THE BAYESIAN ANALYSIS OF STRUCTURAL CHANGE IN MULTIVARIATE LINEAR MODELS

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CHAPTER I

INTRODUCTION AND REVIEW OF THE LITERATURE

Introduction

The study of structural change in the parameter values of a linear model has been a problem of interest for many years. Using both Bayesian and non-Bayesian approaches, primary interest has focused on estimating and making inferences about the change point as well as the other parameters of the model, and on testing the null hypothesis of no change against the alternative hypothesis that a change has occurred.

Applications of structural change are numerous. Some of the more recent applied articles, along with the area of application discussed, include: Chin Choy and Broemeling (1980) (engineering), Smith and Cook (1980) (medicine), and Tsurumi (1977, 1978, 1980) (economics).

Much of the work that has been done in the area of structural change has involved univariate linear models, and it has only been recently that multivariate linear models have been studied to any great extent. This particular research concentrates on the structural change of multivariate linear models from a Bayesian point of view.

Before beginning this analysis, let us first review some of the work that has been done for a Bayesian analysis of a multivariate linear model in the no change situation. This is followed by a historical review of the developments in the area of structural change, initially from a non-Bayesian viewpoint and then from a Bayesian viewpoint.

Review of the Literature

Bayesian Analysis of a Multivariate Linear Model With No Change

As a special case of the multivariate linear model, Geisser and Cornfield (1963) considered the Bayesian analysis of the multivariate normal process. Using an improper prior distribution, they obtained the marginal posterior distributions for the mean vector and the covariance matrix. The posterior distribution of the mean vector was shown to have a multivariate t distribution, while the posterior distribution of the covariance matrix was shown to have an inverse Wishart distribution.

Ando and Kaufman (1965) worked with a multivariate normal process in which the mean vector and the precision matrix were both unknown, but, unlike Geisser and Cornfield (1963), they used a natural conjugate prior distribution, namely the normal-Wishart distribution. They found the joint posterior distribution of the parameters, and for fixed sample size, derived various sampling distributions of some statistics.

With various assumptions about the parameters and using an improper prior distribution, Geisser (1965) derived a $(1-\alpha)$ posterior region for the mean vector, and also gave some estimation procedures based on the marginal posterior distribution of the covariance matrix.

Tiao and Zellner (1964) were primarily interested in conducting a Bayesian analysis of the traditional multivariate regression model. Using an improper prior distribution to express little knowledge about the prior parameters, they discussed the posterior distributions of the parameters and derived several properties of these posterior distributions.

Also using an improper prior distribution, Zellner and Chetty (1965) derived the predictive distribution for a multivariate regression model, indicating its properties and suggesting its possible application in the areas of predicting the return on an investment portfolio or a farmer's crops.

Rossi (1980) derived the posterior odds ratio for testing linear hypotheses in the multivariate regression model.

Non-Bayesian Structural Change

Next, let us consider the non-Bayesian work that has been done in the area of structural change in general. Many articles can be found in the literature, so rather than summarizing all of them, the approach will be taken of reviewing some of them of primary interest and then indicating other articles of related interest.

Page (1955, 1957) devised tests based on cumulative sums to detect a change in the mean of a random sample of observations when the initial population mean was known, but the point of change was unknown.

In 1958, Quandt analyzed independent ordered pairs of observations known to follow exactly two different linear relationships over a particular time period, and he developed a maximum likelihood technique for estimating that point of change. He also indicated a likelihood ratio test that could be used to test the null hypothesis of no change against the alternative hypothesis of exactly one change. In his 1960 article, Quandt looked at some alternative ways of testing the null hypothesis of no change.

Assuming that the point of change was known, Chow (1960) developed a test based on the F-distribution for testing whether or not the regression coefficients of two linear relationships could be considered equal. He also devised a test that tested for equality between subsets of these coefficients.

Hinkley (1969) was concerned with estimating the point of intersection in a two-phase linear regression situation. First, he showed that the maximum likelihood estimator of the point of intersection had an asymptotic normal distribution, and then, using this result, he set up the likelihood ratio test statistic for hypothesis tests concerning the point of intersection. In his 1970 article, Hinkley studied the point of change in a finite sequence of independently and normally distributed random variables with constant variance. A maximum likelihood estimate for the change point was derived, as well as a likelihood ratio test for testing the null hypothesis that the change point was at some hypothesized point. The asymptotic distributions for both the maximum likelihood estimator and the likelihood ratio test statistic were found.

A likelihood ratio test was devised by Farley and Hinich (1970) for testing the null hypothesis of no change in the slope coefficient of a simple linear model against the alternative hypothesis of exactly one shift at some unknown time. They studied the power of this test in some detail, and one of the conclusions reached was that even for relatively small shifts, the power was very good in the middle range of the data, but declined near the endpoints of the data.

In 1975, Farley, Hinich and McGuire conducted a comparison study of the likelihood ratio test, the test developed by Chow in 1960, and a modification of the test proposed by Farley and Hinich in 1970. They wanted to determine the best of the three for detecting whether or not a parameter shift in the slopes had occurred in a linear time series model.

The conclusion that they reached was that Chow's test was the best when the actual shift occurred near the middle of the data, while the Farley-Hinich test was preferable if the shift occurred nearer to the endpoints.

Hsu (1977) presented two different tests for use in testing whether or not a shift in the variance of a sequence of independent and normally distributed random variables had occurred.

Other articles of interest that deal with structural change from a non-Bayesian viewpoint include: Robison (1964), Hudson (1966), Bhattacharyya and Johnson (1968), McGee and Carleton (1970), Quandt (1972), Hawkins (1977), and Worsley (1979).

Bayesian Structural Change

Finally, let us consider the Bayesian work that has been done in the structural change area. As in the non-Bayesian case, only some of the articles will be reviewed with others of interest listed at the end.

Using a Bayesian approach, Chernoff and Zacks (1964) studied the problem of estimating the current mean in a finite sequence of independently and normally distributed random variables with known variance. A Bayes' estimator for the current mean was given for the case of several changes in the mean, and another simpler Bayes' estimator was given for the case of at most one change. They also derived a Bayes' test procedure for testing the null hypothesis studied by Page (1955, 1957) of no change in the mean against the alternative hypothesis of exactly one change at some unknown point, and determined that the Bayes' procedure was, in most cases, slightly more powerful.

Kander and Zacks (1966) extended the hypothesis testing procedure of Chernoff and Zacks (1964) to include the situation where the independent sequence of random variables could belong to the one parameter exponential family.

Using an improper prior distribution, Bacon and Watts (1971) proposed a transition model for estimating a changing linear relationship. This model included a transition parameter that allowed for either a smooth or an abrupt transition at the point of change.

With several different variations concerning parameter assumptions, Holbert (1973) studied the change point problem for an independent normal sequence of random variables with unknown variance and for the two-phase regression situation with both known and unknown error variances. Assuming that a change had occurred and using improper prior distributions, emphasis was given to deriving the posterior distribution of the change point, but posterior distributions for some of the other parameters in the model under discussion were also given. In the case of two-phase regression and again with various assumptions, the posterior distribution of the point of intersection for the two regression lines was also derived.

Broemeling (1974) estimated the change point in a sequence of independent random variables belonging to a one parameter exponential family. He found the joint posterior distributions of all the unknown parameters and also derived the posterior mass functions of the change point for a Bernoulli, an exponential, and a normal sequence.

Sen and Srivastava (1973, 1975a, 1975b, 1975c) considered tests for detecting a change in the mean at some unknown point in a finite sequence of independent and normally distributed random variables. In their 1973 article they discussed the multivariate case, while in their 1975 articles they dealt with different aspects of the univariate case. In both the multivariate and univariate cases they derived exact and asymptotic distributions for many of the test statistics, assuming that the null hypothesis of no change in the means was true.

Broemeling (1977) studied the problem of forecasting future values of changing sequences of independent random variables. Using an improper prior distribution, he derived the Bayesian predictive density of k future observations for a normal sequence, considering the cases of both known and unknown variance. Also, using conjugate prior distributions, he derived the predictive mass function of k future observations for a Bernoulli sequence and the predictive density of k future observations for an exponential sequence.

Also in 1977, Chin Choy generalized the two-phase simple linear regression work done by Holbert in 1973 to include multiple linear regression. With the use of proper prior distributions for all of the unknown parameters, and assuming that one change had occurred at some unknown point, she derived the posterior distribution for the change point as well as for the unknown regression parameters using several different assumptions about the parameters. She also used a proper prior distribution for the simple linear regression case to make inferences about the point of intersection of two regression lines.

Using Bacon and Watts' (1971) proposed transition function to allow for gradual parameter change, Tsurumi (1980) extended their work to that of a simultaneous equation model and to include either a permanent or a temporary shift. He then applied this method to the analysis of the U.S. gasoline market in an attempt to determine the impact of the 1973 oil crisis to supply and demand equations. He concluded that there was a permanent shift in the supply equation, but only a temporary shift in the demand equation.

Salazar (1980) studied the change point problem for a multivariate normal sequence, for a multivariate regression model, and for certain univariate time series models. In all three cases, certain joint and marginal posterior distributions of the parameters were derived. In the case of the multivariate normal sequence, normal-Wishart prior distributions for the mean vectors and precision matrices and a uniform prior distribution for the shift point were used, and consideration was given to a single shift, two shifts, and a temporary shift. For the multivariate regression model, the use of both improper prior distributions and proper prior distributions were considered for the single shift, while only improper prior distributions were used for two shifts and the temporary shift. Whereas Chi (1979) studied time series models with an abrupt change, Salazar used a transition parameter to allow for gradual changes in the parameters. Time series models discussed included the regression model with autocorrelated errors, first and second order autoregressive processes, and distributed lag models. Numerical examples were provided for a multivariate normal sequence and for a regression model with autocorrelated errors.

Broemeling and Chin Choy (1981) derived a Bayesian test, based on the marginal posterior mass function of the change point, to test the null hypothesis of no change in a univariate linear model against the alternative hypothesis of exactly one change.

Salazar, Broemeling and Chi (1981) analyzed a regression model with an autocorrelated error structure assumed to have one change in the parameters. In order to estimate where the change occurred as well as to estimate the parameter values before and after the change, and with the use of a normal-gamma prior distribution, they determined the posterior probability mass function of the change point along with the marginal posterior distributions for the other parameters of the model.

Considering both improper and proper prior distributions, Land (1981) developed a Bayesian forecasting technique for a two-phase regression model by deriving the predictive density of the next k values for both the known and unknown precision parameter situations.

Other articles of interest that deal with structural change from a Bayesian viewpoint include: Ferreira (1975), Swamy and Mehta (1975), Holbert and Broemeling (1977), Tsurumi (1977, 1978), Chin Choy and Broemeling (1980), Smith and Cook (1980), and Menzefricke (1981).

Scope of the Research

As indicated in the review of the literature, the work done by Salazar in 1980 on multivariate linear models makes use of both improper and proper prior distributions in the Bayesian analysis of structural change. In the proper prior distribution case, a normal-Wishart natural conjugate prior distribution was used for the unknown parameters. However, as pointed out by Rothenberg (1963), the use of such a prior distribution places certain restrictions on the variances and covariances of the coefficients of the model. In order to avoid this, the single shift structural change problem for a multivariate linear model is analyzed in Chapter II with the use of a generalized natural conjugate prior distribution. This analysis includes the determination of the posterior distributions for the change point, the regression parameters, and the precision matrix. Then, to determine the effects of parameter changes on the posterior distribution of the shift point, and, in addition, to compare results when using either a natural conjugate prior distribution, or a generalized natural conjugate prior distribution, a computer study is undertaken. Also, the posterior expected values, variances, and co-variances of the model parameters are found.

To detect structural change in a multivariate linear model, a Bayesian test for testing the null hypothesis of no change against the alternative hypothesis of exactly one change is derived in Chapter III, and an example is considered.

A predictive analysis is done in Chapter IV to find the predictive distribution of future observations for a changing multivariate linear model. This is followed by a computer analysis that compares the pre-dictions of a model incorporating a change to that of a model which does not incorporate a change.

Finally, while Salazar (1980) looked at the double and temporary shift problems with the use of an improper prior distribution, Chapter V develops this theory using a natural conjugate prior distribution and includes a numerical study of these results.

CHAPTER II

MULTIVARIATE LINEAR MODELS WITH A SINGLE SHIFT

Posterior Distribution of the Change Point

Consider the multivariate linear model

$$Y = X\beta + e,$$

and suppose there is a shift in β at some point m, a positive integer, such that $1 \leqslant m \leqslant n-1$. In such a case the model can be written as

$$\begin{aligned} Y_1 &= X_1\beta_1 + e_1 \\ Y_2 &= X_2\beta_2 + e_2, \qquad \beta_1 \neq \beta_2 \end{aligned} \tag{2.1}$$

where β_1 and β_2 are k×p matrices of real unknown parameters,

$$Y = (Y_1' \,\vdots\, Y_2')' = (\underset{\sim}{Y}_1, \underset{\sim}{Y}_2, \cdots, \underset{\sim}{Y}_m \,\vdots\, \underset{\sim}{Y}_{m+1}, \cdots, \underset{\sim}{Y}_n)'$$

is an n×p matrix of observations,

$$X = (X_1' \,\vdots\, X_2')' = (\underset{\sim}{X}_1, \underset{\sim}{X}_2, \cdots, \underset{\sim}{X}_m \,\vdots\, \underset{\sim}{X}_{m+1}, \cdots, \underset{\sim}{X}_n)'$$

is an n×k design matrix, and

$$e = (e_1' \,\vdots\, e_2')' = (\underset{\sim}{e}_1, \underset{\sim}{e}_2, \cdots, \underset{\sim}{e}_m \,\vdots\, \underset{\sim}{e}_{m+1}, \cdots, \underset{\sim}{e}_n)'$$

is an n×p matrix of unobservable random variables, with $\underset{\sim}{e}_i'$, i = 1, 2, ⋯, n, being independently and identically distributed as $N_p(\underset{\sim}{0}, P)$. $\underset{\sim}{e}_i'$ is the i^{th} row of the matrix e, and P is a p×p positive definite symmetric precision matrix.

Let $\beta_i = (\underset{\sim}{\beta}_{i1}, \underset{\sim}{\beta}_{i2}, \cdots, \underset{\sim}{\beta}_{ip})$ for i = 1, 2, and define

$$\underset{\sim}{B}_i = (\underset{\sim}{\beta}'_{i1}, \underset{\sim}{\beta}'_{i2}, \cdots, \underset{\sim}{\beta}'_{ip})', \; i = 1, 2,$$

so that $\underset{\sim}{B}_i$ will be a pk×1 vector.

In his dissertation, Salazar (1980) studied this single shift multivariate linear model using a natural conjugate prior distribution; however, the use of this particular prior distribution results in placing certain restrictions on the variances and covariances of the regression coefficients as pointed out by Rothenberg (1963). This is the case, because with a natural conjugate prior distribution, the conditional distribution of β_i given P has a multivariate normal distribution with a variance-covariance matrix in the form $P \otimes R$, where R is some arbitrary parameter matrix. The Kronecker product forces certain ratios of the variances and covariances to be equal, since for $P = (p_{ij})$ which is p×p,

$$P \otimes R = \begin{pmatrix} p_{11}R & p_{12}R & \cdots & p_{1p}R \\ p_{21}R & p_{22}R & \cdots & p_{2p}R \\ \vdots & \vdots & \ddots & \vdots \\ p_{p1}R & p_{p2}R & \cdots & p_{pp}R \end{pmatrix}.$$

To avoid these restrictions on the prior parameters, the use of a generalized natural conjugate prior distribution is proposed instead. Thus, for a single shift multivariate linear model, a generalized natural conjugate prior distribution will be used to find the posterior probability mass function of the change point m if model (2.1) holds, and the following conditions are satisfied. The parameters m, β_1, β_2, and P are unknown, where m is a uniform discrete random variable such that $m \in [k, n-k]$ and m is a positive integer, the marginal distribution of

$\underset{\sim}{B}_i$, i = 1, 2, is a multivariate normal distribution with mean vector $\underset{\sim}{U}_i$, i = 1, 2, and precision matrix F_i, i = 1, 2, such that $\underset{\sim}{U}_i \in R^{pk}$ and F_i is a pk×pk positive definite symmetric precision matrix, and the marginal distribution of P is a Wishart distribution with ν degrees of freedom (ν ⩾ p) and precision matrix Σ (p×p). Furthermore, m, the rows of β_1 and β_2, and P are independent. Then the marginal prior distribution of $\underset{\sim}{B}_i$, i = 1, 2, is

$$\Pi_o(\underset{\sim}{B}_i) \propto \exp\{-\frac{1}{2}(\underset{\sim}{B}_i-\underset{\sim}{U}_i)'F_i(\underset{\sim}{B}_i-\underset{\sim}{U}_i)\}, \tag{2.2}$$

while the marginal prior distribution of P will be

$$\Pi_o(P) \propto |P|^{(\nu-p-1)/2}\exp\{-\frac{1}{2}tr(\Sigma P)\}, \tag{2.3}$$

so the joint prior distribution of β_1, β_2, P, and m is

$$\Pi_o(\beta_1,\beta_2,P,m) \propto |P|^{(\nu-p-1)/2}\exp\{-\frac{1}{2}tr(\Sigma P)\}\cdot \exp\{-\frac{1}{2}(\underset{\sim}{B}_1-\underset{\sim}{U}_1)'F_1(\underset{\sim}{B}_1-\underset{\sim}{U}_1)-\frac{1}{2}(\underset{\sim}{B}_2-\underset{\sim}{U}_2)'F_2(\underset{\sim}{B}_2-\underset{\sim}{U}_2)\}. \tag{2.4}$$

The likelihood function for β_1, β_2, P, and m can be written as

$$L(\beta_1,\beta_2,P,m) \propto |P|^{n/2}\exp\{-\frac{1}{2}tr(Y_1-X_1\beta_1)'(Y_1-X_1\beta_1)P - \frac{1}{2}tr(Y_2-X_2\beta_2)'(Y_2-X_2\beta_2)P\}. \tag{2.5}$$

Combining the joint prior distribution (2.4) with the likelihood function (2.5), the joint posterior distribution of β_1, β_2, P, and m is of the form

$$\Pi(\beta_1,\beta_2,P,m|Y) \propto |P|^{(n+\nu-p-1)/2}\exp\{-\frac{1}{2}tr(\Sigma P)\}\cdot \exp\{-\frac{1}{2}(\underset{\sim}{B}_1-\underset{\sim}{U}_1)'F_1(\underset{\sim}{B}_1-\underset{\sim}{U}_1)-\frac{1}{2}(\underset{\sim}{B}_2-\underset{\sim}{U}_2)'F_2(\underset{\sim}{B}_2-\underset{\sim}{U}_2)\}\cdot \exp\{-(1/2)tr[(Y_1-X_1\beta_1)'(Y_1-X_1\beta_1) + (Y_2-X_2\beta_2)'(Y_2-X_2\beta_2)]P\}. \tag{2.6}$$

By completing the square on β_i,

$(Y_i-X_i\beta_i)'(Y_i-X_i\beta_i)$ can be rewritten as

$(\beta_i-\hat{\beta}_i)'X_i'X_i(\beta_i-\hat{\beta}_i) + S_i$, where

$\hat{\beta}_i = (X_i'X_i)^{-1}X_i'Y_i$ and $S_i = (Y_i-X_i\hat{\beta}_i)'(Y_i-X_i\hat{\beta}_i)$, $i = 1, 2$.

Then the joint posterior distribution (2.6) can be rewritten as

$$\Pi(\beta_1,\beta_2,P,m|Y) \propto |P|^{(n+\nu-p-1)/2}\exp\{-\frac{1}{2}(\underset{\sim}{B}_1-\underset{\sim}{U}_1)'F_1(\underset{\sim}{B}_1-\underset{\sim}{U}_1) - \frac{1}{2}(\underset{\sim}{B}_2-\underset{\sim}{U}_2)'F_2(\underset{\sim}{B}_2-\underset{\sim}{U}_2)\}\exp\{-\frac{1}{2}\mathrm{tr}[\Sigma + S_1 + S_2 + (\beta_1-\hat{\beta}_1)'X_1'X_1(\beta_1-\hat{\beta}_1) + (\beta_2-\hat{\beta}_2)'X_2'X_2(\beta_2-\hat{\beta}_2)]P\}. \quad (2.7)$$

In order to find the posterior distribution of m, (2.7) needs to be integrated with respect to β_1, β_2, and P. Using Wishart distribution properties, (2.7) may be integrated with respect to P to obtain

$$\Pi(\beta_1,\beta_2,m|Y) \propto \exp\{-\frac{1}{2}(\underset{\sim}{B}_1-\underset{\sim}{U}_1)'F_1(\underset{\sim}{B}_1-\underset{\sim}{U}_1) - \frac{1}{2}(\underset{\sim}{B}_2-\underset{\sim}{U}_2)'F_2(\underset{\sim}{B}_2-\underset{\sim}{U}_2)\}\cdot |\Sigma + S_1 + S_2 + (\beta_1-\hat{\beta}_1)'X_1'X_1(\beta_1-\hat{\beta}_1) + (\beta_2-\hat{\beta}_2)'X_2'X_2(\beta_2-\hat{\beta}_2)|^{-(n+\nu)/2}. \quad (2.8)$$

The posterior distribution of β_1, β_2, and m is seen to be the product of one factor which is in the form of a multivariate normal distribution, and a second factor which is in the form of a matrix T-distribution. As it stands, β_1 and β_2 cannot be directly integrated out of (2.8) to obtain the posterior distribution of m; however, a normal approximation can be found for the matrix T factor, and then integration with respect to β_1 and β_2 is possible.

Therefore, consider

$$|\Sigma + S_1 + S_2 + (\beta_1-\hat{\beta}_1)'X_1'X_1(\beta_1-\hat{\beta}_1) + (\beta_2-\hat{\beta}_2)'X_2'X_2(\beta_2-\hat{\beta}_2)|^{-(n+\nu)/2} \tag{2.9}$$

Let $A(m) = \Sigma + S_1 + S_2$, and define

$$(\beta^*-\hat{\beta}^*) = \begin{pmatrix} \beta_1-\hat{\beta}_1 \\ \beta_2-\hat{\beta}_2 \end{pmatrix} \quad \text{and} \quad X^* = \begin{pmatrix} X_1 & \phi \\ \phi & X_2 \end{pmatrix}.$$

$A(m)$ is $p\times p$, $(\beta^*-\hat{\beta}^*)$ is $2k\times p$, the ϕ's are zero matrices of appropriate order, and X^* is $n\times 2k$. Then (2.9) can be rewritten as

$$|A(m) + (\beta^*-\hat{\beta}^*)'X^{*\prime}X^*(\beta^*-\hat{\beta}^*)|^{-(n+\nu)/2}, \tag{2.10}$$

and (2.8) may be rewritten as

$$\Pi(\beta_1,\beta_2,m|Y) \propto \exp\{-\frac{1}{2}(\underset{\sim}{B}_1-\underset{\sim}{U}_1)'F_1(\underset{\sim}{B}_1-\underset{\sim}{U}_1) - \frac{1}{2}(\underset{\sim}{B}_2-\underset{\sim}{U}_2)'F_2(\underset{\sim}{B}_2-\underset{\sim}{U}_2)\}\cdot$$

$$|\overline{A(m)} + (\beta^*-\hat{\beta}^*)'M(\beta^*-\hat{\beta}^*)|^{-(n+\nu)/2} \tag{2.11}$$

where $\overline{A(m)} = A(m)/n$, $M = X^{*\prime}X^*/n$, and n is the number of observations.

By simultaneous diagonalization, there will exist a nonsingular $p\times p$ matrix C such that $C\overline{A(m)}C' = I$ and $C(\beta^*-\hat{\beta}^*)'M(\beta^*-\hat{\beta}^*)C' = D$, where D is a diagonal matrix whose diagonal elements are λ_j, $j = 1,\cdots,p$, the characteristic roots of $|(\beta^*-\hat{\beta}^*)'M(\beta^*-\hat{\beta}^*) - \lambda\overline{A(m)}| = 0$. Thus,

$$|\overline{A(m)} + (\beta^*-\hat{\beta}^*)'M(\beta^*-\hat{\beta}^*)|^{-(n+\nu)/2}$$

$$= |C'C|^{(n+\nu)/2}|C\overline{A(m)}C' + C(\beta^*-\hat{\beta}^*)'M(\beta^*-\hat{\beta}^*)C'|^{-(n+\nu)/2}$$

$$= |\overline{A(m)}|^{-(n+\nu)/2}|I + D|^{-(n+\nu)/2}$$

$$= |\overline{A(m)}|^{-(n+\nu)/2}\exp[\log_e|I + D|^{-(n+\nu)/2}]$$

$$= |\overline{A(m)}|^{-(n+\nu)/2}\exp[-((n+\nu)/2)\sum_{j=1}^{p}\log_e(1 + \lambda_j)]$$

$$= |\overline{A(m)}|^{-(n+\nu)/2} \exp[-((n+\nu)/2)(\sum_{j=1}^{p} \lambda_j - \frac{1}{2}\sum_{j=1}^{p} \lambda_j^2 + \frac{1}{3}\sum_{j=1}^{p} \lambda_j^3 - \cdots)]$$

$$= |\overline{A(m)}|^{-(n+\nu)/2} \exp[-((n+\nu)/2)(\mathrm{tr}D - (1/2)\mathrm{tr}D^2 + (1/3)\mathrm{tr}D^3 - \cdots)]. \quad (2.12)$$

Now, let $E = (\beta^*-\hat{\beta}^*)'X^{*\prime}X^*(\beta^*-\hat{\beta}^*)$ and $\overline{E} = (\beta^*-\hat{\beta}^*)'M(\beta^*-\hat{\beta}^*)$, and note that $D = C\overline{E}C'$, then $\mathrm{tr}D = \mathrm{tr}C\overline{E}C' = \mathrm{tr}C'C\overline{E} = \mathrm{tr}(\overline{A(m)})^{-1}\overline{E} = \frac{1}{n}\mathrm{tr}(\overline{A(m)})^{-1}E$. Similarly, $\mathrm{tr}D^2 = (1/n^2)\mathrm{tr}(\overline{A(m)})^{-1}E(\overline{A(m)})^{-1}E$, and $\mathrm{tr}D^3 = (1/n^3)\mathrm{tr}(\overline{A(m)})^{-1} E(\overline{A(m)})^{-1}E(\overline{A(m)})^{-1}E$.

By making these substitutions into (2.12) and using a Taylor series expansion on the exponential function, the leading term is found to be in the form of a multivariate normal distribution, so that

$$|\overline{A(m)} + (\beta^*-\hat{\beta}^*)'M(\beta^*-\hat{\beta}^*)|^{-(n+\nu)/2} \mathrel{\dot{\propto}} |\overline{A(m)}|^{-(n+\nu)/2} \exp\{-\frac{1}{2}\mathrm{tr}(\overline{A(m)})^{-1} \cdot (\beta^*-\hat{\beta}^*)'X^{*\prime}X^*(\beta^*-\hat{\beta}^*)\} \quad (2.13)$$

where $\dot{\propto}$ means "approximately proportional to".

However, it is possible to rewrite the right hand side of (2.13) by using the fact that

$$\mathrm{tr}(\overline{A(m)})^{-1}(\beta_i-\hat{\beta}_i)'X_i'X_i(\beta_i-\hat{\beta}_i) = (\underset{\sim}{B}_i-\hat{\underset{\sim}{B}}_i)'[(\overline{A(m)})^{-1} \otimes X_i'X_i](\underset{\sim}{B}_i-\hat{\underset{\sim}{B}}_i),$$

where $\underset{\sim}{B}_i = (\underset{\sim}{\beta}_{i1}', \underset{\sim}{\beta}_{i2}', \cdots, \underset{\sim}{\beta}_{ip}')'$ and $\hat{\underset{\sim}{B}}_i = (\hat{\underset{\sim}{\beta}}_{i1}', \hat{\underset{\sim}{\beta}}_{i2}', \cdots, \hat{\underset{\sim}{\beta}}_{ip}')'$, $i = 1, 2$, are $pk\times 1$ vectors.

Therefore,

$$|\overline{A(m)} + (\beta^*-\hat{\beta}^*)'M(\beta^*-\hat{\beta}^*)|^{-(n+\nu)/2} \mathrel{\dot{\propto}} |\overline{A(m)}|^{-(n+\nu)/2} \exp\{-\frac{1}{2}(\underset{\sim}{B}_1-\hat{\underset{\sim}{B}}_1)' \cdot [(\overline{A(m)})^{-1} \otimes X_1'X_1](\underset{\sim}{B}_1-\hat{\underset{\sim}{B}}_1) - \frac{1}{2}(\underset{\sim}{B}_2-\hat{\underset{\sim}{B}}_2)'[(\overline{A(m)})^{-1} \otimes X_2'X_2](\underset{\sim}{B}_2-\hat{\underset{\sim}{B}}_2)\},$$

and the posterior distribution for β_1, β_2, and m becomes

$$\Pi(\beta_1,\beta_2,m|Y) \stackrel{\bullet}{\propto} |\overline{A(m)}|^{-(n+\nu)/2} \exp\{-\frac{1}{2}[(\underset{\sim}{B}_1-\underset{\sim}{U}_1)'F_1(\underset{\sim}{B}_1-\underset{\sim}{U}_1) + (\underset{\sim}{B}_1-\hat{\underset{\sim}{B}}_1)'[(\overline{A(m)})^{-1} \otimes X_1'X_1](\underset{\sim}{B}_1-\hat{\underset{\sim}{B}}_1)]\} \exp\{-\frac{1}{2}[(\underset{\sim}{B}_2-\underset{\sim}{U}_2)'F_2(\underset{\sim}{B}_2-\underset{\sim}{U}_2) + (\underset{\sim}{B}_2-\hat{\underset{\sim}{B}}_2)'[(\overline{A(m)})^{-1} \otimes X_2'X_2](\underset{\sim}{B}_2-\hat{\underset{\sim}{B}}_2)]\}. \quad (2.14)$$

Completing the square on $\underset{\sim}{B}_1$ and $\underset{\sim}{B}_2$,

$$\Pi(\beta_1,\beta_2,m|Y) \stackrel{\bullet}{\propto} |\overline{A(m)}|^{-(n+\nu)/2} \exp\{-\frac{1}{2}[(\underset{\sim}{B}_1-\underset{\sim}{B}_1(m))'G_1(\underset{\sim}{B}_1-\underset{\sim}{B}_1(m)) + H_1]\} \cdot \exp\{-\frac{1}{2}[(\underset{\sim}{B}_2-\underset{\sim}{B}_2(m))'G_2(\underset{\sim}{B}_2-\underset{\sim}{B}_2(m)) + H_2]\} \quad (2.15)$$

where for i = 1, 2,

$$\underset{\sim}{B}_i(m) = \{F_i + [(\overline{A(m)})^{-1} \otimes X_i'X_i]\}^{-1} \cdot \{F_i\underset{\sim}{U}_i + [(\overline{A(m)})^{-1} \otimes X_i'X_i]\hat{\underset{\sim}{B}}_i\},$$

$$G_i = F_i + [(\overline{A(m)})^{-1} \otimes X_i'X_i], \text{ and}$$

$$H_i = \underset{\sim}{U}_i'F_i\underset{\sim}{U}_i + \hat{\underset{\sim}{B}}_i'[(\overline{A(m)})^{-1} \otimes X_i'X_i]\hat{\underset{\sim}{B}}_i - \{F_i\underset{\sim}{U}_i + [(\overline{A(m)})^{-1} \otimes X_i'X_i]\hat{\underset{\sim}{B}}_i\}' \cdot G_i^{-1}\{F_i\underset{\sim}{U}_i + [(\overline{A(m)})^{-1} \otimes X_i'X_i]\hat{\underset{\sim}{B}}_i\}.$$

The posterior distribution of the change point m can now be found from (2.15) by integrating with respect to β_1 and β_2, and it is

$$\Pi(m|Y) \stackrel{\bullet}{\propto} \begin{cases} \dfrac{\exp\{-(1/2)(H_1 + H_2)\}}{|\overline{A(m)}|^{(n+\nu)/2}|G_1G_2|^{1/2}}, & k \leq m \leq n-k \\ 0, & \text{otherwise} \end{cases} \quad (2.16)$$

where G_1, G_2, H_1, and H_2 are defined in (2.15) and $\overline{A(m)}$ is defined in (2.11). The mean and variance of the posterior distribution of m will be

$$E(m|Y) = \sum_{m=k}^{n-k} m \cdot \Pi(m|Y), \text{ and}$$

$$\mathrm{Var}(m|Y) = \sum_{m=k}^{n-k} (m - E(m|Y))^2 \cdot \Pi(m|Y), \text{ respectively.} \tag{2.17}$$

Posterior Distributions of the Other Model Parameters

From (2.15) the marginal posterior distributions of β_1 and β_2 can be found. Integrating with respect to β_2 yields

$$\Pi(\beta_1, m|Y) \mathrel{\dot{\propto}} |\overline{A(m)}|^{-(n+\nu)/2} |G_2|^{-1/2} \exp\{-\frac{1}{2}(H_1 + H_2)\} \cdot$$

$$\exp\{-\frac{1}{2}(\underset{\sim}{B}_1 - \underset{\sim}{B}_1(m))' G_1 (\underset{\sim}{B}_1 - \underset{\sim}{B}_1(m))\}. \tag{2.18}$$

Summing over m, the marginal posterior distribution of β_1 is seen to be a mixture of approximate multivariate normal distributions with a mean of $\underset{\sim}{B}_1(m)$ and a precision matrix of G_1. That is,

$$\Pi(\underset{\sim}{B}_1|Y) \doteq \sum_{m=k}^{n-k} \Pi(m|Y) \cdot N_{pk}(\underset{\sim}{B}_1(m), G_1) \tag{2.19}$$

where $\Pi(m|Y)$ is the marginal posterior distribution of m. Then

$$E(\underset{\sim}{B}_1|m,Y) \doteq \underset{\sim}{B}_1(m) \text{ and}$$

$$\mathrm{Cov}(\underset{\sim}{B}_1|m,Y) \doteq G_1^{-1}, \text{ while} \tag{2.20}$$

$$E(\underset{\sim}{B}_1|Y) = \underset{m}{E}(E(\underset{\sim}{B}_1|m,Y)) \doteq \sum_{m=k}^{n-k} \Pi(m|Y) \cdot \underset{\sim}{B}_1(m) \text{ and}$$

$$\mathrm{Cov}(\underset{\sim}{B}_1|Y) = \underset{m}{E}[\mathrm{Cov}(\underset{\sim}{B}_1|m,Y)] + \underset{m}{\mathrm{Cov}}[E(\underset{\sim}{B}_1|m,Y)].$$

By integrating (2.15) with respect to β_1 it is found that

$$\Pi(\beta_2, m|Y) \mathrel{\dot{\propto}} |\overline{A(m)}|^{-(n+\nu)/2} |G_1|^{-1/2} \exp\{-\frac{1}{2}(H_1 + H_2)\} \cdot$$

$$\exp\{-\frac{1}{2}(\underset{\sim}{B}_2 - \underset{\sim}{B}_2(m))' G_2 (\underset{\sim}{B}_2 - \underset{\sim}{B}_2(m))\}. \tag{2.21}$$

So, the marginal posterior distribution of β_2 is a mixture of approximate multivariate normal distributions with a mean of $\underset{\sim}{B}_2(m)$ and a precision matrix of G_2.

$$\Pi(\underset{\sim}{B}_2|Y) \doteq \sum_{m=k}^{n-k} \Pi(m|Y)\cdot N_{pk}(\underset{\sim}{B}_2(m), G_2), \tag{2.22}$$

so that,

$$E(\underset{\sim}{B}_2|m,Y) \doteq \underset{\sim}{B}_2(m) \text{ and}$$

$$Cov(\underset{\sim}{B}_2|m,Y) \doteq G_2^{-1}, \text{ while} \tag{2.23}$$

$$E(\underset{\sim}{B}_2|Y) = \underset{m}{E}(E(\underset{\sim}{B}_2|m,Y)) \doteq \sum_{m=k}^{n-k} \Pi(m|Y)\cdot \underset{\sim}{B}_2(m) \text{ and}$$

$$Cov(\underset{\sim}{B}_2|Y) = \underset{m}{E}[Cov(\underset{\sim}{B}_2|m,Y)] + \underset{m}{Cov}[E(\underset{\sim}{B}_2|m,Y)].$$

To find the marginal posterior distribution of P, (2.7) can be written as

$$\Pi(\beta_1,\beta_2,P,m|Y) \propto |P|^{(n+\nu-p-1)/2} \exp\{-\frac{1}{2}tr[\Sigma + S_1 + S_2]P\}\cdot$$

$$\exp\{-\frac{1}{2}[(\underset{\sim}{B}_1-\underset{\sim}{U}_1)'F_1(\underset{\sim}{B}_1-\underset{\sim}{U}_1) + (\underset{\sim}{B}_1-\hat{\underset{\sim}{B}}_1)'(P \otimes X_1'X_1)(\underset{\sim}{B}_1-\hat{\underset{\sim}{B}}_1)]\}\cdot$$

$$\exp\{-\frac{1}{2}[(\underset{\sim}{B}_2-\underset{\sim}{U}_2)'F_2(\underset{\sim}{B}_2-\underset{\sim}{U}_2) + (\underset{\sim}{B}_2-\hat{\underset{\sim}{B}}_2)'(P \otimes X_2'X_2)(\underset{\sim}{B}_2-\hat{\underset{\sim}{B}}_2)]\}. \tag{2.24}$$

By completing the square on $\underset{\sim}{B}_i$,

$(\underset{\sim}{B}_i-\underset{\sim}{U}_i)'F_i(\underset{\sim}{B}_i-\underset{\sim}{U}_i) + (\underset{\sim}{B}_i-\hat{\underset{\sim}{B}}_i)'(P \otimes X_i'X_i)(\underset{\sim}{B}_i-\hat{\underset{\sim}{B}}_i)$ can be rewritten as

$(\underset{\sim}{B}_i-\underset{\sim}{B}_i^*(m))'G_i^*(\underset{\sim}{B}_i-\underset{\sim}{B}_i^*(m)) + H_i^*$, where

$\underset{\sim}{B}_i^*(m) = [F_i + (P \otimes X_i'X_i)]^{-1}[F_i\underset{\sim}{U}_i + (P \otimes X_i'X_i)\hat{\underset{\sim}{B}}_i]$,

$G_i^* = [F_i + (P \otimes X_i'X_i)]$, and

$$H_i^* = \underset{\sim}{U}_i{}'F_i\underset{\sim}{U}_i + \hat{\underset{\sim}{B}}_i{}'(P \otimes X_i{}'X_i)\hat{\underset{\sim}{B}}_i - [F_i\underset{\sim}{U}_i + (P \otimes X_i{}'X_i)\hat{\underset{\sim}{B}}_i]' \cdot$$

$$[F_i + (P \otimes X_i{}'X_i)]^{-1}[F_i\underset{\sim}{U}_i + (P \otimes X_i{}'X_i)\hat{\underset{\sim}{B}}_i].$$

Then,

$$\Pi(\beta_1,\beta_2,P,m|Y) \propto |P|^{(n+\nu-p-1)/2}\exp\{-\frac{1}{2}tr[\Sigma + S_1 + S_2]P\}\cdot$$

$$\exp\{-\frac{1}{2}(\underset{\sim}{B}_1-\underset{\sim}{B}_1{}^*(m))'G_1{}^*(\underset{\sim}{B}_1-\underset{\sim}{B}_1{}^*(m))\}\exp\{-\frac{1}{2}(\underset{\sim}{B}_2-\underset{\sim}{B}_2{}^*(m))'\cdot$$

$$G_2{}^*(\underset{\sim}{B}_2-\underset{\sim}{B}_2{}^*(m))\}\exp\{-\frac{1}{2}(H_1{}^* + H_2{}^*)\}. \tag{2.25}$$

The marginal posterior distribution of P can now be found by integrating (2.25) with respect to $\underset{\sim}{B}_1$ and $\underset{\sim}{B}_2$ and summing over m, and it is of the form

$$\Pi(P|Y) \propto \sum_{m=k}^{n-k}|P|^{(n+\nu-p-1)/2}|G_1{}^*|^{-1/2}|G_2{}^*|^{-1/2}\exp\{-\frac{1}{2}(H_1{}^* + H_2{}^*)\}\cdot$$

$$\exp\{-\frac{1}{2}tr[\Sigma + S_1 + S_2]P\}. \tag{2.26}$$

Numerical Study of the Posterior Distribution of the Change Point

In order to make comparisons between the use of a natural conjugate prior distribution and a generalized natural conjugate prior distribution, and to obtain some indication of the effect that changes in sample size and certain population parameters have on the posterior distribution of the change point m, a sensitivity analysis will be conducted using a bivariate regression model.

With the model defined as in (2.1) and k = p = 2, Y becomes an n×2 matrix of observations, β_1 and β_2 are 2×2 matrices of real unknown parameters, X is an n×2 design matrix, and e is an n×2 matrix of unobservable random variables. The rows of e, ($\underset{\sim}{e}_i'$, i = 1, 2, ⋯, n), are independently and identically distributed as $N_2(\underset{\sim}{0}, P)$. P is a 2×2 positive definite symmetric precision matrix, so the variance-covariance matrix is P^{-1}, where

$$P^{-1} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}.$$

Using the results of Salazar (1980) with a natural conjugate prior distribution, the conditional distribution of the rows of β_i, i = 1, 2, ($\underset{\sim}{\beta}_{ij}'$, j = 1, 2, ⋯, k), given P is multivariate normal with mean vector $\underset{\sim}{\mu}_{ij}' \in R^P$, i = 1, 2, j = 1, 2, ⋯, k, and precision matrix $r_{ij}P$, $r_{ij} > 0$, and the marginal distribution of P is distributed as a Wishart distribution with ν degrees of freedom and precision matrix Σ.

Then the posterior distribution of the change point m is given by

$$\Pi_1(m|Y) \propto \begin{cases} |D_1 D_2|^{-p/2} |Q(m)|^{-(n+\nu)/2}, & 1 \leq m \leq n-1 \\ 0, & \text{otherwise} \end{cases} \qquad (2.27)$$

where

$$D_i = X_i'X_i + R_i,$$

$$R_i = \text{Diagonal}(r_{ij}),$$

$$Q(m) = \Sigma + Y_1'Y_1 + Y_2'Y_2 + \mu_1'R_1\mu_1 + \mu_2'R_2\mu_2 - (X_1'Y_1+R_1\mu_1)' \cdot$$

$$D_1^{-1}(X_1'Y_1+R_1\mu_1) - (X_2'Y_2+R_2\mu_2)'D_2^{-1}(X_2'Y_2+R_2\mu_2), \text{ and}$$

$$\mu_i = (\underset{\sim}{\mu}_{i1}, \underset{\sim}{\mu}_{i2}, \cdots, \underset{\sim}{\mu}_{ik})', \text{ for } i = 1, 2.$$

The mean and variance of the posterior distribution of m will be

$$E(m|Y) = \sum_{m=1}^{n-1} m \cdot \Pi_1(m|Y), \text{ and}$$

$$\mathrm{Var}(m|Y) = \sum_{m=1}^{n-1} (m - E(m|Y))^2 \cdot \Pi_1(m|Y), \text{ respectively.} \qquad (2.28)$$

With a generalized natural conjugate prior distribution, the posterior distribution of the change point m is given by (2.16).

For this particular study the following choices were made for the parameters of the model: $\sigma_1^2 = \sigma_2^2 = 1$, $\rho = -.7, -.5, -.2, 0, .2, .5$, and $.7$, $\nu = 2$,

$$\Sigma = \begin{pmatrix} 2 & 2\rho \\ 2\rho & 2 \end{pmatrix}, \qquad \beta_1 = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix},$$

$$\mu_1 = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}, \qquad R_1 = R_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\mu_2 = \mu_1 + \begin{pmatrix} \Delta_1 & 0 \\ \Delta_1 & 0 \end{pmatrix}, \quad \text{where } \Delta_1 = 0, .2, .4, \text{ and } .6, \text{ and}$$

$$\beta_2 = \beta_1 + \begin{pmatrix} \Delta_2 & 0 \\ \Delta_2 & 0 \end{pmatrix}, \quad \text{where } \Delta_2 = .04, .05, .06, \text{ and } .07.$$

The $n\times 2$ design matrix consisted of ones in column one, while column two was a two-digit number selected at random from a random number table. Sample sizes considered were n = 10, 20, 50, and 100, and three different cases were used for the actual change point. That is, the change was constructed to be at the third data point in case one, at n/2 in case two, and at n-3 in case three.

A Fortran program which made use of an IMSL subroutine was used to generate the bivariate normal error terms for a specified ρ, and SAS programs were written for finding the posterior distribution of the change point m when using a natural conjugate prior distribution and a generalized natural conjugate prior distribution.

The results with a natural conjugate prior distribution are found in

Tables I through XII of the Appendix, while Tables XIII through XXIV of the Appendix are with a generalized natural conjugate prior distribution.

In looking at a set of tables for either of the two prior distributions with fixed values for Δ_1, Δ_2, ρ, and location of the point of change, there is some indication that by increasing the sample size the change point is more easily detected; however, the observations that actually make up the sample for a certain sample size also affect the posterior probability of m, so it is difficult to arrive at any definite conclusions concerning increases in the size of the sample from this particular study.

Also, for a particular prior distribution and location of the change point, and with fixed values of n, ρ, and Δ_2, changes in Δ_1 do not seem to have much of an effect on the posterior probability of the actual point of change; however, in all cases, as Δ_2 increases for fixed n, ρ, and Δ_1, the posterior probability also increases. Additionally, it is most often the case that for fixed n, Δ_1, and Δ_2, the posterior probability is smallest when $\rho = 0$ and increases as ρ becomes increasingly more positive or more negative.

Finally, for given values of n, ρ, Δ_1, Δ_2, and location of the change point, the probabilities are quite similar between the two different prior distributions, particularly it seems when the change point is near the middle of the data. For example, when $n = 20$ and the actual change is at 10, with $\rho = 0$, $\Delta_1 = 0$, and $\Delta_2 = .06$, the posterior probability that m equals 10 is .61133 from Table V when using a natural conjugate prior distribution, and it is .56394 from Table XVII when using a generalized natural conjugate prior distribution. One reason for differences in the probabilities between these two prior distributions is that for a natural conjugate prior distribution, m can take values from 1 to n-1, while for the generalized natural conjugate prior distribution the range of m is from k to n-k, or in this particular instance from 2 to n-2. This is true because in the latter case matrix singularity problems arise when m equals 1 or n-1. Another reason for differences is that with the generalized natural conjugate prior distribution only approximate proportionality is obtained when deriving the posterior distribution of m. In general though, it appears that good results are still obtained when using this prior distribution in comparison with the natural conjugate prior distribution, and one must remember that a generalized prior does have the advantage of allowing an unrestricted prior variance-covariance matrix for β_1 and β_2, which is not the case with a natural conjugate prior distribution.

Numerical Study of the Posterior Expected Values, Variances, and Covariances of the Model Parameters

In addition to studying the posterior distribution of the change point m, another area of interest is that of the posterior expected values, variances, and covariances for the parameters of the model. The same changing bivariate regression model as in the previous section is also considered here, although not as many different parameter settings are used. Parameter values considered are $\sigma_1^2 = \sigma_2^2 = 1$, $\rho = -.5$, 0, and .5, $\nu = 2$, n = 20 and 100 with the actual change point m occurring at n/2,

$$\Sigma = \begin{pmatrix} 2 & 2\rho \\ 2\rho & 2 \end{pmatrix}, \qquad \beta_1 = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix},$$

$$\mu_1 = \mu_2 = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}, \qquad R_1 = R_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \text{ and}$$

$$\beta_2 = \beta_1 + \begin{pmatrix} \Delta & 0 \\ \Delta & 0 \end{pmatrix}, \text{ where } \Delta = .04,\ .05,\ .06, \text{ and } .07.$$

Results are obtained when using both a natural conjugate prior distribution and a generalized natural conjugate prior distribution.

As pointed out by Salazar (1980), when using a natural conjugate prior distribution the marginal posterior distributions of β_1 and β_2 will be mixtures of matrix T-distributions, while the marginal posterior distribution of P is a mixture of Wishart distributions.

In particular,

$$\Pi(\beta_1|Y) = \sum_{m=1}^{n-1} \Pi_1(m|Y)\cdot\text{Matrix T}[D_1,\ A_1(m),\ \beta_1(m),\ n+\nu+k], \tag{2.29}$$

where

$$A_1(m) = \Sigma + Z_1 + Z_2,$$

$$\beta_1(m) = (X_1{}'X_1 + R_1)^{-1}(X_1{}'Y_1 + R_1\mu_1), \text{ and for } i = 1,\ 2,$$

$$Z_i = Y_i{}'Y_i + \mu_i{}'R_i\mu_i - (X_i{}'Y_i + R_i\mu_i)'(X_i{}'X_i + R_i)^{-1}(X_i{}'Y_i + R_i\mu_i).$$

$\Pi_1(m|Y)$ and D_1 are defined in (2.27).

Then from the properties of the matrix T-distribution it follows that

$$E(\underset{\sim}{B}_1|m,Y) = \tilde{\underset{\sim}{B}}_1(m),$$

$$\text{Cov}(\underset{\sim}{B}_1|m,Y) = \left[(1/(n+\nu-p-1))(A_1(m) \otimes D_1^{-1})\right], \tag{2.30}$$

$$E(\underset{\sim}{B}_1|Y) = \sum_{m=1}^{n-1} \Pi_1(m|Y)\cdot\tilde{\underset{\sim}{B}}_1(m), \text{ and}$$

$$\text{Cov}(\underset{\sim}{B}_1|Y) = \underset{m}{E}[\text{Cov}(\underset{\sim}{B}_1|m,Y)] + \underset{m}{\text{Cov}}[E(\underset{\sim}{B}_1|m,Y)],$$

where for $\beta_1(m) = (\underset{\sim}{\beta}_{11}(m), \underset{\sim}{\beta}_{12}(m), \cdots, \underset{\sim}{\beta}_{1p}(m))$,

$$\tilde{\underset{\sim}{B}}_1(m) = (\underset{\sim}{\beta}'_{11}(m), \underset{\sim}{\beta}'_{12}(m), \cdots, \underset{\sim}{\beta}'_{1p}(m))'.$$

Similarly for β_2 the marginal posterior distribution is a mixture of matrix T-distributions. That is,

$$\Pi(\beta_2|Y) = \sum_{m=1}^{n-1} \Pi_1(m|Y) \cdot \text{Matrix T}[D_2,\ A_1(m),\ \beta_2(m),\ n+\nu+k], \tag{2.31}$$

where $\Pi_1(m|Y)$ and D_2 are defined in (2.27),

$$\beta_2(m) = (X_2{}'X_2 + R_2)^{-1}(X_2{}'Y_2 + R_2\mu_2),$$

and $A_1(m)$ is the same as in (2.29).

Therefore, for $\beta_2(m) = (\underset{\sim}{\beta}_{21}(m), \underset{\sim}{\beta}_{22}(m), \cdots, \underset{\sim}{\beta}_{2p}(m))$ by defining

$$\tilde{\underset{\sim}{B}}_2(m) = (\underset{\sim}{\beta}'_{21}(m), \underset{\sim}{\beta}'_{22}(m), \cdots, \underset{\sim}{\beta}'_{2p}(m))',$$

$$E(\underset{\sim}{B}_2|m,Y) = \tilde{\underset{\sim}{B}}_2(m),$$

$$\text{Cov}(\underset{\sim}{B}_2|m,Y) = \left[(1/(n+\nu-p-1))(A_1(m) \otimes D_2^{-1})\right], \tag{2.32}$$

$$E(\underset{\sim}{B}_2|Y) = \sum_{m=1}^{n-1} \Pi_1(m|Y) \cdot \tilde{\underset{\sim}{B}}_2(m), \text{ and}$$

$$\text{Cov}(\underset{\sim}{B}_2|Y) = \underset{m}{E}[\text{Cov}(\underset{\sim}{B}_2|m,Y)] + \underset{m}{\text{Cov}}[E(\underset{\sim}{B}_2|m,Y)].$$

The marginal posterior distribution of P is a mixture of Wishart distributions with $n+\nu$ degrees of freedom and a precision matrix of $A_1(m)$. That is,

$$\Pi(P|Y) = \sum_{m=1}^{n-1} \Pi_1(m|Y) \cdot \text{Wishart}(A_1(m),\ n+\nu,\ p). \tag{2.33}$$

Then,

$$E(P|m,Y) = (n+\nu)A_1^{-1}(m), \text{ and} \tag{2.34}$$

$$E(P|Y) = (n+\nu)\sum_{m=1}^{n-1} \Pi_1(m|Y)\cdot A_1^{-1}(m).$$

Now for $P = (p_{ij})$ and $A_1^{-1}(m) = (a_{ij})$,

$$\mathrm{Var}(p_{ij}|m,Y) = (n+\nu)(a_{ij}^2 + a_{ii}a_{jj}) \text{ and} \tag{2.35}$$

$$\mathrm{Cov}(p_{ij},p_{k\ell}|m,Y) = (n+\nu)(a_{ik}a_{j\ell} + a_{i\ell}a_{jk}),$$

while for $P = (\underset{\sim}{p}_1,\underset{\sim}{p}_2,\ldots,\underset{\sim}{p}_p)$ by defining

$$\underset{\sim}{P} = (\underset{\sim}{p}_1',\underset{\sim}{p}_2',\cdots,\underset{\sim}{p}_p')'$$

the covariance of $\underset{\sim}{P}$ given Y is

$$\mathrm{Cov}(\underset{\sim}{P}|Y) = \underset{m}{E}[\mathrm{Cov}(\underset{\sim}{P}|m,Y)] + \underset{m}{\mathrm{Cov}}[E(\underset{\sim}{P}|m,Y)].$$

Tables XXV through XXX in the Appendix present the results when using a natural conjugate prior distribution, while Tables XXXI through XXXVI are with the use of a generalized natural conjugate prior distribution.

In general, for a given value of n, ρ, and Δ there is not too much difference between the two prior distributions. For a given ρ and Δ, as n increases from 20 to 100, the expected values of the parameters are usually closer to the actual values, while at the same time the corresponding variances decrease for the larger sample size in all cases.

CHAPTER III

HYPOTHESIS TESTING PROCEDURE

Testing for a Single Shift

Suppose that the multivariate linear model, $Y = X\beta + e$, is written as

$$Y_1 = X_1\beta_1 + e_1$$
$$Y_2 = X_2\beta_2 + e_2, \tag{3.1}$$

where

$$Y = (Y_1' \vdots Y_2')' = (\underset{\sim}{Y}_1, \underset{\sim}{Y}_2, \cdots, \underset{\sim}{Y}_m \vdots \underset{\sim}{Y}_{m+1}, \cdots, \underset{\sim}{Y}_n)'$$

is an $n\times p$ matrix of observations,

$$X = \begin{pmatrix} X_1' & \phi \\ \phi & X_2' \end{pmatrix}' = \begin{pmatrix} \underset{\sim}{X}_1, \cdots, \underset{\sim}{X}_m & \phi \\ \phi & \underset{\sim}{X}_{m+1}, \cdots, \underset{\sim}{X}_n \end{pmatrix}'$$

is an $n\times 2k$ design matrix with the ϕ's being zero matrices of appropriate order,

$$\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$$

is a $2k\times p$ matrix of real unknown parameters, with β_1 and β_2 each being $k\times p$ matrices, and

$$e = (e_1' \vdots e_2')' = (\underset{\sim}{e}_1, \underset{\sim}{e}_2, \cdots, \underset{\sim}{e}_m \vdots \underset{\sim}{e}_{m+1}, \cdots, \underset{\sim}{e}_n)'$$

is an $n\times p$ matrix of unobservable random variables. The rows of e, ($\underset{\sim}{e}_i'$, $i = 1, 2, \cdots, n$), are assumed to be independently and identically distributed as $N_p(\underset{\sim}{0},P)$, where P is a p×p positive definite symmetric precision matrix.

Let m be a positive integer such that $1 \leqslant m \leqslant n$. If $1 \leqslant m \leqslant n-1$, then a change in the model has occurred and $\beta_1 \neq \beta_2$, while if $m = n$, then no change has occurred and $\beta_1 = \beta_2$.

In testing for a single shift the null and alternative hypotheses can be written as

$$H_o:\ m = n \quad \text{versus} \quad H_a:\ 1 \leqslant m \leqslant n-1.$$

A Bayesian test will be developed based on the posterior probability mass function of m, and then, in order to test H_o against H_a, the probability of no change, $\Pi(n)$, can be compared with the probability of exactly one change, $1 - \Pi(n)$.

Assign prior distributions for the parameters as follows. For a given value of q, $0 < q < 1$, let

$$\Pi_o(m) = \begin{cases} q, & m = n \\ \frac{(1-q)}{(n-1)}, & 1 \leqslant m \leqslant n-1. \end{cases} \tag{3.2}$$

If $m = n$, the unknown parameters are the k×p matrix β_1 and the p×p precision matrix P. Choose the conditional distribution of the rows of β_1 given P to be a multivariate normal distribution with a mean vector $\underset{\sim}{\mu}_j$, $j = 1, 2, \cdots, k$, and a precision matrix of r_jP, $r_j > 0$, and let the marginal prior distribution of P be a Wishart distribution with ν degrees of freedom and precision matrix Σ.

If $1 \leqslant m \leqslant n-1$, then the unknown parameters are the 2k×p matrix $\beta = (\beta_1{}',\beta_2{}')'$, the precision matrix P, and the change point m. In this case, let m be a uniform discrete random variable and choose the conditional distribution of the rows of β given P to be a multivariate normal distribution with mean vector $\underset{\sim}{\mu}_j$, $j = 1, 2, \cdots, 2k$ and precision matrix

r_jP, $r_j > 0$. Also, let the marginal prior distribution of P be a Wishart distribution with ν degrees of freedom and precision matrix Σ. The rows of β are independent.

Thus, when m = n,

$$\Pi_o(\beta_1|P) = \frac{1}{(2\pi)^{kp/2}} \cdot \left(\prod_{j=1}^{k} r_{j1}^{p}\right)|P|^{k/2} \exp\{-\frac{1}{2}\mathrm{tr}(\beta_1-\mu_1)'R_1(\beta_1-\mu_1)P\}, \tag{3.3}$$

while for $1 \leqslant m \leqslant n-1$,

$$\Pi_o(\beta|P) = \frac{1}{(2\pi)^{kp}} \cdot \left(\prod_{j=1}^{k} r_{j1}^{p} r_{j2}^{p}\right)|P|^{k} \exp\{-\frac{1}{2}\mathrm{tr}(\beta-\mu)'R(\beta-\mu)P\}, \tag{3.4}$$

where

$$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \text{ and } R = \begin{pmatrix} R_1 & \phi \\ \phi & R_2 \end{pmatrix} \text{ with } R_i = \mathrm{Diagonal}(r_{ij}),\ i = 1,2,$$

and $j = 1,\cdots,k$.

In both cases the marginal prior distribution of P is

$$\Pi_o(P) = c|\Sigma|^{\nu/2}|P|^{(\nu-p-1)/2}\exp\{-\frac{1}{2}\mathrm{tr}\Sigma P\}, \tag{3.5}$$

where

$$c = \left[2^{\nu p/2}\pi^{p(p-1)/4}\prod_{j=1}^{p}\Gamma((\nu+1-j)/2)\right]^{-1}.$$

The likelihood function can be written as

$$L(\beta_1,\beta_2,P,m) = \begin{cases} \frac{1}{(2\pi)^{np/2}} \cdot |P|^{n/2}\exp\{-\frac{1}{2}\mathrm{tr}(Y_1(n)-X_1(n)\beta_1)' \cdot \\ \qquad (Y_1(n)-X_1(n)\beta_1)P\}, & m = n \\ \frac{1}{(2\pi)^{np/2}} \cdot |P|^{n/2}\exp\{-\frac{1}{2}\mathrm{tr}(Y-X\beta)'(Y-X\beta)P\}, & 1 \leqslant m \leqslant n-1 \end{cases} \tag{3.6}$$

where

$$Y_1(n)-X_1(n)\beta_1 = \begin{pmatrix} \underset{\sim}{Y}_1{}'-\underset{\sim}{X}_1{}'\beta_1 \\ \underset{\sim}{Y}_2{}'-\underset{\sim}{X}_2{}'\beta_1 \\ \cdot \\ \cdot \\ \cdot \\ \underset{\sim}{Y}_n{}'-\underset{\sim}{X}_n{}'\beta_1 \end{pmatrix} \quad \text{and} \quad Y-X\beta = \begin{pmatrix} \underset{\sim}{Y}_1{}'-\underset{\sim}{X}_1{}'\beta_1 \\ \cdot \\ \cdot \\ \cdot \\ \underset{\sim}{Y}_m{}'-\underset{\sim}{X}_m{}'\beta_1 \\ \underset{\sim}{Y}_{m+1}{}'-\underset{\sim}{X}_{m+1}{}'\beta_2 \\ \cdot \\ \cdot \\ \cdot \\ \underset{\sim}{Y}_n{}'-\underset{\sim}{X}_n{}'\beta_2 \end{pmatrix}.$$

The posterior distribution of β_1, β_2, P, and m now can be found by combining the likelihood function and the prior distribution, and it is

$$\Pi(\beta_1,\beta_2,P,m|Y) \propto \begin{cases} \left[\dfrac{qc|\Sigma|^{\nu/2}(\prod_{j=1}^{k} r_{j1}^{p})}{(2\pi)^{(np+kp)/2}}\right]|P|^{(n+k+\nu-p-1)/2}\exp\{-\frac{1}{2}\mathrm{tr}[\Sigma + (\beta_1-\mu_1)'R_1(\beta_1-\mu_1) + (Y_1(n)-X_1(n)\beta_1)'\cdot(Y_1(n)-X_1(n)\beta_1)]P\}, & m = n \\ \left[\dfrac{(1-q)c|\Sigma|^{\nu/2}(\prod_{j=1}^{k} r_{j1}^{p} r_{j2}^{p})}{(n-1)(2\pi)^{(np+2kp)/2}}\right]|P|^{(n+2k+\nu-p-1)/2}\exp\{-\frac{1}{2}\mathrm{tr}[\Sigma + (\beta-\mu)'R(\beta-\mu) + (Y-X\beta)'(Y-X\beta)]P\}, & 1 \leq m \leq n-1 \end{cases} \tag{3.7}$$

To find the posterior probability mass function of m, (3.7) has to be integrated with respect to β_1, β_2, and P.

First, consider the case when m = n. Using the properties of the Wishart distribution,

$$\Pi(\beta_1,m|Y) \propto \left[\frac{qc|\Sigma|^{\nu/2}(\prod_{j=1}^{k} r_{j1}^{p})c'}{(2\pi)^{(np+kp)/2}}\right]|\Sigma + (\beta_1-\mu_1)'R_1(\beta_1-\mu_1) + (Y_1(n)-X_1(n)\beta_1)'(Y_1(n)-X_1(n)\beta_1)|^{-(n+k+\nu)/2},$$

where

$$c' = \left[2^{(n+k+\nu)p/2}\pi^{p(p-1)/4}\prod_{j=1}^{p}\Gamma((n+k+\nu+1-j)/2)\right],$$

and upon completing the square on β_1 this can be written as

$$\Pi(\beta_1,m|Y) \propto \left[\frac{qc|\Sigma|^{\nu/2}(\prod_{j=1}^{k} r_{j1}^{p})c'}{(2\pi)^{(np+kp)/2}}\right]|C(n) + (\beta_1-\beta_1(n))'D(n)\cdot(\beta_1-\beta_1(n))|^{-(n+k+\nu)/2} \quad (3.8)$$

where

$$\beta_1(n) = (X_1'(n)X_1(n) + R_1)^{-1}(X_1'(n)Y_1(n) + R_1\mu_1),$$

$$D(n) = X_1'(n)X_1(n) + R_1, \text{ and}$$

$$C(n) = \Sigma + Y_1'(n)Y_1(n) + \mu_1'R_1\mu_1 - (X_1'(n)Y_1(n) + R_1\mu_1)'(D(n))^{-1}(X_1'(n)Y_1(n) + R_1\mu_1).$$

But (3.8) is in the form of a matrix T-distribution, and so integration with respect to β_1 yields

$$\Pi(m|Y) \propto \left[\frac{qc|\Sigma|^{\nu/2}(\prod_{j=1}^{k} r_{j1}^{p})c'k'}{(2\pi)^{(np+kp)/2}}\right]|C(n)|^{-(n+\nu)/2}|D(n)|^{-p/2}, \text{ for } m = n$$

with

$$k' = \left(\pi^{kp/2}\prod_{j=1}^{p}\Gamma((n+\nu+1-j)/2)\right)/\left(\prod_{j=1}^{p}\Gamma((n+k+\nu+1-j)/2)\right).$$

Next, consider the case for $1 \leq m \leq n-1$. Integrating with respect to P,

$$\Pi(\beta_1,\beta_2,m|Y) \propto \left[\frac{(1-q)c|\Sigma|^{\nu/2}(\prod_{j=1}^{k} r_{j1}^{p} r_{j2}^{p})c''}{(n-1)(2\pi)^{(np+2kp)/2}}\right]|\Sigma + (\beta-\mu)'R(\beta-\mu) + (Y-X\beta)'(Y-X\beta)|^{-(n+2k+\nu)/2} \quad (3.9)$$

where

$$c'' = \left[2^{(n+2k+\nu)p/2}\pi^{p(p-1)/4}\prod_{j=1}^{p}\Gamma((n+2k+\nu+1-j)/2)\right].$$

By completing the square on β, (3.9) becomes

$$\Pi(\beta_1,\beta_2,m|Y) \propto \left[\frac{(1-q)c|\Sigma|^{\nu/2}(\prod_{j=1}^{k} r_{j1}^{p} r_{j2}^{p})c''}{(n-1)(2\pi)^{(np+2kp)/2}}\right]|C(m) + (\beta-\beta(m))'D(m)\cdot(\beta-\beta(m))|^{-(n+2k+\nu)/2} \quad (3.10)$$

where

$$\beta(m) = (X'X + R)^{-1}(X'Y + R\mu),$$

$$D(m) = X'X + R = \begin{pmatrix} X_1'X_1 + R_1 & \phi \\ \phi & X_2'X_2 + R_2 \end{pmatrix} = \begin{pmatrix} D_1 & \phi \\ \phi & D_2 \end{pmatrix}, \text{ and}$$

$$C(m) = \Sigma + Y'Y + \mu'R\mu - (X'Y + R\mu)'(D(m))^{-1}(X'Y + R\mu).$$

Since (3.10) is in the form of a matrix T-distribution, integration with respect to β results in a posterior distribution for m of the form

$$\Pi(m|Y) \propto \left[\frac{(1-q)c|\Sigma|^{\nu/2}(\prod_{j=1}^{k} r_{j1}^{p} r_{j2}^{p})c''k''}{(n-1)(2\pi)^{(np+2kp)/2}}\right]|C(m)|^{-(n+\nu)/2}|D_1D_2|^{-p/2},$$

$$\text{for } 1 \leqslant m \leqslant n-1,$$

where

$$k'' = \left(\pi^{kp}\prod_{j=1}^{p}\Gamma((n+\nu+1-j)/2)\right)/\left(\prod_{j=1}^{p}\Gamma((n+2k+\nu+1-j)/2)\right).$$

Upon simplification, the posterior probability mass function of m can be expressed as

$$\Pi(m|Y) \propto \quad q(\prod_{j=1}^{k} r_{j1}^{p})|C(n)|^{-(n+\nu)/2}|D(n)|^{-p/2}, \quad m = n \quad (3.11)$$

$$\Pi(m|Y) \propto \left[\frac{(1-q)\left(\prod_{j=1}^{k} r_{j1}^{p} r_{j2}^{p}\right)}{(n-1)}\right] |C(m)|^{-(n+\nu)/2} |D_1 D_2|^{-p/2}, \quad 1 \leq m \leq n-1$$

where C(n) and D(n) are given in (3.8), and C(m), D_1, and D_2 are given in (3.10).

Numerical Study of the Test Procedure

For a sample of size 20 and a change point at m = 10, the test procedure of the previous section is used here to test the null hypothesis of no change in the model, (H_o: m = n), against the alternative hypothesis that a change has occurred, (H_a: $1 \leq m \leq n-1$).

The same bivariate regression model is considered as in the numerical studies of Chapter II, but the parameter values chosen for this study are $\sigma_1^2 = \sigma_2^2 = 1$, $\rho = -.5$, 0, and .5, $\nu = 2$,

$$\Sigma = \begin{pmatrix} 2 & 2\rho \\ 2\rho & 2 \end{pmatrix}, \qquad \beta_1 = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix},$$

$$\mu_1 = \mu_2 = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}, \qquad R_1 = R_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\beta_2 = \beta_1 + \begin{pmatrix} \Delta & 0 \\ \Delta & 0 \end{pmatrix}, \quad \text{where } \Delta = 0, .03, .06, \text{ and } .09,$$

and q = .1, .5, and .9, where q is the prior probability of no change. The results are presented in Tables XXXVII through XLV of the Appendix.

As can be seen from the tables, when there is actually no change in the model, that is when $\Delta = 0$, the posterior probability that m = 20 is close to one for any of the choices of ρ and q, indicating that there has not been any change. But as Δ takes on the increasing values of .03, .06, and .09, this probability continually decreases, while the posterior probability that m = 10 increases.

For fixed values of Δ and q the detection of a change when one is actually present is more difficult for $\rho = 0$ then it is for $\rho = -.5$ or $\rho = .5$; however, when $\Delta = .09$ the shift is large enough so that the null hypothesis of no change would be rejected for any of the choices of ρ and q.

The effect that q has on the detection of a shift can also be observed from the tables. For example, when $\Delta = .06$ and $\rho = 0$ the posterior probability that m = 10 is .48619 for q = .1, decreases to .18433 for q = .5, and decreases again to .02798 for q = .9. In other words, it is seen that a shift is more easily detected for small values of q, that is when the probability of no change is small apriori.

CHAPTER IV

PREDICTION FOR MULTIVARIATE LINEAR MODELS WITH A SINGLE SHIFT

Bayesian Predictive Density

In this chapter the Bayesian predictive density is found for ℓ future observations of the unknown dependent variables for a multivariate linear model with a single shift in the β matrix at some point m, where m is a positive integer between 1 and n-1. Then the multivariate linear model, $Y = X\beta + e$, may be written as in (2.1) of Chapter II, with Y, X, β, and e defined as before. Recall that the rows of e are independently and identically distributed as $N_p(\underset{\sim}{0},P)$, with P being a p×p positive definite symmetric precision matrix.

As a first step the Bayesian predictive density will be found using an improper prior distribution for β_1 and β_2. In such a case the posterior distribution of the change point m exists only in the range $k \leq m \leq n-k$, so in order to apply the resulting procedure, the shift cannot occur from 1, ···, (k-1) or from (n-k+1), ···, (n-1).

Denote the ℓ future observations as $W = (\underset{\sim}{w}_1,\underset{\sim}{w}_2,\cdots,\underset{\sim}{w}_\ell)'$ which is an ℓ×p matrix, and assume that these observations are generated by the model

$$W = V\beta_2 + E \tag{4.1}$$

where $V = (\underset{\sim}{v}_1,\underset{\sim}{v}_2,\cdots,\underset{\sim}{v}_\ell)'$ is an ℓ×k known matrix, β_2 is a k×p matrix of real unknown parameters, and $E = (\underset{\sim}{E}_1,\underset{\sim}{E}_2,\cdots,\underset{\sim}{E}_\ell)'$ is an ℓ×p matrix of unobservable random variables, where $\underset{\sim}{E}_i'$, i = 1, ···, ℓ are independently and identically distributed as $N_p(\underset{\sim}{0},P)$.

The likelihood function can be written as

$$L(\beta_1,\beta_2,P,m) \propto |P|^{n/2} \exp\{-\frac{1}{2}tr(S_1 + S_2)P\}\cdot$$
$$\exp\{-\frac{1}{2}tr[(\beta_1-\hat{\beta}_1)'X_1'X_1(\beta_1-\hat{\beta}_1) + (\beta_2-\hat{\beta}_2)'X_2'X_2(\beta_2-\hat{\beta}_2)]P\}, \tag{4.2}$$

where $S_i = Y_i'Y_i - Y_i'X_i(X_i'X_i)^{-1}X_i'Y_i$ and $\hat{\beta}_i = (X_i'X_i)^{-1}X_i'Y_i$, $i = 1, 2$.

Assume that m, P, and the rows of β_1 and β_2 are independent, and assign the prior distribution of m as a uniform discrete random variable on the interval [k, n-k]. Suppose that P has a Wishart distribution with ν degrees of freedom and precision matrix Σ $(p\times p)$, and that β_1 and β_2 have improper prior distributions. Then the joint prior distribution is

$$\Pi_0(\beta_1,\beta_2,P,m) \propto |P|^{(\nu-p-1)/2} \exp\{-\frac{1}{2}tr\Sigma P\}. \tag{4.3}$$

Combining the likelihood function (4.2) with the joint prior distribution (4.3) results in a joint posterior distribution given by

$$\Pi(\beta_1,\beta_2,P,m|Y) \propto |P|^{(n+\nu-p-1)/2} \exp\{-\frac{1}{2}tr(\Sigma + S_1 + S_2)P\}\cdot$$
$$\exp\{-\frac{1}{2}tr[(\beta_1-\hat{\beta}_1)'X_1'X_1(\beta_1-\hat{\beta}_1) + (\beta_2-\hat{\beta}_2)'X_2'X_2(\beta_2-\hat{\beta}_2)]P\}. \tag{4.4}$$

From model (4.1) for the ℓ future observations, the distribution of W given β_2, V, and P is

$$f(W|\beta_2,V,P) \propto |P|^{\ell/2} \exp\{-\frac{1}{2}tr(W-V\beta_2)'(W-V\beta_2)P\}. \tag{4.5}$$

To find the predictive density of W, it is necessary to take the product of (4.4) and (4.5), and then integrate with respect to β_1, β_2, and P, and sum with respect to m.

Thus,

$$g(W,\beta_1,\beta_2,P,m|Y,V) = \Pi(\beta_1,\beta_2,P,m|Y)\cdot f(W|\beta_2,V,P) \propto$$

$$|P|^{(n+\nu+\ell-p-1)/2}\exp\{-\frac{1}{2}tr(\Sigma + S_1 + S_2)P\}\cdot$$

$$\exp\{-\frac{1}{2}tr[(\beta_2-\hat{\beta}_2)'X_2'X_2(\beta_2-\hat{\beta}_2) + (W-V\beta_2)'(W-V\beta_2)]P\}\cdot$$

$$\exp\{-\frac{1}{2}tr[(\beta_1-\hat{\beta}_1)'X_1'X_1(\beta_1-\hat{\beta}_1)]P\}. \quad (4.6)$$

For $\beta_1 = (\underset{\sim}{\beta}_{11}, \underset{\sim}{\beta}_{12}, \cdots, \underset{\sim}{\beta}_{1p})$ and $\hat{\beta}_1 = (\hat{\underset{\sim}{\beta}}_{11}, \hat{\underset{\sim}{\beta}}_{12}, \cdots, \hat{\underset{\sim}{\beta}}_{1p})$, define $\underset{\sim}{B}_1 = (\underset{\sim}{\beta}'_{11}, \underset{\sim}{\beta}'_{12}, \cdots, \underset{\sim}{\beta}'_{1p})'$ and $\hat{\underset{\sim}{B}}_1 = (\hat{\underset{\sim}{\beta}}'_{11}, \hat{\underset{\sim}{\beta}}'_{12}, \cdots, \hat{\underset{\sim}{\beta}}'_{1p})'$, so that $\underset{\sim}{B}_1$ and $\hat{\underset{\sim}{B}}_1$ will both be pk×1 vectors. Then by using the fact that

$$tr(\beta_1-\hat{\beta}_1)'X_1'X_1(\beta_1-\hat{\beta}_1)P = (\underset{\sim}{B}_1-\hat{\underset{\sim}{B}}_1)'(P \otimes X_1'X_1)(\underset{\sim}{B}_1-\hat{\underset{\sim}{B}}_1),$$

along with the properties of the multivariate normal distribution, (4.6) may be integrated with respect to β_1 to obtain

$$g(W,\beta_2,P,m|Y,V) \propto |P|^{(n+\nu+\ell-p-1)/2}|P \otimes X_1'X_1|^{-1/2}\exp\{-\frac{1}{2}tr(\Sigma + S_1 + S_2)P\}\exp\{-\frac{1}{2}tr[(\beta_2-\hat{\beta}_2)'X_2'X_2(\beta_2-\hat{\beta}_2) + (W-V\beta_2)'(W-V\beta_2)]P\}. \quad (4.7)$$

Completing the square on β_2 results in the identity

$$(\beta_2-\hat{\beta}_2)'X_2'X_2(\beta_2-\hat{\beta}_2) + (W-V\beta_2)'(W-V\beta_2) = (\beta_2-\beta^*)'D(\beta_2-\beta^*) + F, \quad (4.8)$$

where

$$\beta^* = (X_2'X_2 + V'V)^{-1}(X_2'Y_2 + V'W),$$

$$D = X_2'X_2 + V'V, \text{ and}$$

$$F = W'W + \hat{\beta}_2'X_2'X_2\hat{\beta}_2 - (X_2'Y_2 + V'W)'D^{-1}(X_2'Y_2 + V'W).$$

By writing the k×p matrices $\beta_2 = (\underset{\sim}{\beta}_{21}, \underset{\sim}{\beta}_{22}, \cdots, \underset{\sim}{\beta}_{2p})$ and $\beta^* = (\underset{\sim}{\beta}^*_{21}, \underset{\sim}{\beta}^*_{22}, \cdots, \underset{\sim}{\beta}^*_{2p})$ as pk×1 vectors, namely, $\underset{\sim}{B}_2 = (\underset{\sim}{\beta}'_{21}, \underset{\sim}{\beta}'_{22}, \cdots, \underset{\sim}{\beta}'_{2p})'$

and $\underset{\sim}{B}^* = (\underset{\sim}{\beta}^{*\prime}_{21}, \underset{\sim}{\beta}^{*\prime}_{22}, \cdots, \underset{\sim}{\beta}^{*\prime}_{2p})'$, respectively, then $tr(\beta_2-\beta^*)'D(\beta_2-\beta^*)P$ can be rewritten as $(\underset{\sim}{B}_2-\underset{\sim}{B}^*)'(P \otimes D)(\underset{\sim}{B}_2-\underset{\sim}{B}^*)$.

Using this result, along with (4.8) and multivariate normal properties, (4.7) may now be integrated with respect to β_2, and so

$$g(W,P,m|Y,V) \propto |P|^{(n+\nu+\ell-p-1)/2}|P \otimes X_1'X_1|^{-1/2}|P \otimes D|^{-1/2}\cdot$$
$$\exp\{-\frac{1}{2}tr(\Sigma + S_1 + S_2 + F)P\}.$$

But $|P \otimes X_1'X_1|^{-1/2} = |P|^{-k/2}|X_1'X_1|^{-p/2}$ and $|P \otimes D|^{-1/2} = |P|^{-k/2}|D|^{-p/2}$ so that

$$g(W,P,m|Y,V) \propto |P|^{(n+\nu+\ell-2k-p-1)/2}|X_1'X_1|^{-p/2}|D|^{-p/2}\cdot$$
$$\exp\{-\frac{1}{2}tr(\Sigma + S_1 + S_2 + F)P\}. \tag{4.9}$$

Integrating (4.9) with respect to P using the properties of the Wishart distribution, it is found that

$$g(W,m|Y,V) \propto |X_1'X_1D|^{-p/2}|\Sigma + S_1 + S_2 + F|^{-(n+\nu+\ell-2k)/2}. \tag{4.10}$$

It is possible to write (4.10) in a slightly different form, since from (4.8),

$$\Sigma + S_1 + S_2 + F = \Sigma + S_1 + S_2 + W'W + \hat{\beta}_2'X_2'X_2\hat{\beta}_2 - (X_2'Y_2 + V'W)'D^{-1}\cdot$$
$$(X_2'Y_2 + V'W).$$

By completing the square on W, and with some simplification, it can be shown that

$$\Sigma + S_1 + S_2 + F = \Sigma + S_1 + S_2 + (W-V\hat{\beta}_2)'(I-V(X_2'X_2 + V'V)^{-1}V')(W-V\hat{\beta}_2).$$

Therefore,

$$g(W,m|Y,V) \propto |X_1'X_1D|^{-p/2}|\Sigma + S_1 + S_2 + (W-V\hat{\beta}_2)'\cdot$$
$$(I-V(X_2'X_2 + V'V)^{-1}V')(W-V\hat{\beta}_2)|^{-(n+\nu+\ell-2k)/2} \tag{4.11}$$

Since $g(W|Y,V) \propto \sum_{m=k}^{n-k} g(m|Y,V)\cdot g(W|m,Y,V)$, it follows that the predictive density will be

$$g(W|Y,V) \propto \sum_{m=k}^{n-k} g(m|Y,V)\cdot \text{Matrix } T(R,Q,V\hat{\beta}_2,n+\nu+\ell-2k), \tag{4.12}$$

where

$$Q = \Sigma + S_1 + S_2, \text{ and } R = I-V(X_2'X_2 + V'V)^{-1}V'.$$

That is, the predictive density is a mixture of matrix T-distributions.

Instead of using an improper prior distribution for β_1 and β_2, suppose that a natural conjugate prior distribution is used for β_i, $i = 1$, 2, and P. That is, the conditional distribution of the rows of β_i, given P is multivariate normal with mean vector $\underset{\sim}{\mu}'_{ij}$, $i = 1, 2$, $j = 1, \cdots, k$, and precision matrix $r_{ij}P$, $r_{ij} > 0$, such that $\underset{\sim}{\mu}'_{ij} \in R^p$, while the marginal distribution of P is a Wishart distribution with ν degrees of freedom and positive definite symmetric precision matrix Σ. Let the change point m be a uniform discrete random variable defined on the integer values over the interval [1, n-1]. Note that when using a natural conjugate prior distribution, m exists for all integer values in the range from 1 to n-1. Then for $i = 1, 2$, the prior distribution for β_i is

$$\Pi_o(\beta_i|P) \propto |P|^{k/2} \exp\{-\frac{1}{2}\text{tr}(\beta_i-\mu_i)'R_i(\beta_i-\mu_i)P\}, \tag{4.13}$$

while the prior distribution for P is

$$\Pi_o(P) \propto |P|^{(\nu-p-1)/2} \exp\{-\frac{1}{2}\text{tr}\Sigma P\}. \tag{4.14}$$

The likelihood function is

$$L(\beta_1,\beta_2,P,m) \propto |P|^{n/2} \exp\{-\frac{1}{2}\text{tr}[(Y_1-X_1\beta_1)'(Y_1-X_1\beta_1) +$$

$$(Y_2-X_2\beta_2)'(Y_2-X_2\beta_2)]P\}, \tag{4.15}$$

and by combining this with the joint prior distribution, the joint posterior distribution is of the form

$$\Pi(\beta_1,\beta_2,P,m|Y) \propto |P|^{(n+\nu+2k-p-1)/2}\exp\{-\frac{1}{2}tr[\Sigma + (\beta_1-\mu_1)'R_1(\beta_1-\mu_1) + (\beta_2-\mu_2)'R_2(\beta_2-\mu_2) + (Y_1-X_1\beta_1)'(Y_1-X_1\beta_1) + (Y_2-X_2\beta_2)'(Y_2-X_2\beta_2)]P\}. \quad (4.16)$$

Now, to find the predictive density of W as given by model (4.1), the product of (4.5) and (4.16) is found and this result is integrated with respect to β_1, β_2, and P, and summed with respect to m. So,

$$g(W,\beta_1,\beta_2,P,m|Y,V) \propto |P|^{(n+\nu+\ell+2k-p-1)/2}\exp\{-\frac{1}{2}tr[\Sigma + (\beta_1-\mu_1)'R_1\cdot(\beta_1-\mu_1) + (\beta_2-\mu_2)'R_2(\beta_2-\mu_2) + (Y_1-X_1\beta_1)'(Y_1-X_1\beta_1) + (Y_2-X_2\beta_2)'(Y_2-X_2\beta_2) + (W-V\beta_2)'(W-V\beta_2)]P\}. \quad (4.17)$$

Completing the square on β_1 results in the identity

$$(\beta_1-\mu_1)'R_1(\beta_1-\mu_1) + (Y_1-X_1\beta_1)'(Y_1-X_1\beta_1) = (\beta_1-\beta_1(m))'D_1(\beta_1-\beta_1(m)) + F_1, \quad (4.18)$$

where

$$\beta_1(m) = (X_1'X_1 + R_1)^{-1}(X_1'Y_1 + R_1\mu_1),$$

$$D_1 = (X_1'X_1 + R_1), \text{ and}$$

$$F_1 = Y_1'Y_1 + \mu_1'R_1\mu_1 - (X_1'Y_1 + R_1\mu_1)'(X_1'X_1 + R_1)^{-1}(X_1'Y_1 + R_1\mu_1),$$

while completing the square on β_2 results in the identity

$$(\beta_2-\mu_2)'R_2(\beta_2-\mu_2) + (Y_2-X_2\beta_2)'(Y_2-X_2\beta_2) + (W-V\beta_2)'(W-V\beta_2) = (\beta_2-\beta_2(m))'D_2(\beta_2-\beta_2(m)) + F_2^*, \quad (4.19)$$

where

$$\beta_2(m) = (X_2'X_2 + V'V + R_2)^{-1}(X_2'Y_2 + V'W + R_2\mu_2),$$

$D_2 = X_2{}'X_2 + V'V + R_2$, and

$$F_2{}^* = Y_2{}'Y_2 + W'W + \mu_2{}'R_2\mu_2 - (X_2{}'Y_2 + V'W + R_2\mu_2)'D_2{}^{-1}\cdot$$

$$(X_2{}'Y_2 + V'W + R_2\mu_2).$$

By using these results, (4.17) can be written as

$$g(W,\beta_1,\beta_2,P,m|Y,V) \propto |P|^{(n+\nu+\ell+2k-p-1)/2}\exp\{-\frac{1}{2}tr[\Sigma + F_1 + F_2{}^* +$$

$$(\beta_1-\beta_1(m))'D_1(\beta_1-\beta_1(m)) + (\beta_2-\beta_2(m))'D_2(\beta_2-\beta_2(m))]P\}, \quad (4.20)$$

and (4.20) can in turn be expressed as

$$g(W,\beta_1,\beta_2,P,m|Y,V) \propto |P|^{(n+\nu+\ell+2k-p-1)/2}\exp\{-\frac{1}{2}tr(A(m))P\}\cdot$$

$$\exp\{-\frac{1}{2}(\underset{\sim}{B}_1-\underset{\sim}{B}_1(m))'(P \otimes D_1)(\underset{\sim}{B}_1-\underset{\sim}{B}_1(m))\}\cdot$$

$$\exp\{-\frac{1}{2}(\underset{\sim}{B}_2-\underset{\sim}{B}_2(m))'(P \otimes D_2)(\underset{\sim}{B}_2-\underset{\sim}{B}_2(m))\}, \quad (4.21)$$

by writing the k×p matrices β_i and $\beta_i(m)$, i = 1, 2, as the pk×1 vectors $\underset{\sim}{B}_i$ and $\underset{\sim}{B}_i(m)$, respectively, by using the fact that

$$tr(\beta_i-\beta_i(m))'D_i(\beta_i-\beta_i(m))P = (\underset{\sim}{B}_i-\underset{\sim}{B}_i(m))'(P \otimes D_i)(\underset{\sim}{B}_i-\underset{\sim}{B}_i(m)),$$

for i = 1, 2, and by letting $A(m) = \Sigma + F_1 + F_2{}^*$.

Integration with respect to β_1 and β_2 is now possible so that

$$g(W,P,m|Y,V) \propto |P|^{(n+\nu+\ell-p-1)/2}|D_1D_2|^{-p/2}\exp\{-\frac{1}{2}tr(A(m))P\}. \quad (4.22)$$

Equation (4.22) may now be integrated with respect to P to obtain

$$g(W,m|Y,V) \propto |D_1D_2|^{-p/2}|A(m)|^{-(n+\nu+\ell)/2}, \quad (4.23)$$

but by completing the square on W and with some algebraic simplification, A(m) can be written as

$$A(m) = \Sigma + F_1 + F_2 + \{W-V\beta_2^*(m)\}'[I-V(X_2{}'X_2+V'V+R_2)^{-1}V']\{W-V\beta_2^*(m)\},$$

where

$$\beta_2^*(m) = (X_2'X_2 + R_2)^{-1}(X_2'Y_2 + R_2\mu_2), \text{ and}$$

$$F_2 = Y_2'Y_2 + \mu_2'R_2\mu_2 - (X_2'Y_2 + R_2\mu_2)'(X_2'X_2 + R_2)^{-1}(X_2'Y_2 + R_2\mu_2).$$

Therefore,

$$g(W,m|Y,V) \propto |D_1D_2|^{-p/2}|\Sigma + F_1 + F_2 + \{W-V\beta_2^*(m)\}'\cdot$$

$$[I-V(X_2'X_2+V'V+R_2)^{-1}V']\{W-V\beta_2^*(m)\}|^{-(n+\nu+\ell)/2}. \qquad (4.24)$$

However, since

$$g(W|Y,V) \propto \sum_{m=1}^{n-1} g(m|Y,V)\cdot g(W|m,Y,V),$$

it follows that the Bayesian predictive density is

$$g(W|Y,V) \propto \sum_{m=1}^{n-1} g(m|Y,V)\cdot \text{Matrix } T(R^*,Q^*,V\beta_2^*(m),n+\nu+\ell) \qquad (4.25)$$

where $g(m|Y,V)$ is the posterior distribution of the change point m when using a natural conjugate prior distribution,

$$Q^* = \Sigma + F_1 + F_2, \text{ and } R^* = [I-V(X_2'X_2 + V'V + R_2)^{-1}V'].$$

That is, the predictive density is a mixture of matrix T-distributions.

By expressing the $\ell\times p$ matrices W and $V\beta_2^*(m)$ as the $\ell p\times 1$ vectors $\underset{\sim}{W}$ and $\underset{\sim}{B}_2^*(m)$, respectively, and by using the properties of the matrix T-distribution,

$$E(W|m,Y,V) = V\beta_2^*(m) \text{ and}$$

$$\text{Cov}(\underset{\sim}{W}|m,Y,V) = (1/(n+\nu-p-1))(Q^* \otimes R^{*^{-1}}), \qquad (4.26)$$

while

$$E(W|Y,V) = \underset{m}{E}[E(W|m,Y,V)] = \sum_{m=1}^{n-1} g(m|Y,V)\cdot V\beta_2^*(m) \text{ and}$$

$$\text{Cov}(\underset{\sim}{W}|Y,V) = \underset{m}{E}[\text{Cov}(\underset{\sim}{W}|m,Y,V)] + \underset{m}{\text{Cov}}[E(\underset{\sim}{W}|m,Y,V)].$$

Numerical Study

When using a natural conjugate prior distribution, the Bayesian predictive density for ℓ future observations of the unknown dependent variables for a multivariate linear model with no shift can be shown to be

$$g(W|Y,V) \propto |D|^{-p/2}|S^* + \{W-V\hat{\beta}^*\}'[I - V(X'X + V'V + R)^{-1}V']\cdot$$
$$\{W-V\hat{\beta}^*\}|^{-(n+\nu+\ell)/2}, \qquad (4.27)$$

where

$$D = X'X + V'V + R,$$

$$S^* = \Sigma + Y'Y + \mu'R\mu - (X'Y + R\mu)'(X'X + R)^{-1}(X'Y + R\mu), \text{ and}$$

$$\hat{\beta}^* = (X'X + R)^{-1}(X'Y + R\mu).$$

That is,

$$g(W|Y,V) \propto \text{Matrix } T(T^*,S^*,V\hat{\beta}^*,n+\nu+\ell)$$

with

$$T^* = [I - V(X'X + V'V + R)^{-1}V'],$$

and from the properties of the matrix T-distribution,

$$E(W|Y,V) = V\hat{\beta}^* = V(X'X + R)^{-1}(X'Y + R\mu), \text{ and}$$

$$\text{Cov}(\underset{\sim}{W}|Y,V) = (1/(n+\nu-p-1))(S^* \otimes T^{*^{-1}}), \qquad (4.28)$$

where, as before, $\underset{\sim}{W}$ is the $\ell p\times 1$ vector created by stacking the columns of the $\ell\times p$ matrix W.

The numerical study in this section compares the two-step ahead forecasts generated by the predictive densities given in (4.25) and

(4.27) for different settings in the amount of the shift. Using a bivariate regression model with a sample of size 30 and a change point at m = 15, the parameter choices are $\sigma_1^2 = \sigma_2^2 = 1$, $\rho = 0$, $\nu = 2$,

$$\Sigma = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \qquad \beta_1 = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix},$$

$$\mu_1 = \mu_2 = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}, \qquad R_1 = R_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \text{ and}$$

$$\beta_2 = \beta_1 + \begin{pmatrix} \Delta & 0 \\ \Delta & 0 \end{pmatrix}, \quad \text{where } \Delta = 0, .03, .06, \text{ and } .09.$$

The 30×2 X matrix was constructed so that the first column consisted of all ones, while the second column was a two-digit number selected at random from a random number table, and the 2×2 V matrix was created in the same way. The bivariate normal error matrices, e and E, were generated with a Fortran program which made use of an IMSL subroutine.

The expected values, variances, and covariances of the future observations are presented in Table XLVI of the Appendix for the model which incorporates a change point, (4.25), and for the model which does not incorporate a change point, (4.27). When $\Delta = 0$, that is when there is actually no shift in the regression matrix, the variances associated with the predicted values when using (4.27) are smaller than they are for (4.25), but just the opposite is true for values of Δ greater than zero. In fact, as Δ increases from .03 to .06 to .09, the variances associated with the predicted values for the model incorporating a change tend to decrease, while those associated with the model not incorporating a change increase. Thus, the study indicates that when there is actually no shift in the regression matrix, prediction with the no change model is preferable in the sense that the variances for the predicted values are smaller; but, when there is a shift, then the model which does incorporate a change point has the smaller variances for the predicted values.

The same results hold when β_2 is changed to be $\beta_2 = \beta_1 + \begin{pmatrix} \Delta & \Delta \\ \Delta & \Delta \end{pmatrix}$, again for $\Delta = 0$, .03, .06, and .09. The expected values, variances, and covariances for this second case are presented in Table XLVII of the Appendix.

CHAPTER V

MULTIVARIATE LINEAR MODELS WITH A DOUBLE SHIFT OR A TEMPORARY SHIFT

Double Shift

Consider the multivariate linear model

$$Y = X\beta + e, \tag{5.1}$$

and suppose there is a shift in β from β_1 to β_2 at some point m_1, and another shift from β_2 to β_3 at some point m_2, where m_1 and m_2 are positive integers such that $1 \leq m_1 < m_2 \leq n-1$. Then the model can be written as

$$\begin{aligned} Y_1 &= X_1\beta_1 + e_1 \\ Y_2 &= X_2\beta_2 + e_2 \\ Y_3 &= X_3\beta_3 + e_3 \end{aligned} \tag{5.2}$$

where β_1, β_2, and β_3 are each k×p matrices of real unknown parameters ($\beta_1 \neq \beta_2 \neq \beta_3$),

$$Y = (Y_1' \vdots Y_2' \vdots Y_3')' = (\underset{\sim}{Y}_1,\cdots,\underset{\sim}{Y}_{m_1} \vdots \underset{\sim}{Y}_{m_1+1},\cdots,\underset{\sim}{Y}_{m_2} \vdots \underset{\sim}{Y}_{m_2+1},\cdots,\underset{\sim}{Y}_n)'$$

is an n×p matrix of observations,

$$X = (X_1' \vdots X_2' \vdots X_3')' = (\underset{\sim}{X}_1,\cdots,\underset{\sim}{X}_{m_1} \vdots \underset{\sim}{X}_{m_1+1},\cdots,\underset{\sim}{X}_{m_2} \vdots \underset{\sim}{X}_{m_2+1},\cdots,\underset{\sim}{X}_n)'$$

is an n×k design matrix, and

$$e = (e_1' \vdots e_2' \vdots e_3')' = (\underset{\sim}{e}_1,\cdots,\underset{\sim}{e}_{m_1} \vdots \underset{\sim}{e}_{m_1+1},\cdots,\underset{\sim}{e}_{m_2} \vdots \underset{\sim}{e}_{m_2+1},\cdots,\underset{\sim}{e}_n)'$$

is an n×p matrix of unobservable random variables, with the rows of e

($\underset{\sim}{e}_i'$, i = 1, ⋯, n) being independently and identically distributed as $N_p(\underset{\sim}{0},P)$ with P being a p×p positive definite symmetric precision matrix.

The likelihood function for this model can be written as

$$L(\beta_1,\beta_2,\beta_3,P,m_1,m_2) \propto |P|^{n/2} \exp\{-\frac{1}{2}tr[(Y_1-X_1\beta_1)'(Y_1-X_1\beta_1) +$$

$$(Y_2-X_2\beta_2)'(Y_2-X_2\beta_2) + (Y_3-X_3\beta_3)'(Y_3-X_3\beta_3)]P\}. \quad (5.3)$$

Let m_1 and m_2 be uniform discrete random variables whose prior distributions are

$$\Pi_o(m_1) = \begin{cases} 1/(n-2), & 1 \leqslant m_1 \leqslant n-2 \\ 0, & \text{otherwise} \end{cases}, \text{ and} \quad (5.4)$$

$$\Pi_o(m_2|m_1) = \begin{cases} 1/(n-m_1-1), & m_1+1 \leqslant m_2 \leqslant n-1 \\ 0, & \text{otherwise.} \end{cases} \quad (5.5)$$

Using a natural conjugate prior distribution, the joint distribution of the β_i's, i = 1, 2, 3, and P is defined as follows: the conditional distribution of the rows of β_i, namely, ($\underset{\sim}{\beta}_{ij}'$, i = 1, 2, 3, j = 1, ⋯, k) given P is multivariate normal with a mean vector $\underset{\sim}{\mu}_{ij}'$, and precision matrix $r_{ij}P$, $r_{ij} > 0$, such that $\underset{\sim}{\mu}_{ij}' \in R^p$, and the marginal distribution of P is a Wishart distribution with ν degrees of freedom and positive definite symmetric precision matrix Σ.

Assuming independence among the $\underset{\sim}{\beta}_{ij}$'s, it is possible to write the conditional distribution of β_i given P as

$$\Pi_o(\beta_i|P) \propto |P|^{k/2} \exp\{-\frac{1}{2}tr(\beta_i-\mu_i)'R_i(\beta_i-\mu_i)P\}, \quad (5.6)$$

where,

$\mu_i = (\underset{\sim}{\mu}_{i1},\cdots,\underset{\sim}{\mu}_{ik})'$ is a k×p matrix, and

R_i = Diagonal(r_{ij}) is a k×k matrix, i = 1, 2, 3, j = 1, ⋯, k.

Since the marginal prior distribution of P is a Wishart distribution with ν degrees of freedom and precision matrix Σ,

$$\Pi_0(P) \propto |P|^{(\nu-p-1)/2}\exp\{-\frac{1}{2}tr\Sigma P\}. \tag{5.7}$$

By combining the likelihood function (5.3) with the joint prior distribution, the joint posterior distribution of β_1, β_2, β_3, P, m_1, and m_2 is found to be

$$\Pi(\beta_1,\beta_2,\beta_3,P,m_1,m_2|Y) \propto (n-m_1-1)^{-1}|P|^{(n+\nu+3k-p-1)/2}\exp\{-\frac{1}{2}tr[\Sigma + \sum_{i=1}^{3}(\beta_i-\mu_i)'R_i(\beta_i-\mu_i) + \sum_{i=1}^{3}(Y_i-X_i\beta_i)'(Y_i-X_i\beta_i)]P\}. \tag{5.8}$$

To find the joint posterior distribution of the change points m_1 and m_2, (5.8) has to be integrated with respect to β_1, β_2, β_3 and P.

By completing the square on β_i for i = 1, 2, 3, (5.8) can be written as

$$\Pi(\beta_1,\beta_2,\beta_3,P,m_1,m_2|Y) \propto (n-m_1-1)^{-1}|P|^{(n+\nu+3k-p-1)/2}\exp\{-\frac{1}{2}tr(\Sigma + F_1 + F_2 + F_3)P\}\exp\{-\frac{1}{2}tr[\sum_{i=1}^{3}(\beta_i-\beta_i(m_1,m_2))'D_i(\beta_i-\beta_i(m_1,m_2))]P\}, \tag{5.9}$$

where for i = 1, 2, 3,

$$\beta_i(m_1,m_2) = (X_i'X_i + R_i)^{-1}(X_i'Y_i + R_i\mu_i),$$

$$D_i = X_i'X_i + R_i, \text{ and}$$

$$F_i = Y_i'Y_i + \mu_i'R_i\mu_i - (X_i'Y_i + R_i\mu_i)'D_i^{-1}(X_i'Y_i + R_i\mu_i).$$

Let $A(m_1,m_2) = \Sigma + F_1 + F_2 + F_3$, and define $\underset{\sim}{B}_i$ and $\underset{\sim}{B}_i(m_1,m_2)$ to be the pk×1 vectors created by stacking the columns of the k×p matrices β_i

and $\beta_i(m_1,m_2)$, respectively, for i = 1, 2, 3. Then since

$$\mathrm{tr}(\beta_i-\beta_i(m_1,m_2))'D_i(\beta_i-\beta_i(m_1,m_2))P =$$

$$(\underset{\sim}{B}_i-\underset{\sim}{B}_i(m_1,m_2))'(P \otimes D_i)(\underset{\sim}{B}_i-\underset{\sim}{B}_i(m_1,m_2)), \tag{5.10}$$

(5.9) becomes

$$\Pi(\beta_1,\beta_2,\beta_3,P,m_1,m_2|Y) \propto (n-m_1-1)^{-1}|P|^{(n+\nu+3k-p-1)/2}\exp\{-\frac{1}{2}\mathrm{tr}(A(m_1,m_2))P\}\cdot$$

$$\exp\{-\frac{1}{2}[\sum_{i=1}^{3}(\underset{\sim}{B}_i-\underset{\sim}{B}_i(m_1,m_2))'(P \otimes D_i)(\underset{\sim}{B}_i-\underset{\sim}{B}_i(m_1,m_2))\}. \tag{5.11}$$

Integrating with respect to β_1, β_2, and β_3 with the use of multivariate normal properties yields

$$\Pi(P,m_1,m_2|Y) \propto (n-m_1-1)^{-1}|P|^{(n+\nu+3k-p-1)/2}|P \otimes D_1|^{-1/2}|P \otimes D_2|^{-1/2}\cdot$$

$$|P \otimes D_3|^{-1/2}\exp\{-\frac{1}{2}\mathrm{tr}(A(m_1,m_2))P\}. \tag{5.12}$$

But $|P \otimes D_i|^{-1/2} = |P|^{-k/2}|D_i|^{-p/2}$, so that

$$\Pi(P,m_1,m_2|Y) \propto (n-m_1-1)^{-1}|P|^{(n+\nu-p-1)/2}|D_1D_2D_3|^{-p/2}\cdot$$

$$\exp\{-\frac{1}{2}\mathrm{tr}(A(m_1,m_2))P\}. \tag{5.13}$$

Finally, by integrating with respect to P, the joint posterior distribution of m_1 and m_2 is found to be

$$\Pi(m_1,m_2|Y) \propto \begin{cases} \dfrac{|D_1D_2D_3|^{-p/2}|A(m_1,m_2)|^{-(n+\nu)/2}}{(n-m_1-1)}, & 1 \leq m_1 < m_2 \leq n-1 \\ 0, \text{ otherwise} & \end{cases} \tag{5.14}$$

where $A(m_1,m_2) = \Sigma + F_1 + F_2 + F_3$, and for i = 1, 2, 3, D_i and F_i are defined in (5.9).

Numerical Study of the Double Shift

A bivariate regression model is used in this section to study the effect that parameter changes have on the joint posterior distribution of the change points m_1 and m_2 as given in (5.14). The same design and random error matrices are used as in the single shift numerical study of Chapter II, while parameter values considered are $\sigma_1^2 = \sigma_2^2 = 1$, $\rho = -.7$, $-.5$, $-.2$, 0, $.2$, $.5$, and $.7$, $\nu = 2$,

$$\Sigma = \begin{pmatrix} 2 & 2\rho \\ 2\rho & 2 \end{pmatrix}, \qquad \beta_1 = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix},$$

$$\mu_1 = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}, \qquad R_1 = R_2 = R_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\mu_2 = \mu_1 + \begin{pmatrix} \Delta_1 & 0 \\ \Delta_1 & 0 \end{pmatrix}, \qquad \mu_3 = \mu_2 + \begin{pmatrix} \Delta_1 & 0 \\ \Delta_1 & 0 \end{pmatrix},$$

$$\beta_2 = \beta_1 + \begin{pmatrix} \Delta_2 & 0 \\ \Delta_2 & 0 \end{pmatrix}, \text{ and } \beta_3 = \beta_2 + \begin{pmatrix} \Delta_2 & 0 \\ \Delta_2 & 0 \end{pmatrix},$$

where $\Delta_1 = 0$, $.2$, $.4$, and $.6$, and $\Delta_2 = .04$, $.05$, $.06$, and $.07$. Sample sizes of 10, 20, and 50 are considered, and the actual change points are constructed so that for a sample of size n, the first shift in the regression matrix occurs at $m_1 = n/2$ while the second shift occurs at $m_2 = 4n/5$.

The results of the study are presented in Tables XLVIII through L in the Appendix. For a sample size of only 10, the posterior probability associated with the actual change points of 5 and 8 is quite small for all of the choices of Δ_1 and Δ_2 with the exception of a few cases when ρ is $-.5$ or $-.7$, but for fixed values of ρ, Δ_1, and Δ_2 as the sample size increases, it is usually true that the posterior probability associated with the actual change points also increases. For fixed values of n, ρ, and Δ_2, changes in Δ_1 do not have too much of an effect on the posterior probability associated with the actual change points for the larger sample sizes of 20 and 50. This is also generally true for a sample of size 10; however, when $\rho = -.7$, increases in Δ_1 actually result in some fairly large decreases in the posterior probability. If n, ρ, and Δ_1 are fixed, increases in Δ_2 result in corresponding increases in the posterior probability. The only exceptions to this occur when n is 10 and ρ is .5 or .7 when there are a couple of instances where an increase in Δ_2 results in a small decrease in the posterior probability. Finally, for fixed values of n, Δ_1, and Δ_2, the posterior probability is most often smallest when ρ is zero and then increases as ρ becomes increasingly more positive or more negative.

As has been noted, the results for n = 10 are somewhat different from that of the larger sample sizes of twenty and fifty. This is most likely due to the fact that there just are not enough observations to do a good job of detecting the location of the shifts or to expect results consistent with larger samples.

Temporary Shift

Instead of a double shift in the β matrix of a multivariate linear model, suppose that there is a shift in β from β_1 to β_2 at some point m_1, and another shift from β_2 back to β_1 at some point m_2. That is, the shift occurring in the β matrix is only a temporary shift. As with the double shift situation, m_1 and m_2 are both positive integers such that $1 \leq m_1 < m_2 \leq n-1$. Then model (5.1) can be written as

$$Y_1 = X_1\beta_1 + e_1$$

$$Y_2 = X_2\beta_2 + e_2 \qquad (5.15)$$

$$Y_3 = X_3\beta_1 + e_3$$

with the same definitions and dimensions for the matrices as in the double shift case.

Thus, the likelihood function in the case of a temporary shift is seen to be

$$L(\beta_1,\beta_2,P,m_1,m_2) \propto |P|^{n/2} \exp\{-\frac{1}{2}\mathrm{tr}[(Y_1-X_1\beta_1)'(Y_1-X_1\beta_1) + (Y_2-X_2\beta_2)'(Y_2-X_2\beta_2) + (Y_3-X_3\beta_1)'(Y_3-X_3\beta_1)]P\}. \quad (5.16)$$

With the same prior distributions for β_1, β_2, P, m_1, and m_2 as in the double shift case and with the likelihood function given by (5.16), the joint posterior distribution for a multivariate linear model with a temporary shift is

$$\Pi(\beta_1,\beta_2,P,m_1,m_2|Y) \propto (n-m_1-1)^{-1} |P|^{(n+\nu+2k-p-1)/2} \exp\{-\frac{1}{2}\mathrm{tr}[\Sigma + \sum_{i=1}^{2}(\beta_i-\mu_i)'R_i(\beta_i-\mu_i) + \sum_{i=1}^{2}(Y_i-X_i\beta_i)'(Y_i-X_i\beta_i) + (Y_3-X_3\beta_1)'(Y_3-X_3\beta_1)]P\}. \quad (5.17)$$

As a first step in finding the joint posterior distribution of the change points m_1 and m_2, the square is completed on β_1 and β_2, and then (5.17) can be written as

$$\Pi(\beta_1,\beta_2,P,m_1,m_2|Y) \propto (n-m_1-1)^{-1} |P|^{(n+\nu+2k-p-1)/2} \exp\{-\frac{1}{2}\mathrm{tr}(\Sigma + F_1^* + F_2)P\}\exp\{-\frac{1}{2}\mathrm{tr}[(\beta_1-\beta_1^*(m_1,m_2))'D_1^*(\beta_1-\beta_1^*(m_1,m_2)) + (\beta_2-\beta_2(m_1,m_2))'D_2(\beta_2-\beta_2(m_1,m_2))]P\}, \quad (5.18)$$

where,

$$\beta_1^*(m_1,m_2) = (X_1'X_1 + X_3'X_3 + R_1)^{-1}(X_1'Y_1 + X_3'Y_3 + R_1\mu_1),$$

$$\overset{*}{D}_1 = X_1'X_1 + X_3'X_3 + R_1, \text{ and}$$

$$\overset{*}{F}_1 = Y_1'Y_1 + Y_3'Y_3 + \mu_1'R_1\mu_1 - (X_1'Y_1 + X_3'Y_3 + R_1\mu_1)'(\overset{*}{D}_1)^{-1}\cdot (X_1'Y_1 + X_3'Y_3 + R_1\mu_1),$$

while $\beta_2(m_1,m_2)$, D_2, and F_2 are given in (5.9). Now, let $\overset{*}{A}(m_1,m_2) = \Sigma + \overset{*}{F}_1 + F_2$, and define $\underset{\sim}{B}_1$, $\underset{\sim}{B}_2$, $\overset{*}{\underset{\sim}{B}}_1(m_1,m_2)$, and $\underset{\sim}{B}_2(m_1,m_2)$ to be pk×l vectors created by stacking the columns of the k×p matrices β_1, β_2, $\overset{*}{\beta}_1(m_1,m_2)$, and $\beta_2(m_1,m_2)$, respectively. Finally, by using (5.10) along with the identity that

$$\mathrm{tr}(\beta_1-\overset{*}{\beta}_1(m_1,m_2))'\overset{*}{D}_1(\beta_1-\overset{*}{\beta}_1(m_1,m_2))P = (\underset{\sim}{B}_1-\overset{*}{\underset{\sim}{B}}_1(m_1,m_2))'(P \otimes \overset{*}{D}_1)(\underset{\sim}{B}_1-\overset{*}{\underset{\sim}{B}}_1(m_1,m_2)),$$

(5.18) can be rewritten as

$$\Pi(\beta_1,\beta_2,P,m_1,m_2|Y) \propto (n-m_1-1)^{-1}|P|^{(n+\nu+2k-p-1)/2}\exp\{-\frac{1}{2}\mathrm{tr}(\overset{*}{A}(m_1,m_2))P\}\cdot \exp\{-\frac{1}{2}(\underset{\sim}{B}_1-\overset{*}{\underset{\sim}{B}}_1(m_1,m_2))'(P \otimes \overset{*}{D}_1)(\underset{\sim}{B}_1-\overset{*}{\underset{\sim}{B}}_1(m_1,m_2))\}\cdot \exp\{-\frac{1}{2}(\underset{\sim}{B}_2-\underset{\sim}{B}_2(m_1,m_2))'(P \otimes D_2)(\underset{\sim}{B}_2-\underset{\sim}{B}_2(m_1,m_2))\}. \quad (5.19)$$

Integrating (5.19) with respect to β_1 and β_2 yields

$$\Pi(P,m_1,m_2|Y) \propto (n-m_1-1)^{-1}|P|^{(n+\nu+2k-p-1)/2}|P \otimes \overset{*}{D}_1|^{-1/2}|P \otimes D_2|^{-1/2}\cdot \exp\{-\frac{1}{2}\mathrm{tr}(\overset{*}{A}(m_1,m_2))P\}. \quad (5.20)$$

However, since $|P \otimes \overset{*}{D}_1|^{-1/2} = |P|^{-k/2}|\overset{*}{D}_1|^{-p/2}$ and $|P \otimes D_2|^{-1/2} = |P|^{-k/2}\cdot |D_2|^{-p/2}$, (5.20) becomes

$$\Pi(P,m_1,m_2|Y) \propto (n-m_1-1)^{-1}|P|^{(n+\nu-p-1)/2}|D_1^*D_2|^{-p/2}\cdot$$

$$\exp\{-\frac{1}{2}tr(A^*(m_1,m_2))P\}. \quad (5.21)$$

By integrating (5.21) with respect to P, the joint posterior distribution of m_1 and m_2 in the case of a temporary shift is

$$\Pi(m_1,m_2|Y) \propto \begin{cases} \dfrac{|D_1^*D_2|^{-p/2}|A^*(m_1,m_2)|^{-(n+\nu)/2}}{(n-m_1-1)}, & 1 \leqslant m_1 < m_2 \leqslant n-1 \\ 0, \text{ otherwise} \end{cases} \quad (5.22)$$

where $A^*(m_1,m_2) = \Sigma + F_1^* + F_2$, D_1^* and F_1^* are given by (5.18), and D_2 and F_2 are given by (5.9).

Numerical Study of the Temporary Shift

This section uses a bivariate regression model to study the effect that parameter changes have on the joint posterior distribution of m_1 and m_2 as given by (5.22). The same design matrix, random error matrices, and parameter settings are used as was the case in the numerical study of the double shift. For sample sizes of 10, 20, and 50, the shift from β_1 to β_2 is constructed to be at $m_1 = n/2$, while the shift from β_2 back to β_1 is at $m_2 = 4n/5$.

Tables LI through LIII in the Appendix present the results of this temporary shift study. One observation that can be made in looking at these tables is that for fixed values of Δ_1, Δ_2, and ρ, by allowing the sample size to increase the corresponding posterior probability associated with the actual change points also increases in almost every case. Also, for fixed values of n, ρ, and Δ_2, there is not much effect on the posterior probability when Δ_1 changes, but for fixed values of n, ρ, and

Δ_1, increases in Δ_2 always result in increases in the posterior probability. As was the case with the double shift when n, Δ_1, and Δ_2 are fixed, the posterior probability is usually the smallest when $\rho = 0$ and then increases as ρ becomes increasingly more positive or more negative. Finally, for given values of n, ρ, Δ_1, and Δ_2 when comparing the results of the double shift with those of the temporary shift, it is seen that the posterior probability associated with the temporary shift is usually larger than the corresponding posterior probability associated with the double shift.

CHAPTER VI

SUMMARY

The research done in the previous chapters has concentrated on structural change in multivariate linear models from a Bayesian viewpoint, with single, double, and temporary shifts each having been analyzed.

For a single shift in the β matrix, say from β_1 to β_2, at some point m in the multivariate linear model, $Y = X\beta + e$, the marginal posterior distributions of the change point m, the regression matrices β_1 and β_2, and the precision matrix P were found when using a generalized natural conjugate prior distribution. Also for a single shift, a Bayesian test based on the posterior distribution of the change point m was developed to test the null hypothesis of no change against the alternative hypothesis of exactly one change, and the Bayesian predictive density was determined for use in predicting future observations of the unknown dependent variables.

Double and temporary shifts in the β matrix of a multivariate linear model were discussed in Chapter V. With a double shift there is a change in β from β_1 to β_2 at some point m_1, and another change from β_2 to β_3 at some point m_2, where m_1 and m_2 are positive integers such that $1 \leqslant m_1 < m_2 \leqslant n-1$ and $\beta_1 \neq \beta_2 \neq \beta_3$, while for a temporary shift there is a change from β_1 to β_2 at some point m_1 and then a change from β_2 back to β_1 at some point m_2. In each case the joint posterior distribution of the change points m_1 and m_2 was found using a natural conjugate prior distribution.

For each of the previously mentioned areas of study, numerical examples were given by making use of a bivariate regression model, and different sample sizes and parameter settings were considered so that the effect of these changes on the posterior distribution of the change point(s) could be observed. The computer programs for the numerical studies were all written using the Matrix procedure in SAS and were run on the IBM 3081D computer at Oklahoma State University.

Besides using the Bayesian test procedure for the numerical study presented in Chapter III, it was also tested on the univariate problem discussed by Chin Choy (1977), who made use of Quandt's 1958 data set. The results were identical to those of Chin Choy. That is, the formula for the posterior probability mass function of m as given in (3.11) for a multivariate linear model, simplifies to the formula given by Chin Choy in the univariate case.

Other numerical work should still be done using different generated data sets and/or different parameter choices, for, in this way, additional information and insight can be gained into the behavior of the posterior distribution of the change point(s) as well as those of the other model parameters. The application of these techniques to actual data also needs to be considered. In addition, one might try to make use of a transition function as proposed by Bacon and Watts (1971) to allow for the possibility of either an abrupt or a gradual change in the regression matrix at the point of change.

In sum, work still remains in the area of structural change, not only with multivariate linear models, but in many other areas as well, and additional work is being published all of the time, as evidenced by a recent special issue of the *Journal of Econometrics* edited by Broemeling (1982) on structural change in linear models.

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TABLE I

POSTERIOR PROBABILITY THAT m = 3 WHEN THE ACTUAL POINT OF CHANGE IS THREE, USING A NATURAL CONJUGATE PRIOR DISTRIBUTION

$$n = 10, \quad \mu_2 = \mu_1 + \begin{pmatrix} \Delta_1 & 0 \\ \Delta_1 & 0 \end{pmatrix}, \quad \beta_2 = \beta_1 + \begin{pmatrix} \Delta_2 & 0 \\ \Delta_2 & 0 \end{pmatrix}$$

Δ_1	Δ_2	ρ = -.7	-.5	-.2	0	.2	.5	.7
0.	.04	.76983	.26653	.07157*	.04838*	.04945*	.15009*	.64761
	.05	.94804	.55264	.15088*	.09831*	.10855*	.40319*	.92326
	.06	.98889	.80636	.29885	.19958*	.23568*	.72357	.98552
	.07	.99740	.93115	.50746	.36841	.44489	.90626	.99696
.2	.04	.81332	.30073	.07806*	.05075*	.04954*	.13784*	.59448
	.05	.96064	.59669	.16364*	.10312*	.10904*	.38016*	.90597
	.06	.99186	.83495	.32025	.20869*	.23719*	.70465	.98210
	.07	.99813	.94338	.53372	.38237	.44787	.89852	.99626
.4	.04	.83331	.32707	.08373*	.05262*	.04896*	.12207*	.51193
	.05	.96571	.62687	.17425*	.10663*	.10772*	.34613*	.87159
	.06	.99298	.85262	.33716	.21495*	.23461*	.67215	.97460
	.07	.99840	.95048	.55337	.39146	.44437	.88365	.99465
.6	.04	.83315	.34311	.08823*	.05386*	.04774*	.10476*	.41031*
	.05	.96509	.64269	.18205*	.10865*	.10468*	.30447*	.81314
	.06	.99278	.86085	.34874	.21801*	.22810*	.62603	.96016
	.07	.99835	.95356	.56598	.39533	.43453	.86014	.99140

*The largest probability occurs at the 9th data point.

TABLE II

POSTERIOR PROBABILITY THAT m = 5 WHEN THE ACTUAL POINT OF CHANGE IS FIVE, USING A NATURAL CONJUGATE PRIOR DISTRIBUTION

$$n = 10, \quad \mu_2 = \mu_1 + \begin{pmatrix} \Delta_1 & 0 \\ \Delta_1 & 0 \end{pmatrix}, \quad \beta_2 = \beta_1 + \begin{pmatrix} \Delta_2 & 0 \\ \Delta_2 & 0 \end{pmatrix}$$

Δ_1	Δ_2 \ ρ	-.7	-.5	-.2	0	.2	.5	.7
0.	.04	.99079	.78506	.10136*	.02855*	.02592*	.30076*	.95600
	.05	.99863	.94928	.32477*	.11768*	.14168*	.81928	.99622
	.06	.99975	.98808	.65289	.36758	.47698	.97032	.99953
	.07	.99995	.99695	.86771	.68764	.80290	.99443	.99992
.2	.04	.99255	.80812	.10841*	.02943*	.02558*	.27852*	.94566
	.05	.99892	.95605	.34183	.12135*	.14072*	.80471	.99535
	.06	.99981	.98985	.67084	.37680	.47688	.96778	.99942
	.07	.99996	.99744	.87734	.69733	.80404	.99397	.99991
.4	.04	.99302	.82122	.11379*	.02987*	.02471*	.24483*	.92266
	.05	.99900	.95971	.35371	.12280*	.13628*	.77633	.99325
	.06	.99982	.99079	.68272	.38046	.46858	.96221	.99916
	.07	.99996	.99769	.88358	.70164	.79973	.99291	.99986
.6	.04	.99238	.82526	.11712*	.02984*	.02341*	.20467*	.87905
	.05	.99890	.96072	.35988	.12193*	.12872*	.73212	.98890
	.06	.99981	.99104	.68853	.37838	.45228	.95259	.99859
	.07	.99996	.99776	.88666	.70056	.78975	.99102	.99976

*The largest probability occurs at the 9th data point.

TABLE III

POSTERIOR PROBABILITY THAT m = 7 WHEN THE ACTUAL POINT OF CHANGE IS SEVEN, USING A NATURAL CONJUGATE PRIOR DISTRIBUTION

$$n = 10, \quad \mu_2 = \mu_1 + \begin{pmatrix} \Delta_1 & 0 \\ \Delta_1 & 0 \end{pmatrix}, \quad \beta_2 = \beta_1 + \begin{pmatrix} \Delta_2 & 0 \\ \Delta_2 & 0 \end{pmatrix}$$

Δ_1	Δ_2	ρ = -.7	-.5	-.2	0	.2	.5	.7
0.	.04	.99809	.95834	.48605	.17159*	.07798*	.16692*	.83696
	.05	.99971	.99122	.75878	.37406*	.21400*	.59309	.98582
	.06	.99994	.99792	.91068	.64261	.48905	.90743	.99848
	.07	.99999	.99944	.96919	.84329	.77189	.98260	.99978
.2	.04	.99841	.96317	.50038	.17366*	.07613*	.15220*	.80180
	.05	.99976	.99233	.77000	.37842	.21029*	.56638	.98210
	.06	.99996	.99820	.91598	.64777	.48442	.89797	.99809
	.07	.99999	.99952	.97125	.84675	.76923	.98072	.99973
.4	.04	.99834	.96456	.50788	.17342*	.07317*	.13247*	.73240
	.05	.99975	.99264	.77541	.37801	.20294*	.52239	.97314
	.06	.99995	.99828	.91843	.64745	.47283	.87953	.99708
	.07	.99999	.99955	.97219	.84669	.76073	.97675	.99958
.6	.04	.99786	.96281	.50834	.17087*	.06932*	.11073*	.62462
	.05	.99968	.99224	.77513	.37283	.19244*	.46384	.95458
	.06	.99994	.99818	.91816	.64166	.45467	.84982	.99485
	.07	.99999	.99952	.97207	.84310	.74622	.96984	.99925

*The largest probability occurs at the 9th data point.

TABLE IV

POSTERIOR PROBABILITY THAT m = 3 WHEN THE ACTUAL POINT OF CHANGE IS THREE, USING A NATURAL CONJUGATE PRIOR DISTRIBUTION

$$n = 20, \quad \mu_2 = \mu_1 + \begin{pmatrix} \Delta_1 & 0 \\ \Delta_1 & 0 \end{pmatrix}, \quad \beta_2 = \beta_1 + \begin{pmatrix} \Delta_2 & 0 \\ \Delta_2 & 0 \end{pmatrix}$$

Δ_1	Δ_2 \ ρ	-.7	-.5	-.2	0	.2	.5	.7
	.04	.60709	.18253*	.07398*	.07374*	.10962*	.41512*	.91379
0.	.05	.91138	.52496	.24787*	.25171*	.37789*	.85267	.99335
	.06	.98507	.83394	.58358	.60240	.76023	.97817	.99941
	.07	.99764	.95520	.84746	.86660	.93980	.99669	.99993
	.04	.61913	.18900*	.07608*	.07516*	.11055*	.41126*	.90975
.2	.05	.91615	.53609	.25404*	.25632*	.38127*	.85168	.99311
	.06	.98612	.84069	.59226	.60917	.76384	.97820	.99939
	.07	.99784	.95760	.85253	.87036	.94125	.99672	.99993
	.04	.61606	.18916*	.07609*	.07451*	.10795*	.39160*	.89395
.4	.05	.91579	.53726	.25416*	.25460*	.37512*	.84118	.99180
	.06	.98610	.84217	.59311	.60765	.75967	.97654	.99928
	.07	.99783	.95827	.85358	.87016	.94027	.99650	.99992
	.04	.59747	.18299*	.07403*	.07184*	.10210*	.35773*	.86235
.6	.05	.91020	.52835	.24819*	.24666*	.35974*	.81991	.98892
	.06	.98500	.83838	.58607	.59775	.74746	.97285	.99903
	.07	.99763	.95728	.85060	.86594	.93673	.99595	.99989

*The largest probability occurs at the 19^{th} data point.

TABLE V

POSTERIOR PROBABILITY THAT m = 10 WHEN THE ACTUAL POINT OF CHANGE IS TEN, USING A NATURAL CONJUGATE PRIOR DISTRIBUTION

$$n = 20, \quad \mu_2 = \mu_1 + \begin{pmatrix} \Delta_1 & 0 \\ \Delta_1 & 0 \end{pmatrix}, \quad \beta_2 = \beta_1 + \begin{pmatrix} \Delta_2 & 0 \\ \Delta_2 & 0 \end{pmatrix}$$

Δ_1	Δ_2 \ ρ	−.7	−.5	−.2	0	.2	.5	.7
0.	.04	.53581	.20797	.08031*	.10368*	.23619*	.59613	.79931
	.05	.75976	.51787	.36097	.42065	.54068	.74536	.91593
	.06	.90374	.70831	.58655	.61133	.68529	.86086	.97162
	.07	.96846	.84290	.72901	.73916	.79954	.93511	.99162
.2	.04	.55664	.21213	.08487*	.11160*	.25076*	.61090	.81380
	.05	.78028	.53525	.37299	.43463	.55492	.75969	.92442
	.06	.91514	.72680	.60236	.62617	.69965	.87180	.97517
	.07	.97288	.85662	.74430	.75340	.81222	.94157	.99282
.4	.04	.57246	.21269	.08727*	.11653*	.25963*	.62255	.82337
	.05	.79476	.54896	.38127	.44509	.56662	.77093	.92921
	.06	.92213	.74148	.61559	.63885	.71182	.87971	.97690
	.07	.97529	.86678	.75696	.76529	.82247	.94590	.99334
.6	.04	.58229	.20952	.08738*	.11809*	.26232*	.63075	.82823
	.05	.80338	.55841	.38542	.45173	.57558	.77902	.93076
	.06	.92537	.75225	.62597	.64914	.72165	.88478	.97712
	.07	.97608	.87363	.76691	.77477	.83032	.94835	.99331

*The largest probability occurs at the 19^{th} data point.

TABLE VI

POSTERIOR PROBABILITY THAT m = 17 WHEN THE ACTUAL POINT OF CHANGE IS SEVENTEEN, USING A NATURAL CONJUGATE PRIOR DISTRIBUTION

$$n = 20, \quad \mu_2 = \mu_1 + \begin{pmatrix} \Delta_1 & 0 \\ \Delta_1 & 0 \end{pmatrix}, \quad \beta_2 = \beta_1 + \begin{pmatrix} \Delta_2 & 0 \\ \Delta_2 & 0 \end{pmatrix}$$

Δ_1	Δ_2	ρ: -.7	-.5	-.2	0	.2	.5	.7
0.	.04	.99903	.96745	.62674	.36332*	.25127*	.39714*	.90392
	.05	.99994	.99679	.92265	.79327	.72147	.91054	.99715
	.06	1.00000	.99966	.98862	.96538	.95687	.99377	.99990
	.07	1.00000	.99996	.99836	.99505	.99473	.99955	1.00000
.2	.04	.99912	.96993	.63778	.36921*	.25168*	.38634*	.89395
	.05	.99995	.99705	.92619	.79793	.72258	.90703	.99683
	.06	1.00000	.99969	.98921	.96644	.95724	.99352	.99989
	.07	1.00000	.99996	.99845	.99522	.99479	.99953	1.00000
.4	.04	.99910	.97069	.64235	.36969*	.24728*	.36419*	.87132
	.05	.99995	.99711	.92737	.79783	.71701	.89777	.99597
	.06	1.00000	.99969	.98937	.96635	.95597	.99276	.99986
	.07	1.00000	.99996	.99847	.99520	.99462	.99948	.99999
.6	.04	.99897	.96986	.64047	.36473*	.23832*	.33235*	.83168
	.05	.99994	.99699	.92627	.79294	.70465	.88170	.99426
	.06	1.00000	.99968	.98914	.96513	.95295	.99133	.99979
	.07	1.00000	.99996	.99843	.99500	.99420	.99936	.99999

*The largest probability occurs at the 19^{th} data point.

TABLE VII

POSTERIOR PROBABILITY THAT m = 3 WHEN THE ACTUAL POINT OF CHANGE IS THREE, USING A NATURAL CONJUGATE PRIOR DISTRIBUTION

$$n = 50, \quad \mu_2 = \mu_1 + \begin{pmatrix} \Delta_1 & 0 \\ \Delta_1 & 0 \end{pmatrix}, \quad \beta_2 = \beta_1 + \begin{pmatrix} \Delta_2 & 0 \\ \Delta_2 & 0 \end{pmatrix}$$

Δ_1	Δ_2 \ ρ	-.7	-.5	-.2	0	.2	.5	.7
0.	.04	.57855	.27883*	.20215*	.23371*	.33169*	.70591	.96284
	.05	.89972	.70037	.59700	.65472	.78027	.96179	.99757
	.06	.98251	.91941	.88444	.91424	.95749	.99532	.99982
	.07	.99760	.98153	.97118	.98062	.99179	.99938	.99999
.2	.04	.58287	.28274*	.20505*	.23647*	.33427*	.70645	.96263
	.05	.90154	.70436	.60159	.65867	.78282	.96214	.99759
	.06	.98296	.92091	.88637	.91568	.95819	.99540	.99982
	.07	.99768	.98197	.97176	.98101	.99196	.99939	.99999
.4	.04	.57895	.28021*	.20331*	.23398*	.32975*	.69808	.96022
	.05	.90134	.70359	.60026	.65677	.78048	.96115	.99748
	.06	.98297	.92117	.88648	.91556	.95798	.99533	.99981
	.07	.99768	.98209	.97191	.98108	.99196	.99939	.99999
.6	.04	.56639	.27134*	.19705*	.22642*	.31837*	.68043	.95515
	.05	.89905	.69794	.59299	.64893	.77313	.95870	.99723
	.06	.98255	.92019	.88473	.91386	.95683	.99510	.99979
	.07	.99761	.98189	.97164	.98082	.99179	.99936	.99998

*The largest probability occurs at the 49th data point.

TABLE VIII

POSTERIOR PROBABILITY THAT m = 25 WHEN THE ACTUAL POINT OF CHANGE IS TWENTY-FIVE, USING A NATURAL CONJUGATE PRIOR DISTRIBUTION

$$n = 50, \quad \mu_2 = \mu_1 + \begin{pmatrix} \Delta_1 & 0 \\ \Delta_1 & 0 \end{pmatrix}, \quad \beta_2 = \beta_1 + \begin{pmatrix} \Delta_2 & 0 \\ \Delta_2 & 0 \end{pmatrix}$$

Δ_1	Δ_2	ρ -.7	-.5	-.2	0	.2	.5	.7
0.	.04	.97704	.80324	.69420	.74136	.84978	.98514	.99943
	.05	.99946	.99046	.97769	.98604	.99451	.99961	.99999
	.06	.99999	.99970	.99906	.99940	.99978	.99999	1.00000
	.07	1.00000	.99999	.99996	.99997	.99999	1.00000	1.00000
.2	.04	.97767	.80840	.70032	.74606	.85218	.98531	.99943
	.05	.99948	.99083	.97849	.98650	.99468	.99962	.99999
	.06	.99999	.99972	.99911	.99942	.99978	.99999	1.00000
	.07	1.00000	.99999	.99996	.99997	.99999	1.00000	1.00000
.4	.04	.97766	.81159	.70414	.74845	.85283	.98510	.99941
	.05	.99947	.99098	.97888	.98669	.99471	.99962	.99999
	.06	.99999	.99972	.99912	.99943	.99979	.99999	1.00000
	.07	1.00000	.99999	.99996	.99997	.99999	1.00000	1.00000
.6	.04	.97699	.81283	.70565	.74856	.85173	.98452	.99935
	.05	.99944	.99090	.97889	.98661	.99462	.99960	.99999
	.06	.99999	.99971	.99911	.99942	.99978	.99999	1.00000
	.07	1.00000	.99999	.99996	.99997	.99999	1.00000	1.00000

TABLE IX

POSTERIOR PROBABILITY THAT m = 47 WHEN THE ACTUAL POINT OF CHANGE IS FORTY-SEVEN, USING A NATURAL CONJUGATE PRIOR DISTRIBUTION

$$n = 50, \quad \mu_2 = \mu_1 + \begin{pmatrix} \Delta_1 & 0 \\ \Delta_1 & 0 \end{pmatrix}, \quad \beta_2 = \beta_1 + \begin{pmatrix} \Delta_2 & 0 \\ \Delta_2 & 0 \end{pmatrix}$$

Δ_1	Δ_2	ρ = -.7	-.5	-.2	0	.2	.5	.7
0.	.04	.99893	.95475	.56625	.32841*	.22615*	.29467*	.78603
	.05	.99998	.99820	.95106	.86687	.80949	.92325	.99736
	.06	1.00000	.99994	.99746	.99203	.98940	.99818	.99998
	.07	1.00000	1.00000	.99989	.99966	.99962	.99997	1.00000
.2	.04	.99895	.95699	.57626	.33496*	.22913*	.29295*	.77926
	.05	.99998	.99828	.95304	.87061	.81265	.92293	.99726
	.06	1.00000	.99994	.99757	.99230	.98964	.99817	.99998
	.07	1.00000	1.00000	.99990	.99968	.99963	.99997	1.00000
.4	.04	.99891	.95785	.58110	.33719*	.22845*	.28493*	.76240
	.05	.99998	.99830	.95382	.87157	.81179	.91974	.99695
	.06	1.00000	.99994	.99760	.99235	.98956	.99808	.99998
	.07	1.00000	1.00000	.99990	.99968	.99963	.99997	1.00000
.6	.04	.99880	.95742	.58078	.33505*	.22415*	.27103*	.73428
	.05	.99997	.99825	.95345	.86979	.80689	.91342	.99636
	.06	1.00000	.99994	.99756	.99218	.98914	.99789	.99997
	.07	1.00000	1.00000	.99990	.99967	.99961	.99996	1.00000

*The largest probability occurs at the 49th data point.

TABLE X

POSTERIOR PROBABILITY THAT m = 3 WHEN THE ACTUAL POINT OF CHANGE IS THREE, USING A NATURAL CONJUGATE PRIOR DISTRIBUTION

$$n = 100, \quad \mu_2 = \mu_1 + \begin{pmatrix} \Delta_1 & 0 \\ \Delta_1 & 0 \end{pmatrix}, \quad \beta_2 = \beta_1 + \begin{pmatrix} \Delta_2 & 0 \\ \Delta_2 & 0 \end{pmatrix}$$

Δ_1	Δ_2	ρ: −.7	−.5	−.2	0	.2	.5	.7
0.	.04	.68624	.30352*	.15446*	.15752*	.22051*	.60388	.95985
	.05	.93983	.74189	.53405	.53770	.66068	.94119	.99758
	.06	.99312	.93507	.85769	.86980	.92614	.99282	.99984
	.07	.99949	.98778	.96294	.96838	.98483	.99905	.99999
.2	.04	.68631	.30242*	.15443*	.15792*	.22141*	.60505	.95982
	.05	.94034	.74263	.53488	.53905	.66227	.94153	.99759
	.06	.99321	.93560	.85851	.87063	.92671	.99288	.99984
	.07	.99950	.98793	.96328	.96866	.98498	.99906	.99999
.4	.04	.68136	.29447*	.15073*	.15481*	.21775*	.59892	.95847
	.05	.94010	.74044	.53032	.53488	.65882	.94062	.99754
	.06	.99316	.93559	.85793	.86994	.92629	.99283	.99984
	.07	.99950	.98794	.96333	.96868	.98498	.99905	.99999
.6	.04	.67038	.27964*	.14352*	.14835*	.20967*	.58525	.95561
	.05	.93906	.73499	.52017	.52505	.65016	.93835	.99743
	.06	.99299	.93501	.85588	.86764	.92484	.99266	.99983
	.07	.99948	.98781	.96311	.96844	.98483	.99903	.99999

*The largest probability occurs at the 99th data point.

TABLE XI

POSTERIOR PROBABILITY THAT m = 50 WHEN THE ACTUAL POINT OF CHANGE IS FIFTY, USING A NATURAL CONJUGATE PRIOR DISTRIBUTION

$$n = 100, \quad \mu_2 = \mu_1 + \begin{pmatrix} \Delta_1 & 0 \\ \Delta_1 & 0 \end{pmatrix}, \quad \beta_2 = \beta_1 + \begin{pmatrix} \Delta_2 & 0 \\ \Delta_2 & 0 \end{pmatrix}$$

Δ_1	Δ_2	ρ = -.7	-.5	-.2	0	.2	.5	.7
0.	.04	.88516	.84227	.76787	.69655	.62894	.60133	.75187
	.05	.95726	.92837	.89747	.86203	.82000	.82652	.94658
	.06	.98714	.97119	.95863	.94832	.93554	.95330	.99298
	.07	.99676	.98972	.98402	.98220	.98097	.99065	.99907
.2	.04	.88607	.84377	.76995	.69873	.63100	.60330	.75394
	.05	.95767	.92917	.89870	.86350	.82162	.82814	.94736
	.06	.98729	.97155	.95922	.94903	.93638	.95397	.99312
	.07	.99681	.98987	.98427	.98249	.98129	.99082	.99909
.4	.04	.88610	.84459	.77139	.70032	.63254	.60476	.75505
	.05	.95759	.92953	.89947	.86447	.82271	.82912	.94759
	.06	.98723	.97168	.95956	.94948	.93689	.95430	.99314
	.07	.99678	.98990	.98441	.98266	.98146	.99089	.99909
.6	.04	.88528	.84472	.77219	.70130	.63356	.60569	.75521
	.05	.95702	.92944	.89981	.86496	.82328	.82947	.94730
	.06	.98697	.97157	.95966	.94965	.93707	.95431	.99304
	.07	.99669	.98983	.98443	.98271	.98150	.99087	.99908

TABLE XII

POSTERIOR PROBABILITY THAT m = 97 WHEN THE ACTUAL POINT OF CHANGE IS NINETY-SEVEN, USING A NATURAL CONJUGATE PRIOR DISTRIBUTION

$$n = 100, \quad \mu_2 = \mu_1 + \begin{pmatrix} \Delta_1 & 0 \\ \Delta_1 & 0 \end{pmatrix}, \quad \beta_2 = \beta_1 + \begin{pmatrix} \Delta_2 & 0 \\ \Delta_2 & 0 \end{pmatrix}$$

Δ_1	Δ_2	ρ −.7	−.5	−.2	0	.2	.5	.7
0.	.04	.99999	.99861	.91035	.64100	.35310*	.23482*	.63712
	.05	1.00000	.99998	.99651	.97029	.89289	.87948	.99482
	.06	1.00000	1.00000	.99992	.99896	.99542	.99711	.99998
	.07	1.00000	1.00000	1.00000	.99998	.99988	.99996	1.00000
.2	.04	.99999	.99866	.91441	.64989	.35850*	.23342*	.62671
	.05	1.00000	.99998	.99667	.97141	.89531	.87891	.99459
	.06	1.00000	1.00000	.99992	.99900	.99554	.99710	.99998
	.07	1.00000	1.00000	1.00000	.99998	.99989	.99996	1.00000
.4	.04	.99999	.99866	.91678	.65496	.36036*	.22856*	.60825
	.05	1.00000	.99998	.99674	.97196	.89595	.87571	.99411
	.06	1.00000	1.00000	.99992	.99901	.99556	.99700	.99998
	.07	1.00000	1.00000	1.00000	.99998	.99989	.99996	1.00000
.6	.04	.99998	.99862	.91757	.65624	.35861*	.22036*	.58126
	.05	1.00000	.99998	.99674	.97196	.89483	.86972	.99330
	.06	1.00000	1.00000	.99992	.99901	.99548	.99681	.99997
	.07	1.00000	1.00000	1.00000	.99998	.99988	.99996	1.00000

*The largest probability occurs at the 99th data point.

TABLE XIII

POSTERIOR PROBABILITY THAT m = 3 WHEN THE ACTUAL POINT OF CHANGE IS THREE, USING A GENERALIZED NATURAL CONJUGATE PRIOR DISTRIBUTION

$$n = 10, \quad \mu_2 = \mu_1 + \begin{pmatrix} \Delta_1 & 0 \\ \Delta_1 & 0 \end{pmatrix}, \quad \beta_2 = \beta_1 + \begin{pmatrix} \Delta_2 & 0 \\ \Delta_2 & 0 \end{pmatrix}$$

Δ_1	Δ_2	ρ: -.7	-.5	-.2	0	.2	.5	.7
0.	.04	.76555	.36135	.19176*	.18258*	.22292*	.48282	.88196
	.05	.93876	.60465	.30397*	.28258*	.35953	.74550	.97784
	.06	.98490	.81430	.46132	.42251	.53677	.90526	.99550
	.07	.99577	.92601	.63391	.58093	.70765	.96763	.99884
.2	.04	.78971	.38178	.19824*	.18535*	.22241*	.47326	.87565
	.05	.94745	.62830	.31504	.28870*	.36199	.74209	.97748
	.06	.98749	.83106	.47716	.43298	.54315	.90572	.99563
	.07	.99659	.93455	.65150	.59463	.71617	.96856	.99891
.4	.04	.80912	.39997	.20369*	.18703*	.22032*	.45950	.86530
	.05	.95416	.64889	.32480	.29344*	.36237	.73533	.97638
	.06	.98943	.84517	.49139	.44187	.54736	.90473	.99563
	.07	.99720	.94156	.66722	.60664	.72287	.96897	.99894
.6	.04	.82465	.41591	.20804*	.18755*	.21658*	.44131	.84983
	.05	.95937	.66662	.33314	.29669*	.36054	.72481	.97436
	.06	.99091	.85701	.50391	.44904	.54926	.90214	.99548
	.07	.99765	.94731	.68109	.61690	.72769	.96886	.99895

*The largest probability occurs at the 8^{th} data point.

TABLE XIV

POSTERIOR PROBABILITY THAT m = 5 WHEN THE ACTUAL POINT OF CHANGE IS FIVE, USING A GENERALIZED NATURAL CONJUGATE PRIOR DISTRIBUTION

$$n = 10, \quad \mu_2 = \mu_1 + \begin{pmatrix} \Delta_1 & 0 \\ \Delta_1 & 0 \end{pmatrix}, \quad \beta_2 = \beta_1 + \begin{pmatrix} \Delta_2 & 0 \\ \Delta_2 & 0 \end{pmatrix}$$

Δ_1	Δ_2	ρ = -.7	-.5	-.2	0	.2	.5	.7
0.	.04	.97529	.76280	.19804*	.08543*	.08781*	.48062	.94220
	.05	.99412	.91355	.39385	.19558*	.23295*	.80817	.98919
	.06	.99843	.96991	.62934	.39552	.48341	.94162	.99764
	.07	.99953	.98925	.81067	.62927	.72697	.98213	.99940
.2	.04	.97751	.77227	.20066*	.08498*	.08616*	.47386	.94101
	.05	.99479	.91892	.40158	.19773*	.23325*	.80780	.98911
	.06	.99864	.97234	.64005	.40254	.48807	.94229	.99764
	.07	.99960	.99029	.81950	.63902	.73318	.98250	.99940
.4	.04	.97921	.77998	.20211*	.08397*	.08391*	.46409	.93867
	.05	.99529	.92331	.40759	.19871*	.23215*	.80574	.98880
	.06	.99879	.97431	.64894	.40785	.49085	.94236	.99757
	.07	.99965	.99112	.82692	.64700	.73785	.98265	.99938
.6	.04	.98046	.78607	.20236*	.08238*	.08107*	.45106	.93491
	.05	.99566	.92686	.41181	.19848*	.22960*	.80179	.98823
	.06	.99890	.97590	.65606	.41138	.49169	.94179	.99744
	.07	.99969	.99177	.83303	.65324	.74099	.98258	.99935

*The largest probability occurs at the 8^{th} data point.

TABLE XV

POSTERIOR PROBABILITY THAT m = 7 WHEN THE ACTUAL POINT OF CHANGE IS SEVEN, USING A GENERALIZED NATURAL CONJUGATE PRIOR DISTRIBUTION

$$n = 10, \quad \mu_2 = \mu_1 + \begin{pmatrix} \Delta_1 & 0 \\ \Delta_1 & 0 \end{pmatrix}, \quad \beta_2 = \beta_1 + \begin{pmatrix} \Delta_2 & 0 \\ \Delta_2 & 0 \end{pmatrix}$$

Δ_1	Δ_2	ρ: -.7	-.5	-.2	0	.2	.5	.7
0.	.04	.99536	.94608	.58236	.30662*	.19768*	.39090	.89607
	.05	.99898	.98392	.77246	.48693	.36660*	.74032	.98401
	.06	.99973	.99484	.89149	.68100	.59499	.92331	.99709
	.07	.99992	.99819	.95099	.82934	.79151	.97866	.99935
.2	.04	.99556	.94727	.57972	.29962*	.18974*	.37452*	.88698
	.05	.99903	.98446	.77270	.48183	.35802*	.72958	.98262
	.06	.99975	.99506	.89265	.67904	.58891	.92010	.99686
	.07	.99992	.99829	.95197	.82946	.78901	.97789	.99930
.4	.04	.99560	.94750	.57449	.29094*	.18081*	.35539*	.87433
	.05	.99905	.98469	.77091	.47434	.34738*	.71545	.98054
	.06	.99976	.99518	.89263	.67480	.58026	.91547	.99649
	.07	.99993	.99834	.95235	.82801	.78458	.97668	.99923
.6	.04	.99549	.94682	.56674	.28073*	.17105*	.33374*	.85729
	.05	.99904	.98462	.76709	.46453	.33482*	.69771	.97759
	.06	.99976	.99520	.89145	.66827	.56904	.90922	.99596
	.07	.99993	.99836	.95215	.82499	.77818	.97497	.99911

*The largest probability occurs at the 8th data point.

TABLE XVI

POSTERIOR PROBABILITY THAT m = 3 WHEN THE ACTUAL POINT OF CHANGE IS THREE, USING A GENERALIZED NATURAL CONJUGATE PRIOR DISTRIBUTION

$$n = 20, \quad \mu_2 = \mu_1 + \begin{pmatrix} \Delta_1 & 0 \\ \Delta_1 & 0 \end{pmatrix}, \quad \beta_2 = \beta_1 + \begin{pmatrix} \Delta_2 & 0 \\ \Delta_2 & 0 \end{pmatrix}$$

Δ_1	Δ_2 \ ρ	-.7	-.5	-.2	0	.2	.5	.7
0.	.04	.67780	.34443	.22370*	.24563*	.34115	.70538	.95523
	.05	.90596	.62227	.43824	.47937	.62881	.92245	.99423
	.06	.97899	.83965	.68423	.72830	.84889	.98252	.99918
	.07	.99569	.94443	.85678	.88505	.94759	.99595	.99987
.2	.04	.68176	.34420	.22018*	.24030*	.33343	.69789	.95407
	.05	.90880	.62553	.43714	.47634	.62510	.92157	.99425
	.06	.97996	.84318	.68642	.72906	.84911	.98266	.99920
	.07	.99595	.94635	.85949	.88680	.94841	.99605	.99987
.4	.04	.68346	.34083	.21414*	.23226*	.32223	.68632	.95188
	.05	.91099	.62654	.43291	.46992	.61801	.91954	.99415
	.06	.98076	.84578	.68650	.72765	.84784	.98259	.99921
	.07	.99616	.94794	.86127	.88768	.94876	.99611	.99987
.6	.04	.68267	.33415	.20555*	.22152*	.30747*	.66998	.94833
	.05	.91253	.62509	.42531	.45984	.60717	.91613	.99392
	.06	.98140	.84741	.68426	.72385	.84490	.98227	.99920
	.07	.99633	.94921	.86208	.88763	.94863	.99612	.99988

*The largest probability occurs at the 18th data point.

TABLE XVII

POSTERIOR PROBABILITY THAT m = 10 WHEN THE ACTUAL POINT OF CHANGE IS TEN, USING A GENERALIZED NATURAL CONJUGATE PRIOR DISTRIBUTION

$$n = 20, \quad \mu_2 = \mu_1 + \begin{pmatrix} \Delta_1 & 0 \\ \Delta_1 & 0 \end{pmatrix}, \quad \beta_2 = \beta_1 + \begin{pmatrix} \Delta_2 & 0 \\ \Delta_2 & 0 \end{pmatrix}$$

Δ_1	Δ_2 \ ρ	−.7	−.5	−.2	0	.2	.5	.7
0.	.04	.48254	.19606	.11050#	.16519*	.30963	.56668	.75540
	.05	.69829	.46291	.32251	.38674	.50578	.70266	.88006
	.06	.85359	.64954	.53304	.56394	.64225	.81900	.95120
	.07	.94072	.78727	.67667	.69156	.75483	.90330	.98245
.2	.04	.49891	.19901	.11265#	.16953	.31740	.57764	.76792
	.05	.71757	.47657	.33084	.39651	.51659	.71426	.88888
	.06	.86744	.66617	.54618	.57627	.65409	.82890	.95582
	.07	.94781	.80165	.69052	.70414	.76601	.91011	.98443
.4	.04	.51500	.20102	.11410#	.17295	.32429	.58837	.77982
	.05	.73595	.48975	.33840	.40567	.52706	.72546	.89697
	.06	.87994	.68221	.55887	.58824	.66560	.83822	.95989
	.07	.95393	.81507	.70383	.71624	.77669	.91637	.98613
.6	.04	.53067	.20196	.11478#	.17534	.33017	.59884	.79114
	.05	.75341	.50236	.34510	.41416	.53719	.73628	.90437
	.06	.89119	.69763	.57109	.59983	.67678	.84700	.96350
	.07	.95919	.82757	.71660	.72787	.78689	.92211	.98759

*The largest probability occurs at the 9th data point.

#The largest probability occurs at the 14th data point.

TABLE XVIII

POSTERIOR PROBABILITY THAT m = 17 WHEN THE ACTUAL POINT OF CHANGE IS SEVENTEEN, USING A GENERALIZED NATURAL CONJUGATE PRIOR DISTRIBUTION

$$n = 20, \quad \mu_2 = \mu_1 + \begin{pmatrix} \Delta_1 & 0 \\ \Delta_1 & 0 \end{pmatrix}, \quad \beta_2 = \beta_1 + \begin{pmatrix} \Delta_2 & 0 \\ \Delta_2 & 0 \end{pmatrix}$$

Δ_1	Δ_2	ρ: -.7	-.5	-.2	0	.2	.5	.7
0.	.04	.99875	.97863	.83055	.69118	.61692	.75656	.96195
	.05	.99988	.99689	.96009	.91253	.89370	.96586	.99787
	.06	.99999	.99952	.99177	.98094	.97927	.99617	.99986
	.07	1.00000	.99992	.99831	.99608	.99625	.99955	.99999
.2	.04	.99876	.97879	.82806	.68379	.60523	.74315	.95818
	.05	.99989	.99694	.95980	.91069	.89017	.96385	.99768
	.06	.99999	.99953	.99179	.98071	.97875	.99598	.99985
	.07	1.00000	.99992	.99833	.99607	.99619	.99953	.99999
.4	.04	.99874	.97868	.82434	.67474	.59176	.72777	.95362
	.05	.99988	.99693	.95914	.90812	.88574	.96140	.99743
	.06	.99999	.99953	.99171	.98029	.97802	.99573	.99984
	.07	1.00000	.99992	.99833	.99602	.99609	.99950	.99999
.6	.04	.99869	.97829	.81929	.66383	.57626	.71000	.94801
	.05	.99988	.99688	.95808	.90473	.88025	.95840	.99711
	.06	.99999	.99953	.99155	.97967	.97705	.99542	.99982
	.07	1.00000	.99992	.99830	.99592	.99595	.99947	.99999

TABLE XIX

POSTERIOR PROBABILITY THAT m = 3 WHEN THE ACTUAL POINT OF CHANGE IS THREE, USING A GENERALIZED NATURAL CONJUGATE PRIOR DISTRIBUTION

$$n = 50, \quad \mu_2 = \mu_1 + \begin{pmatrix} \Delta_1 & 0 \\ \Delta_1 & 0 \end{pmatrix}, \quad \beta_2 = \beta_1 + \begin{pmatrix} \Delta_2 & 0 \\ \Delta_2 & 0 \end{pmatrix}$$

Δ_1	Δ_2	ρ: -.7	-.5	-.2	0	.2	.5	.7
0.	.04	.65479	.43136	.37325	.42074	.52941	.81036	.96922
	.05	.90100	.75145	.70287	.75556	.84439	.96681	.99720
	.06	.98014	.91953	.89616	.92367	.96012	.99466	.99975
	.07	.99694	.97905	.96899	.97889	.99059	.99919	.99998
.2	.04	.65491	.42797	.36768	.41408	.52210	.80606	.96869
	.05	.90206	.75170	.70123	.75358	.84291	.96662	.99722
	.06	.98049	.92025	.89651	.92381	.96020	.99469	.99976
	.07	.99701	.97934	.96928	.97908	.99067	.99920	.99998
.4	.04	.65309	.42183	.35967	.40498	.51231	.80016	.96789
	.05	.90290	.75101	.69822	.75040	.84061	.96626	.99722
	.06	.98080	.92084	.89660	.92375	.96018	.99472	.99976
	.07	.99707	.97961	.96954	.97924	.99075	.99921	.99998
.6	.04	.64894	.41261	.34905	.39327	.49976	.79233	.96674
	.05	.90348	.74922	.69365	.74581	.83735	.96570	.99721
	.06	.98109	.92127	.89639	.92345	.96004	.99473	.99976
	.07	.99713	.97986	.96976	.97936	.99081	.99922	.99998

TABLE XX

POSTERIOR PROBABILITY THAT m = 25 WHEN THE ACTUAL POINT OF CHANGE IS TWENTY-FIVE USING A GENERALIZED NATURAL CONJUGATE PRIOR DISTRIBUTION

$$n = 50, \quad \mu_2 = \mu_1 + \begin{pmatrix} \Delta_1 & 0 \\ \Delta_1 & 0 \end{pmatrix}, \quad \beta_2 = \beta_1 + \begin{pmatrix} \Delta_2 & 0 \\ \Delta_2 & 0 \end{pmatrix}$$

Δ_1	Δ_2	ρ = -.7	-.5	-.2	0	.2	.5	.7
0.	.04	.96882	.77674	.65521	.69815	.81462	.97798	.99897
	.05	.99908	.98630	.96902	.97938	.99133	.99929	.99998
	.06	.99998	.99947	.99841	.99893	.99958	.99998	1.00000
	.07	1.00000	.99998	.99992	.99994	.99998	1.00000	1.00000
.2	.04	.96912	.78026	.65919	.70073	.81538	.97784	.99896
	.05	.99908	.98651	.96949	.97961	.99138	.99929	.99998
	.06	.99997	.99948	.99843	.99894	.99958	.99998	1.00000
	.07	1.00000	.99998	.99992	.99994	.99998	1.00000	1.00000
.4	.04	.96934	.78355	.66302	.70324	.81610	.97767	.99894
	.05	.99907	.98669	.96991	.97983	.99142	.99929	.99998
	.06	.99997	.99948	.99845	.99895	.99958	.99998	1.00000
	.07	1.00000	.99998	.99992	.99994	.99998	1.00000	1.00000
.6	.04	.96947	.78663	.66671	.70567	.81678	.97749	.99892
	.05	.99906	.98684	.97029	.98001	.99146	.99928	.99998
	.06	.99997	.99948	.99846	.99896	.99958	.99998	1.00000
	.07	1.00000	.99998	.99992	.99994	.99998	1.00000	1.00000

TABLE XXI

POSTERIOR PROBABILITY THAT m = 47 WHEN THE ACTUAL POINT OF CHANGE IS FORTY-SEVEN USING A GENERALIZED NATURAL CONJUGATE PRIOR DISTRIBUTION

$$n = 50, \quad \mu_2 = \mu_1 + \begin{pmatrix} \Delta_1 & 0 \\ \Delta_1 & 0 \end{pmatrix}, \quad \beta_2 = \beta_1 + \begin{pmatrix} \Delta_2 & 0 \\ \Delta_2 & 0 \end{pmatrix}$$

Δ_1	Δ_2	ρ = -.7	-.5	-.2	0	.2	.5	.7
0.	.04	.99921	.98510	.88150	.77627	.70472	.78297	.96282
	.05	.99998	.99919	.98934	.97607	.96894	.98880	.99948
	.06	1.00000	.99996	.99929	.99837	.99816	.99966	.99999
	.07	1.00000	1.00000	.99996	.99991	.99991	.99999	1.00000
.2	.04	.99916	.98473	.87808	.76866	.69316	.76998	.95927
	.05	.99997	.99916	.98902	.97515	.96742	.98802	.99943
	.06	1.00000	.99996	.99927	.99831	.99808	.99964	.99999
	.07	1.00000	1.00000	.99996	.99990	.99991	.99999	1.00000
.4	.04	.99909	.98424	.87423	.76053	.68118	.75665	.95552
	.05	.99997	.99912	.98863	.97409	.96573	.98717	.99937
	.06	1.00000	.99996	.99924	.99824	.99798	.99961	.99999
	.07	1.00000	1.00000	.99996	.99990	.99990	.99999	1.00000
.6	.04	.99900	.98360	.86980	.75164	.66847	.74258	.95140
	.05	.99996	.99907	.98815	.97286	.96382	.98621	.99931
	.06	1.00000	.99995	.99921	.99816	.99787	.99959	.99999
	.07	1.00000	1.00000	.99995	.99989	.99990	.99999	1.00000

TABLE XXII

POSTERIOR PROBABILITY THAT m = 3 WHEN THE ACTUAL POINT OF CHANGE IS THREE, USING A GENERALIZED NATURAL CONJUGATE PRIOR DISTRIBUTION

$$n = 100, \quad \mu_2 = \mu_1 + \begin{pmatrix} \Delta_1 & 0 \\ \Delta_1 & 0 \end{pmatrix}, \quad \beta_2 = \beta_1 + \begin{pmatrix} \Delta_2 & 0 \\ \Delta_2 & 0 \end{pmatrix}$$

Δ_1	Δ_2	ρ: −.7	−.5	−.2	0	.2	.5	.7
0.	.04	.71913	.45045	.31904	.33331	.42092	.74901	.96762
	.05	.93778	.77029	.65650	.68225	.77783	.95468	.99748
	.06	.99233	.93365	.87280	.89150	.93847	.99286	.99982
	.07	.99939	.98665	.96193	.96827	.98464	.99896	.99999
.2	.04	.71892	.44550	.31267	.32707	.41496	.74595	.96733
	.05	.93819	.77023	.65403	.67945	.77590	.95448	.99749
	.06	.99240	.93400	.87293	.89145	.93847	.99289	.99982
	.07	.99940	.98675	.96213	.96843	.98472	.99896	.99999
.4	.04	.71791	.43786	.30365	.31817	.40619	.74112	.96689
	.05	.93857	.76969	.65004	.67501	.77274	.95411	.99749
	.06	.99248	.93432	.87284	.89115	.93833	.99290	.99982
	.07	.99940	.98685	.96231	.96856	.98479	.99897	.99999
.6	.04	.71577	.42699	.29172	.30638	.39425	.73411	.96625
	.05	.93893	.76851	.64419	.66859	.76808	.95352	.99749
	.06	.99255	.93461	.87249	.89053	.93800	.99291	.99983
	.07	.99941	.98695	.96248	.96866	.98484	.99898	.99999

TABLE XXIII

POSTERIOR PROBABILITY THAT m = 50 WHEN THE ACTUAL POINT OF CHANGE IS FIFTY, USING A GENERALIZED NATURAL CONJUGATE PRIOR DISTRIBUTION

$$n = 100, \quad \mu_2 = \mu_1 + \begin{pmatrix} \Delta_1 & 0 \\ \Delta_1 & 0 \end{pmatrix}, \quad \beta_2 = \beta_1 + \begin{pmatrix} \Delta_2 & 0 \\ \Delta_2 & 0 \end{pmatrix}$$

Δ_1	Δ_2	ρ = -.7	-.5	-.2	0	.2	.5	.7
0.	.04	.87723	.83235	.75404	.68274	.61762	.59361	.74136
	.05	.95239	.92165	.88781	.85047	.80809	.81579	.93988
	.06	.98499	.96738	.95327	.94162	.92786	.94718	.99138
	.07	.99602	.98791	.98139	.97913	.97761	.98868	.99878
.2	.04	.87803	.83372	.75590	.68466	.61945	.59536	.74328
	.05	.95273	.92236	.88891	.85176	.80950	.81721	.94065
	.06	.98510	.96770	.95380	.94226	.92859	.94778	.99153
	.07	.99605	.98803	.98162	.97939	.97789	.98884	.99880
.4	.04	.87878	.83504	.75772	.68655	.62124	.59708	.74516
	.05	.95305	.92305	.88998	.85302	.81087	.81859	.94139
	.06	.98521	.96800	.95431	.94288	.92930	.94836	.99167
	.07	.99608	.98814	.98183	.97965	.97816	.98900	.99883
.6	.04	.87948	.83632	.75951	.68841	.62300	.59878	.74698
	.05	.95334	.92371	.89102	.85425	.81221	.81993	.94210
	.06	.98530	.96828	.95480	.94348	.92999	.94892	.99180
	.07	.99610	.98825	.98204	.97989	.97842	.98914	.99885

TABLE XXIV

POSTERIOR PROBABILITY THAT m = 97, WHEN THE ACTUAL POINT OF CHANGE IS NINETY-SEVEN USING A GENERALIZED NATURAL CONJUGATE PRIOR DISTRIBUTION

$$n = 100, \quad \mu_2 = \mu_1 + \begin{pmatrix} \Delta_1 & 0 \\ \Delta_1 & 0 \end{pmatrix}, \quad \beta_2 = \beta_1 + \begin{pmatrix} \Delta_2 & 0 \\ \Delta_2 & 0 \end{pmatrix}$$

Δ_1	Δ_2	ρ -.7	-.5	-.2	0	.2	.5	.7
0.	.04	.99999	.99931	.97917	.91543	.79466	.69727	.91754
	.05	1.00000	.99998	.99906	.99465	.98374	.98291	.99912
	.06	1.00000	1.00000	.99997	.99978	.99930	.99960	1.00000
	.07	1.00000	1.00000	1.00000	.99999	.99998	.99999	1.00000
.2	.04	.99999	.99927	.97870	.91355	.78962	.68726	.91230
	.05	1.00000	.99998	.99903	.99448	.98318	.98207	.99905
	.06	1.00000	1.00000	.99997	.99977	.99927	.99958	1.00000
	.07	1.00000	1.00000	1.00000	.99999	.99998	.99999	1.00000
.4	.04	.99998	.99921	.97809	.91141	.78456	.67853	.90798
	.05	1.00000	.99998	.99898	.99428	.98257	.98128	.99899
	.06	1.00000	1.00000	.99996	.99975	.99924	.99956	.99999
	.07	1.00000	1.00000	1.00000	.99999	.99998	.99999	1.00000
.6	.04	.99998	.99915	.97731	.90884	.77910	.67051	.90433
	.05	1.00000	.99998	.99892	.99403	.98186	.98047	.99893
	.06	1.00000	1.00000	.99996	.99974	.99920	.99953	.99999
	.07	1.00000	1.00000	1.00000	.99999	.99997	.99999	1.00000

TABLE XXV

EXPECTED VALUES, VARIANCES, AND COVARIANCES OF THE MODEL PARAMETERS USING A NATURAL CONJUGATE PRIOR DISTRIBUTION

$$n = 20, \quad \rho = -.5, \quad m = 10, \quad \beta_2 = \beta_1 + \begin{pmatrix} \Delta & 0 \\ \Delta & 0 \end{pmatrix}$$

Δ	E(m|Y)	Var(m|Y)	$E(\underset{\sim}{\beta}_1\|Y)$	$Cov(\underset{\sim}{\beta}_1\|Y)$				$E(\underset{\sim}{\beta}_2\|Y)$	$Cov(\underset{\sim}{\beta}_2\|Y)$			
			.9956	3.3E-1	-5.1E-3	-1.4E-1	2.1E-3	1.5155	4.1E-1	-1.2E-2	-1.5E-1	4.1E-3
.04	11.143	12.7235	2.0095	-5.1E-3	1.4E-4	1.8E-3	-4.2E-5	2.0299	-1.2E-2	1.6E-3	4.0E-3	-4.8E-4
			3.1574	-1.4E-1	1.8E-3	2.1E-1	-3.3E-3	2.8444	-1.5E-1	4.0E-3	2.5E-1	-5.5E-3
			3.9979	2.1E-3	-4.2E-5	-3.3E-3	7.8E-5	4.0078	4.1E-3	-4.8E-4	-5.5E-3	4.9E-4
			.9383	3.1E-1	-4.9E-3	-1.3E-1	2.0E-3	1.5551	2.9E-1	-4.2E-3	-1.2E-1	1.7E-3
.05	9.493	2.0420	2.0075	-4.9E-3	1.2E-4	2.0E-3	-4.6E-5	2.0323	-4.2E-3	1.5E-4	1.7E-3	-5.0E-5
			3.2576	-1.3E-1	2.0E-3	2.1E-1	-3.3E-3	2.7360	-1.2E-1	1.7E-3	1.8E-1	-2.6E-3
			3.9964	2.0E-3	-4.6E-5	-3.3E-3	7.7E-5	4.0099	1.7E-3	-5.0E-5	-2.6E-3	6.9E-5
			.9396	2.9E-1	-4.7E-3	-1.2E-1	2.0E-3	1.5911	2.8E-1	-3.8E-3	-1.1E-1	1.6E-3
.06	9.602	.5519	2.0071	-4.7E-3	1.1E-4	2.0E-3	-4.6E-5	2.0418	-3.8E-3	7.8E-5	1.6E-3	-3.2E-5
			3.2802	-1.2E-1	2.0E-3	2.0E-1	-3.2E-3	2.7122	-1.1E-1	1.6E-3	1.8E-1	-2.5E-3
			3.9961	2.0E-3	-4.6E-5	-3.2E-3	7.5E-5	4.0101	1.6E-3	-3.2E-5	-2.5E-3	5.0E-5
			.9470	2.8E-1	-4.6E-3	-1.2E-1	1.9E-3	1.6287	2.8E-1	-3.7E-3	-1.1E-1	1.5E-3
.07	9.791	.2753	2.0070	-4.6E-3	1.1E-4	1.9E-3	-4.6E-5	2.0515	-3.7E-3	7.4E-5	1.5E-3	-3.1E-5
			3.2863	-1.2E-1	1.9E-3	1.9E-1	-3.1E-3	2.7047	-1.1E-1	1.5E-3	1.8E-1	-2.5E-3
			3.9960	1.9E-3	-4.6E-5	-3.1E-3	7.5E-5	4.0102	1.5E-3	-3.1E-5	-2.5E-3	5.0E-5

TABLE XXV (Continued)

Λ	E(P|Y)		Cov($\underset{\sim}{P}$|Y)			
			2.6E-1	1.8E-1	1.8E-1	1.2E-1
.04	1.5481	1.0320	1.8E-1	2.7E-1	2.7E-1	2.9E-1
	1.0320	2.5190	1.8E-1	2.7E-1	2.7E-1	2.9E-1
			1.2E-1	2.9E-1	2.9E-1	6.5E-1
			2.7E-1	1.6E-1	1.6E-1	9.6E-2
.05	1.6182	1.0143	1.6E-1	2.4E-1	2.4E-1	2.3E-1
	1.0143	2.4348	1.6E-1	2.4E-1	2.4E-1	2.3E-1
			9.6E-2	2.3E-1	2.3E-1	5.5E-1
			2.6E-1	1.6E-1	1.6E-1	9.9E-2
.06	1.6466	1.0190	1.6E-1	2.4E-1	2.4E-1	2.3E-1
	1.0190	2.4255	1.6E-1	2.4E-1	2.4E-1	2.3E-1
			9.9E-2	2.3E-1	2.3E-1	5.4E-1
			2.7E-1	1.6E-1	1.6E-1	1.0E-1
.07	1.6640	1.0268	1.6E-1	2.4E-1	2.4E-1	2.3E-1
	1.0268	2.4282	1.6E-1	2.4E-1	2.4E-1	2.3E-1
			1.0E-1	2.3E-1	2.3E-1	5.4E-1

TABLE XXVI

EXPECTED VALUES, VARIANCES, AND COVARIANCES OF THE MODEL PARAMETERS USING A NATURAL CONJUGATE PRIOR DISTRIBUTION

$$n = 100, \quad \rho = -.5, \quad m = 50, \quad \beta_2 = \beta_1 + \begin{pmatrix} \Delta & 0 \\ \Delta & 0 \end{pmatrix}$$

Δ	E(m|Y)	Var(m|Y)	$E(\underset{\sim}{B}_1\|Y)$	$Cov(\underset{\sim}{B}_1\|Y)$				$E(\underset{\sim}{B}_2\|Y)$	$Cov(\underset{\sim}{B}_2\|Y)$			
.04	50.131	.1466	1.1795	7.1E-2	-1.1E-3	-4.0E-2	6.3E-4	1.3747	8.1E-2	-1.1E-3	-4.6E-2	6.4E-4
			2.0021	-1.1E-3	2.5E-5	6.3E-4	-1.4E-5	2.0307	-1.1E-3	2.1E-5	6.4E-4	-1.2E-5
			2.8740	-4.0E-2	6.3E-4	6.7E-2	-1.1E-3	2.8477	-4.6E-2	6.4E-4	7.6E-2	-1.1E-3
			3.9973	6.3E-4	-1.4E-5	-1.1E-3	2.3E-5	4.0029	6.4E-4	-1.2E-5	-1.1E-3	2.0E-5
.05	50.069	.0671	1.1736	7.1E-2	-1.1E-3	-4.0E-2	6.3E-4	1.3892	8.0E-2	-1.1E-3	-4.5E-2	6.4E-4
			2.0021	-1.1E-3	2.5E-5	6.3E-4	-1.4E-5	2.0407	-1.1E-3	2.1E-5	6.4E-4	-1.2E-5
			2.8771	-4.0E-2	6.3E-4	6.7E-2	-1.1E-3	2.8434	-4.5E-2	6.4E-4	7.6E-2	-1.1E-3
			3.9973	6.3E-4	-1.4E-5	-1.1E-3	2.3E-5	4.0030	6.4E-4	-1.2E-5	-1.1E-3	2.0E-5
.06	50.029	.0280	1.1697	7.1E-2	-1.1E-3	-4.0E-2	6.3E-4	1.4012	8.0E-2	-1.1E-3	-4.5E-2	6.4E-4
			2.0022	-1.1E-3	2.5E-5	6.3E-4	-1.4E-5	2.0506	-1.1E-3	2.1E-5	6.4E-4	-1.2E-5
			2.8787	-4.0E-2	6.3E-4	6.7E-2	-1.1E-3	2.8409	-4.5E-2	6.4E-4	7.6E-2	-1.1E-3
			3.9973	6.3E-4	-1.4E-5	-1.1E-3	2.3E-5	4.0030	6.4E-4	-1.2E-5	-1.1E-3	2.0E-5
.07	50.010	.0102	1.1678	7.0E-2	-1.1E-3	-3.9E-2	6.3E-4	1.4116	8.0E-2	-1.1E-3	-4.5E-2	6.4E-4
			2.0022	-1.1E-3	2.5E-5	6.3E-4	-1.4E-5	2.0606	-1.1E-3	2.1E-5	6.4E-4	-1.2E-5
			2.8795	-3.9E-2	6.3E-4	6.7E-2	-1.1E-3	2.8398	-4.5E-2	6.4E-4	7.6E-2	-1.1E-3
			3.9973	6.3E-4	-1.4E-5	-1.1E-3	2.3E-5	4.0030	6.4E-4	-1.2E-5	-1.1E-3	2.0E-5

TABLE XXVI (Continued)

Δ	E(P|Y)		Cov($\underset{\sim}{P}$|Y)			
			4.8E-2	2.9E-2	2.9E-2	1.7E-2
.04	1.5600	.9262	2.9E-2	3.4E-2	3.4E-2	3.0E-2
	.9262	1.6546	2.9E-2	3.4E-2	3.4E-2	3.0E-2
			1.7E-2	3.0E-2	3.0E-2	5.4E-2
			4.8E-2	2.9E-2	2.9E-2	1.7E-2
.05	1.5639	.9280	2.9E-2	3.4E-2	3.4E-2	3.0E-2
	.9280	1.6551	2.9E-2	3.4E-2	3.4E-2	3.0E-2
			1.7E-2	3.0E-2	3.0E-2	5.4E-2
			4.8E-2	2.9E-2	2.9E-2	1.7E-2
.06	1.5667	.9290	2.9E-2	3.4E-2	3.4E-2	3.0E-2
	.9290	1.6551	2.9E-2	3.4E-2	3.4E-2	3.0E-2
			1.7E-2	3.0E-2	3.0E-2	5.4E-2
			4.8E-2	2.9E-2	2.9E-2	1.7E-2
.07	1.5682	.9295	2.9E-2	3.4E-2	3.4E-2	3.0E-2
	.9295	1.6550	2.9E-2	3.4E-2	3.4E-2	3.0E-2
			1.7E-2	3.0E-2	3.0E-2	5.4E-2

TABLE XXVII

EXPECTED VALUES, VARIANCES, AND COVARIANCES OF THE MODEL PARAMETERS USING A NATURAL CONJUGATE PRIOR DISTRIBUTION

$$n = 20, \quad \rho = 0, \quad m = 10, \quad \beta_2 = \beta_1 + \begin{pmatrix} \Delta & 0 \\ \Delta & 0 \end{pmatrix}$$

Δ	E(m|Y)	Var(m|Y)	$E(\underset{\sim}{\beta}_1\|Y)$	$Cov(\underset{\sim}{\beta}_1\|Y)$				$E(\underset{\sim}{\beta}_2\|Y)$	$Cov(\underset{\sim}{\beta}_2\|Y)$			
			.9391	3.1E-1	-5.2E-3	-7.1E-3	2.6E-5	1.2891	8.8E-1	-5.9E-2	-1.4E-2	-9.4E-4
.04	14.738	25.5531	2.0158	-5.2E-3	2.0E-4	-2.5E-4	3.2E-6	2.0857	-5.9E-2	9.4E-3	-1.9E-3	4.5E-4
			3.2051	-7.1E-3	-2.5E-4	1.7E-1	-2.8E-3	3.0527	-1.4E-2	-1.9E-3	4.4E-1	-2.2E-2
			3.9993	2.6E-5	3.2E-6	-2.8E-3	7.4E-5	4.0146	-9.4E-4	4.5E-4	-2.2E-2	2.7E-3
			.9207	3.2E-1	-5.2E-3	8.6E-3	-1.8E-4	1.5397	3.9E-1	-1.4E-2	4.1E-4	-4.4E-4
.05	10.226	7.4405	2.0094	-5.2E-3	1.6E-4	-3.3E-4	3.2E-6	2.0390	-1.4E-2	1.7E-3	-1.7E-4	1.3E-4
			3.2833	8.6E-3	-3.3E-4	2.0E-1	-3.1E-3	3.0516	4.1E-4	-1.7E-4	2.1E-1	-4.7E-3
			3.9993	-1.8E-4	3.2E-6	-3.1E-3	7.5E-5	4.0024	-4.4E-4	1.3E-4	-4.7E-3	3.7E-4
			.9286	2.9E-1	-4.8E-3	1.7E-2	-2.9E-4	1.5858	2.8E-1	-4.3E-3	1.1E-2	-1.9E-4
.06	9.622	.8821	2.0074	-4.8E-3	1.2E-4	-3.0E-4	6.5E-6	2.0421	-4.3E-3	1.6E-4	-1.7E-4	9.4E-6
			3.3034	1.7E-2	-3.0E-4	1.9E-1	-3.1E-3	3.0338	1.1E-2	-1.7E-4	1.8E-1	-2.5E-3
			3.9994	-2.9E-4	6.5E-6	-3.1E-3	7.4E-5	4.0007	-1.9E-4	9.4E-6	-2.5E-3	6.4E-5
			.9378	2.9E-1	-4.7E-3	1.9E-2	-3.2E-4	1.6104	2.8E-1	-3.8E-3	1.5E-2	-2.2E-4
.07	9.713	.3156	2.0071	-4.7E-3	1.1E-4	-3.2E-4	7.5E-6	2.0517	-3.8E-3	8.0E-5	-2.2E-4	5.0E-6
			3.3078	1.9E-2	-3.2E-4	1.9E-1	-3.1E-3	3.0257	1.5E-2	-2.2E-4	1.7E-1	-2.4E-3
			3.9994	-3.2E-4	7.5E-6	-3.1E-3	7.4E-5	4.0007	-2.2E-4	5.0E-6	-2.4E-3	5.0E-5

TABLE XXVII (Continued)

Δ	E(P|Y)		Cov(P|Y)			
			1.2E-1	-9.1E-3	-9.1E-3	1.2E-3
.04	1.0298	.0230	-9.1E-3	9.8E-2	9.8E-2	7.6E-3
	.0230	1.8364	-9.1E-3	9.8E-2	9.8E-2	7.6E-3
			1.2E-3	7.6E-3	7.6E-3	3.1E-1
			1.6E-1	-2.7E-2	-2.7E-2	-4.0E-4
.05	1.1620	-.0810	-2.7E-2	1.1E-1	1.1E-1	-9.9E-3
	-.0810	1.8429	-2.7E-2	1.1E-1	1.1E-1	-9.9E-3
			-4.0E-4	-9.9E-3	-9.9E-3	3.1E-1
			1.5E-1	-1.9E-2	-1.9E-2	1.3E-3
.06	1.2180	-.1218	-1.9E-2	1.1E-1	1.1E-1	-2.0E-2
	-.1218	1.8364	-1.9E-2	1.1E-1	1.1E-1	-2.0E-2
			1.3E-3	-2.0E-2	-2.0E-2	3.1E-1
			1.4E-1	-1.7E-2	-1.7E-2	1.8E-3
.07	1.2263	-.1300	-1.7E-2	1.0E-1	1.0E-1	-2.2E-2
	-.1300	1.8354	-1.7E-2	1.0E-1	1.0E-1	-2.2E-2
			1.8E-3	-2.2E-2	-2.2E-2	3.1E-1

TABLE XXVIII

EXPECTED VALUES, VARIANCES, AND COVARIANCES OF THE MODEL PARAMETERS USING A NATURAL CONJUGATE PRIOR DISTRIBUTION

$$n = 100, \quad \rho = 0, \quad m = 50, \quad \beta_2 = \beta_1 + \begin{pmatrix} \Delta & 0 \\ \Delta & 0 \end{pmatrix}$$

Δ	$E(m\|Y)$	$Var(m\|Y)$	$E(\beta_1\|Y)$	$Cov(\beta_1\|Y)$				$E(\beta_2\|Y)$	$Cov(\beta_2\|Y)$			
.04	49.895	.4067	1.1768	7.2E-2	-1.1E-3	-5.0E-3	7.8E-5	1.3715	8.1E-2	-1.1E-3	-5.7E-3	8.1E-5
			2.0021	-1.1E-3	2.5E-5	7.9E-5	-1.7E-6	2.0307	-1.1E-3	2.1E-5	8.1E-5	-1.5E-6
			2.9558	-5.0E-3	7.9E-5	6.0E-2	-9.5E-4	3.0162	-5.7E-3	8.1E-5	6.8E-2	-9.6E-4
			3.9982	7.8E-5	-1.7E-6	-9.5E-4	2.1E-5	3.9980	8.1E-5	-1.5E-6	-9.6E-4	1.8E-5
.05	49.995	.1466	1.1732	7.1E-2	-1.1E-3	-5.1E-3	8.0E-5	1.3870	8.1E-2	-1.1E-3	-5.7E-3	8.1E-5
			2.0021	-1.1E-3	2.5E-5	8.0E-5	-1.7E-6	2.0407	-1.1E-3	2.1E-5	8.1E-5	-1.5E-6
			2.9565	-5.1E-3	8.0E-5	6.0E-2	-9.5E-4	3.0161	-5.7E-3	8.1E-5	6.8E-2	-9.6E-4
			3.9981	8.0E-5	-1.7E-6	-9.5E-4	2.1E-5	3.9980	8.1E-5	-1.5E-6	-9.6E-4	1.8E-5
.06	50.015	.0519	1.1702	7.1E-2	-1.1E-3	-5.1E-3	8.0E-5	1.4001	8.0E-2	-1.1E-3	-5.7E-3	8.1E-5
			2.0022	-1.1E-3	2.5E-5	8.0E-5	-1.8E-6	2.0506	-1.1E-3	2.1E-5	8.1E-5	-1.5E-6
			2.9569	-5.1E-3	8.0E-5	6.0E-2	-9.5E-4	3.0159	-5.7E-3	8.1E-5	6.8E-2	-9.6E-4
			3.9981	8.0E-5	-1.8E-6	-9.5E-4	2.1E-5	3.9980	8.1E-5	-1.5E-6	-9.6E-4	1.8E-5
.07	50.011	.0177	1.1682	7.0E-2	-1.1E-3	-5.1E-3	8.0E-5	1.4112	8.0E-2	-1.1E-3	-5.7E-3	8.1E-5
			2.0022	-1.1E-3	2.5E-5	8.0E-5	-1.8E-6	2.0606	-1.1E-3	2.1E-5	8.1E-5	-1.5E-6
			2.9572	-5.1E-3	8.0E-5	6.0E-2	-9.5E-4	3.0156	-5.7E-3	8.1E-5	6.8E-2	-9.6E-4
			3.9981	8.0E-5	-1.8E-6	-9.5E-4	2.1E-5	3.9980	8.1E-5	-1.5E-6	-9.6E-4	1.8E-5

TABLE XXVIII (Continued)

Δ	E(P|Y)		Cov(P|Y)			
			2.2E-2	1.8E-3	1.8E-3	1.5E-4
.04	1.0429	.0845	1.8E-3	1.3E-2	1.3E-2	2.1E-3
	.0845	1.2408	1.8E-3	1.3E-2	1.3E-2	2.1E-3
			1.5E-4	2.1E-3	2.1E-3	3.0E-2
			2.2E-2	1.9E-3	1.9E-3	1.6E-4
.05	1.0464	.0868	1.9E-3	1.3E-2	1.3E-2	2.1E-3
	.0868	1.2412	1.9E-3	1.3E-2	1.3E-2	2.1E-3
			1.6E-4	2.1E-3	2.1E-3	3.0E-2
			2.2E-2	1.9E-3	1.9E-3	1.6E-4
.06	1.0499	.0883	1.9E-3	1.3E-2	1.3E-2	2.2E-3
	.0883	1.2414	1.9E-3	1.3E-2	1.3E-2	2.2E-3
			1.6E-4	2.2E-3	2.2E-3	3.0E-2
			2.2E-2	1.8E-3	1.8E-3	1.5E-4
.07	1.0519	.0889	1.8E-3	1.3E-2	1.3E-2	2.2E-3
	.0889	1.2414	1.8E-3	1.3E-2	1.3E-2	2.2E-3
			1.5E-4	2.2E-3	2.2E-3	3.0E-2

TABLE XXIX

EXPECTED VALUES, VARIANCES, AND COVARIANCES OF THE MODEL PARAMETERS USING A NATURAL CONJUGATE PRIOR DISTRIBUTION

$$n = 20, \quad \rho = .5, \quad m = 10, \quad \beta_2 = \beta_1 + \begin{pmatrix} \Delta & 0 \\ \Delta & 0 \end{pmatrix}$$

Δ	E(m|Y)	Var(m|Y)	$E(\underset{\sim}{B}_1\|Y)$	$Cov(\underset{\sim}{B}_1\|Y)$				$E(\underset{\sim}{B}_2\|Y)$	$Cov(\underset{\sim}{B}_2\|Y)$			
			.9280	2.8E-1	-4.5E-3	1.5E-1	-2.5E-3	1.6048	2.8E-1	-6.0E-3	1.4E-1	-2.9E-3
.04	9.731	1.8988	2.0075	-4.5E-3	1.1E-4	-2.4E-3	5.7E-5	2.0235	-6.0E-3	5.2E-4	-2.8E-3	2.5E-4
			3.2353	1.5E-1	-2.4E-3	2.3E-1	-3.8E-3	3.3491	1.4E-1	-2.8E-3	2.2E-1	-4.2E-3
			4.0029	-2.5E-3	5.7E-5	-3.8E-3	9.1E-5	3.9923	-2.9E-3	2.5E-4	-4.2E-3	2.8E-4
			.9378	2.8E-1	-4.5E-3	1.5E-1	-2.5E-3	1.6263	2.6E-1	-3.6E-3	1.4E-1	-1.9E-3
.05	9.734	.2715	2.0071	-4.5E-3	1.1E-4	-2.5E-3	5.9E-5	2.0314	-3.6E-3	8.4E-5	-1.9E-3	4.4E-5
			3.2392	1.5E-1	-2.5E-3	2.3E-1	-3.7E-3	3.3447	1.4E-1	-1.9E-3	2.1E-1	-2.9E-3
			4.0030	-2.5E-3	5.9E-5	-3.7E-3	8.9E-5	3.9910	-1.9E-3	4.4E-5	-2.9E-3	6.5E-5
			.9465	2.7E-1	-4.5E-3	1.5E-1	-2.5E-3	1.6450	2.6E-1	-3.6E-3	1.4E-1	-1.9E-3
.06	9.855	.1377	2.0070	-4.5E-3	1.1E-4	-2.5E-3	5.9E-5	2.0413	-3.6E-3	7.3E-5	-1.9E-3	4.0E-5
			3.2424	1.5E-1	-2.5E-3	2.3E-1	-3.7E-3	3.3372	1.4E-1	-1.9E-3	2.1E-1	-2.9E-3
			4.0030	-2.5E-3	5.9E-5	-3.7E-3	8.8E-5	3.9910	-1.9E-3	4.0E-5	-2.9E-3	5.9E-5
			.9522	2.7E-1	-4.5E-3	1.5E-1	-2.5E-3	1.6637	2.6E-1	-3.6E-3	1.4E-1	-2.0E-3
.07	9.933	.0665	2.0070	-4.5E-3	1.1E-4	-2.5E-3	5.9E-5	2.0512	-3.6E-3	7.2E-5	-2.0E-3	4.0E-5
			3.2441	1.5E-1	-2.5E-3	2.2E-1	-3.7E-3	3.3327	1.4E-1	-2.0E-3	2.1E-1	-2.9E-3
			4.0029	-2.5E-3	5.9E-5	-3.7E-3	8.8E-5	3.9910	-2.0E-3	4.0E-5	-2.9E-3	5.9E-5

TABLE XXIX (Continued)

Δ	E(P|Y)		Cov($\underset{\sim}{P}$|Y)			
			3.9E-1	-2.7E-1	-2.7E-1	1.9E-1
.04	1.9712	-1.3038	-2.7E-1	3.3E-1	3.3E-1	-3.1E-1
	-1.3038	2.3696	-2.7E-1	3.3E-1	3.3E-1	-3.1E-1
			1.9E-1	-3.1E-1	-3.1E-1	5.4E-1
			3.7E-1	-2.5E-1	-2.5E-1	1.7E-1
.05	1.9850	-1.3279	-2.5E-1	3.1E-1	3.1E-1	-3.0E-1
	-1.3279	2.3960	-2.5E-1	3.1E-1	3.1E-1	-3.0E-1
			1.7E-1	-3.0E-1	-3.0E-1	5.3E-1
			3.8E-1	-2.6E-1	-2.6E-1	1.8E-1
.06	1.9933	-1.3406	-2.6E-1	3.1E-1	3.1E-1	-3.0E-1
	-1.3406	2.4107	-2.6E-1	3.1E-1	3.1E-1	-3.0E-1
			1.8E-1	-3.0E-1	-3.0E-1	5.4E-1
			3.8E-1	-2.6E-1	-2.6E-1	1.8E-1
.07	2.0076	-1.3557	-2.6E-1	3.1E-1	3.1E-1	-3.1E-1
	-1.3557	2.4253	-2.6E-1	3.1E-1	3.1E-1	-3.1E-1
			1.8E-1	-3.1E-1	-3.1E-1	5.4E-1

TABLE XXX

EXPECTED VALUES, VARIANCES, AND COVARIANCES OF THE MODEL PARAMETERS USING A NATURAL CONJUGATE PRIOR DISTRIBUTION

$$n = 100, \quad \rho = .5, \quad m = 50, \quad \beta_2 = \beta_1 + \begin{pmatrix} \Delta & 0 \\ \Delta & 0 \end{pmatrix}$$

Δ	$E(m \mid Y)$	$Var(m \mid Y)$	$E(\underset{\sim}{B}_1 \mid Y)$	$Cov(\underset{\sim}{B}_1 \mid Y)$				$E(\underset{\sim}{B}_2 \mid Y)$	$Cov(\underset{\sim}{B}_2 \mid Y)$			
			1.1708	7.1E-2	-1.1E-3	3.1E-2	-4.9E-4	1.3726	8.1E-2	-1.1E-3	3.5E-2	-5.0E-4
.04	49.702	.3384	2.0021	-1.1E-3	2.5E-5	-4.9E-4	1.1E-5	2.0306	-1.1E-3	2.1E-5	-5.0E-4	9.3E-6
			3.0483	3.1E-2	-4.9E-4	5.8E-2	-9.2E-4	3.1838	3.5E-2	-5.0E-4	6.6E-2	-9.3E-4
			3.9995	-4.9E-4	1.1E-5	-9.2E-4	2.0E-5	3.9936	-5.0E-4	9.3E-6	-9.3E-4	1.7E-5
			1.1692	7.1E-2	-1.1E-3	3.1E-2	-4.9E-4	1.3872	8.1E-2	-1.1E-3	3.5E-2	-5.0E-4
.05	49.885	.1614	2.0022	-1.1E-3	2.5E-5	-4.9E-4	1.1E-5	2.0406	-1.1E-3	2.1E-5	-5.0E-4	9.3E-6
			3.0473	3.1E-2	-4.9E-4	5.8E-2	-9.2E-4	3.1857	3.5E-2	-5.0E-4	6.6E-2	-9.3E-4
			3.9995	-4.9E-4	1.1E-5	-9.2E-4	2.0E-5	3.9936	-5.0E-4	9.3E-6	-9.3E-4	1.7E-5
			1.1678	7.1E-2	-1.1E-3	3.1E-2	-4.9E-4	1.4009	8.0E-2	-1.1E-3	3.5E-2	-5.0E-4
.06	49.977	.0462	2.0022	-1.1E-3	2.5E-5	-4.9E-4	1.1E-5	2.0506	-1.1E-3	2.1E-5	-5.0E-4	9.3E-6
			3.0466	3.1E-2	-4.9E-4	5.8E-2	-9.2E-4	3.1867	3.5E-2	-5.0E-4	6.6E-2	-9.3E-4
			3.9995	-4.9E-4	1.1E-5	-9.2E-4	2.0E-5	3.9936	-5.0E-4	9.3E-6	-9.3E-4	1.7E-5
			1.1670	7.0E-2	-1.1E-3	3.1E-2	-4.9E-4	1.4118	8.0E-2	-1.1E-3	3.5E-2	-4.9E-4
.07	49.999	.0094	2.0022	-1.1E-3	2.5E-5	-4.9E-4	1.1E-5	2.0606	-1.1E-3	2.1E-5	-4.9E-4	9.2E-6
			3.0464	3.1E-2	-4.9E-4	5.8E-2	-9.2E-4	3.1870	3.5E-2	-4.9E-4	6.6E-2	-9.3E-4
			3.9994	-4.9E-4	1.1E-5	-9.2E-4	2.0E-5	3.9936	-4.9E-4	9.2E-6	-9.3E-4	1.7E-5

TABLE XXX (Continued)

Δ	E(P|Y)		Cov($\underset{\sim}{P}$|Y)			
			3.6E-2	-1.9E-2	-1.9E-2	1.0E-2
.04	1.3536	-.7260	-1.9E-2	2.7E-2	2.7E-2	-2.4E-2
	-.7260	1.6614	-1.9E-2	2.7E-2	2.7E-2	-2.4E-2
			1.0E-2	-2.4E-2	-2.4E-2	5.4E-2
			3.6E-2	-1.9E-2	-1.9E-2	1.0E-2
.05	1.3546	-.7232	-1.9E-2	2.7E-2	2.7E-2	-2.4E-2
	-.7232	1.6572	-1.9E-2	2.7E-2	2.7E-2	-2.4E-2
			1.0E-2	-2.4E-2	-2.4E-2	5.4E-2
			3.7E-2	-1.9E-2	-1.9E-2	1.0E-2
.06	1.3599	-.7233	-1.9E-2	2.7E-2	2.7E-2	-2.3E-2
	-.7233	1.6553	-1.9E-2	2.7E-2	2.7E-2	-2.3E-2
			1.0E-2	-2.3E-2	-2.3E-2	5.4E-2
			3.7E-2	-1.9E-2	-1.9E-2	1.0E-2
.07	1.3630	-.7241	-1.9E-2	2.7E-2	2.7E-2	-2.4E-2
	-.7241	1.6551	-1.9E-2	2.7E-2	2.7E-2	-2.4E-2
			1.0E-2	-2.4E-2	-2.4E-2	5.4E-2

TABLE XXXI

EXPECTED VALUES, VARIANCES, AND COVARIANCES OF THE MODEL PARAMETERS USING A GENERALIZED NATURAL CONJUGATE PRIOR DISTRIBUTION

$$n = 20, \quad \rho = -.5, \quad m = 10, \quad \beta_2 = \beta_1 + \begin{pmatrix} \Delta & 0 \\ \Delta & 0 \end{pmatrix}$$

Δ	E(m|Y)	Var(m|Y)	E($\underset{\sim}{\beta}_1$|Y)	Cov($\underset{\sim}{\beta}_1$|Y)				E($\underset{\sim}{\beta}_2$|Y)	Cov($\underset{\sim}{\beta}_2$|Y)			
			.9993	3.2E-1	-4.8E-3	-1.5E-1	2.3E-3	1.5362	3.3E-1	-4.8E-3	-1.5E-1	2.5E-3
.04	10.687	10.7964	2.0089	-4.8E-3	1.2E-4	2.0E-3	-4.3E-5	2.0225	-4.8E-3	1.5E-4	2.5E-3	-7.3E-5
			3.1780	-1.5E-1	2.0E-3	2.4E-1	-3.7E-3	2.8453	-1.5E-1	2.5E-3	2.7E-1	-4.3E-3
			3.9974	2.3E-3	-4.3E-5	-3.7E-3	8.1E-5	4.0108	2.5E-3	-7.3E-5	-4.3E-3	1.6E-4
			.9304	3.0E-1	-4.7E-3	-1.3E-1	2.1E-3	1.5559	2.8E-1	-3.8E-3	-1.2E-1	1.7E-3
.05	9.466	2.6343	2.0078	-4.7E-3	1.2E-4	2.0E-3	-4.5E-5	2.0319	-3.8E-3	8.1E-5	1.7E-3	-3.4E-5
			3.2838	-1.3E-1	2.0E-3	2.3E-1	-3.6E-3	2.7275	-1.2E-1	1.7E-3	2.0E-1	-2.7E-3
			3.9959	2.1E-3	-4.5E-5	-3.6E-3	7.9E-5	4.0103	1.7E-3	-3.4E-5	-2.7E-3	6.1E-5
			.9262	2.8E-1	-4.5E-3	-1.2E-1	2.0E-3	1.5955	2.7E-1	-3.7E-3	-1.1E-1	1.6E-3
.06	9.524	.7316	2.0073	-4.5E-3	1.1E-4	2.0E-3	-4.5E-5	2.0416	-3.7E-3	7.3E-5	1.5E-3	-3.0E-5
			3.3163	-1.2E-1	2.0E-3	2.1E-1	-3.4E-3	2.6932	-1.1E-1	1.5E-3	1.9E-1	-2.6E-3
			3.9955	2.0E-3	-4.5E-5	-3.4E-3	7.7E-5	4.0104	1.6E-3	-3.0E-5	-2.6E-3	5.1E-5
			.9332	2.7E-1	-4.4E-3	-1.2E-1	1.9E-3	1.6352	2.7E-1	-3.6E-3	-1.1E-1	1.5E-3
.07	9.714	.3744	2.0072	-4.4E-3	1.0E-4	1.9E-3	-4.4E-5	2.0513	-3.6E-3	7.1E-5	1.5E-3	-3.0E-5
			3.3254	-1.2E-1	1.9E-3	2.1E-1	-3.3E-3	2.6831	-1.1E-1	1.5E-3	1.9E-1	-2.6E-3
			3.9954	1.9E-3	-4.4E-5	-3.3E-3	7.6E-5	4.0105	1.5E-3	-3.0E-5	-2.6E-3	5.0E-5

TABLE XXXII

EXPECTED VALUES, VARIANCES, AND COVARIANCES OF THE MODEL PARAMETERS USING A GENERALIZED NATURAL CONJUGATE PRIOR DISTRIBUTION

$$n = 100, \quad \rho = -.5, \quad m = 50, \quad \beta_2 = \beta_1 + \begin{pmatrix} \Delta & 0 \\ \Delta & 0 \end{pmatrix}$$

Δ	E(m|Y)	Var(m|Y)	$E(\underset{\sim}{B}_1\|Y)$	$Cov(\underset{\sim}{B}_1\|Y)$				$E(\underset{\sim}{B}_2\|Y)$	$Cov(\underset{\sim}{B}_2\|Y)$			
			1.1803	7.1E-2	-1.1E-3	-3.9E-2	6.2E-4	1.3750	8.0E-2	-1.1E-3	-4.5E-2	6.3E-4
.04	50.136	.1570	2.0021	-1.1E-3	2.4E-5	6.2E-4	-1.4E-5	2.0307	-1.1E-3	2.1E-5	6.3E-4	-1.2E-5
			2.8740	-3.9E-2	6.2E-4	6.6E-2	-1.1E-3	2.8500	-4.5E-2	6.3E-4	7.6E-2	-1.1E-3
			3.9973	6.2E-4	-1.4E-5	-1.1E-3	2.3E-5	4.0029	6.3E-4	-1.2E-5	-1.1E-3	2.0E-5
			1.1743	7.0E-2	-1.1E-3	-3.9E-2	6.2E-4	1.3898	8.0E-2	-1.1E-3	-4.4E-2	6.3E-4
.05	50.074	.0730	2.0021	-1.1E-3	2.4E-5	6.2E-4	-1.4E-5	2.0407	-1.1E-3	2.1E-5	6.3E-4	-1.2E-5
			2.8771	-3.9E-2	6.2E-4	6.6E-2	-1.1E-3	2.8457	-4.4E-2	6.3E-4	7.6E-2	-1.1E-3
			3.9973	6.2E-4	-1.4E-5	-1.1E-3	2.3E-5	4.0029	6.3E-4	-1.2E-5	-1.1E-3	2.0E-5
			1.1702	7.0E-2	-1.1E-3	-3.9E-2	6.2E-4	1.4021	7.9E-2	-1.1E-3	-4.4E-2	6.3E-4
.06	50.032	.0316	2.0022	-1.1E-3	2.4E-5	6.2E-4	-1.4E-5	2.0506	-1.1E-3	2.1E-5	6.3E-4	-1.2E-5
			2.8788	-3.9E-2	6.2E-4	6.6E-2	-1.1E-3	2.8431	-4.4E-2	6.3E-4	7.5E-2	-1.1E-3
			3.9973	6.2E-4	-1.4E-5	-1.1E-3	2.3E-5	4.0030	6.3E-4	-1.2E-5	-1.1E-3	2.0E-5
			1.1681	7.0E-2	-1.1E-3	-3.9E-2	6.2E-4	1.4127	7.9E-2	-1.1E-3	-4.4E-2	6.2E-4
.07	50.012	.0119	2.0022	-1.1E-3	2.4E-5	6.2E-4	-1.4E-5	2.0606	-1.1E-3	2.1E-5	6.2E-4	-1.2E-5
			2.8796	-3.9E-2	6.2E-4	6.6E-2	-1.1E-3	2.8420	-4.4E-2	6.2E-4	7.5E-2	-1.1E-3
			3.9972	6.2E-4	-1.4E-5	-1.1E-3	2.3E-5	4.0030	6.2E-4	-1.2E-5	-1.1E-3	2.0E-5

TABLE XXXIII

EXPECTED VALUES, VARIANCES, AND COVARIANCES OF THE MODEL PARAMETERS USING A GENERALIZED NATURAL CONJUGATE PRIOR DISTRIBUTION

$$n = 20, \quad \rho = 0, \quad m = 10, \quad \beta_2 = \beta_1 + \begin{pmatrix} \Delta & 0 \\ \Delta & 0 \end{pmatrix}$$

Δ	$E(m\|Y)$	$Var(m\|Y)$	$E(\underset{\sim}{B}_1\|Y)$	$Cov(\underset{\sim}{B}_1\|Y)$				$E(\underset{\sim}{B}_2\|Y)$	$Cov(\underset{\sim}{B}_2\|Y)$			
			.9870	3.3E-1	-5.4E-3	-1.2E-2	-7.1E-7	1.5424	4.0E-1	-9.2E-3	-1.6E-2	3.8E-4
.04	11.371	17.9939	2.0103	-5.4E-3	1.6E-4	-1.3E-4	9.9E-7	2.0214	-9.2E-3	5.1E-4	6.7E-4	-4.9E-5
			3.2465	-1.2E-2	-1.3E-4	2.5E-1	-4.0E-3	3.1272	-1.6E-2	6.7E-4	3.4E-1	-9.1E-3
			3.9984	-7.1E-7	9.9E-7	-4.0E-3	9.7E-5	4.0073	3.8E-4	-4.9E-5	-9.1E-3	5.9E-4
			.9250	3.1E-1	-4.9E-3	1.8E-3	-1.1E-4	1.5635	3.0E-1	-4.6E-3	-5.2E-3	-1.2E-4
.05	9.913	5.4200	2.0090	-4.9E-3	1.4E-4	-1.9E-4	1.0E-6	2.0319	-4.6E-3	1.6E-4	1.0E-4	-4.5E-7
			3.3164	1.8E-3	-1.9E-4	2.2E-1	-3.5E-3	3.0738	-5.2E-3	1.0E-4	2.2E-1	-3.5E-3
			3.9985	-1.1E-4	1.0E-6	-3.5E-3	8.0E-5	4.0017	-1.2E-4	-4.5E-7	-3.5E-3	1.4E-4
			.9189	2.9E-1	-4.6E-3	1.1E-2	-2.0E-4	1.5905	2.7E-1	-3.7E-3	3.3E-3	-1.1E-4
.06	9.588	1.2121	2.0078	-4.6E-3	1.1E-4	-2.2E-4	4.4E-6	2.0417	-3.7E-3	8.2E-5	-7.3E-5	3.2E-6
			3.3403	1.1E-2	-2.2E-4	2.1E-1	-3.4E-3	3.0362	3.3E-3	-7.3E-5	1.9E-1	-2.6E-3
			3.9987	-2.0E-4	4.4E-6	-3.4E-3	7.7E-5	4.0008	-1.1E-4	3.2E-6	-2.6E-3	5.9E-5
			.9263	2.8E-1	-4.5E-3	1.4E-2	-2.4E-4	1.6166	2.7E-1	-3.7E-3	7.7E-3	-1.3E-4
.07	9.656	.4443	2.0073	-4.5E-3	1.1E-4	-2.4E-4	5.9E-6	2.0515	-3.7E-3	7.3E-5	-1.3E-4	3.6E-6
			3.3476	1.4E-2	-2.4E-4	2.1E-1	-3.3E-3	3.0213	7.7E-3	-1.3E-4	1.9E-1	-2.5E-3
			3.9987	-2.4E-4	5.9E-6	-3.3E-3	7.6E-5	4.0008	-1.3E-4	3.6E-6	-2.5E-3	5.1E-5

TABLE XXXIV

EXPECTED VALUES, VARIANCES, AND COVARIANCES OF THE MODEL PARAMETERS USING A GENERALIZED NATURAL CONJUGATE PRIOR DISTRIBUTION

$$n = 100, \quad \rho = 0, \quad m = 50, \quad \beta_2 = \beta_1 + \begin{pmatrix} \Delta & 0 \\ \Delta & 0 \end{pmatrix}$$

Δ	E(m|Y)	Var(m|Y)	$E(\underset{\sim}{\beta}_1\|Y)$	$Cov(\underset{\sim}{\beta}_1\|Y)$				$E(\underset{\sim}{\beta}_2\|Y)$	$Cov(\underset{\sim}{\beta}_2\|Y)$			
			1.1773	7.1E-2	-1.1E-3	-4.7E-3	7.3E-5	1.3714	8.0E-2	-1.1E-3	-5.3E-3	7.5E-5
.04	49.888	.4381	2.0021	-1.1E-3	2.5E-5	7.4E-5	-1.6E-6	2.0307	-1.1E-3	2.1E-5	7.5E-5	-1.4E-6
			2.9560	-4.7E-3	7.4E-5	6.0E-2	-9.5E-4	3.0186	-5.3E-3	7.5E-5	6.8E-2	-9.7E-4
			3.9982	7.3E-5	-1.6E-6	-9.5E-4	2.1E-5	3.9979	7.5E-5	-1.4E-6	-9.7E-4	1.8E-5
			1.1738	7.1E-2	-1.1E-3	-4.7E-3	7.4E-5	1.3872	8.0E-2	-1.1E-3	-5.3E-3	7.5E-5
.05	49.994	.1605	2.0021	-1.1E-3	2.5E-5	7.5E-5	-1.6E-6	2.0406	-1.1E-3	2.1E-5	7.5E-5	-1.4E-6
			2.9567	-4.7E-3	7.5E-5	6.0E-2	-9.5E-4	3.0186	-5.3E-3	7.5E-5	6.8E-2	-9.6E-4
			3.9981	7.4E-5	-1.6E-6	-9.5E-4	2.1E-5	3.9980	7.5E-5	-1.4E-6	-9.6E-4	1.8E-5
			1.1707	7.0E-2	-1.1E-3	-4.7E-3	7.5E-5	1.4007	7.9E-2	-1.1E-3	-5.3E-3	7.5E-5
.06	50.016	.0587	2.0022	-1.1E-3	2.4E-5	7.5E-5	-1.7E-6	2.0506	-1.1E-3	2.1E-5	7.5E-5	-1.4E-6
			2.9571	-4.7E-3	7.5E-5	6.0E-2	-9.5E-4	3.0185	-5.3E-3	7.5E-5	6.8E-2	-9.6E-4
			3.9981	7.5E-5	-1.7E-6	-9.5E-4	2.1E-5	3.9980	7.5E-5	-1.4E-6	-9.6E-4	1.8E-5
			1.1686	7.0E-2	-1.1E-3	-4.7E-3	7.5E-5	1.4120	7.9E-2	-1.1E-3	-5.3E-3	7.5E-5
.07	50.012	.0207	2.0022	-1.1E-3	2.4E-5	7.5E-5	-1.7E-6	2.0606	-1.1E-3	2.1E-5	7.5E-5	-1.4E-6
			2.9574	-4.7E-3	7.5E-5	6.0E-2	-9.5E-4	3.0182	-5.3E-3	7.5E-5	6.8E-2	-9.6E-4
			3.9981	7.5E-5	-1.7E-6	-9.5E-4	2.1E-5	3.9980	7.5E-5	-1.4E-6	-9.6E-4	1.8E-5

TABLE XXXV

EXPECTED VALUES, VARIANCES, AND COVARIANCES OF THE MODEL PARAMETERS
USING A GENERALIZED NATURAL CONJUGATE PRIOR DISTRIBUTION

$$n = 20, \quad \rho = .5, \quad m = 10, \quad \beta_2 = \beta_1 + \begin{pmatrix} \Delta & 0 \\ \Delta & 0 \end{pmatrix}$$

Δ	$E(m\|Y)$	$Var(m\|Y)$	$E(\underset{\sim}{\beta}_1\|Y)$	$Cov(\underset{\sim}{\beta}_1\|Y)$				$E(\underset{\sim}{\beta}_2\|Y)$	$Cov(\underset{\sim}{\beta}_2\|Y)$			
			.9190	2.7E-1	-4.4E-3	1.5E-1	-2.4E-3	1.6320	2.5E-1	-3.5E-3	1.3E-1	-1.8E-3
.04	9.549	.8192	2.0074	-4.4E-3	1.0E-4	-2.3E-3	5.5E-5	2.0210	-3.5E-3	7.5E-5	-1.8E-3	3.8E-5
			3.2635	1.5E-1	-2.3E-3	2.4E-1	-3.9E-3	3.3684	1.3E-1	-1.8E-3	2.1E-1	-3.0E-3
			4.0024	-2.4E-3	5.5E-5	-3.9E-3	9.0E-5	3.9908	-1.8E-3	3.8E-5	-3.0E-3	6.6E-5
			.9267	2.7E-1	-4.3E-3	1.4E-1	-2.3E-3	1.6442	2.5E-1	-3.4E-3	1.3E-1	-1.8E-3
.05	9.678	.3134	2.0073	-4.3E-3	1.0E-4	-2.3E-3	5.5E-5	2.0310	-3.4E-3	6.9E-5	-1.8E-3	3.7E-5
			3.2682	1.4E-1	-2.3E-3	2.3E-1	-3.8E-3	3.3561	1.3E-1	-1.8E-3	2.1E-1	-2.9E-3
			4.0025	-2.3E-3	5.5E-5	-3.8E-3	8.8E-5	3.9908	-1.8E-3	3.7E-5	-2.9E-3	5.9E-5
			.9356	2.6E-1	-4.3E-3	1.4E-1	-2.3E-3	1.6633	2.5E-1	-3.4E-3	1.3E-1	-1.8E-3
.06	9.807	.1851	2.0072	-4.3E-3	1.0E-4	-2.3E-3	5.5E-5	2.0410	-3.4E-3	6.9E-5	-1.8E-3	3.7E-5
			3.2723	1.4E-1	-2.3E-3	2.3E-1	-3.8E-3	3.3458	1.3E-1	-1.8E-3	2.1E-1	-2.9E-3
			4.0025	-2.3E-3	5.5E-5	-3.8E-3	8.7E-5	3.9909	-1.8E-3	3.7E-5	-2.9E-3	5.8E-5
			.9418	2.6E-1	-4.3E-3	1.4E-1	-2.3E-3	1.6848	2.5E-1	-3.4E-3	1.3E-1	-1.8E-3
.07	9.898	.1021	2.0071	-4.3E-3	1.0E-4	-2.3E-3	5.6E-5	2.0509	-3.4E-3	6.8E-5	-1.8E-3	3.7E-5
			3.2748	1.4E-1	-2.3E-3	2.3E-1	-3.7E-3	3.3385	1.3E-1	-1.8E-3	2.1E-1	-2.9E-3
			4.0024	-2.3E-3	5.6E-5	-3.7E-3	8.7E-5	3.9910	-1.8E-3	3.7E-5	-2.9E-3	5.8E-5

TABLE XXXVI

EXPECTED VALUES, VARIANCES, AND COVARIANCES OF THE MODEL PARAMETERS USING A GENERALIZED NATURAL CONJUGATE PRIOR DISTRIBUTION

$$n = 100, \quad \rho = .5, \quad m = 50, \quad \beta_2 = \beta_1 + \begin{pmatrix} \Delta & 0 \\ \Delta & 0 \end{pmatrix}$$

Δ	E(m|Y)	Var(m|Y)	$E(\underset{\sim}{B}_1 \| Y)$	$Cov(\underset{\sim}{B}_1 \| Y)$				$E(\underset{\sim}{B}_2 \| Y)$	$Cov(\underset{\sim}{B}_2 \| Y)$			
.04	49.701	.3521	1.1712	7.1E-2	-1.1E-3	3.1E-2	-4.9E-4	1.3727	8.0E-2	-1.1E-3	3.5E-2	-5.0E-4
			2.0021	-1.1E-3	2.5E-5	-4.9E-4	1.1E-5	2.0306	-1.1E-3	2.1E-5	-5.0E-4	9.3E-6
			3.0487	3.1E-2	-4.9E-4	5.8E-2	-9.2E-4	3.1859	3.5E-2	-5.0E-4	6.6E-2	-9.4E-4
			3.9995	-4.9E-4	1.1E-5	-9.2E-4	2.0E-5	3.9935	-5.0E-4	9.3E-6	-9.4E-4	1.7E-5
.05	49.880	.1716	1.1695	7.0E-2	-1.1E-3	3.1E-2	-4.9E-4	1.3873	8.0E-2	-1.1E-3	3.5E-2	-5.0E-4
			2.0022	-1.1E-3	2.5E-5	-4.9E-4	1.1E-5	2.0406	-1.1E-3	2.1E-5	-5.0E-4	9.3E-6
			3.0476	3.1E-2	-4.9E-4	5.8E-2	-9.2E-4	3.1877	3.5E-2	-5.0E-4	6.6E-2	-9.4E-4
			3.9995	-4.9E-4	1.1E-5	-9.2E-4	2.0E-5	3.9935	-5.0E-4	9.3E-6	-9.4E-4	1.7E-5
.06	49.975	.0523	1.1681	7.0E-2	-1.1E-3	3.1E-2	-4.9E-4	1.4013	7.9E-2	-1.1E-3	3.5E-2	-4.9E-4
			2.0022	-1.1E-3	2.4E-5	-4.9E-4	1.1E-5	2.0506	-1.1E-3	2.1E-5	-4.9E-4	9.2E-6
			3.0470	3.1E-2	-4.9E-4	5.8E-2	-9.2E-4	3.1889	3.5E-2	-4.9E-4	6.6E-2	-9.4E-4
			3.9995	-4.9E-4	1.1E-5	-9.2E-4	2.0E-5	3.9935	-4.9E-4	9.2E-6	-9.4E-4	1.7E-5
.07	49.999	.0113	1.1672	7.0E-2	-1.1E-3	3.1E-2	-4.9E-4	1.4125	7.9E-2	-1.1E-3	3.5E-2	-4.9E-4
			2.0022	-1.1E-3	2.4E-5	-4.9E-4	1.1E-5	2.0606	-1.1E-3	2.1E-5	-4.9E-4	9.2E-6
			3.0468	3.1E-2	-4.9E-4	5.8E-2	-9.2E-4	3.1892	3.5E-2	-4.9E-4	6.6E-2	-9.4E-4
			3.9994	-4.9E-4	1.1E-5	-9.2E-4	2.0E-5	3.9935	-4.9E-4	9.2E-6	-9.4E-4	1.7E-5

TABLE XXXVII

POSTERIOR PROBABILITY MASS FUNCTION OF m
WHEN $\rho = -.5$ AND $q = .1$

$$\beta_2 = \beta_1 + \begin{pmatrix} \Delta & 0 \\ \Delta & 0 \end{pmatrix}$$

m \ Δ	.00	.03	.06	.09
1	.00171	.00093	.00001	.00000
2	.00018	.00037	.00000	.00000
3	.00010	.00016	.00000	.00000
4	.00004	.00064	.00010	.00000
5	.00012	.00054	.00040	.00000
6	.00012	.00120	.00190	.00000
7	.00016	.00130	.00175	.00000
8	.00018	.00125	.10652	.00723
9	.00024	.00137	.17600	.02388
10	.00024	.00138	.70322	.96889
11	.00004	.00080	.00081	.00000
12	.00005	.00043	.00020	.00000
13	.00005	.00150	.00031	.00000
14	.00007	.00957	.00077	.00000
15	.00008	.00890	.00070	.00000
16	.00022	.00108	.00003	.00000
17	.00016	.00145	.00001	.00000
18	.00161	.00262	.00002	.00000
19	.00833	.02034	.00007	.00000
20	.98629	.94417	.00719	.00000

TABLE XXXVIII

POSTERIOR PROBABILITY MASS FUNCTION OF m
WHEN $\rho = -.5$ AND $q = .5$

$$\beta_2 = \beta_1 + \begin{pmatrix} \Delta & 0 \\ \Delta & 0 \end{pmatrix}$$

m \ Δ	.00	.03	.06	.09
1	.00019	.00011	.00001	.00000
2	.00002	.00004	.00000	.00000
3	.00001	.00002	.00000	.00000
4	.00000	.00008	.00009	.00000
5	.00001	.00006	.00037	.00000
6	.00001	.00014	.00180	.00000
7	.00002	.00015	.00166	.00000
8	.00002	.00015	.10073	.00723
9	.00003	.00016	.16643	.02388
10	.00003	.00016	.66497	.96887
11	.00000	.00009	.00076	.00000
12	.00001	.00005	.00019	.00000
13	.00001	.00017	.00029	.00000
14	.00001	.00112	.00073	.00000
15	.00001	.00104	.00067	.00000
16	.00003	.00013	.00002	.00000
17	.00002	.00017	.00001	.00000
18	.00018	.00031	.00002	.00000
19	.00094	.00238	.00007	.00000
20	.99846	.99347	.06119	.00001

TABLE XXXIX

POSTERIOR PROBABILITY MASS FUNCTION OF m
WHEN $\rho = -.5$ AND $q = .9$

$$\beta_2 = \beta_1 + \begin{pmatrix} \Delta & 0 \\ \Delta & 0 \end{pmatrix}$$

m \ Δ	.00	.03	.06	.09
1	.00002	.00001	.00000	.00000
2	.00000	.00000	.00000	.00000
3	.00000	.00000	.00000	.00000
4	.00000	.00001	.00006	.00000
5	.00000	.00001	.00025	.00000
6	.00000	.00002	.00121	.00000
7	.00000	.00002	.00111	.00000
8	.00000	.00002	.06762	.00723
9	.00000	.00002	.11174	.02387
10	.00000	.00002	.44644	.96876
11	.00000	.00001	.00051	.00000
12	.00000	.00001	.00013	.00000
13	.00000	.00002	.00019	.00000
14	.00000	.00012	.00049	.00000
15	.00000	.00012	.00045	.00000
16	.00000	.00001	.00002	.00000
17	.00000	.00002	.00001	.00000
18	.00002	.00003	.00001	.00000
19	.00010	.00027	.00004	.00000
20	.99983	.99927	.36971	.00013

TABLE XL

POSTERIOR PROBABILITY MASS FUNCTION OF m
WHEN $\rho = 0$ AND $q = .1$

$$\beta_2 = \beta_1 + \begin{pmatrix} \Delta & 0 \\ \Delta & 0 \end{pmatrix}$$

m \ Δ	.00	.03	.06	.09
1	.00171	.00071	.00008	.00000
2	.00018	.00026	.00009	.00000
3	.00010	.00013	.00006	.00000
4	.00004	.00010	.00065	.00000
5	.00012	.00005	.00061	.00000
6	.00012	.00006	.00135	.00001
7	.00016	.00006	.00117	.00001
8	.00018	.00010	.04606	.00711
9	.00024	.00024	.24418	.08121
10	.00024	.00020	.48619	.91154
11	.00004	.00018	.00511	.00002
12	.00005	.00013	.00178	.00000
13	.00005	.00009	.00075	.00000
14	.00007	.00030	.00257	.00000
15	.00008	.00025	.00208	.00000
16	.00022	.00008	.00007	.00000
17	.00016	.00044	.00026	.00000
18	.00161	.00171	.00045	.00000
19	.00833	.02172	.00178	.00000
20	.98629	.97319	.20470	.00009

TABLE XLI

POSTERIOR PROBABILITY MASS FUNCTION OF m
WHEN ρ = 0 AND q = .5

$$\beta_2 = \beta_1 + \begin{pmatrix} \Delta & 0 \\ \Delta & 0 \end{pmatrix}$$

m \ Δ	.00	.03	.06	.09
1	.00019	.00008	.00003	.00000
2	.00002	.00003	.00003	.00000
3	.00001	.00001	.00002	.00000
4	.00000	.00001	.00025	.00000
5	.00001	.00001	.00023	.00000
6	.00001	.00001	.00051	.00001
7	.00002	.00001	.00044	.00001
8	.00002	.00001	.01746	.00710
9	.00003	.00003	.09258	.08115
10	.00003	.00002	.18433	.91092
11	.00000	.00002	.00194	.00002
12	.00001	.00002	.00068	.00000
13	.00001	.00001	.00029	.00000
14	.00001	.00003	.00097	.00000
15	.00001	.00003	.00079	.00000
16	.00003	.00001	.00003	.00000
17	.00002	.00005	.00010	.00000
18	.00018	.00019	.00017	.00000
19	.00094	.00247	.00067	.00000
20	.99846	.99695	.69848	.00077

TABLE XLII

POSTERIOR PROBABILITY MASS FUNCTION OF m WHEN ρ = 0 AND q = .9

$$\beta_2 = \beta_1 + \begin{pmatrix} \Delta & 0 \\ \Delta & 0 \end{pmatrix}$$

m \ Δ	.00	.03	.06	.09
1	.00002	.00001	.00000	.00000
2	.00000	.00000	.00000	.00000
3	.00000	.00000	.00000	.00000
4	.00000	.00000	.00004	.00000
5	.00000	.00000	.00004	.00000
6	.00000	.00000	.00008	.00001
7	.00000	.00000	.00007	.00001
8	.00000	.00000	.00265	.00706
9	.00000	.00000	.01405	.08066
10	.00000	.00000	.02798	.90534
11	.00000	.00000	.00029	.00002
12	.00000	.00000	.00010	.00000
13	.00000	.00000	.00004	.00000
14	.00000	.00000	.00015	.00000
15	.00000	.00000	.00012	.00000
16	.00000	.00000	.00000	.00000
17	.00000	.00001	.00002	.00000
18	.00002	.00002	.00003	.00000
19	.00010	.00028	.00010	.00000
20	.99983	.99966	.95423	.00689

TABLE XLIII

POSTERIOR PROBABILITY MASS FUNCTION OF m
WHEN ρ = .5 AND q = .1

$$\beta_2 = \beta_1 + \begin{pmatrix} \Delta & 0 \\ \Delta & 0 \end{pmatrix}$$

m \ Δ	.00	.03	.06	.09
1	.00171	.00079	.00000	.00000
2	.00018	.00020	.00000	.00000
3	.00010	.00016	.00000	.00000
4	.00004	.00017	.00000	.00000
5	.00012	.00015	.00001	.00000
6	.00012	.00014	.00001	.00000
7	.00016	.00016	.00001	.00000
8	.00018	.00218	.00597	.00019
9	.00024	.01256	.13297	.01007
10	.00024	.01440	.86018	.98974
11	.00004	.00080	.00004	.00000
12	.00005	.00057	.00001	.00000
13	.00005	.00008	.00000	.00000
14	.00007	.00010	.00000	.00000
15	.00008	.00007	.00000	.00000
16	.00022	.00003	.00000	.00000
17	.00016	.00011	.00000	.00000
18	.00161	.00081	.00000	.00000
19	.00833	.02213	.00001	.00000
20	.98629	.94439	.00079	.00000

TABLE XLIV

POSTERIOR PROBABILITY MASS FUNCTION OF m
WHEN ρ = .5 AND q = .5

$$\beta_2 = \beta_1 + \begin{pmatrix} \Delta & 0 \\ \Delta & 0 \end{pmatrix}$$

m \ Δ	.00	.03	.06	.09
1	.00019	.00009	.00000	.00000
2	.00002	.00002	.00000	.00000
3	.00001	.00002	.00000	.00000
4	.00000	.00002	.00000	.00000
5	.00001	.00002	.00001	.00000
6	.00001	.00002	.00001	.00000
7	.00002	.00002	.00001	.00000
8	.00002	.00025	.00593	.00019
9	.00003	.00147	.13213	.01007
10	.00003	.00168	.85475	.98974
11	.00000	.00009	.00004	.00000
12	.00001	.00007	.00001	.00000
13	.00001	.00001	.00000	.00000
14	.00001	.00001	.00000	.00000
15	.00001	.00001	.00000	.00000
16	.00003	.00000	.00000	.00000
17	.00002	.00001	.00000	.00000
18	.00018	.00010	.00000	.00000
19	.00094	.00259	.00001	.00000
20	.99846	.99350	.00709	.00000

TABLE XLV

POSTERIOR PROBABILITY MASS FUNCTION OF m
WHEN ρ = .5 AND q = .9

$$\beta_2 = \beta_1 + \begin{pmatrix} \Delta & 0 \\ \Delta & 0 \end{pmatrix}$$

m \ Δ	.00	.03	.06	.09
1	.00002	.00001	.00000	.00000
2	.00000	.00000	.00000	.00000
3	.00000	.00000	.00000	.00000
4	.00000	.00000	.00000	.00000
5	.00000	.00000	.00001	.00000
6	.00000	.00000	.00001	.00000
7	.00000	.00000	.00001	.00000
8	.00000	.00003	.00562	.00019
9	.00000	.00016	.12504	.01007
10	.00000	.00019	.80887	.98973
11	.00000	.00001	.00004	.00000
12	.00000	.00001	.00001	.00000
13	.00000	.00000	.00000	.00000
14	.00000	.00000	.00000	.00000
15	.00000	.00000	.00000	.00000
16	.00000	.00000	.00000	.00000
17	.00000	.00000	.00000	.00000
18	.00002	.00001	.00000	.00000
19	.00010	.00029	.00001	.00000
20	.99983	.99927	.06039	.00002

TABLE XLVI

BAYESIAN PREDICTIVE DENSITY USING A NATURAL CONJUGATE PRIOR DISTRIBUTION, TWO-STEP AHEAD FORECAST

$$n = 30, \quad \rho = 0, \quad m = 15, \quad \beta_2 = \beta_1 + \begin{pmatrix} \Delta & 0 \\ \Delta & 0 \end{pmatrix}$$

Δ	Actual Future Values		Model Incorporating a Change E(W|Y,V)		Cov($\underset{\sim}{W}$|Y,V)			
					.8335	.1103	-.1279	-.0283
.00	13.547	29.715	14.922	30.862	.1103	.9345	-.0351	-.1097
	40.149	99.566	48.961	98.832	-.1279	-.0351	1.3398	.2277
					-.0283	-.1097	.2277	1.7695
					.8407	.0739	-.1465	-.0129
.03	13.787	29.715	15.171	30.671	.0739	.8051	-.0142	-.1379
	49.899	99.566	49.731	98.965	-.1465	-.0142	1.1505	.1030
					-.0129	-.1379	.1030	1.1127
					.8071	.0728	-.0954	-.0086
.06	14.027	29.715	15.420	30.644	.0728	.7648	-.0086	-.0904
	50.649	99.566	50.547	98.945	-.0954	-.0086	1.1844	.1069
					-.0086	-.0904	.1069	1.1224
					.8042	.0726	-.0949	-.0086
.09	14.267	29.715	15.655	30.644	.0726	.7620	-.0086	-.0900
	51.399	99.566	51.294	98.944	-.0949	-.0086	1.1848	.1069
					-.0086	-.0900	.1069	1.1227

TABLE XLVI (Continued)

Δ	P(m = 15) in the Changing Model	Model Not Incorporating a Change					
		E(W|Y,V)		Cov(W|Y,V)			
				.8058	.0479	-.0939	-.0056
.00	.000729*	15.110	30.879	.0479	.7772	-.0056	-.0906
		49.226	99.047	-.0939	-.0056	1.1996	.0713
				-.0056	-.0906	.0713	1.1572
				1.3171	.0783	.0588	.0035
.03	.110260#	15.275	30.879	.0783	1.2704	.0035	.0567
		49.602	99.047	.0588	.0035	1.1996	.0713
				.0035	.0567	.0713	1.1572
				3.4482	.2049	.2115	.0126
.06	.993152	15.441	30.879	.2049	3.3261	.0126	.2040
		49.978	99.047	.2115	.0126	1.1996	.0713
				.0126	.2040	.0713	1.1572
				7.1991	.4278	.3643	.0216
.09	.999995	15.606	30.879	.4278	6.9441	.0216	.3514
		50.355	99.047	.3643	.0216	1.1996	.0713
				.0216	.3514	.0713	1.1572

*The largest probability occurs at the 1^{st} data point.

#The largest probability occurs at the 11^{th} data point.

TABLE XLVII

BAYESIAN PREDICTIVE DENSITY USING A NATURAL CONJUGATE PRIOR DISTRIBUTION, TWO-STEP AHEAD FORECAST

$$n = 30, \quad \rho = 0, \quad m = 15, \quad \beta_2 = \beta_1 + \begin{pmatrix} \Delta & \Delta \\ \Delta & \Delta \end{pmatrix}$$

Δ	Actual Future Values		Model Incorporating a Change E(W|Y,V)		Cov($\underset{\sim}{W}$|Y,V)			
					.8335	.1103	-.1279	-.0283
.00	13.547	29.715	14.922	30.862	.1103	.9345	-.0351	-.1097
	49.149	99.566	48.961	98.832	-.1279	-.0351	1.3398	.2277
					-.0283	-.1097	.2277	1.7695
					.8276	.0747	-.0799	-.0072
.03	13.787	29.955	15.188	30.883	.0747	.7844	-.0072	-.0752
	49.899	100.316	49.784	99.674	-.0799	-.0072	1.1707	.1058
					-.0072	-.0752	.1058	1.1100
					.8041	.0726	-.0943	-.0085
.06	14.027	30.195	15.420	31.115	.0726	.7620	-.0085	-.0894
	50.649	101.066	50.548	100.438	-.0943	-.0085	1.1829	.1068
					-.0085	-.0894	.1068	1.1209
					.8042	.0726	-.0947	-.0085
.09	14.267	30.435	15.655	31.350	.0726	.7620	-.0085	-.0897
	51.399	101.816	51.294	101.184	-.0947	-.0085	1.1821	.1067
					-.0085	-.0897	.1067	1.1202

TABLE XLVII (Continued)

Δ	P(m = 15) in the Changing Model	Model Not Incorporating a Change E(W|Y,V)		Cov(W|Y,V)			
.00	.000729*	15.110	30.879	.8058	.0479	-.0939	-.0056
		49.226	99.047	.0479	.7772	-.0056	-.0906
				-.0939	-.0056	1.1996	.0713
				-.0056	-.0906	.0713	1.1572
.03	.500198	15.275	31.044	1.3171	.0783	.7194	.0427
		49.602	99.424	.0783	1.2704	.0427	.6939
				.7194	.0427	2.3150	.1376
				.0427	.6939	.1376	2.2330
.06	.999921	15.441	31.210	3.4482	.2049	3.1526	.1873
		49.978	99.800	.2049	3.3261	.1873	3.0409
				3.1526	.1873	5.0502	.3001
				.1873	3.0409	.3001	4.8713
.09	1.000000	15.606	31.375	7.1991	.4278	7.2055	.4282
		50.355	100.176	.4278	6.9441	.4282	6.9503
				7.2055	.4282	9.4052	.5589
				.4282	6.9503	.5589	9.0721

*The largest probability occurs at the 1st data point.

TABLE XLVIII

POSTERIOR PROBABILITY THAT $m_1 = 5$ and $m_2 = 8$ FOR A DOUBLE SHIFT, WHEN THE ACTUAL POINTS OF CHANGE ARE AT FIVE AND EIGHT

$$n = 10, \quad \mu_2 = \mu_1 + \begin{pmatrix} \Delta_1 & 0 \\ \Delta_1 & 0 \end{pmatrix}, \quad \beta_2 = \beta_1 + \begin{pmatrix} \Delta_2 & 0 \\ \Delta_2 & 0 \end{pmatrix}$$

$$\mu_3 = \mu_2 + \begin{pmatrix} \Delta_1 & 0 \\ \Delta_1 & 0 \end{pmatrix}, \quad \beta_3 = \beta_2 + \begin{pmatrix} \Delta_2 & 0 \\ \Delta_2 & 0 \end{pmatrix}$$

Δ_1	Δ_2 \ ρ	-.7	-.5	-.2	0	.2	.5	.7
0.	.04	.58194	.18157*	.01094#	.00426#	.00631#	.02448*	.02457*
	.05	.73772	.32832*	.04000#	.01798#	.02278*	.02558*	.03044*
	.06	.84691	.44814*	.09184*	.04410*	.03548*	.02723*	.04894*
	.07	.91518	.55650	.13868*	.06430*	.03990*	.03331*	.09433*
.2	.04	.48223	.17539*	.01283#	.00525#	.00805#	.03446*	.03412*
	.05	.64941	.30573*	.04576#	.02184#	.02890*	.03385*	.03573*
	.06	.78576	.41471*	.10071*	.05199*	.04361*	.03306*	.04925*
	.07	.87950	.51934	.14632*	.07300*	.04703*	.03713*	.08504*
.4	.04	.33689*	.15480*	.01467#	.00644#	.01003#	.04995*	.05482*
	.05	.48469	.26842*	.05099#	.02628#	.03720*	.05119*	.05075*
	.06	.63772	.36368*	.11065*	.06302*	.05798*	.04720*	.05821*
	.07	.77101	.45867*	.15793*	.08796*	.06157*	.04865*	.08234*
.6	.04	.21943*	.12778*	.01614#	.00772#	.01176#	.06224*	.07822*
	.05	.31999*	.22701*	.05457#	.03048#	.04553#	.07571*	.07051*
	.06	.44200*	.30780*	.12002*	.07582*	.07799*	.06934*	.07153*
	.07	.57968	.38764*	.17221*	.10889*	.08450*	.06732*	.08492*

*The largest probability occurs at (5,9).

#The largest probability occurs at (8,9).

TABLE XLIX

POSTERIOR PROBABILITY THAT $m_1 = 10$ and $m_2 = 16$ FOR A DOUBLE SHIFT, WHEN THE ACTUAL POINTS OF CHANGE ARE AT TEN AND SIXTEEN

$$n = 20, \quad \mu_2 = \mu_1 + \begin{pmatrix} \Delta_1 & 0 \\ \Delta_1 & 0 \end{pmatrix}, \quad \beta_2 = \beta_1 + \begin{pmatrix} \Delta_2 & 0 \\ \Delta_2 & 0 \end{pmatrix}$$

$$\mu_3 = \mu_2 + \begin{pmatrix} \Delta_1 & 0 \\ \Delta_1 & 0 \end{pmatrix}, \quad \beta_3 = \beta_2 + \begin{pmatrix} \Delta_2 & 0 \\ \Delta_2 & 0 \end{pmatrix}$$

Δ_1	Δ_2 \ ρ	-.7	-.5	-.2	0	.2	.5	.7
0.	.04	.42550	.04748 [1]	.00775 [2]	.01015 [2]	.03233 [3]	.36986	.90362
	.05	.66585	.41088	.14354	.15466	.33637	.84151	.97891
	.06	.83892	.63617	.53598	.57703	.73374	.94719	.99442
	.07	.94018	.77531	.71742	.77460	.87173	.97984	.99858
.2	.04	.45383	.05927 [1]	.00948 [2]	.01173 [2]	.03602 [3]	.38118	.90330
	.05	.69924	.44277	.16511	.17163	.35598	.84924	.97999
	.06	.86606	.66377	.56306	.60146	.75055	.95170	.99482
	.07	.95399	.80145	.73861	.79205	.88314	.98207	.99871
.4	.04	.44774	.06004 [1]	.01029 [2]	.01209 [4]	.03480 [3]	.33336	.86196
	.05	.70985	.44774	.16777	.16893	.34092	.83027	.97386
	.06	.87122	.67775	.57136	.60369	.74816	.94804	.99290
	.07	.95523	.81348	.74987	.79939	.88565	.98065	.99811
.6	.04	.39152	.05018 [1]	.00994 [2]	.01118 [4]	.02960 [3]	.24678	.74369
	.05	.69743	.42210	.15138	.14806	.29480	.77694	.95701
	.06	.85782	.67675	.55919	.58171	.72434	.93521	.98759
	.07	.94608	.81213	.75073	.79628	.87930	.97544	.99628

[1] The largest probability occurs at (16,19).

[2] The largest probability occurs at (15,19).

[3] The largest probability occurs at (12,14).

[4] The largest probability occurs at (12,15).

TABLE L

POSTERIOR PROBABILITY THAT $m_1 = 25$ and $m_2 = 40$ FOR A DOUBLE SHIFT, WHEN THE ACTUAL POINTS OF CHANGE ARE AT TWENTY-FIVE AND FORTY

$$n = 50, \quad \mu_2 = \mu_1 + \begin{pmatrix} \Delta_1 & 0 \\ \Delta_1 & 0 \end{pmatrix}, \quad \beta_2 = \beta_1 + \begin{pmatrix} \Delta_2 & 0 \\ \Delta_2 & 0 \end{pmatrix}$$

$$\mu_3 = \mu_2 + \begin{pmatrix} \Delta_1 & 0 \\ \Delta_1 & 0 \end{pmatrix}, \quad \beta_3 = \beta_2 + \begin{pmatrix} \Delta_2 & 0 \\ \Delta_2 & 0 \end{pmatrix}$$

Δ_1	Δ_2 \ ρ	-.7	-.5	-.2	0	.2	.5	.7
0.	.04	.94682	.50640	.24979	.31377	.50830	.90610	.99631
	.05	.99849	.96871	.89904	.91191	.95363	.99629	.99993
	.06	.99995	.99844	.99298	.99372	.99717	.99986	1.00000
	.07	1.00000	.99991	.99950	.99956	.99983	1.00000	1.00000
.2	.04	.94969	.52796	.27128	.33619	.52806	.91144	.99644
	.05	.99857	.97102	.90598	.91784	.95690	.99654	.99993
	.06	.99996	.99856	.99354	.99423	.99740	.99987	1.00000
	.07	1.00000	.99992	.99954	.99960	.99985	1.00000	1.00000
.4	.04	.94654	.53269	.27801	.34160	.52994	.90811	.99574
	.05	.99838	.97048	.90634	.91785	.95638	.99624	.99992
	.06	.99995	.99849	.99344	.99413	.99732	.99986	1.00000
	.07	1.00000	.99992	.99953	.99959	.99984	1.00000	1.00000
.6	.04	.93672	.51999	.26955	.33006	.51419	.89577	.99384
	.05	.99782	.96705	.90027	.91203	.95207	.99529	.99986
	.06	.99992	.99820	.99268	.99342	.99689	.99981	1.00000
	.07	1.00000	.99990	.99946	.99953	.99981	.99999	1.00000

TABLE LI

POSTERIOR PROBABILITY THAT $m_1 = 5$ and $m_2 = 8$ FOR A TEMPORARY SHIFT, WHEN THE ACTUAL POINTS OF CHANGE ARE AT FIVE AND EIGHT

$$n = 10, \quad \mu_2 = \mu_1 + \begin{pmatrix} \Delta_1 & 0 \\ \Delta_1 & 0 \end{pmatrix}, \quad \beta_2 = \beta_1 + \begin{pmatrix} \Delta_2 & 0 \\ \Delta_2 & 0 \end{pmatrix}$$

Δ_1	Δ_2 \ ρ	-.7	-.5	-.2	0	.2	.5	.7
0.	.04	.15791*	.15578*	.01684#	.00688#	.01212#	.22303#	.81844
	.05	.19061*	.22560*	.06556#	.03395#	.07611#	.62857	.94249
	.06	.24619*	.27976*	.18792#	.13322#	.27791#	.84649	.97873
	.07	.33181*	.33745*	.35426	.33037	.54405	.93119	.99144
.2	.04	.17183*	.16383*	.01763#	.00722#	.01272#	.23460#	.84618
	.05	.21752*	.24137*	.06893#	.03559#	.07988#	.65501	.95550
	.06	.28974*	.30367*	.19904#	.14031#	.29254	.86622	.98418
	.07	.39385*	.36962*	.37617	.34837	.56786	.94231	.99375
.4	.04	.19208*	.17177*	.01831#	.00756#	.01330#	.23947#	.85716
	.05	.25166*	.25866*	.07160#	.03698#	.08275#	.66986	.96167
	.06	.33990*	.32950*	.20842#	.14604#	.30362	.87846	.98679
	.07	.45832*	.40312*	.39584	.36346	.58655	.94927	.99484
.6	.04	.21751*	.17888*	.01884#	.00789#	.01384#	.23756#	.85451
	.05	.29071*	.27656*	.07343#	.03806#	.08459#	.67371	.96327
	.06	.39219*	.35609*	.21551#	.15013#	.31068	.88457	.98765
	.07	.51872	.43639*	.41247	.37507	.59991	.95314	.99521

*The largest probability occurs at (5,9).

#The largest probability occurs at (8,9).

TABLE LII

POSTERIOR PROBABILITY THAT m_1 = 10 and m_2 = 16 FOR A TEMPORARY SHIFT, WHEN THE ACTUAL POINTS OF CHANGE ARE AT TEN AND SIXTEEN

$$n = 20, \quad \mu_2 = \mu_1 + \begin{pmatrix} \Delta_1 & 0 \\ \Delta_1 & 0 \end{pmatrix}, \quad \beta_2 = \beta_1 + \begin{pmatrix} \Delta_2 & 0 \\ \Delta_2 & 0 \end{pmatrix}$$

Δ_1	Δ_2 \ ρ	-.7	-.5	-.2	0	.2	.5	.7
0.	.04	.47286	.10258*	.05053*	.11159*	.33558	.75789	.92070
	.05	.68842	.45743	.32417	.44088	.62907	.86186	.97142
	.06	.85173	.63224	.56132	.62960	.74621	.92813	.99098
	.07	.94589	.76406	.67559	.72872	.82886	.96642	.99736
.2	.04	.48531	.10367*	.05390*	.12035*	.35491	.77432	.93168
	.05	.71035	.46912	.33582	.45575	.64396	.87448	.97626
	.06	.86756	.64982	.57474	.64310	.75972	.93634	.99272
	.07	.95298	.78076	.69006	.74216	.84069	.97091	.99792
.4	.04	.49061	.10039*	.05588*	.12647*	.36847	.78800	.93925
	.05	.72736	.47679	.34313	.46697	.65677	.88477	.97928
	.06	.87825	.66504	.58662	.65533	.77188	.94264	.99372
	.07	.95713	.79439	.70315	.75432	.85097	.97418	.99822
.6	.04	.48538	.09308*	.05628*	.12952*	.37593	.79876	.94391
	.05	.73918	.47929	.34577	.47434	.66737	.89282	.98089
	.06	.88447	.67754	.59671	.66612	.78256	.94725	.99417
	.07	.95894	.80497	.71469	.76510	.85970	.97642	.99833

*The largest probability occurs at (6,7).

TABLE LIII

POSTERIOR PROBABILITY THAT m_1 = 25 and m_2 = 40 FOR A TEMPORARY SHIFT, WHEN THE ACTUAL POINTS OF CHANGE ARE AT TWENTY-FIVE AND FORTY

$$n = 50, \quad \mu_2 = \mu_1 + \begin{pmatrix} \Delta_1 & 0 \\ \Delta_1 & 0 \end{pmatrix}, \quad \beta_2 = \beta_1 + \begin{pmatrix} \Delta_2 & 0 \\ \Delta_2 & 0 \end{pmatrix}$$

Δ_1	Δ_2 \ ρ	-.7	-.5	-.2	0	.2	.5	.7
0.	.04	.98685	.89268	.77420	.79054	.88121	.98833	.99939
	.05	.99958	.99547	.98808	.99009	.99540	.99960	.99999
	.06	.99999	.99979	.99939	.99950	.99977	.99998	1.00000
	.07	1.00000	.99999	.99996	.99997	.99999	1.00000	1.00000
.2	.04	.98711	.89717	.77816	.79082	.87970	.98805	.99938
	.05	.99958	.99558	.98835	.99019	.99539	.99959	.99999
	.06	.99999	.99979	.99940	.99951	.99977	.99998	1.00000
	.07	1.00000	.99999	.99996	.99997	.99999	1.00000	1.00000
.4	.04	.98699	.90017	.78029	.78963	.87712	.98748	.99934
	.05	.99956	.99559	.98842	.99012	.99530	.99958	.99999
	.06	.99999	.99978	.99940	.99950	.99977	.99998	1.00000
	.07	1.00000	.99999	.99996	.99997	.99999	1.00000	1.00000
.6	.04	.98650	.90175	.78062	.78698	.87345	.98661	.99926
	.05	.99952	.99550	.98829	.98988	.99511	.99955	.99999
	.06	.99999	.99977	.99938	.99949	.99976	.99998	1.00000
	.07	1.00000	.99999	.99996	.99997	.99999	1.00000	1.00000

VITA

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