CHARACTER FORMULA OF OUTER PLETHYSMS

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PREFACE

It has been known [2], [6] that for each $n \geq 1$, the Grothendieck representation ring $R(S_n)$ of the symmetric group S_n forms a special λ-ring with respect to the exterior power. Since the Hopf ring H of the symmetric function in infinite number of variables is a free λ-ring on the first elementary symmetric function a_1, the graded Hopf ring $R(S) = \{R(S_n)\}$ is also a special λ-ring by pulling back the λ-structure on H through the Frobenius isomorphism $F: R(S) \longrightarrow H$. However, it has remained to be answered what λ-structure on R(S) is, in fact, compatible with F (see p. 135 [6]). In this paper it is shown that the desired structure is derived from the outer plethysm [5].

In Chapter I, it is shown that the divided polynomial Hopf ring $C = \{C(S_n)\}$ of integer valued class functions defined on the symmetric groups is a special ψ-Hopf ring. In Chapter II, a λ-ring structure is introduced in R(S) in terms of outer plethysms so that R(S) forms a ψ-Hopf ring. The character formula of outer plethysms is given in Chapter III, by taking advantage of this formula it is shown that R(S) forms a special ψ-Hopf ring with respect to the Adams operations derived from outer plethysms. In Chapter IV, we answer a question raised by Knutson. A summary of the results in Chapter V concludes this thesis.

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CHAPTER I

THE SPECIAL ψ-HOPF RING C

The aim of this chapter is to show that the divided polynomial Hopf ring $C = \{C(S_n)\}$ of integer valued class functions defined on the symmetric groups S_n is a special ψ-Hopf ring.

Definition 1.1 Let K be a commutative ring with unity. A Hopf algebra A is a K-module A which is both a K-algebra with product $m: A \otimes A \longrightarrow A$ with unit $\eta: K \longrightarrow A$ and a K-coalgebra with comultiplication $\Delta: A \longrightarrow A \otimes A$ with co-unit $\varepsilon: A \longrightarrow K$ such that:

(i) $\eta: K \longrightarrow A$ is a homomorphism of coalgebras.

(ii) $\varepsilon: A \longrightarrow K$ is a homomorphism of algebras.

(iii) $\Delta: A \longrightarrow A \otimes A$ is a homomorphism of algebras.

Definition 1.2 Let K be a commutative ring with unity and let G be a finite group. A K-valued class function is a map $f: G \longrightarrow K$ satisfying $f(a^{-1}ba) = f(b)$ for any $a, b \in G$.

Let S_n denote the symmetric group of degree n. $C(S_n)$ denotes the abelian group of all integer valued class functions on S_n with addition defined by $(f+g)(s) = f(s) + g(s)$ for all $f, g \in C(S_n)$ and $s \in S_n$. Consider the graded connected abelian group $C = \{C(S_n) \mid n = 0,1,2,\dots\}$ where $C(S_0) = Z$. Then it is well known [3] that C forms a graded Hopf ring with respect to the following operations. First we define a multiplication

$m: C \otimes C \longrightarrow C$. Let $i_{p,q}: S_p \times S_q \longrightarrow S_{p+q}$ be an embedding defined by

$$i_{p,q}(\sigma,\tau)(j) = \begin{cases} \sigma(j) & \text{if} \quad 1 \le j \le p, \\ p+\tau(j) & \text{if } p+1 \le j \le p+q, \end{cases}$$

for $(\sigma,\tau) \in S_p \times S_q$. Note that in dividing up the p+q symbols permuted by S_{p+q} into one set of p symbols and another set of q symbols, there are $\binom{p+q}{p}$ ways, but any two such ways of constructing an injection $S_p \times S_q \longrightarrow S_{p+q}$ give conjugate subgroups of S_{p+q}. Hereafter it is assumed that $S_p \times S_q$ is considered as a subgroup embedded by $i_{p,q}$ in S_{p+q}; $i_{p,q}$ induces the restriction map:

$$\mathrm{Res}_{S_p \times S_q}^{S_{p+q}} : C(S_{p+q}) \longrightarrow C(S_p \times S_q)$$

by defining

$$(\mathrm{Res}_{S_p \times S_q}^{S_{p+q}} f)(t) = f(t), \quad \text{for any } f \in C(S_{p+q}) \text{ and for any } t \in S_p \times S_q.$$

Also, we have the induction map:

$$\mathrm{Ind}_{S_p \times S_q}^{S_{p+q}} : C(S_p \times S_q) \longrightarrow C(S_{p+q})$$

defined by

$$(\mathrm{Ind}_{S_p \times S_q}^{S_{p+q}} g)(s) = \frac{1}{p!q!} \sum_{\substack{t \in S_{p+q} \\ t^{-1}st \in S_p \times S_q}} g(t^{-1}st)$$

for any $g \in C(S_p \times S_q)$ and for any $s \in S_{p+q}$.

Let $f_t \in C(S_p)$ and $g_s \in C(S_q)$ be characteristic functions of the conjugacy class $\bar{t}$ in S_p and the class $\bar{s}$ in S_q, respectively, then the characteristic function h of the conjugacy class $\overline{(t,s)}$ in $S_p \times S_q$ is defined by $h(\sigma,\tau) = f_t(\sigma) \cdot g_s(\tau)$. The characteristic functions of the conjugacy classes of S_p form a base for $C(S_p)$. Hence we have an isomorphism

$$\psi_{p,q}\colon C(S_p) \otimes C(S_q) \xrightarrow{\simeq} C(S_p \times S_q).$$

Define the multiplication $m_{p,q}\colon C(S_p) \otimes C(S_q) \longrightarrow C(S_{p+q})$ by the composition

$$\begin{array}{ccc} C(S_p) \otimes C(S_q) & \xrightarrow{m_{p,q}} & C(S_{p+q}) \\ {\scriptstyle \psi_{p,q}} \searrow & & \nearrow {\scriptstyle \mathrm{Ind}_{S_p \times S_q}^{S_{p+q}}} \\ & C(S_p \times S_q) & \end{array}$$

$m_{p,q} = \mathrm{Ind}_{S_p \times S_q}^{S_{p+q}} \circ \psi_{p,q}$.

Setting $\Delta_{p,q}\colon C(S_n) \longrightarrow C(S_p) \otimes C(S_q)$, for each p, q with $p+q=n$ by $\Delta_{p,q} = \psi_{p,q}^{-1} \circ \mathrm{Res}_{S_p \times S_q}^{S_{p+q}}$, we define a comultiplication $\Delta_n\colon C(S_n) \longrightarrow \sum_{p+q=n} C(S_p) \otimes C(S_p)$ by $\Delta_n = \sum_{p+q=n} \Delta_{p,q}$. We also define a unit map $\eta\colon Z \longrightarrow C$ by the injection onto $C(S_0) = Z$, and a co-unit map $\varepsilon\colon C \longrightarrow Z$ by the projection of C onto $C(S_0)$.

Now we introduce some notations on partitions.

<u>Definition 1.3</u> If n is a positive integer, then a <u>partition π of n</u> (in notation, $\pi \vdash n$) is a sequence $\pi = \{n_1, n_2, \ldots, n_k\}$ such that $n = n_1 + n_2 + \ldots + n_k$, $n_i > 0$. If π_1 of the n's are equal to 1, π_2 are equal to 2, etc., this partition is denoted by $\pi = \{1^{\pi_1}, 2^{\pi_2}, \ldots, n^{\pi_n}\}$, where π_i's are non-negative integers.

The length of π, denoted by $\ell(\pi)$ is defined by $\ell(\pi) = \sum_{i=1}^{n} \pi_i$. $\pi!$

stands for $\prod_{i=1}^{n} \pi_i!$ and $|\pi| = \pi! \prod_{j=1}^{n} j^{\pi_j}$.

An element $s \in S_n$ is said to have shape π if the disjoint cycle decomposition of s produces the partition π. A conjugacy class in S_n is said to have shape π if a representative has shape π.

Let K_π be the characteristic function of the conjugacy class of shape π; that is K_π maps a conjugacy class in S_n of shape π into 1 and all other conjugacy classes in S_n into zero. Then $\{K_\pi \mid n \vdash \pi\}$ is a base for $C(S_n)$. $K_{\{n\}}$ will be denoted by c_n. It is known (for example, see [9]) that $C = \{C(S_n) \mid n = 0,1,2,\ldots\}$ is a divided polynomial Hopf ring with generators $c_1, c_2, \ldots, c_n, \ldots$, where the degree of c_n is 2n.

<u>Definition 1.4</u> By a <u>divided polynomial ring</u> D[x] with one generator x of an even degree, we mean a graded abelian group $\{Zx_n \mid n = 0,1,2,\ldots\}$ with a base $x_0 = 1$, $x_1 = x, x_2, \ldots, x_n, \ldots$, such that the multiplication is given by:

$$x_p \cdot x_q = \frac{(p+q)!}{p!q!} x_{p+q}.$$

Given $\pi = \{1^{\pi_1}, 2^{\pi_2}, \ldots, p^{\pi_p}\} \vdash p$, $\sigma = \{1^{\sigma_1}, 2^{\sigma_2}, \ldots, q^{\sigma_q}\} \vdash q$, and $s \in S_{p+q}$, then

$$\begin{aligned}(K_\pi \cdot K_\sigma)(s) &= (\mathrm{Ind}_{S_p \times S_q}^{S_{p+q}} \circ \psi_{p,q}(K_\pi \otimes K_\sigma))(s) \\ &= \frac{1}{p!q!} \sum_{\substack{t \in S_{p+q} \\ t^{-1}st \in S_p \times S_q}} \psi_{p,q}(K_\pi \otimes K_\sigma)(t^{-1}st) \\ &= \begin{cases} 0 & \text{if the shape of s is not } \pi \vee \sigma \\ \dfrac{1}{p!q!} \cdot \dfrac{p!q!}{|\pi|\,|\sigma|} |\pi \vee \sigma| & \text{if the shape of s is } \pi \vee \sigma. \end{cases}\end{aligned}$$

Thus we have $K_\pi \cdot K_\sigma = \frac{(\pi \vee \sigma)!}{\pi!\sigma!} K_{\pi\vee\sigma}$, where $\pi \vee \sigma = \{1^{\pi_1+\sigma_1}, 2^{\pi_2+\sigma_2}, \ldots\} \vdash p+q$.

For each $\pi \vdash n$, by definition we have

$$\Delta_n(K_\pi)(s,t) = \sum_{p+q=n} \psi_{p,q}^{-1} \circ \mathrm{Res}_{S_p \times S_q}^{S_n}(K_\pi)(s,t).$$

Now $\mathrm{Res}_{S_p \times S_q}^{S_n}$ takes value 1 on conjugacy classes with shape π in the embedded subgroup $S_p \times S_q$ of S_n and takes the value 0 otherwise. An element $(s,t) \in S_p \times S_q$ with s and t having shape σ and τ, respectively, is embedded by $i_{p,q}$ as an element with shape $\sigma \vee \tau$, and conversely. Hence

$$\Delta_n(K_\pi) = \sum_{\sigma\vee\tau=\pi} K_\sigma \otimes K_\tau .$$

Definition 1.5 A commutative Hopf ring $R = \{R_n\}$ of even gradings is called a ψ-Hopf ring if there exists a sequence of operations $\psi^k: R_n \longrightarrow R_{kn}$ for $k \geq 1$, satisfying (1) $R_0 = Z$, (2) $\psi^1(x) = x$, and (3) $\psi^k(x+y) = \psi^k(x) + \psi^k(y)$ for $x, y \in R_n$.

Definition 1.6 A special ψ-Hopf ring is a ψ-Hopf ring $R = \{R_n\}$ which satisfies additional conditions:

(1) $\psi^k(1) = 1$,

(2) $\psi^k(xy) = \psi^k(x)\cdot\psi^k(y)$,

(3) $\Delta\psi^k(x) = (\psi^k \otimes \psi^k)\Delta(x)$, and

(4) $\psi^\ell(\psi^k(x)) = \psi^{\ell k}(x)$

for $\ell, k \geq 1$, where Δ is the comultiplication for R. The condition (3) states that $\psi^k: R \longrightarrow R$ is a coalgebra morphism in the sense that the following diagram is commutative

$$\begin{array}{ccc} R & \xrightarrow{\psi^k} & R \\ \Delta \downarrow & & \downarrow \Delta \\ R\otimes R & \xrightarrow{\psi^k \otimes \psi^k} & R\otimes R \end{array}$$

Now we are going to make $C = \{C(S_n) \mid n = 0,1,2,\ldots\}$ a special ψ-Hopf ring. For $k \geq 1$, define

$$\psi^k : C(S_n) \longrightarrow C(S_{kn})$$

by $\psi^k(1) = 1$, $\psi^k(K_\pi) = k^{\ell(\pi)} K_{k\pi}$, and extend linearly, where $k\pi = \{k^{\pi_1}, (2k)^{\pi_2}, \ldots, (nk)^{\pi_n}\}$. Note that $k\pi \vdash kn$, since $\pi_1 k + \pi_2(2k) + \ldots + \pi_n(nk) = kn$, for any $\pi \vdash n$.

<u>Proposition 1.7</u> The divided polynomial Hopf ring $C = \{C(S_n)\}$ is a special ψ-Hopf ring.

<u>Proof</u>. It is sufficient to show that the operations ψ^k satisfies the above axioms for basis elements.

$$\psi^k(K_\pi \cdot K_\sigma) = \psi^k\left(\frac{(\pi\vee\sigma)!}{\pi!\sigma!} K_{\pi\vee\sigma}\right) = k^{\ell(\pi\vee\sigma)} \frac{(\pi\vee\sigma)!}{\pi!\sigma!} K_{k(\pi\vee\sigma)}$$

$$= k^{\ell(\pi)+\ell(\sigma)} \frac{(k\pi\vee k\sigma)!}{(k\pi)!(k\sigma)!} K_{k\pi\vee k\sigma} = \psi^k(K_\pi)\cdot\psi^k(K_\sigma).$$

$$\psi^k(\psi^\ell(K_\pi)) = \psi^k(\ell^{\ell(\pi)} K_{\ell\pi}) = \ell^{\ell(\pi)}\psi^k(K_{\ell\pi}) = (\ell k)^{\ell(\pi)} K_{(k\ell)\pi} = \psi^{k\ell}(K_\pi).$$

It remains to show that ψ^k is a coalgebra morphism. Consider the following diagram

$$\begin{array}{ccc} C & \xrightarrow{\psi^k} & C \\ \Delta \downarrow & & \downarrow \Delta \\ C\otimes C & \xrightarrow{\psi^k \otimes \psi^k} & C\otimes C \end{array}$$

$$(\psi^k \otimes \psi^k) \circ \Delta(K_\pi) = (\psi^k \otimes \psi^k)\Big(\sum_{\sigma\vee\tau=\pi} K_\sigma \otimes K_\tau\Big) = \sum_{\sigma\vee\tau=\pi} \psi^k(K_\sigma) \otimes \psi^k(K_\tau)$$

$$= \sum_{\sigma\vee\tau=\pi} k^{\ell(\sigma)+\ell(\tau)} K_{k\sigma} \otimes K_{k\tau} = k^{\ell(\pi)} \sum_{k\sigma\vee k\tau=k\pi} K_{k\sigma} \otimes K_{k\tau}$$

$$= k^{\ell(\pi)} \sum_{\sigma'\vee\tau'=k\pi} K_{\sigma'} \otimes K_{\tau'} = k^{\ell(\pi)} \Delta(K_{k\pi}) = \Delta \circ \psi^k(K_\pi).$$

Hence the proof is complete.

For each $k \geq 0$, define $\sigma^k: C(S_n) \otimes Q \longrightarrow C(S_{kn}) \otimes Q$ by $k\sigma^k(f) = \sum_{i=1}^{k} \sigma^{k-i}(f)\psi^i(f)$ and $\sigma^0(f)=1$ for any $f \in C(S_n) \otimes Q$ where Q is the field of rationals. Then it is evident that for each $k \geq 1$ and for any $f \in C(S_n) \otimes Q$, we have $\psi^k(f) = (-1)^{k-1} Q_k(\sigma^1(f), \sigma^2(f), \ldots, \sigma^k(f))$, where $Q_k(a_1, a_2, \ldots, a_k)$ is the k-th Newton polynomial in the elementary symmetric functions a_1, $a_2, \ldots, a_k$ of k-variables.

CHAPTER II

OUTER PLETHYSM

In this chapter our objective is to define a λ-structure $\lambda^k\colon R(S_n) \longrightarrow R(S_{kn})$, $k \geq 0$ on the graded Hopf representation ring of symmetric groups $R(S) = \{R(S_n)\}$. It is shown in §3 of this chapter that the desired structure is derived from the outer plethysm. In §1 we introduce basic facts relative to representations of finite groups, and the graded Hopf ring R(S) is constructed. In §2 we define the wreath product of two symmetric groups. In §4 we study the Adams operations $\psi^k\colon R(S_n) \longrightarrow R(S_{kn})$ for $k \geq 1$, and show that R(S) is a ψ-Hopf ring.

§1 Introduction to Representation Theory of Finite Groups

Definition 2.1 Let G be a finite group. A linear representation of G over a complex field $\mathbb{C}$ is a finite dimensional vector space V over $\mathbb{C}$ together with a group homomorphism $\rho\colon G \longrightarrow \operatorname{Aut} V$.

While, strictly speaking, a representation of G is given by a pair (V,ρ), we will often speak of the representation V.

We often write $\rho(g)$ simply as g and $\rho(g)(v) = gv$. If we use this notation then Definition 2.1 implies that for $g,g' \in G$, $v,v' \in V$, and $\alpha \in \mathbb{C}$,

(1) $(g\,g')v = g(g'\,v)$

(2) $g(v + v') = gv + gv'$,

(3) $g(\alpha v) = \alpha(gv)$,

(4) $1\,v = v$.

Hence the representation V of G can be considered as a left G-module.

Definition 2.2 A map of G-modules $f\colon (V,\rho) \longrightarrow (W,\rho')$ is a linear map of vector spaces $f\colon V \longrightarrow W$ such that $f(gv) = gf(v)$ for any $v \in V$ and $g \in G$.

Definition 2.3 Two G-modules are said to be isomorphic if there exists a map of G-modules between them which is also a vector space isomorphism.

Let V and W be two representations of a finite group G. Then the sum of V and W, denoted by $V + W$, is constructed by considering the underlying vector space as the direct sum $V \oplus W$ on which $g \in G$ acts diagonally, that is $g(v,w) = (gv,gw)$ for any $(v,w) \in V \oplus W$.

Let F(G) be the free abelian group generated by the totality of isomorphism classes of G-modules. Let N(G) be the subgroup generated by all elements of the form $[V + W] - [V] - [W]$. Define $R(G) = \frac{F(G)}{N(G)}$. Then R(G) is called the representation group of G or the Grothendieck group of G.

Definition 2.4 A G-module V is said to be reducible if $V = \{0\}$ or if there is a subspace W of V such that W is stable under G (i.e., $GW \subset W$), with $W \neq \{0\}$ and $W \neq V$. If V is not reducible it is said to be irreducible.

An inner product

$$\langle\ ,\ \rangle\colon R(G) \times R(G) \longrightarrow Z,$$

is determined by its value on representations V and W

$$\langle V,W\rangle = \dim_{\mathbb{C}} \mathrm{Hom}_G (V,W).$$

An immediate consequence of Schur's Lemma [7] is that if V and W are irreducible representations of G then

$$\langle V,W\rangle = \begin{cases} 0 & \text{if } V \not\simeq W \\ 1 & \text{if } V \simeq W \end{cases}.$$

Proposition 2.5 The set of isomorphism classes of irreducible representations of G form an orthonormal basis for R(G) with respect to the inner product $\langle\ ,\ \rangle$. The rank of R(G) equals the number of conjugacy classes of G.

The proposition is a basic fact whose proof can be found in any standard textbook on the representation theory.

Let H be a subgroup of a group G. H acts on G as a set of permutations by $h(g) = gh$. Let $\mathbb{C}G$ be the vector space with basis the elements of G. This action of H makes $\mathbb{C}G$ a right H-module. Hence we have

Definition 2.6 Let H be a subgroup of G. Let V be a representation of H. Then the induced representation $\mathrm{Ind}_H^G V$ of G is defined as the vector space $\mathbb{C}G \otimes V$, modulo the subspace generated by all elements of the form $gh \otimes v - g \otimes hv$. G acts on $\mathrm{Ind}_H^G V$ by, for $g_1 \in G$, $g_1(g \otimes v) = (g_1 g) \otimes v$.

For use in later discussion, we state without proof (see for example [7])

Proposition 2.7 (Frobenius Reciprocity Formula) Let H be a subgroup of a finite group G. Let V and W be two complex representations of

G and H respectively. Then

$$\langle \mathrm{Ind}_H^G W, V\rangle = \langle W, \mathrm{Res}_H^G V\rangle .$$

<u>Proposition 2.8</u> (Induction is Transitive) If $H \subset H' \subset G$ are groups, and X is an H-module, then

$$\mathrm{Ind}_{H'}^G(\mathrm{Ind}_H^{H'} X) \simeq \mathrm{Ind}_H^G X .$$

<u>Proof</u>: By associativity of the tensor product, together with the natural isomorphism $\mathbb{C}G \otimes_{H'} \mathbb{C}H' \simeq \mathbb{C}G$ we have

$$\mathrm{Ind}_{H'}^G(\mathrm{Ind}_H^{H'} X) = \mathbb{C}G \otimes_{H'} (\mathbb{C}H' \otimes_H X) \simeq (\mathbb{C}G \otimes_{H'} \mathbb{C}H') \otimes_H X \simeq \mathbb{C}G \otimes_H X = \mathrm{Ind}_H^G X$$

as $\mathbb{C}$-modules. The isomorphisms all commute with the action of G from the left. This completes the proof.

<u>Proposition 2.9</u> If $H \subset G$ and $H' \subset G'$ are groups, X is an H-module and Y is an H'-module, then

$$(\mathrm{Ind}_H^G X) \otimes (\mathrm{Ind}_{H'}^{G'} Y) \simeq \mathrm{Ind}_{H\times H'}^{G\times G'}(X \otimes Y).$$

<u>Proof</u>: We may identify $\mathbb{C}(G \times G')$ with $\mathbb{C}G \otimes \mathbb{C}G'$ as right $H \times H'$-modules. Define a map

$$\phi: \mathbb{C}(G \times G') \otimes_{H\times H'} (X \otimes Y) \longrightarrow (\mathbb{C}G \otimes_H X) \otimes_{\mathbb{C}} (\mathbb{C}G' \otimes_{H'} Y)$$

by $\phi(g \otimes g' \otimes x \otimes y) = g \otimes x \otimes g' \otimes y$. ϕ is well defined since for any $(h,h') \in H \times H'$,

$$\phi(gh \otimes g'h' \otimes x \otimes y) = gh \otimes x \otimes g'h' \otimes y = g \otimes hx \otimes g' \otimes h'y = \phi(g \otimes g' \otimes hx \otimes h'y).$$

It is a routine to show that ϕ is a $G\times G'$-module isomorphism. This completes the proof.

Let $R(S_n)$ be the Grothendieck group of the symmetric group S_n. Consider the graded connected abelian group $R(S) = \{R(S_n) \mid n=0,1,2,\ldots\}$ with even gradings, where $R(S_0) = Z$. As in the case of the graded abelian group of class functions $C = \{C(S_n)\}$, we define multiplication

$$m_{p,q}: R(S_p) \otimes R(S_q) \longrightarrow R(S_{p+q})$$

by the composition

$$\begin{array}{ccc} R(S_p) \otimes R(S_q) & \xrightarrow{\;m_{p,q}\;} & R(S_{p+q}) \\ {\scriptstyle \psi_{p,q}} \searrow \wr & & \nearrow {\scriptstyle \mathrm{Ind}_{S_p \times S_q}^{S_{p+q}}} \\ & R(S_p \times S_q) & \end{array}$$

$m_{p,q} = \mathrm{Ind}_{S_p \times S_q}^{S_{p+q}} \circ \psi_{p,q}$, where for any irreducible representations M and N of S_p and S_q, respectively,

$$\psi_{p,q}: R(S_p) \otimes R(S_q) \xrightarrow{\;\cong\;} R(S_p \times S_q)$$

given by $\psi_{p,q}([M]\otimes[N]) = [M\otimes N]$ is an abelian group isomorphism. A comultiplication $\Delta_n: R(S_n) \longrightarrow \sum_{p+q=n} R(S_p)\otimes R(S_q)$ is defined by $\Delta_n = \sum_{p+q=n} \psi_{p,q}^{-1} \circ \mathrm{Res}_{S_p \times S_q}^{S_{p+q}}$. It is known (for example, see [4], [9] or [10]) that R(S) forms a graded Hopf ring with respect to these operations where the unit map $\eta: Z \longrightarrow R(S)$ is defined by injection onto $R(S_0)$, and the co-unit map $\varepsilon: R(S) \longrightarrow Z$ is defined by projection of R(S) onto $R(S_0)$.

§2 Wreath Products

The wreath product of S_n by S_k, denoted by $S_k[S_n]$, is constructed as follows: Let $S_k[S_n] = S_n^k \times S_k = \{(g_1,\ldots,g_k;h) \mid g_i \in S_n,\ h \in S_k\}$, with multiplication defined by the rule

$$(g_1,\ldots,g_k;h)(g_1',\ldots,g_k';h') = (g_1 g'_{h^{-1}(1)},\ldots,g_k g'_{h^{-1}(k)};hh').$$

Clearly under this multiplication, $S_k[S_n]$ is a group, with $1 = (e_n,\ldots,e_n;e_k)$ and $(g_1,\ldots,g_k;h)^{-1} = (g_{h(1)}^{-1},\ldots,g_{h(k)}^{-1};h^{-1})$, where e_n and e_k are the identity elements of S_n and S_k, respectively.

<u>Remark 2.10</u> By construction, $S_k[S_n]$ is a <u>semi-direct product</u> $S_n^k \times_\theta S_k$, where $\theta\colon S_k \longrightarrow \mathrm{Aut}\,(S_n^k)$ is a group homomorphism given by $\theta(h)((g_1,\ldots,g_k)) = (g_{h^{-1}(1)},\ldots,g_{h^{-1}(k)})$ for $h \in S_k$, $g_i \in S_n$. In other words, the short exact sequence

$$1 \longrightarrow S_n^k \xrightarrow{\alpha} S_k[S_n] \underset{\gamma}{\overset{\beta}{\rightleftarrows}} S_k \longrightarrow 1$$

is split, where

$$\alpha(g_1,\ldots,g_k) = (g_1,\ldots,g_k;e_k),$$

$$\beta(g_1,\ldots,g_k;h) = h, \quad \text{and}$$

$$\gamma(h) = (e_n,\ldots,e_n;h).$$

The map $\phi\colon S_k[S_n] \longrightarrow S_{kn}$ given by

$$\phi((g_1,\ldots,g_k;h)) = \begin{pmatrix} (j-1)n + i \\ (h(j)-1)n + g_{h(j)}(i) \end{pmatrix}_{\substack{1 \le i \le n \\ 1 \le j \le k}}$$

is a canonical embedding of $S_k[S_n]$ into S_{kn}. Hereafter, $S_k[S_n]$ is considered as a subgroup of S_{kn} by the embedding ϕ.

For a representation M of S_n and for $k \geq 1$, the k-th tensor product $M^{\otimes k}$ is a representation of $S_k[S_n]$ with a group action given by

$$(g_1,\dots,g_k;h)(X_1\otimes\dots\otimes X_k) = g_1X_{h^{-1}(1)}\otimes\dots\otimes g_kX_{h^{-1}(k)},$$

where $(g_1,\dots,g_k;h) \in S_k[S_n]$, and $X_i \in M$. The following proposition will be useful in the sequel.

<u>Proposition 2.11</u> Let M_1 and M_2 be S_n-modules. Then for any $k \geq 1$,

$$(M_1 \oplus M_2)^{\otimes k} \simeq \sum_{i=0}^{k} \operatorname{Ind}_{(S_{k-i}\times S_i)[S_n]}^{S_k[S_n]} (M_1^{\otimes(k-i)} \otimes M_2^{\otimes i}).$$

<u>Proof</u>: First note that,

$$(M_1 \oplus M_2)^{\otimes k} \simeq \sum_{i=0}^{k} W_i$$

where $W_i = \sum_{J_i} M_{i_1}\otimes\dots\otimes M_{i_k}$, and the index set $J_i = \{(i_1,\dots,i_k) \mid i_p = 1 \text{ or } 2,\ \sum_\ell i_\ell = 2i\}$. It suffices to show that

$$W_i \simeq \operatorname{Ind}_{(S_{k-i}\times S_i)[S_n]}^{S_k[S_n]} (M_1^{\otimes(k-i)} \otimes M_2^{\otimes i})$$

for each i. By definition

$$\operatorname{Ind}_{(S_{k-i}\times S_i)[S_n]}^{S_k[S_n]} (M_1^{\otimes(k-i)} \otimes M_2^{\otimes i}) = \mathbb{C}(S_k[S_n]) \underset{(S_{k-i}\times S_i)[S_n]}{\otimes} (M_1^{\otimes(k-i)} \otimes M_2^{\otimes i}).$$

Define a map

$$\phi\colon \mathbb{C}(S_k[S_n]) \underset{(S_{k-i}\times S_i)[S_n]}{\otimes} (M_1^{\otimes(k-i)} \otimes M_2^{\otimes i}) \longrightarrow W_i \quad \text{by}$$

$$\phi([g_1,\dots,g_k;h]\otimes(\bigotimes_{\substack{1\leq j\leq k-i\\ k-i+1\leq \ell\leq k}} (X_j\otimes Y_\ell))) = (g_jX_{h^{-1}(j)} \otimes g_\ell Y_{h^{-1}(\ell)}).$$

In order to show that ϕ is well defined let $(g_1',\dots,g_{k-i}',g_{k-i+1}',\dots,g_k',$

$(\sigma,\sigma')) \in (S_{k-i} \times S_i)[S_n]$, and consider,

$$\phi([g_1,\dots,g_k;h](g'_1,\dots,g'_k;(\sigma,\sigma')) \otimes (\bigotimes_{\substack{1\le j\le k-i \\ k-i+1\le \ell\le k}} (X_j \otimes Y_\ell)))$$

$$= \phi([g_1 g'_{h^{-1}(1)},\dots,g_k g'_{h^{-1}(k)};h(\sigma,\sigma')] \otimes (\otimes(X_j \otimes Y_\ell)))$$

$$= \otimes(g_j g'_{h^{-1}(j)} X_{(h\sigma)^{-1}(j)} \otimes g_\ell g'_{h^{-1}(\ell)} Y_{(h\sigma')^{-1}(\ell)}),$$

and,

$$\phi([g_1,\dots,g_k;h] \otimes ((g'_1,\dots,g'_k;(\sigma,\sigma'))(\otimes(X_j \otimes Y_\ell)))$$

$$= \phi([g_1,\dots,g_k;h] \otimes (\otimes(g'_j X_{\sigma^{-1}(j)} \otimes g'_\ell Y_{\sigma'^{-1}(\ell)})))$$

$$= \otimes(g_j g'_{h^{-1}(j)} X_{\sigma^{-1}(h^{-1}(j))} \otimes g_\ell g'_{h^{-1}(\ell)} Y_{\sigma'^{-1}(h^{-1}(\ell))}).$$

Hence ϕ is well defined. It is a routine to show that ϕ is an $S_k[S_n]$-map. Since W_i is a direct sum of $\binom{k}{i}$ copies of M_1 and M_2 which are permuted transitively by $S_k[S_n]$ it follows that ϕ is surjective. However

$$\dim(\mathrm{Ind}_{(S_{k-i}\times S_i)[S_n]}^{S_k[S_n]}(M_1^{\otimes(k-i)} \otimes M_2^{\otimes i})) = \left|\frac{S_k[S_n]}{(S_{k-i} \times S_i)[S_n]}\right| \dim(M_1^{\otimes(k-i)} \otimes M_2^{\otimes i})$$

$$= \binom{k}{i} \dim(M_1^{\otimes(k-i)} \otimes M_2^{\otimes i})$$

$$= \dim(W_i).$$

Hence ϕ is injective. It follows that ϕ is an $S_k[S_n]$-isomorphism. This completes the proof.

§3 Outer Plethysm $\sigma^k([M])$

Let M be an S_n-module, and N an S_k-module. Then $M^{\otimes k} \otimes N$ is an $S_k[S_n]$-module where the group action is given by

$$(g_1,\ldots,g_k;h)(X_1 \otimes \ldots \otimes X_k \otimes Y) = g_1 X_{h^{-1}(1)} \otimes \ldots \otimes g_k X_{h^{-1}(k)} \otimes hY,$$

where $g_i \in S_n$, $h \in S_k$, $X_i \in M$ and $Y \in N$. Because for any $(g_1',\ldots,g_k';h') \in S_k[S_n]$,

$$((g_1',..,g_k';h')(g_1,\ldots,g_k;h))(X_1 \otimes \ldots \otimes X_k \otimes Y)$$

$$= (g_1' g_{h'^{-1}(1)},\ldots,g_k' g_{h'^{-1}(k)};h'h)(X_1 \otimes \ldots \otimes X_k \otimes Y)$$

$$= g_1' g_{h'^{-1}(1)} X_{(h'h)^{-1}(1)} \otimes \ldots \otimes g_k' g_{h'^{-1}(k)} X_{(h'h)^{-1}(k)} \otimes h'hY$$

and

$$(g_1',\ldots,g_k';h')((g_1,\ldots,g_k;h)(X_1 \otimes \ldots \otimes X_k \otimes Y))$$

$$= (g_1',\ldots,g_k';h')(g_1 X_{h^{-1}(1)} \otimes \ldots \otimes g_k X_{h^{-1}(k)} \otimes hY)$$

$$= g_1' g_{h'^{-1}(1)} X_{h^{-1}(h'^{-1}(1))} \otimes \ldots \otimes g_k' g_{h'^{-1}(k)} X_{h^{-1}(h'^{-1}(k))} \otimes h'hY.$$

It is immediate to verify all other conditions.

Definition 2.12 The *outer plethysm* of M and N denoted by $M \odot N$ and is defined by

$$M \odot N = \mathrm{Ind}_{S_k[S_n]}^{S_{kn}} (M^{\otimes k} \otimes N).$$

For $k \geq 1$ and for any S_n-module M define

$$\hat{\sigma}^k(M) = \mathrm{Ind}_{S_k[S_n]}^{S_{kn}} M^{\otimes k} \in R(S_{kn}), \quad \text{and} \quad \hat{\sigma}^0(M) = 1 \in R(S_0).$$

Note that $\hat{\sigma}^k(M) = M \Theta 1_{S_k}$, where 1_{S_k} is the trivial one-dimensional representation of S_k. For any $[M] \in R(S_n)$, we have

$$\hat{\sigma}^k([M]) = [\mathrm{Ind}_{S_k[S_n]}^{S_{kn}} M^{\otimes k}].$$

<u>Proposition 2.13</u> $\hat{\sigma}^k$ satisfies the following:

(1) $\hat{\sigma}^k(\rho_1) = \rho_k$,

(2) $\hat{\sigma}^1([M]) = [M]$,

(3) $\hat{\sigma}^k([M] + [N]) = \sum_{i=0}^{k} \hat{\sigma}^{k-i}([M])\hat{\sigma}^i([N])$,

where $\rho_k = [1_{S_k}]$ and $[M],[N] \in R(S_n)$.

<u>Proof</u>: The first two formulas are immediate from definition. The third formula is verified as follows:

$$\hat{\sigma}^k([M] + [N]) = \hat{\sigma}^k([M \oplus N])$$

$$= [\mathrm{Ind}_{S_k[S_n]}^{S_{kn}} (M \oplus N)^{\otimes k}]$$

by proposition 2.11

$$= \sum_{i=0}^{k} [\mathrm{Ind}_{S_k[S_n]}^{S_{kn}} (\mathrm{Ind}_{S_{k-i}[S_n] \times S_i[S_n]}^{S_k[S_n]} M^{\otimes (k-i)} \otimes N^{\otimes i})]$$

by proposition 2.8

$$= \sum_{i=0}^{k} [\mathrm{Ind}_{S_{k-i}[S_n] \times S_i[S_n]}^{S_{kn}} M^{\otimes (k-i)} \otimes N^{\otimes i}]$$

$$= \sum_{i=0}^{k} [\mathrm{Ind}_{S_{(k-i)n} \times S_{in}}^{S_{kn}} \mathrm{Ind}_{S_{k-i}[S_n] \times S_i[S_n]}^{S_{(k-i)n} \times S_{in}} M^{\otimes (k-i)} \otimes N^{\otimes i}]$$

by proposition 2.9

$$= \sum_{i=0}^{k} [\mathrm{Ind}_{S_{(k-i)n}\times S_{in}}^{S_{kn}} (\mathrm{Ind}_{S_{k-i}[S_n]}^{S_{(k-i)n}} M^{\otimes(k-i)}) \otimes (\mathrm{Ind}_{S_i[S_n]}^{S_{in}} N^{\otimes i})]$$

$$= \sum_{i=0}^{k} \hat{\sigma}^{k-i}([M])\hat{\sigma}^{i}([N]).$$

Hence the proof is complete.

For any S_n representation M, consider a system of linear equations in variables $\lambda^1([M]),\lambda^2([M]),\ldots,\lambda^k([M])$

$$\sum_{i=0}^{k} (-1)^i\hat{\sigma}^{k-i}([M])\lambda^i([M]) = 0,$$

where $\lambda^0([M]) = 1$, $\lambda^0(n) = 1$ and $\lambda^k(n) = 0$ for $k \geq 1$ and for any integer n. By solving this system of equations, we obtain $\lambda^k([M]) \in R(S_{kn})$ for any $k \geq 1$. Let R be a commutative ring with unity 1. Let $1 + R[[t]]^+$ be the totality of formal power series in one variable t, with constant term 1. Then $1 + R[[t]]^+$ is an abelian group with respect to formal power series multiplication.

For any $[M] \in R(S_n)$, let

$$\hat{\sigma}_t([M]) = \sum_{k=0}^{\infty} \hat{\sigma}^k([M])t^k \quad \text{and}$$

$$\lambda_t([M]) = \sum_{k=0}^{\infty} \lambda^k([M])t^k.$$

Then, the defining relations for λ^k is equivalent to the fact that $\lambda_t([M]) = \frac{1}{\hat{\sigma}_{-t}([M])}$, since

$$\lambda_t([M])\hat{\sigma}_{-t}([M]) = \sum_{k=0}^{\infty} \left(\sum_{i=0}^{k} (-1)^i\hat{\sigma}^{k-i}([M])\lambda^i([M])\right)t^k.$$

<u>Proposition 2.14</u> For any $[M],[N] \in R(S_n)$ we have

(1) $\hat{\sigma}_t([M] + [N]) = \hat{\sigma}_t([M])\hat{\sigma}_t([N])$,

(2) $\lambda_t([M] + [N]) = \lambda_t([M])\lambda_t([N])$,

(3) $\lambda^k([M] + [N]) = \sum_{i=0}^{k} \lambda^{k-i}([M])\lambda^i([N])$.

Proof:

(1) $\hat{\sigma}_t([M] + [N]) = \sum_{k=0}^{\infty} \hat{\sigma}^k([M] + [N])t^k$

by Proposition 2.13

$$= \sum_{k=0}^{\infty} (\sum_{i=0}^{k} \hat{\sigma}^{k-i}([M])\hat{\sigma}^i([N]))t^k$$

$$= (\sum_{i=0}^{\infty} \hat{\sigma}^i([M])t^i)(\sum_{j=0}^{\infty} \hat{\sigma}^j([N])t^j)$$

$$= \hat{\sigma}_t([M])\hat{\sigma}_t([N]).$$

(2) $\lambda_t([M] + [N]) = \dfrac{1}{\hat{\sigma}_{-t}([M] + [N])}$

$$= (\frac{1}{\hat{\sigma}_{-t}([M])})(\frac{1}{\hat{\sigma}_{-t}([N])})$$

$$= \lambda_t([M])\lambda_t([N]).$$

(3) It follows immediately from (2).

Hence the proof is complete.

Let $M(S_n) = \{(M,N) \mid M,N \;\; S_n\text{-modules}\}$. Define addition on $M(S_n)$ by

$(M,N) + (M',N') = (M \oplus M',N \oplus N')$.

With respect to this operation $M(S_n)$ is a semi-group. We define an equivalence relation $\sim$ on $M(S_n)$ by $(M,N) \sim (M',N')$ if and only if $M \oplus N' \simeq M' \oplus N$. We denote by $\langle M,N \rangle$ the equivalence class containing (M,N).

Let $\overline{R}(S_n) = M(S_n)/\sim$. $\overline{R}(S_n)$ is a group with $0 = \langle D,D \rangle$ and $-\langle M,N \rangle = \langle N,M \rangle$. It is clear from the construction that the map $h: \overline{R}(S_n) \longrightarrow R(S_n)$ defined by $h(\langle M,N \rangle) = [M] - [N]$ is a group isomorphism. Hereafter, we identify $\overline{R}(S_n)$ with $R(S_n)$ by h.

For each integer $k \geq 1$, we define a map

$$\tilde{\sigma}^k: M(S_n) \longrightarrow R(S_{kn}) \text{ by}$$

$$\tilde{\sigma}^k((M,N)) = \sum_{i=0}^{k} (-1)^i \hat{\sigma}^{k-i}([M])\lambda^i([N]), \text{ and}$$

$$\tilde{\sigma}^0((M,N)) = 1.$$

Because of defining relations for λ^k, it is immediate to see that for any $k \geq 1$,

$$\tilde{\sigma}^k((D,D)) = \sum_{i=0}^{k} (-1)^i \hat{\sigma}^{k-i}([D])\lambda^i([D]) = 0,$$

for any S_n-module D. Then $\tilde{\sigma}^k$ is well defined, because

$$\tilde{\sigma}^k((M \oplus D, N \oplus D)) = \sum_{i=0}^{k} (-1)^i \hat{\sigma}^{k-i}([M] + [D])\lambda^i([N] + [D])$$

by proposition 2.13 and proposition 2.14,

$$= \sum_{i=0}^{k} (-1)^i \{ \sum_{j=0}^{k-i} \hat{\sigma}^{k-i-j}([M])\hat{\sigma}^j([D]) \sum_{\ell=0}^{i} \lambda^{i-\ell}([N])\lambda^{\ell}([D]) \}$$

$$= \sum_{i=0}^{k} \tilde{\sigma}^{k-i}((M,N))\tilde{\sigma}^i((D,D))$$

$$= \tilde{\sigma}^k((M,N)),$$

for all S_n-modules M, N and D. Hence the map $\tilde{\sigma}^k$ preserves equivalence classes; consequently, let

$$\sigma^k: R(S_n) \longrightarrow R(S_{kn}),$$

be the map induced by $\tilde{\sigma}^k$; $\sigma^k([M]-[N]) = \tilde{\sigma}^k((M,N))$ for any $k \geq 0$. If $[M] \in R(S_n)$, then

$$\sigma^k([M]) = \tilde{\sigma}^k((M,0)) = \sum_{i=0}^{k} (-1)^i \hat{\sigma}^{k-i}([M])\lambda^i([0]) = \hat{\sigma}^k([M]).$$

This definition of $\{\sigma^k \mid k \geq 0\}$ is equivalent to the one defined by

$$\sigma_t([M]-[N]) = \hat{\sigma}_t([M])\lambda_{-t}([N]).$$

<u>Proposition 2.15</u> λ^k satisfies the following:

(1) $\lambda^k(\eta_1) = \eta_k$,

(2) $\lambda^1([M]) = [M]$,

(3) $\lambda^k([M]-[N]) = \sum_{i=0}^{k} (-1)^i \lambda^{k-i}([M])\sigma^i([N])$

where $\eta_k \in R(S_k)$ is represented by the sign representation Alt S_k of S_k, and $[M],[N] \in R(S_n)$.

<u>Proof</u>: Since $\sum_{i=0}^{k} (-1)^i \rho_{k-i}\eta_i = 0$ and since $\sigma^{k-i}(\rho_1) = \rho_{k-i}$, we obtain $\lambda^k(\eta_1) = \eta_k$. The rest of the proposition follows from the defining relations. This completes the proof.

<u>Definition 2.16</u> A commutative graded ring $R = \{R_n\}$ of even gradings is called a <u>λ-ring</u> if there exists a set of maps $\lambda^k: R_n \longrightarrow R_{kn}$ for $k \geq 0$,

satisfying the following axioms:

(1) $\lambda^0(x) = 1$,

(2) $\lambda^1(x) = x$ and

(3) $\lambda^k(x+y) = \sum_{i=0}^{k} \lambda^{k-i}(x)\lambda^i(y)$ for $x,y \in R_n$.

Thus we have

Theorem 2.17 The graded Hopf representation ring of symmetric groups $R(S) = \{R(S_n)\}$ is a λ-ring.

§4 Adams Operations

Now, we will investigate certain operations derived from the λ-operations which are easier to handle.

Let R be a λ-ring, we now define operations $\psi^k: R \longrightarrow R$, $k \geq 1$ in terms of λ^k. To do this, set $\psi_t(x) = \sum_{k=1}^{\infty} \psi^k(x)t^k$, for any $x \in R$ and define ψ_t by the formula:

$$\psi_{-t}(x) = \frac{-t \frac{d}{dt}(\lambda_t(x))}{\lambda_t(x)},$$

where $\lambda_t(x) = \sum_{k=0}^{\infty} \lambda^k(x)t^k$.

Proposition 2.18 The map $\psi^k: R \longrightarrow R$ is additive for any $k \geq 1$.

Proof: $\psi_{-t}(x+y) = -t\frac{d}{dt}(\lambda_t(x+y))/\lambda_t(x+y)$

$$= -t[\tfrac{d}{dt}(\lambda_t(x))\lambda_t(y) + \lambda_t(x)\tfrac{d}{dt}(\lambda_t(y))]/\lambda_t(x)\lambda_t(y)$$

$$= -t(\tfrac{d}{dt}(\lambda_t(x)))/\lambda_t(x) - t(\tfrac{d}{dt}(\lambda_t(y)))/\lambda_t(y)$$

$$= \psi_{-t}(x) + \psi_{-t}(y)$$

and therefore, by comparing coefficients, we have $\psi^k(x+y) = \psi^k(x) + \psi^k(y)$. This completes the proof.

If we solve $\psi_{-t}(x) = -t(\frac{d}{dt}(\lambda_t(x)))/\lambda_t(x)$ for $\psi^k(x)$ explicitly then we obtain the following formulae which may serve as a definition of the ψ^k.

$$\psi^1(x) - \lambda^1(x) = 0$$

$$\psi^2(x) - \psi^1(x)\lambda^1(x) + 2\lambda^2(x) = 0$$

$$\vdots$$

$$\psi^k(x) - \psi^{k-1}(x)\lambda^1(x) + \ldots + (-1)^{k-1}\psi^1(x)\lambda^{k-1}(x) + (-1)^k k\lambda^k(x) = 0.$$

If we solve this system of equations for $\psi^k(x)$, we obtain

$$\psi^k(x) = Q_k(\lambda^1(x), \lambda^2(x), \ldots, \lambda^k(x)),$$

where $Q_k(a_1, a_2, \ldots, a_k)$ is the k-th Newton polynomial in the elementary symmetric functions $a_1, a_2, \ldots, a_k$ of k-variables. Note that for any $x \in R$, $\psi^1(x) = \lambda^1(x) = x$. The maps $\psi^k\colon R \longrightarrow R$ are called the Adams ψ-operations in R.

In §3 we have shown that the graded Hopf representation ring $R(S) = \{R(S_n)\}$ is a λ-ring. Hence $R(S)$ is a ψ-ring where for $k \geq 1$, the Adams operation

$$\psi^k\colon R(S_n) \longrightarrow R(S_{nk})$$

is defined by $\psi^k(x) = Q_k(\lambda^1(x), \ldots, \lambda^k(x))$, thus we have

<u>Proposition 2.19</u> The graded Hopf representation ring $R(S) = \{R(S_n)\}$ is a ψ-Hopf ring.

<u>Proposition 2.20</u> $\psi^k(x) = (-1)^{k-1}Q_k(\sigma^1(x),\sigma^2(x),\ldots,\sigma^k(x))$ for any $x \in R(S_n)$.

<u>Proof</u>: Consider,

$$\psi_{-t}(x) = -\lambda\frac{d}{dt}(\lambda_t(x))/\lambda_t(x)$$

$$= t\frac{d}{dt}(\sigma_{-t}(x))/\sigma_{-t}(x),$$

$$(\sum_{i=1}^{\infty}(-1)^i\psi^i(x)t^i)(\sum_{j=0}^{\infty}(-1)^j\sigma^j(x)t^j) = \sum_{k=1}^{\infty}(-1)^k k\sigma^k(x)t^k.$$

Equating coefficients of t^k we obtain the following system of equations:

$$\psi^1(x) - \sigma^1(x) = 0$$

$$\psi^2(x) + \psi^1(x)\sigma^1(x) - 2\sigma^2(x) = 0$$

$$\vdots$$

$$\psi^k(x) + \psi^{k-1}(x)\sigma^1(x) + \ldots + \psi^1(x)\sigma^{k-1}(x) - k\sigma^k(x) = 0.$$

Solving for ψ^k by Cramer's rule we obtain

$$\psi^k(x) = (-1)^{k-1}Q_k(\sigma^1(x),\sigma^2(x),\ldots,\sigma^k(x)).$$

CHAPTER III

CHARACTERS OF OUTER PLETHYSMS

Let χ: $R(S) \to C$ be the character map and let $\rho_\pi = \rho_1^{\pi_1} \rho_2^{\pi_2} \cdots \rho_n^{\pi_n}$ for $\pi \vdash n$ be a basis element in $R(S_n)$. In this chapter we are going to investigate $\chi(\sigma^k(\rho_\pi))$ in terms of a base $\{K_\tau \mid \tau \vdash kn\}$ for $C(S_{kn})$. In §1 we introduce some necessary preliminaries. In §2 we describe the conjugacy classes of a wreath product $S_k[S_n]$. In §3 the character formula of outer plethysms is given. The formula enables us to prove in §4 that $R(S)$ forms a special ψ-Hopf ring with respect to the Adams operations derived from outer plethysms.

§1 Preliminaries

By the usual inner product

$$\langle f,g\rangle = \frac{1}{n!} \sum_{t \in S_n} f(t)g(t^{-1}),$$

for $f,g \in C(S_n) \otimes Q$, the vector space $C(S_n) \otimes Q$ becomes an inner product space over Q.

The Frobenius reciprocity theorem [7] states that for any subgroup H in S_n and for any $f \in C(S_n) \otimes Q$ and $g \in C(H) \otimes Q$,

$$\langle \mathrm{Res}_H^{S_n} f, g\rangle = \langle f, \mathrm{Ind}_H^{S_n} g\rangle.$$

If a bilinear form β is defined on $C \otimes Q$ by the orthogonal sum such that for $f \in C(S_p) \otimes Q$ and $g \in C(S_q) \otimes Q$

$$\beta(f,g) = \begin{cases} 0 & \text{if } p \neq q \\ \langle f,g\rangle & \text{if } p = q \end{cases}$$

then the graded vector space of finite type $C \otimes Q$ becomes an inner product space. By definition,

$$\langle K_\pi, K_{\pi'}\rangle = \frac{1}{n!} \sum_{t \in S_n} K_\pi(t) K_{\pi'}(t^{-1})$$

$$= \begin{cases} 0 & \text{if } \pi \neq \pi' \\ \frac{1}{|\pi|} & \text{if } \pi = \pi' \end{cases},$$

because the cardinality of a conjugacy class of shape π is $\frac{n!}{|\pi|}$.

For each partition $\pi = \{1^{\pi_1}, 2^{\pi_2}, \ldots, n^{\pi_n}\}$ of n, let S_π stand for the subgroup of S_n,

$$S_\pi = \overbrace{S_1 \times \ldots \times S_1}^{\pi_1} \times \ldots \times \overbrace{S_n \times \ldots \times S_n}^{\pi_n}$$

$$= S_1^{\pi_1} \times \ldots \times S_n^{\pi_n}.$$

Then the trivial representation of S_π is denoted by 1_{S_π}, we denote the induced representation by $\rho_\pi = [\mathrm{Ind}_{S_\pi}^{S_n} 1_{S_\pi}]$. It is known that $\{\rho_\pi \mid \pi \vdash n\}$ forms a basis for $R(S_n)$.

Let χ: $R(S) \longrightarrow C$ be the character map. Then it is well known (for example, see [7]) that χ is a ring monomorphism.

§2 The Conjugacy Classes of $S_k[S_n]$

In this section we shall describe the conjugacy classes of a wreath product $S_k[S_n]$. To do this we consider an element $(g_1, g_2, \ldots, g_k; h) \in$

$S_k[S_n]$. Let $h = h_1 \cdots h_\nu \cdots h_{c(h)}$ be a cycle decomposition of h. Then we can associate with the νth cyclic factor h_ν of h a uniquely determined element of S_n denoted by $g(h_\nu)$, which we call the cycle product associated with the cycle h_ν. Let $c_1, c_2, \ldots, c_{p(n)}$ be the conjugacy classes of S_n and let $c(g(h_\nu))$ be the conjugacy class represented by $g(h_\nu)$. If $\ell(h_\nu)$ denotes the length of the cycle h_ν, then we have $\sum_{\nu=1}^{c(h)} \ell(h_\nu) = k$. Define a non-negative integer a_{ij} with $p(n) \geq i \geq 1$ and $k \geq j \geq 1$ by,

$$\sum_\nu \delta_{\ell(h_\nu),j} \cdot \delta(c(g(h_\nu)), c_i),$$

where the first δ denotes the Kronecker delta, and

$$\delta(c(g(h_\nu)), c_i) = \begin{cases} 1 & \text{if } c_i = c(g(h_\nu)) \\ 0 & \text{otherwise} \end{cases} .$$

Then it is immediate to see $\sum_{j=1}^{k} \sum_{i=1}^{p(n)} j \cdot a_{ij} = k$. If M_k denotes the set of all matrices $A = (a_{ij})$ of order $p(n) \times k$ with non-negative integral entries such that $\sum_{j=1}^{k} \sum_{i=1}^{p(n)} j \cdot a_{ij} = k$, a surjective map

$$\alpha \colon S_k[S_n] \longrightarrow M_k$$

is defined by $\alpha((g_1, \ldots, g_k; h))_{ij} = a_{ij}$.

<u>Proposition 3.1</u> Two elements ξ and ε of $S_k[S_n]$ are conjugate if and only if $\alpha(\xi) = \alpha(\varepsilon)$.

This is Theorem 4.2.8 of [5].

<u>Definition 3.2</u> An element ξ of $S_k[S_n]$ is said to be of type $A = (a_{ij})$ if $\alpha(\xi) = A$.

The order of a conjugacy class of $S_k[S_n]$ which is characterized by the type $A = (a_{ij})$ is given by

Proposition 3.3 The class of elements of type $A = (a_{ij})$ in $S_k[S_n]$ is of order

$$\frac{|S_k[S_n]|}{\prod_{i,j} a_{ij}!(jk!/|c_i|)^{a_{ij}}} .$$

This is Lemma 4.2.10 of [5].

For notations's sake, the characteristic map of the conjugacy class consisting of elements of type A will be denoted by K_A, which is a class function of $S_k[S_n]$.

§3 The Character Formula for Outer Plethysms

Let M be an S_n representation. Then $M^{\otimes k}$ is an $S_k[S_n]$ representation. For later purposes it is useful to know the character of the represention $M^{\otimes k}$. If $g(h_\nu)$, $c(h) \geq \nu \geq 1$, are the cycle products of h, we have the following formula

Proposition 3.4 For each $\xi = (g_1,g_2,\ldots,g_k;h) \in S_k[S_n]$,

$$\chi(M^{\otimes k})(\xi) = \prod_{\nu=1}^{c(h)} \chi(M)(g(h_\nu))$$

This is Lemma 4.3.9 of [5].

Let us consider the shape of a conjugacy class of $S_k[S_n]$ in S_{kn}, as is observed in 4.2.17, [5]. As usual, let $P(kn)$ denote the set of the partitions of kn. Consider a map

$$\beta: M_k \longrightarrow P(kn)$$

defined by

$$\beta(A) = \bigvee_{j=1}^{k} \bigvee_{i=1}^{p(n)} a_{ij}*(j\pi_i),$$

where $\pi_i = \{m_1, m_2, \ldots, m_\ell\}$ denotes the shape of the conjugacy class c_i, $j\pi_i = \{jm_1, \ldots, jm_\ell\}$, and $a_{ij}*(j\pi_i)$ stands for the union $\vee$ of a_{ij} number of $j\pi_i$s. Since $a_{ij}*(j\pi_i) \vdash a_{ij}jn$, we have $\sum_{i,j} a_{ij}\, jn = kn$. Hence β is well defined.

Let $\pi = \{n_1, n_2, \ldots, n_\ell\}$ be a partition of n and set $S_\pi = S_{n_1} \times \ldots \times S_{n_\ell}$ be a subgroup of S_n. For each t in a conjugacy class c_i of S_n define $\theta_i(t)$ by the number of elements $s \in S_n$ such that $s^{-1}ts \in S_\pi$. Since $\theta_i(t)$ does not depend upon the choice of t in c_i and only depends upon π, $\theta_i(t)$ will be denoted by $\theta_i(\pi)$.

<u>Definition 3.5</u> For each $\pi \vdash n$ and for each conjugacy class of $S_k[S_n]$ whose type is A, define

$$\Theta(A,\pi) = \prod_{i,j} \left(\frac{\theta_i(\pi)}{|S_\pi|} \right)^{a_{ij}}.$$

<u>Definition 3.6</u> For each conjugacy class of $S_k[S_n]$ whose type is A, define

$$\Phi(A) = \prod_{i,j} a_{ij}!\,(j|\pi_i|)^{a_{ij}},$$

where π_i is the shape of c_i.

Then we have the following

<u>Theorem 3.7</u> For each basis element ρ_π with $\pi \vdash n$ in $R(S_n)$ and for $\tau \vdash kn$ we obtain

$$<\chi\sigma^k(\rho_\pi), K_\tau> = \sum_{\beta(A)=\tau} \frac{\Theta(A,\pi)}{\Phi(A)} .$$

Proof: Consider

$$<\chi\sigma^k(\rho_\pi), K_\tau> = <\mathrm{Ind}_{S_k[S_n]}^{S_{kn}} \chi(\mathrm{Ind}_{S_\pi}^{S_n} 1_{S_\pi})^{\otimes k}, K_\tau>$$

$$= <\chi(\mathrm{Ind}_{S_\pi}^{S_n} 1_{S_\pi})^{\otimes k}, \mathrm{Res}_{S_k[S_n]}^{S_{kn}} K_\tau> = \sum_{\beta(A)=\tau} <\chi(\mathrm{Ind}_{S_\pi}^{S_n} 1_{S_\pi})^{\otimes k}, K_A>.$$

It remains to show that for each A with $\beta(A) = \tau$,

$$<\chi(\mathrm{Ind}_{S_\pi}^{S_n} 1_{S_\pi})^{\otimes k}, K_A> = \frac{\Theta(A,\pi)}{\Phi(A)} .$$

However,

$$<\chi(\mathrm{Ind}_{S_\pi}^{S_n} 1_{S_\pi})^{\otimes k}, K_A> = \frac{1}{|S_k[S_n]|} \sum_{\xi\in S_k[S_n]} \chi(\mathrm{Ind}_{S_\pi}^{S_n} 1_{S_\pi})^{\otimes k}(\xi) K_A(\xi)$$

$$= \frac{1}{|S_k[S_n]|} \sum_{\alpha(\xi)=A} \chi(\mathrm{Ind}_{S_\pi}^{S_n} 1_{S_\pi})^{\otimes k}(\xi).$$

In virtue of Proposition 3.4, we obtain

$$\chi(\mathrm{Ind}_{S_\pi}^{S_n} 1_{S_\pi})^{\otimes k}(\xi) = \prod_{i,j} (\chi(\mathrm{Ind}_{S_\pi}^{S_n} 1_{S_\pi})(t_i))^{a_{ij}},$$

where t_i is a representative of c_i for $p(n) \geq i \geq 1$. By definition,

$$\chi(\mathrm{Ind}_{S_\pi}^{S_n} 1_{S_\pi})(t_i) = \frac{1}{|S_\pi|} \sum_{s\in S_n,\ s^{-1}t_i s\in S_\pi} \chi(1_{S_\pi})(s^{-1}t_i s)$$

$$= \frac{\theta_i(\pi)}{|S_\pi|} .$$

Hence we obtain $\chi(\mathrm{Ind}_{S_\pi}^{S_n} 1_{S_\pi})^{\otimes k}(\xi) = \Theta(A,\pi)$. Note that the order of the conjugacy class $\alpha^{-1}(A)$ is $\frac{|S_k[S_n]|}{\Phi(A)}$ (see Proposition 3.3), it is immediate to see that

$$<\chi(\mathrm{Ind}_{S_\pi}^{S_n} 1_{S_\pi})^{\otimes k}, K_A> = \frac{1}{|S_k[S_n]|} \cdot \frac{|S_k[S_n]|}{\Phi(A)} \cdot \Theta(A,\pi)$$

$$= \frac{\Theta(A,\pi)}{\Phi(A)} .$$

This completes the proof.

<u>Corollary 3.8</u> For any $\pi \vdash n$ we have

$$\chi(\sigma^k(\rho_\pi)) = \sum_{A \in M_k} \frac{\Theta(A,\pi)}{\Phi(A)} |\beta(A)| K_{\beta(A)} .$$

<u>Proof</u>: Notice that if there does not exist $A \in M_k$ such that $\beta(A) = \tau \vdash kn$, then $<\chi\sigma^k(\rho_\pi), K_\tau> = 0$. Hence the corollary is an immediate consequence of Theorem 3.7.

<u>Corollary 3.9</u> $<\chi\sigma^k(\rho_n), \chi(\rho_{kn})> = 1.$

<u>Proof</u>: Since $\Theta(A,\{n\}) = 1$, we see

$$<\chi\sigma^k(\rho_n), \chi(\rho_{kn})> = \sum_{\tau \vdash kn} <\chi\sigma^k(\rho_n), K_\tau>$$

$$= \sum_{A \in M_k} <\chi\sigma^k(\rho_n), K_{\beta(A)}> = \sum_{A \in M_k} \frac{1}{\Phi(A)} = \sum_{A \in M_k} \frac{|\alpha^{-1}(A)|}{|S_k[S_n]|} = 1 .$$

This completes the proof.

§4 Special ψ-Hopf Ring R(S)

Let $M_k^{ij} = \{A=(a_{rs}) \in M_k \mid a_{ij} \neq 0\}$. For each pair (i,j) with $p(n) \geq i \geq 1$ and $k > j \geq 1$, define the maps

$$(i,j)\downarrow : M_k^{ij} \longrightarrow M_{k-j}$$

and $$(i,j)\uparrow : M_{k-j} \longrightarrow M_k$$

by the rule

$$(i,j)\downarrow(A) = (a'_{rs}) \quad \text{for } A = (a_{rs})$$

and $$(i,j)\uparrow(B) = (b'_{rs}) \quad \text{for } B = (b_{rs})$$

satisfying the condition

$$a'_{rs} = \begin{cases} a_{rs} & \text{if } (r,s) \neq (i,j) \\ a_{ij}-1 & \text{if } (r,s) = (i,j) \end{cases},$$

$$b'_{rs} = \begin{cases} b_{rs} & \text{if } (r,s) \neq (i,j) \\ b_{ij}+1 & \text{if } (r,s) = (i,j) \end{cases}.$$

Hereafter, $(i,j)\downarrow(A)$ and $(i,j)\uparrow(B)$ will be denoted by A_{ij} and B^{ij}, respectively.

<u>Lemma 3.10</u> In $M_{k-j} \times M_k$ we have

$$\{(B,B^{ij}) \mid \forall B \in M_{k-j}, \quad p(n) \geq i \geq 1\}$$

$$= \{(A_{ij},A) \mid \forall A \in M_k^j, \quad p(n) \geq i \geq 1\}$$

where $M_k^j = \bigcup_{i=1}^{p(n)} M_k^{ij}$.

<u>Proof</u>: Note that $(B,B^{aj}) = (\underline{B},\underline{B}^{bj})$ with $p(n) \geq a,b \geq 1$ if and only if $B = \underline{B}$ and $a = b$. The same is true for (A_{ij},A). Thus the proof is complete.

<u>Theorem 3.11</u> For any $\pi \vdash n$ and for any $k \geq 1$, we have

$$\chi\sigma^k(\rho_\pi) = \sigma^k\chi(\rho_\pi),$$

where $\rho_\pi = \rho_1^{\pi_1}\cdot\rho_2^{\pi_2}\cdots\rho_n^{\pi_n}$.

<u>Proof</u>: This theorem is proved by induction on k. If k = 1, the equality is true by definition. From Corollary 3.8 and definition of ψ^j for $k > j \geq 1$, we have

$$\sigma^{k-j}(\chi(\rho_\pi))\psi^j(\chi(\rho_\pi))$$

$$= (\sum_{B\in M_{k-j}} \frac{\Theta(B,\pi)}{\Phi(B)} |\beta(B)| K_{\beta(B)})(\psi^j(\sum_{i=1}^{p(n)} \frac{\theta_i(\pi)}{|S_\pi|} K_{\pi_i}))$$

$$= (\sum_{B\in M_{k-j}} \frac{\Theta(B,\pi)}{\Phi(B)} |\beta(B)| K_{\beta(B)})(\sum_{i=1}^{p(n)} j^{\ell(\pi_i)} \frac{\theta_i(\pi)}{|S_\pi|} K_{j\pi_i})$$

$$= \sum_{B\in M_{k-j}} \sum_{i=1}^{p(n)} \Theta(B,\pi) \frac{\theta_i(\pi)}{|S_\pi|} \frac{|\beta(B)|}{\Phi(B)} j^{\ell(\pi_i)} \frac{\beta(B^{ij})!}{\beta(B)!\pi_i!} K_{\beta(B^{ij})} .$$

In view of the fact that $\Theta(B,\pi) \frac{\theta_i(\pi)}{|S_\pi|} = \Theta(B^{ij},\pi)$, and

$$\frac{\Phi(B^{ij})}{\Phi(B)} \frac{|\beta(B)|}{|\beta(B^{ij})|} \frac{\beta(B^{ij})!}{\beta(B)!\pi_i!} j^{\ell(\pi_i)} = (b_{ij}+1)j|\pi_i| \frac{1}{|\pi_i|} = (b_{ij}+1)j,$$

it follows that

$$\sigma^{k-j}(\chi(\rho_\pi))\psi^j(\chi(\rho_\pi)) = \sum_{B\in M_{k-j}} \sum_{i=1}^{p(n)} (b_{ij}+1)j \frac{|\beta(B^{ij})|}{\Phi(B^{ij})} \Theta(B^{ij},\pi)K_{\beta(B^{ij})},$$

where $B = (b_{ij})$. Then, by Lemma 3.10, we obtain

$$\sigma^{k-j}(\chi(\rho_\pi))\psi^j(\chi(\rho_\pi)) = \sum_{A\in M_k} \sum_{i=1}^{p(n)} a_{ij}\, j \frac{|\beta(A)|}{\Phi(A)} \Theta(A,\pi)K_{\beta(A)} .$$

Hence,

$$\begin{aligned} k\sigma^k(\chi(\rho_\pi)) &= \sum_{j=1}^{k} \sigma^{k-j}(\chi(\rho_\pi))\psi^j(\chi(\rho_\pi)) \\ &= \sum_{A\in M_k} \sum_{j=1}^{k} \sum_{i=1}^{p(n)} a_{ij}\, j \frac{|\beta(A)|}{\Phi(A)} \Theta(A,\pi)K_{\beta(A)} \\ &= \sum_{A\in M_k} k \frac{|\beta(A)|}{\Phi(A)} \Theta(A,\pi)K_{\beta(A)} \\ &= k\chi\sigma^k(\rho_\pi). \end{aligned}$$

This completes the proof.

<u>Theorem 3.12</u> The polynomial Hopf ring R(S) of representations of the symmetric groups is a special ψ-Hopf ring with respect to the Adams operations derived from outer plethysms.

<u>Proof</u>: It is sufficient to show that ψ^k for $k \geq 1$ commutes with the character map χ: R(S) $\longrightarrow$ C, because C is a special ψ-Hopf ring. Since ψ^k is additive and $\{\rho_\pi \mid \pi \vdash n\}$ is a base for $R(S_n)$, it suffices to show $\chi\psi^k(\rho_\pi) = \psi^k\chi(\rho_\pi)$. From Theorem 3.11 it follows that

$$\chi\psi^k(\rho_\pi) = (-1)^{k-1}\, Q_k(\chi\sigma^1(\rho_\pi),\chi\sigma^2(\rho_\pi),\ldots,\chi\sigma^k(\rho_\pi))$$

$$= (-1)^{k-1} Q_k(\sigma^1(\chi(\rho_\pi)),\ldots,\sigma^k(\chi(\rho_\pi)))$$

$= \psi^k(\chi(\rho_\pi))$. This completes the proof.

CHAPTER IV

SPECIAL FREE λ-RINGS

In this chapter, we first construct the graded Hopf ring of symmetric functions H. We then turn our attention to the notation of a special λ-ring and the special free λ-ring on one generator is introduced. Finally we answer a question raised by Knutson (see p. 135 [6]).

§1 The Hopf ring of the Symmetric Functions H

Consider the ring $Z[x_1,x_2,\ldots,x_n]$ of polynomials in n independent variables $x_1,x_2,\ldots,x_n$ with integer coefficients. The symmetric group S_n acts on this ring by

$$\sigma(f(x_1,x_2,\ldots,x_n)) = f(x_{\sigma(1)},x_{\sigma(2)},\ldots,x_{\sigma(n)})$$

for all $\sigma \in S_n$ and $f(x_1,x_2,\ldots,x_n) \in Z[x_1,x_2,\ldots,x_n]$, and a polynomial is said to be symmetric if it is invariant under this action. Thus the symmetric polynomials form a subring

$$H_n = Z[x_1,x_2,\ldots,x_n]^{S_n},$$

where $Z[x_1,\ldots,x_n]^{S_n}$ denotes the set of all S_n-fixed polynomials.

The fundamental theorem of symmetric functions states that H_n itself is also a polynomial ring on n algebraically independent generators,

$$H_n = Z[a_1,\ldots,a_n]$$

where $a_{n,k} = a_k(x_1,\ldots,x_n)$ denotes the k-th elementary symmetric function in n variables. H_n is a graded ring: We have

$$H_n = \{H_{n,k} \mid k = 0,1,2,\ldots\},$$

where $H_{n,k}$ consists of the homogenous symmetric polynomials of degree k, together with the zero polynomial. To each partition $\pi = 1^{\pi_1} 2^{\pi_2} \ldots n^{\pi_n}$ of n, we can associate the monomial $a_{1,k}^{\pi_1} a_{2,k}^{\pi_2} \ldots a_{n,k}^{\pi_n}$, this monomial will be denoted by a_π. Thus $H_{n,k}$ is a free abelian group with basis $\{a_\pi \mid \pi \vdash n, |\pi| = k\}$. For non-negative integers m, n with $m \geq n$, and for all $k \geq 0$, consider the homomorphism

$$\pi^m_{n,k}: H_{m,k} \longrightarrow H_{n,k}$$

which sends each of $x_{n+1},\ldots,x_m$ to zero and $x_1,\ldots,x_n$ to themselves. Since $\pi^m_{n,k} \circ \pi^n_{p,k} = \pi^m_{p,k}$ for all integers $m \geq n \geq p$, we have an inverse system of Z-modules. Consider the inverse limit

$$H_{,k} = \lim_{\substack{\leftarrow \\ n}} H_{n,k} .$$

An element of $H_{,k}$ is by definition a sequence $f = (f_n)_{n\geq 0}$, where each $f_n = f_n(x_1,\ldots,x_n)$ is a homogeneous symmetric polynomial of degree k in $x_1,\ldots,x_n$, and $f_m(x_1,\ldots,x_n,0,\ldots,0) = f_n(x_1,\ldots,x_n)$ whenever $m \geq n$. Since $\pi^m_{n,k}$ is an isomorphism for all $m \geq n \geq k$, it follows that the projection

$$\pi_{n,k}: H_{,k} \longrightarrow H_{n,k}$$

which sends f to f_n is an isomorphism for all $n \geq k$, and hence that $H_{,k}$ has a Z-basis consisting of the monomial symmetric functions a_π (for all partitions π of k) defined by $\pi_{n,k}(a_\pi) = a_\pi(x_1,\ldots,x_n)$, for all $n \geq k$.

Hence $H_{,k}$ is a free Z-module of rank P(k), the number of partitions of k.

Now let $H = \{H_{,k} \mid k = 0,1,2,\ldots\}$, so that H is the free Z-module generated by a_π for all partitions π. The graded ring H thus defined is called the ring of symmetric functions in infinitely many variables $x_1, x_2, \ldots, x_n, \ldots$.

The graded Z-module $H = \{H_{,k} \mid k = 0,1,\ldots\}$ becomes a Z-algebra by defining

$$\pi_n^{p+q}(f \cdot g) = \pi_n^p(f) \cdot \pi_n^q(g)$$

for $f \in H_{,p}$ and $g \in H_{,q}$. It is well known (for example see [3]) that H is a polynomial Hopf ring $P_Z[a_1, a_2, \ldots, a_n, \ldots]$ if we define a comultiplication by $\Delta(a_n) = \sum_{i+j=n} a_i \otimes a_j$.

<u>Remark 4.1</u> A different and quite useful basis for $H_{n,k}$ is $\{h_\pi \mid \pi \vdash n, |\pi| = k\}$, where for a partition $\pi = 1^{\pi_1} 2^{\pi_2} \ldots n^{\pi_n}$ of n, $h_\pi = h_{1,k}^{\pi_1} \ldots h_{n,k}^{\pi_n}$ and $h_{n,k} = \sum_{1 \leq i_1 \leq i_2 \leq \ldots \leq i_k \leq n} x_{i_1} x_{i_2} \ldots x_{i_k}$, is the k-th homogeneous symmetric function in n variables. It is well known [3] that $H = P_Z[h_1, h_2, \ldots, h_n, \ldots]$ where h_k is the k-th homogeneous symmetric function in infinite number of variables $x_1, x_2, \ldots$.

<u>Proposition 4.2</u> The Frobenius isomorphism $F: R(S) \longrightarrow H$ maps Z-basis elements ρ_π into h_π and η_π into a_π.

This is proposition 4.4 of [9].

§2 Special Free λ-Rings

In §3 of Chapter II, we have shown that the graded Hopf representation ring of symmetric groups $R(S) = \{R(S_n)\}$ is a λ-ring. In the present section, it is shown that R(S) is in fact a special free λ-Hopf ring, and this λ-structure is compatible with the Frobenius isomorphism F: $R(S) \longrightarrow H$.

Definition 4.1 A special λ-ring R is a λ-ring in which

(i) $\lambda^k(xy) = P_k(\lambda^1(x),\ldots,\lambda^k(x),\lambda^1(y),\ldots,\lambda^k(y))$, where P_k is the unique polynomial for which

$$\prod_{i,j}(1+\xi_i\varepsilon_j t) = \sum_k P_k[a_1(\xi_1,\ldots,\xi_m),\ldots,a_k(\xi_1,\ldots,\xi_m),a_1(\varepsilon_1,\ldots,\varepsilon_n),\ldots,a_k(\varepsilon_1,\ldots,\varepsilon_n)]t^k$$

in the ring $Z[\xi_1,\ldots,\xi_m,\varepsilon_1,\ldots,\varepsilon_n,t]$ for all m, n.

(ii) $\lambda^k(\lambda^\ell(x)) = P_{k,\ell}(\lambda^1(x),\ldots,\lambda^{k\ell}(x))$, where $P_{k,\ell}$ is the unique polynomial for which

$$\prod_{1\leq i_1<\ldots<i_\ell\leq n}(1+\xi_{i_1}\xi_{i_2}\ldots\xi_{i_\ell}t) = \sum_k P_{k,\ell}[a_1(\xi_1,\ldots,\xi_n),\ldots,a_{k\ell}(\xi_1,\ldots,\xi_n)]t^k$$

in $Z[\xi_1,\ldots,\xi_n,t]$ for all n.

Remark 4.2 By the fundamental theorem of symmetric functions, P_k and $P_{k,\ell}$ are polynomials with integer coefficients, and are independent of n and m as long as $n \geq k$ and $m \geq k$ in the first case, and $n \geq k\ell$ in the second.

Since P_k and $P_{k\ell}$ have integer coefficients, they are well-defined over any ring with unity, and so are sometimes referred to as universal polynomials.

In §1 of this chapter we have seen that

$$H = P_Z[a_1, a_2, \ldots, a_n, \ldots] = P_Z[h_1, h_2, \ldots, h_n, \ldots].$$

In [2], a λ-structure is defined on H such that $\lambda^k(a_1) = a_k$ for all $k \geq 1$. Using the universal polynomials, we can evaluate any expression of the form $\lambda^k(f(a_1, a_2, \ldots))$, where $f(a_1, a_2, \ldots) \in H$. With respect to this λ-structure, H becomes a special λ-ring. Atiyah called H the special free λ-ring generated by one element a_1.

It is known [2], [6] that if R is a special λ-ring then R is a special ψ-ring. The converse of this is not always true, but if R is torsion free it has a converse which will be useful in verifying that the λ-Hopf ring of representation R(S) is in fact a special λ-ring.

<u>Proposition 4.3</u> Let R be a torsion free λ-ring. Let operations ψ^n, be defined by, for $x \in R$,

$$\frac{d}{dt}(\log \lambda_t(x)) = \sum_{n=1}^{\infty} (-1)^{n+1} \psi^n(x)\, t^n.$$

Suppose $\psi^n(1) = 1$, $\psi^n(xy) = \psi^n(x)\psi^n(y)$, and $\psi^n(\psi^m(x)) = \psi^{nm}(x)$ for all $x, y \in R$ and integers $n, m \geq 1$. Then R is a special λ-ring.

Now it is known [9] that R(S) is a polynomial Hopf ring $P_Z[\eta_1, \eta_2, \ldots, \eta_n, \ldots]$ with $\eta_n = [\text{Aut } S_n]$. Moreover, R(S) is a torsion free λ-ring and a special ψ-Hopf ring from Theorem 3.12, thus we have the following

<u>Theorem 4.4</u> The polynomial Hopf ring $R(S) = P_Z[\eta_1, \ldots, \eta_n, \ldots]$ is a special free λ-ring on one generator η_1 such that $\lambda^k(\eta_1) = \eta_k$.

Finally we answer a question raised by Knutson.

<u>Theorem 4.5</u> Let R(S) be a special free λ-Hopf ring of representations of the symmetric groups and let H be a special free λ-Hopf ring of

symmetric functions in infinite number of variables. If $F\colon R(S) \longrightarrow H$ is the Frobenius isomorphism, then F is an isomorphism of Hopf rings preserving λ-structure.

Proof: That F is an isomorphism of Hopf rings is known [9]. It remains to show that F preserves the λ-structures. First we show that for a basis element η_π of $R(S_n)$ with $\pi \vdash n$, $F\lambda^k(\eta_\pi) = \lambda^k F(\eta_\pi)$ by induction on $\ell(\pi)$. If $\ell(\pi) = 1$ we have

$$F\lambda^k(\eta_\pi) = F(\lambda^k(\eta_n)) = F(\lambda^k(\lambda^n(\eta_1)) = F(P_{kn}(\lambda^1(\eta_1),\ldots,\lambda^{kn}(\eta_1))$$

$$= F(P_{kn}(\eta_1,\ldots,\eta_{kn})) = P_{kn}(F(\eta_1),\ldots,F(\eta_{kn}))$$

$$= P_{kn}(a_1,\ldots,a_{kn}).$$

$$\lambda^k F(\eta_\pi) = \lambda^k(F(\eta_n)) = \lambda^k(a_n) = \lambda^k(\lambda^n(a_1)) = P_{kn}(\lambda^1(a_1),\ldots,\lambda^{kn}(a_1))$$

$$= P_{kn}(a_1,\ldots,a_{kn}).$$

Thus we have $F\lambda^k(\eta_n) = \lambda^k F(\eta_n)$.

Now suppose $\ell(\pi) \neq 1$. Let $\pi = \{\beta\}\vee\pi'$, then we have $\eta_\pi = \eta_{\pi'}\cdot\eta_{\{\beta\}}$.

$$F\lambda^k(\eta_\pi) = F\lambda^k(\eta_{\pi'}\cdot\eta_{\{\beta\}}) = F(P_k(\lambda^1(\eta_{\pi'}),\ldots\lambda^k(\eta_{\pi'}),\lambda^1(\eta_{\{\beta\}}),$$

$$\ldots,\lambda^k(\eta_{\{\beta\}})))$$

$$= P_k(\lambda^1(F(\eta_{\pi'})),\ldots,\lambda^k(F(\eta_{\pi'})),\lambda^1(F(\eta_{\{\beta\}})),\ldots,\lambda^k(F(\eta_{\{\beta\}})))$$

$$= \lambda^k(F(\eta_{\pi'}\cdot F(\eta_{\{\beta\}}))$$

$$= \lambda^k F(\eta_\pi).$$

Now, we show that $F\lambda^k(x+y) = \lambda^k F(x+y)$, where x, y are monomials.

$$F\lambda^k(x+y) = \sum_{i=0}^{k} F(\lambda^{k-i}(x) \cdot \lambda^i(y)) = \sum_{i=0}^{k} F(\lambda^{k-i}(x))F(\lambda^i(y))$$

$$= \sum_{i=0}^{k} (\lambda^{k-i}(F(x)))(\lambda^i(F(y)))$$

$$= \lambda^k F(x+y).$$

It follows that F commutes with the λ-structures. Hence the proof is complete.

Corollary 4.6 The λ-structure on R(S) which is derived from the outer plethysm coincides with the pull back λ-structure induced by F^{-1}.

Proof: It is evident.

CHAPTER V

SUMMARY AND CONCLUSIONS

This paper's aim is to construct a λ-structure on the graded Hopf representation ring $R(S) = \{R(S_n)\}$ of the symmetric group S_n which is compatible with the Frobenius isomorphism $F: R(S) \longrightarrow F$.

In Chapter I, it is shown that the divided polynomial Hopf ring $C = \{C(S_n)\}$ of integer valued class functions defined on the symmetric groups is a special ψ-Hopf ring. In Chapter II, a λ-ring structure is introduced in $R(S)$ in terms of outer plethysms so that $R(S)$ forms a ψ-Hopf ring. The character formula of outer plethysms is given in Chapter III, the formula enables us to prove that $R(S)$ forms a special ψ-Hopf ring with respect to the Adams operations derived from outer plethysms. Finally, in Chapter IV, we answer a question raised by Knutson.

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Essam A. Abotteen

Candidate for the Degree of

Doctor of Philosophy

Thesis: CHARACTER FORMULA OF OUTER PLETHYSMS

Major Field: Mathematics

Biographical:

Personal Data: Born in Irtah, Jordan, May 11, 1953, the son of Mr. and Mrs. Abdel-Latif Abotteen.

Education: Graduated from Keifan Secondary School, Kuwait, 1972; received Bachelor of Science degree in Mathematics from Kuwait University, Kuwait, 1975; received Master of Science degree in Mathematics from Sam Houston State University, Huntsville, Texas, 1978; completed requirements for the Doctor of Philosophy degree at Oklahoma State University in May, 1983.

Professional Experience: Served as a high school mathematics teacher, Ministry of Education, Kuwait, 1975-1976; served as Lab Assistant, Department of Mathematics, Sam Houston State University, 1977-1978; served as Graduate Teaching Associate, Department of Mathematics, Oklahoma State University, 1978-1983.

Professional and Honorary Organizations: Received E. K. McLachlan Award in mathematics, Department of Mathematics, Oklahoma State University, 1981; member of the American Mathematical Society, The Mathematical Association of America and Pi Mu Epsilon.