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AXISYMMETRIC FREE VIBRATIONS OF SANDWICH SHELLS OF REVOLUTION

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AXISYMMETRIC FREE VIBRATIONS OF

SANDWICH SHELLS OF REVOLUTION

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Jamal Joseph Azar

TABLE OF CONTENTS

TABLE OF CONTENTS
(Continued)

LIST OF TABLES

LIST OF ILLUSTRATIONS

LIST OF ILLUSTRATIONS
(Continued)

AXISYMMETRIC FREE VIBRATIONS OF SANDWICH

SHELLS OF REVOLUTION

CHAPTER 1

SURVEY OF DEVELOPMENTS IN THE ANALYSIS OF SANDWICH SHELL STRUCTURES

1.1 Introduction

The structural-weight efficiency of missiles and spacecraft structures is one of the major factors in the over-all design of such air vehicles. One of the best methods yet known for improving this structural efficiency is the use of what is called sandwich construction. It consists of a low density, thick core and two relatively thin facings spaced to obtain high bending stiffness.

The concept of sandwich structures is not new. During World War II and thereafter this concept has become increasingly important. For example, as early as 1947 sandwich-type panels were used for more than 75 per cent of the airfoil surface and for nearly all of the internal shear webs of the Chance Vought XF5U-1 Navy fighter and more than 95 per cent of the structure of the XF6U-1. In recent years, with the emergence of the Aerospace Age, where the weight penalties on space vehicles are severe, there has been even greater emphasis on the use of sandwich structures. For example sandwich construction was used on the B-58 and

XB-70 bombers and will be used extensively on the Apollo lunar spacecraft.

In the course of a literature survey on sandwich structures in general and sandwich shells in particular, the writer has found that a limited amount of information is available. Most of the work which has been published in this area is related to static analyses. Thus, from the dynamic point of view, sandwich shell structures pose a real challenge to the structural analyst and designer. A brief resume is presented of the developments pertaining to the static and the vibrational analysis of sandwich shell structures.

1.2 Developments in the Static Analysis of Sandwich Shell Structures

Before discussing sandwich shell structures, it is worthwhile to give a brief background on the development of the linear theory of shells in general.

The first attempt in formulating a bending theory of thin shells from the general equations of elasticity was made by Aron in 1874, and was followed by Love's first approximation theory in 1888. Since that era, shells of various geometrical configurations have been extensively explored by many investigators (1,2,3,4)*.

The basic assumptions employed in their analyses were:

1. The thickness of the shell is very small compared to the radii of curvature.
2. The normals of the middle surface of an undeformed shell remain straight, unstretched, and normal to the middle surface even after deformation takes place in the shell. (This assumption is known as the Kirchhoff hypothesis.)

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*Numbers in parenthesis denote references listed at the end.

3. The strains and displacements are sufficiently small, so that higher order nonlinear terms in the strain-displacement relations may be neglected in comparison with the first order linear terms.
4. The normal stresses in the thickness direction are considered small compared to the other stresses and may be neglected.

Most of the existing theories on shells, which were evolved from the above basic assumptions, belong to what has been termed classical or linear shell theory.

Love's first approximation was a notable contribution to the classical theory of shells. In his work he defined the stress resultants in the following manner:

$$\begin{Bmatrix} F_s \\ M_s \\ Q_s \end{Bmatrix} = \int_{-h}^{h} \begin{Bmatrix} \sigma_s \\ z\sigma_s \\ \sigma_{sz} \end{Bmatrix} (1 + z/R_2)\, dz$$

$$\begin{Bmatrix} F_{\theta s} \\ M_{\theta s} \end{Bmatrix} = \int_{-h}^{h} \sigma_{\theta s} \begin{Bmatrix} 1 \\ z \end{Bmatrix} (1 + z/R_1)\, dz$$

Where σ_s, σ_{sz} and $\sigma_{\theta s}$ were stress components, h was half the shell thickness, R_1 and R_2 were the radii of curvature and z was the normal coordinate to the middle surface of the shell. Love's first approximation was based on the four assumptions which were stated previously and in addition he neglected the z/R term in comparison with unity. The first assumption defined what was meant by <u>thin</u> shells. The second assumption made the transverse-shear deformation negligible. The third assumption ensured linearity in the resulting differential equations.

Following Love's first approximation, many investigators tried to obtain improved derivations of the classical theory of shells. Flugge (4) in 1943 and Byrne (5) in 1944 tried to retain terms of the order $(z/R)^2$ in the stress-and strain-displacement relations. Timoshenko and many others (6,7,8) included transverse-shear deformation in their analyses.

In sandwich constructions, while the investigators were much attracted by the analysis of sandwich plates and beams, very little was done in the area of sandwich shell structures. In 1949 Reissner (9) formulated the basic equations for the static analysis of sandwich-type shells. His theory was an extension of the existing theory of homogeneous thin elastic shells. The following assumptions were employed in his analysis:

(a) The thickness ratio t_f/h_c is small compared with unity.

(b) The ratio $E_f t_f/E_c h_c$ is assumed large compared with unity. (This assumption means the facings are so much stiffer than the core that the contribution of the core to stress couples and tangential stress resultants of the sandwich shell are negligible.)

(c) The effect of transverse-shear deformation is incorporated.

(d) The effect of transverse normal stress deformation of the core is considered in the analysis.

(e) The facings are assumed to be subjected to membrane stresses only.

(f) The core is assumed to behave like a three-dimensional elastic continuum in which those stresses which are parallel to the facings are negligible compared with the transverse shear and normal stresses.

The resulting system of equations was applied to specific problems of circular cylindrical and spherical shells. Results showed that the effects of both transverse-shear and transverse-normal-stress deformation were of such magnitude that an analysis which neglected them resulted in values of deflection and stresses which were appreciably in error.

Wang (10), in 1952, used the method of complementary energy in the derivation of the stress-displacement relations of sandwich shells. The method has been shown to be derivable from the principle of potential energy by a Legendre type of transformation. This procedure is known as "Friedrich's Method" (11).

Crouzet-Pascal, Mahoney, and Salerno (12) formulated twenty-seven equations for the static analysis of a circular cylindrical shell of sandwich construction. The derived equations comprised five equilibrium equations in terms of the stress resultants, sixteen stress resultant-displacement equations, and six equations that described the force interactions between the core and the facings of the shell. The facings were analyzed within the membrane theory, while the core was analyzed on the basis of thick shell theory. Their analysis was limited to the following ranges of shell parameters:

(a) $\frac{t_1}{h} \leq \frac{1}{5}$ (b) $\frac{t_3}{h} \leq \frac{1}{5}$ (c) $\frac{h}{a} \leq \frac{1}{5}$

where t_1 and t_3 were the thicknesses of the outer and inner facings respectively, h was the thickness of the core, and a was the radius of curvature of the middle surface of the sandwich circular cylindrical shell.

Yao (27) applied Reissner's theory of elastic sandwich shells to set up a linear eigenvalue problem for axially symmetric buckling of sandwich spherical shells subjected to uniform external pressure. The flexural rigidity of the facings and the membrane action of the core were neglected. The analysis was based on the shallow shell theory. A closed form solution in terms of Bessel's functions was obtained.

Rutecki (28) analyzed a conical shell consisting of two facings of equal thickness made of aluminum and spaced by wooden ribs, and a core of spongy foam glued to the facings only. He used a variational method to derive the equations of stress and strain and an Airy stress function to solve the derived differential equation.

Kazimi (29) studied experimentally various conventional types of cylindrical sandwich construction under various loading conditions. The growth and nature of fatigue failures were discussed and correlated to hoop stresses and stiffening ratio when using honeycomb cores. It was shown that improved results could be obtained with honeycomb construction in structures exposed to vibrations and shock.

Most of the investigators on sandwich shell structures have treated the facings as membranes. As mentioned before, this assumption is adequate in the design of pressure vessels. A theory which includes bending stiffness of the facings was presented by Grigolyuk (13) and further treated by Fulton (14). Their method of analysis was based on the small-deflection theory.

The first attempt to solve a sandwich shell problem based on large-deflection theory with face bending stiffness included, was made by Wempner (15). The work was an extension of previous analysis by Wempner and Baylor (16) on elastic sandwich shells with weak cores. Another analysis of large-deflection of sandwich shells was made by Grigolyuk (26).

Schmit (17) presented some static analyses of sandwich shells of arbitrary shape. Flexural rigidity of the facings and the transverse deformations of the core were included in his analysis. In the development of the stress-strain and displacement relations, the Kirchhoff hypothesis was employed. The core stresses acting in the tangential directions were neglected. Further, the core was considered to be orthotropic, while the facings were assumed to be isotropic and of the same thickness and material.

In a review of recent Russian work on sandwich structures compiled by Habip (30) it was mentioned that considerable research effort dating from 1957 appears to have been concentrated on the development of both design data and of engineering theories for various types of sandwich panels and shallow shells. Most of the work published in this area was confined to the static part of the problem. Ambartsumian (31) derived nonlinear large-deflection equations for the static analysis of shallow sandwich shells, but only indicated the application to dynamical problems.

1.3 Developments in the Vibrational Analysis of Sandwich Shell Structures

As mentioned in Section 1.1, very limited information is available on the vibrational analysis of sandwich shell structures. The availability of such analysis covers only circular sandwich cylinders. In fact, the first vibrational theory pertaining to sandwich shells appeared in early 1960.

Chu (18) presented a Timoshenko-type theory[1] of vibrations for a circular cylindrical sandwich shell with honeycomb core. He neglected the flexural rigidity, transverse-shear and rotatory inertia in the facings. Thus, the obtained equations of motion were very much simplified. The two identical facings were assumed to be isotropic. The displacements were assumed to vary linearly across the thickness of each layer. As a result the cut-off frequencies[2] were determined for a sandwich cylinder vibrating freely in the following modes: circumferential, longitudinal, breathing, circumferential thickness-shear, and longitudinal thickness-shear.

The dynamic shear factor was taken to be $\sqrt{0.9975}$. In recent unpublished work by Bert, Crisman, and Nordby, (25) it was shown that the dynamic shear factor in sandwich construction is approximately 2.0 depending upon the geometry and material of the structure under consideration. They extended the work of Mindlin and Deresiewicz (19) on homogeneous members to sandwich structures.

Yu in (20, 21) formulated the basic equations for the vibrational analysis of sandwich plates. In (22) he extended the theory and deduced

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[1]Timoshenko-type theory differs from the classical theory in that the normals of the undeformed middle surface of the shell do not remain normal but suffer an extension after deformation takes place in the middle surface.

[2]Cut-off frequencies are defined as the lowest frequencies of the various modes of vibrations of a structure. The circumferential cut-off frequency corresponds to the rotation of the shell as a whole. The breathing cut-off frequency corresponds to the purely radial motion of the shell vibrating in the mode of a ring.

simplified equations for the treatment of sandwich-type, circular cylindrical shells. Based on the latter equations, the axisymmetric and torsional vibrations of an infinite sandwich cylindrical shell were investigated. It was found that the cut-off frequency of the radial mode is much higher than the thickness-shear mode. For axisymmetrical vibrations the cut-off frequencies for a sandwich cylinder were found for the following modes:

(a) For an infinite wavelength:

 (1) Longitudinal

 (2) Radial

 (3) Thickness-shear

(b) For short wavelength:

 (1) Flexural

 (2) Extensional

In 1962, Bieniek and Freudenthal (23) developed a method for the determination of stresses and deformations in forced vibrations of cylindrical orthotropic sandwich shells. In the analysis, transverse-shear deformation of the core and the material damping of both the core and the facings were taken into account. The final equations were applied to a sandwich cylindrical shell loaded by normal pressure. The material damping was introduced by taking the shear and Young's moduli of elasticity as complex moduli:

$$\bar{G} = G\,(1 + i\mu): \qquad \bar{E}_{nm} = E_{nm}\,(1 + i\eta)$$

where G and E_{nm} are the real shear and Young's moduli of elasticity respectively and were assumed to be constants rather than varying with the frequency; μ and η are the damping coefficients.

In the review of Russian vibrational research by Habip (30), it is mentioned that Il'gamov (32,33) considered the small vibrations of sandwich shells and applied his results to the free vibrations of a rectangular sandwich panel simply supported along the edges.

1.4 Summary of the Literature

From the foregoing literature survey one concludes that most of the investigators of sandwich shell structures have confined their work to static analyses. In general, with the exception of sandwich-type, circular cylindrical shells, the vibrational analysis of sandwich shell structures has been unexplored.

With the emergence of the Space Age, the author believes that the vibrational analysis of sandwich shell structures should be pursued. Therefore, in this dissertation the author develops a general method for determining the axisymmetric free vibrational characteristics of sandwich shells of revolution.

CHAPTER 2

BASIC DERIVATIONS AND THE DIFFERENTIAL EQUATIONS OF SANDWICH SHELLS OF REVOLUTION

2.1 Coordinate System

An element abcd (Figure 1), which is cut from a shell of revolution by two meridians and two parallel circles, is considered. The line tangent to the meridian at a point on the surface of the shell, d for instance, is called the meridian coordinate or the x-axis; the line tangent to the parallel circle at the same point is called the parallel circle coordinate or the y-axis; and the outer normal to the surface at that point is the normal coordinate or the z-axis. The coordinates x, y, and z are mutually perpendicular and thus constitute an orthogonal coordinate system.

The displacements along the x, y, and z axes are denoted by u, v, and w respectively and are shown in their positive senses in Figure 1.

Figure 2 shows a meridian of the shell. Radius R_c is the distance of a typical point to the axis of rotation and R_{1c} is the meridional radius of curvature. The distance measured on a normal to the meridian between the intersection with the axis of rotation and the middle surface of the shell is R_{2c} and usually called the tangential radius of curvature. From the geometry of surfaces of revolution, R_{1c} and R_{2c} are the principal radii of curvature.

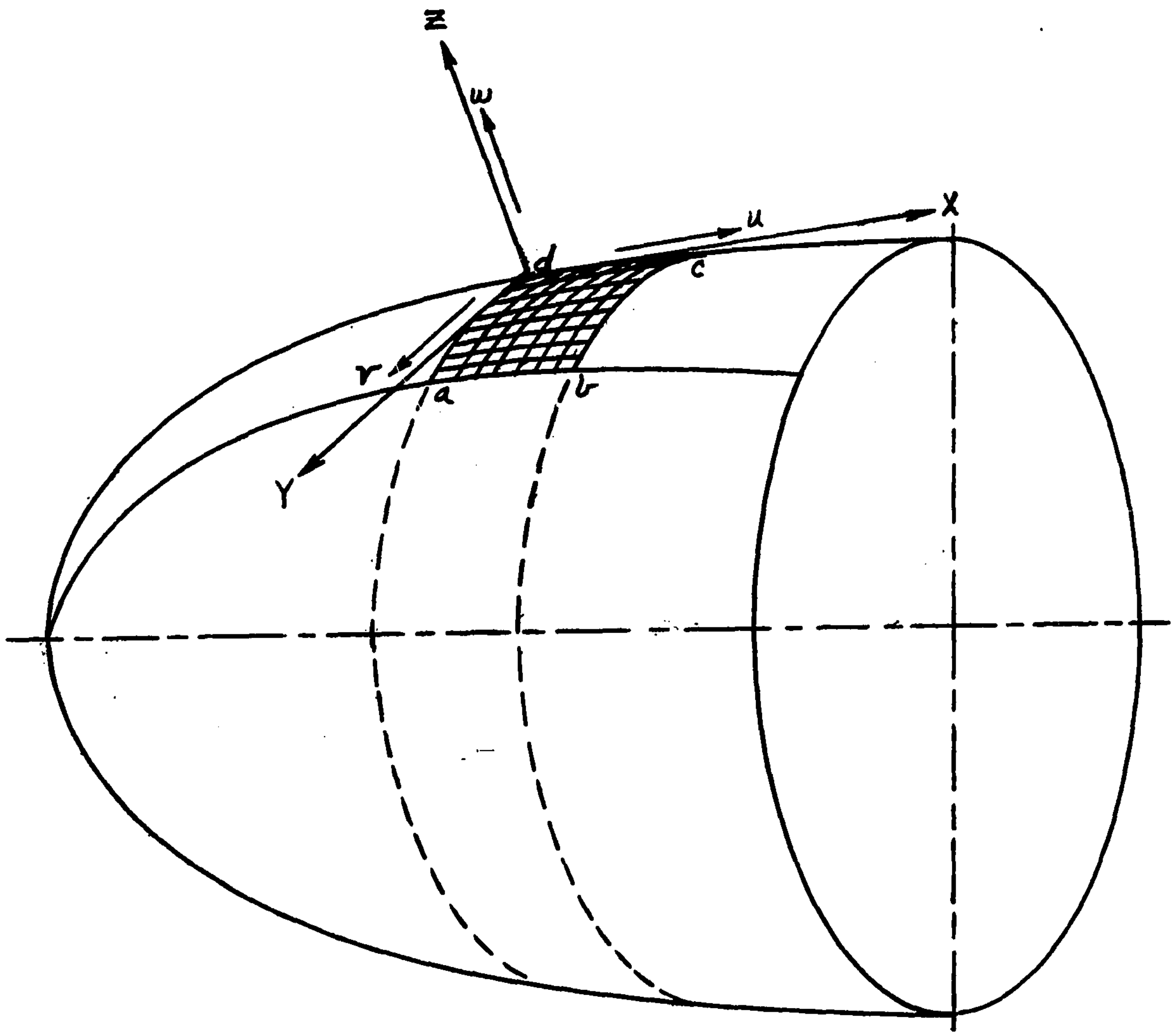

Figure 1--A shell of revolution showing the coordinate system.

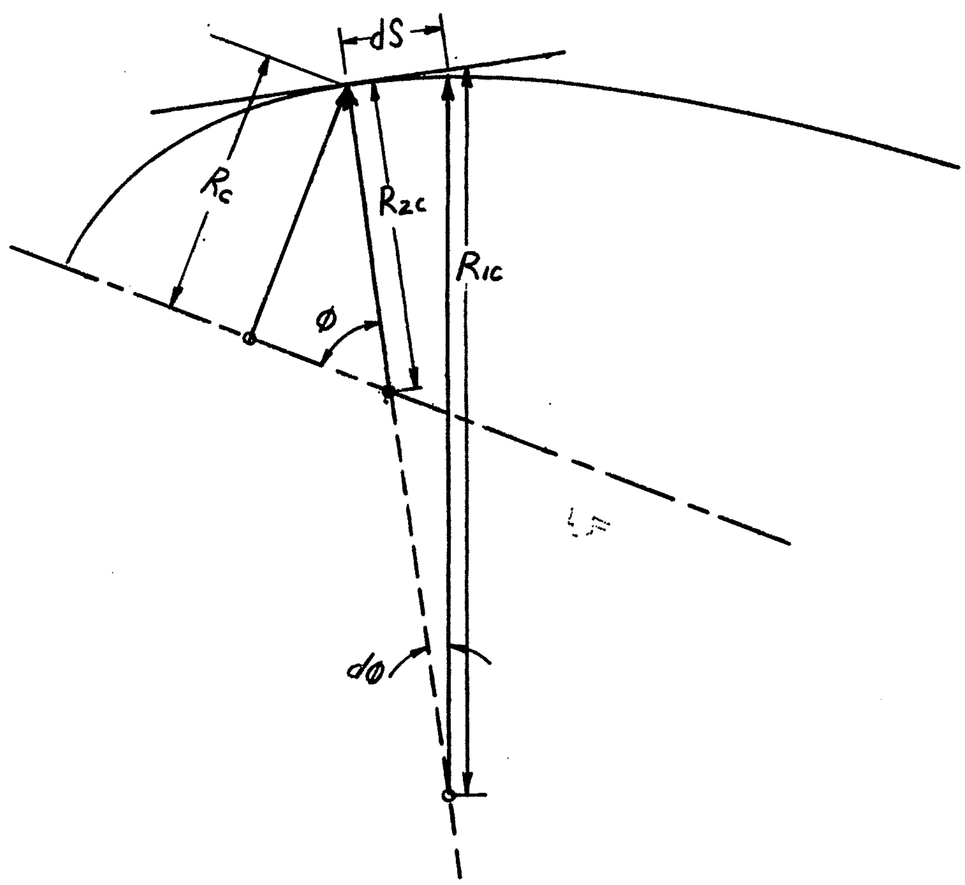

Figure 2--Meridional segment showing the radii of curvature of a shell of revolution.

From Figure 2, the following geometrical relations are obtained:

$$R_c = R_{2c} \sin \phi$$

$$ds = R_{1c}\, d\, \phi$$

2.2 Assumptions

The present analysis of sandwich shells of revolution is based on the following assumptions:

1. The facings are made of the same linearly elastic material and are of equal thicknesses, that is the construction is symmetrical. The computer program allows for the facings to be isotropic or orthotropic.
2. The deflections are small, so that the strain-displacement relations can be assumed to be linear.
3. The total thickness of the sandwich composite is very small compared to the smallest radius of curvature of the shell.
4. The core is made of linearly elastic orthotropic material.
5. The facings are subject to membrane and bending stresses only.
6. The core has no resistance to bending stresses; thus it is subject to transverse shearing stresses only.
7. The contributions of the facings and the core to the inertial effects (rotatory and translational) are incorporated.
8. Only axisymmetric free vibrations are considered.
9. Damping effects in the facings and core are neglected.
10. All thermal effects are neglected.
11. All residual stresses and preload effects are neglected.

Assumptions 1, 2, 3, and 4 imply linear shell theory. Assumption 5 implies that the facings are thin and very stiff so that shearing stresses can be neglected in comparison to bending and membrane stresses.

Assumption 6 infers that the core is very flexible and has no resistance to bending stresses.

Assumption 7, the translational inertial effect, has been always taken into account in vibrational analyses of shells. On the other hand, the rotatory inertial effect, in most cases, has been neglected. The inclusion of rotatory inertia increases the dynamic loading on a structure and neglecting it can lead to frequencies which are appreciably in error.

Assumptions 9 and 11 have been made in almost all previous vibrational analyses of shells, and they are reasonable assumptions from an engineering standpoint.

2.3 Internal Static Stress Resultants

A. Outer facing of the shell:

An element cut from the middle surface of the outer facing of the shell by two adjacent meridians and two parallel circles is considered. The element with the membrane and bending forces is shown in Figures 3 and 4 respectively.

Due to the symmetry of the loading conditions on the shell, all of the inplane shearing forces and the transverse shearing force on the side x=constant are neglected. This is justified by assumming that the facings are thin compared to the core (assumption 2).

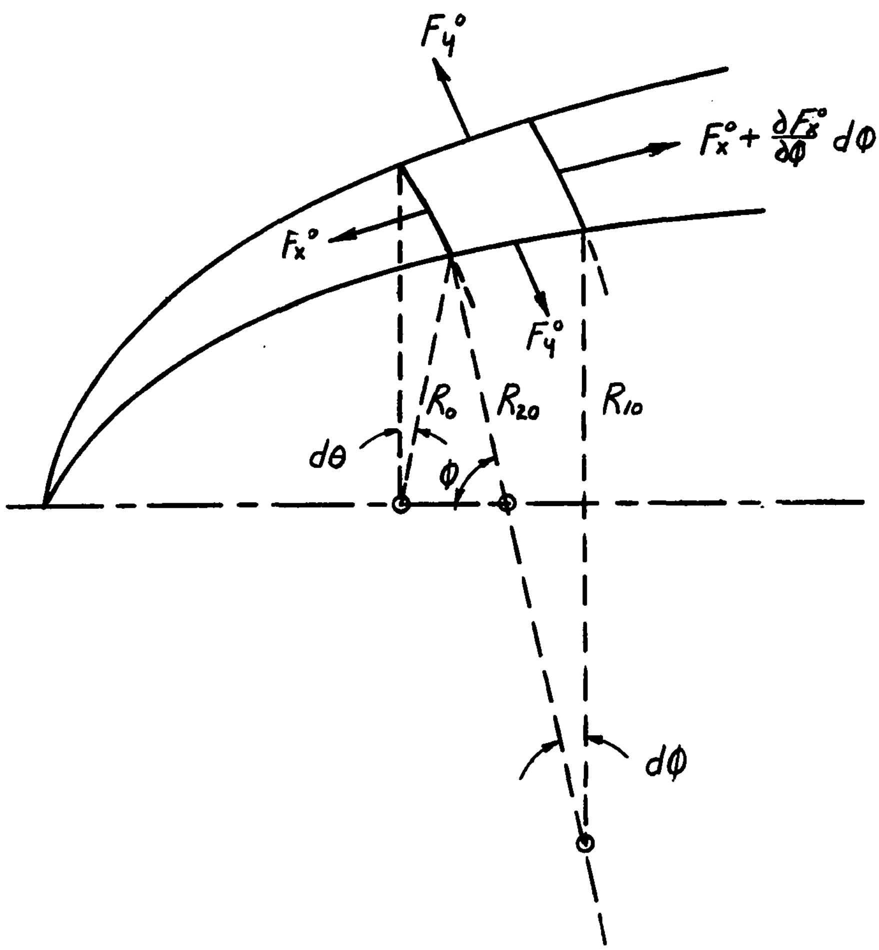

Figure 3--A shell of revolution element showing the normal stress resultants in their positive senses.

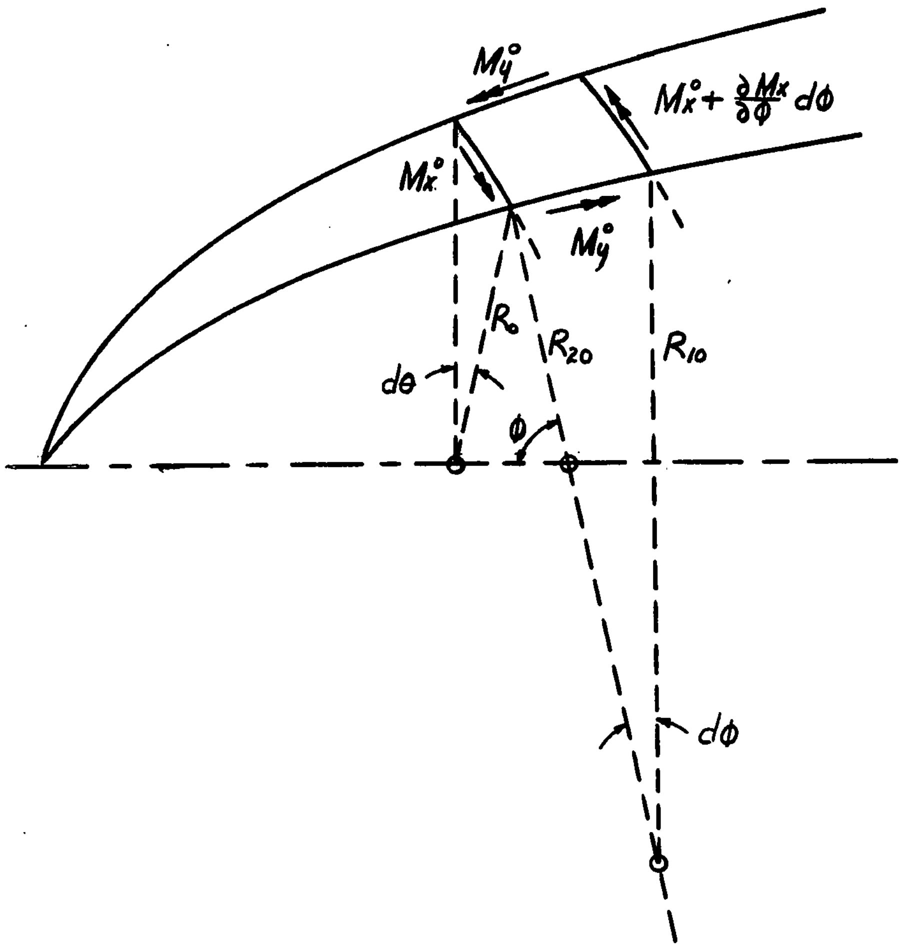

Figure 4--A shell of revolution element showing the stress resultant couples in their positive senses.

A.1 <u>Force components in the x-direction</u>:

The stress resultant, F_x^o, if multiplied by the length on which it acts, will give a force which can be resolved into two components along the x and z axes. See Figure 5.

From Figure 5 the force components which act along the x-axis are:

$$-F_x^o \, R_o d\theta \cos \frac{d\phi}{2}$$

and

$$\left(F_x^o + \frac{\partial F_x^o}{\partial \phi} d\phi\right)\left(R_o + \frac{\partial R_o}{\partial \phi} d\phi\right) d\theta \cdot \cos \frac{d\phi}{2}$$

For small angles $\cos \frac{d\phi}{2}$ is equal approximately to one. Therefore, the above expressions can be written as

$$\left(F_x^o + \frac{\partial F_x^o}{\partial \phi} d\phi\right)\left(R_o + \frac{\partial R_o}{\partial \phi} d\phi\right) d\theta - F_x^o R_o d\theta. \qquad (a)$$

Each of the stress resultants, F_y^o, on the upper and lower sides of the element shown in Figure 3 will give rise to a force component of magnitude $F_y^o \, R_{1o} d\phi$. Since these two forces are not quite parallel to each other, they both form a resultant force which lies in the plane of the parallel circles and is directed along its radius toward the axis of rotation. See Figures 6 and 7.

From Figure 6 the total vertical force is given by:

$$2 \, F_y^o \, R_{1o} d\phi \sin \frac{d\theta}{2} \, ,$$

However, for small angles $\sin \frac{d\phi}{2} = \frac{d\phi}{2}$; therefore, the component becomes $F_y^o \, R_{1o} d\phi d\theta$.

The force, $F_y^o \, R_{1o} d\phi d\theta$, has two components (Figure 7), one acting along the x-axis and the other along the z-axis.

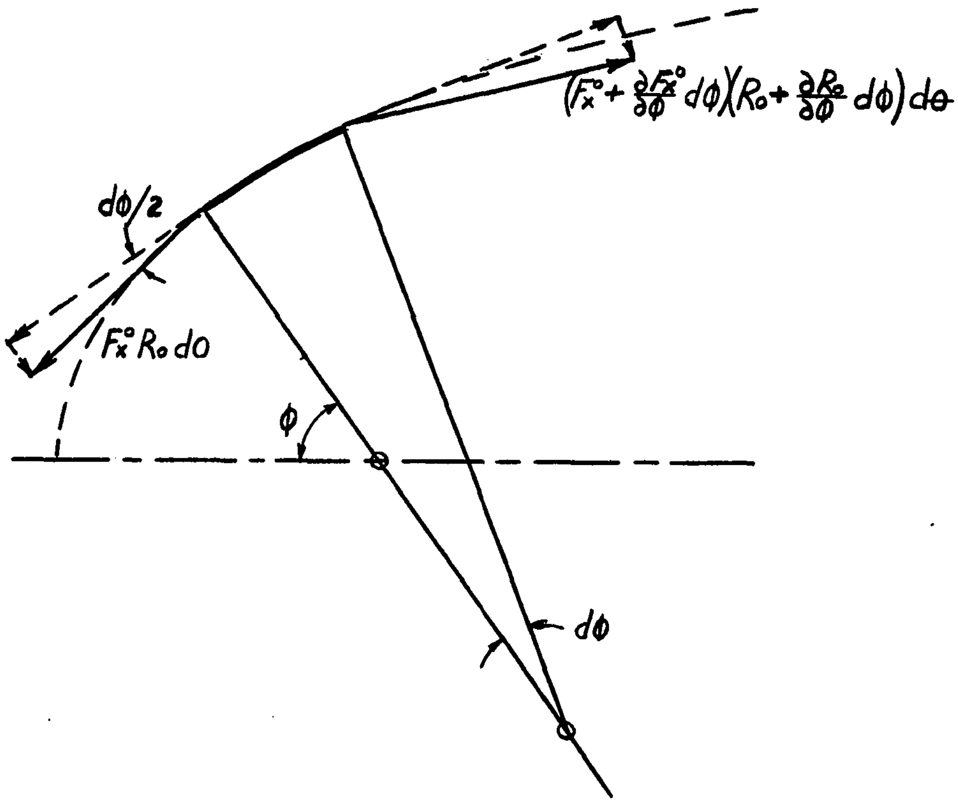

Figure 5--A meridional segment showing the force resultants with their components along the tangent to the meridian and along the normal at that point.

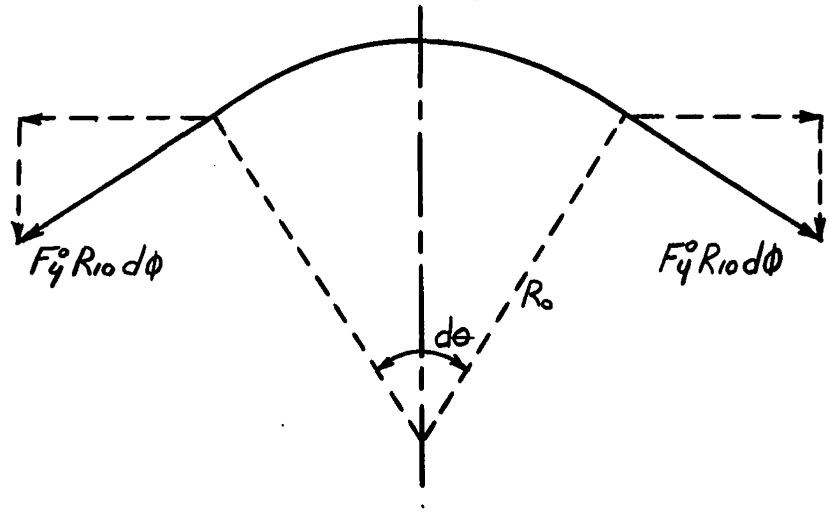

Figure 6--A parallel circle segment showing the resultant forces and their components.

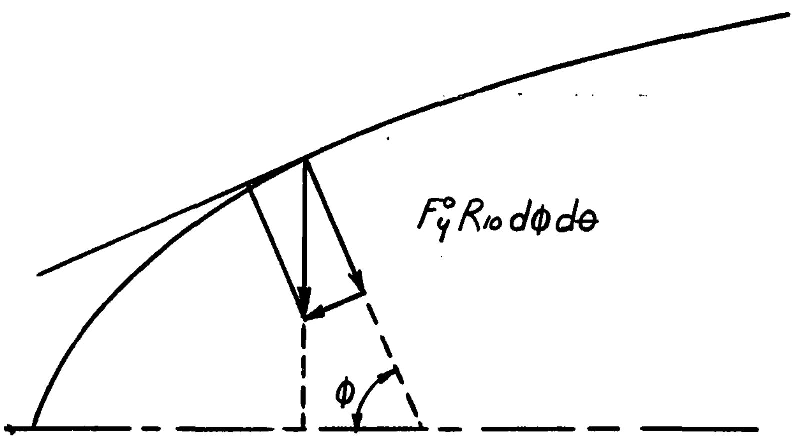

Figure 7--A meridional segment showing the components of the resultant force, F_y^o R_{1o} $d\phi$ $d\theta$.

The component along the x-axis is given by:

$$-F_y^o \, R_{1o} \, d\phi \, d\theta \cos \phi. \qquad \text{(b)}$$

A.2 Force components in the z-direction:

From Figures 5 and 7 the force components acting in the z-direction are:

$$-F_x^o \, R_o d\theta \sin \frac{d\phi}{2} - \left(F_x^o + \frac{\partial F_x^o}{\partial \phi} d\phi\right) \left(R_o + \frac{\partial R_o}{\partial \phi} d\phi\right) d\theta \sin \frac{d\phi}{2} - F_y^o \, R_{1o} d\phi d\theta \sin \phi$$

and for small angles $\sin \frac{d\phi}{2} = \frac{d\phi}{2}$, therefore the above expression reduces to:

$$-F_x^o \, R_o d\theta \frac{d\phi}{2} - \left(F_x^o + \frac{\partial F_x^o}{\partial \phi} d\phi\right) \left(R_o + \frac{\partial R_o}{\partial \phi} d\phi\right) d\theta \frac{d\phi}{2} - F_y^o \, R_{1o} d\phi d\theta \sin \phi \quad \text{(a)}$$

A.3 Bending force components with respect to the y-axis:

From Figure 4 the sum of the vector components of the moments acting on the left and right side of the element is given by:

$$-M_x^o \, R_o \, d\theta + \left(M_x^o + \frac{\partial M_x^o}{\partial \phi} d\phi\right) \left(R_o + \frac{\partial R_o}{\partial \phi} d\phi\right) d\theta \qquad \text{(a)}$$

The moments, $M_y^o \, R_{1o} \, d\phi$, in Figure 4 are not quite parallel to each other. Therefore, their resultant contributes to the total moments with respect to the y-axis.

From Figure 8 the vector component acting along the radius of the parallel circle is given by:

$$M_y^o \, R_{1o} \, d\phi \cos \phi.$$

From Figure 9 the resultant component of moment with respect to the y-axis is therefore given by:

$$-2M_y \, R_{1o} \, d\phi \cos \phi \sin \frac{d\theta}{2}$$

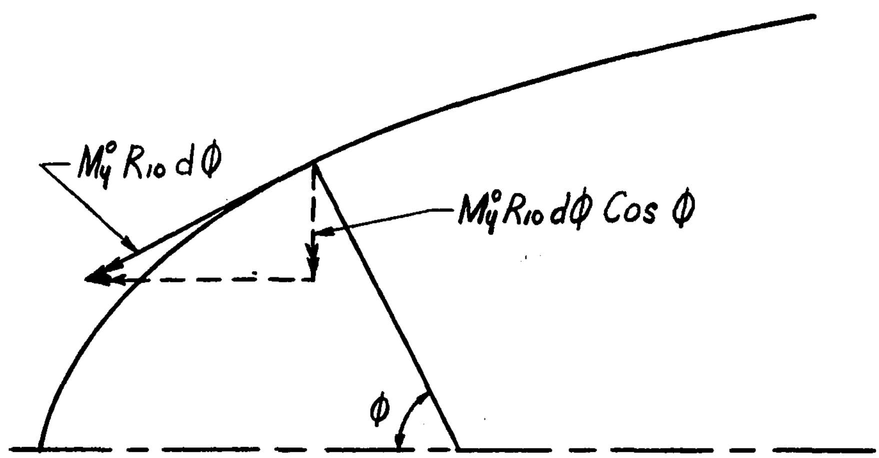

Figure 8--A meridional segment showing the vector components of the resultant vector moment, $M_y^o R_{1o} d\phi$.

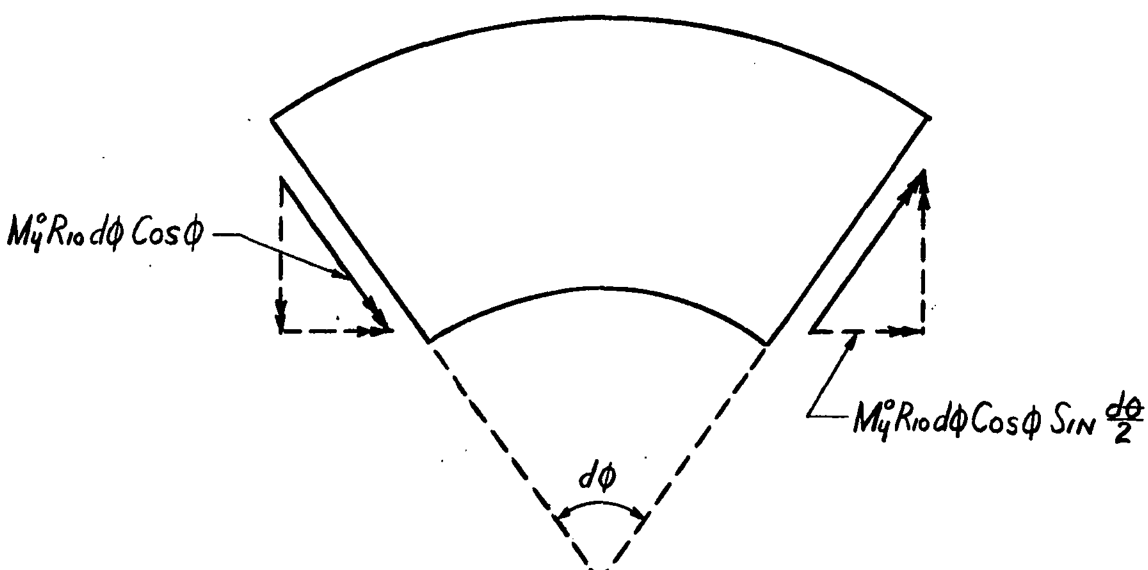

Figure 9--A parallel circles element showing the vector components of the resultant vector moment, $M_y R_{1o} d\phi \cos \phi$.

and for small angles this reduces to:

$$-M_y R_{1o} d\phi d\theta \cos\phi \qquad \text{(b)}$$

B. Inner facing of the shell

The force and moment components which were developed for the outer facing can be written in like manner for the inner facing with the change of the subscripts and superscripts from (o) to (i).

Replacing (o) with (i) yields the following components for the inner facing:

B.1 Force components in the x-direction

$$\left(F_x^i + \frac{\partial F_x^i}{\partial \phi} d\phi\right)\left(R_i + \frac{\partial R_i}{\partial \phi} d\phi\right) d\theta - F_x^i R_i \, d\theta \qquad \text{(a)}$$

$$- F_y^i R_{1i} \, d\phi \, d\theta \cos\phi \qquad \text{(b)}$$

B.2 Force components in the z-direction

$$-F_x^i R_i \, d\phi \frac{d\phi}{2} - \left(F_x^i + \frac{\partial F_x^i}{\partial \phi} d\phi\right)\left(R_i + \frac{\partial R_i}{\partial \phi} d\phi\right) d\theta \frac{d\phi}{2} - F_y^i R_{1i} \, d\phi \, d\theta \sin\phi \qquad \text{(a)}$$

B.3 Bending force components with respect to the y-axis

$$-M_x^i R_i \, d\theta + \left(M_x^i + \frac{\partial M_x^i}{\partial \phi} d\phi\right)\left(R_i + \frac{\partial R_i}{\partial \phi} d\phi\right) d\theta \qquad \text{(a)}$$

$$-M_y^i R_{1i} \, d\phi \, d\theta \cos\phi \qquad \text{(b)}$$

C. Core of the shell

The core of the shell is assumed to carry transverse shearing forces only. Because of symmetry the transverse shearing force acting on the side for which y = constant vanishes. Therefore, the only forces acting on the core are those shown in Figure 10.

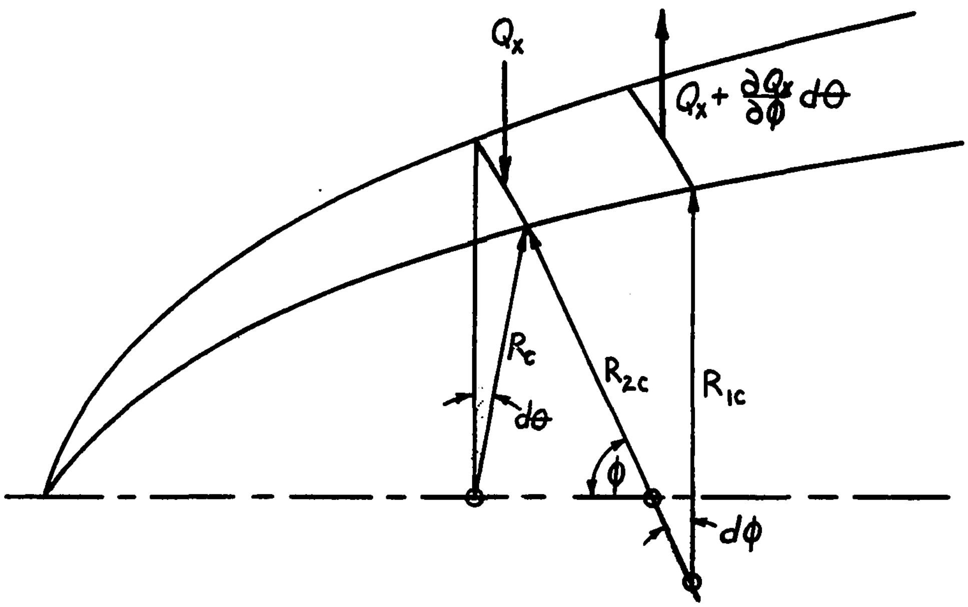

Figure 10--Element of a shell of revolution showing the transverse-shear stress resultants.

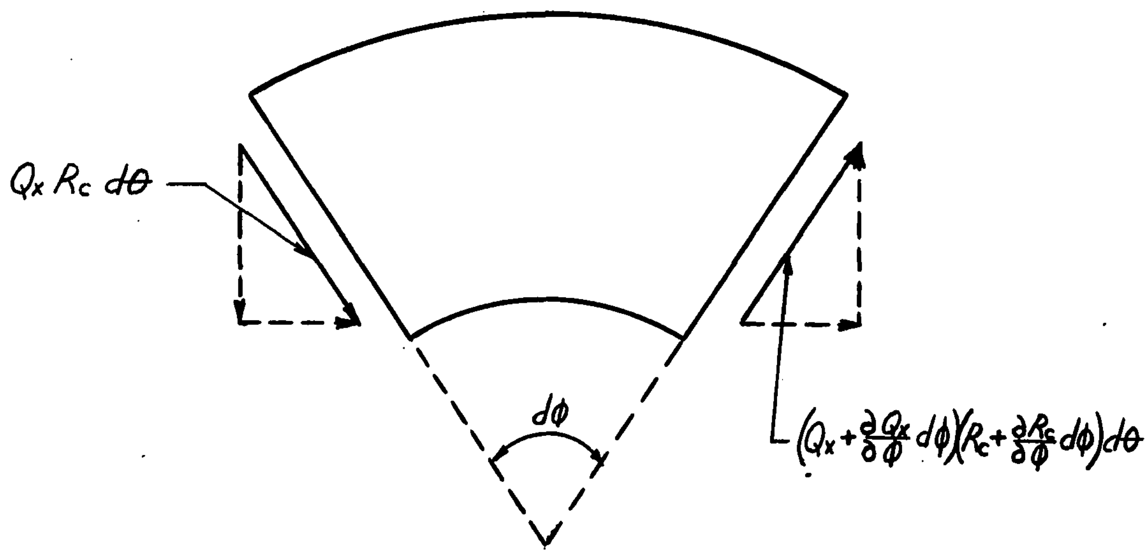

Figure 11--A meridional element showing the components of the transverse-shear forces.

C.1 Force components in the x-direction

From Figure 11 the total force acting in the x-direction is:

$$Q_x R_c d\theta \sin\frac{d\phi}{2} + (Q_x + \frac{\partial Q_x}{\partial \phi} d\phi)\ (R_c + \frac{\partial R_c}{\partial \phi} d\phi)\ d\theta \sin\frac{d\phi}{2}$$

For small angles this reduces to:

$$Q_x R_c d\theta \frac{d\phi}{2} + (Q_x + \frac{\partial Q_x}{\partial \phi} d\phi)\ (R_c + \frac{\partial R_c}{\partial \phi} d\phi)\ d\theta \qquad \text{(a)}$$

C.2 Force components in the z-direction

Figure 11 gives the total force acting along the z-axis as:

$$\left[(Q_x + \frac{\partial Q_x}{\partial \phi} d\phi)\ (R_c + \frac{\partial R_c}{\partial \phi} d\phi)\ d\theta - Q_x R_c d\theta\right] \left[\cos\frac{d\phi}{2}\right]$$

and for small angles this reduces to:

$$(Q_x + \frac{\partial Q_x}{\partial \phi} d\phi)\ (R_c + \frac{\partial R_c}{\partial \phi} d\phi)\ d\theta \ - \ Q_x R_c d\theta \qquad \text{(a)}$$

C.3 Moment with respect to the y-axis:

The moment due to transverse shear is given by:

$$-\left[(Q_x + \frac{\partial Q_x}{\partial \phi} d\phi)\ (R_c + \frac{\partial R_c}{\partial \phi} d\phi)\ d\theta\ Q_x R_c d\theta\right] \left[\frac{R_{1c}\ d\phi}{2}\right] \qquad \text{(a)}$$

2.4 Inertial Forces

The translational and the rotational (rotatory) inertia forces of the facings and the core are included in the present analysis. Many investigators have neglected the effect of rotatory inertia in their analyses. In (34), (35) and (24) extensive studies on the effect of the rotatory inertia on the frequency response of various types of homogeneous structures was presented. Generally, the rotatory inertia increases the dynamic loading on the structure and neglecting it can lead to erroneous frequency results.

The effective inertia forces of the facings and core are given as follows:

$$\left.\begin{array}{l} \left.\begin{array}{l} -m^f \dfrac{\partial^2 u}{\partial t^2} R_o \, R_{1o} \, d\theta \, d\phi \\ -m^f \dfrac{\partial^2 u}{\partial t^2} R_i \, R_{1i} \, d\theta \, d\phi \\ -m^c \dfrac{\partial^2 u}{\partial t^2} R_c \, R_{1c} \, d\theta \, d\phi \end{array}\right\} \quad (1) \\ \left.\begin{array}{l} m^f \dfrac{\partial^2 w}{\partial t^2} R_o \, R_{1o} \, d\theta \, d\phi \\ m^f \dfrac{\partial^2 w}{\partial t^2} R_i \, R_{1i} \, d\theta \, d\phi \\ m^c \dfrac{\partial^2 w}{\partial t^2} R_c \, R_{1c} \, d\theta \, d\phi \end{array}\right\} \quad (2) \end{array}\right\} \text{Translational inertia forces (a)}$$

$$\begin{array}{l} -I^f \, (R_o \, R_{1o} + R_i \, R_{1i}) \dfrac{\partial^2 \psi_x}{\partial t^2} \, d\theta \, d\phi \\ -I^c \dfrac{\partial^2 \psi_x}{\partial t^2} R_c \, R_{1c} \, d\theta \, d\phi \end{array} \qquad \text{Rotatory inertia couples} \quad \text{(b)}$$

2.5 Equilibrium Equations of the Sandwich Shell

To obtain the equilibrium equations of the composite shell, all of the forces acting on the outer and inner facings need to be transferred to the middle surface of the shell, which for a sandwich shell of symmetrical construction is the middle surface of the core. The following relations between the facing and core radii are needed: (See Figure 12.)

$$R_o = R_c + a \sin \phi; \qquad R_i = R_c - a \sin \phi$$

$$R_{1o} = R_{1c} + a \quad ; \qquad R_{1i} = R_{1c} - a$$

$$R_{2o} = R_{2c} + a \quad ; \qquad R_{2i} = R_{2c} - a$$

The first equation of equilibrium of the sandwich shell is obtained by adding the force components: A.1 (a-b), B.1 (a-b) and C.1 (a) of Section 2.3 and (1.a) of Section 2.4. Making the substitution for the inner and outer radii yields:

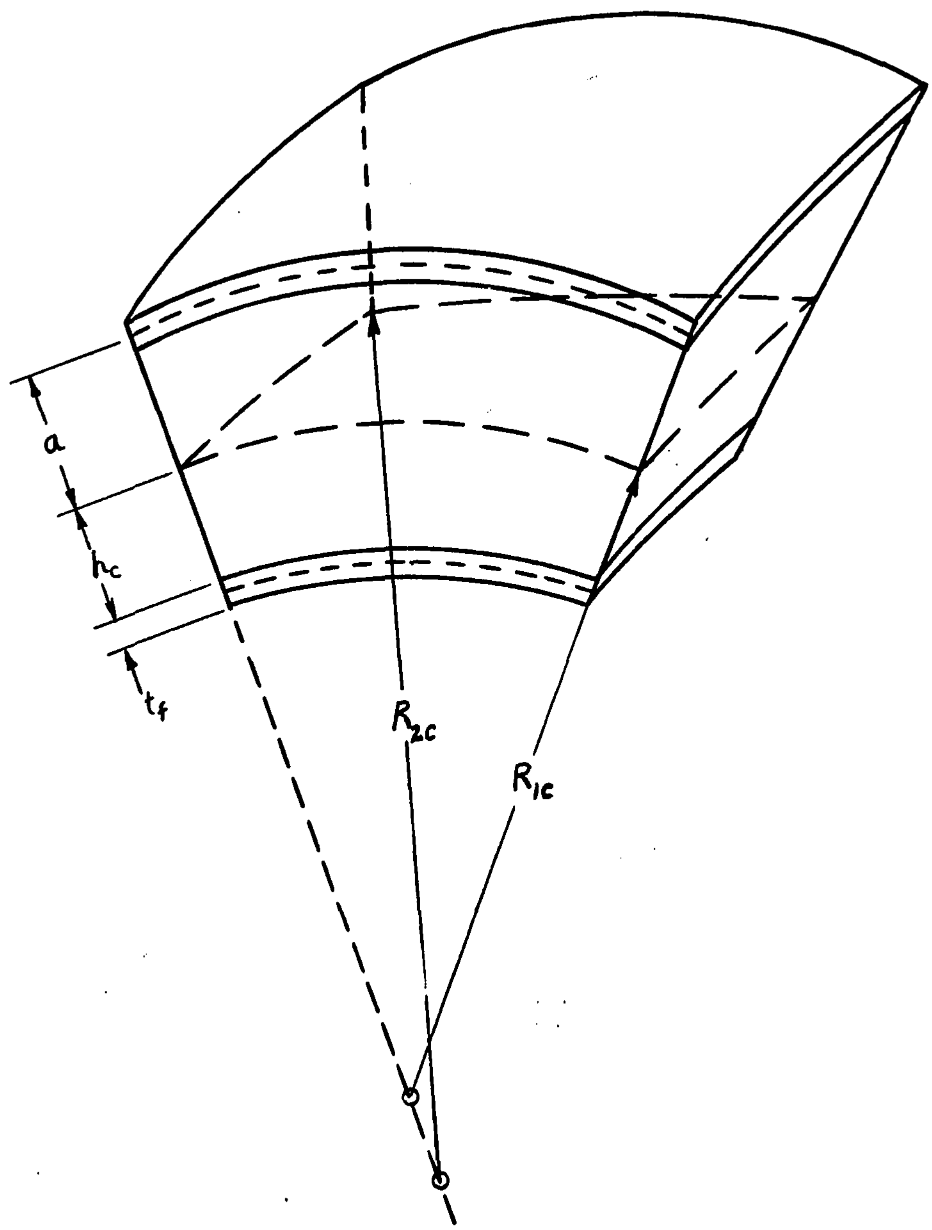

Figure 12-- Section of the sandwich shell of revolution showing the dimensions of the facings and core. Thicknesses are exaggerated.

$$\left[F_x^o + \frac{\partial F_x^o}{\partial \phi} d\phi\right]\left[(R_c + a\sin\phi) + \frac{\partial(R_c + a\sin\phi)}{\partial \phi} d\phi\right] d\theta - F_x^o (R_c + a\sin\phi)\, d\theta$$
$$-F_y^o (R_{1c} + a)\, d\phi\, d\theta \cos\phi + (F_x^i + \frac{\partial F_x^i}{\partial \phi} d\phi)\left[(R_c - a\sin\phi) + \frac{\partial(R_c - a\sin\phi)}{\partial \phi} d\phi\right] d\theta$$
$$-F_x^i (R_c - a\sin\phi)\, d\theta - F_y^i (R_{1c} - a)\, d\phi\, d\theta \cos\phi + Q_x R_c\, d\theta \frac{d\phi}{2}$$
$$+ (Q_x + \frac{\partial Q_x}{\partial \phi} d\phi)(R_c + \frac{\partial R_c}{\partial \phi} d\phi)\, d\theta \frac{d\phi}{2} - m^f \frac{\partial^2 u}{\partial t^2}(R_c + a\sin\phi)(R_{1c} + a)\, d\theta\, d\phi$$
$$-m^f \frac{\partial^2 u}{\partial t^2}(R_c - a\sin\phi)(R_{1c} - a)\, d\theta\, d\phi - m^c \frac{\partial^2 u}{\partial t^2} R_c R_{1c}\, d\theta\, d\phi = 0$$

Expanding the above equation, neglecting higher order terms and dividing by $d\theta\, d\phi$ yields:

$$\frac{\partial}{\partial\phi}\left[(R_c + a\sin\phi) F_x^o + (R_c - a\sin\phi) F_x^i\right] - \left[(R_{1c} + a) F_y^o + (R_{1c} - a) F_y^i\right]\cos\phi$$
$$+ R_c Q_x = \left[(2m^f + m^c) R_c R_{1c} + 2m^f a^2 \sin\phi\right] \frac{\partial^2 u}{\partial t^2} \qquad 2.5.1$$

The second equilibrium equation of the sandwich shell is obtained by adding the force components: A.2 (a), B.2 (a) and C.2 (a) of Section 2.3 and 2.a of Section 2.4. Making the substitution for the inner and outer radii yields:

$$-F_x^o (R_c + a\sin\phi)\, d\theta \frac{d\phi}{2} - \left[(F_x^o + \frac{\partial F_x^o}{\partial\phi} d\phi)\right]\left[(R_c + a\sin\phi) + \frac{\partial(R_c + a\sin\phi)}{\partial\phi} d\phi\right] d\theta \frac{d\phi}{2}$$
$$-F_y^o (R_{1c} + a)\, d\theta\, d\phi \sin\phi - F_x^i (R_c - a\sin\phi)\, d\theta \frac{d\phi}{2} - F_y^i (R_{1c} - a)\, d\theta\, d\phi \sin\phi$$
$$-(F_x^i + \frac{\partial F_x^i}{\partial\phi} d\phi)\left[(R_c - a\sin\phi) + \frac{\partial(R_c - a\sin\phi)}{\partial\phi} d\phi\right] d\theta \frac{d\phi}{2} + (Q_x + \frac{\partial Q_x}{\partial\phi} d\phi)(R_c + \frac{\partial R_c}{\partial\phi} d\phi)\, d\theta$$
$$-Q_x R_c\, d\theta + m^f (R_c + a\sin\phi)(R_{1c} + a) \frac{\partial^2 w}{\partial t^2} d\theta\, d\phi + m^f (R_c - a\sin\phi)(R_{1c} - a) \frac{\partial^2 w}{\partial t^2} d\theta\, d\phi$$
$$+ m^c R_c R_{1c} \frac{\partial^2 w}{\partial t^2} d\theta\, d\phi = 0$$

Expanding the above equation, neglecting higher order terms and dividing by $d\theta\, d\phi$ yields:

$$\left[(R_c + a\sin\phi)F_x^o + (R_c - a\sin\phi)F_x^i\right] + \left[(R_{1c} + a)F_y^o + (R_{1c} - a)F_y^i\right] \sin\phi$$
$$- \frac{\partial}{\partial\phi}(R_c Q_x) = \left[(2m^f + m^c)\ (R_c R_{1c} + 2m^f\ a^2 \sin\phi\right] \frac{\partial^2 w}{\partial t^2} \qquad 2.5.2$$

The third equilibrium equation is obtained by adding the bending force components: A.3 (a-b), B.3 (a-b) and C.3 (a) of Section 2.3 and (b) of Section 2.4, plus all the components contributed by transferring the normal forces acting on the inner and outer facings to the middle surface of the shell.

$$(M_x^o + \frac{\partial M_x^o}{\partial\phi} d\phi) \left[(R_c + a\sin\phi) + \frac{\partial(R_c + a\sin\phi)}{\partial\phi} d\phi\right] d\theta - M_x^o\ (R_c + a\sin\phi)\ d\theta$$
$$-M_y^o(R_{1c}+a)\ d\theta d\phi\cos\phi + (M_x^i + \frac{\partial M_x^i}{\partial\phi} d\theta) \left[(R_c - a\sin\phi) + \frac{\partial(R_c - a\sin\phi)}{\partial\phi} d\phi\right] d\theta$$
$$-M_x^i(R_c - a\sin\phi)\ d\theta - M_y^i\ (R_{1c}-a)\ d\theta d\phi\cos\phi + a \frac{\partial}{\partial\phi}\left[(R_c + a\sin\phi)F_x^o\right] d\theta\ d\phi$$
$$- \left[(Q_x + \frac{\partial Q_x}{\partial\phi} d\phi)(R_c + \frac{\partial R_c}{\partial\phi} d\phi) + Q_x R_c d\theta\right] \frac{R_{1c} d\phi}{2} - a \frac{\partial}{\partial\phi}\left[(R_c - a\sin\phi)F_x^i\right] d\theta\ d\phi$$
$$-a(R_{1c}+a)F_y^o\ d\phi d\theta\cos\phi + a(R_{1c}-a)F_y^i\ d\phi d\theta\cos\phi - I^f(R_c + a\sin\phi)(R_{1c}+a)\frac{\partial^2 \gamma_x}{\partial t^2} d\theta d\phi$$
$$-I^f\ (R_c - a\sin\phi)\ (R_{1c} - a)\ \frac{\partial^2 \gamma_x}{\partial t^2} d\theta d\phi - I^c\ R_c R_{1c} \frac{\partial^2 \gamma_x}{\partial t^2} d\theta\ d\phi$$
$$-a\ m^f\ (R_c + a\sin\phi)(R_{1c}+a)\frac{\partial^2 u}{\partial t^2} d\theta d\phi + a\ m^f\ (R_c - a\sin\phi)(R_{1c}-a)\frac{\partial^2 u}{\partial t^2} d\theta\ d\phi = 0$$

Expanding the above equations, neglecting higher order terms and dividing by $d\theta\ d\phi$ yields:

$$\frac{\partial}{\partial\phi}\left[(R_c + a\sin\phi)M_x^o + (R_c - a\sin\phi)M_x^i + a(R_c + a\sin\phi)F_x^o - a(R_c - a\sin\phi)F_x^i\right]$$
$$\left[(R_{1c}+a)M_y^o + (R_{1c}-a)M_y^i + a(R_{1c}+a)F_y^o - a(R_{1c}-a)F_y^i\right] \cos\phi \qquad 2.5.3$$
$$-R_c R_{1c} Q_x = \left[(2I^f + I^c)R_c R_{1c} + 2I^f a^2 \sin\phi\right] \frac{\partial^2 \gamma_x}{\partial t^2} + 2m^f a^2 (R_c + R_{1c}\sin\phi) \frac{\partial^2 u}{\partial t^2}$$

For a homogeneous shell of revolution $a = 0$. If this substitution is made and the inertia effects are omitted in Equations 2.5.1, 2.5.2 and

2.5.3 they reduce to those derived by Timoshenko(1), page 530, with the signs in front of Q_x in Equations 2.5.1 and 2.5.2 reversed. This is due to the sign convention for the shear force.

Equations 2.5.1, 2.5.2, and 2.5.3 can be expressed in terms of arc length as follows: (See relations 2.1.1)

$$\frac{\partial}{\partial s}\left[(R_c+a\sin\phi)F_s^o + (R_c-a\sin\phi)F_s^i\right] - \frac{1}{R_{1c}}\left[(R_{1c}-a)F_\theta^o + (R_{1c}-a)F_\theta^i\right]\cos\phi + \frac{R_c}{R_{1c}}Q_s = \left[(2m^f+m^c)R_c + 2m^f\frac{a^2}{R_{1c}}\sin\phi\right]\frac{\partial^2 u}{\partial t^2} \qquad 2.5.4$$

$$\frac{1}{R_{1c}}\left[(R_c+a\sin\phi)F_s^o + (R_c-a\sin\phi)F_s^i\right] + \frac{1}{R_{1c}}\left[(R_{1c}+a)F_\theta^o + (R_{1c}-a)F_\theta^i\right]\sin\phi - \frac{\partial}{\partial s}(R_cQ_s) = \left[(2m^f+m^c)R_c + 2m^f\frac{a^2}{R_{1c}}\sin\phi\right]\frac{\partial^2 w}{\partial t^2} \qquad 2.5.5$$

$$\frac{\partial}{\partial s}\left[(R_c+a\sin\phi)M_s^o + (R_c-a\sin\phi)M_s^i + a(R_c+a\sin\phi)F_s^o - a(R_c-a\sin\phi)F_s^i\right] - \frac{1}{R_{1c}}\left[(R_{1c}+a)M_\theta^o + (R_{1c}-a)M_\theta^i + a(R_{1c}+a)F_\theta^o - a(R_{1c}-a)F_\theta^i\right]\cos\phi - R_cQ_s = \left[(2I^f+I^c)R_c + 2I^f\frac{a^2}{R_{1c}}\sin\phi\right]\frac{\partial^2\gamma_s}{\partial t^2} + \left[2m^fa^2\left(\frac{R_c}{R_{1c}}+\sin\phi\right)\right]\frac{\partial^2 u}{\partial t^2} \qquad 2.5.6$$

Where the following subscript changes are made for the sake of conveniency in later discussion: $x \longrightarrow s$

$y \longrightarrow \theta$

From Equations 2.5.1, 2.5.2 and 2.5.3 the following relations can be easily deduced:

$$R_{2c}F_x = (R_{2c}+a)F_x^o + (R_{2c}-a)F_x^i \qquad 2.5.7$$

$$R_{1c}F_y = (R_{1c}+a)F_y^o + (R_{1c}-a)F_y^i$$

$$R_{2c}M_x = (R_{2c}+a)M_x^o + (R_{2c}-a)M_x^i + a(R_{2c}+a)F_x^o - a(R_{2c}-a)F_x^i$$
$$R_{1c}M_y = (R_{1c}+a)M_y^o + (R_{1c}-a)M_y^i + a(R_{1c}+a)F_y^o - a(R_{1c}-a)F_y^i \quad 2.5.4$$

The above equations are similar to those obtained by Reissner (9).

The equilibrium equations which are obtained in summing forces in the y-direction and moments with respect to the x-axis are identically satisfied due to symmetry.

2.6 Differential Equations of the Sandwich Shell

It is assumed that the displacements in the meridional and normal directions at any point of the sandwich shell are as follows (12, 22, 24):

$$U\,(s,z,t) = u(s,t) + z\,\Upsilon_s(s,t)$$
$$W\,(s,z,t) = w(s,t) \quad 2.6.1$$

For axisymmetric deformations, the strains pertaining to the assumed displacements in Equation 2.6.1 are (7):

$$\bar{\epsilon}_s = \epsilon_s + z\mathcal{X}_s$$
$$\bar{\epsilon}_\theta = \epsilon_\theta + z\mathcal{X}_\theta \quad 2.6.2$$
$$\bar{\epsilon}_{sz} = \frac{\partial w}{\partial s} + \Upsilon_s$$

Where ϵ_s and ϵ_θ are the median fiber strains and are defined by:

$$\epsilon_s = \frac{\partial u}{\partial s} + \frac{w}{R_{1c}}$$
$$\epsilon_\theta = \frac{u}{R_{2c}\tan\phi} + \frac{u}{R_{2c}} \quad 2.6.3$$

The change of curvature corresponding to the displacements in Equation 2.6.1 according to the shear theory are:

$$\mathcal{H}_s = \frac{\partial \psi_s}{\partial s} + \frac{\partial}{\partial s}\left(\frac{u}{R_{1c}}\right) \qquad 2.6.4$$

$$\mathcal{H}_\theta = \frac{\psi_s}{R_{2c}\tan\phi} + \frac{u}{R_{1c}R_{2c}\tan\phi}$$

The stress resultants are defined as follows:

$$F_s^o = \int_{h_c}^{h_c+2t_f} \sigma_s^m \left(1 + \frac{z-a}{R_{2c}+a}\right) dz$$

$$F_\theta^o = \int_{h_c}^{h_c+2t_f} \sigma_\theta^m \left(1 + \frac{z-a}{R_{1c}+a}\right) dz$$

$$M_s^o = \int_{h_c}^{h_c+2t_f} (z-a)\sigma_s^b \left(1 + \frac{z-a}{R_{2c}+a}\right) dz$$

$$M_\theta^o = \int_{h_c}^{h_c+2t_f} (z-a)\, \sigma_\theta^b \left(a + \frac{z-a}{R_{1c}+a}\right) dz$$

$$F_s^i = \int_{-(h_c+2t_f)}^{-h_c} \sigma_s^m \left(a + \frac{z+a}{R_{2c}-a}\right) dz \qquad 2.6.5$$

$$F_\theta^i = \int_{-(h_c+2t_f)}^{-h_c} \sigma_\theta^m \left(a + \frac{z+a}{R_{1c}-a}\right) dz$$

$$M_s^i = \int_{-(h_c+2t_f)}^{-h_c} (z+a)\, \sigma_s^b \left(1 + \frac{z+a}{R_{2c}-a}\right) dz$$

$$M_\theta^i = \int_{-(h_c+2t_f)}^{-h_c} (z+a)\, \sigma_\theta^b \left(1 + \frac{z+a}{R_{1c}-a}\right) dz$$

$$Q_s = \int_{-h_c}^{h_c} \sigma_{sz} \left(1 + \frac{z}{R_{2c}}\right) dz$$

If the integrations in Equations 2.6.5 are carried out with the terms z/R retained, the results are of the Flügge-Vlasov type (4,36). On the other hand if the terms z/R are neglected when compared to unity, and then the integrations carried out, the results are that of Love's first approximation (3). On the basis of what Goldenveiser (37) calls "authentic" accuracy, the author has concluded that the latter approach is the most appropriate here. Also, Garnet and Kempner (24) have shown that the difference is negligible in the case of a homogeneous conical frustum. Basically, the fundamental hypothesis is inexact; therefore, one cannot expect to improve the accuracy of Kirchhoff-type theory of shells by merely adopting the more complicated Flügge-Vlasov type analysis.

The stress-strain relations are defined as follows:

$$\begin{aligned}
\sigma_s^{m} &= \frac{E_s}{1 - \nu_s\nu_\theta}(\epsilon_s + \nu_s\epsilon_\theta) \\
\sigma_\theta^{m} &= \frac{E_\theta}{1 - \nu_s\nu_\theta}(\epsilon_\theta + \nu_\theta\,\epsilon_s) \\
\sigma_s^{b} &= \frac{E_s\ z}{1 - \nu_s\nu_\theta}(\varkappa_s + \nu_s\varkappa_\theta) \\
\sigma_\theta^{b} &= \frac{E_\theta\ z}{1 - \nu_s\nu_\theta}(\varkappa_\theta + \nu_\theta\varkappa_s) \\
\sigma_{sz} &= G_{sz}\,\epsilon_{sz}
\end{aligned} \qquad 2.6.6$$

If Equations 2.6.6, with z replaced by (z - a) for the outer facing and (z + a) for the inner facing, are substituted in Equations 2.6.5 and the integration is carried out, the following will be obtained with the relations 2.6.3 and 2.6.4 taken into consideration.

$$\begin{aligned}
F_s^{o},\ F_s^{i} &= \frac{2t_fE_s}{1 - \nu_s\nu_\theta}\left[\frac{\partial u}{\partial s} + \frac{w}{R_{1c}} + \nu_s\left(\frac{u}{R_{2c}\tan\phi} + \frac{w}{R_{2c}}\right)\right] \\
F_\theta^{o},\ F_\theta^{i} &= \frac{2t_fE_\theta}{1 - \ {}_{s}\ {}_{\theta}}\left[\frac{u}{R_{2c}\tan\phi} + \frac{w}{R_{2c}} + \nu_\theta\left(\frac{\partial u}{\partial s} + \frac{w}{R_{1c}}\right)\right]
\end{aligned} \qquad 2.6.7$$

$$M_s^o,\ M_s^i = \frac{2t_f^3\ E_s}{1 - \nu_s\nu_\theta}\left[\frac{\partial\psi_s}{\partial s} + \frac{\partial}{\partial s}\left(\frac{u}{R_{1c}}\right) + \nu_s\left(\psi_s + \frac{u}{R_{1c}}\right)/\ R_{2c}\tan\phi\right]$$

$$M_\theta^o,\ M_\theta^i = \frac{2t_f^3\ E_\theta}{1 - \ {}_{s}\ {}_{\theta}}\left[\left(\psi_s + \frac{u}{R_{1c}}\right)/R_{2c}\tan\phi + \nu_\theta\left\{\frac{\partial\psi_s}{\partial s} + \frac{\partial}{\partial s}\left(\frac{u}{R_{1c}}\right)\right\}\right] \qquad 2.6.8$$

$$Q_s = 2h_c\ K_s\ Gsz\ \left(\frac{\partial w}{\partial s} + \psi_s\right) \qquad 2.6.9$$

where K_s accounts for the transverse shear stress distribution through the thickness.

Substituting Equations 2.6.7, 2.6.8 and 2.6.9 into Equations 2.5.4, 2.5.5 and 2.5.6, the following three differential equations are obtained:

$$\frac{\partial}{\partial s}\left\{\eta_s R_c\left[\frac{\partial u}{\partial s} + \frac{w}{R_{1c}} + \nu_s\left(\frac{u\cot\phi}{R_{2c}} + \frac{w}{R_{2c}}\right)\right]\right\} - \left\{\eta_\theta\left[\frac{u\cot\phi}{R_{2c}} + \frac{w}{R_{2c}} + \nu_\theta\left(\frac{\partial u}{\partial s} + \frac{w}{R_{1c}}\right)\right]\right\}\cos\phi + \eta\frac{R_c}{R_{1c}}\left(\frac{\partial w}{\partial s} + \psi_s\right) = \left[(2m^f + m^c)R_c + 2m^f\frac{a^2}{R_{1c}}\sin\phi\right]\frac{\partial^2 u}{\partial t^2} \qquad 2.6.10$$

$$\left\{\eta_s\frac{R_c}{R_{1c}}\left[\frac{\partial u}{\partial s} + \frac{w}{R_{1c}} + \nu_s\left(\frac{u\cot\phi}{R_{2c}} + \frac{w}{R_{2c}}\right)\right]\right\} + \left\{\eta_\theta\left[\frac{u\cot\phi}{R_{2c}} + \frac{w}{R_{2c}} + \nu_\theta\left(\frac{\partial u}{\partial s} + \frac{w}{R_{1c}}\right)\right]\right\}\sin\phi - \frac{\partial}{\partial s}\left[\eta R_c\left(\frac{\partial w}{\partial s} + \psi_s\right)\right] = \left[(2m^f + m^c)\ R_c + 2m^f\ \frac{a^2}{R_{1c}}\ \sin\phi\right]\frac{\partial^2 w}{\partial t^2} \qquad 2.6.11$$

$$\frac{\partial}{\partial s}\left\{ \eta_s^* R_c \left[\frac{\partial \psi_s}{\partial s} + \frac{\partial}{\partial s}\left(\frac{u}{R_{1c}}\right) + \nu_s \frac{\cot\phi}{R_{2c}}\left(\psi_s + \frac{u}{R_{1c}}\right)\right] + \eta_s a^2 \sin\phi \left[\frac{\partial u}{\partial s} + \frac{w}{R_{1c}}\right.\right.$$

$$\left.\left. + \nu_s \left(\frac{u\cot\phi}{R_{2c}} + \frac{w}{R_{2c}}\right)\right]\right\} - \left\{\eta_\theta^* \left[\frac{\psi_s \cot\phi}{R_{2c}} + \frac{\cot\phi}{R_{2c}}\left(\frac{u}{R_{1c}}\right) + \nu_\theta\left(\frac{\partial \psi_s}{\partial s} + \frac{\partial}{\partial s}\left(\frac{u}{R_{1c}}\right)\right)\right]\right. \quad 2.6.12$$

$$\left. + \eta_\theta \frac{a^2}{R_{1c}}\left[\frac{u\cot\phi}{R_{2c}} + \frac{w}{R_{2c}} + \nu_\theta\left(\frac{\partial u}{\partial s} + \frac{w}{R_{1c}}\right)\right]\right\}\cos\phi - \eta R_c\left(\frac{\partial w}{\partial s} + \psi_s\right) =$$

$$\left[(2I^f + I^c)R_c + 2I^f \frac{a^2}{R_{1c}}\sin\phi\right]\frac{\partial^2 \psi_s}{\partial t^2} + 2m^f a^2\left(\frac{R_c}{R_{1c}} + \sin\phi\right)\frac{\partial^2 u}{\partial t^2}$$

where $\eta = 2h_c K_s G_{sz}$; $\eta_s = 4t_f E_s/(1-\nu_s\nu_\theta)$; $\eta_\theta = 4t_f E_\theta/(1-\nu_s\nu_\theta)$;

$\eta_s^* = 4t_f^3 E_s/3(1-\nu_s\nu_\theta)$; $\eta_\theta^* = 4\ t_f^3\ E_\theta/3(1-\nu_s\nu_\theta)$. 2.6.13

It is assumed that

$$w = \bar{W}(s)\cos\Omega t$$

$$\psi_s = \bar{\Psi}_s(s)\cos\Omega t \qquad 2.6.14$$

and $$u = \bar{U}(s)\cos\Omega t$$

where Ω is the frequency.

Substituting Equations 2.6.14 into Equations 2.6.10, 2.6.11, and 2.6.12 and collecting similar terms, the following basic differential equations for a sandwich shell of revolution are obtained:

$$\eta_s R_c \frac{d^2\bar{U}}{ds^2} + \left[\eta_s \frac{dR_c}{ds} + (\eta_s\nu_s - \eta_\theta\nu_\theta)\cos\phi\right]\frac{d\bar{U}}{ds} + \left[\eta_s\nu_s\frac{d}{ds}(\cos\phi) - \eta_\theta\frac{\sin\phi}{R_{2c}}\right.$$

$$\left. - m_e\Omega^2\right]\bar{U} + \left[\eta_s\frac{R_c}{R_{1c}} + \eta_s\nu_s\sin\phi + \eta\frac{R_c}{R_{1c}}\right]\frac{d\bar{W}}{ds} + \left[\eta_s\frac{d}{ds}\left(\frac{R_c}{R_{1c}}\right) + \eta_s\nu_s\frac{d}{ds}(\sin\phi)\right. \quad 2.6.15$$

$$\left. - \eta_\theta\cos\phi/R_{2c} - \eta_\theta\nu_\theta\cos\phi/R_{1c}\right]\bar{W} + \left[\eta R_c/R_{1c}\right]\bar{\Psi}_s = 0$$

$$\left[\eta_s\frac{R_c}{R_{1c}}+\eta_\theta\nu_\theta\sin\phi\right]\frac{d\bar{U}}{ds}+\left[\eta_s\nu_s\cos\phi/R_{1c}+\eta_\theta\cos\phi/R_{2c}\right]\bar{U}-\left(\eta R_c\right)\frac{d^2\bar{W}}{ds^2}$$

$$-\left(\eta\frac{dR_c}{ds}\right)\frac{d\bar{W}}{ds}+\left[\eta_s\frac{R_c}{R_{1c}^2}+\eta_s\nu_sR_c/R_{1c}R_{2c}+\eta_\theta\frac{\sin\phi}{R_{2c}}+\eta_\theta\nu_\theta\sin\phi/R_{1c}\right. \qquad 2.6.16$$

$$\left.-m_e\Omega^2\right]\bar{W}-\left(\eta R_c\right)\frac{d\bar{\Psi}_s}{ds}-\left(\eta\frac{dR_c}{ds}\right)\bar{\Psi}_s=0$$

$$\left[\eta_s^*\frac{R_c}{R_{1c}}+\eta_s a^2\sin\phi\right]\frac{d^2\bar{U}}{ds^2}+\left[\eta_s^*\frac{d}{ds}\left(\frac{R_c}{R_{1c}}\right)+\eta_s^*R_c\frac{d}{ds}\left(\frac{1}{R_{1c}}\right)+\cos\phi/R_{1c}+\left[\eta_s a^2\frac{d}{ds}\left(\right.\right.\right.$$

$$\left.\left.\left.(\sin\phi)\right)\right]+\eta_s\nu_s a^2\cos\phi/R_{2c}-\eta_\theta^*\nu_\theta\cos\phi/R_{1c}-\eta_\theta\nu_\theta a^2\cos\phi/R_{1c}\right]\frac{d\bar{U}}{ds}+$$

$$\left[\eta_s^*R_c\frac{d^2}{ds^2}\left(\frac{1}{R_{1c}}\right)+\eta_s^*\frac{dR_c}{ds}\frac{d}{ds}\left(\frac{1}{R_{1c}}\right)+\frac{d}{ds}\left(\frac{\cos\phi}{R_{1c}}\right)+\eta_s\nu_s a^2\frac{d}{ds}\left(\frac{\cos\phi}{R_{2c}}\right)-\right.$$

$$\left.\eta_\theta^*\cot\phi\cos\phi/R_{1c}R_{2c}-\eta_\theta^*\nu_\theta\cos\phi\frac{d}{ds}\left(\frac{1}{R_{1c}}\right)-\eta_\theta a^2\cot\phi\cos\phi/R_{1c}R_{2c}-\bar{m}_e\Omega^2\right]\bar{U}$$

$$2.6.17$$

$$+\left[\eta_s a^2\sin\phi/R_{1c}+\eta_s\nu_s a^2\sin\phi/R_{2c}-\eta R_c\right]\frac{d\bar{W}}{ds}+\left[\eta_s a^2\frac{d}{ds}\left(\frac{\sin\phi}{R_{1c}}\right)+\right.$$

$$\left.\eta_s\nu_s a^2\frac{d}{ds}\left(\frac{\sin\phi}{R_{2c}}\right)-\eta_\theta a^2\cos\phi/R_{1c}R_{2c}-\eta_\theta\nu_\theta a^2\cos\phi/R_{1c}^2\right]\bar{W}+\eta_s^*R_c\frac{d^2\bar{\Psi}_s}{ds^2}$$

$$+\left[\eta_s^*\frac{dR_c}{ds}+\eta_s^*\nu_s\cos\phi-\eta_\theta^*\nu_\theta\cos\phi\right]\frac{d\bar{\Psi}_s}{ds}+\left[\eta_s^*\nu_s\frac{d}{ds}(\cos\phi)-\right.$$

$$\left.\eta_s^*\cot\phi\cos\phi/R_{2c}-\eta R_c-I_e\Omega^2\right]\bar{\Psi}_s=0$$

where $m_e = (2m^f + m^c)R_c + 2m^f\dfrac{a^2}{R_{1c}}\sin\phi$

$\bar{m}_e = 2m^f a^2\left(\dfrac{R_c}{R_{1c}} + \sin\phi\right)$

$I_e = (2I^f + I^c)R_c + 2I^f\dfrac{a^2}{R_{1c}}\sin\phi$

Equations 2.6.15, 2.6.16 and 2.6.17 are written in terms of ordinary derivatives because there is only one independent variable.

2.7 Discussion

In general, the quantities, R_c, R_{1c}, and ϕ are all functions of the arc length, s; therefore, Equations 2.6.15, 2.6.16 and 2.6.17 are ordinary differential equations with variable coefficients. To the author's knowledge there is no closed form solution in existence for such equations. An attempt was made to uncouple the equations in the partials of $\bar{U}$, $\bar{W}$, and $\bar{\Psi}_s$. After considerable effort it appeared to the author that even if such uncoupling were to be achieved, the resulting differential equations in $\bar{U}$, $\bar{W}$, and $\bar{\Psi}_s$ would be very lengthy and impractical for calculation purposes. With this in mind the author has chosen the Rayleigh-Ritz method to analyze the vibrational characteristics of sandwich shells of revolution.

CHAPTER 3

RAYLEIGH-RITZ ANALYSIS

3.1 Introductory Formulation of the Stress-Strain Relations

It is assumed, as in Chapter 2, that the displacements in the meridional and normal directions at any point of the composite shell are as follows:

$$\begin{aligned} U\ (s,z,t) &= u(s,t) + z\,\psi_s(s,t) \\ W\ (s,z,t) &= w(s,t) \end{aligned} \qquad 3.1.1$$

where u and w are the effective displacements in the meridional and normal directions on the shell respectively, and ψ_s is the effective angle of rotation of the normal to the middle surface of the composite shell in the meridional direction.

For axisymmetric deformations, the strains pertaining to the assumed displacements in Equations 3.1.1 are:

$$\begin{aligned} \bar{\epsilon}_s &= \epsilon_s + z\,\mathcal{H}_s \\ \bar{\epsilon}_\theta &= \epsilon_\theta + z\,\mathcal{H}_\theta \\ \bar{\epsilon}_{sz} &= \frac{\partial w}{\partial s} + \psi_s \end{aligned} \qquad 3.1.2.$$

where ϵ_s and ϵ_θ are the median fiber strains and are defined for a conical shell as follows:

$$\epsilon_s = \frac{\partial u}{\partial s}$$
$$\epsilon_\theta = \frac{u}{s} + \frac{w}{s\tan\alpha} \qquad 3.1.3$$

The change of curvature corresponding to the displacements in Equations 3.1.1 for a conical shell are:

$$\mathcal{H}_s = \frac{\partial \psi_s}{\partial s}$$
$$\mathcal{H}_\theta = \frac{\psi_s}{s} \qquad 3.1.4$$

From Chapter 2 the stress resultants per unit length are given as follows:

$$\left\{\begin{matrix} F_s \\ M_s \\ Q_s \\ F_\theta \\ M_\theta \end{matrix}\right\} = \int_{z_1}^{z_2} \left\{\begin{matrix} \sigma_s^{\ m} \\ z\sigma_s^{\ b} \\ \sigma_{sz} \\ \sigma_\theta^{\ m} \\ z\sigma_\theta^{\ b} \end{matrix}\right\} dz \qquad 3.1.5$$

In Equations 3.1.5 the term z/R is neglected in comparison to unity. This has been explained in Chapter 2, Section 2.6.

The stress-strain relations are given as follows:

(a) For membrane stresses:

$$\sigma_s^{\ m} = \frac{E_s}{1 - \nu_s\nu_\theta}(\epsilon_s + \nu_s\epsilon_\theta)$$
$$\sigma_\theta^{\ m} = \frac{E_\theta}{1 - \nu_s\nu_\theta}(\epsilon_\theta + \nu_\theta\epsilon_s) \qquad 3.1.6$$

(b) For bending stresses:

$$\sigma_s^{\ b} = \frac{E_s\, z}{1 - \nu_s\nu_\theta}(\mathcal{H}_s + \nu_s\mathcal{H}_\theta)$$
$$\sigma_\theta^{\ b} = \frac{E_\theta\, z}{1 - \nu_s\nu_\theta}(\mathcal{H}_\theta + \nu_\theta\mathcal{H}_s) \qquad 3.1.7$$

(c) For shearing stresses (transverse):

$$\sigma_{sz} = G_{sz}\,\epsilon_{sz} \qquad 3.1.8$$

The limits of integrations in Equations 3.1.5 are categorized as follows:

(a) For the core:

$$z_1 = -h_c$$

$$z_2 = h_c$$

(b) For the outer facing:

$$z_1 = h_c$$

$$z_2 = h_c + 2t_f$$

where the transformation of the normal coordinate from the middle surface of the outer facing to the middle surface of the composite is given by:

$$z^o = z - h_c - t_f = z - a$$

(c) For the inner facing:

$$z_1 = -(h_c + 2t_f)$$

$$z_2 = -h_c$$

with the transformation:

$$z^i = z + h_c + t_f = z + a$$

Equations 2.5.4 can be written as follows:

$$\begin{aligned}
F_s &= (1 + \frac{a}{R_{2c}})\,F_s^o + (1 - \frac{a}{R_{2c}})\,F_s^i \\
F_\theta &= (1 + \frac{a}{R_{1c}})\,F_\theta^o + (1 - \frac{a}{R_{1c}})\,F_\theta^i \\
M_s &= (1 + \frac{a}{R_{2c}})M_s^o + (1 - \frac{a}{R_{2c}})M_s^i + a\,(1 + \frac{a}{R_{2c}})F_s^o - a\,(1 - \frac{a}{R_{2c}})F_s^i \\
M_\theta &= (1 + \frac{a}{R_{1c}})M_\theta^o + (1 - \frac{a}{R_{1c}})M_\theta^i + a(1 + \frac{a}{R_{1c}})F_\theta^o - a\,(1 - \frac{a}{R_{1c}})F_\theta^i
\end{aligned} \qquad 3.1.9$$

For a conical shell, where $R_{2c} = s \tan\alpha$ and $R_{1c} = \infty$, (see Figure 13), the following sandwich conical shell relations are obtained from Equation 3.1.9:

$$
\begin{aligned}
F_s &= (1 + \frac{a}{s\tan\alpha})F_s^o + (1 + \frac{a}{s\tan\alpha})F_s^i \\
F_\theta &= F_\theta^o + F_\theta^i \\
M_s &= (1 + \frac{a}{s\tan\alpha})M_s^o + (1 - \frac{a}{s\tan\alpha})M_s^i + \\
&\quad a\,(1 + \frac{a}{s\tan\alpha})F_s^o - a\,(1 - \frac{a}{s\tan\alpha})F_s^i \\
M_\theta &= M_\theta^o + M_\theta^i\, v + a(F_\theta^o - F_\theta^i)
\end{aligned}
\qquad 3.1.10
$$

For a sandwich conical shell Equations 2.6.7, 2.6.8 and 2.6.9 reduce to the following:

$$
\begin{aligned}
F_s^o,\ F_s^i &= \frac{2t_f\, E_s}{1 - \nu_s\nu_\theta}\left[\frac{\partial u}{\partial s} + \nu_s(\frac{u}{s} + \frac{w}{s\tan\alpha})\right] \\
F_\theta^o,\ F_\theta^i &= \frac{2t_f\, E_\theta}{1 - \nu_s\nu_\theta}\left[\frac{u}{s} + \frac{w}{s\tan\alpha} + \nu_\theta\frac{\partial u}{\partial s}\right] \\
M_s^o,\ M_s^i &= \frac{2t_f^3\, E_s}{3(1 - \nu_s\nu_\theta)}\left[\frac{\partial \psi_s}{\partial s} + \nu_s\frac{\psi_s}{s}\right] \\
M_\theta^o,\ M_\theta^i &= \frac{2t_f^3\, E_\theta}{3(1 - \nu_s\nu_\theta)}\left[\frac{\psi_s}{s} + \nu_\theta\frac{\partial \psi_s}{\partial s}\right] \\
Q_s &= 2h_c K_s G_{sz}(\frac{\partial w}{\partial s} + \psi_s)
\end{aligned}
\qquad 3.1.11
$$

3.2 Rayleigh-Ritz Method

The Rayleigh method (38) of obtaining the frequencies of a conservative elastic system is based on the theory of conservation of energy. For a conservative system vibrating in a simple harmonic motion

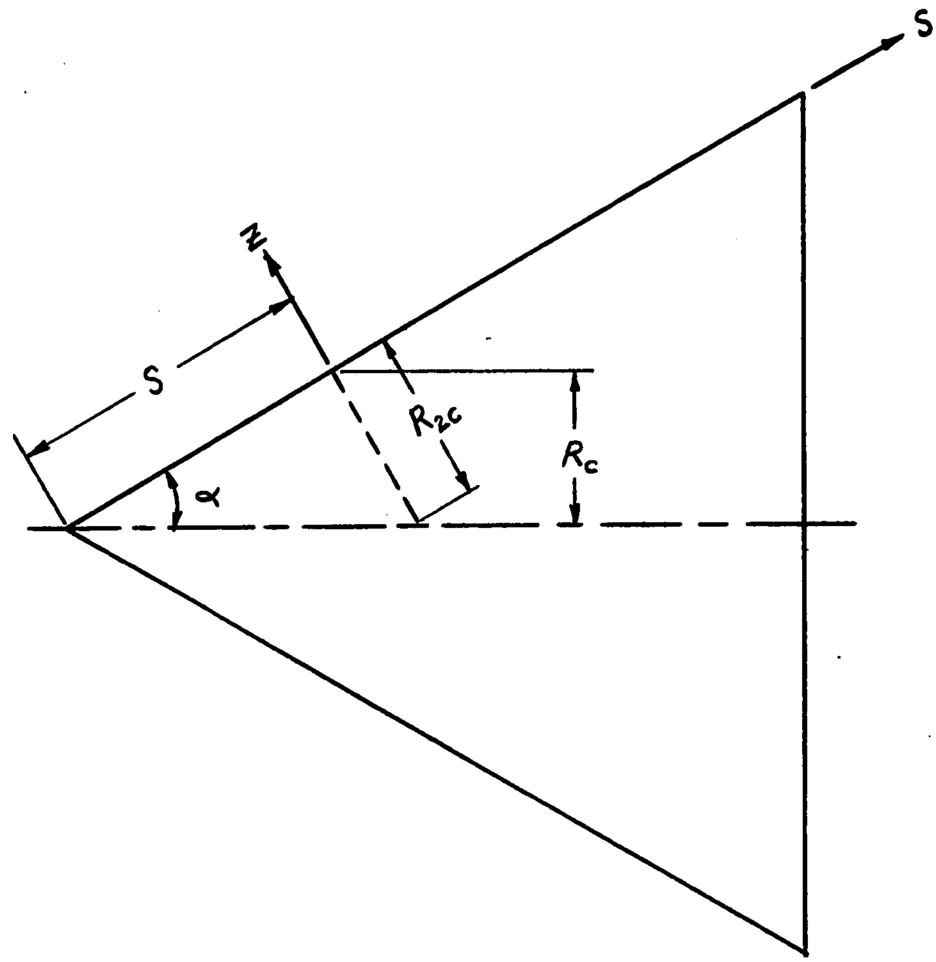

Figure 13--Coordinate system of a conical shell.

the maximum potential[1] energy is equal to the maximum kinetic[2] energy. In using this method a deflection mode is assumed. The corresponding frequencies are then found by equating the maximum potential energy to the maximum kinetic energy. The frequencies obtained using this method are always somewhat higher than the exact values.

A better approximation in calculating the frequencies of conservative systems is obtained by introducing the Ritz method which is a further development of the Rayleigh method. The Ritz modification to the Rayleigh approach represents the deflection mode as a function of several parameters, the magnitude of which should be chosen in such a manner as to reduce to a minimum the vibrational frequency of the system. This modified version of Rayleigh's method is what is called the Rayleigh-Ritz method.

Basically, Ritz suggested that the deflection mode be taken as follows:

$$q_j = \sum_{k=1}^{n} \left(C_{jk} f_{jk} \right) \qquad 3.2.1$$

where q_j is generalized coordinate, C_{jk} are undetermined parameters, and f_{jk} are an arbitrarily chosen set of functions which satisfy the same boundary conditions as q_j. The f_{jk} are functions of time,

---

[1]The potential energy of a conservative system attains its maximum value at the instant the system reaches its maximum displacement.

[2]The kinetic energy of a conservative system attains its maximum value at the instant the system passes through its equilibrium position.

frequency, and the coordinates which define the geometry of the system. If Equation 3.2.1 is substituted into the expressions for the maximum strain and kinetic energies and the undetermined parameters, C_{jk}, chosen in such a manner as to make the following expression hold:

$$\frac{\partial}{C_{jk}} (V_{max} - T_{max}) = 0 \qquad 3.2.2$$

Then an infinite system of homogeneous linear equations are obtained. Substituting for V_{max} and T_{max} their corresponding expressions and carrying out the differentiation with respect to each C_{jk}, the following matrix equation is obtained:

$$\left[Z \right] \left\{ V \right\} = \lambda \left[A \right] \left\{ V \right\} \qquad 3.2.3$$

where Z = n x n square stiffness matrix

V = n x 1 column displacement matrix

λ = eigenvalues

A = n x n square inertia matrix

3.3 Strain Energy of the Composite Shell

The total strain energy of the composite shell consists of the following quantities:

(1) Strain energy due to membrane stresses on the outer facing.

(2) Strain energy due to bending stresses on the outer facing.

(3) Strain energy due to membrane stresses on the inner facing.

(4) Strain energy due to bending stresses on the inner facing.

(5) Strain energy due to transverse shear stresses on the core.

Denoting the total strain energy by V^t then;

$$V^t = V^{o,m} + V^{o,b} + V^{i,m} + V^{i,b} + V^{c,s}$$

In general V is defined as:

$$V = \frac{1}{2}\int_v \sigma_{ij}\, e_{ij}\, dv$$

and for a conical shell this integral extends over its entire volume. The aim set up in this dissertation is to extend the analysis to any sandwich shell of revolution. This will be accomplished by approximating the given shell of revolution by n conical-shell segments.

Thus, the total strain energy of the n conical segments can be written as:

$$V = \sum_{j=1}^{n} \left(V_j^t \right) (j = \text{number of the conical segment}).$$

where V_j^t now takes the form:

$$V_j^t = V_j^{o,m} + V_j^{o,b} + V_j^{i,m} + V_j^{o,b} + V_j^{c,s} \qquad (j = 1,2,\ldots n)$$

Consider a shell of revolution as shown in Figure 14 composed of n conical segments. The strain energy can be written as:

$$\bar{V} = \sum_{j=1}^{n} \left[\frac{1}{2} \int_v \sigma_{ik} e_{ik}\, dv \right]_j \qquad 3.3.1$$

The j determines the limits on the integral in brackets.

Using relations 3.1.6, 3.1.7, and 3.1.8 in 3.3.1 yields

$$\bar{V} = \sum_{j=1}^{n} \Big\{ \frac{1}{2} \int_v \Big[2\bar{\eta}_s(\epsilon_s + \nu_s \epsilon_\theta)\, \epsilon_s + 2\,\bar{\eta}_\theta(\epsilon_\theta + \nu_\theta \epsilon_s)\, \epsilon_\theta \qquad 3.3.2$$

$$+ 2\bar{\eta}_s(z\mathcal{H}_s + \nu_s \mathcal{H}_\theta z)\mathcal{H}_s + 2\bar{\eta}_\theta(z\mathcal{H}_\theta + \nu_\theta \mathcal{H}_s z)\mathcal{H}_\theta + K_s G_{sz}(\epsilon_{sz})^2 \Big]\, dv \Big\}_j$$

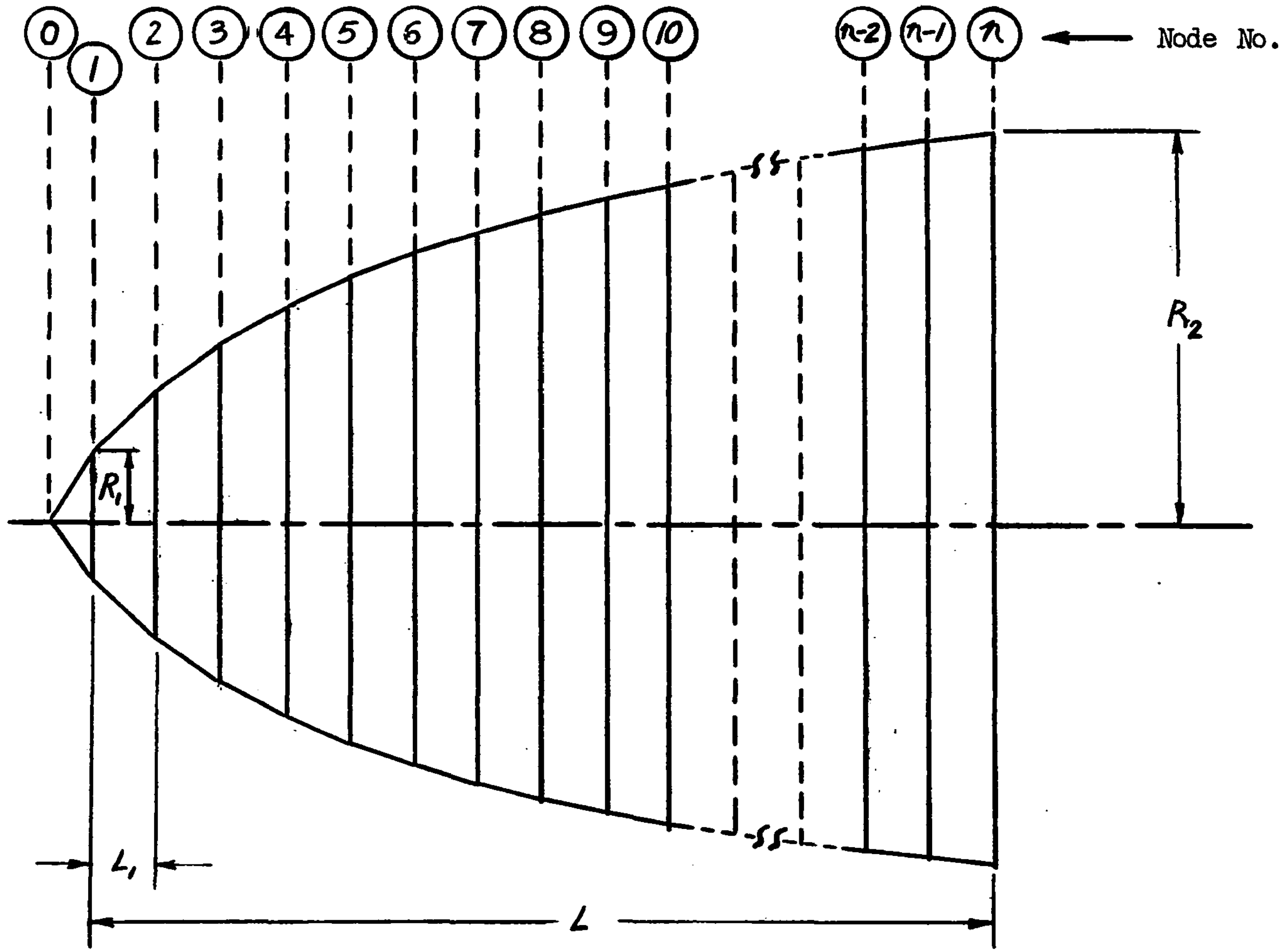

Figure 14--A shell of revolution composed of n conical-shell segments.

where

$$\bar{\eta}_s = \frac{E_s}{1 - \nu_s \nu_\theta} \quad , \quad \bar{\eta}_\theta = \frac{E_\theta}{1 - \nu_s \nu_\theta} \quad .$$

The factor, 2, entered into the above equation because of the outer and inner facing effects on the composite shell.

For a uniform thickness shell, Equation 3.3.1 can be integrated through the thickness and therefore becomes:

$$\bar{V} = \sum_{j=1}^{n} \left(\frac{1}{2} \int_A f_{ik} \, \epsilon_{ik} \, dA \right)_j \qquad 3.3.3$$

where f_{ik} are stress resultants and A is the surface area of the shell.

If Equation 3.3.3 is expanded for the various stress resultants of the composite shell the result is as follows:

$$\bar{V} = \sum_{j=1}^{n} \left\{ \frac{1}{2} \int_s \int_\theta \left[F_s \epsilon_s + F_\theta \epsilon_\theta + M_s \chi_s + M_\theta \chi_\theta + Q_s \epsilon_{sz} \right] \sin\alpha \, d\theta s ds \right\}_j \qquad 3.3.4$$

where $(\sin\alpha)\, d\theta \, s \, ds$ is an area element.

From Equations 3.1.10 and 3.1.11 the following relations are obtained:

$$F_s = \eta_s \left[\frac{\partial u}{\partial s} + \nu_s \left(\frac{u}{s} + \frac{w}{s \tan\alpha} \right) \right]$$

$$F_\theta = \eta_\theta \left[\frac{u}{s} + \frac{w}{s \tan\alpha} + \nu_\theta \frac{\partial u}{\partial s} \right] \qquad 3.3.5$$

$$M_s = \eta_s^* \left[\frac{\partial \psi_s}{\partial s} + \nu_s \frac{\psi_s}{s} + \eta_s a^2 \frac{\partial u}{\partial s} + \nu_s \left(\frac{u}{s} + \frac{w}{s \tan\alpha} \right) \right] \frac{1}{s \tan\alpha}$$

$$M_\theta = \eta_\theta^* \left[\frac{\psi_s}{s} + \nu_\theta \frac{\partial \psi_s}{\partial s} \right]$$

3.3.5 (Cont.)

$$Q_s = \eta \left[\frac{\partial w}{\partial s} + \psi_s \right]$$

Upon substituting Equations 3.1.3, 3.1.4, 3.3.5 into Equation 3.3.4 the following expression is obtained:

$$\begin{aligned}
\bar{V} = \sum_{j=1}^{n} \Bigg\{ \frac{1}{2} \int_{s_{j-1,j}}^{s_{jj}} \int_0^{2\pi} \Bigg[& \eta_s \left(\frac{\partial u}{\partial s}\right)^2 + \eta_s \nu_s \frac{\partial u}{\partial s}\left(\frac{u}{s} + \frac{w}{s \tan\alpha}\right) \\
& + \eta_\theta \left(\frac{u}{s} + \frac{w}{s\tan\alpha}\right)^2 + \eta_\theta \nu_\theta \frac{\partial u}{\partial s}\left(\frac{u}{s} + \frac{w}{s\tan\alpha}\right) + \eta_s^* \left(\frac{\partial \psi_s}{\partial s}\right)^2 \\
& + \eta_s^* \nu_s \frac{\partial \psi_s}{\partial s} \frac{\psi_s}{s} + \eta_{s_s} \frac{a^2}{\tan\alpha} \frac{\partial u}{\partial s} \frac{\partial \psi_s}{\partial s} + \eta_s \nu_{s_s} \frac{a^2}{\tan\alpha} \frac{\partial \psi_s}{\partial s}\left(\frac{u}{s} + \frac{w}{s\tan\alpha}\right) \\
& + \eta_\theta^* \left(\frac{\psi_s}{s}\right)^2 + \eta_\theta^* \nu_\theta \frac{\partial \psi_s}{\partial s} \frac{\psi_s}{s} + \eta \left(\frac{\partial w}{\partial s} + \psi_s\right)^2 \Bigg] d\theta(s) \sin\alpha \, ds \Bigg\}_j
\end{aligned} \qquad 3.3.6$$

where

$$\eta = 2h_c k_s G_{sz}$$

$$\eta_s = \frac{4 \, t_f \, E_s}{1 - \nu_s \nu_\theta}$$

$$\eta_\theta = \frac{4 \, t_f \, E_\theta}{1 - \nu_s \nu_\theta}$$

$$\eta_s^* = \frac{4 \, t_f^3 \, E_s}{3(1 - \nu_s \nu_\theta)}$$

$$\eta_\theta^* = \frac{4 \, t_f^3 \, E_\theta}{3(1 - \nu_s \nu_\theta)}$$

and the first subscript on the parameter, s, denotes the node number, while the second subscript denotes the cone number.

The following deflection modes are assumed in the present analysis (24):

$$w = \sum_{m=1}^{\infty} A_m \sin(\beta_m y) \cos(\Omega t)$$

$$u = \sum_{m=1}^{\infty} B_m \cos(\beta_m y) \cos(\Omega t) \qquad 3.3.7$$

$$\Psi_s = \sum_{m=1}^{\infty} \frac{1}{s_{01} \sin\alpha} D_m \cos(\beta_m y) \cos(\Omega t)$$

where

$$y = \ln\left(\frac{s}{s_{01}}\right) \text{ or } s = s_{01} e^{y}$$

$$\beta_m = \frac{m\pi}{\mathcal{G}} \qquad 3.3.8$$

$$\mathcal{G} = \ln\left(\frac{s_{nn}}{s_{01}}\right)$$

At the edges where $s = s_{01}$ and $s = s_{nn}$ the shell is simply supported. This requires that the following boundary conditions be met at each of the two edges:

$$w = 0$$

$$\partial\Psi_s / \partial s = 0$$

These boundary conditions are met by the model shapes given above. To handle the case of clamped edges, the second boundary condition would be

$$\Psi_s = 0$$

which would require changing $\cos(\beta_m y)$ to $\sin(\beta_m y)$ in the expression for Ψ_s.

From the transformation in Equation 3.3.8 the following partial derivatives of u, w, and ψ_s are obtained:

$$\frac{\partial u}{\partial s} = \frac{e^{-y}}{s_{01}}\frac{\partial u}{\partial y}$$

$$\frac{\partial \psi_s}{\partial s} = \frac{e^{-y}}{s_{01}}\frac{\partial \psi_s}{\partial y} \qquad 3.3.9$$

$$\frac{\partial w}{\partial s} = \frac{e^{-y}}{s_{01}}\frac{\partial w}{\partial y}$$

Substituting Equation 3.3.9 into Equation 3.3.6 and integrating with respect to θ yields:

$$\bar{V} = \sum_{j=1}^{n}\Bigg[\pi\int_{\theta_j}^{\delta_j}\Bigg(\eta_s\left(\frac{\partial u}{\partial y}\right)^2 + (\eta_s^*\nu_s + \eta_\theta^*\nu_\theta)\frac{\partial \psi_s}{\partial y}\psi_s$$

$$+ (\eta_s\nu_s + \eta_\theta\nu_\theta)(u + w\cot\alpha_j)\frac{\partial u}{\partial y} + \eta_\theta(u + w\cot\alpha_j)^2 + \eta_s^*\left(\frac{\partial \psi_s}{\partial y}\right)^2 \qquad 3.3.10$$

$$+ \frac{\eta_s\, a^2\cot\alpha_j}{s_{01}}e^{-y}\frac{\partial u}{\partial y}\frac{\partial \psi_s}{\partial y} + \frac{\eta_s\nu_s\, a^2\cot\alpha_j}{s_{01}}e^{-y}(u + w\cot\alpha_j)\frac{\partial \psi_s}{\partial y}$$

$$+ \eta_\theta^*(\psi_s)^2 + \eta(s_{01})^2 e^{2y}\left(\frac{e^{-y}}{s_{01}}\frac{\partial w}{\partial y} + \psi_s\right)^2\Bigg)\sin\alpha_j\, dy\Bigg]_j$$

where

$$\delta_j = \ln\frac{s_{jj}}{s_{01}}$$

$$\theta_j = \ln\frac{s_{j-1,j}}{s_{01}}$$

Substituting Equation 3.3.7 into 3.3.10 and carrying out the integration with respect to y the following expression for the strain energy of the system is obtained:

$$\overline{V} = \sum_{j=1}^{n} \pi \sin\alpha_j \cos^2(\Omega t) \Bigg\{ \eta_s \sum_{m=1}^{\infty} \beta_m^2 B_m^2 \left[\frac{\delta_j}{2} - \frac{\theta_j}{2} - \frac{\sin(2\beta_m\delta_j)}{4\beta_m} + \frac{\sin(2\beta_m\theta_j)}{4\beta_m} \right]$$

$$+ \eta_s \sum_{\substack{m=1 \\ m\neq\bar{m}}}^{\infty} \sum_{\bar{m}=1}^{\infty} \beta_m \beta_{\bar{m}} B_m B_{\bar{m}} \left[\frac{\sin(\beta_{\bar{m}}-\beta_m)\delta_j}{2(\beta_{\bar{m}}-\beta_m)} - \frac{\sin(\beta_{\bar{m}}-\beta_m)\theta_j}{2(\beta_{\bar{m}}-\beta_m)} - \frac{\sin(\beta_{\bar{m}}+\beta_m)\delta_j}{2(\beta_{\bar{m}}+\beta_m)} \right.$$

$$\left. + \frac{\sin(\beta_{\bar{m}}+\beta_m)\theta_j}{2(\beta_{\bar{m}}+\beta_m)} \right] + \frac{1}{2}\left(\eta_s \nu_s + \eta_\theta \nu_\theta\right) \sum_{m=1}^{\infty} B_m^2 \left[\cos(2\beta_m\delta_j) - \cos(2\beta_m\theta_j) \right]$$

$$+ \frac{1}{2}\left(\eta_s \nu_s + \eta_\theta \nu_\theta\right) \sum_{\substack{m=1 \\ m\neq\bar{m}}}^{\infty} \sum_{\bar{m}=1}^{\infty} \beta_{\bar{m}} B_{\bar{m}} B_m \left[\frac{\cos(\beta_{\bar{m}}-\beta_m)\delta_j}{(\beta_{\bar{m}}-\beta_m)} - \frac{\cos(\beta_{\bar{m}}-\beta_m)\theta_j}{(\beta_{\bar{m}}-\beta_m)} \right.$$

$$\left. + \frac{\cos(\beta_{\bar{m}}+\beta_m)\delta_j}{(\beta_{\bar{m}}+\beta_m)} - \frac{\cos(\beta_{\bar{m}}+\beta_m)\theta_j}{(\beta_{\bar{m}}+\beta_m)} \right] - \frac{1}{2}\left(\eta_s \nu_s + \eta_\theta \nu_\theta\right) \cot\alpha_j \sum_{m=1} \beta_m B_m A_m \left[\delta_j - \theta_j \right.$$

$$\left. - \frac{\sin(2\beta_m\delta_j)}{2\beta_m} + \frac{\sin(2\beta_m\theta_j)}{2\beta_m} \right] - \frac{1}{2}\left(\eta_s \nu_s + \eta_\theta \nu_\theta\right) \cot\alpha_j \sum_{\substack{m=1 \\ m\neq\bar{m}}}^{\infty} \sum_{\bar{m}=1}^{\infty} \beta_{\bar{m}} B_m A_m \left[\frac{\sin(\beta_{\bar{m}}-\beta_m)\delta_j}{(\beta_{\bar{m}}-\beta_m)} \right.$$

$$\left. - \frac{\sin(\beta_{\bar{m}}-\beta_m)\theta_j}{(\beta_{\bar{m}}-\beta_m)} - \frac{\sin(\beta_{\bar{m}}+\beta_m)\delta_j}{(\beta_{\bar{m}}+\beta_m)} + \frac{\sin(\beta_{\bar{m}}+\beta_m)\theta_j}{(\beta_{\bar{m}}+\beta_m)} \right] + \eta_\theta \sum_{m=1} B_m^2 \left[\frac{\delta_j}{2} - \frac{\theta_j}{2} \right.$$

$$\left. + \frac{\sin(2\beta_m\delta_j)}{4\beta_m} - \frac{\sin(2\beta_m\theta_j)}{4\beta_m} \right] + \eta_\theta \sum_{\substack{m=1 \\ m\neq\bar{m}}}^{\infty} \sum_{\bar{m}=1}^{\infty} B_{\bar{m}} B_m \left[\frac{\sin(\beta_{\bar{m}}-\beta_m)\delta_j}{2(\beta_{\bar{m}}-\beta_m)} - \frac{\sin(\beta_{\bar{m}}-\beta_m)\theta_j}{2(\beta_{\bar{m}}-\beta_m)} \right.$$

$$\left. + \frac{\sin(\beta_{\bar{m}}+\beta_m)\delta_j}{2(\beta_{\bar{m}}+\beta_m)} - \frac{\sin(\beta_{\bar{m}}+\beta_m)\theta_j}{2(\beta_{\bar{m}}+\beta_m)} \right] - \eta_\theta \cot\alpha_j \sum_{m=1}^{\infty} \frac{1}{\beta_m} B_m A_m \left[\cos(2\beta_m\delta_j) - \cos(2\beta_m\theta_j) \right]$$

$$- \eta_\theta \cot\alpha_j \sum_{\substack{m=1 \\ m\neq\bar{m}}}^{\infty} \sum_{\bar{m}=1}^{\infty} B_{\bar{m}} A_m \left[\frac{\cos(\beta_m-\beta_{\bar{m}})\delta_j}{(\beta_m-\beta_{\bar{m}})} - \frac{\cos(\beta_m-\beta_{\bar{m}})\theta_j}{(\beta_m-\beta_{\bar{m}})} + \frac{\cos(\beta_m+\beta_{\bar{m}})\delta_j}{(\beta_m+\beta_{\bar{m}})} - \frac{\cos(\beta_m+\beta_{\bar{m}})\theta_j}{(\beta_m+\beta_{\bar{m}})} \right]$$

(Continued)

$$+\,\eta_\theta \cot^2\alpha_j \sum_{m=1}^{\infty} A_m^2 \left[\frac{\delta_j}{2} - \frac{\theta_j}{2} - \frac{\sin(2\beta_m\delta_j)}{4\beta_m} + \frac{\sin(2\beta_m\theta_j)}{4\beta_m} \right]$$

$$+\,\eta_\theta \cot^2\alpha_j \sum_{\substack{m=1 \\ m\neq\bar m}}^{\infty} \sum_{\bar m=1}^{\infty} A_m A_{\bar m} \left[\frac{\sin(\beta_{\bar m}-\beta_m)\delta_j}{2(\beta_{\bar m}-\beta_m)} - \frac{\sin(\beta_{\bar m}-\beta_m)\theta_j}{2(\beta_{\bar m}-\beta_m)} - \frac{\sin(\beta_{\bar m}+\beta_m)\delta_j}{2(\beta_{\bar m}+\beta_m)} + \frac{\sin(\beta_{\bar m}+\beta_m)\theta_j}{2(\beta_{\bar m}+\beta_m)} \right]$$

$$+\,\frac{\eta_s^*}{(S_{01}\sin\alpha_j)^2} \sum_{m=1}^{\infty} \beta_m^2 D_m^2 \left[\frac{\delta_j}{2} - \frac{\theta_j}{2} - \frac{\sin(2\beta_m\delta_j)}{4\beta_m} + \frac{\sin(2\beta_m\theta_j)}{4\beta_m} \right]$$

$$+\,\frac{\eta_s^*}{(S_{01}\sin\alpha_j)^2} \sum_{\substack{m=1 \\ m\neq\bar m}}^{\infty} \sum_{\bar m=1}^{\infty} \beta_{\bar m}\beta_m D_{\bar m} D_m \left[\frac{\sin(\beta_{\bar m}-\beta_m)\delta_j}{2(\beta_{\bar m}-\beta_m)} - \frac{\sin(\beta_{\bar m}-\beta_m)\theta_j}{2(\beta_{\bar m}-\beta_m)} - \frac{\sin(\beta_{\bar m}+\beta_m)\delta_j}{2(\beta_{\bar m}+\beta_m)} + \frac{\sin(\beta_{\bar m}+\beta_m)\theta_j}{2(\beta_{\bar m}+\beta_m)} \right]$$

$$+\,\frac{(\eta_s^*\nu_s + \eta_\theta^*\nu_\theta)}{2(S_{01}\sin\alpha_j)^2} \sum_{m=1}^{\infty} D_m^2 \left[\cos(2\beta_m\delta_j) - \cos(2\beta_m\theta_j)\right] + \frac{(\eta_s^*\nu_s + \eta_\theta^*\nu_\theta)}{2(S_{01}\sin\alpha_j)^2} \sum_{\substack{m=1 \\ m\neq\bar m}}^{\infty} \sum_{\bar m=1}^{\infty} \beta_{\bar m} D_{\bar m} D_m \left[\frac{\cos(\beta_{\bar m}-\beta_m)\delta_j}{(\beta_{\bar m}-\beta_m)} \right.$$

$$\left. - \frac{\cos(\beta_{\bar m}-\beta_m)\theta_j}{(\beta_{\bar m}-\beta_m)} + \frac{\cos(\beta_{\bar m}+\beta_m)\delta_j}{(\beta_{\bar m}+\beta_m)} - \frac{\cos(\beta_{\bar m}+\beta_m)\theta_j}{(\beta_{\bar m}+\beta_m)} \right] + \frac{\eta_s a^2 \cot^2\alpha_j}{(S_{01})^2 \sin\alpha_j} \sum_{m=1}^{\infty} \beta_m^2 B_m D_m \left[-e^{-\delta_j}\sin^2(\beta_m\delta_j) \right.$$

$$\left. + e^{-\theta_j}\sin^2(\beta_m\theta_j) - \frac{\beta_m e^{-\delta_j}(\sin 2\beta_m\delta_j + 2\beta_m\cos 2\beta_m\delta_j)}{(1+4\beta_m^2)} + \frac{\beta_m e^{-\theta_j}(\sin 2\beta_m\theta_j + 2\beta_m\cos 2\beta_m\theta_j)}{(1+4\beta_m^2)} \right]$$

$$+\,\frac{\eta_s a^2 \cot^2\alpha_j}{2(S_{01})^2 \sin\alpha_j} \sum_{\substack{m=1 \\ m\neq\bar m}}^{\infty} \sum_{\bar m=1}^{\infty} \beta_{\bar m}\beta_m B_{\bar m} D_m \left[\frac{e^{-\delta_j}\sin(\beta_{\bar m}-\beta_m)\delta_j}{(\beta_{\bar m}-\beta_m)} - \frac{e^{-\theta_j}\sin(\beta_{\bar m}-\beta_m)\theta_j}{(\beta_{\bar m}-\beta_m)} - \frac{e^{-\delta_j}\sin(\beta_{\bar m}+\beta_m)\delta_j}{(\beta_{\bar m}+\beta_m)} \right.$$

$$+\,\frac{e^{-\theta_j}\sin(\beta_{\bar m}+\beta_m)\theta_j}{(\beta_{\bar m}+\beta_m)} - \frac{e^{-\delta_j}\left\{\sin(\beta_{\bar m}-\beta_m)\delta_j + (\beta_{\bar m}-\beta_m)\cos(\beta_{\bar m}-\beta_m)\delta_j\right\}}{(\beta_{\bar m}-\beta_m)\left[1+(\beta_{\bar m}-\beta_m)^2\right]}$$

$$+\,\frac{\left\{e^{-\theta_j}\right\}\left\{\sin(\beta_{\bar m}-\beta_m)\theta_j + (\beta_{\bar m}-\beta_m)\cos(\beta_{\bar m}-\beta_m)\theta_j\right\}}{(\beta_{\bar m}-\beta_m)\left[1+(\beta_{\bar m}-\beta_m)^2\right]} + \frac{e^{-\delta_j}\left\{\sin(\beta_{\bar m}+\beta_m)\delta_j + (\beta_{\bar m}+\beta_m)\cos(\beta_{\bar m}+\beta_m)\delta_j\right\}}{(\beta_{\bar m}+\beta_m)\left[1+(\beta_{\bar m}+\beta_m)^2\right]}$$

$$\left. - \frac{\left\{e^{-\theta_j}\right\}\left\{\sin(\beta_{\bar m}+\beta_m)\theta_j + (\beta_{\bar m}+\beta_m)\cos(\beta_{\bar m}+\beta_m)\theta_j\right\}}{(\beta_{\bar m}+\beta_m)\left[1+(\beta_{\bar m}+\beta_m)^2\right]} \right]$$

(Continued)

$$-\frac{\eta_s \nu_s a^2 \cot^2\alpha_j}{2(s_{01})^2 \sin\alpha_1} \sum_{m=1}^{\infty} \beta_m B_m D_m \left[\frac{e^{-\delta_j}(-\sin 2\beta_m\delta_j - 2\beta_m \cos 2\beta_m\delta_j)}{(1+4\beta_m^2)} - \frac{e^{-\theta_j}(-\sin 2\beta_m\theta_j - 2\beta_m\cos 2\beta_m\theta_j)}{(1+4\beta_m^2)} \right]$$

$$+ \frac{\eta_s \nu_s a^2 \cot^2\alpha_j}{2(s_{01})^2 \sin\alpha_1} \sum_{\substack{m=1 \\ m\neq\bar{m}}}^{\infty} \sum_{\bar{m}=1}^{\infty} \beta_{\bar{m}} D_{\bar{m}} B_m \left[\frac{\{e^{-\delta_j}\}\{(\beta_{\bar{m}}-\beta_m)\sin(\beta_{\bar{m}}-\beta_m)\delta_j - \cos(\beta_{\bar{m}}-\beta_m)\delta_j\}}{(\beta_{\bar{m}}-\beta_m)[1+(\beta_{\bar{m}}-\beta_m)^2]} \right.$$

$$- \frac{\{e^{-\theta_j}\}\{(\beta_{\bar{m}}-\beta_m)\sin(\beta_{\bar{m}}-\beta_m)\theta_j - \cos(\beta_{\bar{m}}-\beta_m)\theta_j\}}{(\beta_{\bar{m}}-\beta_m)[1+(\beta_{\bar{m}}-\beta_m)^2]} + \frac{\{e^{-\delta_j}\}\{(\beta_{\bar{m}}+\beta_m)\sin(\beta_{\bar{m}}+\beta_m)\delta_j - \cos(\beta_{\bar{m}}+\beta_m)\delta_j\}}{(\beta_{\bar{m}}+\beta_m)[1+(\beta_{\bar{m}}+\beta_m)^2]}$$

$$- \frac{\{e^{-\theta_j}\}\{(\beta_{\bar{m}}+\beta_m)\sin(\beta_{\bar{m}}+\beta_m)\theta_j - \cos(\beta_{\bar{m}}+\beta_m)\theta_j\}}{(\beta_{\bar{m}}+\beta_m)[1+(\beta_{\bar{m}}+\beta_m)^2} + \frac{e^{-\delta_j}\cos(\beta_{\bar{m}}-\beta_m)\delta_j}{(\beta_{\bar{m}}-\beta_m)} - \frac{e^{-\theta_j}\cos(\beta_{\bar{m}}-\beta_m)\theta_j}{(\beta_{\bar{m}}-\beta_m)}$$

$$\left. + \frac{e^{-\delta_j}\cos(\beta_{\bar{m}}+\beta_m)\delta_j}{(\beta_{\bar{m}}+\beta_m)} - \frac{e^{-\theta_j}\cos(\beta_{\bar{m}}+\beta_m)\theta_j}{(\beta_{\bar{m}}+\beta_m)} \right] - \frac{\eta_s \nu_s a^2 \cot^2\alpha_j}{(s_{01})^2 \sin\alpha_1} \sum \beta_m D_m A_m \left[-e^{-\delta_j} \sin^2\beta_m\delta_j \right.$$

$$\left. + e^{-\theta_j}\sin^2\beta_m\theta_j - \frac{\beta_m e^{-\delta_j}(\sin 2\beta_m\delta_j + 2\beta_m\cos 2\beta_m\delta_j)}{1+4\beta_m^2} + \frac{\beta_m e^{-\theta_j}(\sin 2\beta_m\theta_j + 2\beta_m\cos 2\beta_m\theta_j)}{1+4\beta_m^2} \right]$$

$$- \frac{\eta_s \nu_s a^2 \cot^2\alpha_j}{2(s_{01})^2 \sin\alpha_1} \sum_{\substack{m=1 \\ m\neq\bar{m}}}^{\infty} \sum_{\bar{m}=1}^{\infty} \beta_{\bar{m}} D_{\bar{m}} A_m \left[\frac{e^{-\delta_j}\sin(\beta_{\bar{m}}-\beta_m)\delta_j}{(\beta_{\bar{m}}-\beta_m)} - \frac{e^{-\theta_j}\sin(\beta_{\bar{m}}-\beta_m)\theta_j}{(\beta_{\bar{m}}-\beta_m)} - \frac{e^{-\delta_j}\sin(\beta_{\bar{m}}+\beta_m)\delta_j}{(\beta_{\bar{m}}+\beta_m)} \right.$$

$$+ \frac{e^{-\theta_j}\sin(\beta_{\bar{m}}+\beta_m)\theta_j}{(\beta_{\bar{m}}+\beta_m)} - \frac{\{e^{-\delta_j}\}\{\sin(\beta_{\bar{m}}-\beta_m)\delta_j + (\beta_{\bar{m}}-\beta_m)\cos(\beta_{\bar{m}}-\beta_m)\delta_j\}}{(\beta_{\bar{m}}-\beta_m)[1+(\beta_{\bar{m}}-\beta_m)^2]}$$

$$+ \frac{\{e^{-\theta_j}\}\{\sin(\beta_{\bar{m}}-\beta_m)\theta_j + (\beta_{\bar{m}}-\beta_m)\cos(\beta_{\bar{m}}-\beta_m)\theta_j\}}{(\beta_{\bar{m}}-\beta_m)[1+(\beta_{\bar{m}}-\beta_m)^2]} + \frac{\{e^{-\delta_j}\}\{\sin(\beta_{\bar{m}}+\beta_m)\delta_j + (\beta_{\bar{m}}+\beta_m)\cos(\beta_{\bar{m}}+\beta_m)\delta_j\}}{(\beta_{\bar{m}}+\beta_m)[1+(\beta_{\bar{m}}+\beta_m)^2]}$$

$$\left. - \frac{\{e^{-\theta_j}\}\{\sin(\beta_{\bar{m}}+\beta_m)\theta_j + (\beta_{\bar{m}}+\beta_m)\cos(\beta_{\bar{m}}+\beta_m)\theta_j\}}{(\beta_{\bar{m}}+\beta_m)[1+(\beta_{\bar{m}}+\beta_m)^2]} \right] + \frac{\eta_\theta^*}{2(s_{01}\sin\alpha_1)^2} \sum_{m=1}^{\infty} D_m^2 \left[\delta_j - \theta_j \right.$$

$$\left. + \frac{\sin(2\beta_m\delta_j)}{2\beta_m} - \frac{\sin(2\beta_m\theta_j)}{2\beta_m} \right]$$

(Continued)

$$+\frac{\eta_\theta^*}{2(S_{01}\sin\alpha_1)^2}\sum_{m=1}^{\infty}\sum_{\substack{\bar{m}=1\\ m\neq\bar{m}}}^{\infty} D_{\bar{m}} D_m\left[\frac{\sin(\beta_{\bar{m}}-\beta_m)\delta_j}{(\beta_{\bar{m}}-\beta_m)}-\frac{\sin(\beta_{\bar{m}}-\beta_m)\theta_j}{(\beta_{\bar{m}}-\beta_m)}+\frac{\sin(\beta_{\bar{m}}+\beta_m)\delta_j}{(\beta_{\bar{m}}+\beta_m)}-\frac{\sin(\beta_{\bar{m}}+\beta_m)\theta_j}{(\beta_{\bar{m}}+\beta_m)}\right]$$

$$+\eta\sum_{m=1}^{\infty}\beta_m^2 A_m^2\left[\frac{\delta_j}{2}-\frac{\theta_j}{2}+\frac{\sin(2\beta_m\delta_j)}{4\beta_m}-\frac{\sin(2\beta_m\theta_j)}{4\beta_m}\right]+\eta\sum_{m=1}^{\infty}\sum_{\substack{\bar{m}=1\\ m\neq\bar{m}}}^{\infty}\beta_{\bar{m}}\beta_m A_{\bar{m}} A_m\left[\frac{\sin(\beta_{\bar{m}}-\beta_m)\delta_j}{2(\beta_{\bar{m}}-\beta_m)}\right.$$

$$\left.-\frac{\sin(\beta_{\bar{m}}-\beta_m)\theta_j}{2(\beta_{\bar{m}}-\beta_m)}+\frac{\sin(\beta_{\bar{m}}+\beta_m)\delta_j}{2(\beta_{\bar{m}}+\beta_m)}-\frac{\sin(\beta_{\bar{m}}+\beta_m)\theta_j}{2(\beta_{\bar{m}}+\beta_m)}\right]+\frac{2\eta}{\sin\alpha_1}\sum_{m=1}^{\infty}\beta_m A_m D_m\left[e^{\delta_j}\cos^2\beta_m\delta_j\right.$$

$$\left.-e^{\theta_j}\cos^2\beta_m\theta_j+\frac{\beta_m e^{\delta_j}(\sin 2\beta_m\delta_j-2\beta_m\cos 2\beta_m\delta_j)}{(1+4\beta_m^2)}-\frac{\beta_m e^{\theta_j}(\sin 2\beta_m\theta_j-2\beta_m\cos 2\beta_m\theta_j)}{(1+4\beta_m^2)}\right]$$

$$+\frac{\eta}{\sin\alpha_1}\sum_{m=1}^{\infty}\sum_{\substack{\bar{m}=1\\ m\neq\bar{m}}}^{\infty}\beta_m A_m D_{\bar{m}}\left[\frac{e^{\delta_j}\sin(\beta_{\bar{m}}-\beta_m)\delta_j}{(\beta_{\bar{m}}-\beta_m)}-\frac{e^{\theta_j}\sin(\beta_{\bar{m}}-\beta_m)\theta_j}{(\beta_{\bar{m}}-\beta_m)}+\frac{e^{\delta_j}\sin(\beta_{\bar{m}}+\beta_m)\delta_j}{(\beta_{\bar{m}}+\beta_m)}-\frac{e^{\theta_j}\sin(\beta_{\bar{m}}+\beta_m)\theta_j}{(\beta_{\bar{m}}+\beta_m)}\right.$$

$$+\frac{\{e^{\delta_j}\}\{\sin(\beta_{\bar{m}}-\beta_m)\delta_j-(\beta_{\bar{m}}-\beta_m)\cos(\beta_{\bar{m}}-\beta_m)\delta_j\}}{(\beta_{\bar{m}}-\beta_m)[1+(\beta_{\bar{m}}-\beta_m)^2]}-\frac{\{e^{\theta_j}\}\{\sin(\beta_{\bar{m}}-\beta_m)\theta_j-(\beta_{\bar{m}}-\beta_m)\cos(\beta_{\bar{m}}-\beta_m)\theta_j\}}{(\beta_{\bar{m}}-\beta_m)[1+(\beta_{\bar{m}}-\beta_m)^2]}$$

$$\left.+\frac{\{e^{\delta_j}\}\{\sin(\beta_{\bar{m}}+\beta_m)\delta_j-(\beta_{\bar{m}}+\beta_m)\cos(\beta_{\bar{m}}+\beta_m)\delta_j\}}{(\beta_{\bar{m}}+\beta_m)[1+(\beta_{\bar{m}}+\beta_m)^2]}-\frac{\{e^{\theta_j}\}\{\sin(\beta_{\bar{m}}+\beta_m)\theta_j-(\beta_{\bar{m}}+\beta_m)\cos(\beta_{\bar{m}}+\beta_m)\theta_j\}}{(\beta_{\bar{m}}+\beta_m)[1+(\beta_{\bar{m}}+\beta_m)^2]}\right]$$

$$+\frac{\eta}{2\sin^2\alpha_1}\sum_{m=1}^{\infty}D_m^2\left[e^{2\delta_j}\cos^2\beta_m\delta_j-e^{2\theta_j}\cos^2\beta_m\theta_j+\frac{\beta_m e^{2\delta_j}(\sin 2\beta_m\delta_j-\beta_m\cos 2\beta_m\delta_j)}{2(1+\beta_m^2)}\right.$$

$$\left.-\frac{\beta_m e^{2\theta_j}(\sin 2\beta_m\theta_j-\beta_m\cos 2\beta_m\theta_j)}{2(1+\beta_m^2)}\right]+\frac{\eta}{2\sin^2\alpha_1}\sum_{m=1}^{\infty}\sum_{\substack{\bar{m}=1\\ m\neq\bar{m}}}^{\infty}D_{\bar{m}} D_m\left[\frac{e^{2\delta_j}\sin(\beta_{\bar{m}}-\beta_m)\delta_j}{(\beta_{\bar{m}}-\beta_m)}\right.$$

$$-\frac{e^{2\theta_j}\sin(\beta_{\bar{m}}-\beta_m)\theta_j}{(\beta_{\bar{m}}-\beta_m)}+\frac{e^{2\delta_j}\sin(\beta_{\bar{m}}+\beta_m)\delta_j}{(\beta_{\bar{m}}+\beta_m)}-\frac{e^{2\theta_j}\sin(\beta_{\bar{m}}+\beta_m)\theta_j}{(\beta_{\bar{m}}+\beta_m)}$$

(Continued)

$$- \frac{2\{e^{2\delta_j}\}\{2\sin(\beta_{\bar{m}}-\beta_m)\delta_j - (\beta_{\bar{m}}-\beta_m)\cos(\beta_{\bar{m}}-\beta_m)\delta_j\}}{(\beta_{\bar{m}}-\beta_m)[4+(\beta_{\bar{m}}-\beta_m)^2]}$$

$$+ \frac{2\{e^{2\theta_j}\}\{2\sin(\beta_{\bar{m}}-\beta_m)\theta_j - (\beta_{\bar{m}}-\beta_m)\cos(\beta_{\bar{m}}-\beta_m)\theta_j\}}{(\beta_{\bar{m}}-\beta_m)[4+(\beta_{\bar{m}}-\beta_m)^2]}$$

$$- \frac{2\{e^{2\delta_j}\}\{2\sin(\beta_{\bar{m}}+\beta_m)\delta_j - (\beta_{\bar{m}}+\beta_m)\cos(\beta_{\bar{m}}+\beta_m)\delta_j\}}{(\beta_{\bar{m}}+\beta_m)[4+(\beta_{\bar{m}}+\beta_m)^2]}$$

$$\left. + \frac{2\{e^{2\theta_j}\}\{2\sin(\beta_{\bar{m}}+\beta_m)\theta_j - (\beta_{\bar{m}}+\beta_m)\cos(\beta_{\bar{m}}+\beta_m)\theta_j\}}{(\beta_{\bar{m}}+\beta_m)[4+(\beta_{\bar{m}}+\beta_m)^2]} \right]_j \Bigg\} \qquad 3.3.11$$

Equation 3.3.11 defines the total strain energy in a sandwich shell of revolution. For maximum strain energy in the system the term, $\cos^2\Omega t$, is unity.

3.4 Kinetic Energy for the Composite Shell

As in the case of strain energy, the kinetic energy of the composite shell can be written as:

$$\overline{T} = \sum_{j=1}^{n} T_j^t$$

The total kinetic energy consists of two translations and one rotation,

or
$$\overline{T} = \sum_{j=1}^{n} \frac{1}{2}\left\{\left((m^c + 2m^f)\int_{S_{j-1,j}}^{S_{j,j}} \int_0^{2\pi} \left[(\dot{u})^2 + (\dot{w})^2\right]\right.\right.$$

$$\left.\left. + (I^c + 2I^f)\int_{S_{j-1,j}}^{S_{j,j}} \int_0^{2\pi} (\dot{\psi}_s)^2 \right) d\theta \sin\alpha_j \, s\, ds\right\}_j \qquad 3.4.1$$

Where M and I are the mass and the mass moment of inertia per unit area respectively.

Substituting Equations 3.3.7, using the transformation 3.3.8 and integrating with respect to θ, Equation 3.4.1. becomes:

$$\overline{T} = \sum_{j=1}^{n}\left\{\int_{\theta_j}^{b_j}\left[\pi \Omega^2 (s_{01})^3 \sin\alpha_j \sin^2\Omega t \,(m^c + 2m^f)\left(\sum_{m=1}^{\infty} A_m^2 e^{2y} \sin^2\beta_{md} y\right.\right.\right.$$

$$+ \sum_{\substack{m=1 \\ m\neq\bar{m}}}^{\infty}\sum_{\bar{m}=1}^{\infty} A_m A_{\bar{m}} e^{2y} \sin\beta_{md} y \sin\beta_{\bar{m}d} y + \sum_{m=1}^{\infty} B_m^2 e^{2y} \cos^2\beta_{md} y + \sum_{\substack{m=1 \\ m\neq\bar{m}}}^{\infty}\sum_{\bar{m}=1}^{\infty} B_m B_{\bar{m}} e^{2y} \cos\beta_{md} y \cos\beta_{\bar{m}d} y\Bigg)$$

$$+ \frac{s_{01}^3 \pi \Omega^2 \sin^2\Omega t}{(s_{01}\sin\alpha_1)^3 \sin\alpha_j}(I^c + 2I^f)\left(\sum_{m=1}^{\infty} D_m^2 e^{2y} \cos^2\beta_{md} y\right.$$

$$\left.\left.\left. + \sum_{\substack{m=1 \\ m\neq\bar{m}}}^{\infty}\sum_{\bar{m}=1}^{\infty} D_m D_{\bar{m}} e^{2y} \cos\beta_{md} y \cos\beta_{\bar{m}d} y\right)\right] dy\right\}_j \qquad 3.4.2$$

Integrating Equation 3.4.2 yields:

$$\overline{\overline{T}} = \sum_{j=1}^{n} \Bigg\{ \pi\, \Omega^2 (s_{01})^2 \sin\gamma_j \sin^2\Omega t\, (m^c + 2m^f) \Bigg[\frac{1}{2}\sum_{m=1}^{\infty} A_m^2 \Bigg(e^{2\delta_j}\sin^2\beta_m\delta_j - e^{2\theta_j}\sin^2\beta_m\theta_j$$

$$- \frac{\beta_m e^{2\delta_j}(\sin 2\beta_m\delta_j - \beta_m\cos 2\beta_m\delta_j)}{2(1+\beta_m^2)} + \frac{\beta_m e^{2\theta_j}(\sin 2\beta_m\theta_j - \beta_m\cos 2\beta_m\theta_j)}{2(1+\beta_m^2)} \Bigg)$$

$$+ \sum_{\substack{m=1 \\ m\neq\bar{m}}}^{\infty} \sum_{\bar{m}=1}^{\infty} A_m A_{\bar{m}} \Bigg(\frac{e^{2\delta_j}\sin(\beta_{\bar{m}}-\beta_m)\delta_j}{2(\beta_{\bar{m}}-\beta_m)} - \frac{e^{2\theta_j}\sin(\beta_{\bar{m}}-\beta_m)\theta_j}{2(\beta_{\bar{m}}-\beta_m)} - \frac{e^{2\delta_j}\sin(\beta_{\bar{m}}+\beta_m)\delta_j}{2(\beta_{\bar{m}}+\beta_m)} + \frac{e^{2\theta_j}\sin(\beta_{\bar{m}}+\beta_m)\theta_j}{2(\beta_{\bar{m}}+\beta_m)}$$

$$- \frac{\{e^{2\delta_j}\}\{2\sin(\beta_{\bar{m}}-\beta_m)\delta_j - (\beta_{\bar{m}}-\beta_m)\cos(\beta_{\bar{m}}-\beta_m)\delta_j\}}{(\beta_{\bar{m}}-\beta_m)[4+(\beta_{\bar{m}}-\beta_m)^2]} + \frac{\{e^{2\theta_j}\}\{2\sin(\beta_{\bar{m}}-\beta_m)\theta_j - (\beta_{\bar{m}}-\beta_m)\cos(\beta_{\bar{m}}-\beta_m)\theta_j\}}{(\beta_{\bar{m}}-\beta_m)[4+(\beta_{\bar{m}}-\beta_m)^2]}$$

$$+ \frac{\{e^{2\delta_j}\}\{2\sin(\beta_{\bar{m}}+\beta_m)\delta_j - (\beta_{\bar{m}}+\beta_m)\cos(\beta_{\bar{m}}+\beta_m)\delta_j\}}{(\beta_{\bar{m}}+\beta_m)[4+(\beta_{\bar{m}}+\beta_m)^2]} - \frac{\{e^{2\theta_j}\}\{2\sin(\beta_{\bar{m}}+\beta_m)\theta_j - (\beta_{\bar{m}}+\beta_m)\cos(\beta_{\bar{m}}+\beta_m)\theta_j\}}{(\beta_{\bar{m}}+\beta_m)[4+(\beta_{\bar{m}}+\beta_m)^2]} \Bigg)$$

$$+ \frac{1}{2}\sum_{m=1}^{\infty} B_m^2 \Bigg(e^{2\delta_j}\cos^2\beta_m\delta_j - e^{2\theta_j}\cos^2\beta_m\theta_j + \frac{\beta_m e^{2\delta_j}(\sin 2\beta_m\delta_j - \beta_m\cos 2\beta_m\delta_j)}{2(1+\beta_m^2)} - \frac{\beta_m e^{2\theta_j}(\sin 2\beta_m\theta_j - \beta_m\cos 2\beta_m\theta_j)}{2(1+\beta_m^2)} \Bigg)$$

$$+ \sum_{\substack{m=1 \\ m\neq\bar{m}}}^{\infty} \sum_{\bar{m}=1}^{\infty} D_m D_{\bar{m}} \Bigg(\frac{e^{2\delta_j}\sin(\beta_{\bar{m}}-\beta_m)\delta_j}{2(\beta_{\bar{m}}-\beta_m)} - \frac{e^{2\theta_j}\sin(\beta_{\bar{m}}-\beta_m)\theta_j}{2(\beta_{\bar{m}}-\beta_m)} + \frac{e^{2\delta_j}\sin(\beta_{\bar{m}}+\beta_m)\delta_j}{2(\beta_{\bar{m}}+\beta_m)}$$

$$- \frac{e^{2\theta_j}\sin(\beta_{\bar{m}}+\beta_m)\theta_j}{2(\beta_{\bar{m}}+\beta_m)} - \frac{\{e^{2\delta_j}\}\{2\sin(\beta_{\bar{m}}-\beta_m)\delta_j - (\beta_{\bar{m}}-\beta_m)\cos(\beta_{\bar{m}}-\beta_m)\delta_j\}}{(\beta_{\bar{m}}-\beta_m)[4+(\beta_{\bar{m}}-\beta_m)^2]}$$

$$+ \frac{\{e^{2\theta_j}\}\{2\sin(\beta_{\bar{m}}-\beta_m)\theta_j - (\beta_{\bar{m}}-\beta_m)\cos(\beta_{\bar{m}}-\beta_m)\theta_j\}}{(\beta_{\bar{m}}-\beta_m)[4+(\beta_{\bar{m}}-\beta_m)^2]} - \frac{\{e^{2\delta_j}\}\{2\sin(\beta_{\bar{m}}+\beta_m)\delta_j - (\beta_{\bar{m}}+\beta_m)\cos(\beta_{\bar{m}}+\beta_m)\delta_j\}}{(\beta_{\bar{m}}+\beta_m)[4+(\beta_{\bar{m}}+\beta_m)^2]}$$

$$+ \frac{\{e^{2\theta_j}\}\{2\sin(\beta_{\bar{m}}+\beta_m)\theta_j - (\beta_{\bar{m}}+\beta_m)\cos(\beta_{\bar{m}}+\beta_m)\theta_j\}}{(\beta_{\bar{m}}+\beta_m)[4+(\beta_{\bar{m}}+\beta_m)^2]} \Bigg) \Bigg]$$

(Continued)

$$
\begin{aligned}
&+\frac{s_{o1}^2\pi\Omega^2\sin^2\Omega t}{(s_{o1}\sin\alpha_1)^2/\sin\alpha_j}\left(I^c+2I^f\right)\Bigg[\frac{1}{2}\sum_{m=1}^{\infty}D_m^2\Bigg(e^{2\delta_j}\cos^2\beta_m\delta_j-e^{2\theta_j}\cos^2\beta_m\theta_j+\frac{\beta_m e^{2\delta_j}(\sin 2\beta_m\delta_j-\beta_m\cos 2\beta_m\delta_j)}{2(1+\beta_m^2)}\\
&-\frac{\beta_m e^{2\theta_j}(\sin 2\beta_m\theta_j-\beta_m\cos 2\beta_m\theta_j)}{2(1+\beta_m^2)}\Bigg)+\sum_{\substack{m=1\\ m\neq\bar{m}}}^{\infty}\sum_{\bar{m}=1}^{\infty}D_m D_{\bar{m}}\Bigg(\frac{e^{2\delta_j}\sin(\beta_{\bar{m}}-\beta_m)\delta_j}{2(\beta_{\bar{m}}-\beta_m)}\\
&-\frac{e^{2\theta_j}\sin(\beta_{\bar{m}}-\beta_m)\theta_j}{2(\beta_{\bar{m}}-\beta_m)}+\frac{e^{2\delta_j}\sin(\beta_{\bar{m}}+\beta_m)\delta_j}{2(\beta_{\bar{m}}+\beta_m)}-\frac{e^{2\theta_j}\sin(\beta_{\bar{m}}+\beta_m)\theta_j}{2(\beta_{\bar{m}}+\beta_m)}\\
&-\frac{\{e^{2\delta_j}\}\{2\sin(\beta_{\bar{m}}-\beta_m)\delta_j-(\beta_{\bar{m}}-\beta_m)\cos(\beta_{\bar{m}}-\beta_m)\delta_j\}}{(\beta_{\bar{m}}-\beta_m)[4+(\beta_{\bar{m}}-\beta_m)^2]}+\frac{\{e^{2\theta_j}\}\{2\sin(\beta_{\bar{m}}-\beta_m)\theta_j-(\beta_{\bar{m}}-\beta_m)\cos(\beta_{\bar{m}}-\beta_m)\theta_j\}}{(\beta_{\bar{m}}-\beta_m)[4+(\beta_{\bar{m}}-\beta_m)^2]}\\
&-\frac{\{e^{2\delta_j}\}\{2\sin(\beta_{\bar{m}}+\beta_m)\delta_j-(\beta_{\bar{m}}+\beta_m)\cos(\beta_{\bar{m}}+\beta_m)\delta_j\}}{(\beta_{\bar{m}}+\beta_m)[4+(\beta_{\bar{m}}+\beta_m)^2]}\\
&+\frac{\{e^{2\theta_j}\}\{2\sin(\beta_{\bar{m}}+\beta_m)\theta_j-(\beta_{\bar{m}}+\beta_m)\cos(\beta_{\bar{m}}+\beta_m)\theta_j\}}{(\beta_{\bar{m}}+\beta_m)[4+(\beta_{\bar{m}}+\beta_m)^2]}\Bigg)\Bigg]\Bigg\}_j \qquad 3.4.3
\end{aligned}
$$

Equation 3.4.3 defines the total kinetic energy in a sandwich shell of revolution. For maximum kinetic energy in the system the term, $\sin^2\Omega t$, is unity.

3.5 Frequency Equation and the Method of Obtaining the Eigenvalues

It is assumed that the shell under consideration is a conservative system and vibrating in simple harmonic motion, so that the maximum strain and kinetic energies associated with the assumed deflection modes are equal.

Using Equation 3.2.2 with reference to the assumed deflection modes in Equations 3.3.7 the following expressions are obtained:

$$\frac{\partial}{\partial A_m}(\bar{V}_{max} - \bar{T}_{max}) = 0 \quad \text{or} \quad \frac{\partial \bar{V}_{max}}{\partial A_m} = \frac{\partial \bar{T}_{max}}{\partial A_m}$$

$$\frac{\partial}{\partial B_m}(\bar{V}_{max} - \bar{T}_{max}) = 0 \quad \text{or} \quad \frac{\partial \bar{V}_{max}}{\partial B_m} = \frac{\partial \bar{T}_{max}}{\partial B_m} \qquad 3.5.1$$

$$\frac{\partial}{\partial D_m}(\bar{V}_{max} - \bar{T}_{max}) = 0 \quad \text{or} \quad \frac{\partial \bar{V}_{max}}{\partial D_m} = \frac{\partial \bar{T}_{max}}{\partial D_m}$$

Equations 3.5.1 lead to an infinite system of homogeneous linear equations in A_m, B_m, and D_m. In matrix notation this can be written as:

$$\begin{bmatrix} Z \end{bmatrix} \begin{Bmatrix} V \end{Bmatrix} = \lambda \begin{bmatrix} A \end{bmatrix} \begin{Bmatrix} V \end{Bmatrix} \qquad 3.5.2$$

where the Z-matrix can be obtained by differentiating the expression for the maximum strain energy with respect to the parameters, A_m, B_m and D_m respectively; while, the A-matrix can be obtained by differentiating the expression for the maximum kinetic energy with respect to the same parameters. The V-column matrix is defined as follows:

$$V = \begin{Bmatrix} A_1 \\ A_2 \\ \cdot \\ \cdot \\ \cdot \\ A_m \\ B_1 \\ B_2 \\ \cdot \\ \cdot \\ \cdot \\ B_m \\ D_1 \\ D_2 \\ \cdot \\ \cdot \\ \cdot \\ D_m \end{Bmatrix}$$

The matrix Equation 3.5.2 was solved with the aid of a digital computer (IBM 1410) for the lower eigenvalues.

If Equations 3.3.11 and 3.4.3 are substituted correspondingly into Equations 3.5.1 the following results are obtained:

$$
\begin{aligned}
\frac{\partial \bar{V}_{max}}{\partial A_m} = \sum_{j=1}^{n} \Bigg\{ \sin\alpha_j \Bigg[& -(\eta_s \nu_s + \eta_\theta \nu_\theta)\cot\alpha_j \sum_{m=1}^{\infty} \beta_m B_m \left(\frac{\delta_j}{2} - \frac{\theta_j}{2} - \frac{\sin 2\beta_m\delta_j}{4\beta_m} + \frac{\sin 2\beta_m\theta_j}{4\beta_m} \right) \\
& -(\eta_s \nu_s + \eta_\theta \nu_\theta)\cot\alpha_j \sum_{\substack{m=1 \\ m\neq \bar{m}}}^{\infty} \sum_{\bar{m}=1}^{\infty} \beta_{\bar{m}} B_{\bar{m}} \left(\frac{\sin(\beta_{\bar{m}}-\beta_m)\delta_j}{2(\beta_{\bar{m}}-\beta_m)} - \frac{\sin(\beta_{\bar{m}}-\beta_m)\theta_j}{2(\beta_{\bar{m}}-\beta_m)} + \frac{\sin(\beta_{\bar{m}}+\beta_m)\delta_j}{2(\beta_{\bar{m}}+\beta_m)} \right. \\
& \left. - \frac{\sin(\beta_{\bar{m}}+\beta_m)\theta_j}{2(\beta_{\bar{m}}+\beta_m)} \right) - \eta_\theta \cot\alpha_j \sum_{m=1}^{\infty} \frac{B_m}{\beta_m} \left(\cos 2\beta_m\delta_j - \cos 2\beta_m\theta_j \right) \\
& - \eta_\theta \cot\alpha_j \sum_{\substack{m=1 \\ m\neq \bar{m}}}^{\infty} \sum_{\bar{m}=1}^{\infty} B_{\bar{m}} \left(\frac{\cos(\beta_m-\beta_{\bar{m}})\delta_j}{(\beta_m-\beta_{\bar{m}})} - \frac{\cos(\beta_m-\beta_{\bar{m}})\theta_j}{(\beta_m-\beta_{\bar{m}})} + \frac{\cos(\beta_m+\beta_{\bar{m}})\delta_j}{(\beta_m+\beta_{\bar{m}})} \right. \\
& \left. - \frac{\cos(\beta_m+\beta_{\bar{m}})\theta_j}{(\beta_m+\beta_{\bar{m}})} \right) + \eta_\theta \cot^2\alpha_j \sum_{m=1}^{\infty} A_m \left(\delta_j - \theta_j - \frac{\sin 2\beta_m\delta_j}{2\beta_m} + \frac{\sin 2\beta_m\theta_j}{2\beta_m} \right) \\
& + \eta_\theta \cot^2\alpha_j \sum_{\substack{m=1 \\ m\neq \bar{m}}}^{\infty} \sum_{\bar{m}=1}^{\infty} A_{\bar{m}} \left(\frac{\sin(\beta_{\bar{m}}-\beta_m)\delta_j}{(\beta_{\bar{m}}-\beta_m)} - \frac{\sin(\beta_{\bar{m}}-\beta_m)\theta_j}{(\beta_{\bar{m}}-\beta_m)} - \frac{\sin(\beta_{\bar{m}}+\beta_m)\delta_j}{(\beta_{\bar{m}}+\beta_m)} \right. \\
& \left. + \frac{\sin(\beta_{\bar{m}}+\beta_m)\theta_j}{(\beta_{\bar{m}}+\beta_m)} \right) - \frac{\eta_s \nu_s a^2 \cot^2\alpha_j}{(s_{01})^2 \sin\alpha_j} \sum_{m=1}^{\infty} \beta_m D_m \left(-e^{-\delta_j} \sin^2\beta_m\delta_j + e^{-\theta_j} \sin^2\beta_m\theta_j \right. \\
& \left. - \frac{\beta_m e^{-\delta_j}(\sin 2\beta_m\delta_j + 2\beta_m \cos 2\beta_m\delta_j)}{(1+4\beta_m^2)} + \frac{\beta_m e^{-\theta_j}(\sin 2\beta_m\theta_j + 2\beta_m \cos 2\beta_m\theta_j)}{(1+4\beta_m^2)} \right) \quad \text{(Continued)}
\end{aligned}
$$

$$- \frac{\eta_s \nu_s a^2 \cot^2\alpha_j}{2(s_{o1})^2 \sin\alpha_i} \sum_{\substack{m=1\\ m\neq\bar{m}}}^{\infty} \sum_{\bar{m}=1}^{\infty} \beta_m D_{\bar{m}} \left(\frac{e^{-\delta_j}\sin(\beta_{\bar{m}}-\beta_m)\delta_j}{(\beta_{\bar{m}}-\beta_m)} - \frac{e^{-\theta_j}\sin(\beta_{\bar{m}}-\beta_m)\theta_j}{(\beta_{\bar{m}}-\beta_m)} - \frac{e^{-\delta_j}\sin(\beta_{\bar{m}}+\beta_m)\delta_j}{(\beta_{\bar{m}}+\beta_m)} \right.$$

$$+ \frac{e^{-\theta_j}\sin(\beta_{\bar{m}}+\beta_m)\theta_j}{(\beta_{\bar{m}}+\beta_m)} - \frac{\{e^{-\delta_j}\}\{\sin(\beta_{\bar{m}}-\beta_m)\delta_j + (\beta_{\bar{m}}-\beta_m)\cos(\beta_{\bar{m}}-\beta_m)\delta_j\}}{(\beta_{\bar{m}}-\beta_m)[1+(\beta_{\bar{m}}-\beta_m)^2]}$$

$$+ \frac{\{e^{-\theta_j}\}\{\sin(\beta_{\bar{m}}-\beta_m)\theta_j + (\beta_{\bar{m}}-\beta_m)\cos(\beta_{\bar{m}}-\beta_m)\theta_j\}}{(\beta_{\bar{m}}-\beta_m)[1+(\beta_{\bar{m}}-\beta_m)^2]} + \frac{\{e^{-\delta_j}\}\{\sin(\beta_{\bar{m}}+\beta_m)\delta_j + (\beta_{\bar{m}}+\beta_m)\cos(\beta_{\bar{m}}+\beta_m)\delta_j\}}{(\beta_{\bar{m}}+\beta_m)[1+(\beta_{\bar{m}}+\beta_m)^2]}$$

$$\left. - \frac{\{e^{-\theta_j}\}\{\sin(\beta_{\bar{m}}+\beta_m)\theta_j + (\beta_{\bar{m}}+\beta_m)\cos(\beta_{\bar{m}}+\beta_m)\theta_j\}}{(\beta_{\bar{m}}+\beta_m)[1+(\beta_{\bar{m}}+\beta_m)^2]} \right) + \eta \sum_{m=1}^{\infty} \beta_m^2 A_m \left(\delta_j - \theta_j + \frac{\sin 2\beta_m \delta_j}{2\beta_m} - \frac{\sin 2\beta_m \theta_j}{2\beta_m} \right)$$

$$+ \eta \sum_{\substack{m=1\\ m\neq\bar{m}}}^{\infty} \sum_{\bar{m}=1}^{\infty} \beta_{\bar{m}} \beta_m A_{\bar{m}} \left(\frac{\sin(\beta_{\bar{m}}-\beta_m)\delta_j}{(\beta_{\bar{m}}-\beta_m)} - \frac{\sin(\beta_{\bar{m}}-\beta_m)\theta_j}{(\beta_{\bar{m}}-\beta_m)} + \frac{\sin(\beta_{\bar{m}}+\beta_m)\delta_j}{(\beta_{\bar{m}}+\beta_m)} - \frac{\sin(\beta_{\bar{m}}+\beta_m)\theta_j}{(\beta_{\bar{m}}+\beta_m)} \right)$$

$$+ \frac{2\eta}{\sin\alpha_i} \sum_{m=1}^{\infty} \beta_m D_m \left(e^{\delta_j}\cos^2\beta_m\delta_j - e^{\theta_j}\cos^2\beta_m\theta_j + \frac{\beta_m e^{\delta_j}(\sin 2\beta_m\delta_j - 2\beta_m\cos 2\beta_m\delta_j)}{(1+4\beta_m^2)} \right.$$

$$\left. - \frac{\beta_m e^{\theta_j}(\sin 2\beta_m\theta_j - 2\beta_m\cos 2\beta_m\theta_j)}{(1+4\beta_m^2)} \right) + \frac{\eta}{\sin\alpha_i} \sum_{\substack{m=1\\ m\neq\bar{m}}}^{\infty} \sum_{\bar{m}=1}^{\infty} \beta_m D_{\bar{m}} \left(\frac{e^{\delta_j}\sin(\beta_{\bar{m}}-\beta_m)\delta_j}{(\beta_{\bar{m}}-\beta_m)} - \frac{e^{\theta_j}\sin(\beta_{\bar{m}}-\beta_m)\theta_j}{(\beta_{\bar{m}}-\beta_m)} \right.$$

$$+ \frac{e^{\delta_j}\sin(\beta_{\bar{m}}+\beta_m)\delta_j}{(\beta_{\bar{m}}+\beta_m)} - \frac{e^{\theta_j}\sin(\beta_{\bar{m}}+\beta_m)\theta_j}{(\beta_{\bar{m}}+\beta_m)} + \frac{\{e^{\delta_j}\}\{\sin(\beta_{\bar{m}}-\beta_m)\delta_j - (\beta_{\bar{m}}-\beta_m)\cos(\beta_{\bar{m}}-\beta_m)\delta_j\}}{(\beta_{\bar{m}}-\beta_m)[1+(\beta_{\bar{m}}-\beta_m)^2]}$$

$$- \frac{\{e^{\theta_j}\}\{\sin(\beta_{\bar{m}}-\beta_m)\theta_j - (\beta_{\bar{m}}-\beta_m)\cos(\beta_{\bar{m}}-\beta_m)\theta_j\}}{(\beta_{\bar{m}}-\beta_m)[1+(\beta_{\bar{m}}-\beta_m)^2]} + \frac{\{e^{\delta_j}\}\{\sin(\beta_{\bar{m}}+\beta_m)\delta_j - (\beta_{\bar{m}}+\beta_m)\cos(\beta_{\bar{m}}+\beta_m)\delta_j\}}{(\beta_{\bar{m}}+\beta_m)[1+(\beta_{\bar{m}}+\beta_m)^2]}$$

$$\left. \left. \left. - \frac{\{e^{\theta_j}\}\{\sin(\beta_{\bar{m}}+\beta_m)\theta_j - (\beta_{\bar{m}}+\beta_m)\cos(\beta_{\bar{m}}+\beta_m)\theta_j\}}{(\beta_{\bar{m}}+\beta_m)[1+(\beta_{\bar{m}}+\beta_m)^2]} \right) \right] \right\}_j \qquad 3.5.3$$

$$\frac{\partial \bar{V}_{max}}{\partial B_m} = \sum_{j=1}^{n} \Bigg\{ \sin\alpha_j \Bigg[\eta_s \sum_{m=1}^{\infty} \beta_m^2 B_m^2 \left(\delta_j - \theta_j - \frac{\sin 2\beta_m \delta_j}{2\beta_m} + \frac{\sin 2\beta_m \theta_j}{2\beta_m} \right)$$

$$+ \eta_s \sum_{\substack{m=1 \\ m \neq \bar{m}}}^{\infty} \sum_{\bar{m}=1}^{\infty} \beta_m \beta_{\bar{m}} B_{\bar{m}} \left(\frac{\sin(\beta_{\bar{m}} - \beta_m)\delta_j}{(\beta_{\bar{m}} - \beta_m)} - \frac{\sin(\beta_{\bar{m}} - \beta_m)\theta_j}{(\beta_{\bar{m}} - \beta_m)} - \frac{\sin(\beta_{\bar{m}} + \beta_m)\delta_j}{(\beta_{\bar{m}} + \beta_m)} + \frac{\sin(\beta_{\bar{m}} + \beta_m)\theta_j}{(\beta_{\bar{m}} + \beta_m)} \right)$$

$$+ (\eta_s \nu_s + \eta_\theta \nu_\theta) \sum_{m=1}^{\infty} B_m \left(\cos 2\beta_m \delta_j - \cos 2\beta_m \theta_j \right) + (\eta_s \nu_s + \eta_\theta \nu_\theta) \sum_{\substack{m=1 \\ m \neq \bar{m}}}^{\infty} \sum_{\bar{m}=1}^{\infty} \beta_{\bar{m}} B_{\bar{m}} \left(\frac{\cos(\beta_{\bar{m}} - \beta_m)\delta_j}{2(\beta_{\bar{m}} - \beta_m)} \right.$$

$$\left. - \frac{\cos(\beta_{\bar{m}} - \beta_m)\theta_j}{2(\beta_{\bar{m}} - \beta_m)} + \frac{\cos(\beta_{\bar{m}} + \beta_m)\delta_j}{2(\beta_{\bar{m}} + \beta_m)} - \frac{\cos(\beta_{\bar{m}} + \beta_m)\theta_j}{2(\beta_{\bar{m}} + \beta_m)} \right)$$

$$+ (\eta_s \nu_s + \eta_\theta \nu_\theta) \sum_{\substack{m=1 \\ m \neq \bar{m}}}^{\infty} \sum_{\bar{m}=1}^{\infty} \beta_m B_{\bar{m}} \left(\frac{\cos(\beta_m - \beta_{\bar{m}})\delta_j}{2(\beta_m - \beta_{\bar{m}})} - \frac{\cos(\beta_m - \beta_{\bar{m}})\theta_j}{2(\beta_m - \beta_{\bar{m}})} + \frac{\cos(\beta_m + \beta_{\bar{m}})\delta_j}{2(\beta_m + \beta_{\bar{m}})} \right.$$

$$\left. - \frac{\cos(\beta_m + \beta_{\bar{m}})\theta_j}{2(\beta_m + \beta_{\bar{m}})} \right) - (\eta_s \nu_s + \eta_\theta \nu_\theta) \cot\alpha_j \sum_{m=1}^{\infty} \beta_m A_m \left(\frac{\delta_j}{2} - \frac{\theta_j}{2} - \frac{\sin 2\beta_m \delta_j}{4\beta_m} + \frac{\sin 2\beta_m \theta_j}{4\beta_m} \right)$$

$$- (\eta_s \nu_s + \eta_\theta \nu_\theta) \cot\alpha_j \sum_{\substack{m=1 \\ m \neq \bar{m}}}^{\infty} \sum_{\bar{m}=1}^{\infty} \beta_m A_{\bar{m}} \left(\frac{\sin(\beta_{\bar{m}} - \beta_m)\delta_j}{2(\beta_{\bar{m}} - \beta_m)} - \frac{\sin(\beta_{\bar{m}} - \beta_m)\theta_j}{2(\beta_{\bar{m}} - \beta_m)} - \frac{\sin(\beta_{\bar{m}} + \beta_m)\delta_j}{2(\beta_{\bar{m}} + \beta_m)} \right.$$

$$\left. + \frac{\sin(\beta_{\bar{m}} + \beta_m)\theta_j}{2(\beta_{\bar{m}} + \beta_m)} \right) + \eta_\theta \sum_{m=1}^{\infty} B_m \left(\delta_j - \theta_j + \frac{\sin 2\beta_m \delta_j}{2\beta_m} - \frac{\sin 2\beta_m \theta_j}{2\beta_m} \right)$$

$$+ \eta_\theta \sum_{\substack{m=1 \\ m \neq \bar{m}}}^{\infty} \sum_{\bar{m}=1}^{\infty} B_{\bar{m}} \left(\frac{\sin(\beta_{\bar{m}} - \beta_m)\delta_j}{(\beta_{\bar{m}} - \beta_m)} - \frac{\sin(\beta_{\bar{m}} - \beta_m)\theta_j}{(\beta_{\bar{m}} - \beta_m)} + \frac{\sin(\beta_{\bar{m}} + \beta_m)\delta_j}{(\beta_{\bar{m}} + \beta_m)} - \frac{\sin(\beta_{\bar{m}} + \beta_m)\delta_j}{(\beta_{\bar{m}} + \beta_m)} \right)$$

$$- \eta_\theta \cot\alpha_j \sum_{m=1}^{\infty} \frac{A_m}{\beta_m} \left(\cos 2\beta_m \delta_j - \cos 2\beta_m \theta_j \right)$$

(Continued)

$$-\eta_\theta \cot\alpha_j \sum_{\substack{m=1\\ m\neq\bar m}}^{\infty}\sum_{\bar m=1}^{\infty} A_{\bar m}\Bigg(\frac{\cos(\beta_{\bar m}-\beta_m)\delta_j}{(\beta_{\bar m}-\beta_m)} - \frac{\cos(\beta_{\bar m}-\beta_m)\theta_j}{(\beta_{\bar m}-\beta_m)} + \frac{\cos(\beta_{\bar m}+\beta_m)\delta_j}{(\beta_{\bar m}+\beta_m)} - \frac{\cos(\beta_{\bar m}+\beta_m)\theta_j}{(\beta_{\bar m}+\beta_m)}\Bigg)$$

$$+ \frac{\eta_s a^2\cot^2\alpha_j}{(s_{01})^2\sin\alpha_1}\sum_{m=1}^{\infty} \beta_m^2 D_m\Bigg(-e^{-\delta_j}\sin^2\beta_m\delta_j + e^{-\theta_j}\sin^2\beta_m\theta_j - \frac{\beta_m e^{-\delta_j}(\sin 2\beta_m\delta_j + 2\beta_m\cos 2\beta_m\delta_j)}{(1+4\beta_m^2)}$$

$$+ \frac{\beta_m e^{-\theta_j}(\sin 2\beta_m\theta_j + 2\beta_m\cos 2\beta_m\theta_j)}{(1+4\beta_m^2)}\Bigg) + \frac{\eta_s a^2\cot^2\alpha_j}{2(s_{01})^2\sin\alpha_1}\sum_{\substack{m=1\\ m\neq\bar m}}^{\infty}\sum_{\bar m=1}^{\infty} \beta_m\beta_{\bar m} D_{\bar m}\Bigg(\frac{e^{-\delta_j}\sin(\beta_{\bar m}-\beta_m)\delta_j}{(\beta_{\bar m}-\beta_m)}$$

$$- \frac{e^{-\theta_j}\sin(\beta_{\bar m}-\beta_m)\theta_j}{(\beta_{\bar m}-\beta_m)} - \frac{e^{-\delta_j}\sin(\beta_{\bar m}+\beta_m)\delta_j}{(\beta_{\bar m}+\beta_m)} + \frac{e^{-\theta_j}\sin(\beta_{\bar m}+\beta_m)\theta_j}{(\beta_{\bar m}+\beta_m)}$$

$$- \frac{\{e^{-\delta_j}\}\{\sin(\beta_{\bar m}-\beta_m)\delta_j + (\beta_{\bar m}-\beta_m)\cos(\beta_{\bar m}-\beta_m)\delta_j\}}{(\beta_{\bar m}-\beta_m)[1+(\beta_{\bar m}-\beta_m)^2]} + \frac{\{e^{-\theta_j}\}\{\sin(\beta_{\bar m}-\beta_m)\theta_j + (\beta_{\bar m}-\beta_m)\cos(\beta_{\bar m}-\beta_m)\theta_j\}}{(\beta_{\bar m}-\beta_m)[1+(\beta_{\bar m}-\beta_m)^2]}$$

$$+ \frac{\{e^{-\delta_j}\}\{\sin(\beta_{\bar m}+\beta_m)\delta_j + (\beta_{\bar m}+\beta_m)\cos(\beta_{\bar m}+\beta_m)\delta_j\}}{(\beta_{\bar m}+\beta_m)[1+(\beta_{\bar m}+\beta_m)^2]} - \frac{\{e^{-\theta_j}\}\{\sin(\beta_{\bar m}+\beta_m)\theta_j + (\beta_{\bar m}+\beta_m)\cos(\beta_{\bar m}+\beta_m)\theta_j\}}{(\beta_{\bar m}+\beta_m)[1+(\beta_{\bar m}+\beta_m)^2]}\Bigg)$$

$$- \frac{\eta_s\nu_s a^2\cot^2\alpha_j}{2(s_{01})^2\sin\alpha_1}\sum_{m=1}^{\infty} \beta_m D_m\Bigg(\frac{e^{-\delta_j}(-\sin 2\beta_m\delta_j - 2\beta_m\cos 2\beta_m\delta_j)}{(1+4\beta_m^2)} - \frac{e^{-\theta_j}(-\sin 2\beta_m\theta_j - 2\beta_m\cos 2\beta_m\theta_j)}{(1+4\beta_m^2)}\Bigg)$$

$$+ \frac{\eta_s\nu_s a^2\cot^2\alpha_j}{2(s_{01})^2\sin\alpha_1}\sum_{\substack{m=1\\ m\neq\bar m}}^{\infty}\sum_{\bar m=1}^{\infty} \beta_{\bar m} D_{\bar m}\Bigg(\frac{e^{-\delta_j}\cos(\beta_{\bar m}-\beta_m)\delta_j}{(\beta_{\bar m}-\beta_m)} - \frac{e^{-\theta_j}\cos(\beta_{\bar m}-\beta_m)\theta_j}{(\beta_{\bar m}-\beta_m)} + \frac{e^{-\delta_j}\cos(\beta_{\bar m}+\beta_m)\delta_j}{(\beta_{\bar m}+\beta_m)} - \frac{e^{-\theta_j}\cos(\beta_{\bar m}+\beta_m)\theta_j}{(\beta_{\bar m}+\beta_m)}$$

$$+ \frac{\{e^{-\delta_j}\}\{(\beta_{\bar m}-\beta_m)\sin(\beta_{\bar m}-\beta_m)\delta_j - \cos(\beta_{\bar m}-\beta_m)\delta_j\}}{(\beta_{\bar m}-\beta_m)[1+(\beta_{\bar m}-\beta_m)^2]} - \frac{\{e^{-\theta_j}\}\{(\beta_{\bar m}-\beta_m)\sin(\beta_{\bar m}-\beta_m)\theta_j - \cos(\beta_{\bar m}-\beta_m)\theta_j\}}{(\beta_{\bar m}-\beta_m)[1+(\beta_{\bar m}-\beta_m)^2}$$

$$+ \frac{\{e^{-\delta_j}\}\{(\beta_{\bar m}+\beta_m)\sin(\beta_{\bar m}+\beta_m)\delta_j - \cos(\beta_{\bar m}+\beta_m)\delta_j\}}{(\beta_{\bar m}+\beta_m)[1+(\beta_{\bar m}+\beta_m)^2]} - \frac{\{e^{-\theta_j}\}\{(\beta_{\bar m}+\beta_m)\sin(\beta_{\bar m}+\beta_m)\theta_j - \cos(\beta_{\bar m}+\beta_m)\theta_j\}}{(\beta_{\bar m}+\beta_m)[1+(\beta_{\bar m}+\beta_m)^2]}\Bigg)\Bigg]\Bigg\}_j \qquad 3.5.4$$

$$\frac{\partial \bar{V}_{max}}{\partial D_m} = \sum_{m=1}^{\infty}\Bigg\{\sin\alpha_j\Bigg[\frac{\eta_s^*}{(s_{01}\sin\alpha_1)^2}\sum_{m=1}^{\infty}\beta_m^2 D_m\left(\delta_j - \theta_j - \frac{\sin 2\beta_m\delta_j}{2\beta_m} + \frac{\sin 2\beta_m\theta_j}{2\beta_m}\right)$$

$$+ \frac{\eta_s^*}{(s_{01}\sin\alpha_1)^2}\sum_{\substack{m=1\\ m\neq\bar{m}}}^{\infty}\sum_{\bar{m}=1}^{\infty}\beta_{\bar{m}}\beta_m D_{\bar{m}}\left(\frac{\sin(\beta_{\bar{m}}-\beta_m)\delta_j}{(\beta_{\bar{m}}-\beta_m)} - \frac{\sin(\beta_{\bar{m}}-\beta_m)\theta_j}{(\beta_{\bar{m}}-\beta_m)} - \frac{\sin(\beta_{\bar{m}}+\beta_m)\delta_j}{(\beta_{\bar{m}}+\beta_m)} + \frac{\sin(\beta_{\bar{m}}+\beta_m)\theta_j}{(\beta_{\bar{m}}+\beta_m)}\right)$$

$$+ \frac{(\eta_s^*\nu_s + \eta_\theta^*\nu_\theta)}{(s_{01}\sin\alpha_1)^2}\sum_{m=1}^{\infty}D_m\left(\cos 2\beta_m\delta_j - \cos 2\beta_m\theta_j\right) + \frac{(\eta_s^*\nu_s + \eta_\theta^*\nu_\theta)}{2(s_{01}\sin\alpha_1)^2}\sum_{\substack{m=1\\ m\neq\bar{m}}}^{\infty}\sum_{\bar{m}=1}^{\infty}\beta_{\bar{m}} D_{\bar{m}}\left(\frac{\cos(\beta_{\bar{m}}-\beta_m)\delta_j}{(\beta_{\bar{m}}-\beta_m)}\right.$$

$$\left. - \frac{\cos(\beta_{\bar{m}}-\beta_m)\theta_j}{(\beta_{\bar{m}}-\beta_m)} + \frac{\cos(\beta_{\bar{m}}+\beta_m)\delta_j}{(\beta_{\bar{m}}+\beta_m)} - \frac{\cos(\beta_{\bar{m}}+\beta_m)\theta_j}{(\beta_{\bar{m}}+\beta_m)}\right)$$

$$+ \frac{(\eta_s^*\nu_s + \eta_\theta^*\nu_\theta)}{2(s_{01}\sin\alpha_1)^2}\sum_{\substack{m=1\\ m\neq\bar{m}}}^{\infty}\sum_{\bar{m}=1}^{\infty}\beta_m D_{\bar{m}}\left(\frac{\cos(\beta_m-\beta_{\bar{m}})\delta_j}{(\beta_m-\beta_{\bar{m}})} - \frac{\cos(\beta_m-\beta_{\bar{m}})\theta_j}{(\beta_m-\beta_{\bar{m}})} + \frac{\cos(\beta_{\bar{m}}+\beta_m)\delta_j}{(\beta_{\bar{m}}+\beta_m)}\right.$$

$$\left. - \frac{\cos(\beta_{\bar{m}}+\beta_m)\theta_j}{(\beta_{\bar{m}}+\beta_m)}\right) + \frac{\eta_s a^2\cot^2\alpha_1}{(s_{01})^2\sin\alpha_1}\sum_{m=1}^{\infty}\beta_m^2 B_m\left(-e^{-\delta_j}\sin^2\beta_m\delta_j + e^{-\theta_j}\sin^2\beta_m\theta_j\right.$$

$$\left. - \frac{\beta_m e^{-\delta_j}(\sin 2\beta_m\delta_j + 2\beta_m\cos 2\beta_m\delta_j)}{(1+4\beta_m^2)} + \frac{\beta_m e^{-\theta_j}(\sin 2\beta_m\theta_j + 2\beta_m\cos 2\beta_m\theta_j)}{(1+4\beta_m^2)}\right) + \frac{\eta_s a^2\cot^2\alpha_1}{2(s_{01})^2\sin\alpha_1}\sum_{\substack{m=1\\ m\neq\bar{m}}}^{\infty}\sum_{\bar{m}=1}^{\infty}\beta_m\beta_{\bar{m}}B_{\bar{m}}\left(\frac{e^{-\delta_j}\sin(\beta_{\bar{m}}-\beta_m)\delta_j}{(\beta_{\bar{m}}-\beta_m)}\right.$$

$$- \frac{e^{-\theta_j}\sin(\beta_{\bar{m}}-\beta_m)\theta_j}{(\beta_{\bar{m}}-\beta_m)} - \frac{e^{-\delta_j}\sin(\beta_{\bar{m}}+\beta_m)\delta_j}{(\beta_{\bar{m}}+\beta_m)} + \frac{e^{-\theta_j}\sin(\beta_{\bar{m}}+\beta_m)\theta_j}{(\beta_{\bar{m}}+\beta_m)} - \frac{\{e^{-\delta_j}\}\{\sin(\beta_{\bar{m}}-\beta_m)\delta_j + (\beta_{\bar{m}}-\beta_m)\cos(\beta_{\bar{m}}-\beta_m)\delta_j\}}{(\beta_{\bar{m}}-\beta_m)[1+(\beta_{\bar{m}}-\beta_m)^2]}$$

$$+ \frac{\{e^{-\theta_j}\}\{\sin(\beta_{\bar{m}}-\beta_m)\theta_j + (\beta_{\bar{m}}-\beta_m)\cos(\beta_{\bar{m}}-\beta_m)\theta_j\}}{(\beta_{\bar{m}}-\beta_m)[1+(\beta_{\bar{m}}-\beta_m)^2]} + \frac{\{e^{-\delta_j}\}\{\sin(\beta_{\bar{m}}+\beta_m)\delta_j + (\beta_{\bar{m}}+\beta_m)\cos(\beta_{\bar{m}}+\beta_m)\delta_j\}}{(\beta_{\bar{m}}+\beta_m)[1+(\beta_{\bar{m}}+\beta_m)^2]}$$

$$\left. - \frac{\{e^{-\theta_j}\}\{\sin(\beta_{\bar{m}}+\beta_m)\theta_j + (\beta_{\bar{m}}+\beta_m)\cos(\beta_{\bar{m}}+\beta_m)\theta_j\}}{(\beta_{\bar{m}}+\beta_m)[1+(\beta_{\bar{m}}+\beta_m)^2]}\right)$$

(Continued)

$$- \frac{\eta_s \nu_s a^2 \cot^2\alpha_j}{2(s_{01})^2 \sin\alpha_1} \sum_{m=1}^{\infty} \beta_m B_m \left(\frac{e^{-\delta_j}(-\sin 2\beta_m\delta_j - 2\beta_m \cos 2\beta_m\delta_j)}{(1+4\beta_m^2)} - \frac{e^{-\theta_j}(-\sin 2\beta_m\theta_j - 2\beta_m \cos 2\beta_m\theta_j)}{(1+4\beta_m^2)} \right)$$

$$+ \frac{\eta_s \nu_s a^2 \cot^2\alpha_j}{2(s_{0i})^2 \sin\alpha_1} \sum_{\substack{m=1 \\ m\neq\bar{m}}}^{\infty} \sum_{\bar{m}=1}^{\infty} \beta_m B_{\bar{m}} \left(\frac{e^{-\delta_j}\cos(\beta_m-\beta_{\bar{m}})\delta_j}{(\beta_m-\beta_{\bar{m}})} - \frac{e^{-\theta_j}\cos(\beta_m-\beta_{\bar{m}})\theta_j}{(\beta_m-\beta_{\bar{m}})} + \frac{e^{-\delta_j}\cos(\beta_m+\beta_{\bar{m}})\delta_j}{(\beta_m+\beta_{\bar{m}})}\right.$$

$$- \frac{e^{-\theta_j}\cos(\beta_m+\beta_{\bar{m}})\theta_j}{(\beta_m+\beta_{\bar{m}})} + \frac{\{e^{-\delta_j}\}\{(\beta_m-\beta_{\bar{m}})\sin(\beta_m-\beta_{\bar{m}})\delta_j - \cos(\beta_m-\beta_{\bar{m}})\delta_j\}}{(\beta_m-\beta_{\bar{m}})[1+(\beta_m-\beta_{\bar{m}})^2]}$$

$$- \frac{\{e^{-\theta_j}\}\{(\beta_m-\beta_{\bar{m}})\sin(\beta_m-\beta_{\bar{m}})\theta_j - \cos(\beta_m-\beta_{\bar{m}})\theta_j\}}{(\beta_m-\beta_{\bar{m}})[1+(\beta_m-\beta_{\bar{m}})^2]} + \frac{\{e^{-\delta_j}\}\{(\beta_m+\beta_{\bar{m}})\sin(\beta_m+\beta_{\bar{m}})\delta_j - \cos(\beta_m+\beta_{\bar{m}})\delta_j\}}{(\beta_m+\beta_{\bar{m}})[1+(\beta_m+\beta_{\bar{m}})^2]}$$

$$\left. - \frac{\{e^{-\theta_j}\}\{(\beta_m+\beta_{\bar{m}})\sin(\beta_m+\beta_{\bar{m}})\theta_j - \cos(\beta_m+\beta_{\bar{m}})\theta_j\}}{(\beta_m+\beta_{\bar{m}})[1+(\beta_m+\beta_{\bar{m}})^2]} \right) - \frac{\eta_s \nu_s a^2 \cot^2\alpha_j}{(s_{01})^2 \sin\alpha_1} \sum_{m=1}^{\infty} \beta_m A_m \left(-e^{-\delta_j}\sin^2\beta_m\delta_j + e^{-\theta_j}\sin^2\beta_m\theta_j \right.$$

$$\left. - \frac{\beta_m e^{-\delta_j}(\sin 2\beta_m\delta_j + 2\beta_m \cos 2\beta_m\delta_j)}{(1+4\beta_m^2)} + \frac{\beta_m e^{-\theta_j}(\sin 2\beta_m\theta_j + 2\beta_m \cos 2\beta_m\theta_j)}{(1+4\beta_m^2)} \right)$$

$$- \frac{\eta_s \nu_s a^2 \cot^2\alpha_j}{2(s_{01})^2 \sin\alpha_1} \sum_{\substack{m=1 \\ m\neq\bar{m}}}^{\infty} \sum_{\bar{m}=1}^{\infty} \beta_m A_{\bar{m}} \left(\frac{e^{-\delta_j}\sin(\beta_{\bar{m}}-\beta_m)\delta_j}{(\beta_{\bar{m}}-\beta_m)} - \frac{e^{-\theta_j}\sin(\beta_{\bar{m}}-\beta_m)\theta_j}{(\beta_{\bar{m}}-\beta_m)} - \frac{e^{-\delta_j}\sin(\beta_{\bar{m}}+\beta_m)\delta_j}{(\beta_{\bar{m}}+\beta_m)}\right.$$

$$+ \frac{e^{-\theta_j}\sin(\beta_{\bar{m}}+\beta_m)\theta_j}{(\beta_{\bar{m}}+\beta_m)} - \frac{\{e^{-\delta_j}\}\{\sin(\beta_{\bar{m}}-\beta_m)\delta_j + (\beta_{\bar{m}}-\beta_m)\cos(\beta_{\bar{m}}-\beta_m)\delta_j\}}{(\beta_{\bar{m}}-\beta_m)[1+(\beta_{\bar{m}}-\beta_m)^2]}$$

$$+ \frac{\{e^{-\theta_j}\}\{\sin(\beta_{\bar{m}}-\beta_m)\theta_j + (\beta_{\bar{m}}-\beta_m)\cos(\beta_{\bar{m}}-\beta_m)\theta_j\}}{(\beta_{\bar{m}}-\beta_m)[1+(\beta_{\bar{m}}-\beta_m)^2]} + \frac{\{e^{-\delta_j}\}\{\sin(\beta_{\bar{m}}+\beta_m)\delta_j + (\beta_{\bar{m}}+\beta_m)\cos(\beta_{\bar{m}}+\beta_m)\delta_j\}}{(\beta_{\bar{m}}+\beta_m)[1+(\beta_{\bar{m}}+\beta_m)^2]}$$

$$\left. - \frac{\{e^{-\theta_j}\}\{\sin(\beta_{\bar{m}}+\beta_m)\theta_j + (\beta_{\bar{m}}+\beta_m)\cos(\beta_{\bar{m}}+\beta_m)\theta_j\}}{(\beta_{\bar{m}}+\beta_m)[1+(\beta_{\bar{m}}+\beta_m)^2]} \right)$$

(Continued)

$$+\frac{\eta_\theta^*}{(s_{01}\sin\alpha_1)^2}\sum_{m=1}^{\infty} D_m\left(\delta_j-\theta_j+\frac{\sin 2\beta_m\delta_j}{2\beta_m}-\frac{\sin 2\beta_m\theta_j}{2\beta_m}\right)+\frac{\eta_\theta^*}{(s_{01}\sin\alpha_1)^2}\sum_{\substack{m=1\\ m\neq\bar m}}^{\infty}\sum_{\bar m=1}^{\infty} D_{\bar m}\left(\frac{\sin(\beta_{\bar m}-\beta_m)\delta_j}{(\beta_{\bar m}-\beta_m)}\right.$$

$$\left.-\frac{\sin(\beta_{\bar m}-\beta_m)\theta_j}{(\beta_{\bar m}-\beta_m)}+\frac{\sin(\beta_{\bar m}+\beta_m)\delta_j}{(\beta_{\bar m}+\beta_m)}-\frac{\sin(\beta_{\bar m}+\beta_m)\theta_j}{(\beta_{\bar m}+\beta_m)}\right)+\frac{2\eta}{\sin\alpha_1}\sum_{m=1}^{\infty}\beta_m A_m\left(e^{\delta_j}\cos^2\beta_m\delta_j\right.$$

$$\left.-e^{\theta_j}\cos^2\beta_m\theta_j+\frac{\beta_m e^{\delta_j}(\sin 2\beta_m\delta_j-2\beta_m\cos 2\beta_m\delta_j)}{(1+4\beta_m^2)}-\frac{\beta_m e^{\theta_j}(\sin 2\beta_m\theta_j-2\beta_m\cos 2\beta_m\theta_j)}{(1+4\beta_m^2)}\right)$$

$$+\frac{\eta}{\sin\alpha_1}\sum_{\substack{m=1\\ m\neq\bar m}}^{\infty}\sum_{\bar m=1}^{\infty}\beta_{\bar m}A_{\bar m}\left(\frac{e^{\delta_j}\sin(\beta_{\bar m}-\beta_m)\delta_j}{(\beta_{\bar m}-\beta_m)}-\frac{e^{\theta_j}\sin(\beta_{\bar m}-\beta_m)\theta_j}{(\beta_{\bar m}-\beta_m)}+\frac{e^{\delta_j}\sin(\beta_{\bar m}+\beta_m)\delta_j}{(\beta_{\bar m}+\beta_m)}-\frac{e^{\theta_j}\sin(\beta_{\bar m}+\beta_m)\theta_j}{(\beta_{\bar m}+\beta_m)}\right.$$

$$+\frac{\{e^{\delta_j}\}\{\sin(\beta_{\bar m}-\beta_m)\delta_j-(\beta_{\bar m}-\beta_m)\cos(\beta_{\bar m}-\beta_m)\delta_j\}}{(\beta_{\bar m}-\beta_m)[1+(\beta_{\bar m}-\beta_m)^2]}-\frac{\{e^{\theta_j}\}\{\sin(\beta_{\bar m}-\beta_m)\theta_j-(\beta_{\bar m}-\beta_m)\cos(\beta_{\bar m}-\beta_m)\theta_j\}}{(\beta_{\bar m}-\beta_m)[1+(\beta_{\bar m}-\beta_m)^2]}$$

$$\left.+\frac{\{e^{\delta_j}\}\{\sin(\beta_{\bar m}+\beta_m)-(\beta_{\bar m}+\beta_m)\cos(\beta_{\bar m}+\beta_m)\delta_j\}}{(\beta_{\bar m}+\beta_m)[1+(\beta_{\bar m}+\beta_m)^2]}-\frac{\{e^{\theta_j}\}\{\sin(\beta_{\bar m}+\beta_m)\theta_j-(\beta_{\bar m}+\beta_m)\cos(\beta_{\bar m}+\beta_m)\theta_j\}}{(\beta_{\bar m}+\beta_m)[1+(\beta_{\bar m}+\beta_m)^2]}\right)$$

$$+\frac{\eta}{\sin^2\alpha_1}\sum_{m=1}^{\infty}D_m\left(e^{2\delta_j}\cos^2\beta_m\delta_j-e^{2\theta_j}\cos^2\beta_m\theta_j+\frac{\beta_m e^{2\delta_j}(\sin 2\beta_m\delta_j-\beta_m\cos 2\beta_m\delta_j)}{2(1+\beta_m^2)}-\frac{\beta_m e^{2\theta_j}(\sin 2\beta_m\theta_j-\beta_m\cos 2\beta_m\theta_j)}{2(1+\beta_m^2)}\right)$$

$$+\frac{\eta}{\sin^2\alpha_1}\sum_{\substack{m=1\\ m\neq\bar m}}^{\infty}\sum_{\bar m=1}^{\infty}D_{\bar m}\left(\frac{e^{2\delta_j}\sin(\beta_{\bar m}-\beta_m)\delta_j}{(\beta_{\bar m}-\beta_m)}-\frac{e^{2\theta_j}\sin(\beta_{\bar m}-\beta_m)\theta_j}{(\beta_{\bar m}-\beta_m)}+\frac{e^{2\delta_j}\sin(\beta_{\bar m}+\beta_m)\delta_j}{(\beta_{\bar m}+\beta_m)}-\frac{e^{2\theta_j}\sin(\beta_{\bar m}+\beta_m)\theta_j}{(\beta_{\bar m}+\beta_m)}\right.$$

$$-\frac{2\{e^{2\delta_j}\}\{2\sin(\beta_{\bar m}-\beta_m)\delta_j-(\beta_{\bar m}-\beta_m)\cos(\beta_{\bar m}-\beta_m)\delta_j\}}{(\beta_{\bar m}-\beta_m)[4+(\beta_{\bar m}-\beta_m)^2]}+\frac{2\{e^{2\theta_j}\}\{2\sin(\beta_{\bar m}-\beta_m)\theta_j-(\beta_{\bar m}-\beta_m)\cos(\beta_{\bar m}-\beta_m)\theta_j\}}{(\beta_{\bar m}-\beta_m)[4+(\beta_{\bar m}-\beta_m)^2]}$$

$$\left.\left.\left.-\frac{2\{e^{2\delta_j}\}\{2\sin(\beta_{\bar m}+\beta_m)\delta_j-(\beta_{\bar m}+\beta_m)\cos(\beta_{\bar m}+\beta_m)\delta_j\}}{(\beta_{\bar m}+\beta_m)[4+(\beta_{\bar m}+\beta_m)^2]}+\frac{2\{e^{2\theta_j}\}\{2\sin(\beta_{\bar m}+\beta_m)\theta_j-(\beta_{\bar m}+\beta_m)\cos(\beta_{\bar m}+\beta_m)\theta_j\}}{(\beta_m+\beta_m)[4+(\beta_{\bar m}+\beta_m)^2]}\right)\right]\right\}_j$$

3.5.5

$$\frac{\partial \overline{T}_{max}}{\partial A_m} = \sum_{j=1}^{n} \Bigg\{ \Omega^2 (s_{01})^2 (m^c + 2m^f) \sin\alpha_j \Bigg[\sum_{m=1}^{\infty} A_m \Bigg(e^{2\delta_j} \sin^2\beta_m\delta_j - e^{2\theta_j} \sin^2\beta_m\theta_j$$

$$- \frac{\beta_m e^{2\delta_j} (\sin 2\beta_m\delta_j - \beta_m \cos 2\beta_m\delta_j)}{2(1+\beta_m^2)} + \frac{\beta_m e^{2\theta_j} (\sin 2\beta_m\theta_j - \beta_m \cos 2\beta_m\theta_j)}{2(1+\beta_m^2)} \Bigg)$$

$$+ \sum_{\substack{m=1 \\ m\neq\overline{m}}}^{\infty} \sum_{\overline{m}=1}^{\infty} A_{\overline{m}} \Bigg(\frac{e^{2\delta_j} \sin(\beta_{\overline{m}} - \beta_m)\delta_j}{(\beta_{\overline{m}} - \beta_m)} - \frac{e^{2\theta_j} \sin(\beta_{\overline{m}} - \beta_m)\theta_j}{(\beta_{\overline{m}} - \beta_m)} - \frac{e^{2\delta_j} \sin(\beta_{\overline{m}} + \beta_m)\delta_j}{(\beta_{\overline{m}} + \beta_m)} + \frac{e^{2\theta_j} \sin(\beta_{\overline{m}} + \beta_m)\theta_j}{(\beta_{\overline{m}} + \beta_m)}$$

$$- \frac{2\{e^{2\delta_j}\}\{2\sin(\beta_{\overline{m}} - \beta_m)\delta_j - (\beta_{\overline{m}} - \beta_m)\cos(\beta_{\overline{m}} - \beta_m)\delta_j\}}{(\beta_{\overline{m}} - \beta_m)[4 + (\beta_{\overline{m}} - \beta_m)^2]} + \frac{2\{e^{2\theta_j}\}\{2\sin(\beta_{\overline{m}} - \beta_m)\theta_j - (\beta_{\overline{m}} - \beta_m)\cos(\beta_{\overline{m}} - \beta_m)\theta_j\}}{(\beta_{\overline{m}} - \beta_m)[4 + (\beta_{\overline{m}} - \beta_m)^2]}$$

$$+ \frac{2\{e^{2\delta_j}\}\{2\sin(\beta_{\overline{m}} + \beta_m)\delta_j - (\beta_{\overline{m}} + \beta_m)\cos(\beta_{\overline{m}} + \beta_m)\delta_j\}}{(\beta_{\overline{m}} + \beta_m)[4 + (\beta_{\overline{m}} + \beta_m)^2]} - \frac{2\{e^{2\theta_j}\}\{2\sin(\beta_{\overline{m}} + \beta_m)\theta_j - (\beta_{\overline{m}} + \beta_m)\cos(\beta_{\overline{m}} + \beta_m)\theta_j\}}{(\beta_{\overline{m}} + \beta_m)[4 + (\beta_{\overline{m}} + \beta_m)^2]} \Bigg) \Bigg] \Bigg\}_j$$

3.5.6

$$\frac{\partial \overline{T}_{max}}{\partial B_m} = \sum_{j=1}^{n} \Bigg\{ \Omega^2 (s_{01})^2 (m^c + 2m^f) \sin\alpha_j \Bigg[\sum_{m=1}^{\infty} B_m \Bigg(e^{2\delta_j} \cos^2\beta_m\delta_j - e^{2\theta_j} \cos^2\beta_m\theta_j$$

$$+ \frac{\beta_m e^{2\delta_j} (\sin 2\beta_m\delta_j - \beta_m \cos 2\beta_m\delta_j)}{2(1+\beta_m^2)} - \frac{\beta_m e^{2\theta_j} (\sin 2\beta_m\theta_j - \beta_m \cos 2\beta_m\theta_j)}{2(1+\beta_m^2)} \Bigg)$$

$$+ \sum_{\substack{m=1 \\ m\neq\overline{m}}}^{\infty} \sum_{\overline{m}=1}^{\infty} B_{\overline{m}} \Bigg(\frac{e^{2\delta_j} \sin(\beta_{\overline{m}} - \beta_m)\delta_j}{(\beta_{\overline{m}} - \beta_m)} - \frac{e^{2\theta_j} \sin(\beta_{\overline{m}} - \beta_m)\theta_j}{(\beta_{\overline{m}} - \beta_m)} + \frac{e^{2\delta_j} \sin(\beta_{\overline{m}} + \beta_m)\delta_j}{(\beta_{\overline{m}} + \beta_m)} - \frac{e^{2\theta_j} \sin(\beta_{\overline{m}} + \beta_m)\theta_j}{(\beta_{\overline{m}} + \beta_m)}$$

$$- \frac{2\{e^{2\delta_j}\}\{2\sin(\beta_{\overline{m}} - \beta_m)\delta_j - (\beta_{\overline{m}} - \beta_m)\cos(\beta_{\overline{m}} - \beta_m)\delta_j\}}{(\beta_{\overline{m}} - \beta_m)[4 + (\beta_{\overline{m}} - \beta_m)^2]} + \frac{2\{e^{2\theta_j}\}\{2\sin(\beta_{\overline{m}} - \beta_m)\theta_j - (\beta_{\overline{m}} - \beta_m)\cos(\beta_{\overline{m}} - \beta_m)\theta_j\}}{(\beta_{\overline{m}} - \beta_m)[4 + (\beta_{\overline{m}} - \beta_m)^2]}$$

$$- \frac{2\{e^{2\delta_j}\}\{2\sin(\beta_{\overline{m}} + \beta_m)\delta_j - (\beta_{\overline{m}} + \beta_m)\cos(\beta_{\overline{m}} + \beta_m)\delta_j\}}{(\beta_{\overline{m}} + \beta_m)[4 + (\beta_{\overline{m}} + \beta_m)^2]} + \frac{2\{e^{2\theta_j}\}\{2\sin(\beta_{\overline{m}} + \beta_m)\theta_j - (\beta_{\overline{m}} + \beta_m)\cos(\beta_{\overline{m}} + \beta_m)\theta_j\}}{(\beta_{\overline{m}} + \beta_m)[4 + (\beta_{\overline{m}} + \beta_m)^2]} \Bigg) \Bigg] \Bigg\}_j$$

3.5.7

$$\frac{\partial \bar{T}_{max}}{\partial D_m} = \sum_{j=1}^{n} \left\{ \frac{\Omega^2 (S_{o1})^2 (I^c + 2I^f) \sin\alpha_1}{(S_{o1})^2 \sin^2\alpha_1} \left[\sum_{m=1}^{\infty} D_m \left(e^{2\delta_j} \cos^2 \beta_m \delta_j - e^{2\Theta_j} \cos^2 \beta_m \Theta_j \right.\right.\right.$$

$$+ \frac{\beta_m e^{2\delta_j} (\sin 2\beta_m \delta_j - \beta_m \cos 2\beta_m \delta_j)}{2(1+\beta_m^2)} - \frac{\beta_m e^{2\Theta_j} (\sin 2\beta_m \Theta_j - \beta_m \cos 2\beta_m \Theta_j)}{2(1+\beta_m^2)} \Bigg)$$

$$+ \sum_{m=1}^{\infty} \sum_{\substack{\bar{m}=1 \\ m \neq \bar{m}}}^{\infty} D_{\bar{m}} \left(\frac{e^{2\delta_j} \sin(\beta_{\bar{m}} - \beta_m)\delta_j}{(\beta_{\bar{m}} - \beta_m)} - \frac{e^{2\Theta_j} \sin(\beta_{\bar{m}} - \beta_m)\Theta_j}{(\beta_{\bar{m}} - \beta_m)} + \frac{e^{2\delta_j} \sin(\beta_{\bar{m}} + \beta_m)\delta_j}{(\beta_{\bar{m}} + \beta_m)} \right.$$

$$- \frac{e^{2\Theta_j} \sin(\beta_{\bar{m}} + \beta_m)\Theta_j}{(\beta_{\bar{m}} + \beta_m)} - \frac{2\{e^{2\delta_j}\}\{2\sin(\beta_{\bar{m}} - \beta_m)\delta_j - (\beta_{\bar{m}} - \beta_m)\cos(\beta_{\bar{m}} - \beta_m)\delta_j\}}{(\beta_{\bar{m}} - \beta_m)[4 + (\beta_{\bar{m}} - \beta_m)^2]}$$

$$+ \frac{2\{e^{2\Theta_j}\}\{2\sin(\beta_{\bar{m}} - \beta_m)\Theta_j - (\beta_{\bar{m}} - \beta_m)\cos(\beta_{\bar{m}} - \beta_m)\Theta_j\}}{(\beta_{\bar{m}} - \beta_m)[4 + (\beta_{\bar{m}} - \beta_m)^2]} - \frac{2\{e^{2\delta_j}\}\{2\sin(\beta_{\bar{m}} + \beta_m)\delta_j - (\beta_{\bar{m}} + \beta_m)\cos(\beta_{\bar{m}} + \beta_m)\delta_j\}}{(\beta_{\bar{m}} + \beta_m)[4 + (\beta_{\bar{m}} + \beta_m)^2]}$$

$$\left.\left.\left. + \frac{2\{e^{2\Theta_j}\}\{2\sin(\beta_{\bar{m}} + \beta_m)\Theta_j - (\beta_{\bar{m}} + \beta_m)\cos(\beta_{\bar{m}} + \beta_m)\Theta_j\}}{(\beta_{\bar{m}} + \beta_m)[4 + (\beta_{\bar{m}} + \beta_m)^2]} \right) \right] \right\}_j \qquad 3.5.8$$

Since the lower axisymmetric natural frequencies of vibrations are sought, only three-term series solution were considered. For higher frequencies more terms should be taken to obtain accurate results. Thus, for m = 1, 2, and 3, the following matrices were obtained:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} & a_{19} \\ & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} & a_{28} & a_{29} \\ & & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} & a_{38} & a_{39} \\ & & & a_{44} & a_{45} & a_{46} & a_{47} & a_{48} & a_{49} \\ & & & & a_{55} & a_{56} & a_{57} & a_{58} & a_{59} \\ & & & & & a_{66} & a_{67} & a_{68} & a_{69} \\ & & & & & & a_{77} & a_{78} & a_{79} \\ & & & & & & & a_{88} & a_{89} \\ & & & & & & & & a_{99} \end{bmatrix} \qquad \begin{array}{c} 3.5.10 \\ \text{(Stiffness matrix)} \end{array}$$

where

$$a_{ij} = a_{ji}$$

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\ & b_{22} & b_{23} & 0 & 0 & 0 & 0 & 0 & 0 \\ & & b_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & b_{44} & b_{45} & b_{46} & 0 & 0 & 0 \\ & & & & b_{55} & b_{56} & 0 & 0 & 0 \\ & & & & & b_{66} & 0 & 0 & 0 \\ & & & & & & b_{77} & b_{78} & b_{79} \\ & & & & & & & b_{88} & b_{89} \\ & & & & & & & & b_{99} \end{bmatrix}$$

3.5.11
(Inertia matrix)

$(b_{ij} = b_{ji})$

$$V = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ B_1 \\ B_2 \\ B_3 \\ D_1 \\ D_2 \\ D_3 \end{bmatrix}$$ (Displacement column matrix) 3.5.12

From Equations 3.5.3, 3.5.4 and 3.5.5 the elements, a_{ij}, of matrix Equation 3.5.10 are obtained. Those elements are as follows:

$$a_{11} = \sum_{j=1}^{n} \left\{ \left(\bar{\eta}_1 \cot^2\alpha_j + \bar{\eta}_2 \beta_i^2\right)\left(\delta_j - \theta_j\right) + \left(\bar{\eta}_1 \cot^2\alpha_j - \bar{\eta}_2 \beta_i^2\right)\left(-\sin 2\beta_i\delta_j + \sin 2\beta_i\theta_j\right)/2\beta_i \right\} .$$

$$a_{12} = \sum_{j=1}^{n} \left\{ \left[\left(\bar{\eta}_1 \cot^2\alpha_j + 2\bar{\eta}_2 \beta_i^2\right)\left(\sin\beta_i\delta_j - \sin\beta_i\theta_j\right) + \frac{1}{3}\left(\bar{\eta}_1 \cot^2\alpha_j - 2\bar{\eta}_2 \beta_i^2\right)\left(-\sin 3\beta_i\delta_j + \sin 3\beta_i\theta_j\right)\right]/\beta_i \right\} .$$

$$a_{13} = \sum_{j=1}^{n} \left\{ \left[\left(\bar{\eta}_1 \cot^2\alpha_j + 3\bar{\eta}_2 \beta_i^2\right)\left(\sin 2\beta_i\delta_j - \sin 2\beta_i\theta_j\right) + 0.5\left(\bar{\eta}_1 \cot^2\alpha_j - 3\bar{\eta}_2 \beta_i^2\right)\left(-\sin 4\beta_i\delta_j + \sin 4\beta_i\theta_j\right)\right]/2\beta_i \right\} .$$

$$a_{14} = \sum_{j=1}^{n} \left\{ \left[-0.5\,\bar{\eta}_3 \beta_i \left(\delta_j - \theta_j - \{\sin 2\beta_i\delta_j - \sin 2\beta_i\theta_j\}/2\beta_i\right) - \bar{\eta}_1 \left(\cos 2\beta_i\delta_j - \cos 2\beta_i\theta_j\right)/\beta_i\right]/\tan\alpha_j \right\} .$$

$$a_{15} = \sum_{j=1}^{n} \left\{ \left[-\bar{\eta}_3 \left(\sin\beta_i\delta_j - \sin\beta_i\theta_j - \{\sin 3\beta_i\delta_j - \sin 3\beta_i\theta_j\}/3\right) - \bar{\eta}_1 \left(-\cos\beta_i\delta_j + \cos\beta_i\theta_j \right.\right.\right.$$

$$\left.\left.\left. + \{\cos 3\beta_i\delta_j - \cos 3\beta_i\theta_j\}/3\right)/\beta_i\right]/\tan\alpha_j \right\} .$$

$$a_{16} = \sum_{j=1}^{n} \left\{ \left[-0.75\,\bar{\eta}_3 \left(\sin 2\beta_i\delta_j - \sin 2\beta_i\theta_j - 0.5\{\sin 4\beta_i\delta_j - \sin 4\beta_i\theta_j\}\right) - 0.5\,\bar{\eta}_1 \left(-\cos 2\beta_i\delta_j + \cos 2\beta_i\theta_j \right.\right.\right.$$

$$\left.\left.\left. + 0.5\{\cos 4\beta_i\delta_j - \cos 4\beta_i\theta_j\}\right)/\beta_i\right]/\tan\alpha_j \right\} .$$

$$a_{17} = \sum_{j=1}^{n} \left\{ \frac{\beta_1}{\sin\alpha_1} \left[-\bar{\eta}_4 \left(-e^{-\delta_j} \sin^2\beta_1\delta_j + e^{-\theta_j} \sin^2\beta_1\theta_j - \frac{\beta_1}{1+4\beta_1^2} \{ e^{-\delta_j}(\sin 2\beta_1\delta_j + 2\beta_1 \cos 2\beta_1\delta_j) \right.\right.\right.$$

$$\left. - e^{-\theta_j}(\sin 2\beta_1\theta_j + 2\beta_1 \cos 2\beta_1\theta_j) \} \right) \Big/ \tan^2\alpha_j + 2\bar{\eta}_2 \left(e^{\delta_j} \cos^2\beta_1\delta_j - e^{\theta_j} \cos^2\beta_1\theta_j \right.$$

$$\left.\left.\left. + \frac{\beta_1}{1+4\beta_1^2} \{ e^{\delta_j}(\sin 2\beta_1\delta_j - 2\beta_1 \cos 2\beta_1\delta_j) - e^{\theta_j}(\sin 2\beta_1\theta_j - 2\beta_1 \cos 2\beta_1\theta_j) \} \right) \right] \right\} .$$

$$a_{18} = \sum_{j=1}^{n} \left\{ \frac{1}{\sin\alpha_j} \left[- \frac{\bar{\eta}_4}{\tan^2\alpha_j} \left(e^{-\delta_j}(\sin\beta_1\delta_j - \tfrac{1}{3}\sin 3\beta_1\delta_j) - e^{-\theta_j}(\sin\beta_1\theta_j - \tfrac{1}{3}\sin 3\beta_1\theta_j) \right.\right.\right.$$

$$- \frac{1}{1+\beta_1^2} \{ e^{-\delta_j}(\sin\beta_1\delta_j + \beta_1\cos\beta_1\delta_j) - e^{-\theta_j}(\sin\beta_1\theta_j + \beta_1\cos\beta_1\theta_j) \}$$

$$\left. + \frac{1}{3(1+9\beta_1^2)} \{ e^{-\delta_j}(\sin 3\beta_1\delta_j + 3\beta_1\cos 3\beta_1\delta_j) - e^{-\theta_j}(\sin 3\beta_1\theta_j + 3\beta_1\cos 3\beta_1\theta_j) \} \right)$$

$$+ \bar{\eta}_2 \left(e^{\delta_j}(\sin\beta_1\delta_j + \tfrac{1}{3}\sin 3\beta_1\delta_j) - e^{\theta_j}(\sin\beta_1\theta_j + \tfrac{1}{3}\sin 3\beta_1\theta_j) + \frac{1}{1+\beta_1^2} \{ e^{\delta_j}(\sin\beta_1\delta_j - \beta_1\cos\beta_1\delta_j) \right.$$

$$\left.\left.\left. - e^{\theta_j}(\sin\beta_1\theta_j - \beta_1\cos\beta_1\theta_j) \} + \frac{1}{3(1+9\beta_1^2)} \{ e^{\delta_j}(\sin 3\beta_1\delta_j - 3\beta_1\cos 3\beta_1\delta_j) - e^{\theta_j}(\sin 3\beta_1\theta_j - 3\beta_1\cos 3\beta_1\theta_j) \} \right) \right] \right\} .$$

$$a_{19} = \sum_{j=1}^{n} \Bigg\{ \frac{1}{\sin\alpha_j} \Bigg[- \frac{0.75\,\bar{\eta}_4}{\tan^2\alpha_j} \Bigg(e^{-\delta_j} (\sin 2\beta_i\delta_j - 0.5 \sin 4\beta_i\delta_j) - e^{-\theta_j} (\sin 2\beta_i\theta_j - 0.5 \sin 4\beta_i\theta_j)$$

$$- \frac{1}{1+4\beta_i^2} \{ e^{-\delta_j} (\sin 2\beta_i\delta_j + 2\beta_i \cos 2\beta_i\delta_j) - e^{-\theta_j} (\sin 2\beta_i\theta_j + 2\beta_i \cos 2\beta_i\theta_j) \}$$

$$+ \frac{1}{2(1+16\beta_i^2)} \{ e^{-\delta_j} (\sin 4\beta_i\delta_j + 4\beta_i \cos 4\beta_i\delta_j) - e^{-\theta_j} (\sin 4\beta_i\theta_j + 4\beta_i \cos 4\beta_i\theta_j) \} \Bigg)$$

$$+ 0.5\,\bar{\eta}_2 \Bigg(e^{\delta_j} (\sin 2\beta_i\delta_j + 0.5 \sin 4\beta_i\delta_j) - e^{\theta_j} (\sin 2\beta_i\theta_j + 0.5 \sin 4\beta_i\theta_j)$$

$$+ \frac{1}{1+4\beta_i^2} \{ e^{\delta_j} (\sin 2\beta_i\delta_j - 2\beta_i \cos 2\beta_i\delta_j) - e^{\theta_j} (\sin 2\beta_i\theta_j - 2\beta_i \cos 2\beta_i\theta_j) \}$$

$$+ \frac{1}{2(1+16\beta_i^2)} \{ e^{\delta_j} (\sin 4\beta_i\delta_j - 4\beta_i \cos 4\beta_i\delta_j) - e^{\theta_j} (\sin 4\beta_i\theta_j - 4\beta_i \cos 4\beta_i\theta_j) \} \Bigg) \Bigg] \Bigg\} .$$

$$a_{22} = \sum_{j=1}^{n} \Bigg\{ \left(\bar{\eta}_1 \cot^2\alpha_j + 4\bar{\eta}_2 \beta_i^2 \right) \left(\delta_j - \theta_j \right) + \frac{1}{4\beta_i} \left(\bar{\eta}_1 \cot^2\alpha_j - 4\bar{\eta}_2 \beta_i^2 \right) \left(-\sin 4\beta_i\delta_j + \sin 4\beta_i\theta_j \right) \Bigg\} .$$

$$a_{23} = \sum_{j=1}^{n} \Bigg\{ \frac{1}{\beta_i} \Bigg[\left(\bar{\eta}_1 \cot^2\alpha_j + 6\bar{\eta}_2 \beta_i^2 \right) \left(\sin\beta_i\delta_j - \sin\beta_i\theta_j \right) + 0.2 \left(\bar{\eta}_1 \cot^2\alpha_j - 6\bar{\eta}_2 \beta_i^2 \right) \left(-\sin 5\beta_i\delta_j + \sin 5\beta_i\theta_j \right) \Bigg] \Bigg\} .$$

$$a_{24} = \sum_{j=1}^{n} \Bigg\{ \frac{1}{\tan\alpha_j} \Bigg[-0.5\,\bar{\eta}_3 \left(\sin\beta_i\delta_j - \sin\beta_i\theta_j - \tfrac{1}{3} (\sin 3\beta_i\delta_j - \sin 3\beta_i\theta_j) \right) - \frac{\bar{\eta}_1}{\beta_i} \Bigg(\cos\beta_i\delta_j - \cos\beta_i\theta_j$$

$$+ \tfrac{1}{3} (\cos 3\beta_i\delta_j - \cos 3\beta_i\theta_j) \Bigg) \Bigg] \Bigg\} .$$

$$a_{25} = \sum_{j=1}^{n} \left\{ \frac{1}{\tan\alpha_j} \left[-\bar{\eta}_3 \beta_i \left(\delta_j - \theta_j - \frac{1}{4\beta_i}(\sin 4\beta_i\delta_j - \sin 4\beta_i\theta_j) \right) - \frac{\bar{\eta}_1}{2\beta_i} \left(\cos 4\beta_i\delta_j - \cos 4\beta_i\theta_j \right) \right] \right\} .$$

$$a_{26} = \sum_{j=1}^{n} \frac{1}{\tan\alpha_j} \left[-1.5\bar{\eta}_3 \left(\sin\beta_i\delta_j - \sin\beta_i\theta_j - 0.2(\sin 5\beta_i\delta_j - \sin 5\beta_i\theta_j) \right) - \frac{\bar{\eta}_1}{\beta_i} \left(-\cos\beta_i\delta_j + \cos\beta_i\theta_j \right.\right.$$

$$\left.\left. + 0.2(\cos 5\beta_i\delta_j - \cos 5\beta_i\theta_j) \right) \right] \Bigg\} .$$

$$a_{27} = \sum_{j=1}^{n} \left\{ \frac{1}{\sin\alpha_1} \left[-\frac{0.5\,\bar{\eta}_4}{\tan^2\alpha_j} \left(e^{-\delta_j}(\sin\beta_i\delta_j - \tfrac{1}{3}\sin 3\beta_i\delta_j) - e^{-\theta_j}(\sin\beta_i\theta_j - \tfrac{1}{3}\sin 3\beta_i\theta_j) \right.\right.\right.$$

$$- \frac{1}{1+\beta_i^2} \{ e^{-\delta_j}(\sin\beta_i\delta_j + \beta_i\cos\beta_i\delta_j) - e^{-\theta_j}(\sin\beta_i\theta_j + \beta_i\cos\beta_i\theta_j) \}$$

$$\left. + \frac{1}{3(1+9\beta_i^2)} \{ e^{-\delta_j}(\sin 3\beta_i\delta_j + 3\beta_i\cos 3\beta_i\delta_j) - e^{-\theta_j}(\sin 3\beta_i\theta_j + 3\beta_i\cos 3\beta_i\theta_j) \} \right)$$

$$+ 2\bar{\eta}_2 \left(e^{\delta_j}(\sin\beta_i\delta_j + \tfrac{1}{3}\sin 3\beta_i\delta_j) - e^{\theta_j}(\sin\beta_i\theta_j + \tfrac{1}{3}\sin 3\beta_i\theta_j) \right.$$

$$+ \frac{1}{1+\beta_i^2} \{ e^{\delta_j}(\sin\beta_i\delta_j - \beta_i\cos\beta_i\delta_j) - e^{\theta_j}(\sin\beta_i\theta_j - \beta_i\cos\beta_i\theta_j) \}$$

$$\left.\left.\left. + \frac{1}{3(1+9\beta_i^2)} \{ e^{\delta_j}(\sin 3\beta_i\delta_j - 3\beta_i\cos 3\beta_i\delta_j) - e^{\theta_j}(\sin 3\beta_i\theta_j - 3\beta_i\cos 3\beta_i\theta_j) \} \right) \right] \right\} .$$

$$a_{28} = \sum_{j=1}^{n} \left\{ \frac{2\beta_i}{\sin\alpha_1} \left[- \frac{\bar{\eta}_4}{\tan^2\alpha_j} \left(- e^{-\delta_j} \sin^2 2\beta_i\delta_j + e^{-\theta_j} \sin^2 2\beta_i\theta_j - \frac{2\beta_i}{1+16\beta_i^2} \{ e^{-\delta_j} (\sin 4\beta_i\delta_j + 4\beta_i \cos 4\beta_i\delta_j) \right.\right.\right.$$

$$\left. - e^{-\theta_j} (\sin 4\beta_i\theta_j + 4\beta_i \cos 4\beta_i\theta_j) \} \right) + 2\bar{\eta}_2 \left(e^{\delta_j} \cos^2 2\beta_i\delta_j - e^{\theta_j} \cos^2 2\beta_i\theta_j + \frac{2\beta_i}{1+16\beta_i^2} \{ e^{\delta_j} (\sin 4\beta_i\delta_j - 4\beta_i \cos 4\beta_i\delta_j) \right.$$

$$\left.\left.\left. - e^{\theta_j} (\sin 4\beta_i\theta_j - 4\beta_i \cos 4\beta_i\theta_j) \} \right) \right] \right\} .$$

$$a_{29} = \sum_{j=1}^{\infty} \left\{ \frac{1}{\sin\alpha_1} \left[- \frac{1.5\,\bar{\eta}_4}{\tan^2\alpha_j} \left(e^{-\delta_j} (\sin\beta_i\delta_j - 0.2 \sin 5\beta_i\delta_j) - e^{-\theta_j} (\sin\beta_i\theta_j - 0.2 \sin 5\beta_i\theta_j) \right.\right.\right.$$

$$- \frac{1}{1+\beta_i^2} \{ e^{-\delta_j} (\sin\beta_i\delta_j + \beta_i \cos\beta_i\delta_j) - e^{-\theta_j} (\sin\beta_i\theta_j + \beta_i \cos\beta_i\theta_j) \}$$

$$\left. + \frac{1}{5(1+25\beta_i^2)} \{ e^{-\delta_j} (\sin 5\beta_i\delta_j + 5\beta_i \cos 5\beta_i\delta_j) - e^{-\theta_j} (\sin 5\beta_i\theta_j + 5\beta_i \cos 5\beta_i\theta_j) \} \right)$$

$$+ 2\bar{\eta}_2 \left(e^{\delta_j} (\sin\beta_i\delta_j + .2 \sin 5\beta_i\delta_j) - e^{\theta_j} (\sin\beta_i\theta_j + .2 \sin 5\beta_i\theta_j) + \frac{1}{1+\beta_i^2} \{ e^{\delta_j} (\sin\beta_i\delta_j - \beta_i \cos\beta_i\delta_j) \right.$$

$$\left.\left.\left. - e^{\theta_j} (\sin\beta_i\theta_j - \beta_i \cos\beta_i\theta_j) \} + \frac{1}{5(1+25\beta_i^2)} \{ e^{\delta_j} (\sin 5\beta_i\delta_j - 5\beta_i \cos 5\beta_i\delta_j) - e^{\theta_j} (\sin 5\beta_i\theta_j - 5\beta_i \cos 5\beta_i\theta_j) \} \right) \right] \right\} .$$

$$a_{33} = \sum_{j=1}^{\infty} \left\{ \left(\bar{\eta}_1 \cot^2\alpha_j + 9\bar{\eta}_2 \beta_i^2 \right) \left(\delta_j - \theta_j \right) + \frac{1}{6\beta_i} \left(\bar{\eta}_1 \cot^2\alpha_j - 9\bar{\eta}_2 \beta_i^2 \right) \left(-\sin 6\beta_i\delta_j + \sin 6\beta_i\theta_j \right) \right\} .$$

$$a_{34} = \sum_{j=1}^{n} \left\{ \frac{1}{2\tan\alpha_j} \left[-0.5\,\bar{\eta}_3 \left(\sin 2\beta_1\delta_j - \sin 2\beta_1\theta_j - 0.5(\sin 4\beta_1\delta_j - \sin 4\beta_1\theta_j) \right) \right.\right.$$

$$\left.\left. - \frac{\bar{\eta}_1}{\beta_1} \left(\cos 2\beta_1\delta_j - \cos 2\beta_1\theta_j + 0.5(\cos 4\beta_1\delta_j - \cos 4\beta_1\theta_j) \right) \right] \right\} .$$

$$a_{35} = \sum_{j=1}^{n} \left\{ \frac{1}{\tan\alpha_j} \left[-\bar{\eta}_3 \left(\sin\beta_1\delta_j - \sin\beta_1\theta_j - 0.2(\sin 5\beta_1\delta_j - \sin 5\beta_1\theta_j) \right) \right.\right.$$

$$\left.\left. - \frac{\bar{\eta}_1}{\beta_1} \left(\cos 2\beta_1\delta_j - \cos 2\beta_1\theta_j + 0.5(\cos 4\beta_1\delta_j - \cos 4\beta_1\theta_j) \right) \right] \right\} .$$

$$a_{36} = \sum_{j=1}^{n} \left\{ \frac{1}{\tan\alpha_j} \left[-1.5\,\bar{\eta}_3\beta_1 \left(\delta_j - \theta_j - \frac{1}{6\beta_1}(\sin 6\beta_1\delta_j - \sin 6\beta_1\theta_j) \right) - \frac{\bar{\eta}_1}{3\beta_1} \left(\cos 6\beta_1\delta_j - \cos 6\beta_1\theta_j \right) \right] \right\} .$$

$$a_{37} = \sum_{j=1}^{n} \left\{ \frac{1}{\sin\alpha_1} \left[-\frac{0.25\,\bar{\eta}_4}{\tan^2\alpha_j} \left(e^{-\delta_j}(\sin 2\beta_1\delta_j - 0.5\sin 4\beta_1\delta_j) - e^{-\theta_j}(\sin 2\beta_1\theta_j - 0.5\sin 4\beta_1\theta_j) \right.\right.\right.$$

$$- \frac{1}{1+4\beta_1^2} \{ e^{-\delta_j}(\sin 2\beta_1\delta_j + 2\beta_1\cos 2\beta_1\delta_j) - e^{-\theta_j}(\sin 2\beta_1\theta_j + 2\beta_1\cos 2\beta_1\theta_j) \}$$

$$\left. + \frac{1}{2(1+16\beta_1^2)} \{ e^{-\delta_j}(\sin 4\beta_1\delta_j + 4\beta_1\cos 4\beta_1\delta_j) - e^{-\theta_j}(\sin 4\beta_1\theta_j + 4\beta_1\cos 4\beta_1\theta_j) \} \right)$$

$$+ 1.5\,\bar{\eta}_2 \left(e^{\delta_j}(\sin 2\beta_1\delta_j + 0.5\sin 4\beta_1\delta_j) - e^{\theta_j}(\sin 2\beta_1\theta_j + 0.5\sin 4\beta_1\theta_j) + \frac{1}{1+4\beta_1^2} \{ e^{\delta_j}(\sin 2\beta_1\delta_j - 2\beta_1\cos 2\beta_1\delta_j) \right.$$

$$- e^{\theta_j}(\sin 2\beta_1\theta_j - 2\beta_1\cos 2\beta_1\theta_j) \} + \frac{1}{2(1+16\beta_1^2)} \{ e^{\delta_j}(\sin 4\beta_1\delta_j - 4\beta_1\cos 4\beta_1\delta_j)$$

$$\left.\left.\left. - e^{\theta_j}(\sin 4\beta_1\theta_j - 4\beta_1\cos 4\beta_1\theta_j) \} \right) \right] \right\} .$$

$$a_{38} = \sum_{j=1}^{n} \left\{ \frac{1}{\sin\alpha_1} \left[-\frac{\bar{\eta}_4}{\tan^2\alpha_j} \left(e^{-\delta_j}(\sin\beta_1\delta_j - 0.2\sin 5\beta_1\delta_j) - e^{-\theta_j}(\sin\beta_1\theta_j - 0.2\sin 5\beta_1\theta_j) \right.\right.\right.$$

$$- \frac{1}{(1+\beta_1^2)} \{ e^{-\delta_j}(\sin\beta_1\delta_j + \beta_1\cos\beta_1\delta_j) - e^{-\theta_j}(\sin\beta_1\theta_j + \beta_1\cos\beta_1\theta_j) \}$$

$$\left. + \frac{1}{5(1+25\beta_1^2)} \{ e^{-\delta_j}(\sin 5\beta_1\delta_j + 5\beta_1\cos 5\beta_1\delta_j) - e^{-\theta_j}(\sin 5\beta_1\theta_j + 5\beta_1\cos 5\beta_1\theta_j) \} \right)$$

$$+ 3\bar{\eta}_2 \left(e^{\delta_j}(\sin\beta_1\delta_j + 0.2\sin 5\beta_1\delta_j) - e^{\theta_j}(\sin\beta_1\theta_j + 0.2\sin 5\beta_1\theta_j) \right.$$

$$+ \frac{1}{1+\beta_1^2} \{ e^{\delta_j}(\sin\beta_1\delta_j - \beta_1\cos\beta_1\delta_j) - e^{\theta_j}(\sin\beta_1\theta_j - \beta_1\cos\beta_1\theta_j) \}$$

$$\left.\left.\left. + \frac{1}{5(1+25\beta_1^2)} \{ e^{\delta_j}(\sin 5\beta_1\delta_j - 5\beta_1\cos 5\beta_1\delta_j) - e^{\theta_j}(\sin 5\beta_1\theta_j - 5\beta_1\cos 5\beta_1\theta_j) \} \right) \right] \right\} ;$$

$$a_{39} = \sum_{j=1}^{n} \left\{ \frac{3\beta_1}{\sin\alpha_1} \left[-\frac{\bar{\eta}_4}{\tan^2\alpha_j} \left(-e^{-\delta_j}\sin^2 3\beta_1\delta_j + e^{-\theta_j}\sin^2 3\beta_1\theta_j - \frac{3\beta_1}{1+36\beta_1^2} \{ e^{-\delta_j}(\sin 6\beta_1\delta_j + 6\beta_1\cos 6\beta_1\delta_j) \right.\right.\right.$$

$$\left. - e^{-\theta_j}(\sin 6\beta_1\theta_j + 6\beta_1\cos 6\beta_1\theta_j) \} \right) + \bar{\eta}_2 \left(e^{\delta_j}\cos^2 3\beta_1\delta_j - e^{\theta_j}\cos^2 3\beta_1\theta_j \right.$$

$$\left.\left.\left. + \frac{3\beta_1}{1+36\beta_1^2} \{ e^{\delta_j}(\sin 6\beta_1\delta_j - 6\beta_1\cos 6\beta_1\delta_j) - e^{\theta_j}(\sin 6\beta_1\theta_j - 6\beta_1\cos 6\beta_1\theta_j) \} \right) \right] \right\} .$$

$$a_{44} = \sum_{j=1}^{n} \left\{ \left(\bar{\eta}_1 + \beta_1^2\right)\left(\delta_j - \theta_j\right) + \left(\beta_1^2 - \bar{\eta}_1\right)\left(-\sin 2\beta_1\delta_j + \sin 2\beta_1\theta_j\right)/2\beta_1 + \bar{\eta}_3\left(\cos 2\beta_1\delta_j - \cos 2\beta_1\theta_j\right) \right\} .$$

$$a_{45} = \sum_{j=1}^{n} \Bigg\{ \Bigg[\left(2\beta_1^2 + \bar{\eta}_1\right)\left(\sin\beta_1\delta_j - \sin\beta_1\theta_j\right) + \left(2\beta_1^2 - \bar{\eta}_1\right)\left(-\sin 3\beta_1\delta_j + \sin 3\beta_1\theta_j\right)/3 \Bigg] / \beta_1$$

$$+ 0.5\bar{\eta}_3 \left(\cos\beta_1\delta_j - \cos\beta_1\theta_j + \cos 3\beta_1\delta_j - \cos 3\beta_1\theta_j\right) \Bigg\} .$$

$$a_{46} = \sum_{j=1}^{n} \Bigg\{ \frac{1}{2\beta_1} \Bigg[\left(3\beta_1^2 + \bar{\eta}_1\right)\left(\sin 2\beta_1\delta_j - \sin 2\beta_1\theta_j\right) + 0.5\left(3\beta_1^2 - \bar{\eta}_1\right)\left(-\sin 4\beta_1\delta_j + \sin 4\beta_1\theta_j\right) \Bigg] -$$

$$+ 0.5\bar{\eta}_3 \left(\cos 2\beta_1\delta_j - \cos 2\beta_1\theta_j + \cos 4\beta_1\delta_j - \cos 4\beta_1\theta_j\right) \Bigg\} .$$

$$a_{47} = \sum_{j=1}^{n} \Bigg\{ \frac{\cot\alpha_j}{\sin\alpha_1} \Bigg[\bar{\eta}_5 \beta_1^2 \left(-e^{-\delta_j}\sin^2\beta_1\delta_j + e^{-\theta_j}\sin^2\beta_1\theta_j\right) - \frac{\beta_1}{1+4\beta_1^2}\left(\bar{\eta}_5\beta_1^2 - 0.5\bar{\eta}_4\right)\Big(e^{-\delta_j}(\sin 2\beta_1\delta_j + 2\beta_1\cos 2\beta_1\delta_j)$$

$$- e^{-\theta_j}(\sin 2\beta_1\theta_j + 2\beta_1\cos 2\beta_1\theta_j)\Big)\Bigg]\Bigg\} .$$

$$a_{48} = \sum_{j=1}^{\infty} \Bigg\{ \frac{\cot\alpha_j}{\sin\alpha_1} \Bigg[\bar{\eta}_4 \Big(e^{-\delta_j}(\cos\beta_1\delta_j + \tfrac{1}{3}\cos 3\beta_1\delta_j) - e^{-\theta_j}(\cos\beta_1\theta_j + \tfrac{1}{3}\cos 3\beta_1\theta_j)$$

$$+ \frac{1}{1+\beta_1^2}\{ e^{-\delta_j}(\beta_1\sin\beta_1\delta_j - \cos\beta_1\delta_j) - e^{-\theta_j}(\beta_1\sin\beta_1\theta_j - \cos\beta_1\theta_j)\}$$

$$+ \frac{1}{3(1+9\beta_1^2)}\{ e^{-\delta_j}(3\beta_1\sin 3\beta_1\delta_j - \cos 3\beta_1\delta_j) - e^{-\theta_j}(3\beta_1\sin 3\beta_1\theta_j - \cos 3\beta_1\theta_j)\}\Big)$$

$$+ \bar{\eta}_5 \beta_1 \Big(e^{-\delta_j}(\sin\beta_1\delta_j - \tfrac{1}{3}\sin 3\beta_1\delta_j) - e^{-\theta_j}(\sin\beta_1\theta_j - \tfrac{1}{3}\sin 3\beta_1\theta_j)$$

$$-\frac{1}{1+\beta_1^2}\left\{e^{-\delta_j}(\sin\beta_1\delta_j+\beta_1\cos\beta_1\delta_j)-e^{-\theta_j}(\sin\beta_1\theta_j+\beta_1\cos\beta_1\theta_j)\right\}$$

$$+\frac{1}{3(1+9\beta_1^2)}\left\{e^{-\delta_j}(\sin3\beta_1\delta_j+3\beta_1\cos3\beta_1\delta_j)-e^{-\theta_j}(\sin3\beta_1\theta_j+3\beta_1\cos3\beta_1\theta_j)\right\}\Bigg)\Bigg]\Bigg\}.$$

$$a_{49}=\sum_{j=1}^{n}\Bigg\{\frac{\cot\alpha_j}{\sin\alpha_j}\Bigg[0.75\,\bar{\eta}_4\Bigg(e^{-\delta_j}(\cos2\beta_1\delta_j+0.5\cos4\beta_1\delta_j)-e^{-\theta_j}(\cos2\beta_1\theta_j+0.5\cos4\beta_1\theta_j)$$

$$+\frac{1}{1+4\beta_1^2}\left\{e^{-\delta_j}(2\beta_1\sin2\beta_1\delta_j-\cos2\beta_1\delta_j)-e^{-\theta_j}(2\beta_1\sin2\beta_1\theta_j-\cos2\beta_1\theta_j)\right\}$$

$$+\frac{1}{2(1+16\beta_1^2)}\left\{e^{-\delta_j}(4\beta_1\sin4\beta_1\delta_j-\cos4\beta_1\delta_j)-e^{-\theta_j}(4\beta_1\sin4\beta_1\theta_j-\cos4\beta_1\theta_j)\right\}\Bigg)$$

$$+0.75\,\bar{\eta}_5\beta_1\Bigg(e^{-\delta_j}(\sin2\beta_1\delta_j+0.5\sin4\beta_1\delta_j)-e^{-\theta_j}(\sin2\beta_1\theta_j+0.5\sin4\beta_1\theta_j)$$

$$-\frac{1}{1+4\beta_1^2}\left\{e^{-\delta_j}(\sin2\beta_1\delta_j+2\beta_1\cos2\beta_1\delta_j)-e^{-\theta_j}(\sin2\beta_1\theta_j+2\beta_1\cos2\beta_1\theta_j)\right\}$$

$$+\frac{1}{2(1+16\beta_1^2)}\left\{e^{-\delta_j}(\sin4\beta_1\delta_j+4\beta_1\cos4\beta_1\delta_j)-e^{-\theta_j}(\sin4\beta_1\theta_j+4\beta_1\cos4\beta_1\theta_j)\right\}\Bigg)\Bigg]\Bigg\}.$$

$$a_{55}=\sum_{j=1}^{n}\Bigg\{\left(4\beta_1^2+\bar{\eta}_1\right)\left(\delta_j-\theta_j\right)+\left(4\beta_1^2-\bar{\eta}_1\right)\left(-\sin4\beta_1\delta_j+\sin4\beta_1\theta_j\right)/4\beta_1$$

$$+\bar{\eta}_3\left(\cos4\beta_1\delta_j-\cos4\beta_1\theta_j\right)\Bigg\}.$$

$$a_{56} = \sum_{j=1}^{n} \left\{ \frac{1}{\beta_i} \left[\left(6\beta_i^2 + \bar{\eta}_1\right)\left(\delta_j - \theta_j\right) + \left(6\beta_i^2 - \bar{\eta}_1\right)\left(-\sin 5\beta_i\delta_j + \sin 5\beta_i\theta_j\right)/5 \right] \right.$$

$$\left. + 0.5\,\bar{\eta}_3 \left(\cos\beta_i\delta_j - \cos\beta_i\theta_j + \cos 5\beta_i\delta_j - \cos 5\beta_i\theta_j \right) \right\} .$$

$$a_{57} = \sum_{j=1}^{n} \left\{ \frac{\cot\alpha_j}{\sin\alpha_1} \left[0.5\,\bar{\eta}_4 \left(e^{-\delta_j}\left(-\cos\beta_i\delta_j + \tfrac{1}{3}\cos 3\beta_i\delta_j\right) - e^{-\theta_j}\left(-\cos\beta_i\theta_j + \tfrac{1}{3}\cos 3\beta_i\theta_j\right) \right. \right. \right.$$

$$+ \frac{1}{1+\beta_i^2} \left\{ e^{-\delta_j}\left(-\beta_i \sin\beta_i\delta_j + \cos\beta_i\delta_j\right) - e^{-\theta_j}\left(-\beta_i\sin\beta_i\theta_j + \cos\beta_i\theta_j\right) \right\}$$

$$\left. + \frac{1}{3(1+9\beta_i^2)} \left\{ e^{-\delta_j}\left(3\beta_i\sin 3\beta_i\delta_j - \cos 3\beta_i\delta_j\right) - e^{-\theta_j}\left(3\beta_i\sin 3\beta_i\theta_j - \cos 3\beta_i\theta_j\right) \right\} \right)$$

$$+ \bar{\eta}_5 \beta_i \left(e^{-\delta_j}\left(\sin\beta_i\delta_j - \tfrac{1}{3}\sin 3\beta_i\delta_j\right) - e^{-\theta_j}\left(\sin\beta_i\theta_j - \tfrac{1}{3}\sin 3\beta_i\theta_j\right) \right.$$

$$- \frac{1}{1+\beta_i^2} \left\{ e^{-\delta_j}\left(\sin\beta_i\delta_j + \beta_i\cos\beta_i\delta_j\right) - e^{-\theta_j}\left(\sin\beta_i\theta_j + \beta_i\cos\beta_i\theta_j\right) \right\}$$

$$\left. \left. \left. + \frac{1}{3(1+9\beta_i^2)} \left\{ e^{-\delta_j}\left(\sin 3\beta_i\delta_j + 3\beta_i\cos 3\beta_i\delta_j\right) - e^{-\theta_j}\left(\sin 3\beta_i\theta_j + 3\beta_i\cos 3\beta_i\theta_j\right) \right\} \right) \right] \right\} .$$

$$a_{58} = \sum_{j=1}^{n} \left\{ \frac{\cot\alpha_j}{\sin\alpha_1} \left[4\,\bar{\eta}_5\,\beta_i^2 \left(-e^{-\delta_j}\sin^2 2\beta_i\delta_j + e^{-\theta_j}\sin^2 2\beta_i\theta_j \right) \right. \right.$$

$$\left. \left. - \frac{2\beta_i}{1+16\beta_i^2} \left(4\bar{\eta}_5\beta_i^2 - 0.5\,\bar{\eta}_4 \right) \left(e^{-\delta_j}\left(\sin 4\beta_i\delta_j + 4\beta_i\cos 4\beta_i\delta_j\right) - e^{-\theta_j}\left(\sin 4\beta_i\theta_j + 4\beta_i\cos 4\beta_i\theta_j\right) \right) \right] \right\} .$$

$$a_{59} = \sum_{j=1}^{n} \left\{ \frac{\cot\alpha_j}{\sin\alpha_1} \left[1.5\bar{\eta}_4 \left(e^{-\delta_j} (\cos\beta_1\delta_j + 0.2\cos 5\beta_1\delta_j) - e^{-\theta_j} (\cos\beta_1\theta_j + 0.2\cos 5\beta_1\theta_j) \right.\right.\right.$$

$$+ \frac{1}{1+\beta_1^2} \{ e^{-\delta_j} (\beta_1 \sin\beta_1\delta_j - \cos\beta_1\delta_j) - e^{-\theta_j} (\beta_1 \sin\beta_1\theta_j - \cos\beta_1\theta_j) \}$$

$$\left. + \frac{1}{5(1+25\beta_1^2)} \{ e^{-\delta_j} (5\beta_1 \sin 5\beta_1\delta_j - \cos 5\beta_1\delta_j) - e^{-\theta_j} (5\beta_1 \sin 5\beta_1\theta_j - \cos 5\beta_1\theta_j) \} \right)$$

$$+ 3\bar{\eta}_5 \beta_1 \left(e^{-\delta_j} (\sin\beta_1\delta_j - 0.2\sin 5\beta_1\delta_j) - e^{-\theta_j} (\sin\beta_1\theta_j - 0.2\sin 5\beta_1\theta_j) \right.$$

$$- \frac{1}{1+\beta_1^2} \{ e^{-\delta_j} (\sin\beta_1\delta_j + \beta_1\cos\beta_1\delta_j) - e^{-\theta_j} (\sin\beta_1\theta_j + \beta_1\cos\beta_1\theta_j) \}$$

$$\left.\left.\left. + \frac{1}{5(1+25\beta_1^2)} \{ e^{-\delta_j} (\sin 5\beta_1\delta_j + 5\beta_1\cos 5\beta_1\delta_j) - e^{-\theta_j} (\sin 5\beta_1\theta_j + 5\beta_1\cos 5\beta_1\theta_j) \} \right) \right] \right\} .$$

$$a_{66} = \sum_{j=1}^{n} \left\{ \left(9\beta_1^2 + \bar{\eta}_1\right)\left(\delta_j - \theta_j\right) + \left(9\beta_1^2 - \bar{\eta}_1\right)\left(-\sin 6\beta_1\delta_j + \sin 6\beta_1\theta_j\right)/6\beta_1 + \bar{\eta}_3 \left(\cos 6\beta_1\delta_j - \cos 6\beta_1\theta_j\right)] \right\} .$$

$$a_{67} = \sum_{j=1}^{n} \left\{ \frac{\cot\alpha_j}{\sin\alpha_1} \left[0.25\bar{\eta}_4 \left(e^{-\delta_j} (-\cos 2\beta_1\delta_j + 0.5\cos 4\beta_1\delta_j) - e^{-\theta_j} (-\cos 2\beta_1\theta_j + 0.5\cos 4\beta_1\theta_j) \right.\right.\right.$$

$$+ \frac{1}{1+4\beta_1^2} \{ e^{-\delta_j} (-2\beta_1 \sin 2\beta_1\delta_j + \cos 2\beta_1\delta_j) - e^{-\theta_j} (-2\sin 2\beta_1\theta_j + \cos 2\beta_1\theta_j) \}$$

$$\left. + \frac{1}{2(1+16\beta_1^2)} \{ e^{-\delta_j} (4\beta_1 \sin 4\beta_1\delta_j - \cos 4\beta_1\delta_j) - e^{-\theta_j} (4\beta_1 \sin 4\beta_1\theta_j - \cos 4\beta_1\theta_j) \} \right)$$

$$+ 0.75\,\bar{\eta}_5\,\beta_1\Big(e^{-\delta_j}(\sin 2\beta_1\delta_j - 0.5\sin 4\beta_1\delta_j) - e^{-\theta_j}(\sin 2\beta_1\theta_j - 0.5\sin 4\beta_1\theta_j)$$

$$-\frac{1}{1+4\beta_1^2}\{e^{-\delta_j}(\sin 2\beta_1\delta_j + 2\beta_1\cos 2\beta_1\delta_j) - e^{-\theta_j}(\sin 2\beta_1\theta_j + 2\beta_1\cos 2\beta_1\theta_j)\}$$

$$+\frac{1}{2(1+16\beta_1^2)}\{e^{-\delta_j}(\sin 4\beta_1\delta_j + 4\beta_1\cos 4\beta_1\delta_j) - e^{-\theta_j}(\sin 4\beta_1\theta_j + 4\beta_1\cos 4\beta_1\theta_j)\}\Big)\Big]\Big\} \,.$$

$$a_{68} = \sum_{j=1}^{n}\Big\{\frac{\cot\alpha_j}{\sin\alpha_1}\Big[\bar{\eta}_4\Big(e^{-\delta_j}(-\cos\beta_1\delta_j + 0.2\cos 5\beta_1\delta_j) - e^{-\theta_j}(-\cos\beta_1\theta_j + 0.2\cos 5\beta_1\theta_j)$$

$$+\frac{1}{1+\beta_1^2}\{e^{-\delta_j}(-\beta_1\sin\beta_1\delta_j + \cos\beta_1\delta_j) - e^{-\theta_j}(-\beta_1\sin\beta_1\theta_j + \cos\beta_1\theta_j)\}$$

$$+\frac{1}{5(1+25\beta_1^2)}\{e^{-\delta_j}(5\beta_1\sin 5\beta_1\delta_j - \cos 5\beta_1\delta_j) - e^{-\theta_j}(5\beta_1\sin 5\beta_1\theta_j - \cos 5\beta_1\theta_j)\}\Big)$$

$$+3\bar{\eta}_5\beta_1\Big(e^{-\delta_j}(\sin\beta_1\delta_j - 0.2\sin 5\beta_1\delta_j) - e^{-\theta_j}(\sin\beta_1\theta_j - 0.2\sin 5\beta_1\theta_j)$$

$$-\frac{1}{1+\beta_1^2}\{e^{-\delta_j}(\sin\beta_1\delta_j + \beta_1\cos\beta_1\delta_j) - e^{-\theta_j}(\sin\beta_1\theta_j + \beta_1\cos\beta_1\theta_j)\}$$

$$+\frac{1}{5(1+25\beta_1^2)}\{e^{-\delta_j}(\sin 5\beta_1\delta_j + 5\beta_1\cos 5\beta_1\delta_j) - e^{-\theta_j}(\sin 5\beta_1\theta_j + 5\beta_1\cos 5\beta_1\theta_j)\}\Big)\Big]\Big\} \,.$$

$$a_{69} = \sum_{j=1}^{n} \left\{ \frac{\cot\alpha_1}{\sin\alpha_1} \left[9\bar{\eta}_5 \beta_1^2 \left(-e^{-\delta_j} \sin^2 3\beta_1 \delta_j + e^{-\theta_j} \sin^2 3\beta_1 \theta_j \right) - \left(9\bar{\eta}_5 \beta_1^2 - 0.5\bar{\eta}_4 \right) \left(e^{-\delta_j} (\sin 6\beta_1 \delta_j + 6\beta_1 \cos 6\beta_1 \delta_j) 3\beta_1 \right.\right.\right.$$

$$\left.\left.\left. - 3\beta_1 e^{-\theta_j} (\sin 6\beta_1 \theta_j + 6\beta_1 \cos 6\beta_1 \theta_j) \right) \Big/ (1+36\beta_1^2) \right] \right\} .$$

$$a_{77} = \sum_{j=1}^{n} \left\{ \frac{1}{\sin^2\alpha_1} \left[\bar{\eta}_6 \left((\beta_1^2 + \bar{\eta}_1)(\delta_j - \theta_j) + (\beta_1^2 - \bar{\eta}_1)(-\sin 2\beta_1 \delta_j + \sin 2\beta_1 \theta_j)/2\beta_1 \right.\right.\right.$$

$$\left. + \bar{\eta}_3 (\cos 2\beta_1 \delta_j - \cos 2\beta_1 \theta_j) \right) + \bar{\eta}_2 \left(e^{2\delta_j} \cos^2 \beta_1 \delta_j - e^{2\theta_j} \cos \beta_1 \theta_j \right.$$

$$\left.\left.\left. + \frac{0.5\beta_1}{1+\beta_1^2} \left\{ e^{2\delta_j} (\sin 2\beta_1 \delta_j - \beta_1 \cos 2\beta_1 \delta_j) - e^{2\theta_j} (\sin 2\beta_1 \theta_j - \beta_1 \cos \beta_1 \theta_j) \right\} \right) \right] \right\} .$$

$$a_{78} = \sum_{j=1}^{n} \left\{ \frac{1}{\sin^2\alpha_1} \left[\frac{\bar{\eta}_6}{\beta_1} \left((2\beta_1^2 + \bar{\eta}_1)(\sin \beta_1 \delta_j - \sin \beta_1 \theta_j) + (2\beta_1^2 - \bar{\eta}_1)(-\sin 3\beta_1 \delta_j + \sin 3\beta_1 \theta_j)/3 \right) \right.\right.$$

$$+ 0.5\bar{\eta}_3 \bar{\eta}_6 \left(\cos \beta_1 \delta_j - \cos \beta_1 \theta_j + \cos 3\beta_1 \delta_j - \cos 3\beta_1 \theta_j \right) + \frac{\bar{\eta}_2}{\beta_1} \left(e^{2\delta_j} (\sin \beta_1 \delta_j + \tfrac{1}{3} \sin 3\beta_1 \delta_j) \right.$$

$$- e^{2\theta_j} (\sin \beta_1 \theta_j + \tfrac{1}{3} \sin 3\beta_1 \theta_j) - \frac{2}{4+\beta_1^2} \left\{ e^{2\delta_j} (2\sin \beta_1 \delta_j - \beta_1 \cos \beta_1 \delta_j) - e^{2\theta_j} (2\sin \beta_1 \theta_j - \beta_1 \cos \beta_1 \theta_j) \right\}$$

$$\left.\left.\left. - \frac{2}{3(4+9\beta_1^2)} \left\{ e^{2\delta_j} (2\sin 3\beta_1 \delta_j - 3\beta_1 \cos 3\beta_1 \delta_j) - e^{2\theta_j} (2\sin 3\beta_1 \theta_j - 3\beta_1 \cos 3\beta_1 \theta_j) \right\} \right) \right] \right\} .$$

$$a_{79} = \sum_{j=1}^{n} \left\{ \frac{1}{\sin^2\alpha_1} \left[\frac{\bar{\eta}_6}{2\beta_1} \left((3\beta_1^2 + \bar{\eta}_1)(\sin 2\beta_1\delta_j - \sin 2\beta_1\theta_j) + 0.5\,(3\beta_1^2 - \bar{\eta}_1)(-\sin 4\beta_1\delta_j + \sin 4\beta_1\theta_j) \right) \right. \right.$$

$$+ 0.5\,\bar{\eta}_3\bar{\eta}_6 \left(\cos 2\beta_1\delta_j - \cos 2\beta_1\theta_j + \cos 4\beta_1\delta_j - \cos 4\beta_1\theta_j \right) + \frac{\bar{\eta}_2}{2\beta_1} \left(e^{2\delta_j}(\sin 2\beta_1\delta_j + 0.5\sin 4\beta_1\delta_j) \right.$$

$$- e^{2\theta_j}(\sin 2\beta_1\theta_j + 0.5\sin 4\beta_1\theta_j) - \frac{1}{2(1+\beta_1^2)} \{ e^{2\delta_j}(2\sin 2\beta_1\delta_j - 2\beta_1\cos 2\beta_1\delta_j)$$

$$- e^{2\theta_j}(2\sin 2\beta_1\theta_j - 2\beta_1\cos 2\beta_1\theta_j) \} - \frac{1}{4(1+4\beta_1^2)} \{ e^{2\delta_j}(2\sin 4\beta_1\delta_j - 4\beta_1\cos 4\beta_1\delta_j)$$

$$\left. \left. \left. - e^{2\theta_j}(2\sin 4\beta_1\theta_j - 4\beta_1\cos 4\beta_1\theta_j) \} \right) \right] \right\} .$$

$$a_{88} = \sum_{j=1}^{n} \left\{ \frac{1}{\sin^2\alpha_1} \left[\bar{\eta}_6 \left((4\beta_1^2 + \bar{\eta}_1)(\delta_j - \theta_j) + \frac{1}{4\beta_1}(4\beta_1^2 - \bar{\eta}_1)(-\sin 4\beta_1\delta_j + \sin 4\beta_1\theta_j) \right. \right. \right.$$

$$\left. + \bar{\eta}_3(\cos 4\beta_1\delta_j - \cos 4\beta_1\theta_j) \right) + \bar{\eta}_2 \left(e^{2\delta_j}\cos^2 2\beta_1\delta_j - e^{2\theta_j}\cos^2 2\beta_1\theta_j + \frac{\beta_1}{1+4\beta_1^2} \{ e^{2\delta_j}(\sin 4\beta_1\delta_j - 2\beta_1\cos 4\beta_1\delta_j) \right.$$

$$\left. \left. \left. - e^{2\theta_j}(\sin 4\beta_1\theta_j - 2\beta_1\cos 4\beta_1\theta_j) \} \right) \right] \right\} .$$

$$a_{89} = \sum_{j=1}^{n} \left\{ \frac{1}{\sin^2\alpha_1} \left[\bar{\eta}_6 \left((6\beta_1^2 + \bar{\eta}_1)(\sin\beta_1\delta_j - \sin\beta_1\theta_j) + (6\beta_1^2 - \bar{\eta}_1)(-\sin 5\beta_1\delta_j + \sin 5\beta_1\theta_j)/5 \right) / \beta_1 \right. \right.$$

$$+ 0.5\,\bar{\eta}_3\bar{\eta}_6 \left(\cos\beta_1\delta_j - \cos\beta_1\theta_j + \cos 5\beta_1\delta_j - \cos 5\beta_1\theta_j \right) + \frac{\bar{\eta}_2}{\beta_1} \left(e^{2\delta_j}(\sin\beta_1\delta_j + 0.2\sin 5\beta_1\delta_j) \right)$$

$$- e^{2\theta_j}(\sin\beta_1\theta_j + 0.2\sin 5\beta_1\theta_j) - \frac{2}{4+\beta_1^2}\{ e^{2\delta_j}(2\sin\beta_1\delta_j - \beta_1\cos\beta_1\delta_j) - e^{2\theta_j}(2\sin\beta_1\theta_j - \beta_1\cos\beta_1\theta_j)\}$$

$$- \frac{2}{5(4+25\beta_1^2)}\{ e^{2\delta_j}(2\sin 5\beta_1\delta_j - 5\beta_1\cos 5\beta_1\delta_j) - e^{2\theta_j}(2\sin 5\beta_1\theta_j - 5\beta_1\cos 5\beta_1\theta_j)\}\Big)\Big]\Big\} .$$

$$a_{99} = \sum_{j=1}^{n}\Big\{\frac{1}{\sin^2\alpha_1}\Big[\ \bar\eta_6\Big((9\beta_1^2 + \bar\eta_1)(\delta_j - \theta_j) + \frac{1}{6\beta_1}(9\beta_1^2 - \bar\eta_1)(-\sin 6\beta_1\delta_j + \sin 6\beta_1\theta_j)$$

$$+ \bar\eta_3(\cos 6\beta_1\delta_j - \cos 6\beta_1\theta_j)\Big) + \bar\eta_2\Big(e^{2\delta_j}\cos^2 3\beta_1\delta_j - e^{2\theta_j}\cos^2 3\beta_1\theta_j$$

$$+ \frac{1.5\beta_1}{1+9\beta_1^2}\{(\sin 6\beta_1\delta_j - 3\beta_1\cos 6\beta_1\delta_j)e^{2\delta_j} - (\sin 6\beta_1\theta_j - 3\beta_1\cos 6\beta_1\theta_j)e^{2\theta_j}\}\Big)\Big]\Big\} .$$

From Equations 3.5.6, 3.5.7 and 3.5.8 the elements, b_{ij}, of matrix Equation 3.5.11 are obtained. These elements are as follows:

$$b_{11} = \sum_{j=1}^{n} \left\{ \bar{\eta}_7 \left[e^{2\delta_j} \sin^2\beta_1\delta_j - e^{2\theta_j} \sin^2\beta_1\theta_j - \frac{\beta_1}{2(1+\beta_1^2)} \left(e^{2\delta_j}(\sin 2\beta_1\delta_j - \beta_1 \cos 2\beta_1\delta_j) \right.\right.\right.$$

$$\left.\left.\left. - e^{2\theta_j}(\sin 2\beta_1\theta_j - \beta_1 \cos 2\beta_1\theta_j) \right) \right] \right\} .$$

$$b_{12} = \sum_{j=1}^{n} \left\{ \frac{\bar{\eta}_7}{\beta_1} \left[e^{2\delta_j}(\sin\beta_1\delta_j - \tfrac{1}{3}\sin 3\beta_1\delta_j) - e^{2\theta_j}(\sin\beta_1\theta_j - \tfrac{1}{3}\sin 3\beta_1\theta_j) \right.\right.$$

$$- \frac{2}{4+\beta_1^2} \left(e^{2\delta_j}(2\sin\beta_1\delta_j - \beta_1\cos\beta_1\delta_j) - e^{2\theta_j}(2\sin\beta_1\theta_j - \beta_1\cos\beta_1\theta_j) \right)$$

$$\left.\left. + \frac{2}{3(4+9\beta_1^2)} \left(e^{2\delta_j}(2\sin 3\beta_1\delta_j - 3\beta_1\cos 3\beta_1\delta_j) - e^{2\theta_j}(2\sin 3\beta_1\theta_j - 3\beta_1\cos 3\beta_1\theta_j) \right) \right] \right\}$$

$$b_{13} = \sum_{j=1}^{n} \left\{ \frac{\bar{\eta}_7}{2\beta_1} \left[e^{2\delta_j}(\sin 2\beta_1\delta_j - 0.5\sin 4\beta_1\delta_j) - e^{2\theta_j}(\sin 2\beta_1\theta_j - 0.5\sin 4\beta_1\theta_j) \right.\right.$$

$$- \frac{1}{2(1+\beta_1^2)} \left(e^{2\delta_j}(2\sin 2\beta_1\delta_j - 2\beta_1\cos 2\beta_1\delta_j) - e^{2\theta_j}(2\sin 2\beta_1\theta_j - 2\beta_1\cos 2\beta_1\theta_j) \right)$$

$$\left.\left. + \frac{1}{4(1+4\beta_1^2)} \left(e^{2\delta_j}(2\sin 4\beta_1\delta_j - 4\beta_1\cos 4\beta_1\delta_j) - e^{2\theta_j}(2\sin 4\beta_1\theta_j - 4\beta_1\cos 4\beta_1\theta_j) \right) \right] \right\} .$$

$$b_{14} = b_{15} = b_{16} = b_{17} = b_{18} = b_{19} = 0 \quad .$$

$$b_{22} = \sum_{j=1}^{n} \left\{ \bar{\eta}_7 \left[e^{2\delta_j} \sin^2 2\beta_i\delta_j - e^{2\Theta_j} \sin^2 2\beta_i\Theta_j - \frac{\beta_i}{1+4\beta_i^2} \left(e^{2\delta_j} (\sin 4\beta_i\delta_j - 2\beta_i \cos 4\beta_i\delta_j) \right.\right.\right.$$

$$\left.\left.\left. - e^{2\Theta_j} (\sin 4\beta_i\Theta_j - 2\beta_i \cos 4\beta_i\Theta_j) \right) \right] \right\} .$$

$$b_{23} = \sum_{j=1}^{n} \left\{ \frac{\bar{\eta}_7}{\beta_i} \left[e^{2\delta_j} (\sin\beta_i\delta_j - 0.2 \sin 5\beta_i\delta_j) - e^{2\Theta_j} (\sin\beta_i\Theta_j - 0.2 \sin 5\beta_i\Theta_j) \right.\right.$$

$$- \frac{2}{4+\beta_i^2} \left(e^{2\delta_j} (2\sin\beta_i\delta_j - \beta_i \cos\beta_i\delta_j) - e^{2\Theta_j} (2\sin\beta_i\Theta_j - \beta_i \cos\beta_i\Theta_j) \right)$$

$$\left.\left. + \frac{2}{5(4+25\beta_i^2)} \left(e^{2\delta_j} (2\sin 5\beta_i\delta_j - 5\beta_i \cos 5\beta_i\delta_j) - e^{2\Theta_j} (2\sin 5\beta_i\Theta_j - 5\beta_i \cos 5\beta_i\Theta_j) \right) \right] \right\} .$$

$$b_{24} = b_{25} = b_{26} = b_{27} = b_{28} = b_{29} = 0 .$$

$$b_{33} = \sum_{j=1}^{n} \left\{ \bar{\eta}_7 \left[e^{2\delta_j} \sin^2 3\beta_i\delta_j - e^{2\Theta_j} \sin^2 3\beta_i\Theta_j - \frac{1.5\beta_i}{1+9\beta_i^2} \left(e^{2\delta_j} (\sin 6\beta_i\delta_j - 3\beta_i \cos 6\beta_i\delta_j) \right.\right.\right.$$

$$\left.\left.\left. - e^{2\Theta_j} (\sin 6\beta_i\Theta_j - 3\beta_i \cos 6\beta_i\Theta_j) \right) \right] \right\} .$$

$$b_{34} = b_{35} = b_{36} = b_{37} = b_{38} = b_{39} = 0 .$$

$$b_{44} = \sum_{j=1}^{n} \left\{ \bar{\eta}_7 \left[e^{2\delta_j} \cos^2\beta_i\delta_j - e^{2\Theta_j} \cos^2\beta_i\Theta_j + \frac{0.5\beta_i}{1+\beta_i^2} \left(e^{2\delta_j} (\sin 2\beta_i\delta_j - \beta_i \cos 2\beta_i\delta_j) - e^{2\Theta_j} (\sin 2\beta_i\Theta_j - \beta_i \cos 2\beta_i\Theta_j) \right) \right] \right\} .$$

$$b_{45} = \sum_{j=1}^{n} \left\{ \frac{\bar{\eta}_7}{\beta_i} \left[e^{2\delta_j}\left(\sin\beta_i\delta_j + \tfrac{1}{3}\sin 3\beta_i\delta_j\right) - e^{2\theta_j}\left(\sin\beta_i\theta_j + \tfrac{1}{3}\sin 3\beta_i\theta_j\right) - \frac{2}{4+\beta_i^2}\left(e^{2\delta_j}\left(2\sin 2\beta_i\delta_j - \beta_i\cos\beta_i\delta_j\right) \right.\right.\right.$$

$$\left.\left.\left. - e^{2\theta_j}\left(2\sin\beta_i\theta_j - \beta_i\cos\beta_i\theta_j\right)\right) - \frac{2}{3(4+9\beta_i^2)}\left(e^{2\delta_j}\left(2\sin 3\beta_i\delta_j - 3\beta_i\cos 3\beta_i\delta_j\right) - e^{2\theta_j}\left(2\sin 3\beta_i\theta_j - 3\beta_i\cos 3\beta_i\theta_j\right)\right)\right]\right\}.$$

$$b_{46} = \sum_{j=1}^{n} \left\{ \frac{\bar{\eta}_7}{2\beta_i} \left[e^{2\delta_j}\left(\sin 2\beta_i\delta_j + 0.5\sin 4\beta_i\delta_j\right) - e^{2\theta_j}\left(\sin 2\beta_i\theta_j + 0.5\sin 4\beta_i\theta_j\right) - \frac{1}{2(1+\beta_i^2)}\left(e^{2\delta_j}\left(2\sin 2\beta_i\delta_j - 2\beta_i\cos 2\beta_i\delta_j\right) \right.\right.\right.$$

$$\left.\left.\left. - e^{2\theta_j}\left(2\sin 2\beta_i\theta_j - 2\beta_i\cos 2\beta_i\theta_j\right)\right) - \frac{1}{4(1+4\beta_i^2)}\left(e^{2\delta_j}\left(2\sin 4\beta_i\delta_j - 4\beta_i\cos 4\beta_i\delta_j\right) - e^{2\theta_j}\left(2\sin 4\beta_i\theta_j - 4\beta_i\cos 4\beta_i\theta_j\right)\right)\right]\right\}.$$

$$b_{47} = b_{48} = b_{49} = 0 \quad .$$

$$b_{55} = \sum_{j=1}^{n} \left\{ \bar{\eta}_7 \left[e^{2\delta_j}\cos^2 2\beta_i\delta_j - e^{2\theta_j}\cos^2 2\beta_i\theta_j + \frac{\beta_i}{1+4\beta_i^2}\left(e^{2\delta_j}\left(\sin 4\beta_i\delta_j - 2\beta_i\cos 4\beta_i\delta_j\right) \right.\right.\right.$$

$$\left.\left.\left. - e^{2\theta_j}\left(\sin 4\beta_i\theta_j - 2\beta_i\cos 4\beta_i\theta_j\right)\right)\right]\right\}.$$

$$b_{56} = \sum_{j=1}^{n} \left\{ \frac{\bar{\eta}_7}{\beta_i} \left[e^{2\delta_j}\left(\sin\beta_i\delta_j + 0.2\sin 5\beta_i\delta_j\right) - e^{2\theta_j}\left(\sin\beta_i\theta_j + 0.2\sin 5\beta_i\theta_j\right) - \frac{2}{4+\beta_i^2}\left(e^{2\delta_j}\left(2\sin\beta_i\delta_j - \beta_i\cos\beta_i\delta_j\right) \right.\right.\right.$$

$$\left.\left.\left. - e^{2\theta_j}\left(2\sin\beta_i\theta_j - \beta_i\cos\beta_i\theta_j\right)\right) - \frac{2}{5(4+25\beta_i^2)}\left(e^{2\delta_j}\left(2\sin 5\beta_i\delta_j - 5\beta_i\cos 5\beta_i\delta_j\right) - e^{2\theta_j}\left(2\sin 5\beta_i\theta_j - 5\beta_i\cos 5\beta_i\theta_j\right)\right)\right]\right\}.$$

$$b_{57} = b_{58} = b_{59} = 0 \quad .$$

$$b_{66} = \sum_{j=1}^{n}\left\{\bar{\eta}_7\left[e^{2\delta_j}\cos^2 3\beta_1\delta_j - e^{2\theta_j}\cos^2 3\beta_1\theta_j + \frac{1.5\beta_1}{1+9\beta_1^2}\left(e^{2\delta_j}(\sin 6\beta_1\delta_j - 3\beta_1\cos 3\beta_1\delta_j) - e^{2\theta_j}(\sin 6\beta_1\theta_j - 3\beta_1\cos 3\beta_1\theta_j)\right)\right]\right\} .$$

$$b_{67} = b_{68} = b_{69} = 0 .$$

$$b_{77} = \sum_{j=1}^{n}\left\{\frac{\bar{\eta}_8}{\sin^2\alpha_1}\left[e^{2\delta_j}\cos^2\beta_1\delta_j - e^{2\theta_j}\cos^2\beta_1\theta_j + \frac{\beta_1}{2(1+\beta_1^2)}\left(e^{2\delta_j}(\sin 2\beta_1\delta_j - \beta_1\cos\beta_1\delta_j) - e^{2\theta_j}(\sin 2\beta_1\theta_j - \beta_1\cos\beta_1\theta_j)\right)\right]\right\} .$$

$$b_{78} = \sum_{j=1}^{n}\left\{\frac{\bar{\eta}_8}{\beta_1\sin^2\alpha_1}\left[e^{2\delta_j}(\sin\beta_1\delta_j + \tfrac{1}{3}\sin 3\beta_1\delta_j) - e^{2\theta_j}(\sin\beta_1\theta_j + \tfrac{1}{3}\sin 3\beta_1\theta_j) - \frac{2}{4+\beta_1^2}\left(e^{2\delta_j}(2\sin\beta_1\delta_j - \beta_1\cos\beta_1\delta_j)\right.\right.\right.$$

$$\left.\left.\left. - e^{2\theta_j}(2\sin\beta_1\theta_j - \beta_1\cos\beta_1\theta_j)\right) - \frac{2}{3(4+9\beta_1^2)}\left(e^{2\delta_j}(2\sin 3\beta_1\delta_j - 3\beta_1\cos 3\beta_1\delta_j) - e^{2\theta_j}(2\sin 3\beta_1\theta_j - 3\beta_1\cos 3\beta_1\theta_j)\right)\right]\right\} .$$

$$b_{79} = \sum_{j=1}^{n}\left\{\frac{\bar{\eta}_8}{2\beta_1\sin^2\alpha_1}\left[e^{2\delta_j}(\sin 2\beta_1\delta_j + 0.5\sin 4\beta_1\delta_j) - e^{2\theta_j}(\sin 2\beta_1\theta_j + 0.5\sin 4\beta_1\theta_j)\right.\right.$$

$$- \frac{1}{2(1+\beta_1^2)}\left(e^{2\delta_j}(2\sin 2\beta_1\delta_j - 2\beta_1\cos 2\beta_1\delta_j) - e^{2\theta_j}(2\sin 2\beta_1\theta_j - 2\beta_1\cos 2\beta_1\theta_j)\right)$$

$$\left.\left. - \frac{1}{4(1+4\beta_1^2)}\left(e^{2\delta_j}(2\sin 4\beta_1\delta_j - 4\beta_1\cos 4\beta_1\delta_j) - e^{2\theta_j}(2\sin 4\beta_1\theta_j - 4\beta_1\cos 4\beta_1\theta_j)\right)\right]\right\} .$$

$$b_{88} = \sum_{j=1}^{n}\left\{\frac{\bar{\eta}_8}{\sin^2\alpha_1}\left[e^{2\delta_j}\cos^2 2\beta_1\delta_j - e^{2\theta_j}\cos^2 2\beta_1\theta_j + \frac{\beta_1}{1+4\beta_1^2}\left(e^{2\delta_j}(\sin 4\beta_1\delta_j - 2\beta_1\cos 4\beta_1\delta_j)\right.\right.\right.$$

$$\left.\left.\left. - e^{2\theta_j}(\sin 4\beta_1\theta_j - 2\beta_1\cos 4\beta_1\theta_j)\right)\right]\right\} ,$$

$$b_{89} = \sum_{j=1}^{n} \left\{ \frac{\bar{\eta}_\theta}{\beta_1 \sin^2\alpha_1} \left[e^{2\delta_j} (\sin\beta_1\delta_j + 0\cdot 2 \sin 5\beta_1\delta_j) - e^{2\theta_j} (\sin\beta_1\theta_j + 0\cdot 2 \sin 5\beta_1\theta_j) \right. \right.$$

$$- \frac{2}{4+\beta_1^2} \left(e^{2\delta_j} (2\sin\beta_1\delta_j - \beta_1 \cos\beta_1\delta_j) - e^{2\theta_j} (2\sin\beta_1\theta_j - \beta_1\cos\beta_1\theta_j) \right)$$

$$\left. \left. - \frac{2}{5(4+25\beta_1^2)} \left(e^{2\delta_j} (2\sin 5\beta_1\delta_j - 5\beta_1 \cos 5\beta_1\delta_j) - e^{2\theta_j} (2\sin 5\beta_1\theta_j - 5\beta_1\cos 5\beta_1\theta_j) \right) \right] \right\} .$$

$$b_{99} = \sum_{j=1}^{n} \left\{ \frac{\bar{\eta}_\theta}{\sin^2\alpha_1} \left[e^{2\delta_j} \cos^2 3\beta_1\delta_j - e^{2\theta_j} \cos 3\beta_1\theta_j + \frac{1.5\beta_1}{1+9\beta_1^2} \left(e^{2\delta_j} (\sin 6\beta_1\delta_j - 3\beta_1 \cos 6\beta_1\delta_j) \right. \right. \right.$$

$$\left. \left. \left. - e^{2\theta_j} (\sin 6\beta_1\theta_j - 3\beta_1 \cos 6\beta_1\theta_j) \right) \right] \right\} .$$

CHAPTER 4

APPLICATION TO SPECIFIC PROBLEMS

The method developed in Chapter 3 was applied to the following specific problems:

a. Homogeneous conical shell frustrum.

b. Sandwich conical shell frustrum.

c. Sandwich paraboloidal shell.

All the numerical evaluations for the above problems were made on a digital computer (IBM 1410). Appendix A gives the program which evaluated all the elements, a_{ij} and b_{ij} of Equation 3.5.2. Appendix B contains the program for the frequency evaluations.

4.1 Formulation of Parameters

From Figure 15 the parameters of a conical segment are given as follows:

$$\mu_j = \frac{1}{2}\left(\frac{L_j}{\bar{R}_j}\right)\tan\alpha_j$$

where for $\mu_j = 1$ the cone is complete, and for $0 < \mu_j < 1$ the cone is truncated.

$$\begin{aligned} s_{jj} &= (1 + \mu_j)\,\bar{R}_j \csc\alpha_j \\ s_{j-1,j} &= (1 - \mu_j)\,\bar{R}_j \csc\alpha_j \end{aligned} \qquad 4.1.1$$

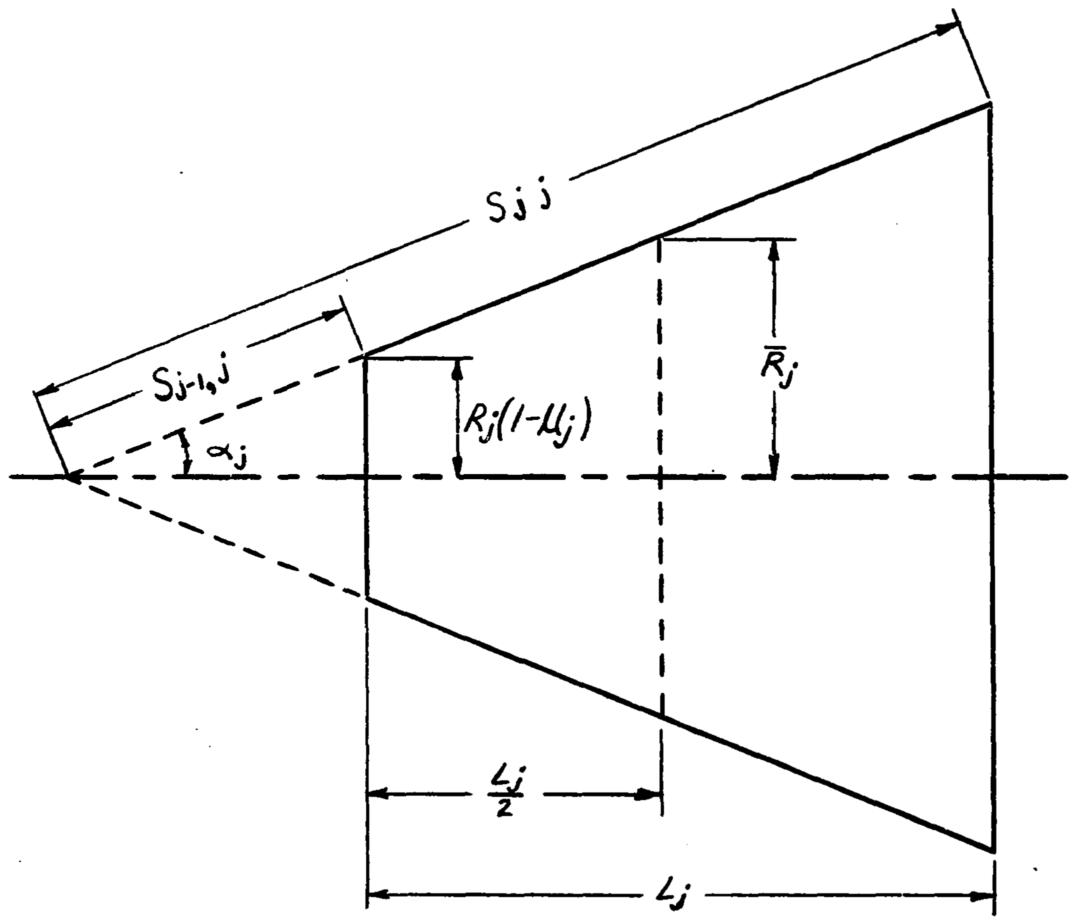

Figure 15--The jth segment of a shell of revolution showing the parameters of the conical segment.

$$\delta_j = \ln\left(\frac{s_{ji}}{s_{01}}\right)$$

$$\theta_j = \ln\left(\frac{s_{j-1i}}{s_{01}}\right) \qquad 4.1.1\ \text{(Cont.)}$$

$$\beta_1 = \frac{\pi}{\ln(s_{nn}/s_{01})}$$

For sandwich shells of revolution the following parameters are used:

$$\bar{\eta}_1 = E_\theta/E_s$$

$$\bar{\eta}_2 = \left(\frac{h_c}{t_f}\right)\left(\frac{G_{sz}}{E_s}\right)\left(\frac{K_s}{2}\right)(1 - \nu_s\nu_\theta)$$

$$\bar{\eta}_3 = \nu_s + E_\theta\nu_\theta/E_s$$

$$\bar{\eta}_4 = \frac{\nu_s}{4}\left(\frac{2t_f}{s_{01}}\right)^2\left(1 + \frac{h_c}{t_f}\right)^2$$

$$\bar{\eta}_5 = \bar{\eta}_4/\nu_s \qquad 4.1.2$$

$$\bar{\eta}_6 = \frac{1}{12}\left(\frac{2t_f}{2_{01}}\right)^2$$

$$\bar{\eta}_7 = \frac{1}{2}\left(\frac{\rho_c}{\rho_f}\right)\left(\frac{h_c}{t_f}\right) + 1$$

$$\bar{\eta}_8 = \bar{\eta}_6\left[\frac{1}{2}\left(\frac{\rho_c}{\rho_f}\right)\left(\frac{h_c}{t_f}\right)^3 + 3\left(1 + \frac{h_c}{t_f}\right)^2 + 1\right]$$

$$= \left(\frac{1 - \nu_s\nu_\theta}{E_s}\right)\rho_f\, s_{01}^2\,\Omega^2$$

4.2 Free Vibrations of Homogeneous Conical Shells

The lower axisymmetrical natural frequencies of a uniform thickness, linearly elastic homogeneous cone were computed. Transverse-shear and rotatory inertia effects were included. Calculations for this particular cone had been made previously by Garnet and Kempner (24).

The fundamental data for this cone as given in (24) were:

$$\nu = 0.3, \quad \alpha_1 = 5^{\circ}, \quad h/\bar{R}_1 = 0.05, \quad L_1/\bar{R}_1 = 0.25.$$

From the theory developed in Chapter 3, the following equation is obtained:

$$\Big[[A] - \lambda [B] \Big] \Big\{ V \Big\} = 0 \qquad 4.2.1$$

For nontrivial solutions the matrix, $\Big[[A] - \lambda [B] \Big]$, must vanish.

or

$$\Big[[A] - \lambda [B] \Big] = 0 \qquad 4.2.2$$

The numerical values of λ which satisfy Equation 4.2.2 are the sought eigenvalues.

The computer input data for this specific problem are given in Table 1.

The first two frequency parameters found are:

$$\sqrt{\lambda_1} = 26.191$$

$$\sqrt{\lambda_2} = 83.366$$

Garnet and Kempner (24) found only the first frequency parameter for this problem. The value they obtained for this problem is:

$$\sqrt{\lambda_1} = 26.233$$

The small difference between the two results is due to the increased accuracy of the present analysis due to the term in the series solution used. This difference is negligible for the fundamental frequency but it increases appreciably for higher frequencies.

4.3 Free Vibrations of Sandwich Conical Shells

The lower axisymmetric natural frequencies of a uniform thickness, linearly elastic sandwich conical shell were computed. Transverse shear and rotatory inertia were included.

The composite was assumed to have the following properties (25):

Material: Fiberglass Facings and Core

$$\frac{G_{sz}}{E_s} = 0.00186$$

$$\frac{\rho_c}{\rho_f} = 0.0388$$

$$\frac{h_c}{t_f} = 25$$

$$\frac{2t_f}{\bar{R}_1} = 0.002$$

$$\frac{L_1}{\bar{R}_1} = 0.25$$

$$\alpha_1 = 5^\circ$$

$$K_s = 1.935$$

The computer input data is given in Table 2.

The first two frequency parameters obtained were:

$$\sqrt{\lambda_1} = 6.6616$$

$$\sqrt{\lambda_2} = 20.286$$

These two frequency parameters were found to be much lower than those for the homogeneous conical shell. These lower frequencies were expected because of the flexibility effects of the core on the composite.

4.4 Free Vibrations of Sandwich Paraboloidal Shells of Revolution

A sandwich paraboloidal shell of revolution having an equation of the form:

$$y^2 = R_1^2 \left(15 \frac{x}{L} + 1\right)$$

was considered, where

$$\frac{L}{R_1} = 8.$$

The paraboloidal shell was approximated by 2, 4, 6, 8, 10 and 12 conical segments successively (Figure 16). In each case the method developed in Chapter 3 was applied and the first two eigenvalues calculated.

The fundamental data used for the shell are as follows:

$$\frac{\rho_c}{\rho_f} = 0.0388$$

$$\frac{G_{sz}}{E_s} = 0.00186$$

$$\frac{h_c}{t_f} = 25$$

$$\frac{2t_f}{\bar{R}} = 0.00129$$

$$K_s = 1.935$$

$$\nu_s = 0.12$$

The input data are given in Tables 4,5,6,7,8 and 9 for each case respectively.

The obtained eigenvalues are given in Table 3 for all six cases and the results plotted in Figures 17 and 18.

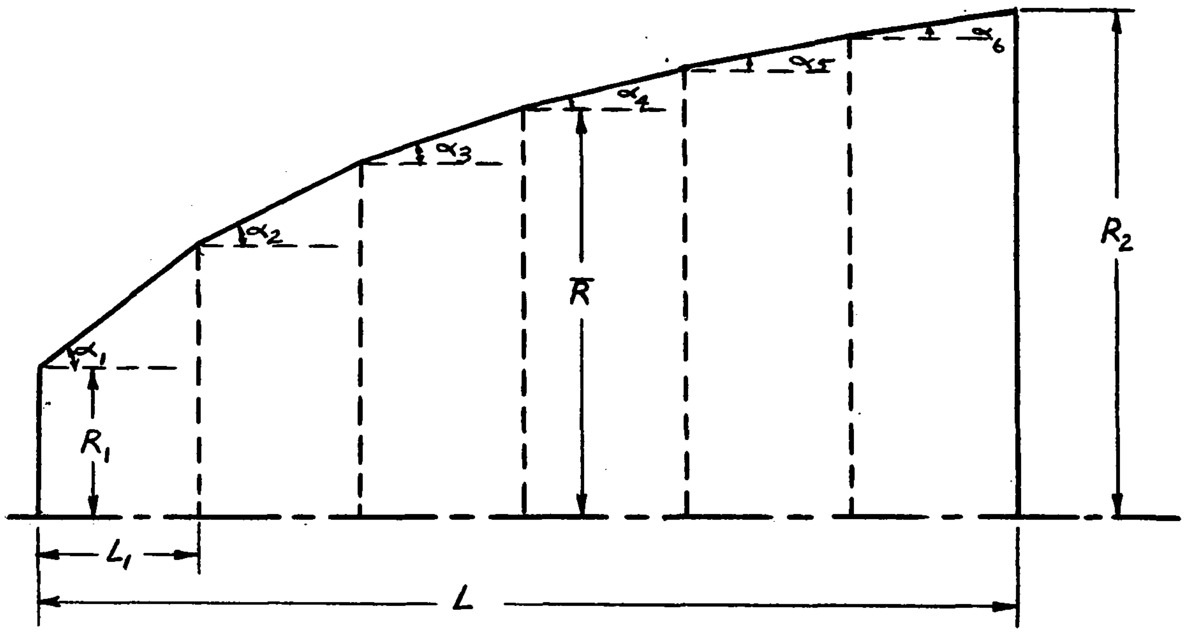

Figure 16--A sandwich paraboloidal shell of revolution composed of six conical-shell segments.

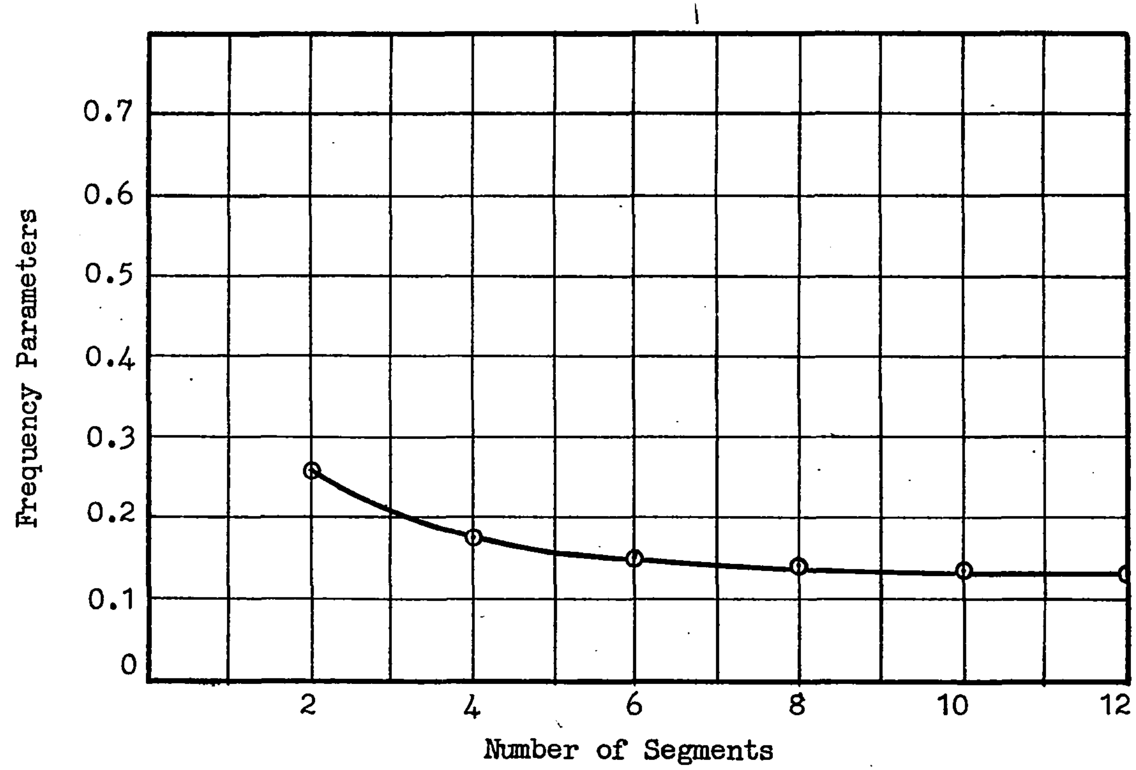

Figure 17--Graph showing the number of conical-shell segments versus the first frequency parameter of a paraboloidal shell of revolution.

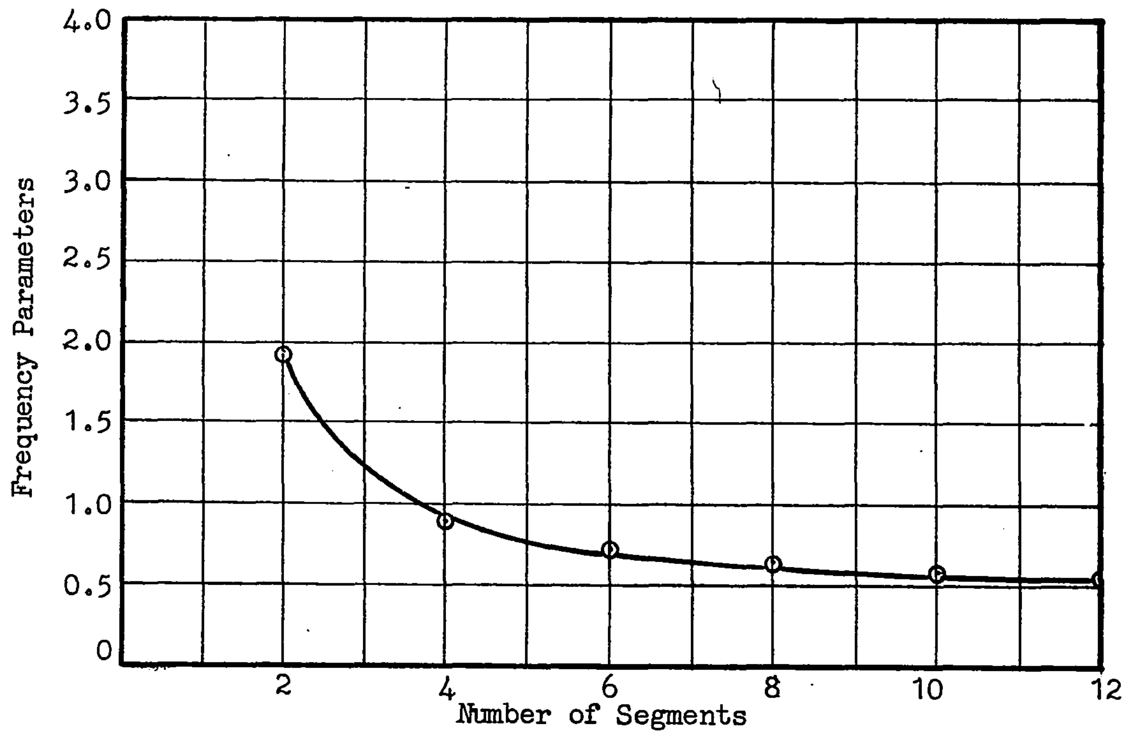

Figure 18--Graph showing the number of conical-shell segments versus the second frequency of a paraboloidal shell of revolution.

TABLE 1

Computer Input Data for a Homogeneous Conical Shell

n=1

j	1
α_j	0.218729244 (10^{-1})
δ_j	0
θ_j	0.8727 (10^{-1})

β	143.6292923
$\bar{\eta}_1$	1
$\bar{\eta}_2$	0.2916666666
$\bar{\eta}_3$	0.6
$\bar{\eta}_4$	0
$\bar{\eta}_5$	0
$\bar{\eta}_6$	0.16177 (10^{-5})
$\bar{\eta}_7$	1
$\bar{\eta}_8$	0.16177 (10^{-5})

TABLE 2

Computer Input Data for a Sandwich Conical Shell

n=1

j	1
α_j	$0.218729244\ (10^{-1})$
δ_j	0
θ_j	$0.8727\ (10^{-1})$

β	143.6292923
$\bar{\eta}_1$	1
$\bar{\eta}_2$	$0.443852972\ (10^{-1})$
$\bar{\eta}_3$	0.24
$\bar{\eta}_4$	$0.6286\ (10^{-6})$
$\bar{\eta}_5$	$0.5239\ (10^{-5})$
$\bar{\eta}_6$	$0.26\ (10^{-8})$
$\bar{\eta}_7$	1.484709
$\bar{\eta}_8$	$0.56929\ (10^{-5})$

TABLE 3

First and Second Eigenvalues for a Sandwich Paraboloidal Shell Approximated by Different Numbers of Conical-Shell Segments

n	2	4	6	8	10	12
$\sqrt{\lambda_1}$	0.26833	0.17088	0.16244	0.13416	0.13420	0.12166
$\sqrt{\lambda_2}$	1.9527	0.89764	0.68562	0.59599	0.57061	0.52230

TABLE 4

Computer Input Data for a Sandwich Paraboloidal Shell
Approximated by Two Conical-Shell Segments

n=2

j	1	2
α_j	0.64344	0.32841
δ_j	0.40541982	2.00258804
θ_j	0	1.01528458

β	1.568766312
$\bar{\eta}_1$	1
$\bar{\eta}_2$	0.44385297 (10^{-1})
$\bar{\eta}_3$	0.24
$\bar{\eta}_4$	0.10264422 (10^{-3})
$\bar{\eta}_5$	0.8553685 (10^{-3})
$\bar{\eta}_6$	0.42177938 (10^{-6})
$\bar{\eta}_7$	1.484615
$\bar{\eta}_8$	0.10286539 (10^{-2})

TABLE 5

Computer Input Data for a Sandwich Paraboloidal Shell Approximated by Four Conical-Shell Segments

n=4

j	1	2	3	4
α_j	0.64344	0.42877	0.30106	0.25074
δ_j	0.40541982	1.32549092	1.92131112	2.26822071
θ_j	0	0.7662923	1.66679289	2.09789493

β	1.38504716
$\bar{\eta}_1$	1
$\bar{\eta}_2$	0.443851972 (10^{-1})
$\bar{\eta}_3$	0.24
$\bar{\eta}_4$	0.10264422 (10^{-3})
$\bar{\eta}_5$	0.8553685 (10^{-3})
$\bar{\eta}_6$	0.42177938 (10^{-6})
$\bar{\eta}_7$	1.484615
$\bar{\eta}_8$	0.10286539 (10^{-2})

TABLE 6

Computer Input Data for a Sandwich Paraboloidal Shell Approximated by Six Conical-Shell Segments

n=6

j	1	2	3	4	5	6
α_j	0.64344	0.46367	0.35866	0.30427	0.26384	0.24115
δ_j	0.40541982	1.097329	1.558985	1.871051	2.122739	2.306632
θ_j	0	0.6989819	1.338760	1.717919	2.007214	2.21208

β	1.361982814
$\bar{\eta}_1$	1
$\bar{\eta}_2$	0.443852972 (10^{-1})
$\bar{\eta}_3$	0.24
$\bar{\eta}_4$	0.10264422 (10^{-3})
$\bar{\eta}_5$	0.8553685 (10^{-3})
$\bar{\eta}_6$	0.42177938 (10^{-6})
$\bar{\eta}_7$	1.484615
$\bar{\eta}_8$	0.10286539 (10^{-2})

TABLE 7

Computer Input Data for a Sandwich Paraboloidal
Approximated by Eight Segments

n=8

j	α_j	δ_j	θ_j
1	0.64344	0.40541982	0
2	0.46913	1.01229776	0.71743922
3	0.39270	1.36327662	1.17227310
4	0.33481	1.65121218	1.51480365
5	0.29438	1.88078424	1.77536492
6	0.27401	2.03852858	1.94977062
7	0.25831	2.17241551	2.09555734
8	0.24958	2.2737372	2.20470688

β	1.38168679
$\bar{\eta}_1$	1
$\bar{\eta}_2$	0.443852972 (10^{-1})
$\bar{\eta}_3$	0.24
$\bar{\eta}_4$	0.10264422 (10^{-3})
$\bar{\eta}_5$	0.8553685 (10^{-3})
$\bar{\eta}_6$	0.42177938 (10^{-6})
$\bar{\eta}_7$	1.484615
$\bar{\eta}_8$	0.10286539 (10^{-2})

TABLE 8

Computer Input Data for a Sandwich Paraboloidal
Approximated by Ten Segments

n=10

j	α_j	δ_j	θ_j		β	1.362306271
1	0.64344	0.40541982	0		$\bar{\eta}_1$	1
2	0.50208	0.88309924	0.62151671		$\bar{\eta}_2$	0.443852972 (10^{-1})
3	0.40957	1.24700091	1.08112755		$\bar{\eta}_3$	0.24
4	0.36477	1.47735398	1.35041816		$\bar{\eta}_4$	0.10264422 (10^{-3})
5	0.33219	1.67090957	1.56878764		$\bar{\eta}_5$	0.8553685 (10^{-3})
6	0.30776	1.83474502	1.74946432		$\bar{\eta}_6$	0.42177938 (10^{-6})
7	0.27925	2.0013828	1.93032362		$\bar{\eta}_7$	1.484615
8	0.25976	2.1315022	2.06971901		$\bar{\eta}_8$	0.10286539 (10^{-2})
9	0.24319	2.24811962	2.19358798			
10	0.24144	2.30608397	2.25489033			

TABLE 9

Computer Input Data for a Sandwich Paraboloidal Shell Approximated by Twelve Segments

n=12

j	α_j	δ_j	θ_j
1	0.64344	0.40541982	0
2	0.5105	0.82937042	0.60847214
3	0.43109	1.1440109	0.99292534
4	0.38484	1.36960354	1.25345483
5	0.34558	1.55989037	1.46630059
6	0.31416	1.73098114	1.65358174
7	0.30514	1.82943568	1.75975552
8	0.29147	1.93504297	1.87284208
9	0.26849	2.06832756	2.01447148
10	0.25452	2.16933358	2.12096275
11	0.24493	2.25391884	2.2096743
12	0.24492	2.29313471	2.25063316

β	1.369999172
$\bar{\eta}_1$	1
$\bar{\eta}_2$	0.443852972 (10^{-1})
$\bar{\eta}_3$	0.24
$\bar{\eta}_4$	0.10264422 (10^{-3})
$\bar{\eta}_5$	0.8553685 (10^{-3})
$\bar{\eta}_6$	0.42177938 (10^{-6})
$\bar{\eta}_7$	1.484615
$\bar{\eta}_8$	0.10286539 (10^{-2})

CHAPTER 5

CONCLUSIONS

A unified method of analysis for the vibrational behavior of sandwich shells of revolution has been developed by utilizing the Rayleigh-Ritz energy approach. Throughout the development, engineering shell theory assumptions have been employed. In addition transverse shear and rotatory inertia effects were taken into consideration.

The developed method was generalized in such a manner to be applicable to sandwich shells with orthotropic or isotropic cores and facings, as well as to orthotropic or isotropic homogeneous shells.

The validity of the analysis was established by applying the present method to a homogeneous conical frustrum which was considered previously by Garnet and Kempner. The percent difference between the frequency they obtained and the frequency obtained in the present analysis was less than 0.17%. This small difference resulted from the increased accuracy of the present analysis due to the number of terms in the series solution used. The percent difference is of negligible magnitude for the fundamental frequency but increases rapidly for higher frequencies. Thus, if higher frequencies, third frequency and above, are sought, more terms in the series solution should be considered.

The frequencies obtained for the sandwich conical frustrum were about 75% lower than those obtained for the homogeneous conical frustrum of the same geometrical dimensions. From a structural efficiency standpoint, sandwich panels are manufactured in such a manner that their cores are very flexible. Primarily, the shear flexibility effect of the core on the composite explains the frequency difference between the two shells.

The developed method was also applied to sandwich paraboloidal shells of revolution. The analysis has shown that approximating the paraboid by twelve conical segments yields good results. The curve of frequency versus number of segments showed that the eigenvalues for this particular shell reached their leveling point around $n=10$, where n is the number of segments.

The method can be applied to any shell of revolution, if such a shell can be approximated by a finite number of conical shells. The number of segments required to yield accurate results depends on the type of shell under consideration.

The equilibrium and the basic differential equations of motion for sandwich shells of revolution were also developed and presented in this dissertation. The differential equations of motion were of the linear homogeneous type with variable coefficients. The author has found no closed form solution of such equations in the literature.

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NOMENCLATURE

A_m, B_m, D_m	=	arbitrary parameters
a	=	$h_c + t_f$
E	=	Young's modulus
G	=	shear modulus
K_s	=	dynamic shear factor
F_x, F_y, Q_{zx}, M_x, M_y	=	stress resultants per unit length
m	=	mass per unit surface area
I	=	mass moment of inertia per unit surface area
t_f, h_c	=	half thicknesses of facings and core respectively
R	=	radius of the parallel circles of the shell measured from the axis of rotation
R_1, R_2	=	principal radii of curvature of the shell
o,i,c	=	subscripts and superscripts used to denote the outer facing, inner facing and core respectively
x,y,z	=	coordinate axes on the middle surface of the shell in the meridional, circumferential and normal directions respectively
u,v,w	=	translational displacements of the middle surface of the shell along the x, y and z axes respectively.
ρ	=	mass density
t	=	time

V, T = strain and kinetic energies respectively

Ω = frequency

λ = eigenvalue parameter = $(1 - \nu_s \nu_\theta)\, \rho_f s_{01}^2 \Omega^2 / E_s$

ν = Poisson's ratio

η = $2h_c K_s G_{sz}$

η_s = $(4t_f E_s)/(1 - \nu_s \nu_\theta)$

η_θ = $(4t_f E_\theta)/(1 - \nu_s \nu_\theta)$

η_s^* = $(4t_f^3 E_s)/3(1 - \nu_s \nu_\theta)$

η_θ^* = $(4t_f^3 E_\theta)/3(1 - \nu_s \nu_\theta)$

$\left.\begin{array}{l} \bar{\eta}_1 \\ \bar{\eta}_2 \\ \bar{\eta}_3 \\ \bar{\eta}_4 \\ \bar{\eta}_5 \\ \bar{\eta}_6 \\ \bar{\eta}_7 \\ \bar{\eta}_8 \end{array}\right\}$ = shell parameters defined in Equations 4.1.2

σ = stresses

ϵ = strains

ψ_x = change of slope of the normal to the middle surface of the shell in the meridional direction

α = cone half angle

μ = conical shell parameter

β_m = $\frac{m\pi}{\mathcal{J}}$

μ_j = $\frac{1}{2}\left(\frac{L_j}{R_j}\right) \tan \alpha_j$

$$\zeta = \ln\left(\frac{s_{nn}}{s_{01}}\right)$$

$$\delta_j = \ln\left(\frac{s_{jj}}{s_{01}}\right)$$

$$\theta_j = \ln \frac{s_{j-1,j}}{s_{01}}$$

$$s_{jj} = (1 + \mu_j)\, \bar{R}_j \csc \alpha_j$$

$$s_{j-1,j} = (1 - \mu_j)\, \bar{R}_j \csc \alpha_j$$

j = 1, 2, ---n

m = a parameter used in the infinite series (m=1, 2, 3---   )

APPENDIX A

Computer Program for the Evaluation of the Matrix Elements a_{ij} and b_{ij}

APPENDIX A

```
      DIMENSION AA(5,9)
    1 FORMAT(3E16.10)
    4 FORMAT(1X,3E20.8)
    2 FORMAT(I5)
   28 FORMAT(1H1,I5)
 1492 READ 2,N
      PRINT 28,N
      READ 1,B
      PRINT 4,B
      READ 1,BN1,BN2,BN3,BN4,BN5,BN6,BN7,BN8
      PRINT 4,BN1,BN2,BN3,BN4,BN5,BN6,BN7,BN8
      DO 3 I=1,5
      DO 3 J=1,9
    3 AA(I,J)=0.0
      B2=B*B
      BN2B2=BN2*B2
      DO 999 J=1,N
      READ 1,A,D,T
      PRINT 4,A,D,T
      TAN=SINF(A)/COSF(A)
      BN1C2=BN1*(1./TAN)*(1./TAN)
      X=B*T
      ST1=SINF(X)
      CT1=COSF(X)
      Y=2.*X
      ST2=SINF(Y)
      CT2=COSF(Y)
      Y=3.*X
      ST3=SINF(Y)
      CT3=COSF(Y)
      Y=4.*X
      ST4=SINF(Y)
      CT4=COSF(Y)
      Y=5.*X
      ST5=SINF(Y)
      CT5=COSF(Y)
      Y=6.*X
      ST6=SINF(Y)
      CT6=COSF(Y)
      X=B*D
      SD1=SINF(X)
      CD1=COSF(X)
      Y=2.*X
      SD2=SINF(Y)
      CD2=COSF(Y)
      Y=3.*X
      SD3=SINF(Y)
      CD3=COSF(Y)
```

```
      Y=4.*X
      SD4=SINF(Y)
      CD4=COSF(Y)
      Y=5.*X
      SD5=SINF(Y)
      CD5=COSF(Y)
      Y=6.*X
      SD6=SINF(Y)
      CD6=COSF(Y)
      IF(J-N)1493,1494,1493
 1494 SD1=0.
      SD2=0.
      SD3=0.
      SD4=0.
      SD5=0.
      SD6=0.
 1493 SSD1=SD1*SD1
      SSD2=SD2*SD2
      SST1=ST1*ST1
      SST2=ST2*ST2
      CSD1=CD1*CD1
      CSD2=CD2*CD2
      CST1=CT1*CT1
      CST2=CT2*CT2
      ED=EXPF(D)
      ET=EXPF(T)
      IF(J-1)34,33,34
   33 SA=SINF(A)
      RSS=1./(SA*SA)
   34 COSS=1./(TAN*SA)
      TAN2=TAN*TAN
      E2D=EXPF(2.*D)
      E2T=EXPF(2.*T)
      DMT=D-T
      SST3=ST3*ST3
      SSD3=SD3*SD3
      CST3=CT3*CT3
      CSD3=CD3*CD3
      AA(1,1)=AA(1,1)+(BN1C2+BN2B2)*DMT+(BN1C2-BN2B2)*(ST2-SD2)/(2.*B)
      AA(1,2)=AA(1,2)+((BN1C2+2.*BN2B2)*(SD1-ST1)+(BN1C2-2.*BN2B2)*(ST3-
     1SD3)/3.)/B
      AA(1,3)=AA(1,3)+((BN1C2+3.*BN2B2)*(SD2-ST2)+(BN1C2-3.*BN2B2)*(ST4-
     1SD4)/2.)/(2.*B)
      A(1,4)=AA(1,4)+(-.5*BN3*B*(DMT-(SD2-ST2)/(2.*B))-BN1*(CD2-CT2)/B)
     1/TAN
      AA(1,5)=AA(1,5)+(-BN3*((SD1-ST1)-(SD3-ST3)/3.)-BN1*((-CD1+CT1)+(CD
     13-CT3)/3.)/B)/TAN
      AA(1,6)=AA(1,6)+(-.75*BN3*((SD2-ST2)-(SD4-ST4)*.5)-.5*BN1*((-CD2+C
     1T2)+(CD4-CT4)*.5)/B)/TAN
      AA(1,7)=AA(1,7)+(B/SA)*(-BN4*(-SSD1/ED+SST1/ET-B*((SD2+2.*B*CD2)/E
```

```
1D-(ST2+2.*B*CT2)/ET)/(1.+4.*B2))/TAN2+2.*BN2*(ED*CSD1-ET*CST1+B*((
2SD2-2.*B*CD2)*ED-(ST2-2.*B*CT2)*ET)/(1.+4.*B2)))
 AA(1,8)=AA(1,8)+(1./SA)*(-(BN4/TAN2)*((SD1-SD3/3.)/ED-(ST1-ST3/3.)
1/ET-((SD1+B*CD1)/ED-(ST1+B*CT1)/ET)/(1.+B2)+((SD3+3.*B*CD3)/ED-(ST
23+3.*B*CT3)/ET)/(3.+27.*B2))+BN2*((SD1+SD3/3.)*ED-(ST1+ST3/3.)*ET+
3((SD1-B*CD1)*ED-(ST1-B*CT1)*ET)/(1.+B2)+((SD3-3.*B*CD3)*ED-(ST3-3.
4*B*CT3)*ET)/(3.+27.*B2)))
 AA(1,9)=AA(1,9)+(1./SA)*((-.75*BN4/TAN2)*((SD2-.5*SD4)/ED-(ST2-.5*
1ST4)/ET-((SD2+2.*B*CD2)/ED-(ST2+2.*B*CT2)/ET)/(1.+4.*B2)+((SD4+4.*
2B*CD4)/ED-(ST4+4.*B*CT4)/ET)/(2.+32.*B2))+.5*BN2*((SD2+.5*SD4)*ED-
3(ST2+.5*ST4)*ET+((SD2-2.*B*CD2)*ED-(ST2-2.*B*CT2)*ET)/(1.+4.*B2)+(
4(SD4-4.*B*CD4)*ED-(ST4-4.*B*CT4)*ET)/(2.+32.*B2)))
 AA(2,2)=AA(2,2)+(BN1C2+4.*BN2B2)*DMT+(BN1C2-4.*BN2B2)*(-SD4+ST4)/(
14.*B)
 AA(2,3)=AA(2,3)+((BN1C2+6.*BN2B2)*(SD1-ST1)+(BN1C2-6.*BN2B2)*(-SD5
1+ST5)/5.)/B
 AA(2,4)=AA(2,4)+(-5.*BN3*((SD1-ST1)-(SD3-ST3)/3.)-BN1*(CD1-CT1+(CD
13-CT3)/3.)/B)/TAN
 AA(2,5)=AA(2,5)+(-BN3*B*( DMT-(SD4-ST4)/(4.*B))-BN1*(CD4-CT4)/(2.*
1B))/TAN
 AA(2,6)=AA(2,6)+(-1.5*BN3*((SD1-ST1)-(SD5-ST5)/5.)-BN1*(-CD1+CT1 +
1(CD5-CT5)/5.)/B)/TAN
 AA(2,7)=AA(2,7)+(1./SA)*((-.5*BN4/TAN2)*((SD1-SD3/3.)/ED-(ST1-ST3/
13.)/ET-((SD1+B*CD1)/ED-(ST1+B*CT1)/ET)/(1.+B2)+((SD3+3.*B*CD3)/ED-
2(ST3+3.*B*CT3)/ET)/(3.+27.*B2))+2.*BN2*((SD1+SD3/3.)*ED-(ST1+ST3/3
3.)*ET+((SD1-B*CD1)*ED-(ST1-B*CT1)*ET)/(1.+B2)+((SD3-3.*B*CD3)*ED-(
4ST3-3.*B*CT3)*ET)/(3.+27.*B2)))
 AA(2,8)=AA(2,8)+(2.*B/SA)*((-BN4/TAN2)*(-SSD2/ED+SST2/ET-2.*B*((SD
14+4.*B*CD4)/ED-(ST4+4.*B*CT4)/ET)/(1.+16.*B2))+2.*BN2*(ED*CSD2-ET*
2CST2+2.*B*((SD4-4.*B*CD4)*ED-(ST4-4.*B*CT4)*ET)/(1.+16.*B2)))
 AA(2,9)=AA(2,9)+(1./SA)*((-1.5*BN4/TAN2)*((SD1-SD5/5.)/ED-(ST1-ST5
1/5.)/ET-((SD1+B*CD1)/ED-(ST1+B*CT1)/ET)/(1.+B2)+((SD5+5.*B*CD5)/ED
2-(ST5+5.*B*CT5)/ET)/(5.+125.*B2))+2.*BN2*((SD1+SD5/5.)*ED-(ST1+ST5
3/5.)*ET+((SD1-B*CD1)*ED-(ST1-B*CT1)*ET)/(1.+B2)+((SD5-5.*B*CD5)*ED
4-(ST5-5.*B*CT5)*ET)/(5.+125.*B2)))
 AA(3,3)=AA(3,3)+(BN1C2+9.*BN2B2)*DMT+(BN1C2-9.*BN2B2)*(-SD6+ST6)/(
1*.*B)
 AA(3,4)=AA(3,4)+(-.5*BN3*((SD2-ST2)-.5*(SD4-ST4))-BN1*((CD2-CT2)+.
15*(CD4-CT4))/B)/(2.*TAN)
 AA(3,5)=AA(3,5)+(-BN3*(SD1-ST1-(SD5-ST5)/5.)
1                                             -BN1*(CD1-CT1+(CD5-CT5)
2/5.)/B)/TAN
 AA(3,6)=AA(3,6)+(-1.5*BN3*B *(DMT-(SD6-ST6)/(6.*B))-BN1*(CD6-CT6)/
1(3.*B))/TAN
 AA(3,7)=AA(3,7)+(1./SA)*((-.25*BN4/TAN2)*((SD2-.5*SD4)/ED-(ST2-.5*
1ST4)/ET-((SD2+2.*B*CD2)/ED-(ST2+2.*B*CT2)/ET)*(1.+4.*B2)+((SD4+4.*
2B*CD4)/ED-(ST4+4.*B*CT4)/ET)/(2.+32.*B2))+1.5*BN2*((SD2+.5*SD4)*ED
3-(ST2+.5*ST4)*ET+((SD2-2.*B*CD2)*ED-(ST2-2.*B*CT2)*ET)/(1.+4.*B2)+
4((SD4-4.*B*CD4)*ED-(ST4-4.*B*CT4)*ET)/(2.+32.*B2)))
```

```
      AA(3,8)=AA(3,8)+(1./SA)*((-BN4/TAN2)*((SD1-SD5/5.)/ED-(ST1-ST5/5.)
     1/ET-((SD1+B*CD1)/ED-(ST1+B*CT1)/ET)/(1.+B2)+((SD5+5.*B*CD5)/ED-(ST
     25+5.*B*CT5)/ET)/(5.+125.*B2))+3.*BN2*((SD1+SD5/5.)*ED-(ST1+ST5/5.)
     3*ET+((SD1-B*CD1)*ED-(ST1-B*CT1)*ET)/(1.+B2)+((SD5-5.*B*CD5)*ED-(ST
     45-5.*B*CT5)*ET)/(5.+125.*B2)))
      AA(3,9)=AA(3,9)+(3.*B/SA)*((-BN4/TAN2)*(-SSD3/ED+SST3/ET-3.*B*((SD
     16+6.*B*CD6)/ED-(ST6+6.*B*CT6)/ET)/(1.+36.*B2))+BN2*(ED*CSD3-ET*CST
     23+3.*B*((SD6-6.*B*CD6)*ED-(ST6-6.*B*CT6)*ET)/(1.+36.*B2)))
      AA(4,4)=AA(4,4)+(B2+BN1)*DMT+(B2-BN1)*(-SD2+ST2)/(2.*B)+BN3*(CD2-C
     1T2)
      AA(4,5)=AA(4,5)+((2.*B2+BN1)*(SD1-ST1)+(2.*B2-BN1)*(-SD3+ST3)/3.)/
     1B+.5*BN3*(CD1-CT1+CD3-CT3)
      AA(4,6)=AA(4,6)+((3.*B2+BN1)*(SD2-ST2)+(3.*B2-BN1)*(-SD4+ST4)/2.)/
     1(2.*B)+.5*BN3*(CD2-CT2+CD4-CT4)
      AA(4,7)=AA(4,7)+COSS*(BN5*B2*(-SSD1/ED+SST1/ET)-B*
     1                                                   (BN5*B2-.5*BN4)*((
     1SD2+2.*B*CD2)/ED-(ST2+2.*B*CT2)/ET)/(1.+4.*B2))
      AA(4,8)=AA(4,8)+COSS*(BN4*((CD1+CD3/3.)/ED-(CT1+CT3/3.)/ET+((B*SD1
     1-CD1)/ED-(B*ST1-CT1)/ET)/(1.+B2)+((3.*B*SD3-CD3)/ED-(3.*B*ST3-CT3)
     2/ET)/(3.+27.*B2))+BN5*B*
     2                         ((SD1-SD3/3.)/ED-(ST1-ST3/3.)/ET-((SD1+B*CD1
     3)/ED-(ST1+B*CT1)/ET)/(1.+B2)+((SD3+3.*B*CD3)/ED-(ST3+3.*B*CT3)/ET)
     4/(3.+27.*B2)))
      AA(4,9)=AA(4,9)+COSS*(.75*BN4*((CD2+.5*CD4)/ED-(CT2+.5*CT4)/ET+((2
     1.*B*SD2-CD2)/ED-(2.*B*ST2-CT2)/ET)/(1.+4.*B2)+((4.*B*SD4-CD4)/ED-(
     24.*B*ST4-CT4)/ET)/(2.+32.*B2))+.75*BN5*B*((SD2-.5*SD4)/ED-(ST2-.5*
     3ST4)/ET-((SD2+2.*B*CD2)/ED-(ST2+2.*B*CT2)/ET)/(1.+4.*B2)+((SD4+4.*
     4B*CD4)/ED-(ST4+4.*B*CT4)/ET)/(2.+32.*B2)))
      AA(5,5)=AA(5,5)+(4.*B2+BN1)*DMT+(4.*B2-BN1)*(-SD4+ST4)/(4.*B)+BN3*
     1(CD4-CT4)
      AA(5,6)=AA(5,6)+((6.*B2+BN1)*(SD1-ST1)+(6.*B2-BN1)*(-SD5+ST5)/5.)/
     1B+.5*BN3*(CD1-CT1+CD5-CT5)
      AA(5,7)=AA(5,7)+COSS*(.5*BN4*((-CD1+CD3/3.)/ED-(-CT1+CT3/3.)/ET+((
     1-B*SD1+CD1)/ED-(-B*ST1+CT1)/ET)/(1.+B2)+((3.*B*SD3-CD3)/ED-(3.*B*S
     2T3-CT3)/ET)/(3.+27.*B2))+BN5*B*((SD1-SD3/3.)/ED-(ST1-ST3)/ET-((SD1
     3+B*CD1)/ED-(ST1+B*CT1)/ET)/(1.+B2)+((SD3+3.*B*CD3)/ED-(ST3+3.*B*CT
     43)/ET)/(3.+27.*B2)))
      AA(5,8)=AA(5,8)+COSS*(4.*BN5*B2*(-SSD2/ED+SST2/ET)-2.*B*
     1                                                   (4.*BN5*B2-.5*B
     1N4)*((SD4+4.*B*CD4)/ED-(ST4+4.*B*CT4)/ET)/(1.+16.*B2))
  999 AA(5,9)=AA(5,9)+COSS*(1.5*BN4*((CD1+CD5/5.)/ED-(CT1+CT5/5.)/ET+((B
     1*SD1-CD1)/ED-(B*ST1-CT1)/ET)/(1.+B2)+((5.*B*SD5-CD5)/ED-(5.*B*ST5-
     2CT5)/ET)/(5.+125.*B2))+3.*BN5*B*((SD1-SD5/5.)/ED-(ST1-ST5/5.)/ET-(
     3(SD1+B*CD1)/ED-(ST1+B*CT1)/ET)/(1.+B2)+((SD5+5.*B*CD5)/ED-(ST5+5.*
     4B*CT5)/ET)/(5.+125.*B2)))
    7 FORMAT(1X,6E20.8)
    8 FORMAT(E16.10,10X,1HA)
      PRINT 21
   21 FORMAT(////1X)
```

```
      DO 6 I=1,5
      PRINT 7,(AA(I,J),J=1,9)
    6 PUNCH 8,(AA(I,J),J=I,9)
      PAUSE
      GO TO 1492
      END
```

```
      DIMENSION AA(9,9),BB(9,9)
      EQUIVALENCE (E2T,ET2),(E2D,ED2)
    1 FORMAT(3E16.10)
    4 FORMAT(1X,3E20.8)
    2 FORMAT(I5)
   28 FORMAT(1H1,I5)
 1492 READ 2,N
      PRINT 28,N
      READ 1,B
      PRINT 4,B
      READ 1,BN1,BN2,BN3,BN4,BN5,BN6,BN7,BN8
      PRINT 4,BN1,BN2,BN3,BN4,BN5,BN6,BN7,BN8
      DO 3 J=1,9
      DO 3 I=1,9
    3 AA(I,J)=0.0
      B2=B*B
      BN2B2=BN2*B2
      DO 999 J=1,N
      READ 1,A,D,T
      PRINT 4,A,D,T
      TAN=SINF(A)/COSF(A)
      BN1C2=BN1*(1./TAN)*(1./TAN)
      X=B*T
      ST1=SINF(X)
      CT1=COSF(X)
      Y=2.*X
      ST2=SINF(Y)
      CT2=COSF(Y)
      Y=3.*X
      ST3=SINF(Y)
      CT3=COSF(Y)
      Y=4.*X
      ST4=SINF(Y)
      CT4=COSF(Y)
      Y=5.*X
      ST5=SINF(Y)
      CT5=COSF(Y)
      Y=6.*X
      ST6=SINF(Y)
      CT6=COSF(Y)
      X=B*D
      SD1=SINF(X)
      CD1=COSF(X)
      Y=2.*X
      SD2=SINF(Y)
      CD2=COSF(Y)
      Y=3.*X
      SD3=SINF(Y)
```

```
      CD3=COSF(Y)
      Y=4.*X
      SD4=SINF(Y)
      CD4=COSF(Y)
      Y=5.*X
      SD5=SINF(Y)
      CD5=COSF(Y)
      Y=6.*X
      SD6=SINF(Y)
      CD6=COSF(Y)
      IF(J-N)1493,1494,1493
 1494 SD1=0.
      SD2=0.
      SD3=0.
      SD4=0.
      SD5=0.
      SD6=0.
 1493 SSD1=SD1*SD1
      SSD2=SD2*SD2
      SST1=ST1*ST1
      SST2=ST2*ST2
      CSD1=CD1*CD1
      CSD2=CD2*CD2
      CST1=CT1*CT1
      CST2=CT2*CT2
      ED=EXPF(D)
      ET=EXPF(T)
      IF(J-1)34,33,34
   33 SA=SINF(A)
      RSS=1./(SA*SA)
   34 COSS=1./(TAN*SA)
      TAN2=TAN*TAN
      E2D=EXPF(2.*D)
      E2T=EXPF(2.*T)
      DMT=D-T
      SST3=ST3*ST3
      SSD3=SD3*SD3
      CST3=CT3*CT3
      CSD3=CD3*CD3
      AA(6,6)=AA(6,6)+((9.*B2+BN1)*(DMT)+(9.*B2-BN1)*(ST6-SD6)/6./B+BN3*
     1(CD6-CT6))
      AA(6,7)=AA(6,7)+COSS*(.25*BN4*((.5*CD4-CD2)/ED-(.5*CT4-CT2)/ET+((C
     1D2-2.*B*SD2)/ED-(CT2-2.*B*ST2)/ET)/(1.+4.*B2)+((4.*B*SD4-CD4)/ED-(
     24.*B*ST4-CT4)/ET)/2./(1.+16.*B2))+.75*BN5*B*((SD2-.5*SD4)/ED-(ST2-
     3.5*ST4)/ET-((SD2+2.*B*CD2)/ED-(ST2+2.*B*CT2)/ET)/(1.+4.*B2)+((SD4+
     44.*B*CD4)/ED-(ST4+4.*B*CT4)/ET)/2./(1.+16.*B2)))
      AA(6,8)=AA(6,8)+COSS*(BN4*((CD5/5.-CD1)/ED-(CT5/5.-CT1)/ET
     1+((CD1-B*SD1)/ED+(B*ST1-CT1)/ET)/(1.+B2)+((5.*B*SD5-CD5)/ED-(5.*B*
     2ST5-CT5)/ET)/5./(1.+25.*B2))+3.*BN5*B*((SD1-SD5/5.)/ED-(ST1-ST5/5.)
```

```
2)
3/ET-((SD1+B*CD1)/ED-(ST1+B*CT1)/ET)/(1.+B2)+((SD5+5.*B*CD5)/ED-(ST5
35
4+5.*B*CT5)/ET)/5./(1.+25.*B2)))
 AA(6,9)=AA(6,9)+COSS*(9.*BN5*B2*(SST3/ET-SSD3/ED)-3.*B*(9.*BN5*B2
1                                                                    .5*BN4
1)*((SD6+6.*B*CD6)/ED-(ST6+6.*B*CT6)/ET)/(1.+36.*B2))
 AA(7,7)=AA(7,7)+RSS*(BN6*((B2+BN1)*DMT+(B2-BN1)*(-SD2+ST2)/2./B+BN
1**(CD2-CT2))+BN2*(E2D*CSD1-E2T*CST1+.5*B*((SD2-B*CD2)*ED2-(ST2-B*C
2T2)*ET2)/(1.+B2)))
 AA(7,8)=AA(7,8)+RSS*(BN6*((2.*B2+BN1)*(SD1-ST1)+(2.*B2-BN1)*(ST3-S
1D3)/3.)/B+.5*BN3*BN6*(CD1-CT1+CD3-CT3)+BN2*((SD1+SD3/3.)*ED2-(ST1+
2ST3/3.)*ET2-2.*((2.*SD1-B*CD1)*ED2-(2.*ST1-B*CT1)*ET2)/(4.+B2)-2.*
3((2.*SD3-3.*B*CD3)*ED2-(2.*ST3-3.*B*CT3)*ET2)/3./(4.+9.*B2))/B)
 AA(7,9)=RSS*(BN6*((3.*B2+BN1)*(SD2-ST2)+(3.*B2-BN1)*(ST4-SD4)/2.)/
12./B+.5*BN3*BN6*(CD2-CT2+CD4-CT4)+BN2*((SD2+.5*SD4)*E2D-(ST2+.5*ST
24)*E2T-2.*((2.*SD2-2.*B*CD2)*E2D-(2.*ST2-2.*B*CT2)*E2T)/4./(1.+B2)
3-2.*((2.*SD4-4.*B*CD4)*E2D-(2.*ST4-4.*B*CT4)*E2T)/8./(1.+4.*B2))/2
4./B)+AA(7,9)
 AA(8,8)=AA(8,8)+RSS*(BN6*((4.*B2+BN1)*DMT+(4.*B2-BN1)*(ST4-SD4)/4.
1/B+BN3*(CD4-CT4))+BN2   *(E2D*CSD2
1                                      -E2T*CST2+B*((SD4-2.*B*CD4)*E2D-(S
2T4-2.*B*CT4)*E2T)/(1.+4.*B2)))
 AA(8,9)=AA(8,9)+RSS*(BN6*((6.*B2+BN1)*(SD1-ST1)+(6.*B2-BN1)*(ST5-S
1D5)/5.)/B+.5*BN3*BN6*(CD1-CT1+CD5-CT5)+BN2*((SD1+SD5/5.)*E2D-(ST1+
2ST5/5.)*E2T-2.*((2.*SD1-B*CD1)*E2D-(2.*ST1-B*CT1)*E2T)/(4.+B2)-2.*
3((2.*SD5-5.*B*CD5)*E2D-(2.*ST5-5.*B*CT5)*E2T)/5./(4.+25.*B2))/B)
 AA(9,9)=AA(9,9)+RSS*(BN6*((9.*B2+BN1)*DMT+(9.*B2-BN1)*(ST6-SD6)/6.
1/B+BN3*(CD6-CT6))+BN2*(E2D*CSD3-E2T*CST3+1.5*B*((SD6-3.*B*CD6)*E2D
2-E2T*
3 (ST6-3.*B*CT6))/(1.+9.*B2)))
 BB(1,1)=BB(1,1)+BN7*(E2D*SSD1-E2T*SST1-B*((SD2-B*CD2)*E2D-(ST2-B*C
1T2)*E2T)/2./(1.+B2))
 BB(1,2)=BB(1,2)+BN7*((SD1-SD3/3.)*E2D-(ST1-ST3/3.)*E2T-2.*((2.*SD1
1-B*CD1)*E2D-(2.*ST1-B*CT1)*E2T)/(4.+B2
1                                            )+2.*((2.*SD3-3.*B*CD3)*E2D-(
22.*ST3-3.*B*CT3)*E2T)/3./(4.+9.*B2))/B
 BB(1,3)=BB(1,3)+BN7*((SD2-.5*SD4)*E2D-(ST2-.5*ST4)*E2T-2.*((2.*SD2
1-2.*B*CD2)*E2D-(2.*ST2-2.*B*CT2)*E2T)/4./(1.+B2)+2.*((2.*SD4-4.*B*
2CD4)*E2D-(2.*ST4-4.*B*CT4)*E2T)/8./(1.+4.*B2))/2./B
 BB(1,4)=0.
 BB(1,5)=0.
 BB(1,6)=0.
 BB(1,7)=0.
 BB(1,8)=0.
 BB(1,9)=0.
 BB(2,2)=BB(2,2)+BN7*(E2D*SSD2-E2T*SST2-B*((SD4-2.*B*CD4)*E2D-(ST4-
12.*B*CT4)*E2T)/(1.+4.*B2))
 BB(2,3)=BB(2,3)+BN7*((SD1-SD5/5.)*E2D-(ST1-ST5/5.)*E2T-2.*((2.*SD1
```

```
 1-B*CD1)*E2D-(2.*ST1-B*CT1)*E2T)/(4.+B2)+2.*((2.*SD5-5.*B*CD5)*E2D-
 2(2.*ST5-5.*B*CT5)*E2T)/5./(4.+25.*B2))/B
  BB(2,4)=0.
  BB(2,5)=0.
  BB(2,6)=0.
  BB(2,7)=0.
  BB(2,8)=0.
  BB(2,9)=0.
  BB(3,3)=BB(3,3)+BN7*(E2D*SSD3-E2T*SST3-1.5*B*((SD6-3.*B*CD6)*E2D-(
 1ST6-3.*B*CT6)*E2T)/(1.+9.*B2))
  DO 10 I=4,9
10 BB(3,I)=0.
  BB(4,4)=BB(4,4)+BN7*(E2D*CSD1-E2T*CST1+.5*B*((SD2-B*CD2)*E2D-(ST2-
 2B*CT2)*E2T)/(1.+B2))
  BB(4,5)=BB(4,5)+BN7*((SD1+SD3/3.)*E2D-(ST1+ST3/3.)*E2T-2.*((2.*SD1
 1-B*CD1)*E2D-(2.*ST1-B*CT1)*E2T)/(4.+B2)-2.*((2.*SD3-3.*B*CD3)*E2D-
 2(2.*ST3-3.*B*CT3)*E2T)/3./(4.+9.*B2))/B
  BB(4,6)=BB(4,6)+BN7*((SD2+.5*SD4)*E2D-(ST2+.5*ST4)*E2T-2.*((2.*SD2
 1-2.*B*CD2)*E2D-(2.*ST2-2.*B*CT2)*E2T)/4./(1.+B2)-2.*((2.*SD4-4.*B*
 2CD4)*E2D-(2.*ST4-4.*B*CT4)*E2T)/8./(1.+4.*B2))/2./B
  BB(4,7)=0.
  BB(4,8)=0.
  BB(4,9)=0.
  BB(5,5)=BB(5,5)+BN7*(E2D*CSD2-E2T*CST2+B*((SD4-2.*B*CD4)*E2D-(ST4-
 12.*B*CT4)*E2T)/(1.+4.*B2))
  BB(5,6)=BB(5,6)+BN7*((SD1+SD5/5.)*E2D-(ST1+ST5/5.)*E2T-2.*((2.*SD1-
 1-
 1B*CD1)*E2D-(2.*ST1-B*CT1)*E2T)      /(4.+B2)-2.*((2.*SD5-5.*B*CD5)*
 2E2D-(2.*ST5-5.*B*CT5)*E2T)/5./(4.+25.*B2))/B
  BB(5,7)=0.0
  BB(5,8)=0.0
  BB(5,9)=0.0
  BB(6,6)=BB(6,6)+BN7*(E2D*CSD3-E2T*CST3+1.5*B*((SD6-3.*B*CD6)*E2D-(
 1ST6-3.*B*CT6)*E2T)/(1.+9.*B2))
  BB(6,7)=0.0
  BB(6,8)=0.0
  BB(6,9)=0.0
  BB(7,7)=BB(7,7)+BN8*(E2D*CSD1-E2T*CST1+B*((SD2-B*CD2)*E2D-(ST2-B*C
 1T2)*E2T)/(2.+2.*B2))*RSS
  BB(7,8)=BB(7,8)+(BN8*RSS/B)*((SD1+SD3/3.)*E2D-(ST1+ST3/3.)*E2T-2.*
 1((2.*SD1-B*CD1)*E2D-(2.*ST1-B*CT1)*E2T)/(4.+B2)-2.*((2.*SD3-3.*B*C
 2D3)*E2D-(2.*ST3-3.*B*CT3)*E2T)/(12.+27.*B2))
  BB(7,9)=BB(7,9)+((BN8*RSS)/(2.*B))*((SD2+.5*SD4)*E2D-(ST2+.5*ST4)*
 1E2T-2.*((2.*SD2-2.*B*CD2)*E2D-(2.*ST2-2.*B*CT2)*E2T)/(4.+4.*B2)-2.
 2*((2.*SD4-4.*B*CD4)*E2D-(2.*ST4-4.*B*CT4)*E2T)/(8.+32.*B2))
  BB(8,8)=BB(8,8)+BN8*RSS*(E2D*CSD2-E2T*CST2+B*((SD4-2.*B*CD4)*E2D-(
 1ST4-2.*B*CT4)*E2T)/(1.+4.*B2))
  BB(8,9)=BB(8,9)+(BN8*RSS/B)*((SD1+SD5/5.)*E2D-(ST1+ST5/5.)*E2T-2.*
 1((2.*SD2-B*CD1)*E2D-(2.*ST1-B*CT1)*E2T)/(4.+B2)-2.*((2.*SD5-5.*B*C
```

```
   2D5)*E2D-(2.*ST5-5.*B*CT5)*E2T)/(20.+125.*B2))
999 BB(9,9)=BB(9,9)+BN8*RSS*(E2D*CSD3-E2T*CST3+1.5*B*((SD6-3.*B*CD6)*E
   12D-(ST6-3.*B*CT6)*E2T)/(1.+9.*B2))
  7 FORMAT (1X,6E20.8)
  8 FORMAT(E16.10,10X,1HA)
    PRINT 21
    DO 6 I=6,9
    PRINT 7,(AA(I,J),J=1,9)
  6 PUNCH 8,(AA(I,J),J=I,9)
    PRINT 21
 21 FORMAT(////1X)
    DO 22 I=1,9
    PRINT 7,(BB(I,J),J=1,9)
 22 PUNCH 18,(BB(I,J),J=I,9)
 18 FORMAT(E16.10,10X,1HB)
    PAUSE
    GO TO 1492
    END
```

APPENDIX B

Computer Program for the Evaluation of the Natural Frequencies

```
      DIMENSION A(9,9),B(9,9),C(9,9),IPVT(9)
   20 FORMAT(E16.10)
   40 DO 21 I=1,9
      DO 21 J=I,9
      READ 20,A(I,J)
      IF( ABSF(A(I,J))-.5E-8)22,21,21
   22 A(I,J)=0.
   21 A(J,I)=A(I,J)
   30 FORMAT(1H1,1X)
   31 FORMAT(///1X)
   32 FORMAT(1X,6E20.10)
      PRINT 30
      DO 33 I=1,9
   33 PRINT 32,(A(I,J),J=1,9)
      DO 23 I=1,9
      DO 23 J=I,9
      READ 20,B(I,J)
      IF( ABSF(B(I,J))-.5E-8)24,23,23
   24 B(I,J)=0.
   23 B(J,I)=B(I,J)
      PRINT 31
      DO 34 I=1,9
   34 PRINT 32,(B(I,J),J=1,9)
      NF=1
      PRINT 31
      U2=0.
   12 DELTA=.01
   42 DO 25 I=1,9
      DO 25 J=1,9
   25 C(I,J)=B(I,J)
      DO 26 I=1,3
      DO 26 J=1,3
   26 C(I,J)=B(I,J)-U2*A(I,J)
      DO 27I=4,6
      DO 27 J=4,6
   27 C(I,J)=B(I,J)-U2*A(I,J)
      DO 28 I=7,9
      DO 28 J=7,9
   28 C(I,J)=B(I,J)-U2*A(I,J)
      CALL DETER(C,9,IPVT,E1)
      DO 35 I=1,9
   35 E1=E1*C(I,I)
   16 U2=U2+DELTA
      DO 36 I=1,9
      DO 36 J=1,9 .
   36 C(I,J)=B(I,J)
      DO 37 I=4,6
```

```
      DO 37 J=4,6
   37 C(I,J)=B(I,J)-U2*A(I,J)
      DO 38 I=7,9
      DO 38 J=7,9
   38 C(I,J)=B(I,J)-U2*A(I,J)
      DO 39 I=1,3
      DO 39 J=1,3
   39 C(I,J)=B(I,J)-U2*A(I,J)
      CALL DETER(C,9,IPVT,E2)
      DO 5 I=1,9
    5 E2=E2*C(I,I)
      PRINT 32,E1,E2,U2
      IF(ABSF(E2)-.1E-20)7,8,8
    7 U=SQRTF(U2)
      PRINT 9,NF,U2,U,E1,E2,DELTA
    9 FORMAT(///5X,4HN.F.,I2//5X,15HOMEGA SQUARED =E16.8,5X,7HOMEGA =
     1E16.8//5X,4HE1 =E16.8,5X,4HE2 =E16.8//5X,7HDELTA =E16.8/1H1)
      IF(NF-1)10,11,11
   10 NF=NF+1
      DELTA=.05
      U2=U2+.01
      GO TO 42
   11 CONTINUE
      GO TO 40
    8 IF(E1*E2)15,7,14
   14  E1=E2
      GO TO 16
   15 IF(ABSF(DELTA)-.1E-5)7,17,17
   17 U2=U2-DELTA
      DELTA=DELTA/5.
      GO TO 16
      END
```

```
      SUBROUTINE DETER(A,N,IPVT,DET)
C     CROUT REDUCTION OF A MATRIX WITH PIVOTING
C     THIS SUBROUTINE TRANSFORMS THE MATRIX A INTO A LOWER TRIANGULAR
C     MATRIX L AND AN UPPER TRIANGULAR MATRIX U SUCH THAT A=LU.
C     FURTHER THE DIAGONAL ELEMENTS OF L EQUAL 1.
C     THE ROWS OF A ARE INTERCHANGED SO THAT ROUND-OFF ERRORS ARE
C     IN GENERAL MINIMIZED.
      DIMENSION A(9,9),IPVT(9)
      DET=1.0
      DO 1 K=1,N
      KP1=K+1
      KM1=K-1
C     COMPUTE U(K,K) AND THE ELEMENTS OF THE KTH COLUMN OF L
      DO 16 I=K,N
      T=A(I,K)
      IF(KM1)16,16,14
   14 DO 15 J=1,KM1
   15 T=T-A(I,J)*A(J,K)
   16 A(I,K)=T
C     SEARCH FOR LARGEST ELEMENT IN COLUMN K ON OR BELOW A(K,K)
      T=0.0
      DO 2 I=K,N
      S=ABSF(A(I,K))
      IF(T-S)3,2,2
C     IF THE ELEMENT A(I,K) IS LARGER THAN THE CURRENT MAXIMUM,
C     CHOOSE IT
    3 T=S
      IMAX=I
    2 CONTINUE
C     IF T=0.0, THE MATRIX IS SINGULAR
      IF(T)4,4,5
    4 DET=0.0
      RETURN
C     OTHERWISE RECORD THE THE LOCATION OF THE MAXIMUM AND INTERCHANGE
C     ROWS IF NECESSARY
    5 IPVT(K)=IMAX
      IF(IMAX-K)6,7,6
    6 DO 8 J=1,N
      T=A(IMAX,J)
      A(IMAX,J)=A(K,J)
    8 A(K,J)=T
      DET=-DET
C     DIVIDE THE KTH COLUMN OF L BY U(K,K)
    7 T=A(K,K)
      IF(KP1-N)12,12,1
   12 DO 13 I=KP1,N
   13 A(I,K)=A(I,K)/T
C     COMPUTE THE ELEMENTS OF THE KTH ROW OF U
      DO 9 J=KP1,N
```

```
      T=A(K,J)
      IF(KM1)9,9,10
   10 DO 11 I=1,KM1
   11 T=T-A(K,I)*A(I,J)
    9 A(K,J)=T
    1 CONTINUE
      RETURN
      END
```