \bar{\eta}_{52} + \sigma_{23,j} (\bar{\eta}_{52} + n^2 \bar{\eta}_{53}) - \bar{\eta}_{52} (\beta_3 \sigma_{27,j} + \beta_1 \sigma_{31,j})] + F_1 n [\beta_1 \sigma_{46,j} \bar{\eta}_{45} - \beta_1 \sigma_{52,j} \bar{\eta}_{40} + \sigma_{58,j} (\bar{\eta}_{38} - \mu \bar{\eta}_{40})] + F_2 n [-\beta_1 \sigma_{49,j} \bar{\eta}_{45} - \beta_2 \sigma_{55,j} \bar{\eta}_{40} + \sigma_{59,j} (\bar{\eta}_{38} - \mu \bar{\eta}_{40})] + F_3 n [-\beta_1 \sigma_{50,j} \bar{\eta}_{45} - \beta_3 \sigma_{56,j} \bar{\eta}_{40} + \sigma_{60,j} (\bar{\eta}_{38} - \mu \bar{\eta}_{40})] + G_1 n [\beta_1 \sigma_{46,j} \bar{\eta}_{43} - \beta_1 \sigma_{52,j} \bar{\eta}_{46} + \sigma_{58,j} (\bar{\eta}_{39} - \mu \bar{\eta}_{46})] + G_2 n [\beta_1 \sigma_{49,j} \bar{\eta}_{43} - \beta_2 \sigma_{55,j} \bar{\eta}_{46} + \sigma_{59,j} (\bar{\eta}_{39} - \mu \bar{\eta}_{46})] + G_3 n [\beta_1 \sigma_{50,j} \bar{\eta}_{43} - \beta_3 \sigma_{56,j} \bar{\eta}_{46} + \sigma_{60,j} (\bar{\eta}_{39} - \mu \bar{\eta}_{46})] \Big\}$$

$$
\begin{aligned}
\frac{\partial V_M}{\partial D_2} = \pi \sum \Big\{ & -B_1 X_1 \sigma_{37,j} \bar{\eta}_{13} - B_2 X_1 \sigma_{35,j} \bar{\eta}_{13} - B_3 X_1 \sigma_{39,j} \bar{\eta}_{13} - C_1 X_1 n \sigma_{37,j} \bar{\eta}_{19} \\
& - C_2 X_1 n \sigma_{35,j} \bar{\eta}_{19} - C_3 X_1 n \sigma_{39,j} \bar{\eta}_{19} + D_1 [2\sigma_{10,j} \bar{\eta}_{51} + 2\beta_1 \beta_2 \sigma_{16,j} \bar{\eta}_{48} + 2\sigma_{22,j} (\bar{\eta}_{49} \\
& + n^2 \bar{\eta}_{50}) - \beta_2 \sigma_{26,j} \bar{\eta}_{48} - \beta_1 \sigma_{28,j} \bar{\eta}_{48}] + 2 D_2 [\sigma_{8,j} \bar{\eta}_{51} + \beta_2^2 \sigma_{14,j} \bar{\eta}_{48} + \sigma_{20,j} (\bar{\eta}_{49} + n^2 \bar{\eta}_{50}) \\
& - \beta_2 \sigma_{29,j} \bar{\eta}_{48}] + D_3 [2\sigma_{12,j} \bar{\eta}_{51} + 2\beta_2 \beta_3 \sigma_{18,j} \bar{\eta}_{48} + 2\sigma_{24,j} (\bar{\eta}_{49} + n^2 \bar{\eta}_{50}) - \bar{\eta}_{48} (\beta_3 \sigma_{30,j} \\
& + \beta_2 \sigma_{32,j})] + E_1 [\beta_1 \beta_2 \sigma_{16,j} \bar{\eta}_{52} + \sigma_{22,j} (\bar{\eta}_{52} + n^2 \bar{\eta}_{53}) - \bar{\eta}_{52} (\beta_2 \sigma_{26,j} + \beta_1 \sigma_{28,j})] \\
& + E_2 [\beta_2^2 \sigma_{14,j} \bar{\eta}_{52} + \sigma_{20,j} (\bar{\eta}_{52} + n^2 \bar{\eta}_{53}) - 2\beta_2 \sigma_{29,j} \bar{\eta}_{52}] + E_3 [\beta_2 \beta_3 \sigma_{18,j} \bar{\eta}_{52} \\
& + \sigma_{24,j} (\bar{\eta}_{52} + n^2 \bar{\eta}_{53}) - \bar{\eta}_{52} (\beta_3 \sigma_{30,j} + \beta_2 \sigma_{32,j})] + F_1 n [-\beta_2 \sigma_{49,j} \bar{\eta}_{45} - \beta_1 \sigma_{55,j} \bar{\eta}_{40} \\
& + \sigma_{61,j} (\bar{\eta}_{38} - \mu \bar{\eta}_{40})] + F_2 n [-\beta_2 \sigma_{47,j} \bar{\eta}_{45} - \beta_2 \sigma_{53,j} \bar{\eta}_{40} + \sigma_{62,j} (\bar{\eta}_{38} - \mu \bar{\eta}_{40})] \\
& + F_3 n [-\beta_2 \sigma_{51,j} \bar{\eta}_{45} - \beta_3 \sigma_{57,j} \bar{\eta}_{40} + \sigma_{63,j} (\bar{\eta}_{38} - \mu \bar{\eta}_{40})] + G_1 n [\beta_2 \sigma_{49,j} \bar{\eta}_{43} \\
& - \beta_1 \sigma_{55,j} \bar{\eta}_{46} + \sigma_{61,j} (\bar{\eta}_{39} - \mu \bar{\eta}_{46})] + G_2 n [\beta_2 \sigma_{47,j} \bar{\eta}_{43} - \beta_2 \sigma_{53,j} \bar{\eta}_{46} \\
& + \sigma_{62,j} (\bar{\eta}_{39} - \mu \bar{\eta}_{46})] + G_3 n [\beta_2 \sigma_{51,j} \bar{\eta}_{43} - \beta_3 \sigma_{57,j} \bar{\eta}_{46} + \sigma_{63,j} (\bar{\eta}_{39} - \mu \bar{\eta}_{46})] \Big\}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial V_M}{\partial D_3} = \pi \sum \Big\{ & -B_1 X_1 \sigma_{38,j} \bar{\eta}_{13} - B_2 X_1 \sigma_{39,j} \bar{\eta}_{13} - B_3 X_1 \sigma_{36,j} \bar{\eta}_{13} - C_1 X_1 n \sigma_{38,j} \bar{\eta}_{19} \\
& - C_2 X_1 n \sigma_{39,j} \bar{\eta}_{19} - C_3 X_1 n \sigma_{36,j} \bar{\eta}_{19} + D_1 [2\sigma_{11,j} \bar{\eta}_{51} + 2\beta_1 \beta_3 \sigma_{17,j} \bar{\eta}_{48} + 2\sigma_{23,j} (\bar{\eta}_{49} \\
& + n^2 \bar{\eta}_{50}) - \bar{\eta}_{48} (\beta_3 \sigma_{27,j} + \beta_1 \sigma_{31,j})] + D_2 [2\sigma_{12,j} \bar{\eta}_{51} + 2\beta_2 \beta_3 \sigma_{18,j} \bar{\eta}_{48} + 2\sigma_{24,j} (\bar{\eta}_{49} \\
& + n^2 \bar{\eta}_{50}) - \bar{\eta}_{48} (\beta_3 \sigma_{30,j} + \beta_2 \sigma_{32,j})] + 2 D_3 [\sigma_{9,j} \bar{\eta}_{51} + \beta_3^2 \sigma_{15,j} \bar{\eta}_{48} + \sigma_{21,j} (\bar{\eta}_{49} + n^2 \bar{\eta}_{50}) \\
& - \beta_3 \sigma_{33,j} \bar{\eta}_{48}] + E_1 [\beta_1 \beta_3 \sigma_{17,j} \bar{\eta}_{52} + \sigma_{23,j} (\bar{\eta}_{52} + n^2 \bar{\eta}_{53}) - \bar{\eta}_{52} (\beta_3 \sigma_{27,j} + \beta_1 \sigma_{31,j})] \\
& + E_2 [\beta_2 \beta_3 \sigma_{18,j} \bar{\eta}_{52} + \sigma_{24,j} (\bar{\eta}_{52} + n^2 \bar{\eta}_{53}) - \bar{\eta}_{52} (\beta_3 \sigma_{30,j} + \beta_2 \sigma_{32,j})] \\
& + E_3 [\beta_3^2 \sigma_{15,j} \bar{\eta}_{52} + \sigma_{21,j} (\bar{\eta}_{52} + n^2 \bar{\eta}_{53}) - 2\sigma_{33,j} \beta_3 \bar{\eta}_{52}] + F_1 n [-\beta_3 \sigma_{50,j} \bar{\eta}_{45} \\
& - \beta_1 \sigma_{56,j} \bar{\eta}_{40} + \sigma_{64,j} (\bar{\eta}_{38} - \mu \bar{\eta}_{40})] + F_2 n [-\beta_3 \sigma_{51,j} \bar{\eta}_{45} - \beta_2 \sigma_{57,j} \bar{\eta}_{40} + \sigma_{65,j} (\bar{\eta}_{38} \\
& - \mu \bar{\eta}_{40})] + F_3 n [-\beta_3 \sigma_{48,j} \bar{\eta}_{45} - \beta_3 \sigma_{54,j} \bar{\eta}_{40} + \sigma_{66,j} (\bar{\eta}_{38} - \mu \bar{\eta}_{40})] + G_1 n [\beta_3 \sigma_{50,j} \bar{\eta}_{43} \\
& - \beta_1 \sigma_{56,j} \bar{\eta}_{46} + \sigma_{64,j} (\bar{\eta}_{39} - \mu \bar{\eta}_{46})] + G_2 n [\beta_3 \sigma_{51,j} \bar{\eta}_{43} - \beta_2 \sigma_{57,j} \bar{\eta}_{46} + \sigma_{65,j} (\bar{\eta}_{39}
\end{aligned}
$$

(Continued)

$$-\mu\bar{\eta}_{46})] + G_3 n\left[\beta_3\sigma_{48,j}\bar{\eta}_{43} - \beta_3\sigma_{54,j}\bar{\eta}_{46} + \sigma_{66,j}(\bar{\eta}_{39} - \mu\bar{\eta}_{46})\right]\}$$

$$\frac{dV_M}{dE_1} = \pi\sum\{-B_1X_1\sigma_{34,j}\bar{\eta}_{14} - B_2X_1\sigma_{37,j}\bar{\eta}_{14} - B_3X_1\sigma_{38,j}\bar{\eta}_{14} - C_1X_1n\sigma_{34,j}\bar{\eta}_{20}$$

$$-C_2X_1n\sigma_{37,j}\bar{\eta}_{20} - C_3X_1n\sigma_{38,j}\bar{\eta}_{20} + D_1[\beta_1^2\sigma_{13,j}\bar{\eta}_{52} + \sigma_{19,j}(\bar{\eta}_{52} + n^2\bar{\eta}_{53})$$

$$-2\beta_1\sigma_{25,j}\bar{\eta}_{52}] + D_2[\beta_1\beta_2\sigma_{16,j}\bar{\eta}_{52} + \sigma_{22,j}(\bar{\eta}_{52} + n^2\bar{\eta}_{53}) - \bar{\eta}_{52}(\beta_2\sigma_{26,j} + \beta_1\sigma_{28,j})]$$

$$+D_3[\beta_1\beta_3\sigma_{17,j}\bar{\eta}_{52} + \sigma_{23,j}(\bar{\eta}_{52} + n^2\bar{\eta}_{53}) - \bar{\eta}_{52}(\beta_3\sigma_{27,j} + \beta_1\sigma_{31,j})] + 2E_1[\sigma_{7,j}\bar{\eta}_{56}$$

$$+\beta_1^2\sigma_{13,j}\bar{\eta}_{54} + \sigma_{19,j}(\bar{\eta}_{54} + n^2\bar{\eta}_{55}) + \beta_1\sigma_{25,j}\bar{\eta}_{57}] + E_2[2\sigma_{10,j}\bar{\eta}_{56} + 2\beta_1\beta_2\sigma_{16,j}\bar{\eta}_{54}$$

$$+2\sigma_{22,j}(\bar{\eta}_{54} + n^2\bar{\eta}_{55}) + (\beta_2\sigma_{26,j} + \beta_1\sigma_{28,j})\bar{\eta}_{57}] + E_3[2\sigma_{11,j}\bar{\eta}_{56} + 2\beta_1\beta_3\sigma_{17,j}\bar{\eta}_{54}$$

$$+2\sigma_{23,j}(\bar{\eta}_{54} + n^2\bar{\eta}_{55}) + (\beta_3\sigma_{27,j} + \beta_1\sigma_{31,j})\bar{\eta}_{57}] + F_1n[\beta_1\sigma_{46,j}\bar{\eta}_{43} - \beta_1\sigma_{52,j}\bar{\eta}_{41}$$

$$+\sigma_{58,j}(\bar{\eta}_{39} - \mu\bar{\eta}_{41})] + F_2n[\beta_1\sigma_{49,j}\bar{\eta}_{43} - \beta_2\sigma_{55,j}\bar{\eta}_{41} + \sigma_{59,j}(\bar{\eta}_{39} - \mu\bar{\eta}_{41})]$$

$$+F_3n[\beta_1\sigma_{50,j}\bar{\eta}_{43} - \beta_3\sigma_{56,j}\bar{\eta}_{41} + \sigma_{60,j}(\bar{\eta}_{39} - \mu\bar{\eta}_{40})] + G_1n[\beta_1\sigma_{46,j}\bar{\eta}_{44}$$

$$-\beta_1\sigma_{52,j}\bar{\eta}_{47} + \sigma_{58,j}(\bar{\eta}_{42} - \mu\bar{\eta}_{47})] + G_2n[+\beta_1\sigma_{49,j}\bar{\eta}_{44} - \beta_2\sigma_{55,j}\bar{\eta}_{47} + \sigma_{59,j}(\bar{\eta}_{42}$$

$$-\mu\bar{\eta}_{47})] + G_3n[+\beta_1\sigma_{50,j}\bar{\eta}_{44} - \beta_3\sigma_{56,j}\bar{\eta}_{47} + \sigma_{60,j}(\bar{\eta}_{42} - \mu\bar{\eta}_{47})]\}$$

$$\frac{dV_M}{dE_2} = \pi\sum\{-B_1X_1\sigma_{37,j}\bar{\eta}_{14} - B_2X_1\sigma_{35,j}\bar{\eta}_{14} - B_3X_1\sigma_{39,j}\bar{\eta}_{14} - C_1X_1n\sigma_{37,j}\bar{\eta}_{20}$$

$$-C_2X_1n\sigma_{35,j}\bar{\eta}_{20} - C_3X_1n\sigma_{39,j}\bar{\eta}_{20} + D_1[\beta_1\beta_2\sigma_{16,j}\bar{\eta}_{52} + \sigma_{22,j}(\bar{\eta}_{52} + n^2\bar{\eta}_{53})$$

$$-\bar{\eta}_{52}(\beta_2\sigma_{26,j} + \beta_1\sigma_{28,j})] + D_2[\beta_2^2\sigma_{14,j}\bar{\eta}_{52} + \sigma_{20,j}(\bar{\eta}_{52} + n^2\bar{\eta}_{53}) - 2\beta_2\sigma_{29,j}\bar{\eta}_{52}]$$

$$+D_3[\beta_2\beta_3\sigma_{18,j}\bar{\eta}_{52} + \sigma_{24,j}(\bar{\eta}_{52} + n^2\bar{\eta}_{53}) - \bar{\eta}_{52}(\beta_3\sigma_{30,j} + \beta_2\sigma_{32,j})] + E_1[2\sigma_{10,j}\bar{\eta}_{56}$$

$$+2\beta_1\beta_2\sigma_{16,j}\bar{\eta}_{54} + 2\sigma_{22,j}(\bar{\eta}_{54} + n^2\bar{\eta}_{55}) + \bar{\eta}_{57}(\beta_2\sigma_{26,j} + \beta_1\sigma_{28,j})]$$

$$+2E_2[\sigma_{8,j}\bar{\eta}_{56} + \beta_2^2\sigma_{14,j}\bar{\eta}_{54} + \sigma_{20,j}(\bar{\eta}_{54} + n^2\bar{\eta}_{55}) + \beta_2\sigma_{29,j}\bar{\eta}_{57}] + E_3[2\sigma_{12,j}\bar{\eta}_{56}$$

$$+2\beta_2\beta_3\sigma_{18,j}\bar{\eta}_{54} + 2\sigma_{24,j}(\bar{\eta}_{54} + n^2\bar{\eta}_{55}) + \bar{\eta}_{57}(\beta_3\sigma_{34,j} + \beta_2\sigma_{32,j})] + F_1n[\beta_2\sigma_{44,j}\bar{\eta}_{43}$$

$$-\beta_1\sigma_{55,j}\bar{\eta}_{41} + \sigma_{61,j}(\bar{\eta}_{39} - \mu\bar{\eta}_{41})] + F_2n[\beta_2\sigma_{47,j}\bar{\eta}_{43} - \beta_2\sigma_{53,j}\bar{\eta}_{41} + \sigma_{62,j}(\bar{\eta}_{39}$$

(Continued)

$$-\mu\bar{\eta}_{41})] + F_3 n[\beta_2\sigma_{51,j}\bar{\eta}_{43} - \beta_3\sigma_{53,j}\bar{\eta}_{41} + \sigma_{63,j}(\bar{\eta}_{39} - \mu\bar{\eta}_{41})] + G_1 n[\beta_2\sigma_{49,j}\bar{\eta}_{44}$$
$$-\beta_1\sigma_{55,j}\bar{\eta}_{47} + \sigma_{61,j}(\bar{\eta}_{42} - \mu\bar{\eta}_{47})] + G_2 n[\beta_2\sigma_{47,j}\bar{\eta}_{44} - \beta_2\sigma_{53,j}\bar{\eta}_{47} + \sigma_{62,j}(\bar{\eta}_{42}$$
$$-\mu\bar{\eta}_{47})] + G_3 n[\beta_2\sigma_{51,j}\bar{\eta}_{44} - \beta_3\sigma_{57,j}\bar{\eta}_{47} + \sigma_{63,j}(\bar{\eta}_{42} - \mu\bar{\eta}_{47})]\}$$

$$\frac{\partial V_M}{\partial E_3} = \pi\Sigma\{-B_1X_1\sigma_{38,j}\bar{\eta}_{14} - B_2X_1\sigma_{39,j}\bar{\eta}_{14} - B_3X_1\sigma_{36,j}\bar{\eta}_{14} - C_1X_1 n\sigma_{38,j}\bar{\eta}_{20}$$
$$-C_2X_1 n\sigma_{39,j}\bar{\eta}_{20} - C_3X_1 n\sigma_{36,j}\bar{\eta}_{20} + D_1[\beta_1\beta_3\sigma_{17,j}\bar{\eta}_{52} + \sigma_{23,j}(\bar{\eta}_{52} + n^2\bar{\eta}_{53})$$
$$-\bar{\eta}_{52}(\beta_1\sigma_{31,j} + \beta_3\sigma_{27,j})] + D_2[\beta_2\beta_3\sigma_{18,j}\bar{\eta}_{52} + \sigma_{24,j}(\bar{\eta}_{52} + n^2\bar{\eta}_{53}) - \bar{\eta}_{52}(\beta_3\sigma_{30,j}$$
$$+\beta_2\sigma_{32,j})] + D_3[\beta_3^2\sigma_{15,j}\bar{\eta}_{52} + \sigma_{21,j}(\bar{\eta}_{52} + n^2\bar{\eta}_{53}) - 2\beta_3\sigma_{33,j}\bar{\eta}_{52}] + E_1[2\sigma_{11,j}\bar{\eta}_{56}$$
$$+2\beta_1\beta_3\sigma_{17,j}\bar{\eta}_{54} + 2\sigma_{23,j}(\bar{\eta}_{54} + n^2\bar{\eta}_{55}) + \bar{\eta}_{57}(\beta_3\sigma_{27,j} + \beta_1\sigma_{31,j})] + E_2[2\sigma_{12,j}\bar{\eta}_{56}$$
$$+2\beta_2\beta_3\sigma_{18,j}\bar{\eta}_{54} + 2\sigma_{24,j}(\bar{\eta}_{54} + n^2\bar{\eta}_{55}) + \bar{\eta}_{57}(\beta_3\sigma_{30,j} + \beta_2\sigma_{32,j})] + 2E_3[\sigma_{9,j}\bar{\eta}_{56}$$
$$+\beta_3^2\sigma_{15,j}\bar{\eta}_{54} + \sigma_{21,j}(\bar{\eta}_{54} + n^2\bar{\eta}_{55}) + \beta_3\sigma_{33,j}\bar{\eta}_{57}] + F_1 n[\beta_3\sigma_{50,j}\bar{\eta}_{43}$$
$$-\beta_1\sigma_{56,j}\bar{\eta}_{41} + \sigma_{64,j}(\bar{\eta}_{39} - \mu\bar{\eta}_{41})] + F_2 n[\beta_3\sigma_{51,j}\bar{\eta}_{43} - \beta_2\sigma_{57,j}\bar{\eta}_{41} + \sigma_{65,j}(\bar{\eta}_{39}$$
$$-\mu\bar{\eta}_{41})] + F_3 n[\beta_3\sigma_{48,j}\bar{\eta}_{43} - \beta_3\sigma_{54,j}\bar{\eta}_{41} + \sigma_{66,j}(\bar{\eta}_{39} - \mu\bar{\eta}_{41})] + G_1 n[\beta_3\sigma_{50,j}\bar{\eta}_{44}$$
$$-\beta_1\sigma_{56,j}\bar{\eta}_{47} + \sigma_{64,j}(\bar{\eta}_{42} - \mu\bar{\eta}_{47})] + G_2 n[\beta_3\sigma_{51,j}\bar{\eta}_{44} - \beta_2\sigma_{57,j}\bar{\eta}_{47} + \sigma_{65,j}(\bar{\eta}_{42}$$
$$-\mu\bar{\eta}_{47})] + G_3 n[\beta_3\sigma_{48,j}\bar{\eta}_{44} - \beta_3\sigma_{54,j}\bar{\eta}_{47} + \sigma_{66,j}(\bar{\eta}_{42} - \mu\bar{\eta}_{47})]\}$$

$$\frac{\partial V_M}{\partial F_1} = \pi\Sigma\{C_1X_1\beta_1\sigma_{40,j}\bar{\eta}_{23} + C_2X_1\beta_2\sigma_{43,j}\bar{\eta}_{23} + C_3X_1\beta_3\sigma_{44,j}\bar{\eta}_{23} + D_1 n[-\beta_1\sigma_{46,j}\bar{\eta}_{45}$$
$$-\beta_1\sigma_{52,j}\bar{\eta}_{40} + \sigma_{58,j}(\bar{\eta}_{38} - \mu\bar{\eta}_{40})] + D_2 n[-\beta_2\sigma_{49,j}\bar{\eta}_{45} - \beta_1\sigma_{55,j}\bar{\eta}_{40} + \sigma_{61,j}(\bar{\eta}_{38}$$
$$-\mu\bar{\eta}_{40})] + D_3 n[-\beta_3\sigma_{50,j}\bar{\eta}_{45} - \beta_1\sigma_{56,j}\bar{\eta}_{40} + \sigma_{64,j}(\bar{\eta}_{38} - \mu\bar{\eta}_{40})] + E_1 n[\beta_1\sigma_{46,j}\bar{\eta}_{43}$$
$$-\beta_1\sigma_{52,j}\bar{\eta}_{41} + \sigma_{58,j}(\bar{\eta}_{39} - \mu\bar{\eta}_{41})] + E_2 n[\beta_2\sigma_{49,j}\bar{\eta}_{43} - \beta_1\sigma_{55,j}\bar{\eta}_{41} + \sigma_{61,j}(\bar{\eta}_{39}$$
$$-\mu\bar{\eta}_{41})] + E_3 n[\beta_3\sigma_{50,j}\bar{\eta}_{43} - \beta_1\sigma_{56,j}\bar{\eta}_{41} + \sigma_{64,j}(\bar{\eta}_{39} - \mu\bar{\eta}_{41})] + F_1[2\sigma_{1,j}\bar{\eta}_{26}$$
$$+2\sigma_{67,j}(\bar{\eta}_{24} + n^2\bar{\eta}_{25} + \mu^2\bar{\eta}_{27} - \mu\bar{\eta}_{28}) + 2\beta_1\sigma_{73,j}(2\mu\bar{\eta}_{27} - \bar{\eta}_{28})$$

(Continued)

$$+2\beta_1^2\sigma_{82,j}\bar{\eta}_{27}] + F_2[2\sigma_{4,j}\bar{\eta}_{26} + 2\sigma_{70,j}(\bar{\eta}_{24}+n^2\bar{\eta}_{25}+\mu^2\bar{\eta}_{27}-\mu\bar{\eta}_{28}) + (\beta_1\sigma_{74,j}$$

$$+\beta_2\sigma_{76,j})(2\mu\bar{\eta}_{27}-\bar{\eta}_{28}) + 2\beta_1\beta_2\sigma_{85,j}\bar{\eta}_{27}] + F_3[2\sigma_{5,j}\bar{\eta}_{26}+2\sigma_{71,j}(\bar{\eta}_{24}+n^2\bar{\eta}_{25}$$

$$+\mu^2\bar{\eta}_{27}-\mu\bar{\eta}_{28}) + (\beta_1\sigma_{75,j}+\beta_3\sigma_{79,j})(2\mu\bar{\eta}_{27}-\bar{\eta}_{28}) + 2\beta_1\beta_3\sigma_{86,j}\bar{\eta}_{27}]$$

$$+G_1[\sigma_{67,j}(\bar{\eta}_{29}+n^2\bar{\eta}_{30}-\mu\bar{\eta}_{31}-\mu\bar{\eta}_{32}+\mu^2\bar{\eta}_{33}) + \beta_1\sigma_{73,j}(-\bar{\eta}_{31}-\bar{\eta}_{32}+2\mu\bar{\eta}_{33})$$

$$+\beta_1^2\sigma_{82,j}\bar{\eta}_{33}] + G_2[\sigma_{70,j}(\bar{\eta}_{29}+n^2\bar{\eta}_{30}-\mu\bar{\eta}_{31}-\mu\bar{\eta}_{32}+\mu^2\bar{\eta}_{33})+\beta_1\sigma_{74,j}(-\bar{\eta}_{32}$$

$$+\mu\bar{\eta}_{33}) + \beta_2\sigma_{76,j}(-\bar{\eta}_{31}+\mu\bar{\eta}_{33}) + \beta_1\beta_2\sigma_{85,j}\bar{\eta}_{33}] + G_3[\sigma_{71,j}(\bar{\eta}_{29}+n^2\bar{\eta}_{30}$$

$$-\mu\bar{\eta}_{31}-\mu\bar{\eta}_{32}+\mu^2\bar{\eta}_{33}) + \beta_1\sigma_{75,j}(-\bar{\eta}_{32}+\mu\bar{\eta}_{33}) + \beta_3\sigma_{79,j}(-\bar{\eta}_{31}+\mu\bar{\eta}_{33})$$

$$+\beta_1\beta_3\sigma_{86,j}\bar{\eta}_{33}]\}$$

$$\frac{\partial V_M}{\partial F_2} = \pi\Sigma\{C_1X_1\beta_1\sigma_{43,j}\bar{\eta}_{23} + C_2X_1\beta_2\sigma_{41,j}\bar{\eta}_{23} + C_3X_1\beta_3\sigma_{45,j}\bar{\eta}_{23} + D_1n[-\beta_1\sigma_{49,j}\bar{\eta}_{45}$$

$$-\beta_2\sigma_{55,j}\bar{\eta}_{40} + \sigma_{59,j}(\bar{\eta}_{38}-\mu\bar{\eta}_{40})] + D_2n[-\beta_2\sigma_{47,j}\bar{\eta}_{45} - \beta_2\sigma_{53,j}\bar{\eta}_{40} + \sigma_{62,j}(\bar{\eta}_{38}$$

$$-\mu\bar{\eta}_{40})] + D_3n[-\beta_3\sigma_{51,j}\bar{\eta}_{45} - \beta_2\sigma_{57,j}\bar{\eta}_{40} + \sigma_{65,j}(\bar{\eta}_{38}-\mu\bar{\eta}_{40})] + E_1n[\beta_1\sigma_{49,j}\bar{\eta}_{43}$$

$$-\beta_2\sigma_{55,j}\bar{\eta}_{41} + \sigma_{59,j}(\bar{\eta}_{39}-\mu\bar{\eta}_{41})] + E_2n[\beta_2\sigma_{47,j}\bar{\eta}_{43} - \beta_2\sigma_{53,j}\bar{\eta}_{41} + \sigma_{62,j}(\bar{\eta}_{39}$$

$$-\mu\bar{\eta}_{41})] + E_3n[\beta_3\sigma_{51,j}\bar{\eta}_{43} - \beta_2\sigma_{57,j}\bar{\eta}_{41} + \sigma_{65,j}(\bar{\eta}_{39}-\mu\bar{\eta}_{41})] + F_1[2\sigma_{4,j}\bar{\eta}_{26}$$

$$+2\sigma_{70,j}(\bar{\eta}_{24}+n^2\bar{\eta}_{25}+\mu^2\bar{\eta}_{27}-\mu\bar{\eta}_{28}) + (\beta_1\sigma_{74,j}+\beta_2\sigma_{76,j})(2\mu\bar{\eta}_{27}-\bar{\eta}_{28})$$

$$+2\beta_1\beta_2\sigma_{85,j}\bar{\eta}_{27}] + 2F_2[\sigma_{2,j}\bar{\eta}_{26} + \sigma_{68,j}(\bar{\eta}_{24}+n^2\bar{\eta}_{25}+\mu^2\bar{\eta}_{27}-\mu\bar{\eta}_{28})$$

$$+\beta_2\sigma_{77,j}(2\mu\bar{\eta}_{27}-\bar{\eta}_{28}) + \beta_2^2\sigma_{83,j}\bar{\eta}_{27}] + F_3[2\sigma_{6,j}\bar{\eta}_{26}+2\sigma_{72,j}(\bar{\eta}_{24}+n^2\bar{\eta}_{25}$$

$$+\mu^2\bar{\eta}_{27}-\mu\bar{\eta}_{28}) + (\beta_2\sigma_{78,j}+\beta_3\sigma_{80,j})(2\mu\bar{\eta}_{27}-\bar{\eta}_{28}) + 2\beta_2\beta_3\sigma_{87,j}\bar{\eta}_{27}]$$

$$+G_1[\sigma_{70,j}(\bar{\eta}_{29}+n^2\bar{\eta}_{30}-\mu\bar{\eta}_{31}-\mu\bar{\eta}_{32}+\mu^2\bar{\eta}_{33}) + \beta_1\sigma_{74,j}(-\bar{\eta}_{31}+\mu\bar{\eta}_{32})$$

$$+\beta_2\sigma_{76,j}(-\bar{\eta}_{32}+\mu\bar{\eta}_{33}) + \beta_1\beta_2\sigma_{85,j}\bar{\eta}_{33}] + G_2[\sigma_{68,j}(\bar{\eta}_{29}+n^2\bar{\eta}_{30}-\mu\bar{\eta}_{31}$$

$$-\mu\bar{\eta}_{32}+\mu^2\bar{\eta}_{33}) + \beta_2\sigma_{77,j}(-\bar{\eta}_{31}-\bar{\eta}_{32}+2\mu\bar{\eta}_{33}) + \beta_2^2\sigma_{83,j}\bar{\eta}_{33}]$$

$$+G_3[\sigma_{72,j}(\bar{\eta}_{29}+n^2\bar{\eta}_{30}-\mu\bar{\eta}_{31}-\mu\bar{\eta}_{32}+\mu^2\bar{\eta}_{33}) + \beta_2\sigma_{78,j}(-\bar{\eta}_{32}+\mu\bar{\eta}_{33})$$

(Continued)

$$+\beta_3\sigma_{80,j}(-\bar{\eta}_{31}+\mu\bar{\eta}_{33}) + \beta_2\beta_3\sigma_{87,j}\bar{\eta}_{33}]\}$$

$$\frac{\partial V_M}{\partial F_3} = \pi\sum\{C_1X_1\beta_1\sigma_{44,j}\bar{\eta}_{23} + C_2X_1\beta_2\sigma_{45,j}\bar{\eta}_{23} + C_3X_1\beta_3\sigma_{42,j}\bar{\eta}_{23} + D_1n[-\beta_1\sigma_{50,j}\bar{\eta}_{45}$$

$$-\beta_3\sigma_{56,j}\bar{\eta}_{40} + \sigma_{60,j}(\bar{\eta}_{38}-\mu\bar{\eta}_{40})] + D_2n[-\beta_2\sigma_{51,j}\bar{\eta}_{45} - \beta_3\sigma_{57,j}\bar{\eta}_{40} + \sigma_{63,j}(\bar{\eta}_{38}$$

$$-\mu\bar{\eta}_{40})] + D_3n[-\beta_3\sigma_{48,j}\bar{\eta}_{45} - \beta_3\sigma_{54,j}\bar{\eta}_{40} + \sigma_{66,j}(\bar{\eta}_{38}-\mu\bar{\eta}_{40})] + E_1n[\beta_1\sigma_{50,j}\bar{\eta}_{43}$$

$$-\beta_3\sigma_{56,j}\bar{\eta}_{41} + \sigma_{60,j}(\bar{\eta}_{39}-\mu\bar{\eta}_{41})] + E_2n[\beta_2\sigma_{51,j}\bar{\eta}_{43} - \beta_3\sigma_{57,j}\bar{\eta}_{41} + \sigma_{63,j}(\bar{\eta}_{39}$$

$$-\mu\bar{\eta}_{41})] + E_3n[\beta_3\sigma_{48,j}\bar{\eta}_{43} - \beta_3\sigma_{54,j}\bar{\eta}_{41} + \sigma_{66,j}(\bar{\eta}_{39}-\mu\bar{\eta}_{41})] + F_1[2\sigma_{5,j}\bar{\eta}_{26}$$

$$+2\sigma_{71,j}(\bar{\eta}_{24} + n^2\bar{\eta}_{25} + \mu^2\bar{\eta}_{27} - \mu\bar{\eta}_{28}) + (\beta_1\sigma_{75,j} + \beta_3\sigma_{79,j})(2\mu\bar{\eta}_{27} - \bar{\eta}_{28})$$

$$+2\beta_1\beta_3\sigma_{86,j}\bar{\eta}_{27}] + F_2[2\sigma_{6,j}\bar{\eta}_{26} + 2\sigma_{72,j}(\bar{\eta}_{24} + n^2\bar{\eta}_{25} + \mu^2\bar{\eta}_{27} - \mu\bar{\eta}_{28})$$

$$+(\beta_2\sigma_{78,j} + \beta_3\sigma_{80,j})(2\mu\bar{\eta}_{27} - \bar{\eta}_{28}) + 2\beta_2\beta_3\sigma_{87,j}\bar{\eta}_{27}] + 2F_3[\sigma_{3,j}\bar{\eta}_{26}$$

$$+\sigma_{69,j}(\bar{\eta}_{24} + n^2\bar{\eta}_{25} + \mu^2\bar{\eta}_{27} - \mu\bar{\eta}_{28}) + \beta_3\sigma_{81,j}(2\mu\bar{\eta}_{27} - \bar{\eta}_{28}) + \beta_3^2\sigma_{84,j}\bar{\eta}_{27}]$$

$$+G_1[\sigma_{71,j}(\bar{\eta}_{29} + n^2\bar{\eta}_{30} - \mu\bar{\eta}_{31} - \mu\bar{\eta}_{32} + \mu^2\bar{\eta}_{33}) + \beta_1\sigma_{75,j}(-\bar{\eta}_{31} + \mu\bar{\eta}_{33})$$

$$+\beta_3\sigma_{79,j}(-\bar{\eta}_{32} + \mu\bar{\eta}_{33}) + \beta_1\beta_3\sigma_{86,j}\bar{\eta}_{33}] + G_2[\sigma_{72,j}(\bar{\eta}_{29} + n^2\bar{\eta}_{30} - \mu\bar{\eta}_{31}$$

$$-\mu\bar{\eta}_{32} + \mu^2\bar{\eta}_{33}) + \beta_2\sigma_{78,j}(-\bar{\eta}_{31} + \mu\bar{\eta}_{33}) + \beta_3\sigma_{80,j}(-\bar{\eta}_{32} + \mu\bar{\eta}_{33})$$

$$+\beta_2\beta_3\sigma_{87,j}\bar{\eta}_{33}] + G_3[\sigma_{69,j}(\bar{\eta}_{29} + n^2\bar{\eta}_{30} - \mu\bar{\eta}_{31} - \mu\bar{\eta}_{32} + \mu^2\bar{\eta}_{33})$$

$$+\beta_3\sigma_{81,j}(-\bar{\eta}_{31} - \bar{\eta}_{32} + 2\mu\bar{\eta}_{33}) + \beta_3^2\sigma_{84,j}\bar{\eta}_{33}]\}$$

$$\frac{\partial V_M}{\partial G_1} = \pi\sum\{C_1X_1\beta_1\sigma_{40,j}\bar{\eta}_{22} + C_2X_1\beta_2\sigma_{43,j}\bar{\eta}_{22} + C_3X_1\beta_3\sigma_{44,j}\bar{\eta}_{22} + D_1n[\beta_1\sigma_{46,j}\bar{\eta}_{43}$$

$$-\beta_1\sigma_{52,j}\bar{\eta}_{46} + \sigma_{58,j}(\bar{\eta}_{39}-\mu\bar{\eta}_{46})] + D_2n[\beta_3\sigma_{49,j}\bar{\eta}_{43} - \beta_1\sigma_{55,j}\bar{\eta}_{46} + \sigma_{61,j}(\bar{\eta}_{39}$$

$$-\mu\bar{\eta}_{46})] + D_3n[\beta_3\sigma_{50,j}\bar{\eta}_{43} - \beta_1\sigma_{56,j}\bar{\eta}_{46} + \sigma_{64,j}(\bar{\eta}_{39}-\mu\bar{\eta}_{46})] + E_1n[\beta_1\sigma_{46,j}\bar{\eta}_{44}$$

$$-\beta_1\sigma_{52,j}\bar{\eta}_{47} + \sigma_{58,j}(\bar{\eta}_{42}-\mu\bar{\eta}_{47})] + E_2n[\beta_2\sigma_{49,j}\bar{\eta}_{44} - \beta_1\sigma_{55,j}\bar{\eta}_{47} + \sigma_{61,j}(\bar{\eta}_{42}$$

$$-\mu\bar{\eta}_{47})] + E_3n[\beta_3\sigma_{50,j}\bar{\eta}_{44} - \beta_1\sigma_{56,j}\bar{\eta}_{47} + \sigma_{64,j}(\bar{\eta}_{42}-\mu\bar{\eta}_{47})] + F_1[\sigma_{67,j}(\bar{\eta}_{29}$$

(Continued)

$$+n^2\bar{\eta}_{30}-\mu\bar{\eta}_{31}-\mu\bar{\eta}_{32}+\mu^2\bar{\eta}_{33})+\beta_1\sigma_{73,j}(-\bar{\eta}_{31}-\bar{\eta}_{32}+2\mu\bar{\eta}_{33})+\beta_1^2\sigma_{82,j}\bar{\eta}_{33}]$$

$$+F_2[\sigma_{70,j}(\bar{\eta}_{29}+n^2\bar{\eta}_{30}-\mu\bar{\eta}_{31}-\mu\bar{\eta}_{32}+\mu^2\bar{\eta}_{33})+\beta_1\sigma_{74,j}(-\bar{\eta}_{31}+\mu\bar{\eta}_{33})$$

$$+\beta_2\sigma_{76,j}(-\bar{\eta}_{32}+\mu\bar{\eta}_{33})+\beta_1\beta_2\sigma_{85,j}\bar{\eta}_{33}]+F_3[\sigma_{71,j}(\bar{\eta}_{29}+n^2\bar{\eta}_{30}-\mu\bar{\eta}_{31}-\mu\bar{\eta}_{32}$$

$$+\mu^2\bar{\eta}_{33})+\beta_1\sigma_{75,j}(-\bar{\eta}_{31}+\mu\bar{\eta}_{33})+\beta_3\sigma_{79,j}(-\bar{\eta}_{32}+\mu\bar{\eta}_{33})+\beta_1\beta_3\sigma_{86,j}\bar{\eta}_{33}]$$

$$+2G_1[\sigma_{1,j}\bar{\eta}_{36}+\sigma_{67,j}(\bar{\eta}_{34}+n^2\bar{\eta}_{35}-\mu\bar{\eta}_{37}+\mu^2\bar{\eta}_{58})+\beta_1\sigma_{73,j}(-\bar{\eta}_{37}+2\mu\bar{\eta}_{58})$$

$$+\beta_1^2\sigma_{82,j}\bar{\eta}_{58}]+G_2[2\sigma_{4,j}\bar{\eta}_{36}+2\sigma_{70,j}(\bar{\eta}_{34}+n^2\bar{\eta}_{35}-\mu\bar{\eta}_{37}+\mu^2\bar{\eta}_{58})$$

$$+(\beta_1\sigma_{74,j}+\beta_2\sigma_{76,j})(-\bar{\eta}_{37}+2\mu\bar{\eta}_{58})+2\beta_1\beta_2\sigma_{85,j}\bar{\eta}_{58}]+G_3[2\sigma_{5,j}\bar{\eta}_{36}$$

$$+2\sigma_{71,j}(\bar{\eta}_{34}+n^2\bar{\eta}_{35}-\mu\bar{\eta}_{37}+\mu^2\bar{\eta}_{58})+(\beta_1\sigma_{75,j}+\beta_3\sigma_{79,j})(-\bar{\eta}_{37}$$

$$+2\mu\bar{\eta}_{58})+2\beta_1\beta_3\sigma_{86,j}\bar{\eta}_{58}]\}$$

$$\frac{\partial V_M}{\partial G_2}=\pi\sum\{C_1X_1\beta_1\sigma_{43,j}\bar{\eta}_{22}+C_2X_1\beta_2\sigma_{41,j}\bar{\eta}_{22}+C_3X_1\beta_3\sigma_{45,j}\bar{\eta}_{22}+D_1n[\beta_1\sigma_{49,j}\bar{\eta}_{43}$$

$$-\beta_2\sigma_{55,j}\bar{\eta}_{46}+\sigma_{59,j}(\bar{\eta}_{39}-\mu\bar{\eta}_{46})]+D_2n[\beta_2\sigma_{47,j}\bar{\eta}_{43}-\beta_2\sigma_{55,j}\bar{\eta}_{46}+\sigma_{62,j}(\bar{\eta}_{39}$$

$$-\mu\bar{\eta}_{46})]+D_3n[\beta_3\sigma_{51,j}\bar{\eta}_{43}-\beta_2\sigma_{57,j}\bar{\eta}_{46}+\sigma_{65,j}(\bar{\eta}_{39}-\mu\bar{\eta}_{46})]+E_1n[\beta_1\sigma_{49,j}\bar{\eta}_{44}$$

$$-\beta_2\sigma_{55,j}\bar{\eta}_{47}+\sigma_{59,j}(\bar{\eta}_{42}-\mu\bar{\eta}_{47})]+E_2n[\beta_2\sigma_{47,j}\bar{\eta}_{44}-\beta_2\sigma_{53,j}\bar{\eta}_{47}+\sigma_{62,j}(\bar{\eta}_{42}$$

$$-\mu\bar{\eta}_{47})]+E_3n[\beta_3\sigma_{51,j}\bar{\eta}_{44}-\beta_2\sigma_{57,j}\bar{\eta}_{47}+\sigma_{65,j}(\bar{\eta}_{42}-\mu\bar{\eta}_{47})]+F_1[\sigma_{70,j}(\bar{\eta}_{29}$$

$$+n^2\bar{\eta}_{30}-\mu\bar{\eta}_{31}-\mu\bar{\eta}_{32}+\mu^2\bar{\eta}_{33})+\beta_1\sigma_{74,j}(-\bar{\eta}_{32}+\mu\bar{\eta}_{33})+\beta_2\sigma_{76,j}(-\bar{\eta}_{31}+\mu\bar{\eta}_{33})$$

$$+\beta_1\beta_2\sigma_{85,j}\bar{\eta}_{33}]+F_2[\sigma_{68,j}(\bar{\eta}_{29}+n^2\bar{\eta}_{30}-\mu\bar{\eta}_{31}-\mu\bar{\eta}_{32}+\mu^2\bar{\eta}_{33})$$

$$+\beta_2\sigma_{77,j}(-\bar{\eta}_{31}-\bar{\eta}_{32}+2\mu\bar{\eta}_{33})+\beta_2^2\sigma_{83,j}\bar{\eta}_{33}]+F_3[\sigma_{72,j}(\bar{\eta}_{29}+n^2\bar{\eta}_{30}$$

$$-\mu\bar{\eta}_{31}-\mu\bar{\eta}_{32}+\mu^2\bar{\eta}_{33})+\beta_2\sigma_{78,j}(-\bar{\eta}_{31}+\mu\bar{\eta}_{33})+\beta_3\sigma_{80,j}(-\bar{\eta}_{32}+\mu\bar{\eta}_{33})$$

$$+\beta_2\beta_3\sigma_{87,j}\bar{\eta}_{33}]+G_1[2\sigma_{4,j}\bar{\eta}_{36}+2\sigma_{70,j}(\bar{\eta}_{34}+n^2\bar{\eta}_{35}-\mu\bar{\eta}_{37}+\mu^2\bar{\eta}_{58})$$

$$+(\beta_1\sigma_{74,j}+\beta_2\sigma_{76,j})(-\bar{\eta}_{37}+2\mu\bar{\eta}_{58})+2\beta_1\beta_2\sigma_{85,j}\bar{\eta}_{58}]+2G_2[\sigma_{2,j}\bar{\eta}_{36}$$

$$+\sigma_{68,j}(\bar{\eta}_{34}+n^2\bar{\eta}_{35}-\mu\bar{\eta}_{37}+\mu^2\bar{\eta}_{58})+\beta_2\sigma_{77,j}(-\bar{\eta}_{37}+2\mu\bar{\eta}_{58})$$

(Continued)

$$+\beta_2^2\sigma_{83,j}\bar{\eta}_{58}] + G_3[2\sigma_{6,j}\bar{\eta}_{36} + 2\sigma_{72,j}(\bar{\eta}_{34}+n^2\bar{\eta}_{35}-\mu\bar{\eta}_{37}+\mu^2\bar{\eta}_{58})$$

$$+(\beta_2\sigma_{78,j}+\beta_3\sigma_{80,j})(-\bar{\eta}_{37}+2\mu\bar{\eta}_{58}) + 2\beta_2\beta_3\sigma_{87,j}\bar{\eta}_{58}]\}$$

$$\frac{\partial V_M}{\partial G_3} = \pi\Sigma\{C_1X_1\beta_1\sigma_{44,j}\bar{\eta}_{22} + C_2X_1\beta_2\sigma_{45,j}\bar{\eta}_{22} + C_3X_1\beta_3\sigma_{42,j}\bar{\eta}_{22} + D_1n[\beta_1\sigma_{50,j}\bar{\eta}_{43}$$

$$-\beta_3\sigma_{56,j}\bar{\eta}_{46} + \sigma_{60,j}(\bar{\eta}_{39}-\mu\bar{\eta}_{46})] + D_2n[\beta_2\sigma_{51,j}\bar{\eta}_{43} - \beta_3\sigma_{57,j}\bar{\eta}_{46} + \sigma_{63,j}(\bar{\eta}_{39}$$

$$-\mu\bar{\eta}_{46})] + D_3n[\beta_3\sigma_{48,j}\bar{\eta}_{43} - \beta_3\sigma_{54,j}\bar{\eta}_{46} + \sigma_{66,j}(\bar{\eta}_{39}-\mu\bar{\eta}_{46})] + E_1n[\beta_1\sigma_{50,j}\bar{\eta}_{44}$$

$$-\beta_3\sigma_{56,j}\bar{\eta}_{47} + \sigma_{60,j}(\bar{\eta}_{42}-\mu\bar{\eta}_{47})] + E_2n[\beta_2\sigma_{51,j}\bar{\eta}_{44} - \beta_3\sigma_{57,j}\bar{\eta}_{47} + \sigma_{63,j}(\bar{\eta}_{42}$$

$$-\mu\bar{\eta}_{47})] + E_3n[\beta_3\sigma_{48,j}\bar{\eta}_{44} - \beta_3\sigma_{54,j}\bar{\eta}_{47} + \sigma_{66,j}(\bar{\eta}_{42}-\mu\bar{\eta}_{47})] + F_1[\sigma_{71,j}(\bar{\eta}_{29}$$

$$+n^2\bar{\eta}_{30}-\mu\bar{\eta}_{31}-\mu\bar{\eta}_{32}+\mu^2\bar{\eta}_{33}) + \beta_1\sigma_{75,j}(-\bar{\eta}_{32}+\mu\bar{\eta}_{33}) + \beta_3\sigma_{79,j}(-\bar{\eta}_{31}$$

$$+\mu\bar{\eta}_{33}) + \beta_1\beta_3\sigma_{86,j}\bar{\eta}_{33}] + F_2[\sigma_{72,j}(\bar{\eta}_{29}+n^2\bar{\eta}_{30}-\mu\bar{\eta}_{31}-\mu\bar{\eta}_{32}+\mu^2\bar{\eta}_{33})$$

$$+\beta_2\sigma_{78,j}(-\bar{\eta}_{32}+\mu\bar{\eta}_{33}) + \beta_3\sigma_{80,j}(-\bar{\eta}_{31}+\mu\bar{\eta}_{33}) + \beta_2\beta_3\sigma_{87,j}\bar{\eta}_{33}] + F_3[\sigma_{69,j}(\bar{\eta}_{29}$$

$$+n^2\bar{\eta}_{30}-\mu\bar{\eta}_{31}-\mu\bar{\eta}_{32}+\mu^2\bar{\eta}_{33}) + \beta_3\sigma_{81,j}(-\bar{\eta}_{31}-\bar{\eta}_{32}+2\mu\bar{\eta}_{33})$$

$$+\beta_3^2\bar{\eta}_{33}\sigma_{84,j}] + G_1[2\sigma_{5,j}\bar{\eta}_{36} + 2\sigma_{71,j}(\bar{\eta}_{34}+n^2\bar{\eta}_{35}-\mu\bar{\eta}_{37}+\mu^2\bar{\eta}_{58})$$

$$+(\beta_1\sigma_{75,j}+\beta_3\sigma_{79,j})(-\bar{\eta}_{37}+2\mu\bar{\eta}_{58}) + 2\beta_1\beta_3\sigma_{86,j}\bar{\eta}_{58}] + G_2[2\sigma_{6,j}\bar{\eta}_{36}$$

$$+2\sigma_{72,j}(\bar{\eta}_{34}+n^2\bar{\eta}_{35}-\mu\bar{\eta}_{37}+\mu^2\bar{\eta}_{58}) + (\beta_2\sigma_{78,j}+\beta_3\sigma_{80,j})(-\bar{\eta}_{37}+2\mu\bar{\eta}_{58})$$

$$+2\beta_2\beta_3\sigma_{87,j}\bar{\eta}_{58}] + 2G_3[\sigma_{3,j}\bar{\eta}_{36} + \sigma_{68,j}(\bar{\eta}_{34}+n^2\bar{\eta}_{35}-\mu\bar{\eta}_{37}+\mu^2\bar{\eta}_{58})$$

$$+\beta_3\sigma_{81,j}(-\bar{\eta}_{37}+2\mu\bar{\eta}_{58}) + \beta_3^2\sigma_{84,j}\bar{\eta}_{58}]\}$$

In order to obtain nondimensional elements in the stiffness matrix, the following nondimensional coefficients are introduced:

$$\eta_1 = 2(t/x_{1,1})(E_\theta'/E_x')$$

$$\eta_2 = 2(t/x_{1,1})(G_{x\theta}'/\bar{E}_x')$$

$$\eta_3 = 2(t/x_{1,1})[\mu_x' + \mu_\theta'(E_\theta'/E_x')]$$

$$\eta_4 = 2(t/x_{1,1})$$

$$\eta_5 = 2(t/x_{1,1})(K_\theta' G_{\theta z}'/\bar{E}_x')$$

$$\eta_6 = (h/x_{1,1})(K_\theta G_{\theta z}/\bar{E}_x')$$

$$\eta_7 = 2(t/x_{1,1})(K_x' G_{zx}'/\bar{E}_x')$$

$$\eta_8 = (h/x_{1,1})(K_x G_{zx}/\bar{E}_x')$$

$$\eta_9 = 2(h/x_{1,1})^2 \, (t/x_{1,1})(E_\theta'/E_x')$$

$$\eta_{10} = 2(h/x_{1,1})^2 \, (t/x_{1,1})(G_{x\theta}'/\bar{E}_x')$$

$$\eta_{11} = 2(h/x_{1,1})^2 \, (t/x_{1,1})$$

$$\eta_{12} = 2(h/x_{1,1})^2 \, (t/x_{1,1})[\mu_x' + \mu_\theta'(E_\theta'/E_x')]$$

$$\eta_{13} = 2(h/x_{1,1})(t/x_{1,1})(E_\theta'/E_x')[-2(h/x_{1,1})+(a/x_{1,1})]$$

$$\eta_{14} = 2(h/x_{1,1})(t/x_{1,1})(G_{x\theta}'/\bar{E}_x')[-2(h/x_{1,1})+(a/x_{1,1})]$$

$$\eta_{15} = 2(h/x_{1,1})(t/x_{1,1})[(t/x_{1,1})\mu_x' - (h/x_{1,1})\mu_\theta'(E\ /E\ )]$$

$$\eta_{16} = 2(h/x_{1,1})(t/x_{1,1})[\mu_\theta'(t/x_{1,1})(E_\theta'/E_x') - (h/x_{1,1})\mu_x']$$

$$\eta_{17} = 2(h/x_{1,1})(t/x_{1,1})[-2(h/x_{1,1})+(a/x_{1,1})]$$

$$\eta_{18} = 2(t/x_{1,1})(E_\theta'/E_x')[(h/x_{1,1})^2 + \tfrac{1}{3}(t/x_{1,1})^2 - (h/x_{1,1})(a/x_{1,1})]$$

$$\eta_{19} = 2(t/x_{1,1})(G_{x\theta}'/\bar{E}_x')[\tfrac{1}{3}(t/x_{1,1})^2 + 2(h/x_{1,1})^2 - 2(h/x_{1,1})(a/x_{1,1})]$$

$$\eta_{20} = 2(t/x_{1,1})[\mu_x' + \mu_\theta'(E_\theta'/E_x')][(h/x_{1,1})^2 + \tfrac{1}{3}(t/x_{1,1})^2 - (h/x_{1,1})(a/x_{1,1})]$$

$$\eta_{21} = 4(h/x_{1,1})^2 (t/x_{1,1})[(E_\theta'/E_x') + (G_{x\theta}'/\bar{E}_x')]$$

$$\eta_{22} = 2(h/x_{1,1})(t/x_{1,1})[(a/x_{1,1}) - 2(h/x_{1,1})][(E_\theta'/E_x') + (G_{x\theta}'/\bar{E}_x')]$$

$$\eta_{23} = 4(t/x_{1,1})[(h/x_{1,1})^2 + \tfrac{1}{3}(t/x_{1,1})^2 - (h/x_{1,1})(a/x_{1,1})][(E_\theta'/E_x') + (G_{x\theta}'/\bar{E}_x')$$

$$\eta_{24} = 2(t/x_{1,1})(G'_{x\theta}/\bar{E}'_x)\left[-\tfrac{1}{3}(t/x_{1,1})^2-(h/x_{1,1})^2+(h/x_{1,1})(a/x_{1,1})\right]$$

$$\eta_{25} = 2(h/x_{1,1})^2(t/x_{1,1})(K'_\theta G'_{\theta z}/\bar{E}'_x)$$

$$\eta_{26} = 2(h/x_{1,1})^2(t/x_{1,1})(E'_\theta/E'_x)$$

$$\eta_{27} = 2(t/x_{1,1})\left[(h/x_{1,1})^2+\tfrac{1}{3}(t/x_{1,1})-(h/x_{1,1})(a/x_{1,1})\right]$$

Using these coefficients, the following elements of the stiffness matrix, $\left[A\right]$, can be deduced from the twenty one equations, pp. 52-62.

$$a_{1,1} = \Sigma\left\{2\sigma_{67,j}\left[(\eta_1-\mu\eta_3+\mu^2\eta_4)\sin\alpha_j+n^2\eta_2\csc\alpha_j\right]+2\beta_1\sigma_{73,j}(-\eta_3+2\mu\eta_4)\sin\alpha_j+2\sigma_{82,j}\,\beta_2^2\eta_4\sin\alpha_j\right\}$$

$$a_{1,2} = \Sigma\left\{2\sigma_{70,j}\left[(\eta_1-\mu\eta_3+\mu^2\eta_4)\sin\alpha_j+n^2\eta_2\csc\alpha_j\right]+(\beta_1\sigma_{74,j}+\beta_2\sigma_{76,j})(-\eta_3+2\mu\eta_4)\sin\alpha_j+2\beta_1\beta_2\sigma_{85,j}\,\eta_4\sin\alpha_j\right\}$$

$$a_{1,3} = \Sigma\left\{2\sigma_{71,j}\left[(\eta_1-\mu\eta_3+\mu^2\eta_4)\sin\alpha_j+n^2\eta_2\csc\alpha_j\right]+(\beta_1\sigma_{75,j}+\beta_3\sigma_{79,j})(-\eta_3+2\mu\eta_4)\sin\alpha_j+2\beta_1\beta_3\sigma_{86,j}\,\eta_4\sin\alpha_j\right\}$$

$$a_{1,4} = n\,\Sigma\left\{-2\beta_1\sigma_{46,j}\eta_2-\beta_1\sigma_{52,j}\eta_3+\sigma_{58,j}\left[2(\eta_1+\eta_2)-\mu\eta_3\right]\right\}$$

$$a_{1,5} = n\,\Sigma\left\{-2\beta_2\sigma_{49,j}\eta_2-\beta_1\sigma_{55,j}\eta_3+\sigma_{61,j}\left[2(\eta_1+\eta_2)-\mu\eta_3\right]\right\}$$

$$a_{1,6} = n\,\Sigma\left\{-2\beta_3\sigma_{50,j}\eta_2-\beta_1\sigma_{56,j}\eta_3+\sigma_{64,j}\left[2(\eta_1+\eta_2)-\mu\eta_3\right]\right\}$$

$$a_{1,7} = \Sigma\left[-\beta_1\sigma_{52,j}\eta_3+\sigma_{58,j}(\eta_1-\mu\eta_3)\right]\cos\alpha_j$$

$$a_{1,8} = \Sigma\left[-\beta_1\sigma_{55,j}\eta_3+\sigma_{61,j}(\eta_1-\mu\eta_3)\right]\cos\alpha_j$$

$$a_{1,9} = \Sigma\left[-\beta_1\sigma_{56,j}\eta_3+\sigma_{64,j}(\eta_1-\mu\eta_3)\right]\cos\alpha_j$$

$$a_{1,10} = \ldots = a_{1,21} = 0$$

$$a_{2,2} = \Sigma\left\{2\sigma_{68,j}\left[(\eta_1-\mu\eta_3+\mu^2\eta_4)\sin\alpha_j+n^2\eta_2\csc\alpha_j\right]+2\beta_2\sigma_{77,j}(-\eta_3+2\mu\eta_4)\sin\alpha_j+2\beta_2^2\sigma_{83,j}\,\eta_4\sin\alpha_j\right\}$$

$$a_{2,3} = \Sigma\{2\sigma_{72,j}[(\eta_1 - \mu\eta_3 + \mu^2\eta_4)\sin\alpha_j + n^2\eta_2\csc\alpha_j] + (\beta_2\sigma_{78,j} + \beta_3\sigma_{80,j})(-\eta_3$$
$$+ 2\mu\eta_4)\sin\alpha_j + 2\beta_2\beta_3\sigma_{87,j}\eta_4\sin\alpha_j\}$$

$$a_{2,4} = \Sigma\, n\{-2\beta_1\sigma_{49,j}\eta_2 - \beta_2\sigma_{55,j}\eta_3 + \sigma_{59,j}[2(\eta_1+\eta_2) - \mu\eta_3]\}$$

$$a_{2,5} = \Sigma\, n\{-2\beta_2\sigma_{47,j}\eta_2 - \beta_2\sigma_{53,j}\eta_3 + \sigma_{62,j}[2(\eta_1+\eta_2) - \mu\eta_3]\}$$

$$a_{2,6} = \Sigma\, n\{-2\beta_3\sigma_{51,j}\eta_2 - \beta_2\sigma_{57,j}\eta_3 + \sigma_{65,j}[2(\eta_1+\eta_2) - \mu\eta_3]\}$$

$$a_{2,7} = \Sigma\,[-\beta_2\sigma_{55,j}\eta_3 + \sigma_{59,j}(\eta_1 - \mu\eta_3)]\cos\alpha_j$$

$$a_{2,8} = \Sigma\,[-\beta_2\sigma_{53,j}\eta_3 + \sigma_{62,j}(\eta_1 - \mu\eta_3)]\cos\alpha_j$$

$$a_{2,9} = \Sigma\,[-\beta_2\sigma_{57,j}\eta_3 + \sigma_{65,j}(\eta_1 - \mu\eta_3)]\cos\alpha_j$$

$$a_{2,10} = \cdot\ \cdot\ \cdot = a_{2,21} = 0$$

$$a_{3,3} = \Sigma\{2\sigma_{69,j}[(\eta_1 - \mu\eta_3 + \mu^2\eta_4)\sin\alpha_j + n^2\eta_2\csc\alpha_j] + 2\beta_3\sigma_{81,j}(-\eta_3$$
$$+ 2\mu\eta_4)\sin\alpha_j + 2\beta_3^2\sigma_{84,j}\eta_4\sin\alpha_j\}$$

$$a_{3,4} = n\,\Sigma\{-2\beta_1\sigma_{50,j}\eta_2 - \beta_3\sigma_{56,j}\eta_3 + \sigma_{60,j}[2(\eta_1+\eta_2) - \mu\eta_3]\}$$

$$a_{3,5} = n\,\Sigma\{-2\beta_2\sigma_{51,j}\eta_2 - \beta_3\sigma_{57,j}\eta_3 + \sigma_{63,j}[2(\eta_1+\eta_2) - \mu\eta_3]\}$$

$$a_{3,6} = n\,\Sigma\{-2\beta_3\sigma_{48,j}\eta_2 - \beta_3\sigma_{54,j}\eta_3 + \sigma_{66,j}[2(\eta_1+\eta_2) - \mu\eta_3]\}$$

$$a_{3,7} = \Sigma\,[-\beta_3\sigma_{56,j}\eta_3 + \sigma_{60,j}(\eta_1 - \mu\eta_3)]\cos\alpha_j$$

$$a_{3,8} = \Sigma\,[-\beta_3\sigma_{57,j}\eta_3 + \sigma_{63,j}(\eta_1 - \mu\eta_3)]\cos\alpha_j$$

$$a_{3,9} = \Sigma\,[-\beta_3\sigma_{54,j}\eta_3 + \sigma_{66,j}(\eta_1 - \mu\eta_3)]\cos\alpha_j$$

$$a_{3,10} = \cdot\ \cdot\ \cdot = a_{3,21} = 0$$

$$a_{4,4} = \Sigma\{2(\beta_1^2\sigma_{13,j} - 2\beta_1\sigma_{25,j})\eta_2\sin\alpha_j + 2\sigma_{19,j}[n^2\eta_1\csc\alpha_j + (\eta_5$$
$$+ \eta_6)\csc\alpha_j\cos^2\alpha_j + \eta_2]\}$$

$$a_{4,5} = 2\,\Sigma\{(\beta_1\beta_2\sigma_{16,j} - \beta_2\sigma_{26,j} - \beta_1\sigma_{28,j})\eta_2\sin\alpha_j + \sigma_{22,j}[n^2\eta_1\csc\alpha_j$$
$$+ (\eta_5+\eta_6)\csc\alpha_j\cos^2\alpha_j + \eta_2]\}$$

$$a_{4,6} = 2\Sigma\{(\beta_1\beta_3\sigma_{17,j} - \beta_3\sigma_{27,j} - \beta_1\sigma_{31,j})\eta_2 \sin\alpha_j + \sigma_{23,j}[n^2\eta_1\csc\alpha_j$$
$$+ (\eta_5+\eta_6)\csc\alpha_j\cos^2\alpha_j + \eta_2]\}$$

$$a_{4,7} = 2n\,\Sigma[\sigma_{19,j}(\eta_1+\eta_5+\eta_6)\cot\alpha_j]$$

$$a_{4,8} = 2n\,\Sigma[\sigma_{22,j}(\eta_1+\eta_5+\eta_6)\cot\alpha_j]$$

$$a_{4,9} = 2n\,\Sigma[\sigma_{23,j}(\eta_1+\eta_5+\eta_6)\cot\alpha_j]$$

$$a_{4,10} = -\Sigma(\sigma_{34,j}\eta_6\cos\alpha_j)$$

$$a_{4,11} = -\Sigma(\sigma_{37,j}\eta_6\cos\alpha_j)$$

$$a_{4,12} = -\Sigma(\sigma_{38,j}\eta_6\cos\alpha_j)$$

$$a_{4,13} = -2\Sigma(\sigma_{34,j}\eta_5\cos\alpha_j)$$

$$a_{4,14} = -2\Sigma(\sigma_{37,j}\eta_5\cos\alpha_j)$$

$$a_{4,15} = -2\Sigma(\sigma_{38,j}\eta_5\cos\alpha_j)$$

$$a_{4,16} = \cdots = a_{4,21} = 0$$

$$a_{5,5} = 2\Sigma\{(\beta_2^2\sigma_{14,j} - 2\beta_2\sigma_{29,j})\eta_2\sin\alpha_j + \sigma_{20,j}[n^2\eta_1\csc\alpha_j$$
$$+ (\eta_5+\eta_6)\csc\alpha_j\cos^2\alpha_j + \eta_2]\}$$

$$a_{5,6} = 2\Sigma\{(\beta_2\beta_3\sigma_{18,j} - \beta_3\sigma_{30,j} - \beta_2\sigma_{32,j})\eta_2\sin\alpha_j + \sigma_{24,j}[n^2\eta_1\csc\alpha_j$$
$$+ (\eta_5+\eta_6)\csc\alpha_j\cos^2\alpha_j + \eta_2]\}$$

$$a_{5,7} = 2n\,\Sigma[\sigma_{22,j}(\eta_1+\eta_5+\eta_6)\cot\alpha_j]$$

$$a_{5,8} = 2n\,\Sigma[\sigma_{20,j}(\eta_1+\eta_5+\eta_6)\cot\alpha_j]$$

$$a_{5,9} = 2n\,\Sigma[\sigma_{24,j}(\eta_1+\eta_5+\eta_6)\cot\alpha_j]$$

$$a_{5,10} = -\Sigma(\sigma_{37,j}\eta_6\cos\alpha_j)$$

$$a_{5,11} = -\Sigma(\sigma_{35,j}\eta_6\cos\alpha_j)$$

$$a_{5,12} = -\Sigma(\sigma_{39,j}\eta_6\cos\alpha_j)$$

$$a_{5,13} = -2\,\Sigma\,(\sigma_{37,j}\,\eta_5 \cos\alpha_j)$$

$$a_{5,14} = -2\,\Sigma\,(\sigma_{35,j}\,\eta_5 \cos\alpha_j)$$

$$a_{5,15} = -2\,\Sigma\,(\sigma_{39,j}\,\eta_5 \cos\alpha_j)$$

$$a_{5,16} = \ldots = a_{5,21} = 0$$

$$a_{6,6} = 2\,\Sigma\,\{(\beta_3^2\,\sigma_{15,j} - 2\beta_3\sigma_{33,j})\,\eta_2 \sin\alpha_j + \sigma_{21,j}\,[n^2\eta_1 \csc\alpha_j + (\eta_5+\eta_6)\csc\alpha_j \cos^2\alpha_j + \eta_2]\}$$

$$a_{6,7} = 2n\,\Sigma\,[\sigma_{23,j}\,(\eta_1+\eta_5+\eta_6)\cot\alpha_j]$$

$$a_{6,8} = 2n\,\Sigma\,[\sigma_{24,j}\,(\eta_1+\eta_5+\eta_6)\cot\alpha_j]$$

$$a_{6,9} = 2n\,\Sigma\,[\sigma_{21,j}\,(\eta_1+\eta_5+\eta_6)\cot\alpha_j]$$

$$a_{6,10} = -\Sigma\,(\sigma_{38,j}\,\eta_6 \cos\alpha_j)$$

$$a_{6,11} = -\Sigma\,(\sigma_{39,j}\,\eta_6 \cos\alpha_j)$$

$$a_{6,12} = -\Sigma\,(\sigma_{36,j}\,\eta_6 \cos\alpha_j)$$

$$a_{6,13} = -2\,\Sigma\,(\sigma_{38,j}\,\eta_5 \cos\alpha_j)$$

$$a_{6,14} = -2\,\Sigma\,(\sigma_{39,j}\,\eta_5 \cos\alpha_j)$$

$$a_{6,15} = -2\,\Sigma\,(\sigma_{36,j}\,\eta_5 \cos\alpha_j)$$

$$a_{6,16} = \ldots = a_{6,21} = 0$$

$$a_{7,7} = 2\,\Sigma\,\{\beta_1^2\sigma_{13,j}(\eta_7+\eta_8)\sin\alpha_j + \sigma_{19,j}\,[\eta_1 \csc\alpha_j \cos^2\alpha_j + n^2(\eta_5 + \eta_6)\csc\alpha_j]\}$$

$$a_{7,8} = 2\,\Sigma\,\{\beta_1\beta_2\,\sigma_{16,j}(\eta_7+\eta_8)\sin\alpha_j + \sigma_{22,j}\,[\eta_1 \csc\alpha_j \cos^2\alpha_j + n^2(\eta_5 + \eta_6)\csc\alpha_j]\}$$

$$a_{7,9} = 2\,\Sigma\,\{\beta_1\beta_3\,\sigma_{17,j}(\eta_7+\eta_8)\sin\alpha_j + \sigma_{23,j}\,[\eta_1 \csc\alpha_j \cos^2\alpha_j + n^2(\eta_5 + \eta_6)\csc\alpha_j]\}$$

$$a_{7,10} = -2n \sum (\sigma_{34,j}\,\eta_6)$$

$$a_{7,11} = -2n \sum (\sigma_{37,j}\,\eta_6)$$

$$a_{7,12} = -2n \sum (\sigma_{38,j}\,\eta_6)$$

$$a_{7,13} = -2n \sum (\sigma_{34,j}\,\eta_5)$$

$$a_{7,14} = -2n \sum (\sigma_{37,j}\,\eta_5)$$

$$a_{7,15} = -2n \sum (\sigma_{38,j}\,\eta_5)$$

$$a_{7,16} = 2\sum (\beta_1 \sigma_{40,j}\,\eta_8 \sin\alpha_j)$$

$$a_{7,17} = 2\sum (\beta_1 \sigma_{43,j}\,\eta_8 \sin\alpha_j)$$

$$a_{7,18} = 2\sum (\beta_1 \sigma_{44,j}\,\eta_8 \sin\alpha_j)$$

$$a_{7,19} = 2\sum (\beta_1 \sigma_{40,j}\,\eta_7 \sin\alpha_j)$$

$$a_{7,20} = 2\sum (\beta_1 \sigma_{43,j}\,\eta_7 \sin\alpha_j)$$

$$a_{7,21} = 2\sum (\beta_1 \sigma_{44,j}\,\eta_7 \sin\alpha_j)$$

$$a_{8,8} = 2\sum \{ \beta_2^2 \sigma_{14,j}\,(\eta_7 + \eta_8)\sin\alpha_j + \sigma_{20,j}[\eta_1 \csc\alpha_j \cos^2\alpha_j + n^2(\eta_5 + \eta_6)\csc\alpha_j]\}$$

$$a_{8,9} = 2\sum \{ \beta_2\beta_3 \sigma_{18,j}(\eta_7 + \eta_8)\sin\alpha_j + \sigma_{24,j}[\eta_1 \csc\alpha_j \cos^2\alpha_j + n^2(\eta_5 + \eta_6)\csc\alpha_j]\}$$

$$a_{8,10} = -2n \sum (\sigma_{37,j}\,\eta_6)$$

$$a_{8,11} = -2n \sum (\sigma_{35,j}\,\eta_6)$$

$$a_{8,12} = -2n \sum (\sigma_{39,j}\,\eta_6)$$

$$a_{8,13} = -2n \sum (\sigma_{37,j}\,\eta_5)$$

$$a_{8,14} = -2n \sum (\sigma_{35,j}\,\eta_5)$$

$$a_{8,15} = -2n \sum (\sigma_{39,j}\,\eta_5)$$

$$a_{8,16} = 2\Sigma\,(\beta_2\sigma_{43,j}\eta_8\sin\alpha_j)$$

$$a_{8,17} = 2\Sigma\,(\beta_2\sigma_{41,j}\eta_8\sin\alpha_j)$$

$$a_{8,18} = 2\Sigma\,(\beta_2\sigma_{45,j}\eta_8\sin\alpha_j)$$

$$a_{8,19} = 2\Sigma\,(\beta_2\sigma_{43,j}\eta_7\sin\alpha_j)$$

$$a_{8,20} = 2\Sigma\,(\beta_2\sigma_{41,j}\eta_7\sin\alpha_j)$$

$$a_{8,21} = 2\Sigma\,(\beta_2\sigma_{45,j}\eta_7\sin\alpha_j)$$

$$a_{9,9} = 2\Sigma\{\beta_3^2\sigma_{15,j}(\eta_7+\eta_8)\sin\alpha_j + \sigma_{21,j}[\eta_1\csc\alpha_j\cos^2\alpha_j + n^2(\eta_5 + \eta_6)\csc\alpha_j]\}$$

$$a_{9,10} = -2n\Sigma\,(\sigma_{38,j}\eta_6)$$

$$a_{9,11} = -2n\Sigma\,(\sigma_{39,j}\eta_6)$$

$$a_{9,12} = -2n\Sigma\,(\sigma_{36,j}\eta_6)$$

$$a_{9,13} = -2n\Sigma\,(\sigma_{38,j}\eta_5)$$

$$a_{9,14} = -2n\Sigma\,(\sigma_{39,j}\eta_5)$$

$$a_{9,15} = -2n\Sigma\,(\sigma_{36,j}\eta_5)$$

$$a_{9,16} = 2\Sigma\,(\beta_3\sigma_{44,j}\eta_8\sin\alpha_j)$$

$$a_{9,17} = 2\Sigma\,(\beta_3\sigma_{45,j}\eta_8\sin\alpha_j)$$

$$a_{9,18} = 2\Sigma\,(\beta_3\sigma_{42,j}\eta_8\sin\alpha_j)$$

$$a_{9,19} = 2\Sigma\,(\beta_3\sigma_{44,j}\eta_7\sin\alpha_j)$$

$$a_{9,20} = 2\Sigma\,(\beta_3\sigma_{45,j}\eta_7\sin\alpha_j)$$

$$a_{9,21} = 2\Sigma\,(\beta_3\sigma_{42,j}\eta_7\sin\alpha_j)$$

$$a_{10,10} = 2\Sigma\{[\sigma_{7,j}\eta_6 + (\beta_1^2\sigma_{13,j} - \beta_1\sigma_{25,j})\eta_{10}]\sin\alpha_j + \sigma_{19,j}(\eta_{25}\csc\alpha_j\cos^2\alpha_j + \eta_{10}\sin\alpha_j + n^2\eta_{26}\csc\alpha_j)\}$$

$$a_{11,15} = \Sigma\,[\,(-\beta_2\beta_3\sigma_{18,j} + \beta_3\sigma_{30,j} + \beta_2\sigma_{32,j})\eta_{14}\sin\alpha_j - \sigma_{24,j}(\eta_{14}\sin\alpha_j$$
$$+ n^2\eta_{13}\csc\alpha_j)]$$

$$a_{11,16} = n\,\Sigma\,[-2\beta_2\sigma_{49,j}\eta_{10} - \beta_1\sigma_{55,j}\eta_{12} + \sigma_{61,j}(\eta_{21} - \mu\eta_{12})]$$

$$a_{11,17} = n\,\Sigma\,[-2\beta_2\sigma_{47,j}\eta_{10} - \beta_2\sigma_{53,j}\eta_{12} + \sigma_{62,j}(\eta_{21} - \mu\eta_{12})]$$

$$a_{11,18} = n\,\Sigma\,[-2\beta_2\sigma_{51,j}\eta_{10} - \beta_3\sigma_{57,j}\eta_{12} + \sigma_{63,j}(\eta_{21} - \mu\eta_{12})]$$

$$a_{11,19} = n\,\Sigma\,[-\beta_2\sigma_{49,j}\eta_{14} - \beta_1\sigma_{55,j}\eta_{15} + \sigma_{61,j}(\eta_{22} - \mu\eta_{15})]$$

$$a_{11,20} = n\,\Sigma\,[-\beta_2\sigma_{47,j}\eta_{14} - \beta_2\sigma_{53,j}\eta_{15} + \sigma_{62,j}(\eta_{22} - \mu\eta_{15})]$$

$$a_{11,21} = n\,\Sigma\,[-\beta_2\sigma_{51,j}\eta_{14} - \beta_3\sigma_{57,j}\eta_{15} + \sigma_{63,j}(\eta_{22} - \mu\eta_{15})]$$

$$a_{12,12} = 2\,\Sigma\,\{\,[\,\sigma_{9,j}\eta_6 + (\beta_3^2\sigma_{15,j} - \beta_3\sigma_{33,j})\eta_{10}]\sin\alpha_j + \sigma_{21,j}(\eta_{25}\csc\alpha_j\cos^2\alpha_j$$
$$+ \eta_{10}\sin\alpha_j + n^2\eta_{26}\csc\alpha_j)\}$$

$$a_{12,13} = \Sigma\,[\,(-\beta_1\beta_3\sigma_{17,j} + \beta_3\sigma_{27,j} + \beta_1\sigma_{31,j})\eta_{14}\sin\alpha_j - \sigma_{23,j}(\eta_{14}\sin\alpha_j$$
$$+ n^2\eta_{13}\csc\alpha_j)]$$

$$a_{12,14} = \Sigma\,[\,(-\beta_2\beta_3\sigma_{18,j} + \beta_3\sigma_{30,j} + \beta_2\sigma_{32,j})\eta_{14}\sin\alpha_j - \sigma_{24,j}(\eta_{14}\sin\alpha_j$$
$$+ n^2\eta_{13}\csc\alpha_j)]$$

$$a_{12,15} = \Sigma\,[\,(-\beta_3^2\sigma_{15,j} + 2\beta_3\sigma_{33,j})\eta_{14}\sin\alpha_j - \sigma_{21,j}(\eta_{14}\sin\alpha_j + n^2\eta_{13}\csc\alpha_j)]$$

$$a_{12,16} = n\,\Sigma\,[-2\beta_3\sigma_{50,j}\eta_{10} - \beta_1\sigma_{56,j}\eta_{12} + \sigma_{64,j}(\eta_{21} - \mu\eta_{12})]$$

$$a_{12,17} = n\,\Sigma\,[-2\beta_3\sigma_{51,j}\eta_{10} - \beta_2\sigma_{57,j}\eta_{12} + \sigma_{65,j}(\eta_{21} - \mu\eta_{12})]$$

$$a_{12,18} = n\,\Sigma\,[-2\beta_3\sigma_{48,j}\eta_{10} - \beta_3\sigma_{54,j}\eta_{12} + \sigma_{66,j}(\eta_{21} - \mu\eta_{12})]$$

$$a_{12,19} = n\,\Sigma\,[-\beta_3\sigma_{50,j}\eta_{14} - \beta_1\sigma_{56,j}\eta_{15} + \sigma_{64,j}(\eta_{22} - \mu\eta_{15})]$$

$$a_{12,20} = n\,\Sigma\,[-\beta_3\sigma_{51,j}\eta_{14} - \beta_2\sigma_{57,j}\eta_{15} + \sigma_{65,j}(\eta_{22} - \mu\eta_{15})]$$

$$a_{12,21} = n\,\Sigma\,[-\beta_3\sigma_{48,j}\eta_{14} - \beta_3\sigma_{54,j}\eta_{15} + \sigma_{66,j}(\eta_{22} - \mu\eta_{15})]$$

$$a_{13,13} = 2\,\Sigma\,[\,(\sigma_{7,j}\eta_5 - \beta_1^2\sigma_{13,j}\eta_{24} + 2\beta_1\sigma_{25,j}\eta_{24})\sin\alpha_j + \sigma_{19,j}(-\eta_{24}\sin\alpha_j$$

(Continued)

$$+ n^2 \eta_{18} \csc\alpha_j)]$$

$$a_{13,14} = 2\Sigma[(\sigma_{10,j}\eta_5 - \beta_1\beta_2\sigma_{16,j}\eta_{24} + \beta_2\sigma_{26,j}\eta_{24} + \beta_1\sigma_{28,j}\eta_{24})\sin\alpha_j$$
$$+ \sigma_{22,j}(-\eta_{24}\sin\alpha_j + n^2\eta_{18}\csc\alpha_j)]$$

$$a_{13,15} = 2\Sigma[(\sigma_{11,j}\eta_5 - \beta_1\beta_3\sigma_{17,j}\eta_{24} + \beta_3\sigma_{27,j}\eta_{24} + \beta_1\sigma_{31,j}\eta_{24})\sin\alpha_j$$
$$+ \sigma_{23,j}(-\eta_{24}\sin\alpha_j + n^2\eta_{18}\csc\alpha_j)]$$

$$a_{13,16} = n\Sigma[-\beta_1\sigma_{46,j}\eta_{14} + \beta_1\sigma_{52,j}\eta_{16} + \sigma_{58,j}(\eta_{22} + \mu\eta_{16})]$$

$$a_{13,17} = n\Sigma[-\beta_1\sigma_{49,j}\eta_{14} + \beta_2\sigma_{55,j}\eta_{16} + \sigma_{59,j}(\eta_{22} + \mu\eta_{16})]$$

$$a_{13,18} = n\Sigma[-\beta_1\sigma_{50,j}\eta_{14} + \beta_3\sigma_{56,j}\eta_{16} + \sigma_{60,j}(\eta_{22} + \mu\eta_{16})]$$

$$a_{13,19} = n\Sigma[2\beta_1\sigma_{46,j}\eta_{24} - \beta_1\sigma_{52,j}\eta_{20} + \sigma_{58,j}(\eta_{23} - \mu\eta_{20})]$$

$$a_{13,20} = n\Sigma[2\beta_1\sigma_{49,j}\eta_{24} - \beta_2\sigma_{55,j}\eta_{20} + \sigma_{59,j}(\eta_{23} - \mu\eta_{20})]$$

$$a_{13,21} = n\Sigma[2\beta_1\sigma_{50,j}\eta_{24} - \beta_3\sigma_{56,j}\eta_{20} + \sigma_{60,j}(\eta_{23} - \mu\eta_{20})]$$

$$a_{14,14} = 2\Sigma[(\sigma_{8,j}\eta_5 - \beta_2^2\sigma_{14,j}\eta_{24} + 2\beta_2\sigma_{29,j}\eta_{24})\sin\alpha_j + \sigma_{20,j}(-\eta_{24}\sin\alpha_j$$
$$+ n^2\eta_{18}\csc\alpha_j)]$$

$$a_{14,15} = 2\Sigma[(\sigma_{12,j}\eta_5 - \beta_2\beta_3\sigma_{18,j}\eta_{24} + \beta_3\sigma_{30,j}\eta_{24} + \beta_2\sigma_{32,j}\eta_{24})\sin\alpha_j$$
$$+ \sigma_{24,j}(-\eta_{24}\sin\alpha_j + n^2\eta_{18}\csc\alpha_j)]$$

$$a_{14,16} = n\Sigma[-\beta_2\sigma_{49,j}\eta_{14} + \beta_1\sigma_{55,j}\eta_{16} + \sigma_{61,j}(\eta_{22} + \mu\eta_{16})]$$

$$a_{14,17} = n\Sigma[-\beta_2\sigma_{47,j}\eta_{14} + \beta_2\sigma_{53,j}\eta_{16} + \sigma_{62,j}(\eta_{22} + \mu\eta_{16})]$$

$$a_{14,18} = n\Sigma[-\beta_2\sigma_{51,j}\eta_{14} + \beta_3\sigma_{57,j}\eta_{16} + \sigma_{63,j}(\eta_{22} + \mu\eta_{16})]$$

$$a_{14,19} = n\Sigma[2\beta_2\sigma_{49,j}\eta_{24} - \beta_1\sigma_{55,j}\eta_{20} + \sigma_{61,j}(\eta_{23} - \mu\eta_{20})]$$

$$a_{14,20} = n\Sigma[2\beta_2\sigma_{47,j}\eta_{24} - \beta_2\sigma_{53,j}\eta_{20} + \sigma_{62,j}(\eta_{23} - \mu\eta_{20})]$$

$$a_{14,21} = n\Sigma[2\beta_2\sigma_{51,j}\eta_{24} - \beta_3\sigma_{57,j}\eta_{20} + \sigma_{63,j}(\eta_{23} - \mu\eta_{20})]$$

$$a_{15,15} = 2\Sigma[(\sigma_{9,j}\eta_5 - \beta_3^2\sigma_{15,j}\eta_{24} + 2\beta_3\sigma_{33,j}\eta_{24})\sin\alpha_j + \sigma_{21,j}(-\eta_{24}\sin\alpha_j$$

(Continued)

$$+ n^2 \eta_{18} \csc\alpha_j)]$$

$$a_{15,16} = n \Sigma [-\beta_3 \sigma_{50,j} \eta_{14} + \beta_1 \sigma_{56,j} \eta_{16} + \sigma_{64,j} (\eta_{22} + \mu \eta_{16})]$$

$$a_{15,17} = n \Sigma [-\beta_3 \sigma_{51,j} \eta_{14} + \beta_2 \sigma_{57,j} \eta_{16} + \sigma_{65,j} (\eta_{22} + \mu \eta_{16})]$$

$$a_{15,18} = n \Sigma [-\beta_3 \sigma_{48,j} \eta_{14} + \beta_3 \sigma_{54,j} \eta_{16} + \sigma_{66,j} (\eta_{22} + \mu \eta_{16})]$$

$$a_{15,19} = n \Sigma [2\beta_3 \sigma_{50,j} \eta_{24} - \beta_1 \sigma_{56,j} \eta_{20} + \sigma_{64,j} (\eta_{23} - \mu \eta_{20})]$$

$$a_{15,20} = n \Sigma [2\beta_3 \sigma_{51,j} \eta_{24} - \beta_2 \sigma_{57,j} \eta_{20} + \sigma_{65,j} (\eta_{23} - \mu \eta_{20})]$$

$$a_{15,21} = n \Sigma [2\beta_3 \sigma_{48,j} \eta_{24} - \beta_3 \sigma_{54,j} \eta_{20} + \sigma_{66,j} (\eta_{23} - \mu \eta_{20})]$$

$$a_{16,16} = 2 \Sigma \{ [\sigma_{1,j} \eta_8 + \beta_1 \sigma_{73,j} (2\mu\eta_{11} - \eta_{12}) + \beta_1^2 \sigma_{82,j} \eta_{11}] \sin\alpha_j$$

$$+ \sigma_{67,j} [(\eta_9 + \mu^2 \eta_{11} - \mu\eta_{12}) \sin\alpha_j + n^2 \eta_{10} \csc\alpha_j]\}$$

$$a_{16,17} = \Sigma \{ [2 \sigma_{4,j} \eta_8 + (\beta_1 \sigma_{74,j} + \beta_2 \sigma_{76,j})(2\mu\eta_{11} - \eta_{12}) + 2\beta_1\beta_2 \sigma_{85,j} \eta_{11}] \sin\alpha_j$$

$$+ 2\sigma_{70,j} [(\eta_9 + \mu^2 \eta_{11} - \mu\eta_{12}) \sin\alpha_j + n^2 \eta_{10} \csc\alpha_j]\}$$

$$a_{16,18} = \Sigma \{ [2\sigma_{5,j} \eta_8 + (\beta_1 \sigma_{75,j} + \beta_3 \sigma_{79,j})(2\mu\eta_{11} - \eta_{12}) + 2\beta_1\beta_3 \sigma_{86,j} \eta_{11}] \sin\alpha_j$$

$$+ 2\sigma_{71,j} [(\eta_9 + \mu^2 \eta_{11} - \mu\eta_{12}) \sin\alpha_j + n^2 \eta_{10} \csc\alpha_j]\}$$

$$a_{16,19} = \Sigma \{ \sigma_{67,j} [(\eta_{13} - \mu\eta_{15} - \mu\eta_{16} + \mu^2 \eta_{17}) \sin\alpha_j + n^2 \eta_{14} \csc\alpha_j]$$

$$+ \beta_1 \sigma_{73,j} (-\eta_{15} - \eta_{16} + 2\mu\eta_{17}) \sin\alpha_j + \sigma_{82,j} \beta_1^2 \eta_{17} \sin\alpha_j \}$$

$$a_{16,20} = \Sigma \{ \sigma_{70,j} [(\eta_{13} - \mu\eta_{15} - \mu\eta_{16} + \mu^2 \eta_{17}) \sin\alpha_j + n^2 \eta_{14} \csc\alpha_j] + \beta_1 \sigma_{74,j} (-\eta_{16}$$

$$+ \mu\eta_{17}) \sin\alpha_j + \beta_2 \sigma_{76,j} (-\eta_{15} + \mu\eta_{17}) \sin\alpha_j + \beta_1\beta_2 \sigma_{85,j} \eta_{17} \sin\alpha_j \}$$

$$a_{16,21} = \Sigma \{ \sigma_{71,j} [(\eta_{13} - \mu\eta_{15} - \mu\eta_{16} + \mu^2 \eta_{17}) \sin\alpha_j + n^2 \eta_{14} \csc\alpha_j] + \beta_1 \sigma_{75,j} (-\eta_{16}$$

$$+ \mu\eta_{17}) \sin\alpha_j + \beta_3 \sigma_{79,j} (-\eta_{15} + \mu\eta_{17}) \sin\alpha_j + \beta_1\beta_3 \sigma_{86,j} \eta_{17} \sin\alpha_j \}$$

$$a_{17,17} = 2 \Sigma \{ [\sigma_{2,j} \eta_8 + \beta_2 \sigma_{77,j} (2\mu\eta_{11} - \eta_{12}) + \beta_2^2 \sigma_{83,j} \eta_{11}] \sin\alpha_j$$

$$+ \sigma_{68,j} [(\eta_9 + \mu^2 \eta_{11} - \mu\eta_{12}) \sin\alpha_j + n^2 \eta_{10} \csc\alpha_j]\}$$

$$a_{17,18} = \Sigma \{ [2 \sigma_{6,j} \eta_8 + (\beta_2 \sigma_{78,j} + \beta_3 \sigma_{80,j})(2\mu\eta_{11} - \eta_{12}) + 2\beta_2\beta_3 \sigma_{87,j} \eta_{11}] \sin\alpha_j$$

(Continued)

$$+ 2\sigma_{72,j}\left[(\eta_9 + \mu^2\eta_{11} - \mu\eta_{12})\sin\alpha_j + n^2\eta_{10}\csc\alpha_j\right]\Big\}$$

$$a_{17,19} = \Sigma\Big\{\sigma_{70,j}\left[(\eta_{13} - \mu\eta_{15} - \mu\eta_{16} + \mu^2\eta_{17})\sin\alpha_j + n^2\eta_{14}\csc\alpha_j\right] + \beta_1\sigma_{74,j}(-\eta_{15}$$
$$+ \mu\eta_{17})\sin\alpha_j + \beta_2\sigma_{76,j}(-\eta_{16} + \mu\eta_{17})\sin\alpha_j + \beta_1\beta_2\sigma_{85,j}\eta_{17}\sin\alpha_j\Big\}$$

$$a_{17,20} = \Sigma\Big\{\sigma_{68,j}\left[(\eta_{13} - \mu\eta_{15} - \mu\eta_{16} + \mu^2\eta_{17})\sin\alpha_j + n^2\eta_{14}\csc\alpha_j\right]$$
$$+ \beta_2\sigma_{77,j}(-\eta_{15} - \eta_{16} + 2\mu\eta_{17})\sin\alpha_j + \beta_2^2\sigma_{83,j}\eta_{17}\sin\alpha_j\Big\}$$

$$a_{17,21} = \Sigma\Big\{\sigma_{72,j}\left[(\eta_{13} - \mu\eta_{15} - \mu\eta_{16} + \mu^2\eta_{17})\sin\alpha_j + n^2\eta_{14}\csc\alpha_j\right] + \beta_2\sigma_{78,j}(-\eta_{16}$$
$$+ \mu\eta_{17})\sin\alpha_j + \beta_3\sigma_{80,j}(-\eta_{15} + \mu\eta_{17})\sin\alpha_j + \beta_2\beta_3\sigma_{87,j}\eta_{17}\sin\alpha_j\Big\}$$

$$a_{18,18} = 2\Sigma\Big\{\left[\sigma_{3,j}\eta_8 + \beta_3\sigma_{81,j}(2\mu\eta_{11} - \eta_{12}) + \beta_3^2\sigma_{84,j}\eta_{11}\right]\sin\alpha_j$$
$$+ \sigma_{69,j}\left[(\eta_9 + \mu^2\eta_{11} - \mu\eta_{12})\sin\alpha_j + n^2\eta_{10}\csc\alpha_j\right]\Big\}$$

$$a_{18,19} = \Sigma\Big\{\sigma_{71,j}\left[(\eta_{13} - \mu\eta_{15} - \mu\eta_{16} + \mu^2\eta_{17})\sin\alpha_j + n^2\eta_{14}\csc\alpha_j\right] + \beta_1\sigma_{75,j}(-\eta_{15}$$
$$+ \mu\eta_{17})\sin\alpha_j + \beta_3\sigma_{79,j}(-\eta_{16} + \mu\eta_{17})\sin\alpha_j + \beta_1\beta_3\sigma_{86,j}\eta_{17}\sin\alpha_j\Big\}$$

$$a_{18,20} = \Sigma\Big\{\sigma_{72,j}\left[(\eta_{13} - \mu\eta_{15} - \mu\eta_{16} + \mu^2\eta_{17})\sin\alpha_j + n^2\eta_{14}\csc\alpha_j\right] + \beta_2\sigma_{78,j}(-\eta_{15}$$
$$+ \mu\eta_{17})\sin\alpha_j + \beta_3\sigma_{80,j}(-\eta_{16} + \mu\eta_{17})\sin\alpha_j + \beta_2\beta_3\sigma_{87,j}\eta_{17}\sin\alpha_j\Big\}$$

$$a_{18,21} = \Sigma\Big\{\sigma_{69,j}\left[(\eta_{13} - \mu\eta_{15} - \mu\eta_{16} + \mu^2\eta_{17})\sin\alpha_j + n^2\eta_{14}\csc\alpha_j\right]$$
$$+ \beta_3\sigma_{81,j}(-\eta_{15} - \eta_{16} + 2\mu\eta_{17})\sin\alpha_j + \beta_3^2\sigma_{84,j}\eta_{17}\sin\alpha_j\Big\}$$

$$a_{19,19} = 2\Sigma\Big\{\left[\sigma_{1,j}\eta_7 + \beta_1\sigma_{73,j}(-\eta_{20} + 2\mu\eta_{27}) + \beta_1^2\sigma_{82,j}\eta_{27}\right]\sin\alpha_j$$
$$+ \sigma_{67,j}\left[(\eta_{18} - \mu\eta_{20} + \mu^2\eta_{27})\sin\alpha_j + n^2\eta_{19}\csc\alpha_j\right]\Big\}$$

$$a_{19,20} = \Sigma\Big\{\left[2\sigma_{4,j}\eta_7 + (\beta_1\sigma_{74,j} + \beta_2\sigma_{76,j})(-\eta_{20} + 2\mu\eta_{27}) + 2\beta_1\beta_2\sigma_{85,j}\eta_{27}\right]\sin\alpha_j$$
$$+ 2\sigma_{70,j}\left[(\eta_{18} - \mu\eta_{20} + \mu^2\eta_{27})\sin\alpha_j + n^2\eta_{19}\csc\alpha_j\right]\Big\}$$

$$a_{19,21} = \Sigma\Big\{\left[2\sigma_{5,j}\eta_7 + (\beta_1\sigma_{75,j} + \beta_3\sigma_{79,j})(-\eta_{20} + 2\mu\eta_{27}) + 2\beta_1\beta_3\sigma_{86,j}\eta_{27}\right]\sin\alpha_j$$
$$+ 2\sigma_{71,j}\left[(\eta_{18} - \mu\eta_{20} + \mu^2\eta_{27})\sin\alpha_j + n^2\eta_{19}\csc\alpha_j\right]\Big\}$$

$$a_{20,20} = 2\Sigma\Big\{\left[\sigma_{2,j}\eta_7 + \beta_2\sigma_{77,j}(-\eta_{20} + 2\mu\eta_{27}) + \beta_2^2\sigma_{83,j}\eta_{27}\right]\sin\alpha_j$$

(Continued)

$$+ \sigma_{68,j}\left[(\eta_{18} - \mu\eta_{20} + \mu^2\eta_{27})\sin\alpha_j + n^2\eta_{19}\csc\alpha_j\right]\Big\}$$

$$a_{20,21} = \Sigma\Big\{\left[2\sigma_{6,j}\eta_7 + (\beta_2\sigma_{78,j} + \beta_3\sigma_{80,j})(-\eta_{20} + 2\mu\eta_{27}) + 2\beta_2\beta_3\sigma_{87,j}\eta_{27}\right]\sin\alpha_j$$

$$+ 2\sigma_{72,j}\left[(\eta_{18} - \mu\eta_{20} + \mu^2\eta_{27})\sin\alpha_j + n^2\eta_{19}\csc\alpha_j\right]\Big\}$$

$$a_{21,21} = 2\,\Sigma\Big\{\left[\sigma_{3,j}\eta_7 + \beta_3\sigma_{81,j}(-\eta_{20} + 2\mu\eta_{27}) + \beta_3^2\sigma_{84,j}\eta_{27}\right]\sin\alpha_j$$

$$+ \sigma_{68,j}\left[(\eta_{18} - \mu\eta_{20} + \mu^2\eta_{27})\sin\alpha_j + n^2\eta_{19}\csc\alpha_j\right]\Big\}$$

Thus the right-hand portions of the twenty one equations pp. 52-62 can be written in the following matrix form:

$$\bar{\bar{\lambda}} \begin{bmatrix} \\ & a_{ij} & \\ \\ \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ B_1 \\ \cdot \\ \cdot \\ \cdot \\ G_3 \end{bmatrix}$$

where $\bar{\bar{\lambda}} = \pi x_{1,1}^3 E_x'$, or,

$$\bar{\bar{\lambda}} \, [A] \{ \delta \}$$

3.7 Closure

Using the results from Sections 3.5 and 3.6, Equation (3-3) can be expressed as

$$\left(\bar{\bar{\lambda}} \, [A] - \bar{\lambda} \, [B] \right) \{ \delta \} = 0$$

or,

$$\left([A] - \lambda \, [B] \right) \{ \delta \} = 0 \tag{3-30}$$

where $$\lambda = \bar{\lambda} / \bar{\bar{\lambda}} = \omega^2 x_{1,1}^2 \rho' / E_x' \, .$$

The values of λ , the eigenvalues, which satisfy Equation (3-30), are the required values for any given free vibration problem.

CHAPTER 4

APPLICATION

TO

SPECIFIC PROBLEMS

4.1 Computer Analysis

The coefficients of the stiffness and inertia matrices were programmed for computation on an electronic digital computer (CDC 6600). The $[B]$ matrix was inverted and multiplied times the $[A]$ matrix to put Equation (3-5) into the usual form for solving for the eigenvalues. The equation thus obtained is

$$\left([T] - \lambda [I] \right) \{\delta\} = 0 \qquad (4\text{-}1)$$

where

$$[T] = [B]^{-1} [A]$$

$$[I] = \text{identity matrix}$$

An eigenvalue routine (48) was then used to solve for the eigenvalues, λ , and eigenvectors, $\{\delta\}$. Several checks were included in the eigenvalue routine to check on the quality of the matrix inversion process and to check the accuracy of the eigenvalues. The sum of the absolute values of all elements of the matrix

TABLE 4-1

MATERIAL AND GEOMETRICAL PROPERTIES

OF EXAMPLE PROBLEMS

Data No.	1 Ref.40 CONE	2 Ref.49 CONE	3 Ref.44 CONE	4 Ref.44 PARAB	5 PARAB	6 PARAB
α, DEG.	5	20	5	**	**	**
$x_{1,2}/x_{1,1}$	1.022	2.284	1.022	**	**	**
L/R_1	0.25	2.149	0.25	8.0	8.0	8.0
h/R_1	0	0	0.025	0.0150	0.0150	0.0150
t/R_1	0.0125	0.00233	0.001	0.00065	0.00065	0.00065
E'_θ/E'_x	1.0	1.0	1.0	1.0	0.673	1.490
$G'_{x\theta}/\bar{E}'_x$	0.3500	0.3500	0.242*	0.242*	0.184	0.184
$K'_\theta\ G'_{\theta z}/\bar{E}'_x$	0.2917	0.2877	0.242	0.242	0.184	0.184
$K'_x\ G'_{zx}/\bar{E}'_x$	0.2917	0.2877	0.242	0.242	0.184	0.184
μ'_x	0.3	0.3	0.12	0.12	0.196	0.132
μ'_θ	0.3	00.3	0.12	0.12	0.132	0.196
ρ/ρ'	0	0	0.03877	0.03877	0.03877	0.03877
$K_\theta\ \mathbb{G}_{\theta z}/\bar{E}'_x$	0	0	0.003551	0.003551	0.00441	0.00297
$K_x\ \mathbb{G}_{zx}/\bar{E}'_x$	0	0	0.003551	0.003551	0.00441	0.00297

*Obtained from Reference 50.

**These quantities are determined by the number of segments approximating the paraboloidal shell.

$$\sqrt{\lambda} = 26.233$$

when the effect of transverse shear was included.

Azar (44), also, solved the same problem using a three-term Rayleigh-Ritz approach. However, his analysis, when applied to a homogeneous shell, does not include the transverse shear effect. He found the frequency parameter to be

$$\sqrt{\lambda} = 26.191$$

4.3 Unsymmetric Vibrations of a Homogeneous Conical Shell

The 20° homogeneous conical shell, Data No. 2 in Table 4-1, was next considered. This same shell was previously solved theoretically by Seide (49) using a Rayleigh-Ritz stress-function procedure. Also, experimental data are given in the preceding reference. (The experimental data were attributed to V. I. Weingarten by Seide.) These results are shown in Figure 4-1.

Cohen (25), also, considered the same conical shell but only calculated the frequency for a circumferential wave number of two (n=2). His result is also shown in Figure 4-1.

Seide's theory included only the normal component of the translational inertia. This could account for his frequency being much higher than experimental values at n=2.

The analysis of this investigation is shown in Figure 4-1 for general trend reasons. It cannot be compared directly to Seide's theoretical curve and Weingarten's experimental values because of the use of slightly different boundary conditions.

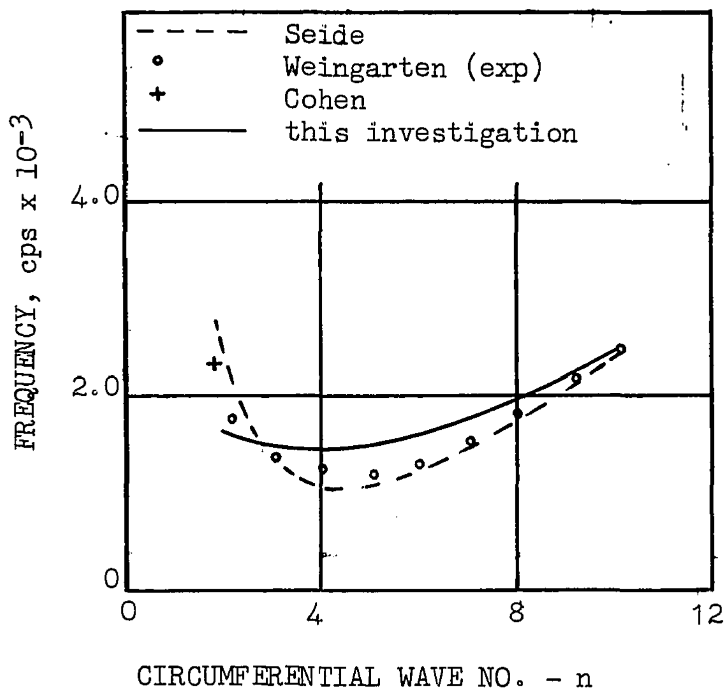

Figure 4-1. Graph showing frequency versus circumferential wave number for a conical shell.

4.4 Symmetric Vibrations of a Sandwich Conical Shell

A 5° sandwich conical shell, Data No. 3 in Table 4-1, was considered next. Only symmetric vibrations were calculated with the first six frequency parameters, $\sqrt{\lambda}$, listed below:

14.772
75.813
155.21
213.63
219.51
233.37

4.5 Symmetric Vibrations of a Sandwich Paraboloidal Shell

A paraboloidal shell was next considered. Approximating this shell of revolution by a series of conical segments as described in Section 3.3, the frequency parameters for symmetric vibrations were calculated to study the effect of the number of conical segments on the frequency parameter.

The specific paraboloidal shell shown was that found by revolving the curve

$$y^2 = R_1^2 \ (15x/L + 1)$$

about the x axis. (The x and y used here should not be confused with those relating to the shell coordinate system.) The material properties are given by Data No. 4, Table 4-1. $L/R_1 = 8$ was chosen with the shell lying on the interval $0 \leq x \leq L$. (L is the axial shell length and R_1 is the shell radius, normal to the axis of revolution at L/2.) The shell was divided at all times into equal segments (Figure 3-2). The results are shown in Table 4-2 and Figure 4-2.

TABLE 4-2

EFFECT OF THE NUMBER OF SEGMENTS

ON THE FREQUENCY PARAMETER

NO. OF SEGMENTS, K	LOWEST FREQUENCY PARAMETER, $\sqrt{\lambda}$
2	0.54113
4	0.49337
6	0.46618
8	0.44014
10	0.44191
12	0.44521

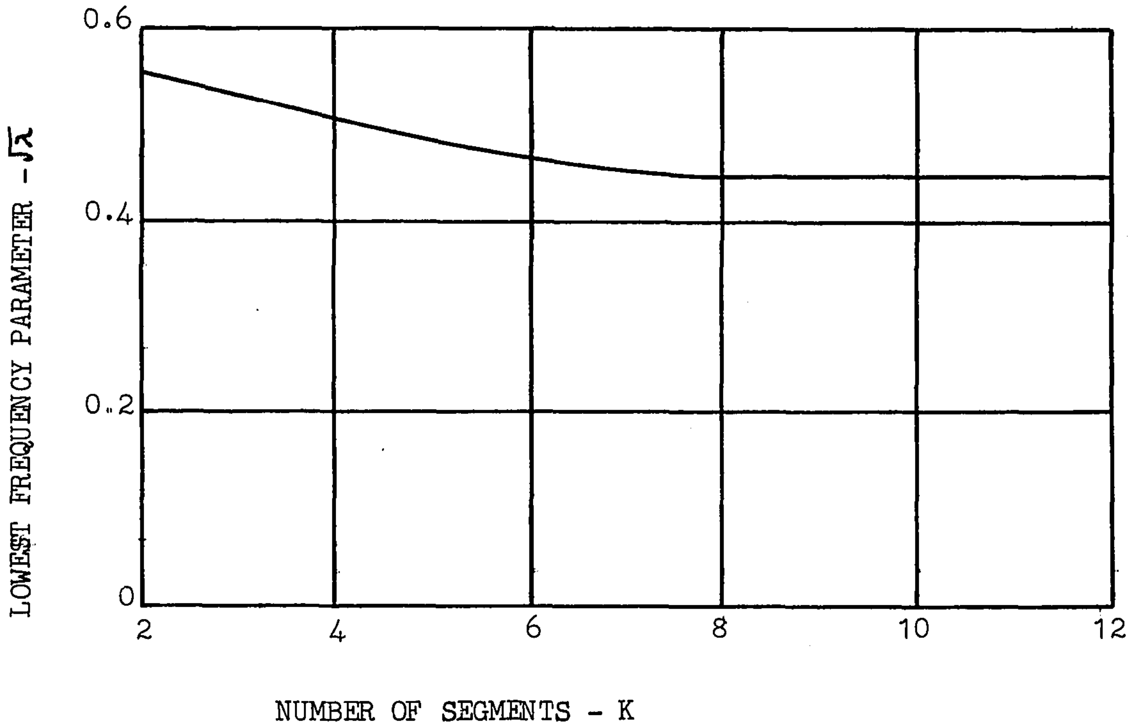

Figure 4-2. Graph showing the effect of the number of segments on the frequency parameter.

Figure 4-2 shows that as the number of conical segments increases, the frequency parameter decreases. The curve becomes relatively flat beyond eight segments. This is the same general effect reported by Azar (44).

4.6 Unsymmetric Vibrations of Two Sandwich Paraboloidal Shells with Orthotropic Facings

The theory of this investigation includes provisions for orthotropic facings and core. Paraboloidal shells with identical geometries to that of Section 4.5, but with orthotropic facings, were next considered. The material properties for the two shells are given by Data No.'s 5 and 6 in Table 4-1. The two shells differ only in the facing orientation. The first problem (Data No. 5) is stiffer in the meridional direction than in the circumferential direction. In the second problem (Data No. 6) the facings have been rotated 90° to make it stiffer in the circumferential direction. (See Appendix B for derivation of the facing properties.) In each problem the paraboloidal shell was divided into twelve segments. The results are tabulated for the first nine frequency parameters in Table 4-3.

Figure 4-3 gives a plot of the first five odd numbered frequency parameters. It shows that the facing orientation has little effect on the frequency parameter. However, modes five and seven show a considerable effect. Mode nine then shows that the effect again diminishes.

The effect of core orthotropy on the natural frequencies was not investigated here. However, the results of a recent analysis by Jacobson (51) indicate that orthotropic ratios $G_{\theta z}/G_{zx}$ in the range of 1.5 to 1.7 (typical of honeycomb cores) result in a lowest frequency (the only one he investigated) for a simply supported sandwich plate which is very close to that for the case of an isotropic core (orthotropic ratio of 1.0).

TABLE 4-3

FIRST NINE FREQUENCY PARAMETERS FOR
TWO SANDWICH PARABOLOIDAL SHELLS
FOR VARIOUS CIRCUMFERENTIAL
WAVE NUMBERS

$E'_\theta / E'_x = 0.673$

$\sqrt{\lambda}$ \ n	2	4	6	8	10	12
1	0.49635	0.55883	0.69124	1.2285	1.6697	2.0790
2	0.61759	1.0775	1.5075	1.9613	2.4274	2.8972
3	0.78036	1.1435	1.7154	2.2374	2.7138	3.1521
4	0.86916	1.4554	2.1870	2.9178	3.6472	4.3764
5	0.91275	1.9299	2.9483	3.9617	4.9724	5.9822
6	1.1754	2.5929	3.9464	5.2457	6.4664	7.6534
7	1.6712	3.1335	4.4786	5.6812	6.7573	7.7287
8	3.7763	4.3724	5.2453	6.3143	7.5238	8.8244
9	4.3485	7.6397	10.323	12.553	14.124	13.908

$E'_\theta / E'_x = 1.490$

$\sqrt{\lambda}$ \ n	2	4	6	8	10	12
1	0.43084	0.55101	0.70350	1.2280	1.6531	2.0453
2	0.67967	1.0017	1.3911	1.8156	2.2564	2.7035
3	0.91725	1.0582	1.5842	2.0625	2.5166	2.9641
4	1.0814	2.1792	3.2623	4.3456	5.4293	6.5130
5	1.3533	2.9118	4.1963	5.3602	6.4532	7.4976
6	1.5185	2.9277	4.4077	5.8735	7.2756	8.5750
7	1.7465	3.7728	5.2911	6.4791	7.7110	9.0769
8	4.0393	5.1272	6.5951	8.5544	10.617	12.670
9	4.3797	7.1934	9.7814	11.979	13.873	13.594

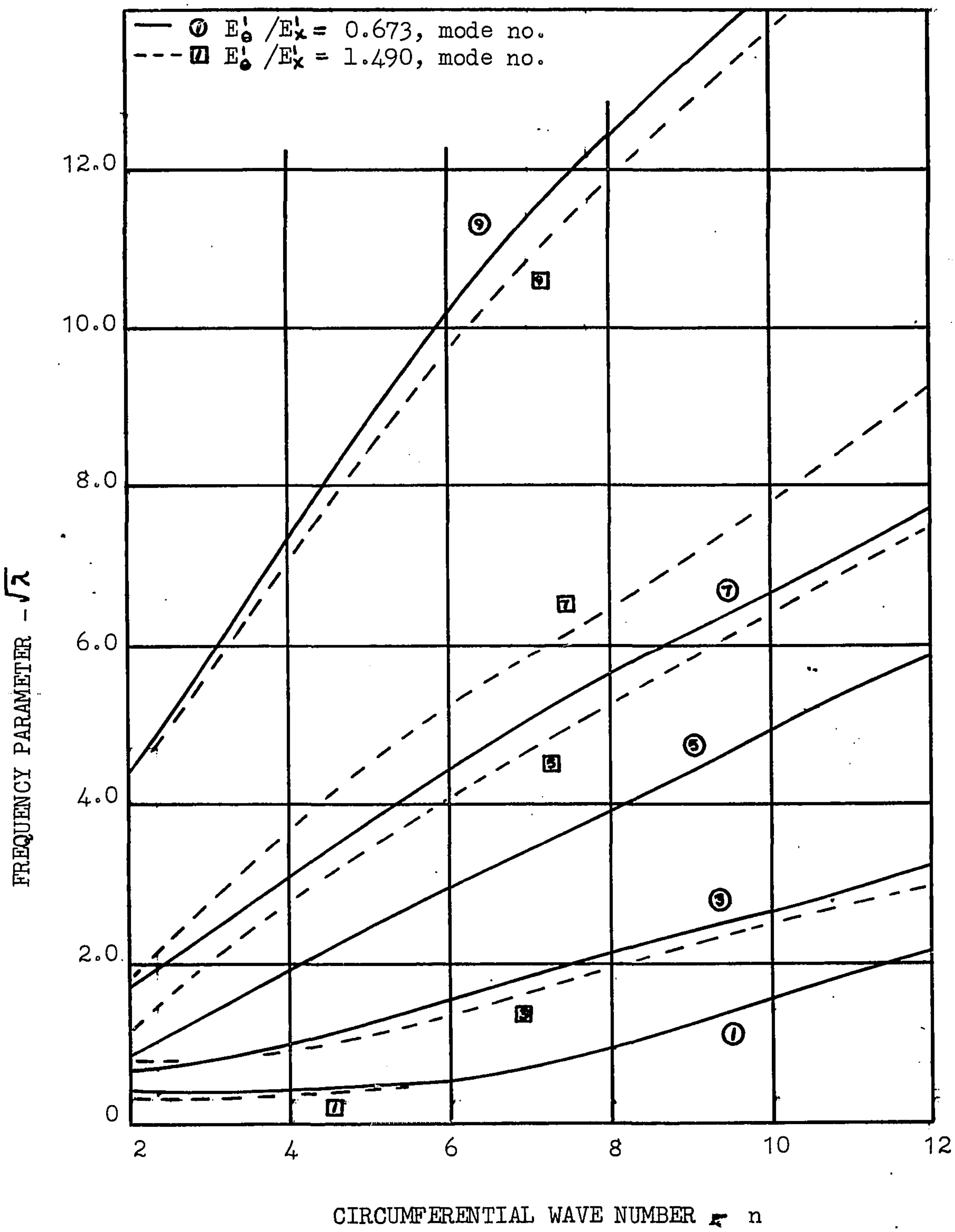

Figure 4-3. Graph showing $\sqrt{\lambda}$ versus the circumferential wave number for a sandwich paraboloidal shell.

CHAPTER 5

CLOSURE

5.1 Conclusions

A very general theory has been developed which can be used to analyze symmetric and unsymmetric free vibrations of sandwich shells of revolution. The theory includes the effect of transverse shear and rotatory inertia in the core and facings, and can accommodate orthotropy in both. A three-term Rayleigh-Ritz procedure is employed. The theoretical results were programmed for solution on a digital computer.

The theory developed is versatile in that by assuming a zero core thickness homogeneous shells can be considered as well as more complex irregular-shaped shells of revolution.

Several example problems were solved with good general agreement with results obtained previously by other authors. However, direct comparisons with the results of other authors is difficult since each one defines his own set of "realistic" boundary conditions. The boundary condition applied in this investigation is that of simply-supported ends in the sense that the normal and circumferential displacements are zero at the ends of the shell. Also, the meridional stress resultant and moment are assumed to be zero at the ends.

Garnet and Kempner used somewhat similar boundary conditions as used in this investigation in solving for the lowest frequency parameter for a homogeneous shell. The theory of this investigation gave a frequency parameter 2.6% lower than that found by Garnet and Kempner. This can be partially attributed to the use of three-term modal functions as opposed to two-term modal functions used by Garnet and Kempner.

Therefore, this dissertation provides a widely applicable theory for treating general sandwich and homogeneous shells of revolution.

5.2 Recommendations for Further Study

As is generally the case, research into a specific problem area raises many more questions than it solves. Such was the case with this investigation. Therefore, a listing of some of the areas which require further study is in order:

1. The effect of other boundary conditions such as clamped-clamped, clamped-free, attached mass, etc.
2. Free vibrations of a shell of revolution closed at the small end, as with a complete missile nose cap or radome.
3. More extensive study into the effect of the variation of parameters such as conicity for sandwich cones.
4. Effect of discrete structural stiffeners such as rings and stringers.
5. Forced vibrations on both homogeneous and sandwich shells of revolution (including conical shells).

LIST OF REFERENCES

1. Aron, H., "Das Gleichgewicht und die Bewegung einer unendlich dunnen beliebig gekrummten, elastischen Schale," J. reine und ang. Math. (Crelle's) 78, (1874).

2. Love, A. E. H., "On the Small Free Vibrations and Deformations of Thin Elastic Shells," Phil. Trans. Roy. Soc. 179(A), (1881) 491-546.

3. Mushtari, K. M. and Sachenkow, A. V., "Stability of Cylindrical and Conical Shells of Circular Cross Section, With Simultaneous Action of Axial Compression and External Normal Pressure," NACA TM 1433, April, 1958.

4. Love, A. E. H., A Treatise on the Mathematical Theory of Elasticity, Dover Publications, Inc., New York, 4th Ed., 1944.

5. Vlasov, V. Z., General Theory of Shells and Its Application to Engineering, NASA Translation TT F-99, 1964.

6. Timoshenko, S. and Woinowsky-Krieger, S., Theory of Plates and Shells, McGraw-Hill Book Co., Inc., New York, 2nd Ed., 1959.

7. Flügge, W., Statik und Dynamik der Schalen, Julius Springer, Berlin, 2nd Ed., 1957.

8. Flügge, W., Stresses in Shells, Springer-Verlag, Berlin, 1960.

9. Novozhilov, V. V., The Theory of Thin Shells, P. Noordhoff Ltd., Groningen, Netherlands, 1959.

10. Hildebrand, F. B., Reissner, E. and Thomas, G. B., "Notes on the Foundations of the Theory of Small Displacements of Orthotropic Shells," NACA Tech. Note 1833, 1949.

11. Vlasov, V. Z., "Basic Differential Equations in General Theory of Elastic Shells," NACA Tech. Memo. 1241, 1951.

12. Reissner, E., "Stress-Strain Relations in the Theory of Thin Elastic Shells," J. Math. Phys. 31 (1952), 109-119.

13. Naghdi, P. M., "On the Theory of Thin Elastic Shells," Quart. Appl. Math. 14 (1957), 369-380.

14. Naghdi, P. M., "The Effect of Transverse Shear Deformation on the Bending of Elastic Shells of Revolution," Quart. Appl. Math. 15 (1957), 41-52.

15. Budiansky, B. and Radkowski, P. P., "Numerical Analysis of Unsymmetrical Bending of Shells of Revolution," AIAA J. 1 (1963), 1833-1842.

16. Cohen, G. A., "Computer Analysis of Asymmetrical Deformation of Orthotropic Shells of Revolution," AIAA J. 2 (1964), 932-934.

17. Epstein, P. S., "On the Theory of Elastic Vibrations in Plates and Shells," J. Math. and Phys. 21 (1942), 198-208.

18. Poisson, S. D., Memoires de l'Academie des Sciences (1829), p. 237.

19. Kirchhoff, G., J. reine und ang. Math. (Crelle's) 40 (1850), p. 51.

20. Love, A. E. H., Proc. London Math. Soc. 21 (1891), p. 119.

21. Saunders, H. and Paslay, P. R., "Inextensional Vibrations of a Sphere-Cone Shell Combination," J. Acous. Soc. Am. 31 (1959), 579-583.

22. Lin, Y. K. and Lee, F. A., "Vibrations of Thin Paraboloidal Shells of Revolution," J. Appl. Mech. 27 (1960), 743-744.

23. Garnet, H., Goldberg, M. A. and Salerno, V. L., "Torsional Vibrations of Shells of Revolution," J. Appl. Mech. 28 (1961), 571-573; discussion, 29 (1962), 595-596.

24. Shiraishi, N. and DiMaggio, F. L., "Perturbation Solution for the Axisymmetric Vibrations of Prolate Spheroidal Shells," J. Acous. Soc. Am. 34 (1962), 1725-1731.

25. Cohen, G. A., "Computer Analysis of Asymmetric Free Vibrations of Orthotropic Shells of Revolution," AIAA Paper 65-109, Jan., 1965.

26. Strutt, M. J. O., "Eigenschwingungen einer Kegelschale," Annalen der Physik, 5th Series, 17 (1933), 729-735.

27. Van Urk, A. T. and Hut, G. B., "Messung der Radialschwingungen von Aluminum Kegelschale," Annalen der Physik, 5th Series, 17 (1933), 915-920.

28. Baron, M. L. and Bleich, H. H., "Tables for Frequencies and Modes of Free Vibration of Infinitely Long Thin Cylindrical Shells," J. Appl. Mech. 21 (1954), 178-184.

43. Chu, H. N., "Vibrations of Honeycomb Sandwich Cylinders," J. Aero. Space Sci. 28 (1961), 930-940.

44. Azar, J. J., "Axisymmetric Free Vibrations of Sandwich Shells of Revolution," Ph.D. Dissertation, University of Oklahoma, 1965.

45. Habip, L. M., "A Survey of Modern Developments in the Analysis of Sandwich Structures," Appl. Mech. Rev. 18 (1965), 93-98.

46. Ref. 4, p. 54.

47. Dar, S. M., "Vibrations of Rectangular Sandwich Plates with Various Edge Conditions," Ph.D. Dissertation, Univ. of Oklahoma, 1964.

48. Ehrlich, L. W., "Eigenvalues and Eigenvectors of Complex Non-Hermitian Matrices Using the Direct and Inverse Power Methods of Matrix Deflation," F4 UTEX MATSUB, Control Data Corporation's Co-op Distribution Center.

49. Seide, P., "On the Free Vibrations of Simply Supported Truncated Conical Shells," Israel J. of Tech. 3, 50-61, 1965.

50. Norris, C. B., "Compressive Buckling Curves for Simply-Supported Sandwich Panels with Glass-Fabric-Laminate Facings with Honeycomb Cores," Forest Products Laboratory, Report No. 1867, Dec., 1958.

51. Jacobson, M. J., "Effects of Orthotropic Cores on the Free Vibrations of Sandwich Plates," paper presented at the 35th Symposium on Shock and Vibration, New Orleans, Louisiana, October 25-28, 1965.

52. Tsai, S. W., "Structural Behavior of Composite Materials," NASA CR-71, July, 1964.

LIST OF SYMBOLS

A_m, B_m, C_m, D_m E_m, F_m, G_m	= undetermined parameters
$[A]$	= stiffness matrix
a	= $h + t$
a_{ij}	= stiffness coefficients
$[B]$	= inertia matrix
b_{ij}	= inertia coefficients
$\underline{C}_1, \underline{C}_2, \underline{C}_3$	= $\cos\beta_1 y$, $\cos\beta_2 y$, $\cos\beta_3 y$
E	= Young's modulus
$\bar{E}'_x$	= $E'_x / (1-\mu'_x \mu'_\theta)$
$\bar{E}'_\theta$	= $E'_\theta / (1-\mu'_x \mu'_\theta)$
e	= total strain
$F_x, F_\theta, F_{x\theta}$	= stress resultants (forces in tangential plane)
G	= shear modulus
h	= half core thickness
I	= mass moment of inertia
K	= number of conical segments
K_x, K_θ	= dynamic shear factor
L	= axial length of shell
M	= mass per unit area of shell composite
m, m'	= mass per unit area of core, facings

$M_x, M_\theta, M_{x\theta}$	= moment stress resultants
n	= circumferential wave number
Q_x, Q_θ	= stress resultants due to transverse shear
R_1	= parallel circle radius of shell at L/2
$\underline{S}_1, \underline{S}_2, \underline{S}_3$	= $\sin\beta_1 y, \sin\beta_2 y, \sin\beta_3 y$
T	= kinetic energy
t	= time; facing half thickness
U	= normal function of u
u	= displacement of middle surface in meridional direction
V	= strain energy; normal function of v
v	= displacement of middle surface in circumferential direction
W	= normal function of w
w	= displacement of middle surface in direction normal to shell
x	= meridional coordinate
y	= $\ln(x/x_{1,1})$
z	= normal coordinate
α	= half cone angle
β_m	= $m\pi/\ln(x_{K,2}/x_{1,1})$
$\{\delta\}$	= displacement vector (eigenvector)
ϵ	= membrane strain
$\eta, \bar{\eta}$	= coefficients defined pp. 63-64, 30-31
θ	= circumferential coordinate
λ	= eigenvalue = $\omega^2 x_{1,1}^2 \rho' / E_x'$
$\bar{\lambda}$	= $\pi\omega^2 x_{1,1}^5 \rho'$

$\bar{\bar{\lambda}}$	$= \pi x_{1,1}^3 E'_x$
μ'_x, μ'_θ	= Poisson's ratio
μ	$\equiv \mu'_x$
ρ	= mass density
Σ	$= \sum_{j=1}^{K}$
σ	= stresses
$\sigma_{i,j}$	= coefficients defined in Appendix A
τ_j	$= \ln(x_{j,2}/x_{1,1})$
ϕ_j	$= \ln(x_{j,1}/x_{1,1})$
χ	= change in curvature due to bending
ψ_x	= meridional rotation of a normal to the middle surface
ψ_θ	= circumferential rotation of a normal to the middle surface
ω	= circular frequency of vibration

APPENDIX A

INTEGRATION COEFFICIENTS

In order to simplify writing Equations (3-24,30), the following integration coefficients are defined:

$$\sigma_{1,j} = \int_{\phi_j}^{\tau_j} e^{2(1-\mu)y} \cos^2 \beta_1 y \, dy$$
$$= \frac{1}{4} \Big((1-\mu)^{-1}(e^{2(1-\mu)\tau_j} - e^{2(1-\mu)\phi_j}) + [(1-\mu)^2+\beta_1^2]^{-1} \Big\{ e^{2(1-\mu)\tau_j}[\beta_1 \sin 2\beta_1\tau_j$$
$$+ (1-\mu)\cos 2\beta_1\tau_j] - e^{2(1-\mu)\phi_j}[\beta_1 \sin 2\beta_1\phi_j + (1-\mu)\cos 2\beta_1\phi_j] \Big\} \Big)$$

$$\sigma_{2,j} = \int_{\phi_j}^{\tau_j} e^{2(1-\mu)y} \cos^2 \beta_2 y \, dy$$
$$= \frac{1}{4} \Big((1-\mu)^{-1}(e^{2(1-\mu)\tau_j} - e^{2(1-\mu)\phi_j}) + [(1-\mu)^2+\beta_2^2]^{-1} \Big\{ e^{2(1-\mu)\tau_j}[\beta_2 \sin 2\beta_2\tau_j$$
$$+ (1-\mu)\cos 2\beta_2\tau_j] - e^{2(1-\mu)\phi_j}[\beta_2 \sin 2\beta_2\phi_j + (1-\mu)\cos 2\beta_2\phi_j] \Big\} \Big)$$

$$\sigma_{3,j} = \int_{\phi_j}^{\tau_j} e^{2(1-\mu)y} \cos^2 \beta_3 y \, dy$$
$$= \frac{1}{4} \Big((1-\mu)^{-1}(e^{2(1-\mu)\tau_j} - e^{2(1-\mu)\phi_j}) + [(1-\mu)^2+\beta_3^2]^{-1} \Big\{ e^{2(1-\mu)\tau_j}[\beta_3 \sin 2\beta_3\tau_j$$
$$+ (1-\mu)\cos 2\beta_3\tau_j] - e^{2(1-\mu)\phi_j}[\beta_3 \sin 2\beta_3\phi_j + (1-\mu)\cos 2\beta_3\phi_j] \Big\} \Big)$$

$$\sigma_{4,j} = \int_{\phi_j}^{\tau_j} e^{2(1-\mu)y} \cos\beta_1 y \cos\beta_2 y \, dy$$
$$= \frac{1}{2} \Big([4(1-\mu)^2 + (\beta_1-\beta_2)^2]^{-1} \Big\{ e^{2(1-\mu)\tau_j}[(\beta_1-\beta_2)\sin(\beta_1-\beta_2)\tau_j + 2(1-\mu)\cos(\beta_1-\beta_2)\tau_j]$$
$$- e^{2(1-\mu)\phi_j}[(\beta_1-\beta_2)\sin(\beta_1-\beta_2)\phi_j + 2(1-\mu)\cos(\beta_1-\beta_2)\phi_j] \Big\}$$

$$+\left[4(1-\mu)^2+(\beta_1+\beta_2)^2\right]^{-1}\left\{e^{2(1-\mu)\Upsilon_j}\left[(\beta_1+\beta_2)\sin(\beta_1+\beta_2)\Upsilon_j+2(1-\mu)\cos(\beta_1+\beta_2)\Upsilon_j\right]\right.$$

$$\left.-e^{2(1-\mu)\phi_j}\left[(\beta_1+\beta_2)\sin(\beta_1+\beta_2)\phi_j+2(1-\mu)\cos(\beta_1+\beta_2)\phi_j\right]\right\}\Big)$$

$$\sigma_{5,j}=\int_{\phi_j}^{\Upsilon_j} e^{2(1-\mu)y}\cos\beta_1 y\cos\beta_3 y\,dy$$

$$=\frac{1}{2}\Big(\left[4(1-\mu)^2+(\beta_1-\beta_3)^2\right]^{-1}\left\{e^{2(1-\mu)\Upsilon_j}\left[(\beta_1-\beta_3)\sin(\beta_1-\beta_3)\Upsilon_j+2(1-\mu)\cos(\beta_1-\beta_3)\Upsilon_j\right]\right.$$

$$\left.-e^{2(1-\mu)\phi_j}\left[(\beta_1-\beta_3)\sin(\beta_1-\beta_3)\phi_j+2(1-\mu)\cos(\beta_1-\beta_3)\phi_j\right]\right\}$$

$$+\left[4(1-\mu)^2+(\beta_1+\beta_3)^2\right]^{-1}\left\{e^{2(1-\mu)\Upsilon_j}\left[(\beta_1+\beta_3)\sin(\beta_1+\beta_3)\Upsilon_j+2(1-\mu)\cos(\beta_1+\beta_3)\Upsilon_j\right]\right.$$

$$\left.-e^{2(1-\mu)\phi_j}\left[(\beta_1+\beta_3)\sin(\beta_1+\beta_3)\phi_j+2(1-\mu)\cos(\beta_1+\beta_3)\phi_j\right]\right\}\Big)$$

$$\sigma_{6,j}=\int_{\phi_j}^{\Upsilon_j} e^{2(1-\mu)y}\cos\beta_2 y\cos\beta_3 y\,dy$$

$$=\frac{1}{2}\Big(\left[4(1-\mu)^2+(\beta_2-\beta_3)^2\right]^{-1}\left\{e^{2(1-\mu)\Upsilon_j}\left[(\beta_2-\beta_3)\sin(\beta_2-\beta_3)\Upsilon_j+2(1-\mu)\cos(\beta_2-\beta_3)\Upsilon_j\right]\right.$$

$$\left.-e^{2(1-\mu)\phi_j}\left[(\beta_2-\beta_3)\sin(\beta_2-\beta_3)\phi_j+2(1-\mu)\cos(\beta_2-\beta_3)\phi_j\right]\right\}$$

$$+\left[4(1-\mu)^2+(\beta_2+\beta_3)^2\right]^{-1}\left\{e^{2(1-\mu)\Upsilon_j}\left[(\beta_2+\beta_3)\sin(\beta_2+\beta_3)\Upsilon_j+2(1-\mu)\cos(\beta_2+\beta_3)\Upsilon_j\right]\right.$$

$$\left.-e^{2(1-\mu)\phi_j}\left[(\beta_2+\beta_3)\sin(\beta_2+\beta_3)\phi_j+2(1-\mu)\cos(\beta_2+\beta_3)\phi_j\right]\right\}\Big)$$

$$\sigma_{7,j}=\int_{\phi_j}^{\Upsilon_j} e^{2y}\sin^2\beta_1 y\,dy$$

$$=\frac{1}{4}\left\{e^{2\Upsilon_j}-e^{2\phi_j}-(1+\beta_1^2)^{-1}\left[e^{2\Upsilon_j}(\beta_1\sin 2\beta_1\Upsilon_j+\cos 2\beta_1\Upsilon_j)\right.\right.$$

$$\left.\left.-e^{2\phi_j}(\beta_1\sin 2\beta_1\phi_j+\cos 2\beta_1\phi_j)\right]\right\}$$

$$\sigma_{8,j}=\int_{\phi_j}^{\Upsilon_j} e^{2y}\sin^2\beta_2 y\,dy$$

$$=\frac{1}{4}\left\{e^{2\Upsilon_j}-e^{2\phi_j}-(1+\beta_2^2)^{-1}\left[e^{2\Upsilon_j}(\beta_2\sin 2\beta_2\Upsilon_j+\cos 2\beta_2\Upsilon_j)\right.\right.$$

$$\left.\left.-e^{2\phi_j}(\beta_2\sin 2\beta_2\phi_j+\cos 2\beta_2\phi_j)\right]\right\}$$

$$\sigma_{9,j} = \int_{\Phi_j}^{\Upsilon_j} e^{2y} \sin^2 \beta_3 y \, dy$$

$$= \frac{1}{4}\Big\{ e^{2\Upsilon_j} - e^{2\Phi_j} - (1+\beta_3^2)^{-1} \big[e^{2\Upsilon_j} (\beta_3 \sin 2\beta_3 \Upsilon_j + \cos 2\beta_3 \Upsilon_j)$$

$$- e^{2\Phi_j} (\beta_3 \sin 2\beta_3 \Phi_j + \cos 2\beta_3 \Phi_j) \big] \Big\}$$

$$\sigma_{10,j} = \int_{\Phi_j}^{\Upsilon_j} e^{2y} \sin \beta_1 y \sin \beta_2 y \, dy$$

$$= \frac{1}{2}\Big([4+(\beta_1-\beta_2)^2]^{-1} \Big\{ e^{2\Upsilon_j} [(\beta_1-\beta_2) \sin(\beta_1-\beta_2)\Upsilon_j + 2\cos(\beta_1-\beta_2)\Upsilon_j]$$

$$- e^{2\Phi_j} [(\beta_1-\beta_2) \sin(\beta_1-\beta_2)\Phi_j + 2\cos(\beta_1-\beta_2)\Phi_j] \Big\}$$

$$- [4+(\beta_1+\beta_2)^2]^{-1} \Big\{ e^{2\Upsilon_j} [(\beta_1+\beta_2) \sin(\beta_1+\beta_2)\Upsilon_j + 2\cos(\beta_1+\beta_2)\Upsilon_j]$$

$$- e^{2\Phi_j} [(\beta_1+\beta_2) \sin(\beta_1+\beta_2)\Phi_j + 2\cos(\beta_1+\beta_2)\Phi_j] \Big\} \Big)$$

$$\sigma_{11,j} = \int_{\Phi_j}^{\Upsilon_j} e^{2y} \sin \beta_1 y \sin \beta_3 y \, dy$$

$$= \frac{1}{2}\Big([4+(\beta_1-\beta_3)^2]^{-1} \Big\{ e^{2\Upsilon_j} [(\beta_1-\beta_3) \sin(\beta_1-\beta_3)\Upsilon_j + 2\cos(\beta_1-\beta_3)\Upsilon_j]$$

$$- e^{2\Phi_j} [(\beta_1-\beta_3) \sin(\beta_1-\beta_3)\Phi_j + 2\cos(\beta_1-\beta_3)\Phi_j] \Big\}$$

$$- [4+(\beta_1+\beta_3)^2]^{-1} \Big\{ e^{2\Upsilon_j} [(\beta_1+\beta_3) \sin(\beta_1+\beta_3)\Upsilon_j + 2\cos(\beta_1+\beta_3)\Upsilon_j]$$

$$- e^{2\Phi_j} [(\beta_1+\beta_3) \sin(\beta_1+\beta_3)\Phi_j + 2\cos(\beta_1+\beta_3)\Phi_j] \Big\} \Big)$$

$$\sigma_{12,j} = \int_{\Phi_j}^{\Upsilon_j} e^{2y} \sin \beta_2 y \sin \beta_3 y \, dy$$

$$= \frac{1}{2}\Big([4+(\beta_2-\beta_3)^2]^{-1} \Big\{ e^{2\Upsilon_j} [(\beta_2-\beta_3) \sin(\beta_2-\beta_3)\Upsilon_j + 2\cos(\beta_2-\beta_3)\Upsilon_j]$$

$$- e^{2\Phi_j} [(\beta_2-\beta_3) \sin(\beta_2-\beta_3)\Phi_j + 2\cos(\beta_2-\beta_3)\Phi_j] \Big\}$$

$$- [4+(\beta_2+\beta_3)^2]^{-1} \Big\{ e^{2\Upsilon_j} [(\beta_2+\beta_3) \sin(\beta_2+\beta_3)\Upsilon_j + 2\cos(\beta_2+\beta_3)\Upsilon_j]$$

$$- e^{2\Phi_j} [(\beta_2+\beta_3) \sin(\beta_2+\beta_3)\Phi_j + 2\cos(\beta_2+\beta_3)\Phi_j] \Big\} \Big)$$

$$\sigma_{13,j} = \int_{\phi_j}^{\tau_j} \cos^2 \beta_1 y \, dy$$

$$= \frac{1}{2}(\tau_j - \phi_j) + (4\beta_1)^{-1}(\sin 2\beta_1\tau_j - \sin 2\beta_1\phi_j)$$

$$\sigma_{14,j} = \int_{\phi_j}^{\tau_j} \cos^2 \beta_2 y \, dy$$

$$= \frac{1}{2}(\tau_j - \phi_j) + (4\beta_2)^{-1}(\sin 2\beta_2\tau_j - \sin 2\beta_2\phi_j)$$

$$\sigma_{15,j} = \int_{\phi_j}^{\tau_j} \cos^2 \beta_3 y \, dy$$

$$= \frac{1}{2}(\tau_j - \phi_j) + (4\beta_3)^{-1}(\sin 2\beta_3\tau_j - \sin 2\beta_3\phi_j)$$

$$\sigma_{16,j} = \int_{\phi_j}^{\tau_j} \cos \beta_1 y \cos \beta_2 y \, dy$$

$$= \frac{1}{2}\Big\{(\beta_1-\beta_2)^{-1}[\sin(\beta_1-\beta_2)\tau_j - \sin(\beta_1-\beta_2)\phi_j] + (\beta_1+\beta_2)^{-1}[\sin(\beta_1+\beta_3)\tau_j - \sin(\beta_1+\beta_3)\phi_j]\Big\}$$

$$\sigma_{17,j} = \int_{\phi_j}^{\tau_j} \cos \beta_1 y \cos \beta_3 y \, dy$$

$$= \frac{1}{2}\Big\{(\beta_1-\beta_3)^{-1}[\sin(\beta_1-\beta_3)\tau_j - \sin(\beta_1-\beta_3)\phi_j] + (\beta_1+\beta_3)^{-1}[\sin(\beta_1+\beta_3)\tau_j - \sin(\beta_1+\beta_3)\phi_j]\Big\}$$

$$\sigma_{18,j} = \int_{\phi_j}^{\tau_j} \cos \beta_2 y \cos \beta_3 y \, dy$$

$$= \frac{1}{2}\Big\{(\beta_2-\beta_3)^{-1}[\sin(\beta_2-\beta_3)\tau_j - \sin(\beta_2-\beta_3)\phi_j] + (\beta_2+\beta_3)^{-1}[\sin(\beta_2+\beta_3)\tau_j - \sin(\beta_2+\beta_3)\phi_j]\Big\}$$

$$\sigma_{19,j} = \int_{\phi_j}^{\tau_j} \sin^2 \beta_1 y \, dy$$

$$= \tfrac{1}{2}(\tau_j - \phi_j) - (4\beta_1)^{-1}(\sin 2\beta_1 \tau_j - \sin 2\beta_1 \phi_j)$$

$$\sigma_{20,j} = \int_{\phi_j}^{\tau_j} \sin^2 \beta_2 y \, dy$$

$$= \tfrac{1}{2}(\tau_j - \phi_j) - (4\beta_2)^{-1}(\sin 2\beta_2 \tau_j - \sin 2\beta_2 \phi_j)$$

$$\sigma_{21,j} = \int_{\phi_j}^{\tau_j} \sin^2 \beta_3 y \, dy$$

$$= \tfrac{1}{2}(\tau_j - \phi_j) - (4\beta_3)^{-1}(\sin 2\beta_3 \tau_j - \sin 2\beta_3 \phi_j)$$

$$\sigma_{22,j} = \int_{\phi_j}^{\tau_j} \sin \beta_1 y \sin \beta_2 y \, dy$$

$$= \tfrac{1}{2}\Big\{(\beta_1 - \beta_2)^{-1}[\sin(\beta_1 - \beta_2)\tau_j - \sin(\beta_1 - \beta_2)\phi_j] - (\beta_1 + \beta_2)^{-1}[\sin(\beta_1 + \beta_2)\tau_j$$
$$- \sin(\beta_1 + \beta_2)\phi_j]\Big\}$$

$$\sigma_{23,j} = \int_{\phi_j}^{\tau_j} \sin \beta_1 y \sin \beta_3 y \, dy$$

$$= \tfrac{1}{2}\Big\{(\beta_1 - \beta_3)^{-1}[\sin(\beta_1 - \beta_3)\tau_j - \sin(\beta_1 - \beta_3)\phi_j] - (\beta_1 + \beta_3)^{-1}[\sin(\beta_1 + \beta_3)\tau_j$$
$$- \sin(\beta_1 + \beta_3)\phi_j]\Big\}$$

$$\sigma_{24,j} = \int_{\phi_j}^{\tau_j} \sin \beta_2 y \sin \beta_3 y \, dy$$

$$= \tfrac{1}{2}\Big\{(\beta_2 - \beta_3)^{-1}[\sin(\beta_2 - \beta_3)\tau_j - \sin(\beta_2 - \beta_3)\phi_j] - (\beta_2 + \beta_3)^{-1}[\sin(\beta_2 + \beta_3)\tau_j$$
$$- \sin(\beta_2 + \beta_3)\phi_j]\Big\}$$

$$\sigma_{25,j} = \int_{\phi_j}^{\tau_j} \sin\beta_1 y \cos\beta_1 y \, dy$$

$$= -(4\beta_1)^{-1}(\cos 2\beta_1\tau_j - \cos 2\beta_1\phi_j)$$

$$\sigma_{26,j} = \int_{\phi_j}^{\tau_j} \sin\beta_1 y \cos\beta_2 y \, dy$$

$$= -\frac{1}{2}\Big\{ (\beta_1-\beta_2)^{-1}\left[\cos(\beta_1-\beta_2)\tau_j - \cos(\beta_1-\beta_2)\phi_j\right] + (\beta_1+\beta_2)^{-1}\left[\cos(\beta_1+\beta_2)\tau_j \right.$$

$$\left. - \cos(\beta_1+\beta_2)\phi_j\right]\Big\}$$

$$\sigma_{27,j} = \int_{\phi_j}^{\tau_j} \sin\beta_1 y \cos\beta_3 y \, dy$$

$$= -\frac{1}{2}\Big\{ (\beta_1-\beta_3)^{-1}\left[\cos(\beta_1-\beta_3)\tau_j - \cos(\beta_1-\beta_3)\phi_j\right] + (\beta_1+\beta_3)^{-1}\left[\cos(\beta_1+\beta_3)\tau_j \right.$$

$$\left. - \cos(\beta_1+\beta_3)\phi_j\right]\Big\}$$

$$\sigma_{28,j} = \int_{\phi_j}^{\tau_j} \sin\beta_2 y \cos\beta_1 y \, dy$$

$$= -\frac{1}{2}\Big\{ (\beta_2-\beta_1)^{-1}\left[\cos(\beta_2-\beta_1)\tau_j - \cos(\beta_2-\beta_1)\phi_j\right] + (\beta_1+\beta_2)^{-1}\left[\cos(\beta_1+\beta_2)\tau_j \right.$$

$$\left. - \cos(\beta_1+\beta_2)\phi_j\right]\Big\}$$

$$\sigma_{29,j} = \int_{\phi_j}^{\tau_j} \sin\beta_2 y \cos\beta_2 y \, dy$$

$$= -(4\beta_2)^{-1}(\cos 2\beta_2\tau_j - \cos 2\beta_2\phi_j)$$

$$\sigma_{30,j} = \int_{\phi_j}^{\tau_j} \sin\beta_2 \cos\beta_3 y \, dy$$

$$= -\frac{1}{2}\Big\{ (\beta_2-\beta_3)^{-1}\left[\cos(\beta_2-\beta_3)\tau_j - \cos(\beta_2-\beta_3)\phi_j\right] + (\beta_2+\beta_3)^{-1}\left[\cos(\beta_2+\beta_3)\tau_j \right.$$

$$\left. - \cos(\beta_2+\beta_3)\phi_j\right]\Big\}$$

$$\sigma_{31,j} = \int_{\phi_j}^{\tau_j} \sin\beta_3 y \cos\beta_1 y \, dy$$

$$= -\tfrac{1}{2}\left\{ (\beta_3-\beta_1)^{-1}[\cos(\beta_3-\beta_1)\tau_j - \cos(\beta_3-\beta_1)\phi_j] + (\beta_1+\beta_3)^{-1}[\cos(\beta_1+\beta_3)\tau_j - \cos(\beta_1+\beta_3)\phi_j]\right\}$$

$$\sigma_{32,j} = \int_{\phi_j}^{\tau_j} \sin\beta_3 y \cos\beta_2 y \, dy$$

$$= -\tfrac{1}{2}\left\{ (\beta_3-\beta_2)^{-1}[\cos(\beta_3-\beta_2)\tau_j - \cos(\beta_3-\beta_2)\phi_j] + (\beta_2+\beta_3)^{-1}[\cos(\beta_2+\beta_3)\tau_j - \cos(\beta_2+\beta_3)\phi_j]\right\}$$

$$\sigma_{33,j} = \int_{\phi_j}^{\tau_j} \sin\beta_3 y \cos\beta_3 y \, dy$$

$$= -(4\beta_3)^{-1}(\cos 2\beta_3\tau_j - \cos 2\beta_3\phi_j)$$

$$\sigma_{34,j} = \int_{\phi_j}^{\tau_j} e^y \sin^2\beta_1 y \, dy$$

$$= \tfrac{1}{2}\left\{ (e^{\tau_j} - e^{\phi_j}) - (1+4\beta_1^2)^{-1}[e^{\tau_j}(2\beta_1 \sin 2\beta_1\tau_j + \cos 2\beta_1\tau_j) + e^{\phi_j}[2\beta_1 \sin 2\beta_1\phi_j + \cos 2\beta_1\phi_j)]\right\}$$

$$\sigma_{35,j} = \int_{\phi_j}^{\tau_j} e^y \sin^2\beta_2 y \, dy$$

$$= \tfrac{1}{2}\left\{ (e^{\tau_j} - e^{\phi_j}) - (1+4\beta_2^2)^{-1}[e^{\tau_j}(2\beta_2 \sin 2\beta_2\tau_j + \cos 2\beta_2\tau_j) + e^{\phi_j}[2\beta_2 \sin 2\beta_2\phi_j + \cos 2\beta_2\phi_j)]\right\}$$

$$\sigma_{36,j} = \int_{\phi_j}^{\tau_j} e^y \sin^2\beta_3 y \, dy$$

$$= \tfrac{1}{2}\left\{ (e^{\tau_j} - e^{\phi_j}) - (1+4\beta_3^2)^{-1}[e^{\tau_j}(2\beta_3 \sin 2\beta_3\tau_j + \cos 2\beta_3\tau_j) + e^{\phi_j}[2\beta_3 \sin 2\beta_3\phi_j + \cos 2\beta_3\phi_j)]\right\}$$

$$\sigma_{37,j} = \int_{\phi_j}^{\tau_j} e^{y} \sin \beta_1 y \sin \beta_2 y \, dy$$

$$= \frac{1}{2}\Big([1+(\beta_1-\beta_2)^2]^{-1}\{e^{\tau_j}[(\beta_1-\beta_2)\sin(\beta_1-\beta_2)\tau_j + \cos(\beta_1-\beta_2)\tau_j]$$

$$- e^{\phi_j}[(\beta_1-\beta_2)\sin(\beta_1-\beta_2)\phi_j + \cos(\beta_1-\beta_2)\phi_j]\}$$

$$- [1+(\beta_1+\beta_2)^2]^{-1}\{e^{\tau_j}[(\beta_1+\beta_2)\sin(\beta_1+\beta_2)\tau_j + \cos(\beta_1+\beta_2)\tau_j]$$

$$- e^{\phi_j}[(\beta_1+\beta_2)\sin(\beta_1+\beta_2)\phi_j + \cos(\beta_1+\beta_2)\phi_j]\}\Big)$$

$$\sigma_{38,j} = \int_{\phi_j}^{\tau_j} e^{y} \sin \beta_1 y \sin \beta_3 y \, dy$$

$$= \frac{1}{2}\Big([1+(\beta_1-\beta_3)^2]^{-1}\{e^{\tau_j}[(\beta_1-\beta_3)\sin(\beta_1-\beta_3)\tau_j + \cos(\beta_1-\beta_3)\tau_j]$$

$$- e^{\phi_j}[(\beta_1-\beta_3)\sin(\beta_1-\beta_3)\phi_j + \cos(\beta_1-\beta_3)\phi_j]\}$$

$$- [1+(\beta_1+\beta_3)^2]^{-1}\{e^{\tau_j}[(\beta_1+\beta_3)\sin(\beta_1+\beta_3)\tau_j + \cos(\beta_1+\beta_3)\tau_j]$$

$$- e^{\phi_j}[(\beta_1+\beta_3)\sin(\beta_1+\beta_3)\phi_j + \cos(\beta_1+\beta_3)\phi_j]\}\Big)$$

$$\sigma_{39,j} = \int_{\phi_j}^{\tau_j} e^{y} \sin \beta_2 y \sin \beta_3 y \, dy$$

$$= \frac{1}{2}\Big([1+(\beta_2-\beta_3)^2]^{-1}\{e^{\tau_j}[(\beta_2-\beta_3)\sin(\beta_2-\beta_3)\tau_j + \cos(\beta_2-\beta_3)\tau_j]$$

$$- e^{\phi_j}[(\beta_2-\beta_3)\sin(\beta_2-\beta_3)\phi_j + \cos(\beta_2-\beta_3)\phi_j]\}$$

$$- [1+(\beta_2+\beta_3)^2]^{-1}\{e^{\tau_j}[(\beta_2+\beta_3)\sin(\beta_2+\beta_3)\tau_j + \cos(\beta_2+\beta_3)\tau_j]$$

$$- e^{\phi_j}[(\beta_2+\beta_3)\sin(\beta_2+\beta_3)\phi_j + \cos(\beta_2+\beta_3)\phi_j]\}\Big)$$

$$\sigma_{40,j} = \int_{\phi_j}^{\tau_j} e^{(1-\mu)y} \cos^2 \beta_1 y \, dy$$

$$= \frac{1}{2}\Big((1-\mu)^{-1}(e^{(1-\mu)\tau_j} - e^{(1-\mu)\phi_j}) + [(1-\mu)^2 + 4\beta_1^2]^{-1}\{e^{(1-\mu)\tau_j}[2\beta_1 \sin 2\beta_1\tau_j$$

$$+ (1-\mu)\cos 2\beta_1\tau_j] - e^{(1-\mu)\phi_j}[2\beta_1 \sin 2\beta_1\phi_j + (1-\mu)\cos 2\beta_1\phi_j]\}\Big)$$

$$\sigma_{41,j} = \int_{\phi_j}^{\tau_j} e^{(1-\mu)y} \cos^2 \beta_2 y \, dy$$

$$= \frac{1}{2}\Big((1-\mu)^{-1}(e^{(1-\mu)\tau_j} - e^{(1-\mu)\phi_j}) + [(1-\mu)^2 + 4\beta_2^2]^{-1}\Big\{ e^{(1-\mu)\tau_j}[2\beta_2 \sin 2\beta_2\tau_j$$

$$+ (1-\mu)\cos 2\beta_2\tau_j] - e^{(1-\mu)\phi_j}[2\beta_2 \sin 2\beta_2\phi_j + (1-\mu)\cos 2\beta_2\phi_j]\Big\}\Big)$$

$$\sigma_{42,j} = \int_{\phi_j}^{\tau_j} e^{(1-\mu)y} \cos^2 \beta_3 y$$

$$= \frac{1}{2}\Big((1-\mu)^{-1}(e^{(1-\mu)\tau_j} - e^{(1-\mu)\phi_j}) + [(1-\mu)^2 + 4\beta_3^2]^{-1}\Big\{ e^{(1-\mu)\tau_j}[2\beta_3 \sin 2\beta_3\tau_j$$

$$+ (1-\mu)\cos 2\beta_3\tau_j] - e^{(1-\mu)\phi_j}[2\beta_3 \sin 2\beta_3\phi_j + (1-\mu)\cos 2\beta_3\phi_j]\Big\}\Big)$$

$$\sigma_{43,j} = \int_{\phi_j}^{\tau_j} e^{(1-\mu)y} \cos\beta_1 y \cos\beta_2 y \, dy$$

$$= \frac{1}{2}\Big([(1-\mu)^2 + (\beta_1-\beta_2)^2]^{-1}\Big\{ e^{(1-\mu)\tau_j}[(\beta_1-\beta_2)\sin(\beta_1-\beta_2)\tau_j + (1-\mu)\cos(\beta_1-\beta_2)\tau_j]$$

$$- e^{(1-\mu)\phi_j}[(\beta_1-\beta_2)\sin(\beta_1-\beta_2)\phi_j + (1-\mu)\cos(\beta_1-\beta_2)\phi_j]\Big\}$$

$$+ [(1-\mu)^2 + (\beta_1+\beta_2)^2]^{-1}\Big\{ e^{(1-\mu)\tau_j}[(\beta_1+\beta_2)\sin(\beta_1+\beta_2)\tau_j + (1-\mu)\cos(\beta_1+\beta_2)\tau_j]$$

$$- e^{(1-\mu)\phi_j}[(\beta_1+\beta_2)\sin(\beta_1+\beta_2)\phi_j + (1-\mu)\cos(\beta_1+\beta_2)\phi_j]\Big\}\Big)$$

$$\sigma_{44,j} = \int_{\phi_j}^{\tau_j} e^{(1-\mu)y} \cos\beta_1 y \cos\beta_3 y \, dy$$

$$= \frac{1}{2}\Big([(1-\mu)^2 + (\beta_1-\beta_3)^2]^{-1}\Big\{ e^{(1-\mu)\tau_j}[(\beta_1-\beta_3)\sin(\beta_1-\beta_3)\tau_j + (1-\mu)\cos(\beta_1-\beta_3)\tau_j]$$

$$- e^{(1-\mu)\phi_j}[(\beta_1-\beta_3)\sin(\beta_1-\beta_3)\phi_j + (1-\mu)\cos(\beta_1-\beta_3)\phi_j]\Big\}$$

$$+ [(1-\mu)^2 + (\beta_1+\beta_3)^2]^{-1}\Big\{ e^{(1-\mu)\tau_j}[(\beta_1+\beta_3)\sin(\beta_1+\beta_3)\tau_j + (1-\mu)\cos(\beta_1+\beta_3)\tau_j]$$

$$- e^{(1-\mu)\phi_j}[(\beta_1+\beta_3)\sin(\beta_1+\beta_3)\phi_j + (1-\mu)\cos(\beta_1+\beta_3)\phi_j]\Big\}\Big)$$

$$\sigma_{45,j} = \int_{\phi_j}^{\tau_j} e^{(1-\mu)y} \cos\beta_2 y \cos\beta_3 y \, dy$$

$$= \frac{1}{2}\Big([(1-\mu)^2 + (\beta_2-\beta_3)^2]^{-1}\Big\{ e^{(1-\mu)\tau_j}[(\beta_2-\beta_3)\sin(\beta_2-\beta_3)\tau_j + (1-\mu)\cos(\beta_2-\beta_3)\tau_j]$$

$$-e^{(1-\mu)\phi_j}\left[(\beta_2-\beta_3)\sin(\beta_2-\beta_3)\phi_j + (1-\mu)\cos(\beta_2-\beta_3)\phi_j\right]\Big\}$$

$$+\left[(1-\mu)^2+(\beta_2+\beta_3)^2\right]^{-1}\Big\{e^{(1-\mu)T_j}\left[(\beta_2+\beta_3)\sin(\beta_2+\beta_3)T_j + (1-\mu)\cos(\beta_2+\beta_3)T_j\right]$$

$$-e^{(1-\mu)\phi_j}\left[(\beta_2+\beta_3)\sin(\beta_2+\beta_3)\phi_j + (1-\mu)\cos(\beta_2+\beta_3)\phi_j\right]\Big\}\Big)$$

$$\sigma_{46,j} = \int_{\phi_j}^{T_j} e^{-\mu y}\cos^2\beta_1 y\, dy$$

$$= \frac{1}{2}\Big\{-\mu^{-1}(e^{-\mu T_j}-e^{-\mu\phi_j}) + (\mu^2+4\beta_1^2)^{-1}\left[e^{-\mu T_j}(2\beta_1\sin 2\beta_1 T_j - \mu\cos 2\beta_1 T_j)\right.$$

$$\left. - e^{-\mu\phi_j}(2\beta_1\sin 2\beta_1\phi_j - \mu\cos 2\beta_1\phi_j)\right]\Big\}$$

$$\sigma_{47,j} = \int_{\phi_j}^{T_j} e^{-\mu y}\cos^2\beta_2 y\, dy$$

$$= \frac{1}{2}\Big\{-\mu^{-1}(e^{-\mu T_j}-e^{-\mu\phi_j}) + (\mu^2+4\beta_2^2)^{-1}\left[e^{-\mu T_j}(2\beta_2\sin 2\beta_2 T_j - \mu\cos 2\beta_2 T_j)\right.$$

$$\left. - e^{-\mu\phi_j}(2\beta_2\sin 2\beta_2\phi_j - \mu\cos 2\beta_2\phi_j)\right]\Big\}$$

$$\sigma_{48,j} = \int_{\phi_j}^{T_j} e^{-\mu y}\cos^2\beta_3 y\, dy$$

$$= \frac{1}{2}\Big\{-\mu^{-1}(e^{-\mu T_j}-e^{-\mu\phi_j}) + (\mu^2+4\beta_3^2)^{-1}\left[e^{-\mu T_j}(2\beta_3\sin 2\beta_3 T_j - \mu\cos 2\beta_3 T_j)\right.$$

$$\left. - e^{-\mu\phi_j}(2\beta_3\sin 2\beta_3\phi_j - \mu\cos 2\beta_3\phi_j)\right]\Big\}$$

$$\sigma_{49,j} = \int_{\phi_j}^{T_j} e^{-\mu y}\cos\beta_1 y\cos\beta_2 y\, dy$$

$$= \frac{1}{2}\Big(\left[\mu^2+(\beta_1-\beta_2)^2\right]^{-1}\Big\{e^{-\mu T_j}\left[(\beta_1-\beta_2)\sin(\beta_1-\beta_2)T_j - \mu\cos(\beta_1-\beta_2)T_j\right]$$

$$-e^{-\mu\phi_j}\left[(\beta_1-\beta_2)\sin(\beta_1-\beta_2)\phi_j - \mu\cos(\beta_1-\beta_2)\phi_j\right]\Big\}$$

$$+\left[\mu^2+(\beta_1+\beta_2)^2\right]^{-1}\Big\{e^{-\mu T_j}\left[(\beta_1+\beta_2)\sin(\beta_1+\beta_2)T_j - \mu\cos(\beta_1+\beta_2)T_j\right]$$

$$-e^{-\mu\phi_j}\left[(\beta_1+\beta_2)\sin(\beta_1+\beta_2)\phi_j - \mu\cos(\beta_1+\beta_2)\phi_j\right]\Big\}\Big)$$

$$\sigma_{50,j} = \int_{\phi_j}^{T_j} e^{-\mu y}\cos\beta_1 y\cos\beta_3 y\,dy$$

$$= \frac{1}{2}\Big([\mu^2+(\beta_1-\beta_3)^2]^{-1}\{ e^{-\mu T_j}[(\beta_1-\beta_3)\sin(\beta_1-\beta_3)T_j - \mu\cos(\beta_1-\beta_3)T_j]$$

$$- e^{-\mu\phi_j}[(\beta_1-\beta_3)\sin(\beta_1-\beta_3)\phi_j - \mu\cos(\beta_1-\beta_3)\phi_j]\}$$

$$+ [\mu^2+(\beta_1+\beta_3)^2]^{-1}\{ e^{-\mu T_j}[(\beta_1+\beta_3)\sin(\beta_1+\beta_3)T_j - \mu\cos(\beta_1+\beta_3)T_j]$$

$$- e^{-\mu\phi_j}[(\beta_1+\beta_3)\sin(\beta_1+\beta_3)\phi_j - \mu\cos(\beta_1+\beta_3)\phi_j]\}\Big)$$

$$\sigma_{51,j} = \int_{\phi_j}^{T_j} e^{-\mu y}\cos\beta_2 y\cos\beta_3 y\,dy$$

$$= \frac{1}{2}\Big([\mu^2+(\beta_2-\beta_3)^2]^{-1}\{ e^{-\mu T_j}[(\beta_2-\beta_3)\sin(\beta_2-\beta_3)T_j - \mu\cos(\beta_2-\beta_3)T_j]$$

$$- e^{-\mu\phi_j}[(\beta_2-\beta_3)\sin(\beta_2-\beta_3)\phi_j - \mu\cos(\beta_2-\beta_3)\phi_j]\}$$

$$+ [\mu^2+(\beta_2+\beta_3)^2]^{-1}\{ e^{-\mu T_j}[(\beta_2+\beta_3)\sin(\beta_2+\beta_3)T_j - \mu\cos(\beta_2+\beta_3)T_j]$$

$$- e^{-\mu\phi_j}[(\beta_2+\beta_3)\sin(\beta_2+\beta_3)\phi_j - \mu\cos(\beta_2+\beta_3)\phi_j]\}\Big)$$

$$\sigma_{52,j} = \int_{\phi_j}^{T_j} e^{-\mu y}\sin^2\beta_1 y\,dy$$

$$= \frac{1}{2}\{ -\mu^{-1}(e^{-\mu T_j} - e^{-\mu\phi_j}) - (\mu^2+4\beta_1^2)^{-1}[e^{-\mu T_j}(2\beta_1\sin 2\beta_1 T_j - \mu\cos 2\beta_1 T_j)$$

$$- e^{-\mu\phi_j}(2\beta_1\sin 2\beta_1\phi_j - \mu\cos 2\beta_1\phi_j)]\}$$

$$\sigma_{53,j} = \int_{\phi_j}^{T_j} e^{-\mu y}\sin^2\beta_2 y\,dy$$

$$= \frac{1}{2}\{ -\mu^{-1}(e^{-\mu T_j} - e^{-\mu\phi_j}) - (\mu^2+4\beta_2^2)^{-1}[e^{-\mu T_j}(2\beta_2\sin 2\beta_2 T_j - \mu\cos 2\beta_2 T_j)$$

$$- e^{-\mu\phi_j}(2\beta_2\sin 2\beta_2\phi_j - \mu\cos 2\beta_2\phi_j)]\}$$

$$\sigma_{54,j} = \int_{\phi_j}^{T_j} e^{-\mu y}\sin^2\beta_3 y\,dy$$

$$= \frac{1}{2}\{ -\mu^{-1}(e^{-\mu T_j} - e^{-\mu\phi_j}) - (\mu^2+4\beta_3^2)^{-1}[e^{-\mu T_j}(2\beta_3\sin 2\beta_3 T_j - \mu\cos 2\beta_3 T_j)$$

$$-e^{-\mu\phi_j}(2\beta_3 \sin 2\beta_3\phi_j - \mu\cos 2\beta_3\phi_j)]\}$$

$$\sigma_{55,j} = \int_{\phi_j}^{\tau_j} e^{-\mu y}\sin\beta_1 y \sin\beta_2 y\, dy$$

$$= \frac{1}{2}\Big([\mu^2+(\beta_1-\beta_2)^2]^{-1}\{e^{-\mu\tau_j}[(\beta_1-\beta_2)\sin(\beta_1-\beta_2)\tau_j - \mu\cos(\beta_1-\beta_2)\tau_j]$$

$$-e^{-\mu\phi_j}[(\beta_1-\beta_2)\sin(\beta_1-\beta_2)\phi_j - \mu\cos(\beta_1-\beta_2)\phi_j]\}$$

$$-[\mu^2+(\beta_1+\beta_2)^2]^{-1}\{e^{-\mu\tau_j}[(\beta_1+\beta_2)\sin(\beta_1+\beta_2)\tau_j - \mu\cos(\beta_1+\beta_2)\tau_j$$

$$-e^{-\mu\phi_j}[(\beta_1+\beta_2)\sin(\beta_1+\beta_2)\phi_j - \mu\cos(\beta_1+\beta_2)\phi_j]\}\Big)$$

$$\sigma_{56,j} = \int_{\phi_j}^{\tau_j} e^{-\mu y}\sin\beta_1 y \sin\beta_3 y\, dy$$

$$= \frac{1}{2}\Big([\mu^2+(\beta_1-\beta_3)^2]^{-1}\{e^{-\mu\tau_j}[(\beta_1-\beta_3)\sin(\beta_1-\beta_3)\tau_j - \mu\cos(\beta_1-\beta_3)\tau_j]$$

$$-e^{-\mu\phi_j}[(\beta_1-\beta_3)\sin(\beta_1-\beta_3)\phi_j - \mu\cos(\beta_1-\beta_3)\phi_j]\}$$

$$-[\mu^2+(\beta_1+\beta_3)^2]^{-1}\{e^{-\mu\tau_j}[(\beta_1+\beta_3)\sin(\beta_1+\beta_3)\tau_j - \mu\cos(\beta_1+\beta_3)\tau_j]$$

$$-e^{-\mu\phi_j}[(\beta_1+\beta_3)\sin(\beta_1+\beta_3)\phi_j - \mu\cos(\beta_1+\beta_3)\phi_j]\}\Big)$$

$$\sigma_{57,j} = \int_{\phi_j}^{\tau_j} e^{-\mu y}\sin\beta_2 y \sin\beta_3 y\, dy$$

$$= \frac{1}{2}\Big([\mu^2+(\beta_2-\beta_3)^2]^{-1}\{e^{-\mu\tau_j}[(\beta_2-\beta_3)\sin(\beta_2-\beta_3)\tau_j - \mu\cos(\beta_2-\beta_3)\tau_j]$$

$$-e^{-\mu\phi_j}[(\beta_2-\beta_3)\sin(\beta_2-\beta_3)\phi_j - \mu\cos(\beta_2-\beta_3)\phi_j]\}$$

$$-[\mu^2+(\beta_2+\beta_3)^2]^{-1}\{e^{-\mu\tau_j}[(\beta_2+\beta_3)\sin(\beta_2+\beta_3)\tau_j - \mu\cos(\beta_2+\beta_3)\tau_j]$$

$$-e^{-\mu\phi_j}[(\beta_2+\beta_3)\sin(\beta_2+\beta_3)\phi_j - \mu\cos(\beta_2+\beta_3)\phi_j]\}\Big)$$

$$\sigma_{58,j} = \int_{\phi_j}^{\tau_j} e^{-\mu y}\sin\beta_1 y \cos\beta_1 y\, dy$$

$$= \frac{1}{2}(\mu^2+4\beta_1^2)^{-1}\{e^{-\mu\tau_j}[-\mu\sin 2\beta_1\tau_j - 2\beta_1\cos 2\beta_1\tau_j] - e^{-\mu\phi_j}[-\mu\sin 2\beta_1\phi_j$$

$$- 2\beta_1 \cos 2\beta_1 \phi_j \big] \Big\}$$

$$\begin{aligned}
\sigma_{59,j} &= \int_{\phi_j}^{\tau_j} e^{-\mu y} \sin \beta_1 y \cos \beta_2 y \, dy \\
&= \tfrac{1}{2} \Big([\mu^2 + (\beta_1 - \beta_2)^2]^{-1} \big\{ e^{-\mu \tau_j} [-\mu \sin (\beta_1 - \beta_2)\tau_j - (\beta_1 - \beta_2) \cos (\beta_1 - \beta_2)\tau_j] \\
&\qquad - e^{-\mu \phi_j} [-\mu \sin (\beta_1 - \beta_2)\phi_j - (\beta_1 - \beta_2) \cos (\beta_1 - \beta_2)\phi_j] \big\} \\
&\quad + [\mu^2 + (\beta_1 + \beta_2)^2]^{-1} \big\{ e^{-\mu \tau_j} [-\mu \sin (\beta_1 + \beta_2)\tau_j - (\beta_1 + \beta_2) \cos (\beta_1 + \beta_2)\tau_j] \\
&\qquad - e^{-\mu \phi_j} [-\mu \sin (\beta_1 + \beta_2)\phi_j - (\beta_1 + \beta_2) \cos (\beta_1 + \beta_2)\phi_j] \big\} \Big)
\end{aligned}$$

$$\begin{aligned}
\sigma_{60,j} &= \int_{\phi_j}^{\tau_j} e^{-\mu y} \sin \beta_1 y \cos \beta_3 y \, dy \\
&= \tfrac{1}{2} \Big([\mu^2 + (\beta_1 - \beta_3)^2]^{-1} \big\{ e^{-\mu \tau_j} [-\mu \sin (\beta_1 - \beta_3)\tau_j - (\beta_1 - \beta_3) \cos (\beta_1 - \beta_3)\tau_j] \\
&\qquad - e^{-\mu \phi_j} [-\mu \sin (\beta_1 - \beta_3)\phi_j - (\beta_1 - \beta_3) \cos (\beta_1 - \beta_3)\phi_j] \big\} \\
&\quad + [\mu^2 + (\beta_1 + \beta_3)^2]^{-1} \big\{ e^{-\mu \tau_j} [-\mu \sin (\beta_1 + \beta_3)\tau_j - (\beta_1 + \beta_3) \cos (\beta_1 + \beta_3)\tau_j] \\
&\qquad - e^{-\mu \phi_j} [-\mu \sin (\beta_1 + \beta_3)\phi_j - (\beta_1 + \beta_3) \cos (\beta_1 + \beta_3)\phi_j] \big\} \Big)
\end{aligned}$$

$$\begin{aligned}
\sigma_{61,j} &= \int_{\phi_j}^{\tau_j} e^{-\mu y} \sin \beta_2 y \cos \beta_1 y \, dy \\
&= \tfrac{1}{2} \Big([\mu^2 + (\beta_2 - \beta_1)^2]^{-1} \big\{ e^{-\mu \tau_j} [-\mu \sin (\beta_2 - \beta_1)\tau_j - (\beta_2 - \beta_1) \cos (\beta_2 - \beta_1)\tau_j] \\
&\qquad - e^{-\mu \phi_j} [-\mu \sin (\beta_2 - \beta_1)\phi_j - (\beta_2 - \beta_1) \cos (\beta_2 - \beta_1)\phi_j] \big\} \\
&\quad + [\mu^2 + (\beta_1 + \beta_2)^2]^{-1} \big\{ e^{-\mu \tau_j} [-\mu \sin (\beta_1 + \beta_2)\tau_j - (\beta_1 + \beta_2) \cos (\beta_1 + \beta_2)\tau_j] \\
&\qquad - e^{-\mu \phi_j} [-\mu \sin (\beta_1 + \beta_2)\phi_j - (\beta_1 + \beta_2) \cos (\beta_1 + \beta_2)\phi_j] \big\} \Big)
\end{aligned}$$

$$\begin{aligned}
\sigma_{62,j} &= \int_{\phi_j}^{\tau_j} e^{-\mu y} \sin \beta_2 y \cos \beta_2 y \, dy \\
&= \tfrac{1}{2} (\mu^2 + 4\beta_2^2)^{-1} \big\{ e^{-\mu \tau_j} [-\mu \sin 2\beta_2 \tau_j - 2\beta_2 \cos 2\beta_2 \tau_j] - e^{-\mu \phi_j} [-\mu \sin 2\beta_2 \phi_j
\end{aligned}$$

$$-2\beta_2 \cos 2\beta_2\phi_j]\Big\}$$

$$\sigma_{63,j} = \int_{\phi_j}^{T_j} e^{-\mu y} \sin\beta_2 y \cos\beta_3 y \, dy$$

$$= \frac{1}{2}\Big([\mu^2 + (\beta_2-\beta_3)^2]^{-1}\Big\{ e^{-\mu T_j}[-\mu \sin(\beta_2-\beta_3)T_j - (\beta_2-\beta_3)\cos(\beta_2-\beta_3)T_j]$$

$$- e^{-\mu\phi_j}[-\mu \sin(\beta_2-\beta_3)\phi_j - (\beta_2-\beta_3)\cos(\beta_2-\beta_3)\phi_j]\Big\}$$

$$+ [\mu^2 + (\beta_2+\beta_3)^2]^{-1}\Big\{ e^{-\mu T_j}[-\mu \sin(\beta_2+\beta_3)T_j - (\beta_2+\beta_3)\cos(\beta_2+\beta_3)T_j]$$

$$- e^{-\mu\phi_j}[-\mu \sin(\beta_2+\beta_3)\phi_j - (\beta_2+\beta_3)\cos(\beta_2+\beta_3)\phi_j]\Big\}\Big)$$

$$\sigma_{64,j} = \int_{\phi_j}^{T_j} e^{-\mu y} \sin\beta_3 y \cos\beta_1 y \, dy$$

$$= \frac{1}{2}\Big([\mu^2 + (\beta_3-\beta_1)^2]^{-1}\Big\{ e^{-\mu T_j}[-\mu \sin(\beta_3-\beta_1)T_j - (\beta_3-\beta_1)\cos(\beta_3-\beta_1)T_j]$$

$$- e^{-\mu\phi_j}[-\mu \sin(\beta_3-\beta_1)\phi_j - (\beta_3-\beta_1)\cos(\beta_3-\beta_1)\phi_j]\Big\}$$

$$+ [\mu^2 + (\beta_1+\beta_3)^2]^{-1}\Big\{ e^{-\mu T_j}[-\mu \sin(\beta_1+\beta_3)T_j - (\beta_1+\beta_3)\cos(\beta_1+\beta_3)T_j]$$

$$- e^{-\mu\phi_j}[-\mu \sin(\beta_1+\beta_3)\phi_j - (\beta_1+\beta_3)\cos(\beta_1+\beta_3)\phi_j]\Big\}\Big)$$

$$\sigma_{65,j} = \int_{\phi_j}^{T_j} e^{-\mu y} \sin\beta_3 y \cos\beta_2 y \, dy$$

$$= \frac{1}{2}\Big([\mu^2 + (\beta_3-\beta_2)^2]^{-1}\Big\{ e^{-\mu T_j}[-\mu \sin(\beta_3-\beta_2)T_j - (\beta_3-\beta_2)\cos(\beta_3-\beta_2)T_j]$$

$$- e^{-\mu\phi_j}[-\mu \sin(\beta_3-\beta_2)\phi_j - (\beta_3-\beta_2)\cos(\beta_3-\beta_2)\phi_j]\Big\}$$

$$+ [\mu^2 + (\beta_2+\beta_3)^2]^{-1}\Big\{ e^{-\mu T_j}[-\mu \sin(\beta_2+\beta_3)T_j - (\beta_2+\beta_3)\cos(\beta_2+\beta_3)T_j]$$

$$- e^{-\mu\phi_j}[-\mu \sin(\beta_2+\beta_3)\phi_j - (\beta_2+\beta_3)\cos(\beta_2+\beta_3)\phi_j]\Big\}\Big)$$

$$\sigma_{66,j} = \int_{\phi_j}^{T_j} e^{-\mu y} \sin\beta_3 y \cos\beta_3 y \, dy$$

$$= \frac{1}{2}(\mu^2 + 4\beta_3^2)^{-1}\Big\{ e^{-\mu T_j}[-\mu \sin 2\beta_3 T_j - 2\beta_3 \cos 2\beta_3 T_j] - e^{-\mu\phi_j}[-\mu \sin 2\beta_3\phi_j$$

$$- e^{-2\mu\phi_j}\left[(\beta_1-\beta_3)\sin(\beta_1-\beta_3)\phi_j - 2\mu\cos(\beta_1-\beta_3)\phi_j\right]\Big\}$$

$$+ \left[4\mu^2+(\beta_1+\beta_3)^2\right]^{-1}\Big\{e^{-2\mu T_j}\left[(\beta_1+\beta_3)\sin(\beta_1+\beta_3)T_j - 2\mu\cos(\beta_1+\beta_3)T_j\right]$$

$$- e^{-2\mu\phi_j}\left[(\beta_1+\beta_3)\sin(\beta_1+\beta_3)\phi_j - 2\mu\cos(\beta_1+\beta_3)\phi_j\right]\Big\}\Big)$$

$$\sigma_{72,j} = \int_{\phi_j}^{T_j} e^{-2\mu y}\cos\beta_2 y\cos\beta_3 y\,dy$$

$$= \frac{1}{2}\Big(\left[4\mu^2+(\beta_2-\beta_3)^2\right]^{-1}\Big\{e^{-2\mu T_j}\left[(\beta_2-\beta_3)\sin(\beta_2-\beta_3)T_j - 2\mu\cos(\beta_2-\beta_3)T_j\right]$$

$$- e^{-2\mu\phi_j}\left[(\beta_2-\beta_3)\sin(\beta_2-\beta_3)\phi_j - 2\mu\cos(\beta_2-\beta_3)\phi_j\right]\Big\}$$

$$+ \left[4\mu^2+(\beta_2+\beta_3)^2\right]^{-1}\Big\{e^{-2\mu T_j}\left[(\beta_2+\beta_3)\sin(\beta_2+\beta_3)T_j - 2\mu\cos(\beta_2+\beta_3)T_j\right]$$

$$- e^{-2\mu\phi_j}\left[(\beta_2+\beta_3)\sin(\beta_2+\beta_3)\phi_j - 2\mu\cos(\beta_2+\beta_3)\phi_j\right]\Big\}\Big)$$

$$\sigma_{73,j} = \int_{\phi_j}^{T_j} e^{-2\mu y}\sin\beta_1 y\cos\beta_1 y\,dy$$

$$= \frac{1}{2}(4\mu^2+4\beta_1^2)^{-1}\Big[e^{-2\mu T_j}(-2\mu\sin 2\beta_1 T_j - 2\beta_1\cos 2\beta_1 T_j)$$

$$- e^{-2\mu\phi_j}(-2\mu\sin 2\beta_1\phi_j - 2\beta_1\cos 2\beta_1\phi_j)\Big]$$

$$\sigma_{74,j} = \int_{\phi_j}^{T_j} e^{-2\mu y}\sin\beta_1 y\cos\beta_2 y\,dy$$

$$= \frac{1}{2}\Big(\left[4\mu^2+(\beta_1-\beta_2)^2\right]^{-1}\Big\{e^{-2\mu T_j}\left[-2\mu\sin(\beta_1-\beta_2)T_j - (\beta_1-\beta_2)\cos(\beta_1-\beta_2)T_j\right]$$

$$- e^{-2\mu\phi_j}\left[-2\mu\sin(\beta_1-\beta_2)\phi_j - (\beta_1-\beta_2)\cos(\beta_1-\beta_2)\phi_j\right]\Big\}$$

$$+ \left[4\mu^2+(\beta_1+\beta_2)^2\right]^{-1}\Big\{e^{-2\mu T_j}\left[-2\mu\sin(\beta_1+\beta_2)T_j - (\beta_1+\beta_2)\cos(\beta_1+\beta_2)T_j\right]$$

$$- e^{-2\mu\phi_j}\left[-2\mu\sin(\beta_1+\beta_2)\phi_j - (\beta_1+\beta_2)\cos(\beta_1+\beta_2)\phi_j\right]\Big\}\Big)$$

$$\sigma_{75,j} = \int_{\phi_j}^{T_j} e^{-2\mu y}\sin\beta_1 y\cos\beta_3 y\,dy$$

$$= \frac{1}{2}\Big(\left[4\mu^2+(\beta_1-\beta_3)^2\right]^{-1}\Big\{e^{-2\mu T_j}\left[-2\mu\sin(\beta_1-\beta_3)T_j - (\beta_1-\beta_3)\cos(\beta_1-\beta_3)T_j\right]$$

$$-e^{-2\mu\phi_j}[-2\mu\sin(\beta_1-\beta_3)\phi_j-(\beta_1-\beta_3)\cos(\beta_1-\beta_3)\phi_j]\}$$

$$+[4\mu^2+(\beta_1+\beta_3)^2]^{-1}\{e^{-2\mu T_j}[-2\mu\sin(\beta_1+\beta_3)T_j-(\beta_1+\beta_3)\cos(\beta_1+\beta_3)T_j]$$

$$-e^{-2\mu\phi_j}[-2\mu\sin(\beta_1+\beta_3)\phi_j-(\beta_1+\beta_3)\cos(\beta_1+\beta_3)\phi_j]\})$$

$$\sigma_{76,j}=\int_{\phi_j}^{T_j}e^{-2\mu y}\sin\beta_2 y\cos\beta_1 y\,dy$$

$$=\frac{1}{2}\Big([4\mu^2+(\beta_2-\beta_1)^2]^{-1}\{e^{-2\mu T_j}[-2\mu\sin(\beta_2-\beta_1)T_j-(\beta_2-\beta_1)\cos(\beta_2-\beta_1)T_j]$$

$$-e^{-2\mu\phi_j}[-2\mu\sin(\beta_2-\beta_1)\phi_j-(\beta_2-\beta_1)\cos(\beta_2-\beta_1)\phi_j]\}$$

$$+[4\mu^2+(\beta_1+\beta_2)^2]^{-1}\{e^{-2\mu T_j}[-2\mu\sin(\beta_1+\beta_2)T_j-(\beta_1+\beta_2)\cos(\beta_1+\beta_2)T_j]$$

$$-e^{-2\mu\phi_j}[-2\mu\sin(\beta_1+\beta_2)\phi_j-(\beta_1+\beta_2)\cos(\beta_1+\beta_2)\phi_j]\}\Big)$$

$$\sigma_{77,j}=\int_{\phi_j}^{T_j}e^{-2\mu y}\sin\beta_2 y\cos\beta_2 y\,dy$$

$$=\frac{1}{2}(4\mu^2+4\beta_2^2)^{-1}[e^{-2\mu T_j}(-2\mu\sin 2\beta_2 T_j-2\beta_2\cos 2\beta_2 T_j)$$

$$-e^{-2\mu\phi_j}(-2\mu\sin 2\beta_2\phi_j-2\beta_2\cos 2\beta_2\phi_j)]$$

$$\sigma_{78,j}=\int_{\phi_j}^{T_j}e^{-2\mu y}\sin\beta_2 y\cos\beta_3 y\,dy$$

$$=\frac{1}{2}\Big([4\mu^2+(\beta_2-\beta_3)^2]^{-1}\{e^{-2\mu T_j}[-2\mu\sin(\beta_2-\beta_3)T_j-(\beta_2-\beta_3)\cos(\beta_2-\beta_3)T_j]$$

$$-e^{-2\mu\phi_j}[-2\mu\sin(\beta_2-\beta_3)\phi_j-(\beta_2-\beta_3)\cos(\beta_2-\beta_3)\phi_j]\}$$

$$+[4\mu^2+(\beta_2+\beta_3)^2]^{-1}\{e^{-2\mu T_j}[-2\mu\sin(\beta_2+\beta_3)T_j-(\beta_2+\beta_3)\cos(\beta_2+\beta_3)T_j]$$

$$-e^{-2\mu\phi_j}[-2\mu\sin(\beta_2+\beta_3)\phi_j-(\beta_2+\beta_3)\cos(\beta_2+\beta_3)]\}\Big)$$

$$\sigma_{79,j}=\int_{\phi_j}^{T_j}e^{-2\mu y}\sin\beta_3 y\cos\beta_1 y\,dy$$

$$=\frac{1}{2}\Big([4\mu^2+(\beta_3-\beta_1)^2]^{-1}\{e^{-2\mu T_j}[-2\mu\sin(\beta_3-\beta_1)T_j-(\beta_3-\beta_1)\cos(\beta_3-\beta_1)T_j]$$

$$-e^{-2\mu\phi_j}[-2\mu\sin(\beta_3-\beta_1)\phi_j-(\beta_3-\beta_1)\cos(\beta_3-\beta_1)\phi_j]\}$$

$$+[4\mu^2+(\beta_3+\beta_1)^2]^{-1}\{e^{-2\mu\tau_j}[-2\mu\sin(\beta_3+\beta_1)\tau_j-(\beta_3+\beta_1)\cos(\beta_3+\beta_1)\tau_j]$$

$$-e^{-2\mu\phi_j}[-2\mu\sin(\beta_3+\beta_1)\phi_j-(\beta_3+\beta_1)\cos(\beta_3+\beta_1)\phi_j]\})$$

$$\sigma_{80,j}=\int_{\phi_j}^{\tau_j}e^{-2\mu y}\sin\beta_3 y\cos\beta_2 y\,dy$$

$$=\frac{1}{2}\Big([4\mu^2+(\beta_3-\beta_2)^2]^{-1}\{e^{-2\mu\tau_j}[-2\mu\sin(\beta_3-\beta_2)\tau_j-(\beta_3-\beta_2)\cos(\beta_3-\beta_2)\tau_j]$$

$$-e^{-2\mu\phi_j}[-2\mu\sin(\beta_3-\beta_2)\phi_j-(\beta_3-\beta_2)\cos(\beta_3-\beta_2)\phi_j]\}$$

$$+[4\mu^2+(\beta_3+\beta_2)^2]^{-1}\{e^{-2\mu\tau_j}[-2\mu\sin(\beta_3+\beta_2)\tau_j-(\beta_3+\beta_2)\cos(\beta_3+\beta_2)\tau_j]$$

$$-e^{-2\mu\phi_j}[-2\mu\sin(\beta_3+\beta_2)\phi_j-(\beta_3+\beta_2)\cos(\beta_3+\beta_2)\phi_j]\}\Big)$$

$$\sigma_{81,j}=\int_{\phi_j}^{\tau_j}e^{-2\mu y}\sin\beta_3 y\cos\beta_3 y\,dy$$

$$=\frac{1}{2}(4\mu^2+4\beta_3^2)^{-1}[e^{-2\mu\tau_j}(-2\mu\sin 2\beta_3\tau_j-2\beta_3\cos 2\beta_3\tau_j)$$

$$-e^{-2\mu\phi_j}(-2\mu\sin 2\beta_3\phi_j-2\beta_3\cos 2\beta_3\phi_j)]$$

$$\sigma_{82,j}=\int_{\phi_j}^{\tau_j}e^{-2\mu y}\sin^2\beta_1 y\,dy$$

$$=\frac{1}{2}\{-(2\mu)^{-1}(e^{-2\mu\tau_j}-e^{-2\mu\phi_j})-(4\mu^2+4\beta_1^2)^{-1}[e^{-2\mu\tau_j}(2\beta_1\sin 2\beta_1\tau_j$$

$$-2\mu\cos 2\beta_1\tau_j)-e^{-2\mu\phi_j}(2\beta_1\sin 2\beta_1\phi_j-2\mu\cos 2\beta_1\phi_j)]\}$$

$$\sigma_{83,j}=\int_{\phi_j}^{\tau_j}e^{-2\mu y}\sin^2\beta_2 y\,dy$$

$$=\frac{1}{2}\{-(2\mu)^{-1}(e^{-2\mu\tau_j}-e^{-2\mu\phi_j})-(4\mu^2+4\beta_2^2)^{-1}[e^{-2\mu\tau_j}(2\beta_2\sin 2\beta_2\tau_j$$

$$-2\mu\cos 2\beta_2\tau_j)-e^{-2\mu\phi_j}(2\beta_2\sin 2\beta_2\phi_j-2\mu\cos 2\beta_2\phi_j)]\}$$

$$\sigma_{B4,j} = \int_{\phi_j}^{T_j} e^{-2\mu y} \sin^2 \beta_3 y \, dy$$

$$= \frac{1}{2} \Big\{ -(2\mu)^{-1} (e^{-2\mu T_j} - e^{-2\mu \phi_j}) - (4\mu^2 + 4\beta_3^2)^{-1} \big[e^{-2\mu T_j} (2\beta_3 \sin 2\beta_3 T_j$$

$$- 2\mu \cos 2\beta_3 T_j) - e^{-2\mu\phi_j} (2\beta_3 \sin 2\beta_3 \phi_j - 2\mu \cos 2\beta_3 \phi_j) \big] \Big\}$$

$$\sigma_{B5,j} = \int_{\phi_j}^{T_j} e^{-2\mu y} \sin \beta_1 y \sin \beta_2 y \, dy$$

$$= \frac{1}{2} \Big(\left[4\mu^2 + (\beta_1 - \beta_2)^2\right]^{-1} \Big\{ e^{-2\mu T_j} \left[(\beta_1 - \beta_2) \sin (\beta_1 - \beta_2) T_j - 2\mu \cos (\beta_1 - \beta_2) T_j \right]$$

$$- e^{-2\mu\phi_j} \left[(\beta_1 - \beta_2) \sin (\beta_1 - \beta_2) \phi_j - 2\mu \cos (\beta_1 - \beta_2) \phi_j \right] \Big\}$$

$$- \left[4\mu^2 + (\beta_1 + \beta_2)^2\right]^{-1} \Big\{ e^{-2\mu T_j} \left[(\beta_1 + \beta_2) \sin (\beta_1 + \beta_2) T_j - 2\mu \cos (\beta_1 + \beta_2) T_j \right]$$

$$- e^{-2\mu\phi_j} \left[(\beta_1 + \beta_2) \sin (\beta_1 + \beta_2) \phi_j - 2\mu \cos (\beta_1 + \beta_2) \phi_j \right] \Big\} \Big)$$

$$\sigma_{B6,j} = \int_{\phi_j}^{T_j} e^{-2\mu y} \sin \beta_1 y \sin \beta_3 y \, dy$$

$$= \frac{1}{2} \Big(\left[4\mu^2 + (\beta_1 - \beta_3)^2\right]^{-1} \Big\{ e^{-2\mu T_j} \left[(\beta_1 - \beta_3) \sin (\beta_1 - \beta_3) T_j - 2\mu \cos (\beta_1 - \beta_3) T_j \right]$$

$$- e^{-2\mu\phi_j} \left[(\beta_1 - \beta_3) \sin (\beta_1 - \beta_3) \phi_j - 2\mu \cos (\beta_1 - \beta_3) \phi_j \right] \Big\}$$

$$- \left[4\mu^2 + (\beta_1 + \beta_3)^2\right]^{-1} \Big\{ e^{-2\mu T_j} \left[(\beta_1 + \beta_3) \sin (\beta_1 + \beta_3) T_j - 2\mu \cos (\beta_1 + \beta_3) T_j \right]$$

$$- e^{-2\mu\phi_j} \left[(\beta_1 + \beta_3) \sin (\beta_1 + \beta_3) \phi_j - 2\mu \cos (\beta_1 + \beta_3) \phi_j \right] \Big\} \Big)$$

$$\sigma_{B7,j} = \int_{\phi_j}^{T_j} e^{-2\mu y} \sin \beta_2 y \sin \beta_3 y \, dy$$

$$= \frac{1}{2} \Big(\left[4\mu^2 + (\beta_2 - \beta_3)^2\right]^{-1} \Big\{ e^{-2\mu T_j} \left[(\beta_2 - \beta_3) \sin (\beta_2 - \beta_3) T_j - 2\mu \cos (\beta_2 - \beta_3) T_j \right]$$

$$- e^{-2\mu\phi_j} \left[(\beta_2 - \beta_3) \sin (\beta_2 - \beta_3) \phi_j - 2\mu \cos (\beta_2 - \beta_3) \phi_j \right] \Big\}$$

$$- \left[4\mu^2 + (\beta_2 + \beta_3)^2\right]^{-1} \Big\{ e^{-2\mu T_j} \left[(\beta_2 + \beta_3) \sin (\beta_2 + \beta_3) T_j - 2\mu \cos (\beta_2 + \beta_3) T_j \right]$$

$$- e^{-2\mu\phi_j} \left[(\beta_2 + \beta_3) \sin (\beta_2 + \beta_3) \phi_j - 2\mu \cos (\beta_2 + \beta_3) \phi_j \right] \Big\} \Big)$$

APPENDIX B

DETERMINATION OF MATERIAL PROPERTIES FOR A TYPICAL ORTHOTROPIC FACING MATERIAL

For the last two problems considered in this dissertation some realistic material properties are required for the facings of an orthotropic sandwich shell.

A typical facing material would be what is commonly known as glass-fiber reinforced plastic. This consists of glass filaments encased in a epoxy resin which is called the matrix. The glass filaments and matrix are individually isotropic; but, when they are combined, they behave structurally as an orthotropic unit ply. Unit plies are then bonded together in layers varying the glass filament alignment in each layer. This is known as a cross-ply composite.

The component materials of typical glass filament-epoxy resin composite have the following material properties (52).

$$E_f = 10.6 \times 10^6 \text{ psi} \qquad E_m = 0.5 \times 10^6 \text{ psi}$$
$$\mu_f = 0.22 \qquad \mu_m = 0.35$$
$$\gamma_f = 2.60 \qquad \gamma_m = 1.15$$

where E is Young's modulus, μ is Poisson's ratio and γ is the specific gravity. The subscripts f and m refer to the filament and the matrix, respectively.

Using the above typical values and the information contained in Figures 3 and 10 of (52), it can be observed that the maximum variation in the ratio of the transverse stiffness to the axial stiffness of a cross-ply composite of three equal thickness plies is 0.65-0.80 approximately. A resin content of 25 percent gives a stiffness ratio near the lower value. Since the practical range of resin content is 15-30 percent, 25 percent is then used to calculate some typical composite material properties.

The following equations are given by Tsai to predict the material properties of the unit ply:

$$E_{11} = k\,[\,E_f - (E_f - E_m)\chi]$$

$$E_{22} = 2\,[1-\mu_f + (\mu_f-\mu_m)\chi][(1-C)\,\frac{K_f\,(2K_m+G_m)-G_m(K_f-K_m)\chi}{(2K_m+G_m)-2(K_f-K_m)\chi}$$

$$+\;C\,\frac{K_f\,(2K_m+G_f)+G_f\,(K_m-K_f)\chi}{(2K_m+G_f)-2(K_m-K_f)\chi}]$$

$$\mu_{12} = (1-C)\left[\frac{K_f\mu_f(2K_m+G_m)(1-\chi)+K_m\mu_m(2K_f+G_m)\chi}{K_f\,(2K_m+G_m)+G_m(K_m-K_f)\chi}\right]$$

$$+\;C\left[\frac{K_m\mu_m(2K_f+G_f)\chi+K_f\mu_f\,(2K_m+G_f)(1-\chi)}{K_f\,(2K_m+G_f)+G_f(K_m-K_f)\chi}\right]$$

$$G = (1-C)\,G_m\left[\frac{2G_f-(G_f-G_m)\chi}{2G_m+(G_f-G_m)\chi}\right] + CG_f\left[\frac{(G_f+G_m)-(G_f-G_m)\chi}{(G_f+G_m)+(G_f-G_m)\chi}\right]$$

$$\chi = \frac{\gamma_f/\gamma_m}{(100/R)+(\gamma_f/\gamma_m)-1}$$

$$K_f = E_f/2(1-\mu_f)$$

$$K_m = E_m/2(1-\mu_m)$$

$$G_f = E_f/2(1+\mu_f)$$

$$G_m = E_m/2(1+\mu_m)$$

The ratio of the principal stiffnesses, F, is given by

$$F = E_{22} / E_{11} = 0.27$$

A typical facing material consists of three unit plies (n=3) of equal thickness. Thus, m = 2 where m is the cross-ply ratio which is defined as the total thickness of the odd layers divided by the total thickness of the even layers.

Thus, the material properties of the cross-ply composite are given by

$$E'_x = \frac{m + F}{1 + m} \quad \frac{E_{11}}{1-\mu_{12}\mu_{21}}$$

$$E'_\theta = \frac{1 + mF}{1 + m} \quad \frac{E_{22}}{1-\mu_{12}\mu_{21}}$$

$$G'_{x\theta} = G$$

The required properties of the cross-ply composite are:

$E'_\theta \;/\; E'_x$	0.673	1.490
$G'_{x\theta} \;/\; \bar{E}'_x$	0.184	0.184
$K'_\theta G'_{\theta z} \;/\; \bar{E}'_x$	0.184	0.184
$K'_x G'_{zx} \;/\; \bar{E}'_x$	0.184	0.184
μ'_x	0.196	0.132
μ'_θ	0.132	0.196

The first column gives the material properties if the odd layers are oriented in the meridional direction of the cone. The second column gives the material properties if the odd layers are rotated 90 degrees to align with the circumferential direction.

Due to general disagreement concerning the proper dynamic value for the shear factors, K'_θ and K'_x , a value of $K'_\theta = K'_x = 1.0$ is used. (The shear factor has been assigned values by various authors in the range 0.9 - 2.0)

Assuming that $K'_\theta \, G'_{\theta z} / \bar{E}'_x = K'_x \, G'_{zx} / \bar{E}'_x = 0.184$ assumes that the transverse shearing rigidity is the same as the in-plane shearing rigidity. This appears to be reasonable since photomicrographs of a typical cross section (perpendicular to the filaments) of a unit ply indicate that the filament spacing is approximately the same in any direction.

APPENDIX C

COMPUTER PROGRAM

The complete computer program, which was used to solve the example problems in Chapter 4, is contained in this Appendix.

The program required for the generation of the a_{ij} and b_{ij} coefficients (pages 64-75 and page 40, respectively) is given on pages 121-161.

On pages 122-163 are the routines required for the inversion of the matrix $[B]$ and to obtain the matrix $[T]$. The eigenvalue-eigenvector routine (48) is found on pages 163-176.

```
      PROGRAM MAC1(INPUT,OUTPUT,PUNCH)
      COMMON/MATS/XINPUT(25,25),XINPUTI(25,25),EIGEN(25),EIGENC(25),
     1VALUR(25,25),VALUI(25,25)
      DIMENSION F(25,60),Q(25)
      DIMENSION ANS(25,25),DIFF(25)
      DIMENSION X(25,25)
      DIMENSION A(25,25),B(25,25),ALPHB(12),X1(12),X2(12),TAU(12),PHI(12
     1),S1(12),S2(12),S3(12),S4(12),S5(12),S6(12),S7(12),S8(12),S9(12),
     2S10(12),S11(12),S13(12),S14(12),S15(12),S16(12),S17(12),S18(12),
     3S19(12),S20(12),S21(12),S22(12),S23(12),S24(12),S25(12),S26(12),
     4S27(12),S28(12),S29(12),S30(12),S31(12),S32(12),S33(12),S34(12),
     5S35(12),S36(12),S37(12),S38(12),S39(12),S40(12),S41(12),S42(12),
     6S43(12),S44(12),S45(12),S46(12),S47(12),S48(12),S49(12),S50(12),
     7S51(12),S52(12),S53(12),S54(12),S55(12),S56(12),S57(12),S58(12),
     8S59(12),S60(12),S61(12),S62(12),S63(12),S64(12),S65(12),S66(12),
     9S67(12),S68(12),S69(12),S70(12),S71(12),S72(12),S73(12),S74(12),
     1S75(12),S76(12),S77(12),S78(12),S79(12),S80(12),S81(12),S82(12),
     2S83(12),S84(12),S85(12),S86(12),S87(12),S(12),TN(12),C(12),CS(12),
     3CT(12),TR1(12),TR2(12),TR3(12),TR4(12),TR5(12),TR6(12),TR7(12),
     4TR8(12),TR9(12),TR10(12),TR11(12),TR12(12)
      DIMENSION TR13(12),TR14(12),TR15(12),TR16(12),TR17(12),TR18(12),
     1TR19(12),TR20(12),TR21(12),TR22(12),TR23(12),TR24(12),TR25(12),
     2TR26(12),TR27(12),TR28(12),TR29(12),TR30(12),TR31(12),TR32(12),
     3TR33(12),TR34(12),TR35(12),TR36(12),TR37(12),TR38(12),TR39(12),
     4TR40(12),TR41(12),EA1(12),EA2(12),EA3(12),EA4(12),EA5(12),EA6(12),
     5EA7(12),EA8(12),EA9(12),EA10(12),EA11(12),EA12(12),SA1(12),SA2(12)
     6,SA3(12),SA4(12),SA5(12),SA6(12),SA7(12),SA8(12),SA9(12),SA10(12),
     8SA11(12),SA12(12),SA13(12),SA14(12),SA15(12),SA16(12),SA17(12),SA
     918(12),SA20(12),SA21(12),SA19(12),SA22(12),SA23(12),SA24(12)
      DIMENSION CA1(12),CA2(12),CA3(12),CA4(12),CA5(12),CA6(12),CA7(12),
     1CA8(12),CA9(12),CA10(12),CA11(12),CA12(12),CA13(12),CA14(12),CA15
     2(12),CA16(12),CA17(12),CA18(12),CA19(12),CA20(12),CA21(12),CA22(12
     3),CA23(12),CA24(12)
      DIMENSION Y(25,25),ATRAN(25,25)
      DIMENSION ALPHA(12),S12(12)
      DIMENSION TEMP(25,25)
      M09999=1
```

```
999   CONTINUE
      READ 1001,X3,XMU,T,E,G,YMU,GY1,GX1,H,GY,GX,AA,RHO,K
1001  FORMAT(8F10.0,/,5F10.0,I10)
      READ 1002,(ALPHB(J),X1(J),X2(J),J=1,K)
1002  FORMAT(3F10.0)
1003  READ 1002,XN
      IF(XN.GT.100.)GO TO 999
      PRINT 1004
1004  FORMAT(1H1,7X,1HK,14X,2HXN,14X,2HX3,14X,3HXMU,13X,1HT,14X,1HE,14X,
     $1HG,13X,3HYMU)
      PRINT 1005,K,XN,X3,XMU,T,E,G,YMU
1005  FORMAT(1H0,I8,7X,7F15.5)
      PRINT 1006
1006  FORMAT(///,7X,3HGY1,13X,3HGX1,13X,1HH,14X,2HGY,13X,2HGX,13X,2HAA,
     $13X,3HRHO)
      PRINT 1007,GY1,GX1,H,GY,GX,AA,RHO
1007  FORMAT(7F15.5)
      PRINT 1015
      PRINT 1014,(ALPHB(J),X1(J),X2(J),J=1,K)
1014  FORMAT(5X,3F15.5)
1015  FORMAT(1H0,5X,5HALPHB,12X,2HX1,15X,2HX2)
      PI=3.14159265
      RADIAN=57.29597795
      XM=2.*(RHO*H*2.*T)
      CI=2./3.*RHO*H**3
      FI=2./3.*T*(3.*H*H+6.*H*T+4.*T*T)
      ETA1=2.*T*E
      ETA2=2.*T*G
      ETA3=2.*T*(XMU+YMU*E)
      ETA4=2.*T
      ETA5=2.*T*GY1
      ETA6=H*GY
      ETA7=2.*T*GX1
      ETA8=H*GX
      ETA9=2.*H*H*T*E
      ETA10=2.*H*H*T*G
      ETA11=2.*H*H*T
      ETA12=2.*H*H*T*(XMU+YMU*E)
      ETA13=2.*H*T*E*(-2.*H+AA)
      ETA14=2.*H*T*G*(-2.*H+AA)
```

```
ETA15=2.*H*T*(T*XMU-H*YMU*E)
ETA16=2.*H*T*(YMU*T*E-H*XMU)
ETA17=2.*H*T*(-2.*H+AA)
ETA18=2.*T*E*(H*H+1./3.*T*T-H*AA)
ETA19=2.*T*G*(1./3.*T*T+2.*H*H-2.*H*AA)
ETA20=2.*T*(XMU+YMU*E)*(H*H+1./3.*T*T-H*AA)
ETA21=4.*H*H*T*(E+G)
ETA22=2.*H*T*(AA-2.*H)*(E+G)
ETA23=4.*T*(H*H+1./3.*T*T-H*AA)*(E+G)
ETA24=2.*T*G*(-1./3.*T*T-H*H+H*AA)
ETA25=2.*H*H*T*GY1
ETA26=2.*H*H*T*E
ETA27=2.*T*(H*H+1./3.*T*T-H*AA)
BETA1=PI/ALOG(X3)
BETA2=2.*BETA1
BETA3=3.*BETA1
BETS12=BETA1+BETA2
BETD12=BETA1-BETA2
BETS13=BETA1+BETA3
BETD13=BETA1-BETA3
BETS23=BETA2+BETA3
BETD23=BETA2-BETA3
BETD21=-1.*BETD12
BETD31=-1.*BETD13
BETD32=-1.*BETD23
BETA12=BETA1*BETA2
BETA13=BETA1*BETA3
BETA23=BETA2*BETA3
BETA11=BETA1*BETA1
BETA22=BETA2*BETA2
BETA33=BETA3*BETA3
BETDD12=BETD12*BETD12
BETSS12=BETS12*BETS12
BETDD13=BETD13*BETD13
BETSS13=BETS13*BETS13
BETDD23=BETD23*BETD23
BETSS23=BETS23*BETS23
CT1=-2.*ETA2
CT2=-1.*BETA1*ETA3
CT3=2.*(ETA1+ETA2)-XMU*ETA3
```

```
CT4=-1.*BETA2*ETA3
CT5=-1.*BETA3*ETA3
CT6=-2.*XN*ETA6
CT7=-2.*XN*ETA5
CT8=-2.*BETA1*ETA10
CT9=-1.*BETA1*ETA12
CT10=-1.*BETA2*ETA12
CT11=-1.*BETA3*ETA12
CT12=ETA21-XMU*ETA12
CT13=-1.*BETA1*ETA14
CT14=-1.*BETA1*ETA15
CT15=-1.*BETA2*ETA15
CT16=-1.*BETA3*ETA15
CT17=ETA22-XMU*ETA15
CT18=-2.*BETA2*ETA10
CT19=-1.*BETA2*ETA14
CT20=-2.*BETA3*ETA10
CT21=-1.*BETA3*ETA14
CT22=BETA1*ETA16
CT23=BETA2*ETA16
CT24=BETA3*ETA16
CT25=ETA22*XMU*ETA16
CT26=2.*BETA1*ETA24
CT27=2.*BETA2*ETA24
CT28=2.*BETA3*ETA24
CT29=BETA1*ETA20
CT30=BETA2*ETA20
CT31=BETA3*ETA20
CT32=ETA23-XMU*ETA20
XMU2=XMU*XMU
FMU=1.-XMU
FMU2=FMU*FMU
DO 500 J=1,K
ALPHA(J)=ALPHB(J)*1./RADIAN
PHI(J)=ALOG(X1(J))
TAU(J)=ALOG(X2(J))
EA1(J)=EXP(2.*FMU*TAU(J))
EA2(J)=EXP(2.*FMU*PHI(J))
EA3(J)=EXP(2.*TAU(J))
EA4(J)=EXP(2.*PHI(J))
```

```
EA5(J)=EXP(TAU(J))
EA6(J)=EXP(PHI(J))
EA7(J)=EXP(FMU*TAU(J))
EA8(J)=EXP(FMU*PHI(J))
EA9(J)=EXP(-1.*XMU*TAU(J))
EA10(J)=EXP(-1.*XMU*PHI(J))
EA11(J)=EXP(-2.*XMU*TAU(J))
EA12(J)=EXP(-2.*XMU*PHI(J))
SA1(J)=SIN(2.*BETA1*TAU(J))
SA2(J)=SIN(2.*BETA1*PHI(J))
SA3(J)=SIN(2.*BETA2*TAU(J))
SA4(J)=SIN(2.*BETA2*PHI(J))
SA5(J)=SIN(2.*BETA3*TAU(J))
SA6(J)=SIN(2.*BETA3*PHI(J))
SA7(J)=SIN(BETD12*TAU(J))
SA8(J)=SIN(BETD12*PHI(J))
SA9(J)=SIN(BETS12*TAU(J))
SA10(J)=SIN(BETS12*PHI(J))
SA11(J)=SIN(BETD13*TAU(J))
SA12(J)=SIN(BETD13*PHI(J))
SA13(J)=SIN(BETS13*TAU(J))
SA14(J)=SIN(BETS13*PHI(J))
SA15(J)=SIN(BETD23*TAU(J))
SA16(J)=SIN(BETD23*PHI(J))
SA17(J)=SIN(BETS23*TAU(J))
SA18(J)=SIN(BETS23*PHI(J))
SA19(J)=SIN(BETD21*TAU(J))
SA20(J)=SIN(BETD21*PHI(J))
SA21(J)=SIN(BETD31*TAU(J))
SA22(J)=SIN(BETD31*PHI(J))
SA23(J)=SIN(BETD32*TAU(J))
SA24(J)=SIN(BETD32*PHI(J))
CA1(J)=COS(2.*BETA1*TAU(J))
CA2(J)=COS(2.*BETA1*PHI(J))
CA3(J)=COS(2.*BETA2*TAU(J))
CA4(J)=COS(2.*BETA2*PHI(J))
CA5(J)=COS(2.*BETA3*TAU(J))
CA6(J)=COS(2.*BETA3*PHI(J))
CA7(J)=COS(BETD12*TAU(J))
CA8(J)=COS(BETD12*PHI(J))
```

```
      CA9(J)=COS(BETS12*TAU(J))
      CA10(J)=COS(BETS12*PHI(J))
      CA11(J)=COS(BETD13*TAU(J))
      CA12(J)=COS(BETD13*PHI(J))
      CA13(J)=COS(BETS13*TAU(J))
      CA14(J)=COS(BETS13*PHI(J))
      CA15(J)=COS(BETD23*TAU(J))
      CA16(J)=COS(BETD23*PHI(J))
      CA17(J)=COS(BETS23*TAU(J))
      CA18(J)=COS(BETS23*PHI(J))
      CA19(J)=COS(BETD21*TAU(J))
      CA20(J)=COS(BETD21*PHI(J))
      CA21(J)=COS(BETD31*TAU(J))
      CA22(J)=COS(BETD31*PHI(J))
      CA23(J)=COS(BETD32*TAU(J))
      CA24(J)=COS(BETD32*PHI(J))
      S1(J)=.25*(1./FMU*(EA1(J)-EA2(J))+1./(FMU2+BETA11)*(EA1(J)*(BETA1*
     $SA1(J)+FMU*CA1(J))-EA2(J)*(BETA1*SA2(J)+FMU*CA2(J))))
      S2(J)=.25*(1./FMU*(EA1(J)-EA2(J))+1./(FMU2+BETA22)*(EA1(J)*(BETA2*
     $SA3(J)+FMU*CA3(J))-EA2(J)*(BETA2*SA4(J)+FMU*CA4(J))))
      S3(J)=.25*(1./FMU*(EA1(J)-EA2(J))+1./(FMU2+BETA33)*(EA1(J)*(BETA3*
     $SA5(J)+FMU*CA3(J))-EA2(J)*(BETA3*SA6(J)+FMU*CA6(J))))
      S4(J)=.5*(1./(4.*FMU2+BETDD12)*(EA1(J)*(BETD12*SA7(J)+2.*FMU*CA7(J
     1))-EA2(J)*(BETD12*SA8(J)+2.*FMU*CA8(J)))+1./(4.*FMU2+BETSS12)*(EA1
     2(J)*(BETS12*SA9(J)+2.*FMU*CA9(J))-EA2(J)*(BETS*SA10(J)+2.*FMU*CA10
     3(J))))
      S5(J)=.5*(1./(4.*FMU2+BETDD13)*(EA1(J)*(BETD13*SA11(J)+2.*FMU*CA11
     1(J))-EA2(J)*(BETD13*SA12(J)+2.*FMU*CA12(J)))+1./(4.*FMU2+BETSS13)*
     2(EA1(J)*(BETS13*SA13(J)+2.*FMU*CA13(J))-EA2(J)*(BETS13*SA14(J)+2.*
     3FMU*CA14(J))))
      S6(J)=.5*(1./(4.*FMU2+BETDD23)*(EA1(J)*(BETD23*SA15(J)+2.*FMU*CA15
     2(J))-EA2(J)*(BETD23*SA16(J)+2.*FMU*CA16(J)))+1./(4.*FMU2+BETSS23)*
     3(EA1(J)*(BETS23*SA17(J)+2.*FMU*CA17(J))-EA2(J)*(BETS23*SA18(J)+2.*
     4FMU*CA18(J))))
      S7(J)=.25*(EA3(J)-EA4(J)-1./(1.+BETA11)*(EA3(J)*(BETA1*SA1(J)+CA1(
     1J))-EA4(J)*(BETA1*SA2(J)+CA2(J))))
      S8(J)=.25*(EA3(J)-EA4(J)-1./(1.+BETA22)*(EA3(J)*(BETA2*SA3(J)+CA3(
     1J))-EA4(J)*(BETA2*SA4(J)+CA4(J))))
      S9(J)=.25*(EA3(J)-EA4(J)-1./(1.+BETA33)*(EA3(J)*(BETA3*SA5(J)+CA5(
     1J))-EA4(J)*(BETA3*SA6(J)+CA6(J))))
```

```
 S10(J)=.5*(1./(4.+BETDD12)*(EA3(J)*(BETD12*SA7(J)+2.*CA7(J))-EA4(J
1)*(BETD12*SA8(J)+2.*CA8(J)))-1./(4.+BETSS12)*(EA3(J)*(BETS12*SA9(J
2)+2.*CA9(J))-EA4(J)*(BETS12*SA10(J)+2.*CA10(J))))
 S11(J)=.5*(1./(4.+BETDD13)*(EA3(J)*(BETD13*SA11(J)+2.*CA11(J))-EA4
1(J)*(BETD13*SA12(J)+2.*CA12(J)))-1./(4.+BETSS13)*(EA3(J)*(BETS13*S
2A13(J)+2.*CA13(J))-EA4(J)*(BETS13*SA14(J)+2.*CA14(J))))
 S12(J)=.5*(1./(4.+BETDD23)*(EA3(J)*(BETD23*SA15(J)+2.*CA15(J))-EA4
2(J)*(BETD23*SA16(J)+2.*CA16(J)))-1./(4.+BETSS23)*(EA3(J)*(BETS23*S
2A17(J)+2.*CA17(J))-EA4(J)*(BETS23*SA18(J)+2.*CA18(J))))
 S13(J)=.5*(TAU(J)-PHI(J))+.25/BETA1*(SA1(J)-SA2(J))
 S14(J)=.5*(TAU(J)-PHI(J))+.25/BETA2*(SA3(J)-SA4(J))
 S15(J)=.5*(TAU(J)-PHI(J))+.25/BETA3*(SA5(J)-SA6(J))
 S16(J)=.5*(1./BETD12*(SA7(J)-SA8(J))+1./BETS12*(SA9(J)-SA10(J)))
 S17(J)=.5*(1./BETD13*(SA11(J)-SA12(J))+1./BETS13*(SA13(J)-SA14(J))
1)
 S18(J)=.5*(1./BETD23*(SA15(J)-SA16(J))+1./BETS23*(SA17(J)-SA18(J))
1)
 S19(J)=.5*(TAU(J)-PHI(J))-.25/BETA1*(SA1(J)-SA2(J))
 S20(J)=.5*(TAU(J)-PHI(J))-.25/BETA2*(SA3(J)-SA4(J))
 S21(J)=.5*(TAU(J)-PHI(J))-.25/BETA3*(SA5(J)-SA6(J))
 S22(J)=.5*(1./BETD12*(SA7(J)-SA8(J))-1./BETS12*(SA9(J)-SA10(J)))
 S23(J)=.5*(1./BETD13*(SA11(J)-SA12(J))-1./BETS13*(SA13(J)-SA14(J))
1)
 S24(J)=.5*(1./BETD23*(SA15(J)-SA16(J))-1./BETS23*(SA17(J)-SA18(J))
1)
 S25(J)=-.25/BETA1*(CA1(J)-CA2(J))
 S26(J)=-.5*(1./BETD12*(CA7(J)-CA8(J))+1./BETS12*(CA9(J)-CA10(J)))
 S27(J)=-.5*(1./BETD13*(CA11(J)-CA12(J))+1./BETS13*(CA13(J)-CA14(J)
1))
 S28(J)=-.5*(1./BETD21*(CA19(J)-CA20(J))+1./BETS12*(CA9(J)-CA10(J))
1)
 S29(J)=-.25/BETA2*(CA3(J)-CA4(J))
 S30(J)=-.5*(1./BETD23*(CA15(J)-CA16(J))+1./BETS23*(CA17(J)-CA18(J)
1))
 S31(J)=-.5*(1./BETD31*(CA21(J)-CA22(J))+1./BETS13*(CA13(J)-CA14(J)
1))
 S32(J)=-.5*(1./BETD32*(CA23(J)-CA24(J))+1./BETS23*(CA17(J)-CA18(J)
1))
 S33(J)=-.25/BETA3*(CA5(J)-CA6(J))
 S34(J)=.5*(EA5(J)-EA6(J)-1./(1.+4.*BETA11)*(EA5(J)*(2.*BETA1*SA1(J
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1)+CA1(J))+EA6(J)*(2.*BETA1*SA2(J)+CA2(J))))
 S35(J)=.5*(EA5(J)-EA6(J)-1./(1.+4.*BETA22)*(EA5(J)*(2.*BETA2*SA3(J
1)+CA3(J))+EA6(J)*(2.*BETA2*SA4(J)+CA5(J))))
 S36(J)=.5*(EA5(J)-EA6(J)-1./(1.+4.*BETA33)*(EA5(J)*(2.*BETA3*SA5(
1J)+CA5(J))+EA6(J)*(2.*BETA3*SA6(J)+CA6(J))))
 S37(J)=.5*(1./(1.+BETDD12)*(EA5(J)*(BETD12*SA7(J)+CA7(J))-EA6(J)*(
1BETD12*SA8(J)+CA8(J)))-1./(1.+BETSS12)*(EA5(J)*(BETS12*SA9(J)+CA9(
1J))-EA6(J)*(BETS12*SA10(J)+CA10(J))))
 S38(J)=.5*(1./(1.+BETDD13)*(EA5(J)*(BETD13*SA11(J)+CA11(J))-EA6(J)
1*(BETD13*SA12(J)+CA12(J)))-1./(1.+BETSS13)*(EA5(J)*(BETS13*SA13(J)
1+CA13(J))-EA6(J)*(BETS13*SA14(J)+CA14(J))))
 S39(J)=.5*(1./(1.+BETDD23)*(EA5(J)*(BETD23*SA15(J)+CA15(J))-EA6(J)
1*(BETD23*SA16(J)+CA16(J)))-1./(1.+BETSS23)*(EA5(J)*(BETS23*SA17(J
2)+CA17(J))-EA6(J)*(BETS23*SA18(J)+CA18(J))))
 S40(J)=.5*(1./FMU*(EA7(J)-EA8(J))+1./(FMU2+4.*BETA11)*(EA7(J)*(2.*
1BETA1*SA1(J)+FMU*CA1(J))-EA8(J)*(2.*BETA1*SA2(J)+FMU*CA2(J))))
 S41(J)=.5*(1./FMU*(EA7(J)-EA8(J))+1./(FMU2+4.*BETA22)*(EA7(J)*(2.*
1BETA2*SA3(J)+FMU*CA3(J))-EA8(J)*(2.*BETA2*SA4(J)+FMU*CA4(J))))
 S42(J)=.5*(1./FMU*(EA7(J)-EA8(J))+1./(FMU2+4.*BETA33)*(EA7(J)*(2.*
1BETA3*SA5(J)+FMU*CA5(J))-EA8(J)*(2.*BETA3*SA6(J)+FMU*CA6(J))))
 S43(J)=.5*(1./(FMU2+BETDD12)*(EA7(J)*(BETD12*SA7(J)+FMU*CA7(J))-EA
18(J)*(BETD12*SA8(J)+FMU*CA8(J)))+1./(FMU2+BETSS12)*(EA7(J)*(BETS12
2*SA9(J)+FMU*CA9(J))-EA8(J)*(BETS12*SA10(J)+FMU*CA10(J))))
 S44(J)=.5*(1./(FMU2+BETDD13)*(EA7(J)*(BETD13*SA11(J)+FMU*CA11(J))-
1EA8(J)*(BETD13*SA12(J)+FMU*CA12(J)))+1./(FMU2+BETSS13)*(EA7(J)*(BE
2TS13*SA13(J)+FMU*CA13(J))-EA8(J)*(BETS13*SA14(J)+FMU*CA14(J))))
 S45(J)=.5*(1./(FMU2+BETDD23)*(EA7(J)*(BETD23*SA15(J)+FMU*CA15(J))-
1EA8(J)*(BETD23*SA16(J)+FMU*CA16(J)))+1./(FMU2+BETSS23)*(EA7(J)*(BE
2TS23*SA17(J)+FMU*CA17(J))-EA8(J)*(BETS23*SA18(J)+FMU*CA18(J))))
 S46(J)=.5*(-1./XMU*(EA9(J)-EA10(J))+1./(XMU2+4.*BETA11)*(EA9(J)*(2
1.*BETA1*SA1(J)-XMU*CA1(J))-EA10(J)*(2.*BETA1*SA2(J)-XMU*CA2(J))))
 S47(J)=.5*(-1./XMU*(EA9(J)-EA10(J))+1./(XMU2+4.*BETA22)*(EA9(J)*(2
1.*BETA2*SA3(J)-XMU*CA3(J))-EA10(J)*(2.*BETA2*SA4(J)-XMU*CA4(J))))
 S48(J)=.5*(-1./XMU*(EA9(J)-EA10(J))+1./(XMU2+4.*BETA33)*(EA9(J)*(2
1.*BETA3*SA5(J)-XMU*CA5(J))-EA10(J)*(2.*BETA3*SA6(J)-XMU*CA6(J))))
 S49(J)=.5*(1./(XMU2+BETDD12)*(EA9(J)*(BETD12*SA7(J)-XMU*CA7(J))-EA
110(J)*(BETD12*SA8(J)-XMU*CA8(J)))+1./(XMU2+BETSS12)*(EA9(J)*(BETS1
22*SA9(J)-XMU*CA9(J))-EA10(J)*(BETS12*SA10(J)-XMU*CA10(J))))
 S50(J)=.5*(1./(XMU2+BETDD13)*(EA9(J)*(BETD13*SA11(J)-XMU*CA11(J))-
1EA10(J)*(BETD13*SA12(J)-XMU*CA12(J)))+1./(XMU2+BETSS13)*(EA9(J)*(B
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2ETS13*SA13(J)-XMU*CA13(J))-EA10(J)*(BETS13*SA14(J)-XMU*CA14(J))))
 S51(J)=.5*(1./(XMU2+BETDD23)*(EA9(J)*(BETD23*SA15(J)-XMU*CA15(J))-
2EA10(J)*(BETD23*SA16(J)-XMU*CA16(J)))+1./(XMU2+BETSS23)*(EA9(J)*(B
3ETS23*SA17(J)-XMU*CA17(J))-EA10(J)*(BETS23*SA18(J)-XMU*CA18(J))))
 S52(J)=.5*(-1./XMU*(EA9(J)-EA10(J))-1./(XMU2+4.*BETA11)*(EA9(J)*(2
1.*BETA1*SA1(J)-XMU*CA1(J))-EA10(J)*(2.*BETA1*SA2(J)-XMU*CA2(J))))
 S53(J)=.5*(-1./XMU*(EA9(J)-EA10(J))-1./(XMU2+4.*BETA22)*(EA9(J)*(2
1.*BETA2*SA3(J)-XMU*CA3(J))-EA10(J)*(2.*BETA2*SA4(J)-XMU*CA4(J))))
 S54(J)=.5*(-1./XMU*(EA9(J)-EA10(J))-1./(XMU2+4.*BETA33)*(EA9(J)*(2
1.*BETA3*SA5(J)-XMU*CA5(J))-EA10(J)*(2.*BETA3*SA6(J)-XMU*CA6(J))))
 S55(J)=.5*(1./(XMU2+BETDD12)*(EA9(J)*(BETD12*SA7(J)-XMU*CA7(J))-EA
110(J)*(BETD12*SA8(J)-XMU*CA8(J)))-1./(XMU2+BETSS12)*(EA9(J)*(BETS1
22*SA9(J)-XMU*CA9(J))-EA10(J)*(BETS12*SA10(J)-XMU*CA10(J))))
 S56(J)=.5*(1./(XMU2+BETDD13)*(EA9(J)*(BETD13*SA11(J)-XMU*CA11(J))-
1EA10(J)*(BETD13*SA12(J)-XMU*CA12(J)))-1./(XMU2+BETSS1
13)*(EA9(J)*(BETS13*SA13(J)-XMU*CA13(J))-EA10(J)*(BETS13*SA14(J)-XM
2U*CA14(J))))
 S57(J)=.5*(1./(XMU2+BETDD23)*(EA9(J)*(BETD23*SA15(J)-XMU*CA15(J))
1-EA10(J)*(BETD23*SA16(J)-XMU*CA16(J)))-1./(XMU2+BETSS23)*(EA9(J)
2*(BETS23*SA17(J)-XMU*CA17(J))-EA10(J)*(BETS23*SA18(J)-XMU*CA18
3(J))))
 S58(J)=.5/(XMU2+4.*BETA11)*(EA9(J)*(-XMU*SA1(J)-2.*BETA1*CA1(J))
2-EA10(J)*(-XMU*SA2(J)-2.*BETA1*CA2(J)))
 S59(J)=.5*(1./(XMU2+BETDD12)*(EA9(J)*(-XMU*SA7(J)-BETD12*CA7(J
2))-EA10(J)*(-XMU*SA8(J)-BETD12*CA8(J)))+1./(XMU2+BETSS12)*(EA
39(J)*(-XMU*SA9(J)-BETS12*CA9(J))-EA10(J)*(-XMU*SA10(J)-BETS12*
4CA10(J))))
 S60(J)=.5*(1./(XMU2+BETDD13)*(EA9(J)*(-XMU*SA11(J)-BETD13*CA11
1(J))-EA10(J)*(-XMU*SA12(J)-BETD13*CA12(J)))+1./(XMU2+BETSS13)*
2(EA9(J)*(-XMU*SA13(J)-BETS13*CA13(J))-EA10(J)*(-XMU*SA14(J)-
3BETS13*CA14(J))))
 S61(J)=.5*(1./(XMU2+BETDD12)*(EA9(J)*(-XMU*SA19(J)-BETD21*CA19
1(J))-EA10(J)*(-XMU*SA20(J)-BETD21*CA20(J)))+1./(XMU2+BETSS12)*
2(EA9(J)*(-XMU*SA9(J)-BETS12*CA9(J))-EA10(J)*(-XMU*SA10(J)-BETS
412*CA10(J))))
 S62(J)=.5/(XMU2+4.*BETA22)*(EA9(J)*(-XMU*SA3(J)-2.*BETA2*CA3(J))
1-EA10(J)*(-XMU*SA4(J)-2.*BETA2*CA4(J)))
 S63(J)=.5*(1./(XMU2+BETDD23)*(EA9(J)*(-XMU*SA15(J)-BETD23*CA15(J))
1-EA10(J)*(-XMU*SA16(J)-BETD23*CA16(J)))+1./(XMU2+BETSS23)*(EA9(J
2)*(-XMU*SA17(J)-BETS23*CA17(J))-EA10(J)*(-XMU*SA18(J)-BETS23*
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3CA18(J))))
 S64(J)=.5*(1./(XMU2+BETDD13)*(EA9(J)*(-XMU*SA21(J)-BETD31*CA21
1(J))-EA10(J)*(-XMU*SA22(J)-BETD31*CA22(J)))+1./(XMU2+BETSS13)*
2(EA9(J)*(-XMU*SA13(J)-BETS13*CA13(J))-EA10(J)*(-XMU*SA14(J)-
3BETS13*CA14(J))))
 S65(J)=.5*(1./(XMU2+BETDD23)*(EA9(J)*(-XMU*SA23(J)-BETD32*CA23(J))
1-EA10(J)*(-XMU*SA24(J)-BETD32*CA24(J)))+1./(XMU2+BETSS23)*(EA9(J)
2*(-XMU*SA17(J)-BETS23*CA17(J))-EA10(J)*(-XMU*SA18(J)-BETS23*
3CA18(J))))
 S66(J)=.5/(XMU2+4.*BETA33)*(EA9(J)*(-XMU*SA5(J)-2.*BETA3*CA5(J))
1-EA10(J)*(-XMU*SA6(J)-2.*BETA3*CA6(J)))
 S67(J)=-.25/XMU*(EA11(J)-EA12(J))+.25/(XMU2+BETA11)*(EA11(J)*
1(BETA1*SA1(J)-XMU*CA1(J))-EA12(J)*(BETA1*SA2(J)-XMU*CA2(J)))
 S68(J)=-.25/XMU*(EA11(J)-EA12(J))+.25/(XMU2+BETA22)*(EA11(J)*
1(BETA2*SA3(J)-XMU*CA3(J))-EA12(J)*(BETA2*SA4(J)-XMU*CA4(J)))
 S69(J)=-.25/XMU*(EA11(J)-EA12(J))+.25/(XMU2+BETA33)*(EA11(J)*
1(BETA3*SA5(J)-XMU*CA5(J))-EA12(J)*(BETA3*SA6(J)-XMU*CA6(J)))
 S70(J)=.5*(1./(4.*XMU2+BETDD12)*(EA11(J)*(BETD12*SA7(J)-2.*XMU*CA7
1(J))-EA12(J)*(BETD12*SA8(J)-2.*XMU*CA8(J)))+1./(4.*XMU2+BETSS12)
2*(EA11(J)*(BETS12*SA9(J)-2.*XMU*CA9(J))-EA12(J)*(BETS12*SA10(J)-2.
3*XMU*CA10(J))))
 S71(J)=.5*(1./(4.*XMU2+BETDD13)*(EA11(J)*(BETD13*SA11(J)-2.*XMU
1*CA11(J))-EA12(J)*(BETD13*SA12(J)-2.*XMU*CA12(J)))+1./(4.*XMU2
3+BETSS13)*(EA11(J)*(BETS13*SA13(J)-2.*XMU*CA13(J))-EA12(J)*(
4BETS13*SA14(J)-2.*XMU*CA14(J))))
 S72(J)=.5*(1./(4.*XMU2+BETDD23)*(EA11(J)*(BETD23*SA15(J)-2.*XMU
1*CA15(J))-EA12(J)*(BETD23*SA16(J)-2.*XMU*CA16(J)))+1./(4.*XMU2
2+BETSS23)*(EA11(J)*(BETS23*SA17(J)-2.*XMU*CA17(J))-EA12(J)*
3(BETS23*SA18(J)-2.*XMU*CA18(J))))
 S73(J)=.25/(XMU2+BETA11)*(EA11(J)*(-XMU*SA1(J)-BETA1*CA1(J))-EA
112(J)*(-XMU*SA2(J)-BETA1*CA2(J)))
 S74(J)=.5*(1./(4.*XMU2+BETDD12)*(EA11(J)*(-2.*XMU*SA7(J)-BETD
112*CA7(J))-EA12(J)*(-2.*XMU*SA8(J)-BETD12*CA8(J)))+1./(4.*XMU2+
2BETSS12)*(EA11(J)*(-2.*XMU*SA9(J)-BETS12*CA9(J))-EA12(J)*(-2.
3*XMU*SA10(J)-BETS12*CA10(J))))
 S75(J)=.5*(1./(4.*XMU2+BETDD13)*(EA11(J)*(-2.*XMU*SA11(J)-BETD13
1*CA11(J))-EA12(J)*(-2.*XMU*SA12(J)-BETD13*CA12(J)))+1./(4.*
2XMU2+BETSS13)*(EA11(J)*(-2.*XMU*SA13(J)-BETS13*CA13(J))-EA12(J)
3*(-2.*XMU*SA14(J)-BETS13*CA14(J))))
 S76(J)=.5*(1./(4.*XMU2+BETDD12)*(EA11(J)*(-2.*XMU*SA19(J)-BETD21
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      CS(J)=1./S(J)
      CT(J)=1./TN(J)
      TR1(J)=2.*((ETA1-XMU*ETA3+XMU2*ETA4)*S(J)+XN*XN*ETA2*CS(J))
      TR2(J)=(2.*XMU*ETA4-ETA3)*S(J)
      TR3(J)=2.*ETA4*S(J)
      TR4(J)=-1.*BETA1*ETA3*C(J)
      TR5(J)=(ETA1-XMU*ETA3)*C(J)
      TR6(J)=-1.*BETA2*ETA3*C(J)
      TR7(J)=-1.*BETA3*ETA3*C(J)
      TR8(J)=2.*ETA2*S(J)
      TR9(J)=2.*(XN*XN*ETA1*CS(J)+(ETA5+ETA6)*CS(J)*C(J)*C(J))
      TR10(J)=2.*XN*(ETA1+ETA5+ETA6)*CT(J)
      TR11(J)=-1.*ETA6*C(J)
      TR12(J)=-2.*ETA5*C(J)
      TR13(J)=2.*(ETA7+ETA8)*S(J)
      TR14(J)=2.*(ETA1*CS(J)*C(J)*C(J)+XN*XN*(ETA5+ETA6)*CS(J))
      TR15(J)=2.*BETA1*ETA8*S(J)
      TR16(J)=2.*BETA1*ETA7*S(J)
      TR17(J)=2.*BETA2*ETA8*S(J)
      TR18(J)=2.*BETA2*ETA7*S(J)
      TR19(J)=2.*BETA3*ETA8*S(J)
      TR20(J)=2.*BETA3*ETA7*S(J)
      TR21(J)=2.*ETA6*S(J)
      TR22(J)=ETA10*S(J)
      TR23(J)=2.*(ETA25*CS(J)*C(J)*C(J)+ETA10*S(J)+XN*XN*ETA26*CS(J))
      TR24(J)=ETA14*S(J)
      TR25(J)=ETA14*S(J)+XN*XN*ETA13*CS(J)
      TR26(J)=2.*ETA5*S(J)
      TR27(J)=2.*ETA24*S(J)
      TR28(J)=2.*(XN*XN*ETA18*CS(J)-ETA24*S(J))
      TR29(J)=2.*ETA8*S(J)
      TR30(J)=(2.*XMU*ETA11-ETA12)*S(J)
      TR31(J)=2.*ETA11*S(J)
      TR32(J)=2.*((ETA9+XMU2*ETA11)*S(J)+XN*XN*ETA10*CS(J))
      TR33(J)=(ETA13-XMU*(ETA15+ETA16)+XMU2*ETA17)*S(J)+XN*XN
     1*ETA14*CS(J)
      TR34(J)=(2.*XMU*ETA17-ETA15-ETA16)*S(J)
      TR35(J)=ETA17*S(J)
      TR36(J)=(XMU*ETA17-ETA16)*S(J)
      TR37(J)=(XMU*ETA17-ETA15)*S(J)
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      TR38(J)=2.*ETA7*S(J)
      TR39(J)=(2.*XMU*ETA27-ETA20)*S(J)
      TR40(J)=2.*ETA27*S(J)
      TR41(J)=2.*((ETA18-XMU*ETA20+XMU2*ETA27)*S(J)+XN*XN*ETA19*CS(J))
  500 CONTINUE
      BSUM=0.
       DO   1 J=1,K
      BSUM=BSUM+S(J)*XM*S1(J)
    1  CONTINUE
       B( 1, 1)=BSUM
      BSUM=0.
       DO   2 J=1,K
      BSUM=BSUM+S(J)*XM*S4(J)
    2  CONTINUE
       B( 1, 2)=BSUM
      BSUM=0.
       DO   3 J=1,K
      BSUM=BSUM+S(J)*XM*S5(J)
    3  CONTINUE
       B( 1, 3)=BSUM
      BSUM=0.
       DO   4 J=1,K
      BSUM=BSUM+S(J)*XM*S2(J)
    4  CONTINUE
       B( 2, 2)=BSUM
      BSUM=0.
       DO   5 J=1,K
      BSUM=BSUM+S(J)*XM*S6(J)
    5  CONTINUE
       B( 2, 3)=BSUM
      BSUM=0.
       DO   6 J=1,K
      BSUM=BSUM+S(J)*XM*S3(J)
    6  CONTINUE
       B( 3, 3)=BSUM
      BSUM=0.
       DO   7 J=1,K
      BSUM=BSUM+S(J)*XM*S7(J)
    7  CONTINUE
       B( 4, 4)=BSUM
```

```
      BSUM=0.
       DO  8 J=1,K
      BSUM=BSUM+S(J)*XM*S10(J)
  8    CONTINUE
       B( 4, 5)=BSUM
      BSUM=0.
       DO  9 J=1,K
      BSUM=BSUM+S(J)*XM*S11(J)
  9    CONTINUE
       B( 4, 6)=BSUM
      BSUM=0.
       DO 10 J=1,K
      BSUM=BSUM+S(J)*XM*S8(J)
 10    CONTINUE
       B( 5, 5)=BSUM
      BSUM=0.
       DO 11 J=1,K
      BSUM=BSUM+S(J)*XM*S9(J)
 11    CONTINUE
       B( 6, 6)=BSUM
      BSUM=0.
       DO 12 J=1,K
      BSUM=BSUM+S(J)*XM*S12(J)
 12    CONTINUE
       B( 5, 6)=BSUM
      BSUM=0.
       DO 13 J=1,K
      BSUM=BSUM+S(J)*XM*S10(J)
 13    CONTINUE
       B( 7, 8)=BSUM
      BSUM=0.
       DO 14 J=1,K
      BSUM=BSUM+S(J)*XM*S11(J)
 14    CONTINUE
       B( 7, 9)=BSUM
      BSUM=0.
       DO 15 J=1,K
      BSUM=BSUM+S(J)*XM*S8(J)
 15    CONTINUE
       B( 8, 8)=BSUM
```

```
      BSUM=0.
       DO  24 J=1,K
      BSUM=BSUM+S(J)*CI*S12(J)
   24  CONTINUE
       B(11,12)=BSUM
      BSUM=0.
       DO  25 J=1,K
      BSUM=BSUM+S(J)*FI*S7(J)*2.
   25  CONTINUE
       B(13,13)=BSUM
      BSUM=0.
       DO  26 J=1,K
      BSUM=BSUM+S(J)*FI*S8(J)*2.
   26  CONTINUE
       B(14,14)=BSUM
      BSUM=0.
       DO  27 J=1,K
      BSUM=BSUM+S(J)*FI*S9(J)*2.
   27  CONTINUE
       B(15,15)=BSUM
      BSUM=0.
       DO  28 J=1,K
      BSUM=BSUM+S(J)*FI*S12(J)*2.
   28  CONTINUE
       B(14,15)=BSUM
      BSUM=0.
       DO  29 J=1,K
      BSUM=BSUM+S(J)*CI*S4(J)
   29  CONTINUE
       B(16,17)=BSUM
      BSUM=0.
       DO  30 J=1,K
      BSUM=BSUM+S(J)*CI*S1(J)
   30  CONTINUE
       B(16,16)=BSUM
      BSUM=0.
       DO  31 J=1,K
      BSUM=BSUM+S(J)*CI*S5(J)
   31  CONTINUE
       B(16,18)=BSUM
```

```
      BSUM=0.
       DO 32 J=1,K
      BSUM=BSUM+S(J)*CI*S2(J)
32     CONTINUE
       B(17,17)=BSUM
      BSUM=0.
       DO 33 J=1,K
      BSUM=BSUM+S(J)*CI*S6(J)
33     CONTINUE
       B(17,18)=BSUM
      BSUM=0.
       DO 34 J=1,K
      BSUM=BSUM+S(J)*CI*S3(J)
       B(18,18)=BSUM
34     CONTINUE
      BSUM=0.
       DO 35 J=1,K
      BSUM=BSUM+2.*S(J)*FI*S1(J)
35     CONTINUE
       B(19,19)=BSUM
      BSUM=0.
       DO 36 J=1,K
      BSUM=BSUM+2.*S(J)*FI*S4(J)
36     CONTINUE
       B(19,20)=BSUM
      BSUM=0.
       DO 37 J=1,K
      BSUM=BSUM+2.*S(J)*FI*S5(J)
37     CONTINUE
       B(19,21)=BSUM
      BSUM=0.
       DO 38 J=1,K
      BSUM=BSUM+2.*S(J)*FI*S2(J)
38     CONTINUE
       B(20,20)=BSUM
      BSUM=0.
       DO 39 J=1,K
      BSUM=BSUM+2.*S(J)*FI*S6(J)
       B(20,21)=BSUM
39     CONTINUE
```

```
      BSUM=0.
       DO 40 J=1,K
      BSUM=BSUM+2.*S(J)*FI*S3(J)
 40    CONTINUE
       B(21,21)=BSUM
      BSUM=0.
       DO 41 J=1,K
      BSUM=BSUM+2.*S(J)*FI*S10(J)
 41    CONTINUE
       B(13,14)=BSUM
      BSUM=0.
       DO 42 J=1,K
      BSUM=BSUM+2.*S(J)*FI*S11(J)
 42    CONTINUE
       B(13,15)=BSUM
      ASUM=0.
       DO 43 J=1,K
      ASUM=ASUM+S67(J)*TR1(J)+2.*S73(J)*BETA1*TR2(J)+S82(J)*BETA22*TR3(J
     1)
 43    CONTINUE
       A( 1, 1)=ASUM
      ASUM=0.
       DO 44 J=1,K
      ASUM=ASUM+S70(J)*TR1(J)+(S74(J)*BETA1+S76(J)*BETA2)*TR2(J)+S85(J)*
     1BETA12*TR3(J)
 44    CONTINUE
       A( 1, 2)=ASUM
      ASUM=0.
       DO 45 J=1,K
      ASUM=ASUM+S71(J)*TR1(J)+(S75(J)*BETA1+S79(J)*BETA2)*TR2(J)+S86(J)*
     1BETA13*TR3(J)
 45    CONTINUE
       A( 1, 3)=ASUM
      ASUM=0.
       DO 46 J=1,K
      ASUM=ASUM+XN*(S46(J)*BETA1*CT1+S52(J)*CT2*S58(J)*CT3)
 46    CONTINUE
       A( 1, 4)=ASUM
      ASUM=0.
       DO 47 J=1,K
```

```
      ASUM=ASUM+XN*(S49(J)*BETA2*CT1+S55(J)*CT2+S61(J)*CT3)
47     CONTINUE
       A( 1, 5)=ASUM
      ASUM=0.
       DO 48 J=1,K
      ASUM=ASUM+XN*(S50(J)*BETA3*CT1+S56(J)*CT2+S64(J)*CT3)
48     CONTINUE
       A( 1, 6)=ASUM
      ASUM=0.
       DO 49 J=1,K
      ASUM=ASUM+S52(J)*TR4(J)+S58(J)*TR5(J)
49     CONTINUE
       A( 1, 7)=ASUM
      ASUM=0.
       DO 50 J=1,K
      ASUM=ASUM+S55(J)*TR4(J)+S61(J)*TR5(J)
50     CONTINUE
       A( 1, 8)=ASUM
      ASUM=0.
       DO 51 J=1,K
      ASUM=ASUM+S56(J)*TR4(J)+S64(J)*TR5(J)
51     CONTINUE
       A( 1, 9)=ASUM
      ASUM=0.
       DO 52 J=1,K
      ASUM=ASUM+S68(J)*TR1(J)+2.*S77(J)*BETA2*TR2(J)+S83(J)*BETA22*TR3(J
     1)
52     CONTINUE
       A( 2, 2)=ASUM
      ASUM=0.
       DO 53 J=1,K
      ASUM=ASUM+S72(J)*TR1(J)+(S78(J)*BETA2+S80(J)*BETA3)*TR2(J)+S87(J)*
     1BETA23*TR3(J)
53     CONTINUE
       A( 2, 3)=ASUM
      ASUM=0.
       DO 54 J=1,K
      ASUM=ASUM+XN*(S49(J)*BETA1*CT1+S55(J)*CT4+S59(J)*CT3)
54     CONTINUE
       A( 2, 4)=ASUM
```

```
      ASUM=0.
       DO 55 J=1,K
      ASUM=ASUM+XN*(S47(J)*BETA2*CT1+S53(J)*CT4+S62(J)*CT3)
55     CONTINUE
       A( 2, 5)=ASUM
      ASUM=0.
       DO 56 J=1,K
      ASUM=ASUM+XN*(S51(J)*BETA3*CT1+S57(J)*CT4+S67(J)*CT3)
56     CONTINUE
       A( 2, 6)=ASUM
      ASUM=0.
       DO 57 J=1,K
      ASUM=ASUM+S55(J)*TR6(J)+S59(J)*TR5(J)
57     CONTINUE
       A( 2, 7)=ASUM
      ASUM=0.
       DO 58 J=1,K
      ASUM=ASUM+S53(J)*TR6(J)+S62(J)*TR5(J)
58     CONTINUE
       A( 2, 8)=ASUM
      ASUM=0.
       DO 59 J=1,K
      ASUM=ASUM+S57(J)*TR6(J)+S65(J)*TR5(J)
59     CONTINUE
       A( 2, 9)=ASUM
      ASUM=0.
       DO 60 J=1,K
      ASUM=ASUM+S69(J)*TR1(J)+2.*S81(J)*BETA3*TR2(J)+S84(J)*BETA33*TR3(J
     1)
60     CONTINUE
       A( 3, 3)=ASUM
      ASUM=0.
       DO 61 J=1,K
      ASUM=ASUM+XN*(S50(J)*BETA1*CT1+S56(J)*CT5+S60(J)*CT3)
61     CONTINUE
       A( 3, 4)=ASUM
      ASUM=0.
       DO 62 J=1,K
      ASUM=ASUM+XN*(S51(J)*BETA2*CT1+S57(J)*CT5+S63(J)*CT3)
62     CONTINUE
```

```
      A( 3, 5)=ASUM
      ASUM=0.
      DO 63 J=1,K
      ASUM=ASUM+XN*(S48(J)*BETA3*CT1+S54(J)*CT5+S66(J)*CT3)
   63 CONTINUE
      A( 3, 6)=ASUM
      ASUM=0.
      DO 64 J=1,K
      ASUM=ASUM+S56(J)*TR7(J)+S60(J)*TR5(J)
   64 CONTINUE
      A( 3, 7)=ASUM
      ASUM=0.
      DO 65 J=1,K
      ASUM=ASUM+S57(J)*TR7(J)+S63(J)*TR5(J)
   65 CONTINUE
      A( 3, 8)=ASUM
      ASUM=0.
      DO 66 J=1,K
      ASUM=ASUM+S54(J)*TR7(J)+S66(J)*TR5(J)
   66 CONTINUE
      A( 3, 9)=ASUM
      ASUM=0.
      DO 67 J=1,K
      ASUM=ASUM+(S13(J)*BETA11-2.*S25(J)*BETA1)*TR8(J)+S19(J)*TR9(J)
   67 CONTINUE
      A( 4, 4)=ASUM
      ASUM=0.
      DO 68 J=1,K
      ASUM=ASUM+(S16(J)*BETA12-S26(J)*BETA2-S28(J)*BETA1)*TR8(J)+S22(J)*
     1TR9(J)
   68 CONTINUE
      A( 4, 5)=ASUM
      ASUM=0.
      DO 69 J=1,K
      ASUM=ASUM+(S17(J)*BETA13-S27(J)*BETA3-S31(J)*BETA1)*TR8(J)+S23(J)*
     1TR9(J)
   69 CONTINUE
      A( 4, 6)=ASUM
      ASUM=0.
      DO 70 J=1,K
```

```
      ASUM=ASUM+S19(J)*TR10(J)
70     CONTINUE
       A( 4, 7)=ASUM
      ASUM=0.
       DO  71 J=1,K
      ASUM=ASUM+S22(J)*TR10(J)
71     CONTINUE
       A( 4, 8)=ASUM
      ASUM=0.
       DO  72 J=1,K
      ASUM=ASUM+S23(J)*TR10(J)
72     CONTINUE
       A( 4, 9)=ASUM
      ASUM=0.
       DO  73 J=1,K
      ASUM=ASUM+S34(J)*TR11(J)
73     CONTINUE
       A( 4,10)=ASUM
      ASUM=0.
       DO  74 J=1,K
      ASUM=ASUM+S37(J)*TR11(J)
74     CONTINUE
       A( 4,11)=ASUM
      ASUM=0.
       DO  75 J=1,K
      ASUM=ASUM+S38(J)*TR11(J)
75     CONTINUE
       A( 4,12)=ASUM
      ASUM=0.
       DO  76 J=1,K
      ASUM=ASUM+S34(J)*TR12(J)
76     CONTINUE
       A( 4,13)=ASUM
      ASUM=0.
       DO  77 J=1,K
      ASUM=ASUM+S37(J)*TR12(J)
77     CONTINUE
       A( 4,14)=ASUM
      ASUM=0.
       DO  78 J=1,K
```

```
      ASUM=ASUM+S38(J)*TR12(J)
78     CONTINUE
       A( 4,15)=ASUM
      ASUM=0.
       DO  79  J=1,K
      ASUM=ASUM+(S14(J)*BETA22-2.*S29(J)*BETA2)*TR8(J)+S20(J)*TR9(J)
79     CONTINUE
       A( 5, 5)=ASUM
      ASUM=0.
       DO  80  J=1,K
      ASUM=ASUM+(S18(J)*BETA23-S30(J)*BETA3-S32(J)*BETA2)*TR8(J)+S24(J)*
     1TR9(J)
80     CONTINUE
       A( 5, 6)=ASUM
      ASUM=0.
       DO  81  J=1,K
      ASUM=ASUM+S22(J)*TR10(J)
81     CONTINUE
       A( 5, 7)=ASUM
      ASUM=0.
       DO  82  J=1,K
      ASUM=ASUM+S20(J)*TR10(J)
82     CONTINUE
       A( 5, 8)=ASUM
      ASUM=0.
       DO  83  J=1,K
      ASUM=ASUM+S24(J)*TR10(J)
83     CONTINUE
       A( 5, 9)=ASUM
      ASUM=0.
       DO  84  J=1,K
      ASUM=ASUM+S37(J)*TR11(J)
84     CONTINUE
       A( 5,10)=ASUM
      ASUM=0.
       DO  85  J=1,K
      ASUM=ASUM+S35(J)*TR11(J)
85     CONTINUE
       A( 5,11)=ASUM
      ASUM=0.
```

```
      DO  94  J=1,K
      ASUM=ASUM+S38(J)*TR11(J)
 94    CONTINUE
       A( 6,10)=ASUM
      ASUM=0.
       DO  95  J=1,K
      ASUM=ASUM+S39(J)*TR11(J)
 95    CONTINUE
       A( 6,11)=ASUM
      ASUM=0.
       DO  96  J=1,K
      ASUM=ASUM+S36(J)*TR11(J)
 96    CONTINUE
       A( 6,12)=ASUM
      ASUM=0.
       DO  97  J=1,K
      ASUM=ASUM+S38(J)*TR12(J)
 97    CONTINUE
       A( 6,13)=ASUM
      ASUM=0.
       DO  98  J=1,K
      ASUM=ASUM+S39(J)*TR12(J)
 98    CONTINUE
       A( 6,14)=ASUM
      ASUM=0.
       DO  99  J=1,K
      ASUM=ASUM+S36(J)*TR12(J)
 99    CONTINUE
       A( 6,15)=ASUM
      ASUM=0.
       DO 100  J=1,K
      ASUM=ASUM+S13(J)*BETA11*TR13(J)+S19(J)*TR14(J)
100    CONTINUE
       A( 7, 7)=ASUM
      ASUM=0.
       DO 101  J=1,K
      ASUM=ASUM+S16(J)*BETA12*TR13(J)+S22(J)*TR14(J)
101    CONTINUE
       A( 7, 8)=ASUM
      ASUM=0.
```

```
      DO 102 J=1,K
      ASUM=ASUM+S17(J)*BETA13*TR13(J)+S23(J)*TR14(J)
102    CONTINUE
       A( 7, 9)=ASUM
      ASUM=0.
       DO 103 J=1,K
      ASUM=ASUM+S34(J)*CT6
103    CONTINUE
       A( 7,10)=ASUM
      ASUM=0.
       DO 104 J=1,K
      ASUM=ASUM+S37(J)*CT6
104    CONTINUE
       A( 7,11)=ASUM
      ASUM=0.
       DO 105 J=1,K
      ASUM=ASUM+S38(J)*CT6
105    CONTINUE
       A( 7,12)=ASUM
      ASUM=0.
       DO 106 J=1,K
      ASUM=ASUM+S34(J)*CT7
106    CONTINUE
       A( 7,13)=ASUM
      ASUM=0.
       DO 107 J=1,K
      ASUM=ASUM+S37(J)*CT7
107    CONTINUE
       A( 7,14)=ASUM
      ASUM=0.
       DO 108 J=1,K
      ASUM=ASUM+S38(J)*CT7
108    CONTINUE
       A( 7,15)=ASUM
      ASUM=0.
       DO 109 J=1,K
      ASUM=ASUM+S40(J)*TR15(J)
109    CONTINUE
       A( 7,16)=ASUM
      ASUM=0.
```

```
      DO 110 J=1,K
      ASUM=ASUM+S43(J)*TR15(J)
110   CONTINUE
      A( 7,17)=ASUM
      ASUM=0.
      DO 111 J=1,K
      ASUM=ASUM+S44(J)*TR15(J)
111   CONTINUE
      A( 7,18)=ASUM
      ASUM=0.
      DO 112 J=1,K
      ASUM=ASUM+S40(J)*TR16(J)
112   CONTINUE
      A( 7,19)=ASUM
      ASUM=0.
      DO 113 J=1,K
      ASUM=ASUM+S43(J)*TR16(J)
113   CONTINUE
      A( 7,20)=ASUM
      ASUM=0.
      DO 114 J=1,K
      ASUM=ASUM+S44(J)*TR16(J)
114   CONTINUE
      A( 7,21)=ASUM
      ASUM=0.
      DO 115 J=1,K
      ASUM=ASUM+S14(J)*BETA22*TR13(J)+S20(J)*TR14(J)
115   CONTINUE
      A( 8, 8)=ASUM
      ASUM=0.
      DO 116 J=1,K
      ASUM=ASUM+S18(J)*BETA23*TR13(J)+S24(J)*TR14(J)
116   CONTINUE
      A( 8, 9)=ASUM
      ASUM=0.
      DO 117 J=1,K
      ASUM=ASUM+S37(J)*CT6
117   CONTINUE
      A( 8,10)=ASUM
      ASUM=0.
```

```
      DO 118 J=1,K
      ASUM=ASUM+S35(J)*CT6
118    CONTINUE
       A( 8,11)=ASUM
      ASUM=0.
       DO 119 J=1,K
      ASUM=ASUM+S39(J)*CT6
119    CONTINUE
       A( 8,12)=ASUM
      ASUM=0.
       DO 120 J=1,K
      ASUM=ASUM+S37(J)*CT7
120    CONTINUE
       A( 8,13)=ASUM
      ASUM=0.
       DO 121 J=1,K
      ASUM=ASUM+S35(J)*CT7
121    CONTINUE
       A( 8,14)=ASUM
      ASUM=0.
       DO 122 J=1,K
      ASUM=ASUM+S39(J)*CT7
122    CONTINUE
       A( 8,15)=ASUM
      ASUM=0.
       DO 123 J=1,K
      ASUM=ASUM+S43(J)*TR17(J)
123    CONTINUE
       A( 8,16)=ASUM
      ASUM=0.
       DO 124 J=1,K
      ASUM=ASUM+S41(J)*TR17(J)
124    CONTINUE
       A( 8,17)=ASUM
      ASUM=0.
       DO 125 J=1,K
      ASUM=ASUM+S45(J)*TR17(J)
125    CONTINUE
       A( 8,18)=ASUM
      ASUM=0.
```

```
      DO 126 J=1,K
      ASUM=ASUM+S43(J)*TR18(J)
  126 CONTINUE
      A( 8,19)=ASUM
      ASUM=0.
      DO 127 J=1,K
      ASUM=ASUM+S41(J)*TR18(J)
  127 CONTINUE
      A( 8,20)=ASUM
      ASUM=0.
      DO 128 J=1,K
      ASUM=ASUM+S45(J)*TR18(J)
  128 CONTINUE
      A( 8,21)=ASUM
      ASUM=0.
      DO 129 J=1,K
      ASUM=ASUM+S15(J)*BETA33*TR13(J)+S21(J)*TR14(J)
  129 CONTINUE
      A( 9, 9)=ASUM
      ASUM=0.
      DO 130 J=1,K
      ASUM=ASUM+S38(J)*CT6
  130 CONTINUE
      A( 9,10)=ASUM
      ASUM=0.
      DO 131 J=1,K
      ASUM=ASUM+S39(J)*CT6
  131 CONTINUE
      A( 9,11)=ASUM
      ASUM=0.
      DO 132 J=1,K
      ASUM=ASUM+S36(J)*CT6
  132 CONTINUE
      A( 9,12)=ASUM
      ASUM=0.
      DO 133 J=1,K
      ASUM=ASUM+S38(J)*CT7
  133 CONTINUE
      A( 9,13)=ASUM
      ASUM=0.
```

```
      DO 134 J=1,K
      ASUM=ASUM+S39(J)*CT7
134   CONTINUE
      A( 9,14)=ASUM
      ASUM=0.
      DO 135 J=1,K
      ASUM=ASUM+S36(J)*CT7
135   CONTINUE
      A( 9,15)=ASUM
      ASUM=0.
      DO 136 J=1,K
      ASUM=ASUM+S44(J)*TR19(J)
136   CONTINUE
      A( 9,16)=ASUM
      ASUM=0.
      DO 137 J=1,K
      ASUM=ASUM+S45(J)*TR19(J)
137   CONTINUE
      A( 9,17)=ASUM
      ASUM=0.
      DO 138 J=1,K
      ASUM=ASUM+S42(J)*TR19(J)
138   CONTINUE
      A( 9,18)=ASUM
      ASUM=0.
      DO 139 J=1,K
      ASUM=ASUM+S44(J)*TR20(J)
139   CONTINUE
      A( 9,19)=ASUM
      ASUM=0.
      DO 140 J=1,K
      ASUM=ASUM+S45(J)*TR20(J)
140   CONTINUE
      A( 9,20)=ASUM
      ASUM=0.
      DO 141 J=1,K
      ASUM=ASUM+S42(J)*TR20(J)
141   CONTINUE
      A( 9,21)=ASUM
      ASUM=0.
```

```
      DO 142 J=1,K
      ASUM=ASUM+S7(J)*TR21(J)+(2.*S13(J)*BETA11-2.*S25(J)*BETA1)*TR22(J)
     1+S19(J)*TR23(J)
  142 CONTINUE
      A(10,10)=ASUM
      ASUM=0.
      DO 143 J=1,K
      ASUM=ASUM+S10(J)*TR21(J)+(2.*S16(J)*BETA12-S26(J)*BETA2-S28(J)*
     1BETA1)*TR22(J)+S22(J)*TR33(J)
  143 CONTINUE
      A(10,11)=ASUM
      ASUM=0.
      DO 144 J=1,K
      ASUM=ASUM+S11(J)*TR21(J)+(2.*S17(J)*BETA13-S27(J)*BETA3-S31(J)*
     1BETA1)*TR22(J)+S23(J)*TR23(J)
  144 CONTINUE
      A(10,12)=ASUM
      ASUM=0.
      DO 145 J=1,K
      ASUM=ASUM+(2.*S25(J)*BETA1-S13(J)*BETA11)*TR24(J)-S14(J)*TR25(J)
     1*TR25(J)
  145 CONTINUE
      A(10,13)=ASUM
      ASUM=0.
      DO 146 J=1,K
      ASUM=ASUM+(S26(J)*BETA2-S16(J)*BETA12+S28(J)*BETA1)*TR24(J)-S22(J)
     1*TR25(J)
  146 CONTINUE
      A(10,14)=ASUM
      ASUM=0.
      DO 147 J=1,K
      ASUM=ASUM+(S27(J)*BETA3-S17(J)*BETA13+S31(J)*BETA1)*TR24(J)-S23(J)
     $*TR25(J)
  147 CONTINUE
      A(10,15)=ASUM
      ASUM=0.
      DO 148 J=1,K
      ASUM=ASUM+XN*(S46(J)*CT8+S52(J)*CT9+S58(J)*CT12)
  148 CONTINUE
      A(10,16)=ASUM
```

```
      ASUM=0.
       DO 150 J=1,K
      ASUM=ASUM+XN*(S50(J)*CT8+S56(J)*CT11+S60(J)*CT12)
150    CONTINUE
       A(10,18)=ASUM
      ASUM=0.
       DO 151 J=1,K
      ASUM=ASUM+XN*(S46(J)*CT13+S52(J)*CT14+S58(J)*CT17)
151    CONTINUE
       A(10,19)=ASUM
      ASUM=0.
       DO 152 J=1,K
      ASUM=ASUM+XN*(S49(J)*CT13+S55(J)*CT15+S59(J)*CT17)
152    CONTINUE
       A(10,20)=ASUM
      ASUM=0.
       DO 153 J=1,K
      ASUM=ASUM+XN*(S50(J)*CT13+S56(J)*CT16+S60(J)*CT17)
153    CONTINUE
       A(10,21)=ASUM
      ASUM=0.
       DO 154 J=1,K
      ASUM=ASUM+S8(J)*TR21(J)+(2.*S14(J)*BETA22-2.*S29(J)*BETA2)*TR22(J)
     1+S20(J)*TR23(J)
154    CONTINUE
       A(11,11)=ASUM
      ASUM=0.
       DO 155 J=1,K
      ASUM=ASUM+S12(J)*TR21(J)+(2.*S18(J)*BETA23-S30(J)*BETA3-S32(J)*
     1BETA2)*TR22(J)+S24(J)*TR23(J)
155    CONTINUE
       A(11,12)=ASUM
      ASUM=0.
       DO 156 J=1,K
      ASUM=ASUM+(S26(J)*BETA2-S16(J)*BETA12+S28(J)*BETA1)*TR24(J)-S22(J)
     1*TR25(J)
156    CONTINUE
       A(11,13)=ASUM
      ASUM=0.
       DO 157 J=1,K
```

```
      ASUM=ASUM+(2.*S29(J)*BETA2-S14(J)*BETA22)*TR24(J)-S20(J)*TR25(J)
157    CONTINUE
       A(11,14)=ASUM
      ASUM=0.
       DO 158 J=1,K
      ASUM=ASUM+(S30(J)*BETA3-S18(J)*BETA23+S32(J)*BETA2)*TR24(J)-S24(J)
     1*TR25(J)
158    CONTINUE
       A(11,15)=ASUM
      ASUM=0.
       DO 159 J=1,K
      ASUM=ASUM+XN*(S49(J)*CT18+S55(J)*CT9+S61(J)*CT12)
159    CONTINUE
       A(11,16)=ASUM
      ASUM=0.
       DO 160 J=1,K
      ASUM=ASUM+XN*(S47(J)*CT18+S53(J)*CT10+S62(J)*CT12)
160    CONTINUE
       A(11,17)=ASUM
      ASUM=0.
       DO 161 J=1,K
      ASUM=ASUM+XN*(S51(J)*CT18+S57(J)*CT11+S63(J)*CT12)
161    CONTINUE
       A(11,18)=ASUM
      ASUM=0.
       DO 162 J=1,K
      ASUM=ASUM+XN*(S49(J)*CT19+S55(J)*CT14+S61(J)*CT17)
162    CONTINUE
       A(11,19)=ASUM
      ASUM=0.
       DO 163 J=1,K
      ASUM=ASUM+XN*(S47(J)*CT19+S53(J)*CT15+S62(J)*CT17)
163    CONTINUE
       A(11,20)=ASUM
      ASUM=0.
       DO 164 J=1,K
      ASUM=ASUM+XN*(S51(J)*CT19+S57(J)*CT16+S63(J)*CT17)
164    CONTINUE
       A(11,21)=ASUM
      ASUM=0.
```

```
      DO 165 J=1,K
      ASUM=ASUM+S9(J)*TR21(J)+(2.*S15(J)*BETA33-2.*S33(J)*BETA3)*TR22(J)
     1+S21(J)*TR23(J)
  165 CONTINUE
      A(12,12)=ASUM
      ASUM=0.
      DO 166 J=1,K
      ASUM=ASUM+(S27(J)*BETA3-S17(J)*BETA13+S31(J)*BETA1)*TR24(J)-S23(J)
     1*TR25(J)
  166 CONTINUE
      A(12,13)=ASUM
      ASUM=0.
      DO 167 J=1,K
      ASUM=ASUM+(S30(J)*BETA3-S18(J)*BETA23+S32(J)*BETA2)*TR24(J)
     1-S24(J)*TR25(J)
  167 CONTINUE
      A(12,14)=ASUM
      ASUM=0.
      DO 168 J=1,K
      ASUM=ASUM+(2.*BETA3*S33(J)-S15(J)*BETA33)*TR24(J)-S21(J)*TR25(J)
  168 CONTINUE
      A(12,15)=ASUM
      ASUM=0.
      DO 169 J=1,K
      ASUM=ASUM+XN*(S50(J)*CT20+S56(J)*CT9+S64(J)*CT12)
  169 CONTINUE
      A(12,16)=ASUM
      ASUM=0.
      DO 170 J=1,K
      ASUM=ASUM+XN*(S51(J)*CT20+S57(J)*CT10+S65(J)*CT12)
  170 CONTINUE
      A(12,17)=ASUM
      ASUM=0.
      DO 171 J=1,K
      ASUM=ASUM+XN*(S48(J)*CT20+S54(J)*CT11+S66(J)*CT12)
  171 CONTINUE
      A(12,18)=ASUM
      ASUM=0.
      DO 172 J=1,K
      ASUM=ASUM+XN*(S50(J)*CT21+S56(J)*CT14+S64(J)*CT17)
```

```
172   CONTINUE
      A(12,19)=ASUM
     ASUM=0.
      DO 173 J=1,K
     ASUM=ASUM+XN*(S51(J)*CT21+S57(J)*CT15+S65(J)*CT17)
173   CONTINUE
      A(12,20)=ASUM
     ASUM=0.
      DO 174 J=1,K
     ASUM=ASUM+XN*(S48(J)*CT21+S54(J)*CT16+S66(J)*CT17)
174   CONTINUE
      A(12,21)=ASUM
     ASUM=0.
      DO 175 J=1,K
     ASUM=ASUM+S7(J)*TR26(J)+(2.*S25(J)*BETA1-S13(J)*BETA11)*TR27(J)+
    1S19(J)*TR28(J)
175   CONTINUE
      A(13,13)=ASUM
     ASUM=0.
      DO 176 J=1,K
     ASUM=ASUM+S10(J)*TR26(J)+(S26(J)*BETA2-S16(J)*BETA12+S28(J)*BETA1)
    1*TR27(J)+S22(J)*TR28(J)
176   CONTINUE
      A(13,14)=ASUM
     ASUM=0.
      DO 177 J=1,K
     ASUM=ASUM+S11(J)*TR26(J)+(S27(J)*BETA3-S17(J)*BETA13+S31(J)*BETA1)
    1*TR27(J)+S23(J)*TR28(J)
177   CONTINUE
      A(13,15)=ASUM
     ASUM=0.
      DO 178 J=1,K
     ASUM=ASUM+XN*(S46(J)*CT13+S52(J)*CT22+S58(J)*CT25)
178   CONTINUE
      A(13,16)=ASUM
     ASUM=0.
      DO 179 J=1,K
     ASUM=ASUM+XN*(S49(J)*CT13+S55(J)*CT23+S59(J)*CT25)
179   CONTINUE
      A(13,17)=ASUM
```

```
      ASUM=0.
       DO 180 J=1,K
      ASUM=ASUM+XN*(S50(J)*CT13+S56(J)*CT24+S60(J)*CT25)
180    CONTINUE
       A(13,18)=ASUM
      ASUM=0.
       DO 181 J=1,K
      ASUM=ASUM+XN*(S46(J)*CT26-S52(J)*CT24+S58(J)*CT32)
181    CONTINUE
       A(13,19)=ASUM
      ASUM=0.
       DO 182 J=1,K
      ASUM=ASUM+XN*(S49(J)*CT26-S55(J)*CT30+S59(J)*CT32)
182    CONTINUE
       A(13,20)=ASUM
      ASUM=0.
       DO 183 J=1,K
      ASUM=ASUM+XN*(S50(J)*CT26-S56(J)*CT31+S60(J)*CT32)
183    CONTINUE
       A(13,21)=ASUM
      ASUM=0.
       DO 184 J=1,K
      ASUM=ASUM+S8(J)*TR26(J)+(2.*S29(J)*BETA2-S14(J)*BETA22)*TR27(J)+
     1S20(J)*TR28(J)
184    CONTINUE
       A(14,14)=ASUM
      ASUM=0.
       DO 185 J=1,K
      ASUM=ASUM+S12(J)*TR26(J)+(S30(J)*BETA3-S18(J)*BETA23*S32(J)*BETA2)
     1*TR27(J)+S24(J)*TR28(J)
185    CONTINUE
       A(14,15)=ASUM
      ASUM=0.
       DO 186 J=1,K
      ASUM=ASUM+XN*(S49(J)*CT19+S55(J)*CT22+S61(J)*CT25)
186    CONTINUE
       A(14,16)=ASUM
      ASUM=0.
       DO 187 J=1,K
      ASUM=ASUM+XN*(S47(J)*CT19+S53(J)*CT23+S62(J)*CT25)
```

```
187   CONTINUE
      A(14,17)=ASUM
     ASUM=0.
      DO 188 J=1,K
     ASUM=ASUM+XN*(S51(J)*CT19+S57(J)*CT24+S63(J)*CT25)
188   CONTINUE
      A(14,18)=ASUM
     ASUM=0.
      DO 189 J=1,K
     ASUM=ASUM+XN*(S49(J)*CT27-S55(J)*CT29+S61(J)*CT32)
189   CONTINUE
      A(14,19)=ASUM
     ASUM=0.
      DO 190 J=1,K
     ASUM=ASUM+XN*(S47(J)*CT27-S53(J)*CT30+S62(J)*CT32)
190   CONTINUE
      A(14,20)=ASUM
     ASUM=0.
      DO 191 J=1,K
     ASUM=ASUM+XN*(S51(J)*CT27-S57(J)*CT31+S63(J)*CT32)
191   CONTINUE
      A(14,21)=ASUM
     ASUM=0.
      DO 192 J=1,K
     ASUM=ASUM+S9(J)*TR26(J)+(2.*S33(J)*BETA3-S15(J)*BETA33)*TR27(J)+
    1S21(J)*TR28(J)
192   CONTINUE
      A(15,15)=ASUM
     ASUM=0.
      DO 193 J=1,K
     ASUM=ASUM+XN*(S50(J)*CT21+S56(J)*CT22+S64(J)*CT25)
193   CONTINUE
      A(15,16)=ASUM
     ASUM=0.
      DO 194 J=1,K
     ASUM=ASUM+XN*(S51(J)*CT21+S57(J)*CT23+S65(J)*CT25)
194   CONTINUE
      A(15,17)=ASUM
     ASUM=0.
      DO 195 J=1,K
```

```
      ASUM=ASUM+XN*(S48(J)*CT21+S54(J)*CT24+S66(J)*CT25)
195    CONTINUE
       A(15,18)=ASUM
      ASUM=0.
       DO 196 J=1,K
      ASUM=ASUM+XN*(S50(J)*CT28-S56(J)*CT29+S64(J)*CT32)
196    CONTINUE
       A(15,19)=ASUM
      ASUM=0.
       DO 197 J=1,K
      ASUM=ASUM+XN*(S51(J)*CT28-S57(J)*CT30+S65(J)*CT32)
197    CONTINUE
       A(15,20)=ASUM
      ASUM=0.
       DO 198 J=1,K
      ASUM=ASUM+XN*(S48(J)*CT28-S54(J)*CT31+S66(J)*CT32)
198    CONTINUE
       A(15,21)=ASUM
      ASUM=0.
       DO 199 J=1,K
      ASUM=ASUM+S1(J)*TR29(J)+2.*S73(J)*BETA1*TR30(J)+S82(J)*BETA11*TR31
     1(J)+S67(J)*TR32(J)
199    CONTINUE
       A(16,16)=ASUM
      ASUM=0.
       DO 200 J=1,K
      ASUM=ASUM+S4(J)*TR29(J)+(S74(J)*BETA1+S76(J)*BETA2)*TR30(J)+S85(J)
     1*BETA12*TR31(J)+S70(J)*TR32(J)
200    CONTINUE
       A(16,17)=ASUM
      ASUM=0.
       DO 201 J=1,K
      ASUM=ASUM+S5(J)*TR29(J)+(S75(J)*BETA1+S79(J)*BETA3)*TR30(J)+S86(J)
     1*BETA13*TR31(J)+S71(J)*TR32(J)
201    CONTINUE
       A(16,18)=ASUM
      ASUM=0.
       DO 202 J=1,K
      ASUM=ASUM+S67(J)*TR33(J)+S73(J)*BETA1*TR34(J)+S82(J)*BETA11*TR35(J
     1)
```

```
202   CONTINUE
      A(16,19)=ASUM
      ASUM=0.
      DO 203 J=1,K
      ASUM=ASUM+S70(J)*TR33(J)+S74(J)*BETA1*TR36(J)+S76(J)*BETA2*TR37(J)
     1+S85(J)*BETA12*TR35(J)
203   CONTINUE
      A(16,20)=ASUM
      ASUM=0.
      DO 204 J=1,K
      ASUM=ASUM+S71(J)*TR33(J)+S75(J)*BETA1*TR36(J)+S79(J)*BETA3*TR37(J)
     1+S86(J)*BETA13*TR35(J)
204   CONTINUE
      A(16,21)=ASUM
      ASUM=0.
      DO 205 J=1,K
      ASUM=ASUM+S2(J)*TR29(J)+2.*S77(J)*BETA2*TR30(J)+S83(J)*BETA22*TR31
     1(J)+S68(J)*TR32(J)
205   CONTINUE
      A(17,17)=ASUM
      ASUM=0.
      DO 206 J=1,K
      ASUM=ASUM+S6(J)*TR29(J)+(S78(J)*BETA2+S80(J)*BETA3)*TR30(J)+S87(J)
     1*BETA23*TR31(J)+S72(J)*TR32(J)
206   CONTINUE
      A(17,18)=ASUM
      ASUM=0.
      DO 207 J=1,K
      ASUM=ASUM+S70(J)*TR33(J)+S74(J)*BETA1*TR37(J)+S76(J)*BETA2*TR36(J)
     1+S85(J)*BETA12*TR35(J)
207   CONTINUE
      A(17,19)=ASUM
      ASUM=0.
      DO 208 J=1,K
      ASUM=ASUM+S68(J)*TR33(J)+S77(J)*BETA2*TR34(J)+S83(J)*BETA22*TR35(J
     1)
208   CONTINUE
      A(17,20)=ASUM
      ASUM=0.
      DO 209 J=1,K
```

```
      ASUM=ASUM+S72(J)*TR33(J)+S78(J)*BETA2*TR36(J)+S80(J)*BETA3*TR37(J)
     1+S87(J)*BETA23*TR35(J)
       A(17,21)=ASUM
209    CONTINUE
      ASUM=0.
       DO 210 J=1,K
      ASUM=ASUM+S3(J)*TR29(J)+2.*S81(J)*BETA3*TR30(J)+S84(J)*BETA33*TR31
     1(J)+S69(J)*TR32(J)
210    CONTINUE
       A(18,18)=ASUM
      ASUM=0.
       DO 211 J=1,K
      ASUM=ASUM+S71(J)*TR33(J)+S75(J)*BETA1*TR37(J)+S79(J)*BETA3*TR36(J)
     1+S86(J)*BETA13*TR35(J)
211    CONTINUE
       A(18,19)=ASUM
      ASUM=0.
       DO 212 J=1,K
      ASUM=ASUM+S72(J)*TR33(J)+S78(J)*BETA2*TR37(J)+S80(J)*BETA3*TR36(J)
     1+S87(J)*BETA23*TR35(J)
212    CONTINUE
       A(18,20)=ASUM
      ASUM=0.
       DO 213 J=1,K
      ASUM=ASUM+S69(J)*TR33(J)+S81(J)*BETA3*TR34(J)+S84(J)*BETA33*TR35(J
     1)
213    CONTINUE
       A(18,21)=ASUM
      ASUM=0.
       DO 214 J=1,K
      ASUM=ASUM+S1(J)*TR38(J)+2.*S73(J)*BETA1*TR39(J)+S82(J)*BETA11*TR40
     1(J)+S67(J)*TR41(J)
214    CONTINUE
       A(19,19)=ASUM
      ASUM=0.
       DO 215 J=1,K
      ASUM=ASUM+S4(J)*TR38(J)+(S74(J)*BETA1+S76(J)*BETA2)*TR39(J)+S85(J)
     1*BETA12*TR40(J)+S70(J)*TR41(J)
215    CONTINUE
       A(19,20)=ASUM
```

```
      ASUM=0.
       DO 216 J=1,K
      ASUM=ASUM+S5(J)*TR38(J)+(S75(J)*BETA1+S79(J)*BETA3)*TR39(J)+S86(J)
     1*BETA13*TR40(J)+S71(J)*TR41(J)
216    CONTINUE
       A(19,21)=ASUM
      ASUM=0.
       DO 217 J=1,K
      ASUM=ASUM+S2(J)*TR38(J)+2.*S77(J)*BETA2*TR39(J)+S83(J)*BETA22*TR40
     1(J)+S68(J)*TR41(J)
217    CONTINUE
       A(20,20)=ASUM
      ASUM=0.
       DO 218 J=1,K
      ASUM=ASUM+S6(J)*TR38(J)+(S78(J)*BETA2+S80(J)*BETA3)*TR39(J)+S87(J)
     1*BETA23*TR40(J)+S72(J)*TR41(J)
218    CONTINUE
       A(20,21)=ASUM
      ASUM=0.
       DO 219 J=1,K
      ASUM=ASUM+S3(J)*TR38(J)+2.*S81(J)*BETA3*TR39(J)+S84(J)*BETA33*TR40
     1(J)+S68(J)*TR41(J)
219    CONTINUE
       A(21,21)=ASUM
      ASUM=0.
       DO 149 J=1,K
      ASUM=ASUM+XN*(S49(J)*CT8+S55(J)*CT10+S59(J)*CT12)
149    CONTINUE
       A(10,17)=ASUM
      MM=21
      NN=MM
      DO 503 I=1,21
      DO 503 J=I,21
      A(J,I)=A(I,J)
503   B(J,I)=B(I,J)
```

```
      DO 501 I=1,21
501   Y(I,I)=1.
      IARDVRK=1HA
      PRINT 1008,IARDVRK
      PRINT 1009,((A(I,J),J=1,MM),I=1,MM)
      IARDVRK=1HB
      PRINT 1008,IARDVRK
      PRINT 1009,((B(I,J),J=1,MM),I=1,MM)
      CALL GERIAC(MM,MM,2,B,Y,X)
      CALL MATMUL(X,A,MM,MM,MM,ATRAN)
      DO 502 I=1,21
      DO 502 J=1,21
502   XINPUT(I,J)=ATRAN(I,J)
      IARDVRK=1HT
      PRINT 1008,IARDVRK
      PRINT 1009,((ATRAN(I,J),J=1,MM),I=1,MM)
      M09999=0
      CALL MATMUL(X,B,MM,MM,MM,ANS)
      IARDVRK=1HI
      PRINT 1008,IARDVRK
      PRINT 1009,((ANS(I,J),J=1,MM),I=1,MM)
      SUM=0.
      DO 510 I=1,MM
      DO 510 J=1,MM
      IF(I.EQ.J)SUM=SUM+ABSF(ANS(I,J)-1.)
      IF(I.EQ.J) GO TO 510
      SUM=SUM+ABSF(ANS(I,J))
510   CONTINUE
      PRINT 1010,SUM
      DO 5101 I=1,NN
      DO 5101 J=1,NN
5101  XINPUT(I,J)=ATRAN(I,J)
525   CONTINUE
      CALL MATSUB(NN,0,1,1.,0.,10.,0.,1,100,100,1.E-4,1.E-10)
      DO 512 L=1,NN
      PRINT 1011,EIGEN(L),EIGENC(L)
      PRINT 1012
      IARDVRK=10HREAL PART
      PRINT 1013,IARDVRK,(VALUR(I,L),I=1,NN)
```

```
      DO 1 I = 1, N
      DO 2 J = 1, N
2     A(I,J) = ORIGA(I,J)
      DO 1 J = 1, M
1     B(I,J) = ORIGB(I,J)
      LIT1 = 8H X(
      LIT2 = 8H,
      LIT3 = 8H)=
500   FORMAT(1X, 28H A DIAGONAL ELEMENT VANISHES)
502   FORMAT( 4( 2X, A3, I2, A1, I2, A2, E18.11))
504   FORMAT (24H AVERAGE RELATIVE ERROR=, E10.3)
505   FORMAT (24H MAXIMUM RELATIVE ERROR=, E10.3)
C
C     SINGLE PRECISION GAUSSIAN ELIMINATION AND RECORDING OF TRANSFORMATION
      DO 20 L = 1, N
      INT(L) = 0
      KP = 0
      GO TO ( 16, 11), IS
11    Z = 0.0
      DO 13 K = L, N
      TZ = ABSF(A(K,L))
      IF(Z - TZ) 12, 13, 13
12    Z = TZ
      KP = K
13    CONTINUE
      IF(L - KP) 14, 16, 16
14    INT(L) = KP
      DO 15 J = L, N
      Z = A(L,J)
      A(L,J) = A(KP,J)
15    A(KP,J) = Z
16    IF(A(L,L))17, 21, 17
21    PRINT 500
      CALL EXIT
17    IF(L - N) 18, 22, 22
18    LP1 = L + 1
      DO 20 K = LP1, N
      RATIO = A(K,L)/A(L,L)
      A(K,L) = RATIO
      DO 19 J = LP1, N
```

```
19    A(K,J) = A(K,J) - RATIO*A(L,J)
20    CONTINUE
22    CONTINUE
C
C     ITERATE TO CORRECT SOLUTION BY USING RESIDUALS
      DO 90 ITERATE = 1, MANY
      IF(ITERATE - 1) 50, 50, 70
C
C     COMPUTE RESIDUAL ARRAY
70    DO 71 J = 1, M
      DO 71 I = 1, N
      TEMP = ORIGB(I,J)
      DO 711 INDX = 1, N
      C = ORIGA(I,INDX)
      D = XN(INDX,J)
 711  TEMP = TEMP-C*D
      B(I,J) = TEMP
 71   CONTINUE
C
C     PERFORM TRANSFORMATION ON ARRAY OF CONSTANTS
50    DO 56 L = 1, N
      KP = INT(L)
      IF(KP) 53, 53, 51
51    DO 52 J = 1, M
      Z = B(KP,J)
      B(KP,J) = B(L,J)
52    B(L,J) = Z
53    LP1 = L + 1
      DO 56 K = LP1, N
      RATIO = A(K,L)
      IF(RATIO) 54, 56, 54
54    DO 55 J = 1, M
55    B(K,J) = B(K,J) - RATIO*B(L,J)
56    CONTINUE
C
C     COMPUTE SOLUTION BY BACK SUBSTITUTION
      DO 63 I = 1, N
      II = N + 1 - I
      DO 63 J = 1, M
      S = 0.0
```

```
      IF(II - N) 61, 63, 63
61    IIP1 = II + 1
      DO 62 K = IIP1, N
62    S = S + A(II,K) * X(K,J)
63    X(II,J) = (B(II,J) - S)/A(II,II)
C
C     ADD CORRECTIONS AND PRINT RESULTS
      AV=0
      REMAX = 0.0
      IF(ITERATE - 1) 81, 81, 83
81    DO 82 I = 1, N
      DO 82 J = 1, M
82    XN(I,J) = X(I,J)
      GO TO 90
83    DO 84 I = 1, N
      DO 84 J = 1, M
      TEMP1=XN(I,J)
      TEMP2=X(I,J)
      IF(TEMP1) 835,84,835
835   TEMP3 = ABSF(TEMP2/TEMP1)
      IF(REMAX - TEMP3) 836, 837, 837
836   REMAX = TEMP3
837   AV = AV + TEMP3
 84   XN(I,J)=TEMP1+TEMP2
      TEMP1=M*N
      AV=AV/TEMP1
      PRINT 504,AV
      PRINT 505, REMAX
86    DO 87 J = 1, M
87    PRINT 502, ( LIT1, I, LIT2, J, LIT3, XN(I,J), I = 1, N)
C
90    CONTINUE
      RETURN
      END
      SUBROUTINE RDIP(B,X,NROW,KR,NO,MM,Y,NCOL,AN)
      DIMENSION X(25,25),Y(25,25)
      TYPE DOUBLE TEMP,C,D
      TEMP = B
      DO 1 J = 1,MM
      C = X(NROW,J)
```

```
      D = Y(J,NCOL)
 1    TEMP = TEMP - C * D
      AN = TEMP
       END
     $SUBROUTINE MATSUB (M,IEG,IVEC,ALRS,ALIS,GBR,GBI,IDET,MIT,MITS,EP1,
     1EP2)
      COMMON/MATS/CR(25,25),CI(25,25),ZR(25),ZI(25),VALUR(25,25),
     1VALUI(25,25)
      DIMENSION AR(25,25),AI(25,25),BR(25,25),BI(25,25),YR(25),YI(25),
     1XR(25),XI(25)
      IAARD=M+1
      IONE=1
      ITWO=2
      N=M
      SUMR=0.0
      SUMI=0.0
      PRDR=1.0
      PRDI=0.0
      TRACER=0.0
      TRACEI=0.0
      PRINT 6911
 6911 FORMAT(//,13H INPUT MATRIX,//)
      DO 7053 I=1,N
 7053 PRINT 7052,(CR(I,J),J=1,N)
 7052 FORMAT(10E12.4)
      DO 450 I=1,N
      TRACER=TRACER+CR(I,I)
  450 TRACEI=TRACEI+CI(I,I)
C     SET UP MATRICES
      DO 519 I=1,N
      DO 519 J=1,N
      BR(I,J)=CR(I,J)
      AR(I,J)=CR(I,J)
      BI(I,J)=CI(I,J)
  519 AI(I,J)=CI(I,J)
C     EVALUATE DETERMINENT
      ASSIGN 520 TO IA
      ASSIGN 811 TO ID
      MM=M
      INTER=0
```

```
      GO TO 535
  520 DETR=1.0
      DETI=0.0
      DO 522 K=1,M
      T1=DETR*AR(K,K)-DETI*AI(K,K)
      DETI=DETR*AI(K,K)+DETI*AR(K,K)
  522 DETR=T1
      INTER=XMODF(INTER,2)
      IF (INTER) 1000,917,810
 1000 STOP
  810 DETR=-DETR
      DETI=-DETI
  917 GO TO ID
  811 PRINT 557,TRACER,TRACEI,DETR,DETI
  557 FORMAT (19H   TRACE OF MATRIX= 2E18.8,
     125H   DETERMINANT OF MATRIX= 2E18.8)
      ASSIGN 912 TO ID
      ASSIGN 530 TO IA
      ASSIGN 40 TO IB
      ASSIGN 523 TO IC
      ISL=1
      GO TO 92
  523 ISL=0
C     EIGENVALUE GUESS OR ORIGIN TRANSLATION
    9 ALR=ALRS
      ALI=ALIS
      IT=1
C     EIGENVECTOR GUESS
  403 DO 504 I=1,N
      XR(I)=1.0
  504 XI(I)=0.0
    4 DO 5 I=1,N
      AR(I,I)=AR(I,I)-ALR
    5 AI(I,I)=AI(I,I)-ALI
C     FIRST ITERATION - POWER METHOD
      IJ=1
   10 BIG=0.
C     COMPUTE Y=(A-ALPHA)*X
      DO 13 I=1,N
      YR(I)=0.
```

```
      DO 15 I=1,N
   15 TS=TS+(YR(I)-AMUR*XR(I)+AMUI*XI(I))**2+
     1(YI(I)-AMUR*XI(I)-AMUI*XR(I))**2
C     NORMALIZATION
      DO 16 I=1,N
      XR(I)=(YR(JJ)*YR(I)+YI(JJ)*YI(I))/BIG
   16 XI(I)=(YR(JJ)*YI(I)-YI(JJ)*YR(I))/BIG
      XR(JJ)=1.0
      XI(JJ)=0.0
  111 IF (TS/RQD-EP1) 20,20,18
   18 IF (IJ-MIT) 19,20,20
   19 IJ=IJ+1
      GO TO 10
C     SECOND ITERATION - INVERSE POWER METHOD
   20 ICT=IJ
      MIT2=MITS+IJ
      ALR=AMUR+ALR
      ALI=AMUI+ALI
      MM=N
      DO 310 I=1,N
      AR(I,I)=AR(I,I)-AMUR
  310 AI(I,I)=AI(I,I)-AMUI
      GO TO 29
   99 DO 100 I=1,N
      AR(I,I)=AR(I,I)-ALR
  100 AI(I,I)=AI(I,I)-ALI
   29 IJ=IJ+1
C     GAUSSIAN ELIMINATION - (A-ALPHA)*Y=X
  535 DO 27 I=2,MM
      IM1=I-1
      DO 27 J=1,IM1
   21 FM=AR(I,J)**2+AI(I,J)**2
      SM=AR(J,J)**2+AI(J,J)**2
      IF (FM-SM) 24,24,22
C     ROW INTERCHANGE - IF NECESSARY
   22 DO 23 K=J,MM
      T1=AR(J,K)
      T2=AI(J,K)
      AR(J,K)=AR(I,K)
      AI(J,K)=AI(I,K)
```

```
      AR(I,K)=T1
   23 AI(I,K)=T2
      T1=XR(J)
      T2=XI(J)
      XR(J)=XR(I)
      XI(J)=XI(I)
      XR(I)=T1
      XI(I)=T2
      T1=FM
      FM=SM
      SM=T1
      INTER=INTER+1
   24 IF (SM) 25,27,25
   25 IF (FM) 90,27,90
C     TRIANGULARIZATION
   90 RR=(AR(I,J)*AR(J,J)+AI(I,J)*AI(J,J))/SM
      RI=(AR(J,J)*AI(I,J)-AR(I,J)*AI(J,J))/SM
      DO 26 K=J,MM
      AR(I,K)=AR(I,K)-RR*AR(J,K)+RI*AI(J,K)
   26 AI(I,K)=AI(I,K)-RR*AI(J,K)-RI*AR(J,K)
      AR(I,J)=0.
      AI(I,J)=0.
      XR(I)=XR(I)-RR*XR(J)+RI*XI(J)
      XI(I)=XI(I)-RR*XI(J)-RI*XR(J)
   27 CONTINUE
      GO TO IA
  530 SMALL=1000.
      DO 28 K=1,MM
      IKK=K
      T1=AR(K,K)**2+AI(K,K)**2
      IF (T1) 750,752,750
  750 IF (T1-SMALL) 751,28,28
  751 SMALL=T1
      IZ=K
   28 CONTINUE
      GO TO IB
  752 IZ=IKK
      IF (ISL) 753,30,30
C     EXACT EIGENVALUE - (A-ALPHA) SINGULAR. FLAG=2000
   30 ISL=1
```

```
      ICT=2000
      DO 974 I=1,MM
      XR(I)=0.0
  974 XI(I)=0.0
  753 YR(IZ)=1.0
      YI(IZ)=0.0
      JJ=IZ
      BIG=1.0
      IF (IZ-MM) 33,32,33
   32 IZZ=2
      GO TO 95
   33 IZZ=IZ+1
      DO 31 I=IZZ,MM
      YR(I)=0.
   31 YI(I)=0.
      IZZ=MM-IZ+2
      IF (IZ-1) 95,49,95
C     BACKWARD SUBSTITUTION
   40 IZZ=1
   41 BIG=0.
   95 DO 46 I=IZZ,MM
      II=MM-I+1
      KK=II+1
      SR=0.
      SI=0.
      IF (I-1) 42,44,42
   42 DO 43 K=KK,MM
      SR=SR+AR(II,K)*YR(K)-AI(II,K)*YI(K)
   43 SI=SI+AR(II,K)*YI(K)+AI(II,K)*YR(K)
   44 T1=AR(II,II)**2+AI(II,II)**2
      YR(II)=(AR(II,II)*(XR(II)-SR)+AI(II,II)*(XI(II)-SI))/T1
      YI(II)=(AR(II,II)*(XI(II)-SI)-AI(II,II)*(XR(II)-SR))/T1
      AM=YR(II)**2+YI(II)**2
      IF (AM-BIG) 46,46,45
   45 JJ=II
      BIG=AM
   46 CONTINUE
C     NORMALIZATION - X=NORMALIZED Y
   49 DO 47 I=1,MM
      XR(I)=(YR(JJ)*YR(I)+YI(JJ)*YI(I))/BIG
```

```
   47 XI(I)=(YR(JJ)*YI(I)-YI(JJ)*YR(I))/BIG
      XR(JJ)=1.0
      XI(JJ)=0.0
   92 DO 601 I=1,N
      DO 601 J=1,N
      AR(I,J)=BR(I,J)
  601 AI(I,J)=BI(I,J)
  116 IF (ISL) 755,50,60
  755 GO TO IC
C     ALPHA RAYLEIGH QUOTIENT = (AX,X)/(X,X)=ALPHA
   50 ALR=0.
      ALI=0.
      SUM=0.0
   55 DO 52 I=1,N
      YR(I)=0.
      YI(I)=0.
      DO 51 K=1,N
      YR(I)=YR(I)+AR(I,K)*XR(K)-AI(I,K)*XI(K)
   51 YI(I)=YI(I)+AR(I,K)*XI(K)+AI(I,K)*XR(K)
      ALR=ALR+XR(I)*YR(I)+XI(I)*YI(I)
      ALI=ALI+XR(I)*YI(I)-XI(I)*YR(I)
   52 SUM=SUM+XR(I)**2+XI(I)**2
      ALR=ALR/SUM
      ALI=ALI/SUM
      AM=ALR**2+ALI**2
      IF (IEG) 1000,83,82
   82 PRINT 300,ITWO,IJ,ALR,ALI
C     TEST SECOND ITERATION
   83 TS=0.
      DO 53 I=1,N
      T1=YR(I)-ALR*XR(I)+ALI*XI(I)
      T2=YI(I)-ALR*XI(I)-ALI*XR(I)
   53 TS=TS+T1**2+T2**2
   93 IF (TS/SUM-EP2)60,60,301
  301 IF (IJ-MIT2) 99,400,400
  400 PRINT 401,IT
  401 FORMAT (54H     INVERSE POWER METHOD NOT CONVERGED ON TRY NUMBER
     1I5)
      IF (IT-3) 402,990,402
  402 ALR=ALR+GBR
```

```
      ALI=ALI+GBI
      IT=IT+1
      PRINT 820,ALR,ALI
  820 FORMAT (11H     ALPHA= 2E20.8)
      GO TO 4
   60 ISL=0
   63 PRINT 64,  N,ALR,ALI,ICT,IJ
   64 FORMAT (I5,15H TH EIGENVALUE= 2E18.8,53X,2I5)
      ZR(N)=ALR
      ZI(N)=ALI
      SUMR=SUMR+ALR
      SUMI=SUMI+ALI
      T1=PRDR*ALR-PRDI*ALI
      PRDI=PRDR*ALI+PRDI*ALR
      PRDR=T1
C     DEFLATION OF MATRIX
      IF (JJ-N) 61,65,61
C     PERMUTATION OPERATION
   61 T1=XR(JJ)
      T2=XI(JJ)
      XR(JJ)=XR(N)
      XI(JJ)=XI(N)
      XR(N)=T1
      XI(N)=T2
      DO 68 K=1,N
      T1=AR(JJ,K)
      T2=AI(JJ,K)
      AR(JJ,K)=AR(N,K)
      AI(JJ,K)=AI(N,K)
      AR(N,K)=T1
   68 AI(N,K)=T2
      DO 62 K=1,N
      T1=AR(K,JJ)
      T2=AI(K,JJ)
      AR(K,JJ)=AR(K,N)
      AI(K,JJ)=AI(K,N)
      AR(K,N)=T1
   62 AI(K,N)=T2
C     DEFLATION
   65 N=N-1
```

```
      DO 66 I=1,N
      DO 66 J=1,N
      AR(I,J)=AR(I,J)-XR(I)*AR(N+1,J)+XI(I)*AI(N+1,J)
   66 AI(I,J)=AI(I,J)-XR(I)*AI(N+1,J)-XI(I)*AR(N+1,J)
      DO 600 I=1,N
      DO 600 J=1,N
      BR(I,J)=AR(I,J)
  600 BI(I,J)=AI(I,J)
C     COMPUTE EIGENVECTOR AND/OR DETERMINANT AS REQUIRED
  910 IF (IDET) 1000,527,700
  527 IF (IVEC) 1000,525,700
  700 DO 702 I=1,M
      DO 702 J=1,M
      AR(I,J)=CR(I,J)
      AI(I,J)=CI(I,J)
      IF (I-J) 702,701,702
  701 AR(I,I)=AR(I,I)-ALR
      AI(I,I)=AI(I,I)-ALI
  702 CONTINUE
      MM=M
      INTER=0
      ASSIGN 911 TO IA
      GO TO 535
  911 ASSIGN 530 TO IA
      IF (IDET) 1000,914,520
  912 PRINT 913,DETR,DETI
  913 FORMAT (58X,12HDETERMINANT= 2E18.8)
      ZLAG=SQRTF(AR(1,1)**2+AI(1,1)**2)
      ZLIT=ZLAG
      DO 923 I=2,M
      ZMAGT=SQRTF(AR(I,I)**2+AI(I,I)**2)
      IF (ZLAG-ZMAGT) 922,920,920
  920 IF (ZLIT-ZMAGT) 923,923,921
  921 ZLIT=ZMAGT
      GO TO 923
  922 ZLAG=ZMAGT
  923 CONTINUE
      PRINT 924, ZLAG,ZLIT
  924 FORMAT (70H  LARGEST AND SMALLEST MAGNITUDES OF DIAGONAL ELEMENTS
     1OF TRI. MATRIX= 2E18.8/)
```

```
 914 ISL=-1
     IF (IVEC) 1000,916,915
 915 DO 703 I=1,M
     XR(I)=0.
 703 XI(I)=0.
     ASSIGN 753 TO IB
     ASSIGN 704 TO IC
     GO TO 530
 916 ASSIGN 525 TO IC
     GO TO 92
 704 PRINT 705, (XR(I),XI(I),I=1,M)
 705 FORMAT (30H     ASSOCIATED EIGENVECTOR IS/(2E20.8))
     IAARD=IAARD-1
     DO 7051 I=1,M
     VALUR(I,IAARD)=XR(I)
7051 VALUI(I,IAARD)=XI(I)
     ASSIGN 40 TO IB
 525 IF (N-1) 526,67,523
  67 ALR=AR(1,1)
     ALI=AI(1,1)
     SUMR=SUMR+ALR
     SUMI=SUMI+ALI
     T1=PRDR*ALR-PRDI*ALI
     PRDI=PRDR*ALI+PRDI*ALR
     PRDR=T1
     ZR(1)=ALR
     ZI(1)=ALI
     PRINT 320,ALR,ALI
 320 FORMAT (20H   FINAL EIGENVALUE= 2E18.8)
     N=0
     GO TO 910
 526 PRINT 321,SUMR,SUMI,PRDR,PRDI
 3210FORMAT (21H  SUM OF EIGENVALUES= 2E18.8,
    125H  PRODUCT OF EIGENVALUES= 2E18.8//)
 990 CONTINUE
     RETURN
     END
1.022     .12       .00008812  1.       .242      .12       .242      .242
.002203   .003551   .003551    .002291  .03877          1
5.        1.        1.022
```