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**NUMERICAL METHODS
FOR
ARCHITECTURAL ENGINEERS**

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INTRODUCTION

Engineers have for centuries used models to test their designs. Physical models have ranged in complexity from full-size wooden beams stacked with lead weights until they broke to intricate scale models fitted with strain gauges and observed under carefully controlled conditions of humidity, temperature, and vibration.

Building physical models, however, is only one way for an engineer to predict the performance of his design. More commonly in recent years, he will build a mathematical model. The advances made in engineering theory and application during the last two centuries have given engineers the ability to construct elaborate mathematical models of almost any design, and in the present generation unparalleled advances in computer technology have given him the ability to solve those models--to actually predict how a structure will perform under given conditions.

Theories and numbers are the materials from which an engineer constructs his models; numerical methods--the practical application of mathematics to engineering--are the tools he uses to put them together. The purpose of this text is to give the engineering student a set of tools with which he can shape the theories he has learned into appropriate mathematical models.

A set of tools becomes truly useful only through practice. To become pro-

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ficient at the craft of model building, an engineer must not only know the methods, he must practice them. Towards this end, this text acknowledges two fundamental steps in learning how to use numerical methods. The first step is to become completely familiar with the methods by making careful hand calculations. The second step is to put the individual methods together and test them on a computer. It is through this second step, through the ability to test many different kinds of models speedily and accurately, that one can see both the possibilities and limitations of mathematical modelling.

ROOTS OF EQUATIONS

An equation commonly found in engineering problems takes the form

$$f(x) = 0$$

where $f(x)$ is any combination of functions (powers, sines, cosines, reciprocals, logarithms, etc.) of the single variable x . The solution of the equation involves finding values of x for which the equation is true; these values of x are called roots of the equation.

Equations of the form shown above are commonly grouped into two broad categories: algebraic equations and transcendental equations. Algebraic equations are those that involve only powers of x . Transcendental equations are those that involve sines and cosines, e raised to the power x , logarithms--in short, any function that is not a power of x . Following are several examples of algebraic and transcendental equations.

ALGEBRAIC EQUATIONS

$$5x^3 - 2x^2 + 10 = 0$$

$$\sqrt{x} - 1/x = 0$$

$$x^{3.25} - \pi = 0$$

TRANSCENDENTAL EQUATIONS

$$\sin x + \cos x = 0$$

$$e^x - 10 = 0$$

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$$\ln(x^2 - 1/x) = 0$$

A particular group of algebraic equations--those involving only positive integer powers of x --are called polynomials. Polynomials, as discussed later, are convenient equations because their roots are generally easier to find than roots of other algebraic and transcendental equations. Moreover, the integration and differentiation of polynomials can be performed by anyone with a calculus book and a secondary education. In the examples of algebraic equations shown above, the first is a polynomial because it involves only positive integer powers of x . The second is not a polynomial because it contains a square root, which is x raised to the power $1/2$, and a reciprocal, which is x raised to a negative power. The third is not a polynomial because it contains x raised to a fractional power.

Methods for solving both algebraic and transcendental equations are presented in the following pages, with special attention given to the direct solution of some polynomials. The more general methods will work for almost any equation, and may even be easier than a direct solution in some cases.

POLYNOMIALS

Polynomials, as already noted, are equations that involve only positive integer powers of x . The order or degree of a polynomial is the highest power of x it contains. The following equation is an example of a fifth degree polynomial.

$$x^5 - 2x^3 + x^2 + 9 = 0$$

A polynomial will always have as many roots as it has degrees; a fifth degree polynomial will thus have five roots. These roots may be real or imaginary, and two or more may be equal.

Roots of second and third degree polynomials--quadratics and cubics--can be found directly using fairly simple procedures. Roots of higher degree polynomials can be found iteratively using Bairstow's method.

Polynomials may also be solved using the procedures presented in the following chapter on transcendental equations; in many cases these techniques may be simpler and easier than a direct solution or Bairstow's method. Or one may choose to use trial and error. The only general rule in equation solving is that there is no general rule.

DESCARTES' RULE OF SIGNS

Descartes' Rule of Signs can be used to predict the number of positive and negative roots a polynomial will have. This can be useful when one is only interested in positive roots (a common restriction in engineering problems) or when one employs an iterative solution and needs to know how many roots are left to be found.

Consider a polynomial in its general form:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_1 x + a_0 = 0$$

The number of positive real roots is equal to the number of times the equation changes signs when $x = 1$ (or fewer by an even number). The number of negative real roots is equal to the number of times the equation changes signs when $x = -1$ (or fewer by an even number).

EXAMPLE

$$f(x) = x^5 + 2x^4 - x^3/3 - 5x^2 + 1$$

Let $x = 1$:

$$f(1) = 1 + 2 - 1/3 - 5 + 1$$

There are two sign changes (after the 2 and after the 5). There must be either two or zero positive real roots.

Let $x = -1$

$$f(-1) = -1 + 2 - 1/3 - 5 + 1$$

There are three sign changes (after the -1, after the 1/3 and after the 5). There must be either three or one negative real roots.

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THE QUADRATIC FORMULA

The quadratic formula is a direct solution for any quadratic equation. It always gives two roots, which may be both real, both imaginary, or equal. A quadratic equation is a polynomial whose highest power of x is x^2 :

$$f(x) = ax^2 + bx + c = 0$$

The roots of this equation are given by the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

EXAMPLE

$$3x^2 - 5x + 2 = 0$$

$$x = \frac{5 \pm \sqrt{25 - 24}}{6}$$
$$= 1 \text{ or } 2/3$$

EXAMPLE

$$2x^2 - 6x + 5 = 0$$

$$x = \frac{6 \pm \sqrt{36 - 40}}{4}$$
$$= 3/2 \pm (1/2)i$$

where i is the square root of -1

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ROOTS OF A CUBIC EQUATION

The roots of a cubic equation, both real and imaginary, may be found by the following method. If the third-order term, A_3 , is not one, the equation should first be divided by A_3 . Consider the following general form of a cubic equation:

$$f(x) = x^3 + A_2x^2 + A_1x + A_0 = 0$$

The second-order term, A_2x^2 , can be removed by the substitution

$$y = x + A_2/3$$

which gives the equivalent equation

$$f(y) = f(x) = y^3 + B_1y + B_0 = 0$$

in which

$$B_0 = 2A_2^3/27 - A_1A_2/3 + A_0$$

$$B_1 = A_1 - A_2^2/3$$

Then let

$$C = -B_0/2$$

$$D = C^2 + B_1^3/27$$

and

$$P = (C + \sqrt{D})^{1/3}$$

$$Q = (C - \sqrt{D})^{1/3}$$

If D is a positive number, there will be one real root and two complex roots (y_1 , y_2 and y_3), given by the following equations:

$$y_1 = P + Q$$

$$y_2 = -(P + Q)/2 + i(P - Q)(\sqrt{3}/2)$$

$$y_3 = -(P + Q)/2 - i(P - Q)(\sqrt{3}/2)$$

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If D is negative, there will be three real roots. While the equations listed above can still be used, the following trigonometric solution, derived from the three equations shown above, avoids a trip into the land of complex numbers:

$$r = (C^2 - D)$$

$$\theta_1 = (\tan^{-1}(\sqrt{-D}/C))$$

$$\theta_2 = \theta_1 + 2\pi \quad \text{or } \theta_2 = \theta_1 + 360^\circ$$

$$\theta_3 = \theta_2 + 2\pi \quad \text{or } \theta_3 = \theta_2 + 360^\circ$$

$$y_1 = 2r^{1/3}\cos(\theta_1/3)$$

$$y_2 = 2r^{1/3}\cos(\theta_2/3)$$

$$y_3 = 2r^{1/3}\cos(\theta_3/3)$$

Once y has been found by either of the above methods, x must still be found by reversing the original substitution:

$$x_1 = y_1 - A_2/3$$

$$x_2 = y_2 - A_2/3$$

$$x_3 = y_3 - A_2/3$$

These are the three roots of the original cubic equation.

EXAMPLE

$$x^3 - 5x^2 - 50x + 5000 = 0$$

$$A_0 = 5000$$

$$A_1 = -50$$

$$A_2 = -5$$

$$B_0 = 2(-5)^3/27 - (-50)(-5)/3 + 5000 = 4907.407407$$

$$B_1 = -50 - (-5)^2/3 = -58.33333333$$

$$C = -4907.407407/2 = -2453.703704$$

$$D = (-2453.703704)^2 + (-58.33333333^3/27) = 6013310.187$$

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$$P = (-2453.703704 + 6013310.187)^{1/3} = -1.144341708$$

$$Q = (-2453.703704 - 6013310.187)^{1/3} = -16.99181713$$

$$y_1 = -1.144341708 - 16.99181713 = -18.13615884$$

$$y_2 = -(-1.44341708 - 16.99181713)/2 + i(-1.44341708 - 16.99181713)(\sqrt{3}/2) \\ = 9.068079420 + 13.72431631(i)$$

$$y_3 = -(-1.44341708 - 16.99181713)/2 - i(-1.44341708 - 16.99181713)(\sqrt{3}/2) \\ = 9.068079420 - 13.72431631(i)$$

$$x_1 = y_1 + 5/3 = -16.46949217$$

$$x_2 = y_2 + 5/3 = 10.73474609 + 13.72431631(i)$$

$$x_3 = y_3 + 5/3 = 10.73474609 - 13.72431631(i)$$

EXAMPLE

$$x^3 - 40x^2 - 6000x + 40000 = 0$$

$$A_0 = 40000$$

$$A_1 = -6000$$

$$A_2 = -40$$

$$B_0 = 2(-40)^3/27 - (-6000)(-40)/3 + 40000 = 44740.74074$$

$$B_1 = -6000 - (-40)^2/3 = -6533.333333$$

$$C = -44740.74074/2 = -22370.37037$$

$$D = (-22370.37037)^2 + (-6533.333333^3/27) = -9828148150$$

$$r = \sqrt{(-22370.37037)^2 + 9828148150} = 101629.6296$$

$$\theta_1 = (\tan^{-1}(9828148150/-22370.37037)) = -1.348862308 \text{ radians}$$

$$\theta_2 = -1.348862308 + 2\pi = 4.934323000 \text{ radians}$$

$$\theta_3 = 4.934323000 + 2\pi = 11.21750831 \text{ radians}$$

$$y_1 = 2(101629.6296)^{1/3} \cos(-1.348862308/3) = 84.05711901$$

$$y_2 = 2(101629.6296)^{1/3} \cos(4.934323/3) = -6.898317766$$

$$y_3 = 2(101629.6296)^{1/3} \cos(11.217508831/3) = -77.15880115$$

$$x_1 = y_1 + 40/3 = 97.39045234$$

$$x_2 = y_2 + 40/3 = 6.435015564$$

$$x_3 = y_3 + 40/3 = -63.82546782$$

Bairstow's method finds roots of a polynomial two-at-a-time, reducing the order of the polynomial by two each time a pair of roots is extracted.

Consider the general polynomial:

$$f(x) = x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0 = 0$$

This polynomial may be reduced by synthetic division to:

$$f(x) = (x^2 + ux + v)(x^{n-1} + b_{n-2}x^{n-2} + \dots + b_1x + b_0) + R = 0$$

where R is simply a remainder. An initial guess for u and v can be improved by iteration until R approaches zero. When R is zero, the quadratic term $(x^2 + ux + v)$ contains two of the polynomial's roots, which can be found by using the quadratic formula.

The coefficients for the reduced polynomial are defined by u, v and the original coefficients:

$$b_i = a_{i+2} - b_{i+1}u - b_{i+2}v$$

$$b_{n-1} = 1$$

$$b_{n-2} = 0$$

A Taylor series provides the means for finding u and v by iteration:

$$b_i(u + \Delta u, v + \Delta v) = (\partial b_i / \partial u)\Delta u + (\partial b_i / \partial v)\Delta v$$

Thus Δu and Δv are found from previously known or guessed values of u and v, added to u and v, and used to calculate a new Δu and Δv . As Δu and Δv approach zero, u and v approach the values needed to calculate the roots. Initial values for u and v are usually taken as zero, but other values may

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be tried if the solution does not converge. Once two roots are found, the polynomial is reduced by two degrees using synthetic division.

While Bairstow's method is long and cumbersome for hand calculation, it is ideally suited for a computer program. An outline of the procedure for either hand or machine calculation is shown below.

PROCEDURE

Given $f(x) = x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0 = 0$

1. Choose starting values for u and v, usually zero
2. Repeat the following sequence until Δu and Δv approach zero:
 - a) Calculate b_i for $i = n - 3$ to $i = -2$ (descending order)

$$b_{n-3} = a_{n-1} - u$$

$$b_{n-4} = a_{n-2} - b_{n-3}u - v$$

$$b_{n-5} = a_{n-3} - b_{n-4}u - b_{n-3}v$$

...

$$b_i = a_{i+2} - b_{i+1}u - b_{i+2}v$$

...

$$b_0 = a_2 - b_1u - b_2v$$

$$b_{-1} = a_1 - b_0u - b_1v$$

$$b_{-2} = a_0 - b_{-1}u - b_0v$$

- b) Calculate c_i for $i = n - 3$ to $i = -1$

$$c_{n-3} = b_{n-3} - u$$

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$$\begin{aligned}
c_{n-4} &= b_{n-4} - c_{n-3}u - v \\
c_{n-5} &= b_{n-5} - c_{n-4}u - c_{n-3}v \\
&\dots \\
c_i &= b_i - c_{i+1}u - c_{i+2}v \\
&\dots \\
c_0 &= b_0 - c_1u - c_2v \\
c_{-1} &= b_{-1} - c_0u - c_1v
\end{aligned}$$

c) Calculate Δu and Δv :

$$\begin{aligned}
\Delta u &= \frac{\det \begin{vmatrix} b_{-2} & c_0 \\ b_{-1} & c_1 \end{vmatrix}}{\det \begin{vmatrix} c_{-1} & c_0 \\ c_0 & c_1 \end{vmatrix}} \\
\Delta v &= \frac{\det \begin{vmatrix} c_{-1} & b_{-2} \\ c_0 & b_{-1} \end{vmatrix}}{\det \begin{vmatrix} c_{-1} & c_0 \\ c_0 & c_1 \end{vmatrix}}
\end{aligned}$$

d) Calculate new values for u and v :

$$\begin{aligned}
u &= u + \Delta u \\
v &= v + \Delta v
\end{aligned}$$

3. Now find two roots by the quadratic formula:

$$x = \frac{-u \pm \sqrt{u^2 - 4v}}{2}$$

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4. Formulate a reduced polynomial, using the latest values found for b_{n-3} to b_0 as coefficients. Remember that n is now reduced by 2.

$$\begin{aligned}
n &= n - 2 \\
f(x) &= x^n + b_{n-1}x^{n-1} + \dots + b_1x + b_0 = 0 \\
b_{n-1} &= \text{old } b_{n-3} \\
b_{n-2} &= \text{old } b_{n-4} \text{ etc.}
\end{aligned}$$

Repeat these four steps until all roots of the polynomial are found.

SYNTHETIC DIVISION

Synthetic division of polynomials can accomplish several purposes. It can divide a polynomial by a linear factor, it can evaluate a polynomial at a given value of x , and it can find any derivative of the polynomial at a given value of x . Consider the general polynomial:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

This polynomial can be divided by the linear factor $x - r$ as shown:

r	$+ \begin{matrix} a_n \\ 0_n \end{matrix}$	$+ \begin{matrix} a_{n-1} \\ r b_{n-1} \end{matrix}$	\dots	$+ \begin{matrix} a_1 \\ r b_1 \end{matrix}$	$+ \begin{matrix} a_0 \\ r b_0 \end{matrix}$
	b_{n-1}	b_{n-2}	\dots	b_0	R

where $b_i = a_{i+1} + r b_{i+1}$

$$b_n = 0$$

R is the remainder and b_i is the coefficient of x to the i power in a new polynomial of one lower degree:

$$g(x) = b_{n-1} x^{n-1} + \dots + b_1 x + b_0$$

$$f(x) = (x-r)g(x) + R$$

After one synthetic division by $x - r$, the remainder R is the value of $f(x)$ at $x = r$.

After two synthetic divisions by $x - r$, the remainder R is the value of the first derivative of $f(x)$ at $x = r$.

Three synthetic divisions give the second derivative, four synthetic divisions give the third derivative, and so on.

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EXAMPLE

Divide $x^5 - 2x^4 - 4x^3 - 8x + 10$ by $x - 4$

Add the columns below, starting at the left. Multiply each result by 4 and add to the next column:

4	$+ 0$	$+ 4$	$+ 8$	$+ 16$	$+ 64$	$+ 224$
	1	2	4	16	56	234

The resulting polynomial is

$$(x - 4)(x^4 + 2x^3 + 4x^2 + 16x + 56) + 234$$

234 is the value of the function at $x = 4$

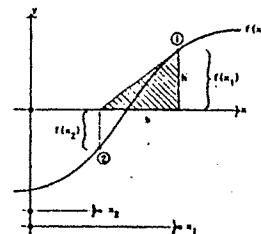
Transcendental equations involve non-algebraic functions such as trigonometric, logarithmic, and exponential functions. Algebraic functions involving roots and inverses (x or $1/x$, for example), while technically not transcendental, are best solved by the same methods as transcendentals.

Roots of transcendentals are usually more elusive than roots of polynomials. The first step in the search for a root of a transcendental is to plot the equation so that the general location of the roots can be seen. Once located on a plot, the roots can be calculated more exactly using one of the two iterative methods presented here. If the derivative of the function is known, Newton's Rule of Tangents usually converges more rapidly; if not, the secant method is simpler, if slower.

NEWTON'S RULE OF TANGENTS

The derivative of a function at any point is the slope of a line tangent to the graph of the function at that point. By knowing the derivative, one can approximate where the function will intersect the x-axis, and thus estimate the location of a root. By using this method again and again, one can increase the accuracy of the approximation as much as one desires.

In the drawing of a function, $f(x)$, at right, both the value of the function, $f(x_1)$, and the derivative of the function, $f'(x_1)$, are known for some arbitrary point x_1 . The derivative gives the ratio of the height, h , to the base, b , of the shaded triangle. By subtracting b from x_1 , one can find a new point, x_2 , that is usually closer to the root of the equation. A simple formula for this iterative procedure is derived in the following manner:



$$\begin{aligned}
 f(x_1) &= h \\
 f'(x_1) &= h/b \\
 x_2 &= x_1 - b \\
 &= x_1 - h/f'(x_1) \\
 &= x_1 - f(x_1)/f'(x_1)
 \end{aligned}$$

As the iteration is continued to point x_3 , x_4 , and beyond, each approximation becomes closer to the actual root:

$$x_{n+1} = x_n - f(x_n)/f'(x_n)$$

There are, of course, equations for which Newton's Rule of Tangents will not converge, and first guesses that will give a derivative of zero or send one looking far away. Since there is no set of rules governing when this will happen, it is usually best to plot the equation so that a good first guess

can be made.

EXAMPLE

Find the root of $e^{x/3} - 50 = 0$

$$f(x) = e^{x/3} - 50$$

$$f'(x) = (1/3)e^{x/3}$$

Initial guess: $x_1 = 12$

$$x_1 = 12$$

$$f(x_1) = 4.589$$

$$f'(x_1) = 18.199$$

$$x_2 = 12 - 4.589/18.199 = 11.747$$

$$f(x_2) = 0.188$$

$$f'(x_2) = 16.729$$

$$x_3 = 11.747 - 0.188/16.729 = 11.736$$

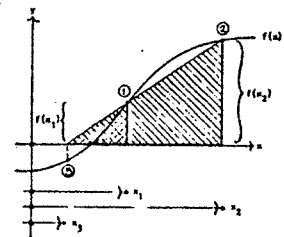
$$f(x_3) = 0.0003 = 0$$

$$x = 11.736$$

THE SECANT METHOD

Given two points on a curve near a root, a line can be drawn connecting the two points that will intersect the x-axis near that root (unless the line happens to be horizontal). This method of connecting points on a curve can be used to iterate towards the exact location of a root.

On the graph at right, a line has been drawn through two arbitrarily chosen points, creating two similar triangles (shown by shading). Since the two triangles are similar, the ratio of the base to the height of each is equal to that of the other, and the point of intersection with the x-axis can be found by combining two equations:



$$h_1 = f(x_1)$$

$$h_2 = f(x_2)$$

$$b_1 = x_1 - x_3$$

$$b_2 = x_2 - x_3$$

$$b_1/h_1 = b_2/h_2$$

$$\frac{x_1 - x_3}{h_1} = \frac{x_2 - x_3}{h_2}$$

$$x_3 = \frac{h_2 x_1 - h_1 x_2}{h_2 - h_1} = \frac{f(x_2)x_1 - f(x_1)x_2}{f(x_2) - f(x_1)}$$

Continued iteration with this equation, always using the two most recently found points, will usually converge upon the root:

$$x_{n+1} = \frac{f(x_n)x_{n-1} - f(x_{n-1})x_n}{f(x_n) - f(x_{n-1})}$$

EXAMPLE :-

Find the root of $e^{x/3} - 50 = 0$

$$f(x) = e^{x/3} - 50$$

Initial guesses: $x_1 = 10$

$$x_2 = 12$$

$$x_1 = 10$$

$$f(x_1) = -21.968$$

$$x_2 = 12$$

$$f(x_2) = 4.598$$

$$x_3 = \frac{(4.598)(10) - (-21.968)(12)}{4.598 + 21.968} = 11.654$$

$$f(x_3) = -1.352$$

$$x_4 = \frac{(-1.352)(12) - (4.598)(11.654)}{-1.352 - 4.598} = 11.732$$

$$f(x_4) = -0.0596$$

$$x_5 = \frac{(-0.0596)(11.654) - (-1.352)(11.732)}{-0.0596 + 1.352} = 11.736$$

$$f(x_5) = 0.0008 = 0$$

$$x = 11.736$$

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MATRICES

Analysis of structures can sometimes require the solution of large numbers of simultaneous equations. Matrix formulation is a means of organizing these equations into blocks of numbers that can be solved by orderly procedures; matrix algebra is essentially a system of bookkeeping.

A set of three simultaneous equations

$$2x + 3y + 4z = 20$$

$$2x - 2y + 5z = 13$$

$$3x - y + 3z = 10$$

may be rewritten in matrix form as

$$\begin{bmatrix} 2 & 3 & 4 \\ 2 & -2 & 5 \\ 3 & -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 20 \\ 13 \\ 10 \end{bmatrix}$$

and solved by one of many available elimination techniques.

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NOTATION :-

- [A], [B], [C] ... denote matrices
- {a}, {b}, {c} ... denote single rows or columns (vectors)
- a_{ij} , b_{ij} , c_{ij} ... denote scalars or individual elements of a matrix
- $[A]^T$ is the transpose of [A]
- $[A]^{-1}$ is the inverse of [A]
- $\det[A]$ is the determinant of [A]
- $[A]_{m \times n}$ is a matrix of m rows and n columns

MATRIX ALGEBRA

MULTIPLICATION

$$[A]_{1 \times m} \times [B]_{m \times n} = [C]_{1 \times n}$$

each element in the resulting matrix, [C], is the sum of row and column products from the first two matrices, as shown in the following equation:

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \dots + a_{im}b_{mj}$$

The associative and distributive properties hold as for scalars:

$$\begin{aligned} ([A] \times [B]) \times [C] &= [A] \times ([B] \times [C]) \\ ([A] + [B]) \times [C] &= [A][C] + [B][C] \\ [C] \times ([A] + [B]) &= [C][A] + [C][B] \end{aligned}$$

The commutative property does not apply to matrices:

$$[A] \times [B] \neq [B] \times [A]$$

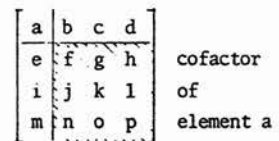
The two sides of the above equation are not equal.

DETERMINANTS OF SQUARE MATRICES

$$\det[A]_{2 \times 2} = a_{11}a_{22} - a_{12}a_{21}$$

$$\begin{aligned} \det[A]_{3 \times 3} &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) \\ &\quad - a_{12}(a_{21}a_{33} - a_{23}a_{31}) \\ &\quad + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \end{aligned}$$

For larger matrices, the determinant is the sum of each top row element times the determinant of its cofactor (negative for every other element in the top row: a_{12} , a_{14} , a_{16} , etc.). The cofactor of element a_{ij} is the matrix that remains when row i and column j are removed (see diagram).



Thus, the determinant of the matrix shown at right is

$$(a)\det \begin{vmatrix} f & g & h \\ j & k & l \\ n & o & p \end{vmatrix} - (b)\det \begin{vmatrix} e & g & h \\ i & k & l \\ m & o & p \end{vmatrix} + (c)\det \begin{vmatrix} e & f & h \\ i & k & l \\ m & n & p \end{vmatrix} - (d)\det \begin{vmatrix} e & f & g \\ i & j & k \\ m & n & o \end{vmatrix}$$

If a pair of rows or columns is exchanged, the determinant will change signs.

If two or more of the equations differ by only a constant, the matrix is singular, i. e., the determinant is zero.

$$\det(a[A]_{n \times n}) = a^n \det[A]$$

$$\det([A]^T[A]) = \det([A][A]^T) = (\det[A])^2$$

TRANSPOSES

The transpose of a square matrix is that matrix flipped about its major diagonal:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}^T = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$$

The following three properties of transposes may be useful:

$$\begin{aligned}
 [A]^T [B]^T &= ([B][A])^T \\
 [A]^T + [B]^T &= ([A] + [B])^T \\
 [A][B] &= ([B][A])^T \text{ only if } [A] \text{ and } [B] \text{ are symmetrical}
 \end{aligned}$$

INVERSES

The inverse of a square matrix is that matrix which, when multiplied by the original matrix, produces the identity matrix:

$$[A]_{3 \times 3} \quad [A]^{-1}_{3 \times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The inverse has the following properties:

$$\begin{aligned}
 [A]^{-1} [B]^{-1} &= ([B][A])^{-1} \\
 ([A]^{-1})^T &= ([A]^T)^{-1} \\
 [A]^{-1} [A] &= [A][A]^{-1} = [I] \quad [I] \text{ is the identity matrix} \\
 (a[A])^{-1} &= (1/a)[A]^{-1} \\
 \det([A]^{-1}) &= 1/\det[A] \\
 \text{if } [A][B] &= [C] \text{ then } [B] = [A]^{-1}[C]
 \end{aligned}$$

PARTITIONING

Partitioning is useful in computer applications where matrix size exceeds available memory size. [A] and [B] are matrices. [A1] to [A4] and [B1] to [B4] are partitions of [A] and [B]. The procedure for determining the inverse is as follows:

$$[A] = \begin{bmatrix} A1 & A2 \\ A3 & A4 \end{bmatrix} \quad [B] = [A]^{-1} = \begin{bmatrix} B1 & B2 \\ B3 & B4 \end{bmatrix}$$

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$$\begin{aligned}
 [B1] &= ([A1] - [A2][A4]^{-1}[A3])^{-1} \\
 [B2] &= -([A1]^{-1}[A2][B4]) \\
 [B3] &= -([A4]^{-1}[A3][B1]) \\
 [B4] &= ([A4] - [A3][A1]^{-1}[A2])^{-1} \\
 [A1] \text{ and } [A4] &\text{ are square matrices} \\
 \det[A] &= \det[A4]\det([A1] - [A2][A4]^{-1}[A3]) \\
 &= \det[A4]/\det[B1] \\
 &= \det[A1]\det([A4] - [A3][A1]^{-1}[A2]) \\
 &= \det[A1]/\det[B4]
 \end{aligned}$$

AUGMENTED MATRICES

In most reduction schemes, entire rows are operated upon to eliminate numbers. Since both the coefficient matrix [A] and the constant vector {C} are operated upon simultaneously, it is convenient to combine them into a single augmented matrix.

Given the set of simultaneous equations

$$[A]\{B\} = \{C\}$$

the augmented matrix is formed by combining [A] and {C}:

$$[A]_{\text{aug}} = [A|C]$$

It has one more column than the original matrix [A].

GAUSSIAN ELIMINATION

Gaussian elimination, the simplest of all reduction schemes, adds rows together to eliminate all values below the diagonal. Unknowns can then be found one at a time by back substitution. The procedure is described below:

Given the set of equations

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{bmatrix}$$

Create the augmented matrix

$$\left[\begin{array}{cccccc} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & c_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & c_2 \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & c_3 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} & c_n \end{array} \right]$$

Using row one as an operator:

multiply row one by a_{21}/a_{11} and subtract from the second row; a_{21} will now equal zero.

multiply row one by a_{31}/a_{11} and subtract from the third row; a_{31} will now equal zero.

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continue to multiply row one by a_{i1}/a_{11} , adding it to each row i until the first column contains only zeroes below a_{11} .

Using row two as an operator:

multiply row two by a_{32}/a_{22} and subtract from the third row; a_{32} will now equal zero.

continue to multiply row two by a_{i2}/a_{22} , adding it to each row i until the second column contains only zeroes below a_{22} .

Continue to use each row as an operator for the rows following it until the matrix is an upper triangular:

$$\left[\begin{array}{cccccc} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & c_1 \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} & c_2 \\ 0 & 0 & a_{33} & \cdots & a_{3n} & c_3 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} & c_n \end{array} \right]$$

Starting with the bottom row, solve the equations one at a time. Since the matrix is an upper triangular, each solution will involve only one unknown:

$$x_n = c_n/a_{nn}$$

$$x_{n-1} = (c_{n-1} - a_{n-1,n}x_n)/a_{n-1,n-1}$$

$$x_i = (c_i - \sum_{j=i+1}^n a_{ij}x_j)/a_{ii}$$

EXAMPLE

$$\begin{bmatrix} 2 & 3 & 4 \\ 2 & -2 & 5 \\ 3 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 20 \\ 13 \\ 10 \end{bmatrix}$$

Form the augmented matrix

$$\begin{bmatrix} 2 & 3 & 4 & 20 \\ 2 & -2 & 5 & 13 \\ 3 & -1 & 3 & 10 \end{bmatrix}$$

Multiply row one by 2/2 and subtract from row two; multiply row one by 3/2 and subtract from row three

$$\begin{bmatrix} 2 & 3 & 4 & 20 \\ 0 & -5 & 1 & -7 \\ 0 & -5.5 & -3 & -20 \end{bmatrix}$$

Multiply row two by -5.5/-5 and subtract from row three

$$\begin{bmatrix} 2 & 3 & 4 & 20 \\ 0 & -5 & 1 & -7 \\ 0 & 0 & -4.1 & -12.3 \end{bmatrix}$$

Back substitute

$$x_3 = -12.3/-4.1 = 3$$

$$x_2 = (-7 - 1x_3)/-5 = 2$$

$$x_1 = (20 - 4x_3 - 3x_2)/2 = 1$$

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GAUSS-JORDAN ELIMINATION

This variation of Gaussian elimination reduces the coefficient matrix not to an upper triangular but to an identity matrix. The augmented matrix thus contains the solution, and no back substitution is necessary.

Using row one as an operator:

divide row one by a_{11} .

multiply row one by a_{21} and subtract from row two; a_{21} will now equal zero.

multiply row one by a_{31} and subtract from row three; a_{31} will now equal zero.

continue this process for each row until the first column has only zeroes and $a_{11} = 1$.

Using row two as an operator:

divide row two by a_{22} .

multiply row two by a_{12} and subtract from row one; a_{12} will now equal zero.

multiply row two by a_{32} (skip a_{22} , since it is part of the operator row) and subtract from row three; a_{32} will now equal zero.

continue this process for each row following until the second column has only zeroes and $a_{22} = 1$.

Continue to use each row as an operator for the rows above and below it until the matrix is an identity matrix with the constant vector on the right side. The constant terms, c_1 to c_n , are now equal to the solution, x_1 to x_n .

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EXAMPLE 4

The augmented matrix is the same as in the previous example:

$$\left[\begin{array}{ccc|c} 2 & 3 & 4 & 20 \\ 2 & -2 & 5 & 13 \\ 3 & -1 & 3 & 10 \end{array} \right]$$

Divide row one by 2

$$\left[\begin{array}{ccc|c} 1 & 1.5 & 2 & 10 \\ 2 & -2 & 5 & 13 \\ 3 & -1 & 3 & 10 \end{array} \right]$$

Multiply row one by 2 and subtract from row two; multiply row one by 3 and subtract from row three

$$\left[\begin{array}{ccc|c} 1 & 1.5 & 2 & 10 \\ 0 & -5 & 1 & -7 \\ 0 & -5.5 & -3 & -20 \end{array} \right]$$

Divide row two by -5

$$\left[\begin{array}{ccc|c} 1 & 1.5 & 2 & 10 \\ 0 & 1 & -0.2 & 1.4 \\ 0 & -5.5 & -3 & -20 \end{array} \right]$$

Multiply row two by 1.5 and subtract from row one; multiply row two by -5.5 and subtract from row three

$$\left[\begin{array}{ccc|c} 1 & 0 & 2.3 & 7.9 \\ 0 & 1 & -0.2 & 1.4 \\ 0 & 0 & -4.1 & -12.3 \end{array} \right]$$

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Divide row three by -4.1

$$\left[\begin{array}{ccc|c} 1 & 0 & 2.3 & 7.9 \\ 0 & 1 & -0.2 & 1.4 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

Multiply row three by 2.3 and subtract from row one; multiply row three by -0.2 and subtract from row two

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

The constant vector contains the solution:

$$x_1 = 1$$

$$x_2 = 2$$

$$x_3 = 3$$

Cholesky's method decomposes the augmented matrix into an upper triangular matrix and a lower triangular matrix; the product of the two is the original augmented matrix. The unknowns are found by back substituting into the upper triangular matrix.

The procedure, which involves operating on the pivotal elements, rows, and columns, is presented in symbolic form:

PROCEDURE

Divide row one by a_{11} .

Subtract $a_{21}a_{12}$ from a_{22} .

For each element to the right of the diagonal in row two:

$$a_{2i} = (1/a_{22})(a_{2i} - a_{21}a_{1i}) \quad \text{for } i = 3 \text{ to } n+1$$

For each element below the diagonal in column two:

$$a_{i2} = a_{i2} - a_{i1}a_{12} \quad \text{for } i = 3 \text{ to } n$$

For each subsequent pivotal element, row, and column: $i = 3$ to n

$$a_{ii} = a_{ii} - \sum_{j=1}^{i-1} (a_{ij}a_{ji})$$

$$a_{ij} = (1/a_{ii})(a_{ij} - \sum_{k=1}^{i-1} (a_{ik}a_{kj})) \quad \text{for } j = i+1 \text{ to } n+1$$

$$a_{ji} = a_{ji} - \sum_{k=1}^{i-1} (a_{jk}a_{ki}) \quad \text{for } j = i+1 \text{ to } n$$

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Set all values on the diagonal to one, set all values below the diagonal to zero, and back substitute.

EXAMPLE

The augmented matrix is from the previous two examples:

$$\left[\begin{array}{ccc|c} 2 & 3 & 4 & 20 \\ 2 & -2 & 5 & 13 \\ 3 & -1 & 3 & 10 \end{array} \right]$$

Divide row one by 2

$$\left[\begin{array}{ccc|c} 1 & 1.5 & 2 & 10 \\ 2 & -2 & 5 & 13 \\ 3 & -1 & 3 & 10 \end{array} \right]$$

Diagonal: $a_{22} = -2 - (2 \times 1.5) = -5$

Row two: $a_{23} = (1/-5)(5 - (2 \times 2)) = -0.2$

$a_{24} = (1/-5)(13 - (2 \times 10)) = 1.4$

Column two: $a_{32} = -1 - (3 \times 1.5) = -5.5$

$$\left[\begin{array}{ccc|c} 1 & 1.5 & 2 & 10 \\ 2 & -5 & -0.2 & 1.4 \\ 3 & -5.5 & 3 & 10 \end{array} \right]$$

Diagonal: $a_{33} = 3 - (3 \times 2) - (5.5 \times 0.2) = -4.1$

Row three: $a_{34} = (1/-4.1)(10 - (3 \times 10) - (-5.5 \times 1.4)) = 3$

$$\begin{bmatrix} 1 & 1.5 & 2 & 10 \\ 2 & -5 & -0.2 & 1.4 \\ 3 & -5.5 & -4.1 & 3 \end{bmatrix}$$

Set values on diagonal to one and values below diagonal to zero

$$\begin{bmatrix} 1 & 1.5 & 2 & 10 \\ 0 & 1 & -0.2 & 1.4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Back substitution

$$x_3 = 3$$

$$x_2 = 1.4 - (-0.2)x_3 = 2$$

$$x_1 = 10 - 1.5x_2 - 2x_3 = 1$$

PIVOTING AND PARTIAL PIVOTING

In elimination schemes, the row currently being subtracted from other rows is called the pivot row, and the column of coefficients being eliminated is called the pivot column. The pivot row and the pivot column intersect at the pivot element (see sketch).

For the sake of accuracy, the pivot element should not be too small relative to the other numbers in the matrix. Pivoting and partial pivoting are sequences of row and column interchanges that increase accuracy.

Partial pivoting only exchanges rows. Each time a new pivot element is needed, the pivot column is scanned below the pivot row for the element with the largest absolute value. Two rows are exchanged so that the largest element found becomes the new pivot element. Since only rows are exchanged, partial pivoting does not affect the order of the unknowns, x_1 to x_n .

Complete pivoting exchanges both rows below the pivot row and columns to the right of the pivot column so that the largest element left in the matrix is always the pivot element. Complete pivoting requires complicated bookkeeping because the unknowns must be reordered each time two columns are exchanged.

No direct methods for solving nonlinear simultaneous equations exist, as they do for the linear sets of equations discussed previously. The following iterative method, based upon a Taylor series expansion, can be used to approximate solutions for nonlinear sets of equations with one or two variables.

Given two nonlinear equations in two variables

$$f(x,y)$$

$$g(x,y)$$

The Taylor series expansion about point i is

$$f(x_{i+h}, y_{i+j}) = f(x_i, y_i) + h \frac{\partial f}{\partial x_i} + j \frac{\partial f}{\partial y_i} = 0$$

$$g(x_{i+h}, y_{i+j}) = g(x_i, y_i) + h \frac{\partial g}{\partial x_i} + j \frac{\partial g}{\partial y_i} = 0$$

Iterating these two equations until h and j approach zero will give an approximate solution.

EXAMPLE

Given the following two nonlinear equations in x and y

$$f(x,y) = x^2 + y^2 - 10 = 0$$

$$g(x,y) = -3x^2 + 2xy - y = 0$$

The partial derivatives are

$$\frac{\partial f}{\partial x} = 2x$$

$$\frac{\partial f}{\partial y} = 2y$$

$$\frac{\partial g}{\partial x} = -6x + 2y$$

$$\frac{\partial g}{\partial y} = 2x - 1$$

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Initial guess:

$$x = 2$$

$$y = 3$$

$$f(2,3) = 3$$

$$g(2,3) = -3$$

$$\frac{\partial f}{\partial x} = 4$$

$$\frac{\partial f}{\partial y} = 6$$

$$\frac{\partial g}{\partial x} = -6$$

$$\frac{\partial g}{\partial y} = 3$$

$$3 + 4h + 6j = 0$$

$$-3 - 6h + 3j = 0$$

$$h = -0.5625$$

$$j = -0.125$$

New values:

$$x = 2 - 0.5625 = 1.4375$$

$$y = 3 - 0.125 = 2.875$$

$$f(x,y) = 0.332$$

$$g(x,y) = -0.8086$$

$$\frac{\partial f}{\partial x} = 2.875$$

$$\frac{\partial f}{\partial y} = 5.75$$

$$\frac{\partial g}{\partial x} = -2.875$$

$$\frac{\partial g}{\partial y} = 1.875$$

$$0.332 + 2.875h + 5.75j = 0$$

$$-0.8086 - 2.875h + 1.875j = 0$$

$$h = -0.24$$

$$j = 0.0625$$

New values:

$$\begin{aligned}x &= 1.4375 - 0.24 = 1.1975 \\y &= 2.875 + 0.0625 = 2.9375 \\f(x,y) &= 0.0629 \\g(x,y) &= -0.2042 \\\partial f/\partial x &= 2.395 \\\partial f/\partial y &= 5.875 \\\partial g/\partial x &= -1.31 \\\partial g/\partial y &= 1.395 \\0.0629 + 2.395h + 5.875j &= 0 \\-0.2042 - 1.31h + 1.395j &= 0 \\h &= -0.1166 \\j &= 0.0368\end{aligned}$$

The approximation becomes more accurate as h and j approach zero.

$$\begin{aligned}x &= 1 \\y &= 3\end{aligned}$$

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EIGENVALUES AND EIGENVECTORS

Eigenvalues and eigenvectors are used to determine dynamic properties of a structure, such as buckled shapes and seismic vibrations. Both involve the solution of a matrix equation with a multiplicity of possible solutions.

An extra variable may be added to a matrix equation along the diagonal as shown:

$$\begin{bmatrix} a_{11} - \lambda & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} - \lambda & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} - \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{bmatrix}$$

or, in matrix notation:

$$([A] - \lambda[I])\{x\} = \{c\}$$

If the constant vector, $\{c\}$, is set to zero, then the set of equations is said to be homogeneous, and lambda (λ) has n possible values, each corresponding to a unique set of unknowns, x_1 to x_n . The set of all solutions for λ is called the set of eigenvalues of the system, and the corresponding sets of unknowns are its eigenvectors.

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The solution of all possible eigenvalues of a system is the solution of the equation .

$$\det([A] - \lambda[I]) = 0$$

which is found by the following procedure:

Find $[B_1]$ to $[B_n]$ by the formula

$$[B_{i+1}] = [A]([B_i] - \rho_i[I])$$

where

$$[B_1] = [A]$$

$$\rho_1 = \text{trace}[A]$$

$$\rho_i = \text{trace}[B_i]/i$$

$$\text{trace}[A] = a_{11} + a_{22} + a_{33} + \dots + a_{nn}$$

= the sum of all the diagonal elements of $[A]$

Solve the polynomial

$$(-1)^n(\lambda^n - \rho_1\lambda^{n-1} - \rho_2\lambda^{n-2} - \dots - \rho_{n-1}\lambda - \rho_n) = 0$$

for the roots λ_1 to λ_n . These roots are the eigenvalues of $[A]$.

To find the eigenvectors corresponding to λ_1 to λ_n , solve the equation

$$([A] - \lambda_i[I])(x) = 0$$

using any reduction scheme. Complete pivoting may be necessary to avoid zeroes on the diagonal.

After the matrix has been reduced, one or more rows will be full of zeroes. To back substitute, it will be necessary to set the unknowns that correspond to these rows equal to some arbitrary value, usually one.

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Being equal to zero, the equations do not have just one unique solution, but an infinity of solutions, each differing by a constant. Setting one or more of the unknowns to an arbitrary value determines what that constant will be.

EXAMPLE

$$[A] = \begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$$

$$[A] - \lambda[I] = \begin{bmatrix} 4 - \lambda & 2 & -2 \\ -5 & 3 - \lambda & 2 \\ -2 & 4 & 1 - \lambda \end{bmatrix}$$

$$\rho_1 = 4 + 3 + 1 = 8$$

$$\begin{aligned} [B_2] &= \begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix} \left(\begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix} - \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} \right) \\ &= \begin{bmatrix} -22 & -10 & 10 \\ 1 & -17 & 2 \\ -14 & -20 & 5 \end{bmatrix} \end{aligned}$$

$$\rho_2 = (1/2)(-22 - 17 + 5) = -17$$

$$\begin{aligned} [B_3] &= \begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix} \left(\begin{bmatrix} -22 & -10 & 10 \\ 1 & -17 & 2 \\ -14 & -20 & 5 \end{bmatrix} + \begin{bmatrix} 17 & 0 & 0 \\ 0 & 17 & 0 \\ 0 & 0 & 17 \end{bmatrix} \right) \\ &= \begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix} \end{aligned}$$

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$$\rho_3 = (1/3)(10 + 10 + 10) = 0$$

solve the cubic

$$(-1)^3(\lambda^3 - 8\lambda^2 + 17\lambda - 10) = 0$$

which gives the three eigenvalues of [A].

$$\lambda_1 = 5$$

$$\lambda_2 = 2$$

$$\lambda_3 = 1$$

To determine the eigenvector for λ_1 , use simple Gaussian elimination:

$$\lambda_1 = 5$$

$$[A] - \lambda_1[I] = \begin{bmatrix} -1 & 2 & -2 \\ -5 & -2 & 2 \\ -2 & 4 & -4 \end{bmatrix}$$

Row reduction:

$$\begin{bmatrix} -1 & 2 & -2 \\ -5 & -2 & 2 \\ -2 & 4 & -4 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 2 & -2 \\ 0 & -12 & 12 \\ -2 & 4 & -4 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 2 & -2 \\ 0 & -12 & 12 \\ 0 & 0 & 0 \end{bmatrix}$$

Back substitution:

since x_3 cannot be found ($x_3 \times 0 = 0$ has infinitely many solutions),
set x_3 arbitrarily to one

$$x_3 = 1$$

$$x_2 = -12x_3 / -12 = 1$$

$$x_1 = (-2x_2 + 2x_3) / -1 = 0$$

$$\{x\} = \{0, 1, 1\}$$

In the same manner the eigenvectors can be found for λ_2 :

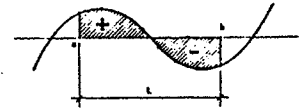
$$\{x\} = \{1/2, 1/2, 1\}$$

and for λ_3 :

$$\{x\} = \{1/2, 1/4, 1\}$$

NUMERICAL INTEGRATION

The integral of a function is simply the area under that function's plotted curve. If the formal integral of a function is unknown, the integral over a specified region may still be approximated using known points on the curve and any of several numerical integration techniques. These techniques are equations for shapes--assumed shapes of the curve between known points, ranging from straight lines (the trapezoidal rule) to parabolas (Simpson's rule). Gaussian quadrature improves upon these two rules by dictating both the shape of the curve and the spacing between points. Integration over two dimensions (double integration) can be performed numerically by assuming the shape of an area rather than just a line.



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THE TRAPEZOIDAL RULE

The area under a curve can be approximated by dividing it into a series of trapezoids, all of equal width. The area under a single trapezoid with base b and heights h_0 and h_1 is

$$A = L(h_0 + h_1)/2$$

Similarly, the area under three trapezoids of base b placed side by side is

$$\begin{aligned} A &= L(h_0 + h_1)/2 + L(h_1 + h_2)/2 + L(h_2 + h_3)/2 \\ &= (L/2)(h_0 + 2h_1 + 2h_2 + h_3) \end{aligned}$$

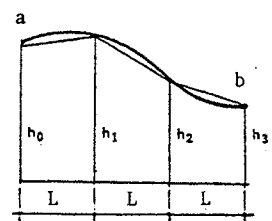
And the area under n trapezoids is

$$A = (L/2)(h_0 + 2h_1 + 2h_2 + \dots + 2h_{n-1} + h_n)$$

In applying this formula to the definite integral of a function $y = f(x)$, one can use the following:

$$\int_a^b f(x)dx = \frac{b-a}{2n} (y_0 + 2y_1 + 2y_2 + \dots + 2y_{n-1} + y_n)$$

where n is the number of spaces, $n+1$ is the number of sampling points, and y_i is the function evaluated at point i .



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SIMPSON'S RULE

Simpson's rule approximates the area under a curve by dividing the curve into parabolas, each parabola determined by three known points. The area under a parabola, with its height, h , known at three evenly spaced points, is

$$A = (L/3)(h_0 + 4h_1 + h_2)$$

Similarly, the area under two adjoining parabolas is

$$\begin{aligned} A &= (L/3)(h_0 + 4h_1 + h_2) + (L/3)(h_2 + 4h_3 + h_4) \\ &= (L/3)(h_0 + 4h_1 + 2h_2 + 4h_3 + h_4) \end{aligned}$$

And the area under q parabolas is

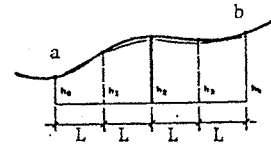
$$A = (L/3)(h_0 + 4h_1 + 2h_2 + 4h_3 + 2h_4 + \dots + 2h_{n-2} + 4h_{n-1} + h_n)$$

where $n = 2q$, the number of spaces, allowing two per parabola.

The integral of a function $y = f(x)$ using Simpson's rule is, therefore

$$\int_b^a f(x) dx = \frac{b-a}{3n} (y_0 + 4y_1 + 2y_2 + 4y_3 + \dots + 4y_{n-1} + y_n)$$

where n is the number of spaces (always even), $n+1$ is the number of sampling points (always odd), and y_i is the function evaluated at point i .



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EXAMPLE

Calculate the following integral using 4 spaces (5 sampling points) by the trapezoidal rule and by Simpson's rule:

$$\int_0^1 \sin \frac{\pi x}{2} \cos \frac{\pi x}{2} dx$$

Evaluating the function at five evenly spaced points gives the following values for x and y :

$x:$	0	0.25	0.5	0.75	1
$y:$	0	0.35355	0.5	0.35355	0

Integrating from 0 to 1 over four spaces by the trapezoidal rule:

$$\begin{aligned} \int_b^a f(x) dx &= (1/8)(0 + 2(0.35355) + 2(0.5) + 2(0.35355) + 0) \\ &= 0.30177 \end{aligned}$$

Integrating from 0 to 1 over four spaces by Simpson's rule:

$$\begin{aligned} \int_b^a f(x) dx &= (1/12)(0 + 4(0.35355) + 2(0.5) + 4(0.35355) + 0) \\ &= 0.31903 \end{aligned}$$

The formal solution to the integral is

$$\begin{aligned} \int_0^1 \sin \frac{\pi x}{2} \cos \frac{\pi x}{2} dx &= -(1/2\pi) \cos(\pi x) \Big|_0^1 \\ &= 1/\pi \\ &= 0.31831 \end{aligned}$$

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DOUBLE INTEGRATION - THE TRAPEZOIDAL RULE

Double integration (integration in two dimensions) may be easily accomplished by imagining the equation for numerical integration to be expanded to two dimensions. The trapezoidal rule counts the first and last values in a series once and all the values in between twice. If this procedure were expanded to two dimensions (a rectangle), the values at the corners of the rectangle would be counted once, the values along the edges twice, and the values in the middle four times. This way of counting is shown graphically in the following operator:

$$\int_a^b \int_c^d f(x,y) dy dx = (hk/4) \begin{array}{cccccccc} 1 & 2 & 2 & 2 & 2 & 2 & \dots & 2 & 2 & 1 \\ & 2 & 4 & 4 & 4 & 4 & \dots & 4 & 4 & 2 \\ & & 2 & 4 & 4 & 4 & \dots & 4 & 4 & 2 \\ & & & 2 & 4 & 4 & \dots & 4 & 4 & 2 \\ & & & & 2 & 4 & \dots & 4 & 4 & 2 \\ & & & & & \vdots & & \vdots & \vdots & \\ & & & & & 2 & & 4 & 4 & 2 \\ & & & & & 2 & & 4 & 4 & 2 \\ & & & & & 1 & & 2 & 2 & 1 \end{array}$$

- h = (b - a)/n is the size of the interval in the x-direction
- k = (d - c)/m is the size of the interval in the y-direction
- n is the number of spaces in the x-direction
- m is the number of spaces in the y-direction

The function f(x,y) is evaluated at each point and multiplied by the corresponding value shown in the operator. The products are added, and the sum is multiplied by hk/4.

DOUBLE INTEGRATION - SIMPSON'S RULE

Double integration by Simpson's rule is similar to double integration by the trapezoidal rule. The operator, which must have an even number of spaces in each direction, is shown below:

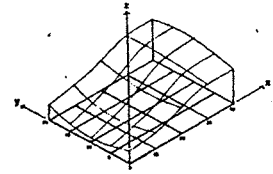
$$\int_a^b \int_c^d f(x,y) dy dx = (hk/9) \begin{array}{cccccccc} 1 & 4 & 2 & 4 & 2 & 4 & \dots & 2 & 4 & 1 \\ & 4 & 16 & 8 & 16 & 8 & \dots & 8 & 16 & 4 \\ & & 2 & 8 & 4 & 8 & \dots & 4 & 8 & 2 \\ & & & 4 & 16 & 8 & \dots & 8 & 16 & 4 \\ & & & & \vdots & \vdots & & \vdots & \vdots & \\ & & & & 2 & 8 & & 4 & 8 & 2 \\ & & & & 4 & 16 & & 8 & 16 & 4 \\ & & & & 1 & 4 & & 2 & 4 & 1 \end{array}$$

- h = (b - a)/n is the size of the interval in the x-direction
- k = (d - c)/m is the size of the interval in the y-direction
- n is the number of spaces in the x-direction (always even)
- m is the number of spaces in the y-direction (always even)

The function f(x,y) is multiplied by each number in the operator and the products are added.

EXAMPLE-

Given the x, y and z coordinates of the surface shown, calculate the volume under the surface by the trapezoidal rule and by Simpson's rule.



x	y	z
0	0	8.0
0	6	6.6
0	12	5.8
0	18	7.3
0	24	10.0
10	0	10.0
10	6	8.5
10	12	7.8
10	18	9.3
10	24	12.0
20	0	18.0
20	6	16.9
20	12	16.2
20	18	16.5
20	24	17.0
30	0	21.0
30	6	22.3
30	12	25.0
30	18	24.4
30	24	24.0
40	0	22.0
40	6	24.2
40	12	26.9
40	18	27.5
40	24	27.0

THE TRAPEZOIDAL RULE

$$h = 10$$

$$k = 6$$

$$\begin{aligned} \int_0^{40} \int_0^{24} f(x,y) dy dx &= (60/4) [1(8.0) + 2(6.6) + 2(5.8) + 2(7.3) + 1(10.0) + \\ & 2(10.0) + 4(8.5) + 4(7.8) + 4(9.3) + 2(12.0) + \\ & 2(18.0) + 4(16.9) + 4(16.2) + 4(16.5) + 2(17.0) + \\ & 2(21.0) + 4(22.3) + 4(25.0) + 4(24.4) + 2(24.0) + \\ & 1(22.0) + 2(24.2) + 2(26.9) + 2(27.5) + 1(27.0)] \\ &= 15828.0 \end{aligned}$$

SIMPSON'S RULE

$$\begin{aligned} \int_0^{40} \int_0^{24} f(x,y) dy dx &= (60/9) [1(8.0) + 4(6.6) + 2(5.8) + 4(7.3) + 1(10.0) + \\ & 4(10.0) + 16(8.5) + 8(7.8) + 16(9.3) + 4(12.0) + \\ & 2(18.0) + 8(16.9) + 4(16.2) + 8(16.5) + 2(17.0) + \\ & 4(21.0) + 16(22.3) + 8(25.0) + 16(24.4) + 4(24.0) + \\ & 1(22.0) + 4(24.2) + 2(26.9) + 4(27.5) + 1(27.0)] \\ &= 15733.333 \end{aligned}$$

From Simpson's rule it can be seen that a polynomial approximation gives different importances to different points on the curve. For example, the second point is four times as important as the first point and twice as important as the third point. Gaussian quadrature dictates both the spacing and importance of each point to establish higher order approximating polynomials.

To derive a quadrature formula using a third degree polynomial, a function is defined in terms of four variables:

$$\int_{-1}^1 f(x) dx = f_1 w_1 + f_2 w_2$$

where f_1 and f_2 are the values of the function at two yet unspecified points, and w_1 and w_2 are the importances, or weight factors, associated with those two values.

A third degree polynomial has four terms: a constant term, a linear term, a quadratic term, and a cubic term:

$$f(x) = Ax^3 + Bx^2 + Cx + D$$

Ax^3 is the cubic term

Bx^2 is the quadratic term

Cx is the linear term

D is the constant term

Integrating each term and substituting it into the original equation in four variables gives four definite integrals, with corresponding illustrations, on the following page. These definite integrals, which have known areas, can be used along with known values of the function at two arbitrary points (x_1 and x_2) to find the values of w_1 and w_2 .

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The definite integrals, between -1 and 1, for the four terms are

$$\int_{-1}^1 D dx = 2D$$

$$f_1 = D$$

$$f_2 = D$$

$$\int_{-1}^1 Cx dx = 0$$

$$f_1 = Cx_1$$

$$f_2 = Cx_2$$

$$\int_{-1}^1 Bx^2 dx = 2B/3$$

$$f_1 = Bx_1^2$$

$$f_2 = Bx_2^2$$

$$\int_{-1}^1 Ax^3 dx = 0$$

$$f_1 = Ax_1^3$$

$$f_2 = Ax_2^3$$

When these integrals are combined with the original expression

$$\int_{-1}^1 f(x) dx = f_1 w_1 + f_2 w_2$$

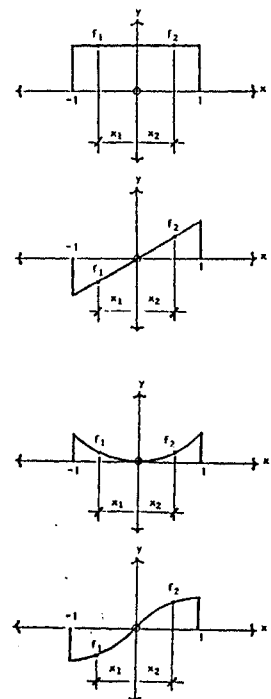
they yield the following four simultaneous equations:

$$Dw_1 + Dw_2 = 2D$$

$$Cx_1 w_1 + Cx_2 w_2 = 0$$

$$Bx_1^2 w_1 + Bx_2^2 w_2 = 2B/3$$

$$Ax_1^3 w_1 + Ax_2^3 w_2 = 0$$



Solving these four equations gives the values of x and w :

$$\begin{aligned}x_1 &= -1/\sqrt{3} \\x_2 &= 1/\sqrt{3} \\w_1 &= 1 \\w_2 &= 1\end{aligned}$$

To approximate the integral of a function from -1 to 1 , the value of the function must be known at x_1 and x_2 :

$$\int_{-1}^1 f(x) dx = f(x_1)w_1 + f(x_2)w_2$$

where $f(x_1)$ is the function evaluated at $x_1 = -1/\sqrt{3}$ and $f(x_2)$ is the function evaluated at $x_2 = 1/\sqrt{3}$.

The more general formula for an integral from a to b is

$$\int_a^b f(x) dx = f[x_1(\frac{b-a}{2})]w_1 + f[x_2(\frac{b-a}{2})]w_2 *$$

The same procedure can be applied to derive a polynomial of any order. Values are listed below for polynomials up to thirteenth order increasing by odd numbers.

* Note that trigonometric functions must be in radians

LOCATION AND WEIGHT VALUES FOR GAUSSIAN QUADRATURE OF ORDER $2n - 1$

$$\int_a^b f(x) dx = (s) \sum_{i=1}^n f(s + sx_i) w_i$$

$$s = (b - a)/2$$

n = number of sampling points

n	i	x_i	w_i	order
2	1	-0.577350269	1	3rd
	2	0.577350269	1	
3	1	-0.774596669	0.555555556	5th
	2	0	0.888888889	
	3	0.774596669	0.555555556	
4	1	-0.861136311	0.347854845	7th
	2	-0.339981044	0.625145155	
	3	0.339981044	0.625145155	
	4	0.861136311	0.047854845	
5	1	-0.906179846	0.236926885	9th
	2	-0.538469310	0.478628670	
	3	0	0.568888889	
	4	0.538469310	0.478628670	
	5	0.906179846	0.236926885	
6	1	-0.932469514	0.171324492	11th
	2	-0.661209386	0.360761573	
	3	-0.238619186	0.467913935	
	4	0.238619186	0.467913935	
	5	0.661209386	0.360761573	
	6	0.932469514	0.171324492	
7	1	-0.949107912	0.129484966	13th
	2	-0.741531186	0.279705391	
	3	-0.405845151	0.381830051	
	4	0	0.417959184	
	5	0.405845151	0.381830051	
	6	0.741531186	0.279705391	
	7	0.949107912	0.129484966	

EXAMPLE

Integrate $\sin(x)$ from 0 to $\pi/2$

TWO-POINT

$$\begin{aligned}
 s &= \pi/4 = 0.785398164 \\
 s + sx_1 &= 0.331948323 \\
 s + sx_2 &= 1.238848005 \\
 f_1 &= 0.325885608 & w_1 &= 1 \\
 f_2 &= 0.945409208 & w_2 &= 1 \\
 \int_0^{\pi/2} \sin(x) dx &= (0.785398164)(0.325885608 + 0.945409208) \\
 &= 0.998472614
 \end{aligned}$$

THREE-POINT

$$\begin{aligned}
 s &= \pi/4 = 0.785398164 \\
 s + sx_1 &= 0.177031362 \\
 s + sx_2 &= 0.785398164 \\
 s + sx_3 &= 1.393764965 \\
 f_1 &= 0.176108113 & w_1 &= 0.555555556 \\
 f_2 &= 0.707106781 & w_2 &= 0.888888889 \\
 f_3 &= 0.984370831 & w_3 &= 0.555555556 \\
 \int_0^{\pi/2} \sin(x) dx &= (0.785398164) \left[\begin{aligned} &(0.555555556)(0.176108113) \\ &+ (0.888888889)(0.707106781) \\ &+ (0.555555556)(0.984370831) \end{aligned} \right] \\
 &= 1.000008122
 \end{aligned}$$

EXACT VALUE

$$\begin{aligned}
 \int_0^{\pi/2} \sin(x) dx &= -\cos(x) \Big|_0^{\pi/2} \\
 &= 1.000000000
 \end{aligned}$$

CURVE FITTING

The least-squares method of curve fitting attempts to reduce the square of the error (the error being the discrepancy between the actual value of a data point and its value when fit on a curve) to a minimum. By reducing the square of the error rather than the error itself, the method insures that no single point is much farther from its actual value than any other point.



Let r be the error between actual and curve-fit values of any point, and Σr^2 be the sum of the squares of all the errors. If Σr^2 is held to a minimum, then its derivative will be zero. Specifically, to fit a particular function with m constants to a set of n data points, the derivative of Σr^2 with respect to each constant must be zero. The function (which defines the curve) and the resulting simultaneous equations are shown below:

$$\begin{aligned}
 f(x) &= c_1 f_1(x) + c_2 f_2(x) + \dots + c_m f_m(x) \\
 \partial \Sigma r^2 / \partial c_1 &= 0 \\
 \partial \Sigma r^2 / \partial c_2 &= 0 \\
 \partial \Sigma r^2 / \partial c_m &= 0 \quad \text{etc.}
 \end{aligned}$$

Solving these equations gives m constants, which define the desired curve.

PROCEDURE

Define the desired curve in terms of x, whether it be a polynomial, transcendental, or some combination of the two:

$$f(x) = c_1 f_1(x) + c_2 f_2(x) + \dots + c_m f_m(x)$$

where c_1 to c_m are constants by which the functions $f_1(x)$ to $f_m(x)$ are multiplied.

The set of data points can be larger or smaller than the number of constants, and is given as a series of pairs (x,y):

$$\begin{aligned} &(x_1, y_1) \\ &(x_2, y_2) \\ &(x_3, y_3) \quad \text{etc.} \end{aligned}$$

To find the constants, formulate the matrix

$$[F] = \begin{bmatrix} f_1(x_1) & f_2(x_1) & f_3(x_1) & \dots & f_m(x_1) \\ f_1(x_2) & f_2(x_2) & f_3(x_2) & \dots & f_m(x_2) \\ f_1(x_3) & f_2(x_3) & f_3(x_3) & \dots & f_m(x_3) \\ \vdots & \vdots & \vdots & & \vdots \\ f_1(x_n) & f_2(x_n) & f_3(x_n) & \dots & f_m(x_n) \end{bmatrix}$$

and solve the following equation for {c}:

$$[F]^T [F] \{c\} = [F]^T \{y\}$$

by any reduction technique. Premultiplication by $[F]^T$ is to insure square matrices, and therefore a solvable system.

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EXAMPLE

Find the constants that will fit the following parabola to the given data points:

$$f(x) = c_1 x^2 + c_2 x + c$$

$$\begin{array}{l} x: \quad 0 \quad 2 \quad 4 \quad 6 \quad 8 \quad 10 \\ y: \quad 4 \quad 6.5 \quad 12.8 \quad 10.3 \quad 9.8 \quad 10.3 \end{array}$$

$$\begin{aligned} f_1 &= x^2 \\ f_2 &= x \\ f_3 &= 1 \end{aligned}$$

$$[F] = \begin{bmatrix} 0 & 0 & 0 \\ 4 & 2 & 1 \\ 16 & 4 & 1 \\ 36 & 6 & 1 \\ 64 & 8 & 1 \\ 100 & 10 & 1 \end{bmatrix} \quad [F]^T = \begin{bmatrix} 0 & 4 & 16 & 36 & 64 & 100 \\ 0 & 2 & 4 & 6 & 8 & 10 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$[F]^T [F] = \begin{bmatrix} 15664 & 1800 & 220 \\ 1800 & 220 & 30 \\ 220 & 30 & 5 \end{bmatrix}$$

$$[F]^T \{y\} = \begin{bmatrix} 0 & 4 & 16 & 36 & 64 & 100 \\ 0 & 2 & 4 & 6 & 8 & 10 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 6.5 \\ 12.8 \\ 10.3 \\ 9.8 \\ 10.3 \end{bmatrix} = \begin{bmatrix} 2258.8 \\ 307.4 \\ 49.7 \end{bmatrix}$$

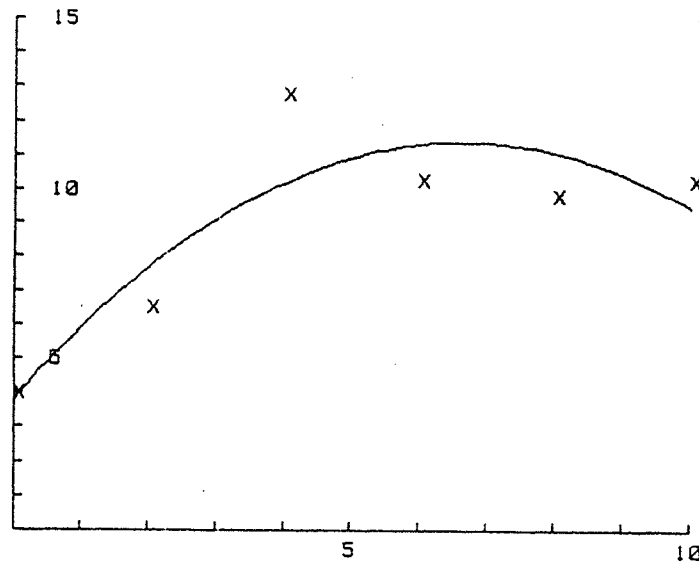
$[F]^T [F] \{c\} = [F]^T \{y\}$; by simple Gaussian elimination

$$\{c\} = \begin{bmatrix} -0.171428 \\ 2.287142 \\ 3.76 \end{bmatrix}$$

$$f(x) = -0.171428x^2 + 2.287142x + 3.76$$

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PLOT OF FITTED CURVE



PLOT OF $F(x) = -0.171423(x^2) + 2.287142(x) + 3.76$

DATA POINTS

POINT #1:	X = 0	Y = 4
POINT #2:	X = 2	Y = 6.5
POINT #3:	X = 4	Y = 12.8
POINT #4:	X = 6	Y = 10.3
POINT #5:	X = 8	Y = 9.8
POINT #6:	X = 10	Y = 10.3

FINITE DIFFERENCES

Often the formal derivative of a function may be difficult or impossible to calculate. In such cases, a finite difference approximation can be used if the value of the function is known at several specific points.

Finite differences are derived from Taylor series expansions of a generalized function, $f(x)$, about points spaced at a given interval to the right and left of a starting point, x_0 . The Taylor series gives the values of $f(x_0 - h)$ and $f(x_0 + h)$ in terms of the function and its derivatives at x_0 :

$$f(x_0 - h) = f(x_0) - hf'(x_0) + (h^2/2!)f''(x_0) - (h^3/3!)f'''(x_0) + \dots$$

$$f(x_0 + h) = f(x_0) + hf'(x_0) + (h^2/2!)f''(x_0) + (h^3/3!)f'''(x_0) + \dots$$

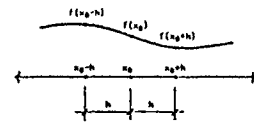
Subtracting the first equation from the second gives

$$f(x_0 + h) - f(x_0 - h) = 2hf'(x_0) + (2h^3/3!)f'''(x_0) + \dots$$

which can be written (neglecting higher order terms h^3, h^5 , etc.) as

$$f'(x_0) = (1/2h)[f(x_0 + h) - f(x_0 - h)]$$

This is the central difference equation for the first derivative of any function of x , based on values of the function at a distance h to the left and right of the "pivotal point"--the value of x for which the derivative is desired.



It is apparent that the error (e) in this approximation is the sum of the truncated terms in the Taylor series from which it is derived:

$$e = (1/2h) [(2h^3/3!)f'''(x_0) + (2h^5/5!)f''''(x_0) + \dots]$$

For small values of h, these terms are very small; where the third and all successive derivatives are zero, these terms are identically zero, and the approximation is exact. In all cases the order of the error will be h^2 , which means that h^2 is the largest magnitude the error can have.

Finite difference approximations can be derived for higher-order derivatives by various simultaneous solutions of the same basic Taylor series expansions about $x \pm h$, $x \pm 2h$, and so on. The approximations for second, third, and fourth derivatives are shown below.

$$\begin{aligned} f'(x_0) &= (1/h^2)[f(x_0 + h) - 2f(x_0) + f(x_0 - h)] \\ f''(x_0) &= (1/2h^3)[f(x_0 + 2h) - 2f(x_0 + h) + 2f(x_0 - h) - f(x_0 - 2h)] \\ f'''(x_0) &= (1/h^4)[f(x_0 + 2h) - 4f(x_0 + h) + 6f(x_0) - 4f(x_0 - h) + f(x_0 - 2h)] \end{aligned}$$

These approximations all have errors of the order h^2 . More accurate approximations can be derived by keeping more terms in the original Taylor series. Approximations with order h^4 are given at the end of this chapter.

The finite difference equations shown above are called central differences because they require values both to the left and to the right of the pivotal point. In many practical applications, such as the analysis of a beam connection, only the values to one side of the pivotal point may be known. In such cases, forward differences and backward differences may be used. Forward differences use only values to the right of (and including) the pivotal point; backward differences use only values to the left of (and including) the pivotal point. Forward and backward differences for the first four derivatives of $f(x_0)$ are given at the end of this chapter.

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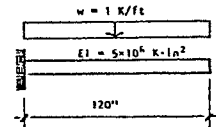
Finite difference equations are usually written in the form of operators, which are simply graphic representations of the equations. The first central difference operator, $f'(x)$, is shown at the right with its pivotal point shaded. The number in the shaded circle is multiplied by the value of $f(x_0)$, where x_0 is the pivotal point; the number to the left is multiplied by $f(x_0 - h)$; the number to the right is multiplied by $f(x_0 + h)$. These products are added together, and the sum is multiplied by $1/2h$ to give the derivative $f'(x_0)$.

$$(1/2h) \text{ (1) } \textcircled{0} \text{ (1)}$$

$$(1/h^2) \text{ (1) } \textcircled{-2} \text{ (1)}$$

As an example, assume that the deflections but not the load are known for the beam shown at right (length is expressed as x, deflection as -y).

x:	0"	20"	40"	60"	80"	100"	120"
y:	0	-0.021"	-0.076"	-0.153"	-0.242"	-0.336"	-0.432"



The moment in the beam is the second derivative of its deflection, as given by the equation

$$M = EI(d^2y/dx^2)$$

The moment at $x = 60''$ can be calculated using the second derivative operator shown above, resulting in the following equation:

$$\begin{aligned} M &= (EI/h^2) [(1)(0.076) - (2)(0.153) + (1)(0.242)] \\ &= (50 \times 10^6/20^2)(0.012) = 150 \text{ kip-inches} \end{aligned}$$

The deflections above were calculated from a one kip/foot load. The actual moment caused by such a load is $w(L - x)^2/2$:

$$M = (0.083333 \text{ K/in})(60'')^2/2 = 150 \text{ kip-inches}$$

For functions of two variables, such as $z = f(x,y)$, two-dimensional operators can be constructed by overlaying vertical and horizontal operators. For example, the operator for d^4z/dx^2dy^2 is constructed by multiplying

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operators for d^2z/dx^2 and d^2z/dy^2 :

$$d^2z/dx^2 = (1/h^2) \begin{array}{ccc} \textcircled{1} & \textcircled{-2} & \textcircled{1} \\ & \textcircled{1} & \\ & & \textcircled{-2} \\ & & \textcircled{1} \end{array}$$

$$d^2z/dy^2 = (1/h^2) \begin{array}{ccc} & \textcircled{1} & \\ & \textcircled{-2} & \\ & \textcircled{1} & \end{array}$$

$\begin{array}{c} y \\ \updownarrow \\ \rightarrow x \end{array}$

The combined operator is made by multiplying the x-operator by each value in the y-operator:

$$d^4z/dx^2dy^2 = (1/h^2) \begin{array}{ccc} \textcircled{1} & \textcircled{-2} & \textcircled{1} \\ & \textcircled{1} & \\ & & \textcircled{-2} \\ & & \textcircled{1} \end{array} \times \begin{array}{c} \textcircled{1} \\ \textcircled{-2} \\ \textcircled{1} \end{array} (1/h^2)$$

$$= (1/h^4) \begin{array}{ccc} \textcircled{1} & \textcircled{-2} & \textcircled{1} \\ \textcircled{-2} & \textcircled{4} & \textcircled{-2} \\ \textcircled{1} & \textcircled{-2} & \textcircled{1} \end{array}$$

$\begin{array}{c} y \\ \updownarrow \\ \rightarrow x \end{array}$

Note that if h is different for x and y directions, the coefficient becomes $(1/h_x^2 h_y^2)$

Operators can also be added. The construction of an operator for

$$\nabla^2 z = d^2z/dx^2 + 2d^2z/dxdy + d^2z/dy^2$$

is shown below:

$$d^2z/dx^2 = (1/h^2) \begin{array}{ccc} \textcircled{1} & \textcircled{-2} & \textcircled{1} \\ & & \\ & & \\ & & \end{array} \quad \rightarrow x$$

$$d^2z/dy^2 = (1/h^2) \begin{array}{ccc} & \textcircled{1} & \\ & \textcircled{-2} & \\ & \textcircled{1} & \end{array} \quad \updownarrow y$$

$$d^2z/dxdy = (1/h^2) \begin{array}{ccc} \textcircled{-1} & \textcircled{0} & \textcircled{1} \\ \textcircled{0} & \textcircled{0} & \textcircled{0} \\ \textcircled{1} & \textcircled{0} & \textcircled{-1} \end{array} \quad \begin{array}{c} y \\ \updownarrow \\ \rightarrow x \end{array}$$

Overlaying d^2z/dx^2 , d^2z/dy^2 , and $2d^2z/dxdy$ gives the complete operator:

$$\nabla^2 z = (1/h^2) \begin{array}{ccc} \textcircled{-1} & \textcircled{1} & \textcircled{1} \\ \textcircled{1} & \textcircled{-4} & \textcircled{1} \\ \textcircled{1} & \textcircled{1} & \textcircled{-1} \end{array}$$

$\begin{array}{c} y \\ \updownarrow \\ \rightarrow x \end{array}$

CENTRAL DIFFERENCE OPERATORS WITH ERROR OF ORDER h^2

$$\begin{aligned}
 f'(x) &= (1/2h) && \begin{array}{c} (-1) \quad 0 \quad 1 \\ \circ \quad \circ \quad \circ \end{array} \\
 f''(x) &= (1/h^2) && \begin{array}{c} 1 \quad -2 \quad 1 \\ \circ \quad \circ \quad \circ \end{array} \\
 f'''(x) &= (1/2h^3) && \begin{array}{c} (-1) \quad 2 \quad 0 \quad -2 \quad 1 \\ \circ \quad \circ \quad \circ \quad \circ \quad \circ \end{array} \\
 f^{(4)}(x) &= (1/h^4) && \begin{array}{c} 1 \quad -4 \quad 6 \quad -4 \quad 1 \\ \circ \quad \circ \quad \circ \quad \circ \quad \circ \end{array}
 \end{aligned}$$

CENTRAL DIFFERENCE OPERATORS WITH ERROR OF ORDER h^4

$$\begin{aligned}
 f'(x) &= (1/12h) && \begin{array}{c} 1 \quad -8 \quad 0 \quad 8 \quad -1 \\ \circ \quad \circ \quad \circ \quad \circ \quad \circ \end{array} \\
 f''(x) &= (1/12h^2) && \begin{array}{c} -1 \quad 16 \quad -30 \quad 16 \quad 1 \\ \circ \quad \circ \quad \circ \quad \circ \quad \circ \end{array} \\
 f'''(x) &= (1/8h^3) && \begin{array}{c} 1 \quad -8 \quad 13 \quad 0 \quad -13 \quad 8 \quad -1 \\ \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \end{array} \\
 f^{(4)}(x) &= (1/6h^4) && \begin{array}{c} -1 \quad 12 \quad -39 \quad 56 \quad -39 \quad 12 \quad -1 \\ \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \end{array}
 \end{aligned}$$

FORWARD DIFFERENCE OPERATORS WITH ERROR OF ORDER h

$$\begin{aligned}
 f'(x) &= (1/h) && \begin{array}{c} (-1) \quad 1 \\ \circ \quad \circ \end{array} \\
 f''(x) &= (1/h^2) && \begin{array}{c} 1 \quad -2 \quad 1 \\ \circ \quad \circ \quad \circ \end{array} \\
 f'''(x) &= (1/h^3) && \begin{array}{c} (-1) \quad 3 \quad -3 \quad 1 \\ \circ \quad \circ \quad \circ \quad \circ \end{array} \\
 f^{(4)}(x) &= (1/h^4) && \begin{array}{c} 1 \quad -4 \quad 6 \quad -4 \quad 1 \\ \circ \quad \circ \quad \circ \quad \circ \quad \circ \end{array}
 \end{aligned}$$

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FORWARD DIFFERENCE OPERATORS WITH ERROR OF ORDER h^2

$$\begin{aligned}
 f'(x) &= (1/2h) && \begin{array}{c} (-3) \quad 4 \quad -1 \\ \circ \quad \circ \quad \circ \end{array} \\
 f''(x) &= (1/h^2) && \begin{array}{c} 2 \quad -5 \quad 4 \quad -1 \\ \circ \quad \circ \quad \circ \quad \circ \end{array} \\
 f'''(x) &= (1/2h^3) && \begin{array}{c} (-5) \quad 18 \quad -24 \quad 14 \quad -3 \\ \circ \quad \circ \quad \circ \quad \circ \quad \circ \end{array} \\
 f^{(4)}(x) &= (1/h^4) && \begin{array}{c} 3 \quad -14 \quad 26 \quad -24 \quad 11 \quad -2 \\ \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \end{array}
 \end{aligned}$$

BACKWARD DIFFERENCE OPERATORS WITH ERROR OF ORDER h

$$\begin{aligned}
 f'(x) &= (1/h) && \begin{array}{c} (-1) \quad 1 \\ \circ \quad \circ \end{array} \\
 f''(x) &= (1/h^2) && \begin{array}{c} 1 \quad -2 \quad 1 \\ \circ \quad \circ \quad \circ \end{array} \\
 f'''(x) &= (1/h^3) && \begin{array}{c} (-1) \quad 3 \quad -3 \quad 1 \\ \circ \quad \circ \quad \circ \quad \circ \end{array} \\
 f^{(4)}(x) &= (1/h^4) && \begin{array}{c} 1 \quad -4 \quad 6 \quad -4 \quad 1 \\ \circ \quad \circ \quad \circ \quad \circ \quad \circ \end{array}
 \end{aligned}$$

BACKWARD DIFFERENCE OPERATORS WITH ERROR OF ORDER h^2

$$\begin{aligned}
 f'(x) &= (1/2h) && \begin{array}{c} 1 \quad -4 \quad 3 \\ \circ \quad \circ \quad \circ \end{array} \\
 f''(x) &= (1/h^2) && \begin{array}{c} (-1) \quad 4 \quad -5 \quad 2 \\ \circ \quad \circ \quad \circ \quad \circ \end{array} \\
 f'''(x) &= (1/2h^3) && \begin{array}{c} 3 \quad -14 \quad 24 \quad -18 \quad 5 \\ \circ \quad \circ \quad \circ \quad \circ \quad \circ \end{array} \\
 f^{(4)}(x) &= (1/h^4) && \begin{array}{c} (-2) \quad 11 \quad -24 \quad 26 \quad -14 \quad 3 \\ \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \end{array}
 \end{aligned}$$

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TWO-DIMENSIONAL OPERATORS

$$\nabla^2 = d^2z/dx^2 + 2d^2z/dxdy + d^2z/dy^2$$

$$= (1/4h^2) \begin{pmatrix} \textcircled{-1} & \textcircled{1} & \textcircled{1} \\ \textcircled{1} & \textcircled{4} & \textcircled{1} \\ \textcircled{1} & \textcircled{1} & \textcircled{-1} \end{pmatrix}$$



$$\nabla^4 = d^4z/dx^4 + 2d^4z/dx^2dy^2 + d^4z/dy^4$$

$$= (1/h^4) \begin{pmatrix} & & \textcircled{1} & & \\ & \textcircled{2} & \textcircled{-8} & \textcircled{2} & \\ \textcircled{1} & \textcircled{-8} & \textcircled{20} & \textcircled{-8} & \textcircled{1} \\ & \textcircled{2} & \textcircled{-8} & \textcircled{2} & \\ & & \textcircled{1} & & \end{pmatrix}$$



SHEAR CONTRIBUTION TO THE STIFFNESS MATRIX

The basic stiffness matrix shown below is derived from bending deformations of a beam element. A beam subjected to shearing force exhibits additional shear deformations that are not accounted for in this basic matrix. A derivation that includes shear deformations yields the enhanced stiffness matrix, also shown below.

STIFFNESS MATRIX WITHOUT SHEAR:

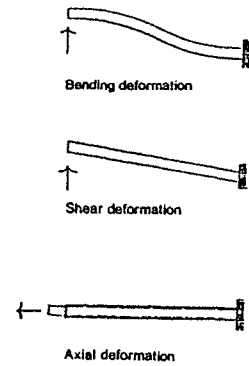
$$[K] = \begin{bmatrix} \frac{AE}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}$$

STIFFNESS MATRIX WITH SHEAR:

$$[K] = \begin{bmatrix} \frac{AE}{L} & 0 & 0 \\ 0 & \frac{12EIAvG}{L^3AvG + 12LEI} & \frac{6EIAvG}{L^2AvG + 12EI} \\ 0 & \frac{6EIAvG}{L^2AvG + 12EI} & \frac{4EIL^3AvG + 12E^2I^2L}{L^4AvG + 12L^2EI} \end{bmatrix}$$

Most ordinary beams and columns are considerably longer than they are deep; in such members the shear deformations are usually negligible. Short, deep members, however, or members with a small shear modulus will exhibit significant shear deformations. In either case, it should be noted that to ignore shear deformations is not necessarily conservative: these deformations cause a redistribution of forces and moments within a frame; in some places the forces and moments are reduced, in some places they are magnified.

As in the derivation of the basic bending stiffness matrix, axial forces and deformations are considered separately from lateral forces and deformations (including moment). Thus, the shear deformation has no effect upon axial forces (see sketches at right, above).



Consider, then, a beam element subjected to both shear and moment, with a known deflection at its free end, and with the rotation of that end set to zero (see sketches at right, below). Equilibrium of the undeformed shape gives the following equations for shear and moment at any point x :

$$V_x = P_{iy}$$

$$M_x = P_{iy}(x) - M_{iz}$$

The derivatives of these with respect to the free-end forces and moments are

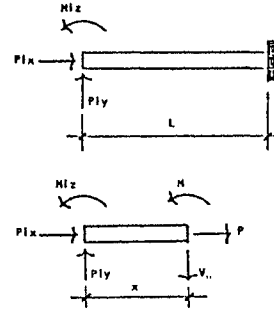
$$\frac{\partial V_x}{\partial P_{iy}} = 1$$

$$\frac{\partial V_x}{\partial M_{iz}} = 0$$

$$\frac{\partial M_x}{\partial P_{iy}} = x$$

$$\frac{\partial M_x}{\partial M_{iz}} = -1$$

Castigliano's theorem gives the following equations for the free-end rotation and deflection:



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$$\Delta_{iy} = \int_0^L (M_x/EI) (\partial M_x / \partial P_{iy}) dx + \int_0^L (V_x/AvG) (\partial V_x / \partial P_{iy}) dx$$

$$\theta_{iz} = \int_0^L (M_x/EI) (\partial M_x / \partial M_{iz}) dx + \int_0^L (V_x/AvG) (\partial V_x / \partial M_{iz}) dx$$

Substituting the appropriate values and setting the rotation equal to zero yields equations for P_{iy} and M_{iz} in terms of the deflection:

$$\Delta_{iy} = \int_0^L (1/EI) [P_{iy}(x) - M_{iz}] (x) dx + \int_0^L (1/AvG) (P_{iy}) dx$$

$$= P_{iy}(L^3/3EI + L/AvG) + M_{iz}(-L^2/2EI)$$

$$\theta_{iz} = \int_0^L (1/EI) [P_{iy}(x) - M_{iz}] (-1) dx$$

$$= P_{iy}(-L^2/2EI) + M_{iz}(L/EI) = 0$$

$$M_{iz} = P_{iy}(L/2)$$

$$\Delta_{iy} = P_{iy}(L^3/3EI - L^3/4EI + L/AvG)$$

$$= P_{iy}(L^3/12EI + L/AvG)$$

Let $\alpha = L^3/12EI$ and $\beta = L/AvG$

$$P_{iy} = \Delta_{iy} \left(\frac{1}{\alpha + \beta} \right)$$

$$M_{iz} = \Delta_{iy} \left(\frac{L}{2(\alpha + \beta)} \right)$$

These are the forces and moments that would create a deflection of Δ_{iy} with no rotation.

Consider now a beam with a known rotation and a deflection equal to zero. The shear and moment equations and their derivatives are the same as for the previous case. Setting Castigliano's equation for Δ_{iy} equal to zero gives P_{iy} and M_{iz} in terms of the rotation:

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$$\begin{aligned} \theta_{ix} &= P_{iy}(-L^2/2EI) + M_{iz}(L/EI) \\ M_{iz} &= P_{iy}(L/2) + \theta_{iz}(EI/L) \\ \Delta_{iy} &= P_{iy}(L^3/3EI + L/AvG) + M_{iz}(-L^2/2EI) \\ &= P_{iy}(L^3/3EI - L^2/4EI + L/AvG) + \theta_{iz}(-L/2) = 0 \\ \theta_{iz} &= P_{iy}(L^2/6EI + 2/AvG) \end{aligned}$$

Again, let $\alpha = L^3/12EI$ and $\beta = L/AvG$

$$\begin{aligned} P_{iy} &= \theta_{iz} \left(\frac{L}{2(\alpha + \beta)} \right) \\ M_{iz} &= \theta_{iz} \left(\frac{L^2}{4(\alpha + \beta)} + \frac{EI}{L} \right) \\ &= \theta_{iz} \left(\frac{EI(4\alpha + \beta)}{L(\alpha + \beta)} \right) \end{aligned}$$

Superimposing these two cases and adding the axial effects gives the i-i member stiffness matrix for a coplanar structure:

$$\begin{bmatrix} P_{ix} \\ P_{iy} \\ M_{iz} \end{bmatrix} = \begin{bmatrix} \frac{AE}{L} & 0 & 0 \\ 0 & \frac{1}{(\alpha + \beta)} & \frac{L}{2(\alpha + \beta)} \\ 0 & \frac{L}{2(\alpha + \beta)} & \frac{EI(4\alpha + \beta)}{L(\alpha + \beta)} \end{bmatrix} \begin{bmatrix} \Delta_{ix} \\ \Delta_{iy} \\ \theta_{iz} \end{bmatrix}$$

Appropriate transformations yield the three-dimensional member stiffness matrix shown on the following page.

It should be noted that when the shear stiffness contribution, β , is set to zero the matrix is identical to the basic bending stiffness matrix.

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ELASTIC STIFFNESS MATRIX INCLUDING SHEAR:

$$[K] = \begin{bmatrix} \frac{EA_x}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{EA_x}{L} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\alpha_x + \beta_x} & 0 & 0 & 0 & \frac{L}{2(\alpha_x + \beta_x)} & 0 & -\frac{1}{\alpha_x + \beta_x} & 0 & 0 & 0 & \frac{L}{2(\alpha_x + \beta_x)} \\ 0 & 0 & \frac{1}{\alpha_y + \beta_y} & 0 & 0 & \frac{L}{2(\alpha_y + \beta_y)} & 0 & 0 & -\frac{1}{\alpha_y + \beta_y} & 0 & 0 & \frac{L}{2(\alpha_y + \beta_y)} \\ 0 & 0 & 0 & \frac{GI_x}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{GI_x}{L} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{GI_y}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{GI_y}{L} & 0 \\ 0 & 0 & \frac{L}{2(\alpha_x + \beta_x)} & 0 & 0 & \frac{EI_x(4\alpha_x + \beta_x)}{L(\alpha_x + \beta_x)} & 0 & \frac{L}{2(\alpha_x + \beta_x)} & 0 & 0 & \frac{EI_x(2\alpha_x - \beta_x)}{L(\alpha_x + \beta_x)} \\ -\frac{EA_x}{L} & 0 & 0 & 0 & 0 & 0 & \frac{EA_x}{L} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{\alpha_x + \beta_x} & 0 & 0 & 0 & \frac{L}{2(\alpha_x + \beta_x)} & 0 & \frac{1}{\alpha_x + \beta_x} & 0 & 0 & 0 & -\frac{L}{2(\alpha_x + \beta_x)} \\ 0 & 0 & -\frac{1}{\alpha_y + \beta_y} & 0 & 0 & \frac{L}{2(\alpha_y + \beta_y)} & 0 & 0 & \frac{1}{\alpha_y + \beta_y} & 0 & 0 & -\frac{L}{2(\alpha_y + \beta_y)} \\ 0 & 0 & 0 & -\frac{GI_x}{L} & 0 & 0 & 0 & 0 & 0 & \frac{GI_x}{L} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{GI_y}{L} & 0 & 0 & 0 & 0 & 0 & \frac{GI_y}{L} & 0 \\ 0 & 0 & \frac{L}{2(\alpha_x + \beta_x)} & 0 & 0 & \frac{EI_x(2\alpha_x - \beta_x)}{L(\alpha_x + \beta_x)} & 0 & \frac{L}{2(\alpha_x + \beta_x)} & 0 & 0 & \frac{EI_x(4\alpha_x + \beta_x)}{L(\alpha_x + \beta_x)} \\ 0 & \frac{L}{2(\alpha_x + \beta_x)} & 0 & 0 & 0 & \frac{EI_x(2\alpha_x - \beta_x)}{L(\alpha_x + \beta_x)} & 0 & -\frac{L}{2(\alpha_x + \beta_x)} & 0 & 0 & \frac{EI_x(4\alpha_x + \beta_x)}{L(\alpha_x + \beta_x)} \end{bmatrix}$$

$\alpha_x = L^3/12EI_x$ $\beta_x = L/AvG_x$
 $\alpha_y = L^3/12EI_y$ $\beta_y = L/AvG_y$

$$\begin{aligned} \alpha_y &= L^3/12EI_y & \alpha_z &= L^3/12EI_z \\ \beta_y &= L/AvG_y & \beta_z &= L/AvG_z \end{aligned}$$

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$$[F] = \begin{bmatrix} P_{ix} \\ P_{iy} \\ P_{iz} \\ M_{ix} \\ M_{iy} \\ M_{iz} \\ P_{jx} \\ P_{jy} \\ P_{jz} \\ M_{jx} \\ M_{jy} \\ M_{jz} \end{bmatrix} \quad [\Delta] = \begin{bmatrix} \delta_{ix} \\ \delta_{iy} \\ \delta_{iz} \\ \theta_{ix} \\ \theta_{iy} \\ \theta_{iz} \\ \delta_{jx} \\ \delta_{jy} \\ \delta_{jz} \\ \theta_{jx} \\ \theta_{jy} \\ \theta_{jz} \end{bmatrix}$$

GEOMETRIC STIFFNESS

An ordinary stiffness analysis, whether it includes shear deformations or not, makes no adjustments for the changing geometry of a loaded structure. Forces and moments are calculated from the original positions of the joints, not from their deformed positions. Elastic buckling, which is a function of joint deformations, is therefore impossible to predict using ordinary stiffness analysis.

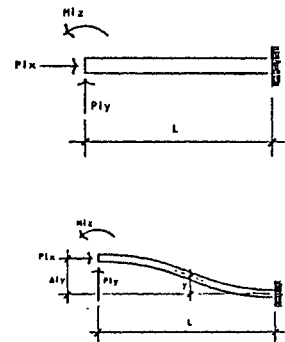
A procedure to include member and joint deformations in force and moment calculations has been developed by assuming a deformed shape and calculating the additional moment such a deformation would cause. Consider the two beams shown at right. The first is a beam used to derive the ordinary elastic stiffness matrix: the moment in the beam is a function of only the end shears and moments, as given by the following equation.

$$M_x = P_{iy}(x) - M_{iz}$$

The second is a beam showing bending deformations: the internal moment is a function of not only the end shears and moments, but of also the axial force multiplied by the beam's lateral deflection, y :

$$M_x = P_{iy}(x) - M_{iz} + P_{ix}(y)$$

This additional moment, the product of axial force (P_{ix}) and lateral deflection (y), is usually called the "P-delta effect."



To derive a stiffness matrix that includes the P-delta effect, one must consider the equilibrium of the deformed beam. Using the principle of superposition, one can first consider a beam with a known deflection while the rotation is held to zero (see sketches at right). The internal bending moment in the beam is found from equilibrium:

$$M_x = P_{ix}(\Delta i_y - y) + P_{iy}(x) - M_{iz}$$

If y is assumed to be a cubic function (it is actually a trigonometric function), the four known boundary conditions give the following solution.

$$y = ax^3 + bx^2 + cx + d$$

$$y(0) = \Delta i_y$$

$$y(L) = 0$$

$$y'(0) = 0$$

$$y'(L) = 0$$

$$y = \Delta i_y(2x^3/L^3 - 3x^2/L^2 + 1)$$

And the internal moment is

$$M_x = P_{ix}\Delta i_y(-2x^3/L^3 + 3x^2/L^2) + P_{iy}(x) - M_{iz}$$

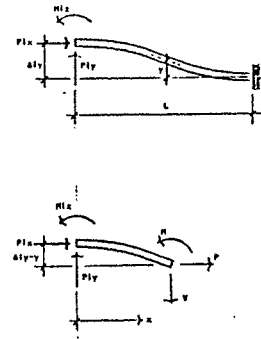
As in the basic elastic stiffness derivation, it is assumed that axial shortening is a separate phenomenon, caused only by the axial force, P_{ix} . This geometric stiffness derivation, therefore, only considers Δi_y and θ_{iz} .

From Castigliano's theorem, the rotation and deflection of the free end are

$$\Delta i_y = \int_0^L (M_x/EI) (\partial M_x / \partial P_{iy}) dx$$

$$\theta_{iz} = \int_0^L (M_x/EI) (\partial M_x / \partial M_{iz}) dx$$

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The derivatives of the expressions with respect to shear force and moment are

$$\partial M_x / \partial P_{iy} = x$$

$$\partial M_x / \partial M_{iz} = -1$$

It should be noted that rotation and deflection are functions of moment only, since shear deformations are ignored in this derivation. They will, however, be considered in the following chapter.

Setting θ_{iz} to zero and solving for P_{iy} and M_{iz} in terms of the deflection gives two terms in the stiffness matrix:

$$\theta_{iz} = \int_0^L (1/EI) [P_{ix}\Delta i_y(2x^3/L^3 - 3x^2/L^2) - P_{iy}(x) + M_{iz}] dx$$

$$= P_{ix}\Delta i_y(-L/2EI) + P_{iy}(-L^2/2EI) + M_{iz}(L/EI) = 0$$

$$M_{iz} = P_{ix}\Delta i_y(L/2) + P_{iy}(L/2)$$

$$\Delta i_y = \int_0^L (1/EI) [P_{ix}\Delta i_y(-2x^4/L^3 + 3x^3/L^2) + P_{iy}(x^2) - M_{iz}(x)] dx$$

$$= P_{ix}\Delta i_y(7L^2/20EI) + P_{iy}(L^3/3EI) + M_{iz}(-L^2/2EI)$$

$$= P_{ix}\Delta i_y(L^2/10EI) + P_{iy}(L^3/12EI)$$

$$P_{iy} = P_{ix}\Delta i_y(-6/5L) + \Delta i_y(12EI/L^3)$$

$$M_{iz} = P_{ix}\Delta i_y(-1/10) + \Delta i_y(6EI/L^2)$$

Now consider a beam with a known rotation while the deflection is held to zero (see sketches next page). Again, assuming y to be a cubic, the internal bending moment is found from the boundary conditions:

$$y = ax^3 + bx^2 + cx + d$$

$$y(0) = 0$$

$$y(L) = 0$$

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$$\begin{aligned}
 y'(\bar{0}) &= \theta_{iz} \\
 y'(L) &= 0 \\
 y &= \theta_{iz}(x^3/L^2 - 2x^2/L + x) \\
 M_x &= -P_{ix}(y) + P_{iy}(x) - M_{iz} \\
 &= P_{ix}\theta_{iz}(-x^3/L^2 + 2x^2/L - x) + P_{iy}(x) - M_{iz}
 \end{aligned}$$

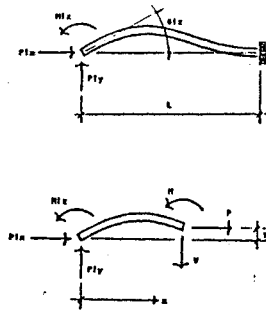
And the rotation and deflection from Castigliano's theorem are

$$\begin{aligned}
 \Delta i_y &= \int_0^L (M_x/EI) (\partial M_x / \partial P_{iy}) dx \\
 \theta_{iz} &= \int_0^L (M_x/EI) (\partial M_x / \partial M_{iz}) dx \\
 \partial M_x / \partial P_{iy} &= x \\
 \partial M_x / \partial M_{iz} &= -1
 \end{aligned}$$

Setting Δi_y to zero provides the other two terms in the stiffness matrix:

$$\begin{aligned}
 \theta_{iz} &= \int_0^L (1/EI) P_{ix}\theta_{iz}(x^3/L^2 - 2x^2/L + x) - P_{iy}(x) + M_{iz} dx \\
 &= P_{ix}\theta_{iz}(L^2/12EI) + P_{iy}(-L^2/2EI) + M_{iz}(L/EI) \\
 M_{iz} &= P_{ix}\theta_{iz}(-L/12) + P_{iy}(L/2) + \theta_{iz}(EI/L) \\
 \Delta i_y &= \int_0^L (1/EI) P_{ix}\theta_{iz}(x^4/L^2 - 2x^3/L + x^2) + P_{iy}(x^2) - M_{iz}(x) dx \\
 &= P_{ix}\theta_{iz}(-L^3/30EI) + P_{iy}(L^3/3EI) + M_{iz}(-L^2/2EI) \\
 &= P_{ix}\theta_{iz}(L^3/120EI) + P_{iy}(L^3/12EI) + \theta_{iz}(-L/2) = 0 \\
 P_{iy} &= P_{ix}\theta_{iz}(-1/10) + \theta_{iz}(6EI/L^2) \\
 M_{iz} &= P_{ix}\theta_{iz}(-2L/15) + \theta_{iz}(4EI/L)
 \end{aligned}$$

Each term in the stiffness matrix contains two values. One value is a function of deformation only (Δi_y or θ_{iz}), and is the same as for the basic stiffness matrix. The second value is a function of both deformation and



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axial force (P_{ix}); it is an additional "geometric" stiffness term. These equations can be written in matrix form as follows:

$$\begin{bmatrix} P_{iy} \\ M_{iz} \end{bmatrix} = \begin{bmatrix} 12EI/L^3 & 6EI/L^2 \\ 6EI/L^2 & 4EI/L \end{bmatrix} + (P_{ix}) \begin{bmatrix} -6/5L & -1/10 \\ -1/10 & -2L/15 \end{bmatrix} \begin{bmatrix} \Delta i_y \\ \theta_{iz} \end{bmatrix}$$

The matrix on the left--the basic elastic stiffness matrix--will be called $[K]$, and the matrix on the right--the geometric stiffness matrix--will be called $[G]$, so that this equation can be rewritten:

$$\{f\} = ([K] + (P_{ix}) [G]) \{\Delta\}$$

where

$\{f\}$ is a vector of forces

$\{\Delta\}$ is a vector of corresponding vector of deformations

Through appropriate transformations, keeping in mind that axial deformation (Δi_x) and torsional rotation (θ_{ix}) are unaffected by the geometric stiffness component, this equation can be expanded to include the 12×12 elastic and geometric stiffness matrices and the force and deformation vectors shown on the following pages.

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ELASTIC STIFFNESS MATRIX

$$[K] = \begin{bmatrix} \frac{EAx}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{EAx}{L} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{12EIz}{L^3} & 0 & 0 & 0 & \frac{6EIz}{L^2} & 0 & -\frac{12EIz}{L^3} & 0 & 0 & 0 & \frac{6EIz}{L^2} \\ 0 & 0 & \frac{12EIy}{L^3} & 0 & -\frac{6EIy}{L^2} & 0 & 0 & 0 & -\frac{12EIy}{L^3} & 0 & -\frac{6EIy}{L^2} & 0 \\ 0 & 0 & 0 & \frac{Gix}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{Gix}{L} & 0 & 0 \\ 0 & 0 & -\frac{6EIy}{L^2} & 0 & \frac{4EIy}{L} & 0 & 0 & 0 & \frac{6EIy}{L^2} & 0 & \frac{2EIy}{L} & 0 \\ 0 & \frac{6EIz}{L^2} & 0 & 0 & 0 & \frac{4EIz}{L} & 0 & -\frac{6EIz}{L^2} & 0 & 0 & 0 & \frac{2EIz}{L} \\ -\frac{EAx}{L} & 0 & 0 & 0 & 0 & 0 & \frac{EAx}{L} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{12EIz}{L^3} & 0 & 0 & 0 & -\frac{6EIz}{L^2} & 0 & \frac{12EIz}{L^3} & 0 & 0 & 0 & -\frac{6EIz}{L^2} \\ 0 & 0 & -\frac{12EIy}{L^3} & 0 & \frac{6EIy}{L^2} & 0 & 0 & 0 & \frac{12EIy}{L^3} & 0 & \frac{6EIy}{L^2} & 0 \\ 0 & 0 & 0 & -\frac{Gix}{L} & 0 & 0 & 0 & 0 & 0 & \frac{Gix}{L} & 0 & 0 \\ 0 & 0 & -\frac{6EIy}{L^2} & 0 & \frac{2EIy}{L} & 0 & 0 & 0 & \frac{6EIy}{L^2} & 0 & \frac{4EIy}{L} & 0 \\ 0 & \frac{6EIz}{L^2} & 0 & 0 & 0 & \frac{2EIz}{L} & 0 & -\frac{6EIz}{L^2} & 0 & 0 & 0 & \frac{4EIz}{L} \end{bmatrix}$$

GEOMETRIC STIFFNESS MATRIX

$$[G] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{6}{5L} & 0 & 0 & 0 & -\frac{1}{10} & 0 & \frac{6}{5L} & 0 & 0 & 0 & -\frac{1}{10} \\ 0 & 0 & -\frac{6}{5L} & 0 & -\frac{1}{10} & 0 & 0 & 0 & \frac{6}{5L} & 0 & \frac{1}{10} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{10} & 0 & -\frac{2L}{15} & 0 & 0 & 0 & -\frac{1}{10} & 0 & \frac{L}{30} & 0 \\ 0 & -\frac{1}{10} & 0 & 0 & 0 & -\frac{2L}{15} & 0 & \frac{1}{10} & 0 & 0 & 0 & \frac{L}{30} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{6}{5L} & 0 & 0 & 0 & \frac{1}{10} & 0 & -\frac{6}{5L} & 0 & 0 & 0 & \frac{1}{10} \\ 0 & 0 & \frac{6}{5L} & 0 & -\frac{1}{10} & 0 & 0 & 0 & -\frac{6}{5L} & 0 & -\frac{1}{10} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{10} & 0 & \frac{L}{30} & 0 & 0 & 0 & -\frac{1}{10} & 0 & -\frac{2L}{15} & 0 \\ 0 & -\frac{1}{10} & 0 & 0 & 0 & \frac{L}{30} & 0 & \frac{1}{10} & 0 & 0 & 0 & -\frac{2L}{15} \end{bmatrix}$$

FORCE AND DEFORMATION VECTORS

$$[F] = \begin{bmatrix} Pix \\ Piy \\ Pilz \\ Mix \\ Miy \\ Miz \\ Pjx \\ Pjy \\ Pjz \\ Mjx \\ Mjy \\ Mjz \end{bmatrix}$$

$$[\Delta] = \begin{bmatrix} \Delta ix \\ \Delta iy \\ \Delta iz \\ \theta ix \\ \theta iy \\ \theta iz \\ \Delta jx \\ \Delta jy \\ \Delta jz \\ \theta jx \\ \theta jy \\ \theta jz \end{bmatrix}$$

It should be apparent now that a geometric stiffness analysis is an iterative technique, since member axial forces must be found before they can be included in the geometric stiffness matrix. An iterative approach to geometric stiffness is outlined below:

- $[k]$ = basic elastic member stiffness matrix
- $[K]$ = basic elastic joint stiffness matrix
- $[g]$ = geometric member stiffness matrix
- $[G]$ = geometric joint stiffness matrix
- $\{\delta\}$ = member deformation vector
- $\{\Delta\}$ = joint deformation vector
- $\{f\}$ = member force and moment vector
- $\{F\}$ = joint force and moment vector
- $\{F_0\}$ = initial joint load vector

PROCEDURE

1. Calculate initial joint deformations using an ordinary elastic stiffness analysis:

$$\{\Delta_0\} = [K]^{-1}\{F\}$$

2. Calculate member forces and moments (for each member) from results of analysis above:

$$\{f_1\} = [k]\{\delta_0\}$$

3. Find new deformations by geometric stiffness analysis, using member forces and moments already found:

$$\{\Delta_1\} = ([K] + (P_{ix})[G])^{-1}\{F\}$$

Note that appropriate values of P_{ix} should be multiplied into individual member stiffness matrices ($[g]$) before they are assembled into the joint

stiffness matrix ($[G]$).

4. Find new member forces and moments (for each member) from results of most recent analysis:

$$\{f_2\} = ([k] + (P_{ix})[g])\{\delta_1\}$$

5. Repeat steps three and four until values for $\{F\}$ and $\{\Delta\}$ become satisfactorily stable.

If at any point in the iteration the determinant of the joint stiffness matrix ($[K] + (P_{ix})[G]$) becomes negative, the structure has buckled. Even though a solution may still be reached as the iteration closes, the deformations and forces found will have no relation to the actual magnitude of the deformations and forces of the buckled structure. The buckled shape, however, may be found by comparing the relative magnitudes of the deformations.

A simple method to check for buckling is to search for negative values on the major diagonal of the reduced stiffness matrix. The presence of even one negative number shows that the determinant is negative and that the structure has buckled.

The absence of negatives on the diagonal is not, however, proof of stability. Negatives may first appear as the structure buckles, but then may later disappear as it buckles further.

In the previous derivations of the geometric stiffness matrix, the deformed shape of the beam was assumed to be a third degree polynomial. The actual deformed shape of a buckled beam, as shown by Euler, is a transcendental equation involving sine and cosine functions. The following derivation uses just such a transcendental shape function.

UNIT DEFLECTION

Shape function

$$y = a \sin(\pi x/2L) + b \cos(\pi x/2L) + cx + d$$

$$y(0) = \Delta iy$$

$$y(L) = 0$$

$$y'(0) = 0$$

$$y'(L) = 0$$

$$y = \Delta iy \left(\frac{2}{4-\pi} \right) (-\sin(\pi x/2L) + \cos(\pi x/2L) + \pi x/2L + 1 - \pi/2)$$

Internal bending moment

$$\Sigma M = 0 = -P_{ix}(\Delta iy - y) - P_{iy}(x) + M_{iz} + M$$

$$M = P_{ix}(\Delta iy - y) + P_{iy}(x) - M_{iz}$$

$$= P_{ix} \Delta iy \left(\frac{2}{4-\pi} \right) (\sin(\pi x/2L) - \cos(\pi x/2L) - \pi x/2L + 1) + P_{iy}(x) - M_{iz}$$

$$\partial M / \partial P_{iy} = x$$

$$\partial M / \partial M_{iz} = -1$$

Castigliano

$$\Delta iy = \int_0^L (M/EI) (\partial M / \partial P_{iy}) dx$$

$$\begin{aligned} \Delta iy &= (1/EI) \int_0^L [P_{ix} \Delta iy \left(\frac{2}{4-\pi} \right) (x \sin(\pi x/2L) - x \cos(\pi x/2L) - \pi x^2/2L + x)] dx \\ &+ (1/EI) \int_0^L [P_{iy}(x^2) + M_{iz}(-x)] dx \\ &= P_{ix} \Delta iy (L^2/EI) \left(\frac{2}{4-\pi} \right) \left(\frac{48 - 12\pi + 3\pi^2 - \pi^3}{6\pi^2} \right) + P_{iy}(L^3/3EI) + M_{iz}(-L^2/2EI) \\ &= P_{ix} \Delta iy (0.3503020371L^2/EI) + P_{iy}(L^3/3EI) + M_{iz}(-L^2/2EI) \end{aligned}$$

$$\begin{aligned} \theta_{iz} &= \int_0^L (M/EI) (\partial M / \partial M_{iz}) dx \\ &= (1/EI) \int_0^L [P_{ix} \Delta iy \left(\frac{2}{4-\pi} \right) (-\sin(\pi x/2L) + \cos(\pi x/2L) + \pi x/2L - 1)] dx \\ &+ (1/EI) \int_0^L [P_{iy}(-x) + M_{iz}] dx \\ &= P_{ix} \Delta iy (-L/2EI) + P_{iy}(-L^2/2EI) + M_{iz}(L/EI) = 0 \end{aligned}$$

$$M_{iz} = P_{ix} \Delta iy (1/2) + P_{iy}(L/2)$$

$$\Delta iy = P_{ix} \Delta iy (0.1003020371L^2/EI) + P_{iy}(L^3/12EI)$$

$$P_{iy} = \Delta iy (12EI/L^3) - P_{ix} \Delta iy (1.203624445/L)$$

$$M_{iz} = \Delta iy (6EI/L^2) - P_{ix} \Delta iy (0.1018122225)$$

UNIT ROTATION

Shape function

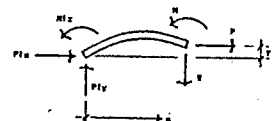
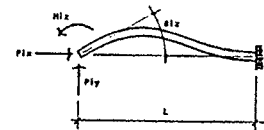
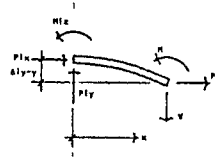
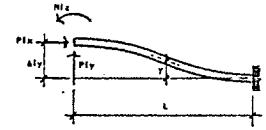
$$y = a \sin(\pi x/2L) + b \cos(\pi x/2L) + cx + d$$

$$y(0) = 0$$

$$y(L) = 0$$

$$y'(0) = \theta_{iz}$$

$$y'(L) = 0$$



$$y = \theta_{iz} \left(\frac{4L}{4\pi - \pi^2} \right) [(1 - \pi/2) \sin(\pi x/2L) + \cos(\pi x/2L) + \pi x/2L - 1]$$

Internal bending moment

$$\Sigma M = 0 = P_{ix}(y) - P_{iy}(x) + M_{iz} + M$$

$$M = P_{ix}(-y) + P_{iy}(x) - M_{iz}$$

$$= P_{ix} \theta_{iz} \left(\frac{4L}{4\pi - \pi^2} \right) [(\pi/2 - 1) \sin(\pi x/2L) - \cos(\pi x/2L) - \pi x/2L + 1] + P_{iy}(x) - M_{iz}$$

$$\partial M / \partial P_{iy} = x$$

$$\partial M / \partial M_{iz} = -1$$

Castigliano

$$\begin{aligned} \Delta i_y &= \int_0^L (M/EI) (\partial M / \partial P_{iy}) dx \\ &= (1/EI) \int_0^L [P_{ix} \theta_{iz} \left(\frac{4L}{4\pi - \pi^2} \right) ((\pi/2 - 1)x \sin(\pi x/2L) - x \cos(\pi x/2L) - \pi x^2/2L + x) + P_{iy}(x^2) + M_{iz}(-x)] dx \\ &= P_{ix} \theta_{iz} (L^3/EI) \left(\frac{6}{12\pi} - \frac{2\pi}{3\pi^2} \right) + P_{iy}(L^3/3EI) + M_{iz}(-L^2/2EI) \\ &= P_{ix} \theta_{iz} (-0.03500307227L^3/EI) + P_{iy}(L^3/3EI) + M_{iz}(-L^2/2EI) \end{aligned}$$

$$\begin{aligned} \theta_{iz} &= \int_0^L (M/EI) (\partial M / \partial M_{iz}) dx \\ &= (1/EI) \int_0^L [P_{ix} \theta_{iz} \left(\frac{4L}{4\pi - \pi^2} \right) ((1 - \pi/2) \sin(\pi x/2L) + \cos(\pi x/2L) + \pi x/2L - 1) + P_{iy}(-x) + M_{iz}] dx \end{aligned}$$

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$$\begin{aligned} \theta_{iz} &= P_{ix} \theta_{iz} (L^2/EI) \left(\frac{4}{\pi} - \frac{\pi}{\pi^2} \right) + P_{iy}(-L^2/2EI) + M_{iz}(L/EI) \\ &= P_{ix} \theta_{iz} (0.08697484832L^2/EI) + P_{iy}(-L^2/2EI) + M_{iz}(L/EI) \end{aligned}$$

$$M_{iz} = P_{ix} \theta_{iz} (-0.08697484832) + P_{iy}(L/2) + \theta_{iz}(EI/L)$$

$$\Delta i_y = P_{ix} \theta_{iz} (0.00848435189L^3/EI) + P_{iy}(L^3/12EI) - \theta_{iz}(L/2) = 0$$

$$P_{iy} = P_{ix} \theta_{iz} (-0.1018122227) + \theta_{iz}(6EI/L^2)$$

$$M_{iz} = P_{ix} \theta_{iz} (-0.1378809597L) + \theta_{iz}(4EI/L)$$

The 3x3 member stiffness matrix for the i end is obtained from superposition of the above two cases. The elastic stiffness is the same as for the previous derivation; only the geometric stiffness matrix is different:

$$\{f\} = ([K] + \{P_{ix}\}[G])\{\Delta\}$$

$$[G] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1.203624445/L & -0.1018122226 \\ 0 & -0.1018122226 & -0.1378809597(L) \end{bmatrix}$$

As always, appropriate transformations yield the 12x12 geometric stiffness matrix shown on the following page.

$$G = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1.2036}{L} & 0 & 0 & 0 & -0.1018 & 0 & \frac{1.2036}{L} & 0 & 0 & 0 & -0.1018 \\ 0 & 0 & -\frac{1.2036}{L} & 0 & 0.1018 & 0 & 0 & 0 & \frac{1.2036}{L} & 0 & 0.1018 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.1018 & 0 & -0.1379L & 0 & 0 & 0 & -0.1018 & 0 & 0.0361L & 0 \\ 0 & -0.1018 & 0 & 0 & 0 & -0.1379L & 0 & 0.1018 & 0 & 0 & 0 & 0.0361L \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1.2036}{L} & 0 & 0 & 0 & 0.1018 & 0 & -\frac{1.2036}{L} & 0 & 0 & 0 & 0.1018 \\ 0 & 0 & \frac{1.2036}{L} & 0 & -0.1018 & 0 & 0 & 0 & -\frac{1.2036}{L} & 0 & -0.1018 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.1018 & 0 & 0.0361L & 0 & 0 & 0 & -0.1018 & 0 & -0.1379L & 0 \\ 0 & -0.1018 & 0 & 0 & 0 & 0.0361L & 0 & 0.1018 & 0 & 0 & 0 & -0.1379L \end{bmatrix}$$

Values to ten significant figures: 1.2036 = 1.203624445
 0.1018 = 0.1018122226
 0.1379 = 0.1378809597
 0.0361 = 0.03606873710

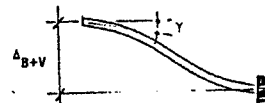
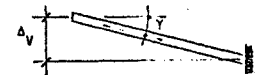
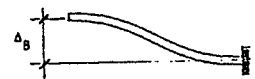
GEOMETRIC STIFFNESS INCLUDING SHEAR

A geometric stiffness matrix that includes the effect of shear deformations may be derived using the procedure already outlined. The difficulty one encounters in such a derivation is that of finding an appropriate shape function. At right are two beams--one showing only bending deformations, the other showing only shear deformations. It would seem that an appropriate shape function should combine these two kinds of deformations, so that the boundary conditions are as follows:

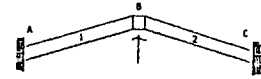
$$\begin{aligned} y(0) &= \Delta_B + \Delta_V \\ y(L) &= 0 \\ y'(0) &= \gamma_0 \\ y'(L) &= \gamma_L \end{aligned}$$

where Δ_B is the deflection due to bending only, Δ_V is the deflection due to shear only, and $\gamma = -Piy/AvG$, the shear strain (below right).

There are two difficulties with such a shape function, however. One problem is that γ is a function of Piy , which greatly complicates the iterative process described in the last chapter. The other problem, a far more serious one, is that while the member rotates through an angle γ as it undergoes shear deformation, the joint does not, and an inconsistency



arises when the joint stiffness matrix is assembled. Consider, for example, two members sharing a common joint (see sketch at right). If only shear deformations were included, the stiffness matrix for member 1 would show a positive rotation of joint B, while the stiffness matrix for member 2 would show a negative rotation for the same joint.



Clearly, a useful shape function must avoid this inconsistency while still closely approximating the deformed shape of the beam. A good approximation is the same cubic polynomial used in the previous derivation. This shape function sets rotation at each end to zero, which is necessary for the analysis to be continuous from joint to joint, but still includes deflections caused by both bending and shear (Δ_B and Δ_V). The boundary conditions for this approximation, then, are

$$\begin{aligned} y(0) &= \Delta_B + \Delta_V \\ y(L) &= 0 \\ y'(0) &= 0 \\ y'(L) &= 0 \end{aligned}$$

It can be shown that the difference between the moments calculated with this shape function and that calculated with a shape function including shear rotation is insignificant: the difference involves factors of $1/(EI)^2$ and $1/(AvG)^2$.

Proceeding with this approximate shape function, the derivation is straightforward. In this derivation, as in all previous stiffness derivations, the axial force is assumed to be constant along the length of the member.

UNIT DEFLECTION

Shape function

$$y = ax^3 + bx^2 + cx + d$$

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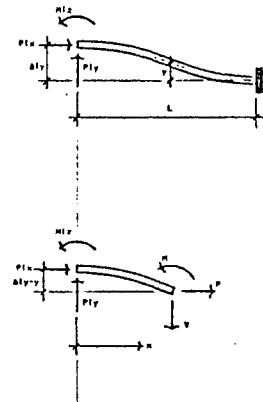
$$\begin{aligned} y(0) &= \Delta_{iy} \\ y(L) &= 0 \\ y'(0) &= 0 \\ y'(L) &= 0 \\ y &= \Delta_{iy}(2x^3/L^3 - 3x^2/L^2 + 1) \end{aligned}$$

Internal shear and moment

$$\begin{aligned} \Sigma M &= 0 = -P_{ix}(\Delta_{iy} - y) - P_{iy}(x) + M_{iz} + M \\ M &= P_{ix}(\Delta_{iy} - y) + P_{iy}(x) - M_{iz} \\ &= P_{ix}\Delta_{iy}(-2x^3/L^3 + 3x^2/L^2) + P_{iy}(x) - M_{iz} \\ V &= \partial M / \partial x = P_{ix}\Delta_{iy}(-6x^2/L^3) + 6x/L^2 + P_{iy} \\ \partial M / \partial P_{iy} &= x \\ \partial M / \partial M_{iz} &= -1 \\ \partial V / \partial P_{iy} &= 1 \\ \partial V / \partial M_{iz} &= 0 \end{aligned}$$

Castigliano

$$\begin{aligned} \Delta_{iy} &= \int_0^L (M/EI) (\partial M / \partial P_{iy}) dx + \int_0^L (V/AvG) (\partial V / \partial P_{iy}) dx \\ &= (1/EI) \int_0^L [P_{ix}\Delta_{iy}(-2x^4/L^3 + 3x^3/L^2) + P_{iy}(x^2) - M_{iz}(x)] dx + \\ &\quad (1/AvG) \int_0^L [P_{ix}\Delta_{iy}(-6x^2/L^3 + 6x/L^2) + P_{iy}] dx \\ &= P_{ix}\Delta_{iy}(7L^2/20EI + 1/AvG) + P_{iy}(L^3/3EI + L/AvG) + M_{iz}(-L^2/2EI) \\ \theta_{iz} &= \int_0^L (M/EI) (\partial M / \partial M_{iz}) dx + \int_0^L (V/AvG) (\partial V / \partial M_{iz}) dx \\ &= (1/EI) \int_0^L [P_{ix}\Delta_{iy}(2x^3/L^3 - 3x^2/L^2) + P_{iy}(-x) + M_{iz}] dx \end{aligned}$$



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$$\theta_{iz} = \text{Pix}\Delta i_y(-L/2EI) + \text{Piy}(-L^2/2EI) + \text{Miz}(L/EI) = 0$$

$$\text{Miz} = \text{Pix}\Delta i_y(L/2) + \text{Piy}(L/2)$$

$$\Delta i_y = \text{Pix}\Delta i_y(L^2/10EI + 1/\text{AvG}) + \text{Piy}(L^3/12EI + L/\text{AvG})$$

$$\text{Let } \alpha = L^3/12EI \text{ and } \beta = L/\text{AvG}$$

$$\Delta i_y = \text{Pix}\Delta i_y(6\alpha/5L + \beta/L) + \text{Piy}(\alpha + \beta)$$

$$\text{Piy} = \text{Pix}\Delta i_y\left(\frac{-6\alpha - 5\beta}{5L(\alpha + \beta)}\right) + \Delta i_y\left(\frac{1}{\alpha + \beta}\right)$$

$$\text{Miz} = \text{Pix}\Delta i_y\left(\frac{-\alpha}{10(\alpha + \beta)}\right) + \Delta i_y\left(\frac{L}{2(\alpha + \beta)}\right)$$

UNIT ROTATION

Shape function

$$y = ax^3 + bx^2 + cx + d$$

$$y(0) = 0$$

$$y(L) = 0$$

$$y'(0) = \theta_{iz}$$

$$y'(L) = 0$$

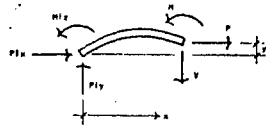
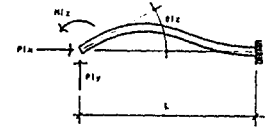
$$y = \theta_{iz}(x^3/L^2 - 2x^2/L + x)$$

Internal shear and moment

$$\Sigma M = 0 = \text{Pix}(y) - \text{Piy}(x) + \text{Miz} + M$$

$$\begin{aligned} M &= -\text{Pix}(y) + \text{Piy}(x) - \text{Miz} \\ &= \text{Pix}\theta_{iz}(-x^3/L^2 + 2x^2/L - x) + \text{Piy}(x) - \text{Miz} \end{aligned}$$

$$V = \partial M / \partial x = \text{Pix}\theta_{iz}(-3x^2/L^2 + 4x/L - 1) + \text{Piy}$$



$$\partial M / \partial \text{Piy} = x$$

$$\partial M / \partial \text{Miz} = -1$$

$$\partial V / \partial \text{Piy} = 1$$

$$\partial V / \partial \text{Miz} = 0$$

Castigliano

$$\begin{aligned} \Delta i_y &= \int_0^L (M/EI) (\partial M / \partial \text{Piy}) dx + \int_0^L (V/\text{AvG}) (\partial V / \partial \text{Piy}) dx \\ &= (1/EI) \int_0^L [\text{Pix}\theta_{iz}(-x^4/L^2 + 2x^3/L - x^2) + \text{Piy}(x^2) + \text{Miz}(-x)] dx + \\ &\quad (1/\text{AvG}) \int_0^L [\text{Pix}\theta_{iz}(-3x^2/L^2 + 4x/L - 1) + \text{Piy}] dx \\ &= \text{Pix}\theta_{iz}(-L^3/30EI) + \text{Piy}(L^3/3EI + L/\text{AvG}) + \text{Miz}(-L^2/2EI) \end{aligned}$$

$$\begin{aligned} \theta_{iz} &= \int_0^L (M/EI) (\partial M / \partial \text{Miz}) dx + \int_0^L (V/\text{AvG}) (\partial V / \partial \text{Miz}) dx \\ &= (1/EI) \int_0^L [\text{Pix}\theta_{iz}(x^3/L^2 - 2x^2/L + x) + \text{Piy}(-x) + \text{Miz}] dx \\ &= \text{Pix}\theta_{iz}(L^2/12EI) + \text{Piy}(-L^2/2EI) + \text{Miz}(L/EI) \end{aligned}$$

$$\text{Miz} = \text{Pix}\theta_{iz}(-L/2) + \text{Piy}(L/2) + \theta_{iz}(EI/L)$$

$$\Delta i_y = \text{Pix}\theta_{iz}(L^3/120EI) + \text{Piy}(L^3/12EI) + L/\text{AvG} + \theta_{iz}(-L/2)$$

$$\text{Again, let } \alpha = L^3/12EI \text{ and } \beta = L/\text{AvG}$$

$$\Delta i_y = \text{Pix}\theta_{iz}(\alpha/10) + \text{Piy}(\alpha + \beta) + \theta_{iz}(-L/2) = 0$$

$$\text{Piy} = \text{Pix}\theta_{iz}\left(\frac{-\alpha}{10(\alpha + \beta)}\right) + \theta_{iz}\left(\frac{L}{2(\alpha + \beta)}\right)$$

$$\text{Miz} = \text{Pix}\theta_{iz}\left(\frac{-8\alpha L - 5\beta L}{60(\alpha + \beta)}\right) + \theta_{iz}\left(\frac{EI(4\alpha + \beta)}{L(\alpha + \beta)}\right)$$

STIFFNESS MATRIX

Using the principle of superposition, P_{iy} and M_{iz} for the two cases (unit deflection and unit rotation) can be added together. These equations describe the general case of combined deflection and rotation:

$$P_{iy} = P_{ix}\Delta i y \left(\frac{-6\alpha - 5\beta}{5L(\alpha + \beta)} \right) + P_{ix}\theta iz \left(\frac{-\alpha}{10(\alpha + \beta)} \right) + \Delta i y \left(\frac{1}{\alpha + \beta} \right) + \theta iz \left(\frac{L}{2(\alpha + \beta)} \right)$$

$$M_{iz} = P_{ix}\Delta i y \left(\frac{-\alpha}{10(\alpha + \beta)} \right) + P_{ix}\theta iz \left(\frac{-8\alpha L - 5\beta L}{60(\alpha + \beta)} \right) + \Delta i y \left(\frac{L}{2(\alpha + \beta)} \right) + \theta iz \left(\frac{EI(4\alpha + \beta)}{L(\alpha + \beta)} \right)$$

Or in matrix form:

$$\begin{bmatrix} P_{iy} \\ M_{iz} \end{bmatrix} = \left(\begin{bmatrix} \frac{1}{\alpha + \beta} & \frac{L}{2(\alpha + \beta)} \\ \frac{L}{2(\alpha + \beta)} & \frac{EI(4\alpha + \beta)}{L(\alpha + \beta)} \end{bmatrix} + (P_{ix}) \begin{bmatrix} \frac{-6\alpha - 5\beta}{5L(\alpha + \beta)} & \frac{-\alpha}{10(\alpha + \beta)} \\ \frac{-\alpha}{10(\alpha + \beta)} & \frac{-8\alpha L - 5\beta L}{60(\alpha + \beta)} \end{bmatrix} \right) \begin{bmatrix} \Delta i y \\ \theta iz \end{bmatrix}$$

By including the axial stiffness (AE/L and GI/L) and applying, as always, the appropriate transformations, the three dimensional member stiffness matrices shown on the following pages can be obtained. Note that the basic elastic stiffness matrix ($[k]$) is the same as in the prior shear stiffness derivation.

$$[K] = \begin{bmatrix} \frac{EAx}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{EAx}{L} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\alpha z + \beta z} & 0 & 0 & 0 & \frac{L}{2(\alpha z + \beta z)} & 0 & \frac{-1}{\alpha z + \beta z} & 0 & 0 & 0 & \frac{L}{2(\alpha z + \beta z)} \\ 0 & 0 & \frac{1}{\alpha y + \beta y} & 0 & \frac{-L}{2(\alpha y + \beta y)} & 0 & 0 & 0 & \frac{-1}{\alpha y + \beta y} & 0 & \frac{-L}{2(\alpha y + \beta y)} & 0 \\ 0 & 0 & 0 & \frac{GJx}{L} & 0 & 0 & 0 & 0 & 0 & \frac{-GJx}{L} & 0 & 0 \\ 0 & 0 & \frac{-L}{2(\alpha y + \beta y)} & \frac{EJy(4\alpha y + \beta y)}{L(\alpha y + \beta y)} & 0 & 0 & 0 & \frac{L}{2(\alpha y + \beta y)} & 0 & \frac{EJy(2\alpha y - \beta y)}{L(\alpha y + \beta y)} & 0 & 0 \\ 0 & \frac{L}{2(\alpha z + \beta z)} & 0 & 0 & 0 & \frac{EJz(4\alpha z + \beta z)}{L(\alpha z + \beta z)} & 0 & \frac{L}{2(\alpha z + \beta z)} & 0 & 0 & 0 & \frac{EJz(2\alpha z - \beta z)}{L(\alpha z + \beta z)} \\ -\frac{EAx}{L} & 0 & 0 & 0 & 0 & 0 & \frac{EAx}{L} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-1}{\alpha z + \beta z} & 0 & 0 & 0 & \frac{-L}{2(\alpha z + \beta z)} & 0 & \frac{1}{\alpha z + \beta z} & 0 & 0 & 0 & \frac{-L}{2(\alpha z + \beta z)} \\ 0 & 0 & \frac{-1}{\alpha y + \beta y} & 0 & \frac{L}{2(\alpha y + \beta y)} & 0 & 0 & 0 & \frac{1}{\alpha y + \beta y} & 0 & \frac{L}{2(\alpha y + \beta y)} & 0 \\ 0 & 0 & 0 & \frac{-GJx}{L} & 0 & 0 & 0 & 0 & 0 & \frac{GJx}{L} & 0 & 0 \\ 0 & 0 & \frac{-L}{2(\alpha y + \beta y)} & \frac{EJy(2\alpha y - \beta y)}{L(\alpha y + \beta y)} & 0 & 0 & 0 & \frac{L}{2(\alpha y + \beta y)} & 0 & \frac{EJy(4\alpha y + \beta y)}{L(\alpha y + \beta y)} & 0 & 0 \\ 0 & \frac{L}{2(\alpha z + \beta z)} & 0 & 0 & 0 & \frac{EJz(2\alpha z - \beta z)}{L(\alpha z + \beta z)} & 0 & \frac{-L}{2(\alpha z + \beta z)} & 0 & 0 & 0 & \frac{EJz(4\alpha z + \beta z)}{L(\alpha z + \beta z)} \end{bmatrix}$$

$$\alpha y = L^3/12EJy \quad \beta y = L/GJy$$

$$\alpha z = L/12EI \quad \beta z = L/GJz$$

0	0	0	0	0	0	0	0	0	0	0	0
0	$-\frac{(6\alpha z - 5\beta z)}{SL(\alpha z + \beta z)}$	0	0	0	$-\frac{\alpha z}{10(\alpha z + \beta z)}$	0	$\frac{(6\alpha z + 5\beta z)}{SL(\alpha z + \beta z)}$	0	0	0	$-\frac{\alpha z}{10(\alpha z + \beta z)}$
0	0	$-\frac{(6\alpha y + 5\beta y)}{SL(\alpha y + \beta y)}$	0	$\frac{\alpha y}{10(\alpha y + \beta y)}$	0	0	0	$\frac{(6\alpha y + 5\beta y)}{SL(\alpha y + \beta y)}$	0	$\frac{\alpha y}{10(\alpha y + \beta y)}$	0
0	0	0	0	0	0	0	0	0	0	0	0
0	0	$\frac{\alpha y}{10(\alpha y + \beta y)}$	0	$-\frac{L(8\alpha y + 5\beta y)}{60(\alpha y + \beta y)}$	0	0	0	$-\frac{\alpha y}{10(\alpha y + \beta y)}$	0	$\frac{L(2\alpha y + 5\beta y)}{60(\alpha y + \beta y)}$	0
0	$-\frac{\alpha z}{10(\alpha z + \beta z)}$	0	0	0	$-\frac{L(8\alpha z + 5\beta z)}{60(\alpha z + \beta z)}$	0	$\frac{\alpha z}{10(\alpha z + \beta z)}$	0	0	0	$\frac{L(2\alpha z + 5\beta z)}{60(\alpha z + \beta z)}$
0	0	0	0	0	0	0	0	0	0	0	0
0	$\frac{6\alpha z + 5\beta z}{SL(\alpha z + \beta z)}$	0	0	0	$\frac{\alpha z}{10(\alpha z + \beta z)}$	0	$-\frac{(6\alpha z + 5\beta z)}{SL(\alpha z + \beta z)}$	0	0	0	$\frac{\alpha z}{10(\alpha z + \beta z)}$
0	0	$\frac{(6\alpha y + 5\beta y)}{SL(\alpha y + \beta y)}$	0	$\frac{\alpha y}{10(\alpha y + \beta y)}$	0	0	0	$-\frac{(6\alpha y + 5\beta y)}{SL(\alpha y + \beta y)}$	0	$\frac{\alpha y}{10(\alpha y + \beta y)}$	0
0	0	0	0	0	0	0	0	0	0	0	0
0	0	$\frac{\alpha y}{10(\alpha y + \beta y)}$	0	$\frac{L(2\alpha y + 5\beta y)}{60(\alpha y + \beta y)}$	0	0	0	$\frac{\alpha y}{10(\alpha y + \beta y)}$	0	$-\frac{L(8\alpha y + 5\beta y)}{60(\alpha y + \beta y)}$	0
0	$-\frac{\alpha z}{10(\alpha z + \beta z)}$	0	0	0	$\frac{L(2\alpha z + 5\beta z)}{60(\alpha z + \beta z)}$	0	$\frac{\alpha z}{10(\alpha z + \beta z)}$	0	0	0	$-\frac{L(8\alpha z + 5\beta z)}{60(\alpha z + \beta z)}$

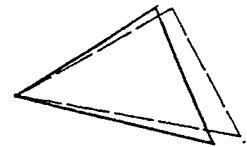
$\alpha y = L/12EI_y$ $\alpha z = L/12MI_z$
 $\beta y = L/GA_y$ $\beta z = L/GA_z$

FINITE ELEMENTS

The finite elements method of structural analysis is similar to the stiffness method in that it relates forces to displacements via a set of simultaneous equations. The distinction between the two methods is in the character of the elements: the stiffness method uses bending theory to derive equations for a one-dimensional beam element, while the finite elements method uses the theory of elasticity to derive equations for two and three dimensional solid elements. Indeed, the stiffness method can be thought of as a special case of the finite elements method using straight line elements.

The object of the finite elements method, then, is to mathematically describe a continuum by modelling it as a set of pieces. A stiffness matrix is formulated to describe each piece, and these are assembled into a larger matrix that describes the entire structure. Finite elements can be used to describe various structures for which beam analysis would be inappropriate, such as membrane structures, plates and shells, and one and two way slab systems.

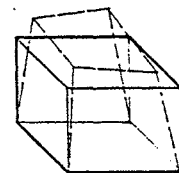
TYPICAL FINITE ELEMENTS



Plane stress



Plate bending

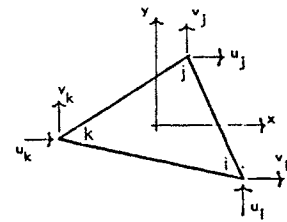


3-D Solid



Beam

The simplest commonly used finite element is a triangular plane stress element. It has three nodes and six possible displacements (a vertical and horizontal displacement at each node). Since it is a coplanar element, it has no resistance to loads normal to its surface; it can only resist forces acting in its own plane.



The horizontal nodal displacements (called "u") and the vertical nodal displacements (called "v") are shown on the sketch at right. The element stiffness matrix ($[k]$) will relate these displacements to forces applied at the nodes:

$$\{f\} = [k]\{u\}$$

where

$\{f\}$ = a vector of nodal forces

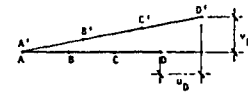
$\{u\}$ = a vector of nodal displacements

Examining first a single straight line, it can be seen that the displacement of any point on the line is a function of its location along the line (below right). Let A, B, C, and D be points spaced equally along the line and let u_D and v_D be the horizontal and vertical displacements of point D. It is apparent that point C moves only two-thirds as far as point D and B moves only one-third as far as point D, while A does not move at all. In general, the displacements u_x and v_x of any point on the line are related to u_D and v_D by the following equations:

$$u_x = (x/L)u_D$$

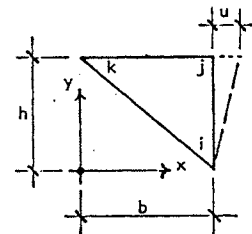
$$v_x = (x/L)v_D$$

where x is measured from point A.



The displacements $u_{x,y}$ and $v_{x,y}$ of any point on a triangular element are analogous to the displacements of points on a straight line. They are functions of the coordinates of the point, but since a triangular element has two dimensions the displacements are functions of both x and y .

Consider the triangular element at right with a single horizontal deformation at node j . Assuming the displacement of any point on the element to be a linear function of the x and y coordinates, a general first-order polynomial and three boundary conditions should be sufficient to describe the displacement function:



$$u_{x,y} = a_1 + a_2x + a_3y$$

$$x_i = b \quad y_i = 0$$

$$x_j = b \quad y_j = h$$

$$x_k = 0 \quad y_k = h$$

$$u_i = a_1 + a_2b = 0$$

$$u_j = a_1 + a_2b + a_3h = u$$

$$u_k = a_1 + a_3h = 0$$

Solving for a_1 , a_2 and a_3 gives:

$$u_{x,y} = u(x/b + y/h - 1)$$

SHAPE FUNCTION

The discussion so far has been limited to the deformation of a single node, with the other nodes held fixed. To completely describe the stiffness of a triangular plane stress element, one must consider both horizontal and vertical displacements of all three nodes simultaneously. The derivation starts with two linear polynomials in x and y :

$$u = a_1 + a_2x + a_3y$$

$$v = a_4 + a_5x + a_6y$$

These polynomials describe the assumed displacements of an arbitrary point on the element and are called shape functions. The choice of a shape function depends upon the number of degrees of freedom, or possible displacements, of the element. Since the triangular plane stress element has six degrees of freedom (two displacements at each node), the assumed shape function must have six unknown constants (a_1 to a_6).

To find these six unknowns, the coordinates of the three nodes ($x_i, x_j, x_k, y_i, y_j, y_k$) are substituted into the shape functions, which will then give the six nodal displacements:

$$u_i = a_1 + a_2x_i + a_3y_i$$

$$v_i = a_4 + a_5x_i + a_6y_i$$

$$u_j = a_1 + a_2x_j + a_3y_j$$

$$v_j = a_4 + a_5x_j + a_6y_j$$

$$u_k = a_1 + a_2x_k + a_3y_k$$

$$v_k = a_4 + a_5x_k + a_6y_k$$

or in matrix notation:

$$\{u\} = [A]\{a\}$$

where

$$\{u\} = \begin{bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_k \\ v_k \end{bmatrix} \quad [A] = \begin{bmatrix} 1 & x_i & y_i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_i & y_i \\ 1 & x_j & y_j & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_j & y_j \\ 1 & x_k & y_k & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_k & y_k \end{bmatrix} \quad \{a\} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix}$$

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The vector of constants, $\{a\}$, is found by inverting $[A]$:

$$\{a\} = [A]^{-1}\{u\}$$

and the general equations for u and v at any point (x,y) are given in matrix form:

$$u = \{1 \ x \ y \ 0 \ 0 \ 0\} [A]^{-1}\{u\}$$

$$v = \{0 \ 0 \ 0 \ 1 \ x \ y\} [A]^{-1}\{u\}$$

STRAIN

From theory of elasticity, the strains are defined as partial derivatives of u and v :

$$\epsilon_x = \partial u / \partial x$$

$$\epsilon_y = \partial v / \partial y$$

$$\gamma_{xy} = \partial u / \partial y + \partial v / \partial x$$

Taking derivatives of the previously defined equations for u and v gives:

$$\partial u / \partial x = \{0 \ 1 \ 0 \ 0 \ 0 \ 0\} [A]^{-1}\{u\}$$

$$\partial u / \partial y = \{0 \ 0 \ 1 \ 0 \ 0 \ 0\} [A]^{-1}\{u\}$$

$$\partial v / \partial x = \{0 \ 0 \ 0 \ 0 \ 1 \ 0\} [A]^{-1}\{u\}$$

$$\partial v / \partial y = \{0 \ 0 \ 0 \ 0 \ 0 \ 1\} [A]^{-1}\{u\}$$

and the strains can be expressed in matrix form:

$$\epsilon_x = \{0 \ 1 \ 0 \ 0 \ 0 \ 0\} [A]^{-1}\{u\}$$

$$\epsilon_y = \{0 \ 0 \ 0 \ 0 \ 0 \ 1\} [A]^{-1}\{u\}$$

$$\gamma_{xy} = \{0 \ 0 \ 1 \ 0 \ 1 \ 0\} [A]^{-1}\{u\}$$

Combining these three strains into a column vector gives the following matrix equation:

$$\{\epsilon\} = [L][A]^{-1}\{u\}$$

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where

$$\{\epsilon\} = \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} \quad [L] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

The matrix that relates strain, $\{\epsilon\}$, to nodal displacements, $\{u\}$, is the product of $[L]$ with the inverse of $[A]$. This matrix will be called $[B]$:

$$[B] = [L][A]^{-1}$$

$$\{\epsilon\} = [B]\{u\}$$

STRESS

Resorting again to the theory of elasticity, stresses are related to strains by the following equations:

$$\begin{aligned} \epsilon_x &= (1/E)\sigma_x + (-\nu/E)\sigma_y \\ \epsilon_y &= (-\nu/E)\sigma_x + (1/E)\sigma_y \\ \gamma_{xy} &= (2(1+\nu)/E)\tau_{xy} \end{aligned}$$

where ν is Poisson's ratio.

This can be written in matrix form

$$\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} = (1/E) \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}$$

and inverted to give stresses in terms of strains:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = (E/(1-\nu^2)) \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix}$$

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The matrix that relates stress to strain is called the elasticity matrix, $[E]$. Symbolically, the previous equation can be written

$$\{\sigma\} = [E]\{\epsilon\} = [E][B]\{u\}$$

where

$$\{\sigma\} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} \quad [E] = (E/(1-\nu^2)) \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}$$

STIFFNESS MATRIX

Thus far, an equation has been derived that relates stresses in the element (which are constant, given the initial assumption of linear displacement) to nodal displacements:

$$\{\sigma\} = [E][B]\{u\}$$

The terms in matrix $[B]$ are the nodal coordinates, and the terms in matrix $[E]$ are the material properties of elastic modulus and Poisson's ratio. It remains to derive an equation that relates these stresses to nodal forces, which will complete the elemental stiffness equation

$$\{f\} = [k]\{u\}$$

To complete the derivation, the external and internal work of the element must be related:

$$W_e = W_i$$

The total external work (W_e) done by the nodal forces is found by multiplying those forces by the corresponding nodal displacements:

$$W_e = u_i f_{ix} + v_i f_{iy} + u_j f_{jx} + v_j f_{jy} + u_k f_{kx} + v_k f_{ky} = \{u\}^T \{f\}$$

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Similarly, the internal work per unit volume (W_i) is the sum of the products of the corresponding strains and stresses:

$$\partial W_i / \partial \text{Vol} = \epsilon_x \sigma_x + \epsilon_y \sigma_y + \gamma_{xy} \tau_{xy} = \{\epsilon\}^T \{\sigma\}$$

The total internal work is the integral of this matrix product over the volume of the element:

$$W_i = \int (\partial W_i / \partial \text{Vol}) d\text{Vol} = \int \{\epsilon\}^T \{\sigma\} d\text{Vol}$$

Since stress and strain are constant in the triangular plane stress element, these two quantities may be moved to the left of the integral sign:

$$W_i = \{\epsilon\}^T \{\sigma\} \int d\text{Vol} = \{\epsilon\}^T \{\sigma\} (\text{Vol})$$

where (Vol) is simply the volume of the element, i. e., its area times its thickness.

It has already been shown that stresses and strains can be expressed in terms of nodal displacements:

$$\begin{aligned} \{\epsilon\} &= [B] \{u\} \\ \{\epsilon\}^T &= \{u\}^T [B]^T \\ \{\sigma\} &= [E] [B] \{u\} \end{aligned}$$

Equating external and internal work finally gives the stiffness matrix for the element:

$$\begin{aligned} W_e &= W_i \\ \{u\}^T \{f\} &= \{\epsilon\}^T \{\sigma\} (\text{Vol}) \\ \{u\}^T \{f\} &= \{u\}^T [B]^T [E] [B] \{u\} (\text{Vol}) \\ \{f\} &= [B]^T [E] [B] \{u\} (\text{Vol}) \\ [k] &= [B]^T [E] [B] (\text{Vol}) \end{aligned}$$

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The volume of the element (Vol) is its thickness times its area, which is found by taking the determinant of the following coordinate matrix:

$$(\text{Vol}) = t \times (1/2) \det \begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{bmatrix}$$

where t is the thickness of the element.

In summary, the element stiffness matrix is found from the following equation:

$$[k] = [B]^T [E] [B] (\text{Vol})$$

where

$$[B] = [L] [A]^{-1}$$

$$[L] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

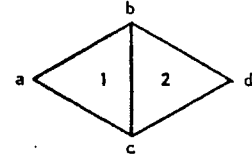
$$[A] = \begin{bmatrix} 1 & x_i & y_i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_i & y_i \\ 1 & x_j & y_j & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_j & y_j \\ 1 & x_k & y_k & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_k & y_k \end{bmatrix}$$

$$[E] = (E/(1 - \nu^2)) \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1 - \nu)/2 \end{bmatrix}$$

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The stiffness matrix, $[k]$, is a 6×6 with two degrees of freedom (u and v) at each node.

The joint stiffness matrix, $[K]$, is assembled in the same manner as for the stiffness method, adding together elemental stiffness values for each node. As a simple example, consider the joint stiffness matrix for a structure of two elements. Let element #1 have nodes a , b and c and let element #2 have nodes b , c and d as shown at right.



The element stiffness matrices are partitioned into nine 2×2 's, each relating forces at a single node to displacements at a single node:

$$[k_1] = \begin{bmatrix} k_{1aa} & k_{1ab} & k_{1ac} \\ k_{1ba} & k_{1bb} & k_{1bc} \\ k_{1ca} & k_{1cb} & k_{1cc} \end{bmatrix} \quad [k_2] = \begin{bmatrix} k_{2bb} & k_{2bc} & k_{2bd} \\ k_{2cb} & k_{2cc} & k_{2cd} \\ k_{2db} & k_{2dc} & k_{2dd} \end{bmatrix}$$

The 2×2 's from each element stiffness matrix are added to the appropriate positions on the joint stiffness matrix:

$$[K] = \begin{bmatrix} k_{1aa} & k_{1ab} & k_{1ac} & \\ k_{1ba} & k_{1bb} + k_{2bb} & k_{1bc} + k_{2bc} & k_{2bd} \\ k_{1ca} & k_{1cb} + k_{2cb} & k_{1cc} + k_{2cc} & k_{2cd} \\ & k_{2db} & k_{2dc} & k_{2dd} \end{bmatrix} \begin{matrix} a \\ b \\ c \\ d \end{matrix}$$

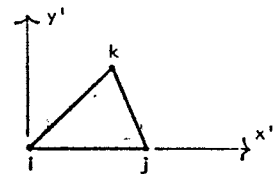
ROTATION OF PLANE STRESS ELEMENTS

The problem of rotation of the element stiffness matrix to the global system is to describe the set of local x , y and z coordinates. A vector analysis will be used to derive the global-to-local rotation matrix $[R]$, and the rotation formula

$$[k_G] = [R]^T [k_L] [R]$$

will be applied to each element before assembly of the global stiffness matrix.

Let the local x -axis (x') be defined as a vector from node i to node j of the element, and the local y -axis (y') be defined as a vector perpendicular to the local x -axis in the direction of node k (see sketch above right). For a right-handed coordinate system, the local z -axis (z') must be perpendicular to the plane of the element and oriented along a vector of positive rotation from x to y (up from the plane of the element--see sketch below right).



Let \bar{U} be a vector directed from i to j (along the positive x' -axis) and \bar{V}' be a vector directed from i to k . By definition

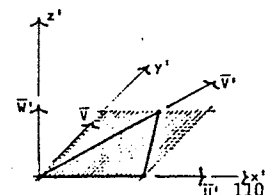
$$\bar{W} = \bar{U} \times \bar{V}'$$

is a vector perpendicular to both \bar{U} and \bar{V}' oriented in the positive z' direction. \bar{W} therefore establishes the local z -axis (z').

Similarly, a vector along the local y -axis can be found from the cross product of \bar{W} and \bar{U} :

$$\bar{V} = \bar{W} \times \bar{U}$$

The vector \bar{V} therefore establishes the local y -axis (y').



The rotation matrix, [R], comprises the cosines of the angles between local and global axes:

$$[R] = \begin{bmatrix} L_1 & M_1 & N_1 \\ L_2 & M_2 & N_2 \\ L_3 & M_3 & N_3 \end{bmatrix}$$

where L_1 is the cosine of the angle between x' and x , L_2 is the cosine of the angle between y' and x , L_3 is the cosine of the angle between z' and x , etc.:

$$\begin{aligned} L_1 &= \cos(x'x) & M_1 &= \cos(x'y) & N_1 &= \cos(x'z) \\ L_2 &= \cos(y'x) & M_2 &= \cos(y'y) & N_2 &= \cos(y'z) \\ L_3 &= \cos(z'x) & M_3 &= \cos(z'y) & N_3 &= \cos(z'z) \end{aligned}$$

The direction cosines can be calculated from the following equations:

$$\begin{aligned} L_1 &= Ux/U & M_1 &= Uy/U & N_1 &= Uz/U \\ L_2 &= Vx/V & M_2 &= Vy/V & N_2 &= Vz/V \\ L_3 &= Wx/W & M_3 &= Wy/W & N_3 &= Wz/W \end{aligned}$$

where

$Ux, Vx, Wx = x'$ components of $\bar{U}, \bar{V}, \bar{W}$

$Uy, Vy, Wy = y'$ components of $\bar{U}, \bar{V}, \bar{W}$

$Uz, Vz, Wz = z'$ components of $\bar{U}, \bar{V}, \bar{W}$

$U, V, W =$ magnitudes of $\bar{U}, \bar{V}, \bar{W}$

To perform the rotation, the 6x6 coplanar stiffness matrix must be expanded to a 9x9 by the addition of zeroes:

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$$[k']_{9 \times 9} = \begin{bmatrix} [k_{ii}] & 0 & [k_{ij}] & 0 & [k_{ik}] & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ [k_{ji}] & 0 & [k_{jj}] & 0 & [k_{jk}] & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ [k_{ki}] & 0 & [k_{kj}] & 0 & [k_{kk}] & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The 9x9 rotation matrix is formed by repeating [R] along the diagonal three times:

$$[R]_{9 \times 9} = \begin{bmatrix} [R]_{3 \times 3} & 0 & 0 \\ 0 & [R]_{3 \times 3} & 0 \\ 0 & 0 & [R]_{3 \times 3} \end{bmatrix}$$

And the rotation of the local member stiffness matrix to global coordinates is performed by applying the following equation:

$$[k_G] = [R]^T [k_L] [R]$$

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CABLE STRUCTURES

A cable, just like any other structural element, has a stiffness: a set of equations can be derived relating forces on the cable to the movement of its ends. The essential task in determining the stiffness of a cable is to describe its shape, which will vary according to the loads on the cable.

THE PARABOLIC SHAPE

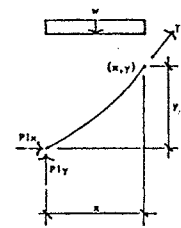
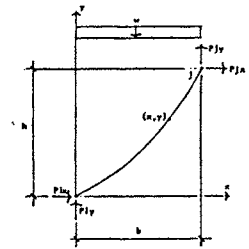
When a cable is loaded uniformly along a straight axis, it assumes a parabolic shape. This kind of uniform load can be found on cables subjected to wind or cables on suspension bridges where evenly spaced ties support a roadway.

Consider a cable loaded uniformly along the global x-axis (see sketch above right). Since a cable can have no internal bending moment, one can cut a free body diagram left of an arbitrary point (x, y) and set the moment about (x, y) to zero. The resulting moment equation can be solved to provide a shape function for the cable in terms of the forces acting upon it (below right).

$$\sum M_{(x,y)} = 0 = P_{iy}(x) - P_{ix}(y) + w(x^2/2)$$

$$y = [P_{iy}(x) + w(x^2/2)]/P_{ix}$$

$$Y(b) = h = [P_{iy}(b) + w(b^2/2)]/P_{ix}$$



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$$P_{iy} = P_{ix}(h/b) - w(b/2)$$

$$y = (w/2P_{ix})x^2 + \left(\frac{h}{b} - \frac{wb}{2P_{ix}}\right)x$$

$$dy/dx = (w/P_{ix})x + \left(\frac{h}{b} - \frac{wb}{2P_{ix}}\right)$$

CABLE LENGTH

Examining an infinitesimal piece of the cable and integrating along x gives s, the cable length (see sketch).

$$\begin{aligned} ds &= \sqrt{dx^2 + dy^2} \\ &= dx\sqrt{dx^2/dx^2 + dy^2/dx^2} \\ &= \sqrt{1 + dy^2/dx^2} dx \end{aligned}$$

$$s = \int_0^b ds$$

By defining u as the derivative dy/dx , the equation may be solved using any standard table of integrals:

$$u = dy/dx = (w/P_{ix})x + \left(\frac{h}{b} - \frac{wb}{2P_{ix}}\right)$$

$$du = (w/P_{ix})dx$$

$$dx = (P_{ix}/w)du$$

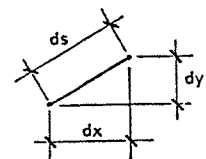
$$s = \int_0^b ds$$

$$= \int_0^b \sqrt{1 + u^2} dx$$

$$= \frac{P_{ix}}{w} \int_0^b \sqrt{1 + u^2} du$$

$$= \frac{P_{ix}}{2w} [u\sqrt{1 + u^2} + \sinh^{-1}(u)] \Big|_0^b$$

$$= \frac{P_{ix}}{2w} [u\sqrt{1 + u^2} + \ln(u\sqrt{1 + u^2})] \Big|_0^b$$



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The limits for u to be substituted into this equation are

$$u_0 = h/b - wb/2Pix$$

$$u_b = h/b + wb/2Pix$$

The length s is the length of the cable after the loads have been applied. The change in length (Δs) is a function of the tension in the cable, which varies over its length. The tension at an arbitrary distance x can be determined from equilibrium (see sketch).

$$\Sigma F_x = 0 = Pix + T(dx/ds)$$

$$T = -Pix(ds/dx)$$

The strain at any point is simply the tension times the unit axial stiffness, ds/AE . Integrating the strain along x gives the total change in length of the cable under the load:

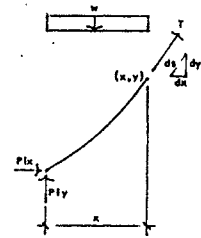
$$\begin{aligned} &= Tds/AE \\ &= (-Pix/AE)(ds^2/dx) \\ &= (-Pix/AE)(1 + u^2) \\ &= (-Pix/AE) \left[1 + \left(\frac{wx}{Pix} + \frac{h}{b} - \frac{wb}{2Pix} \right)^2 \right] \end{aligned}$$

$$\begin{aligned} \Delta s &= \int_0^b \epsilon dx \\ &= -(w^2b^3/12AEPix + Pixh^2/AEb) \end{aligned}$$

The original length of the cable can be found by subtracting the change in length (Δs) from the extended length (s):

$$s_0 = s - \Delta s$$

In summary, the equations describing a cable under a uniform load are as follows:



1. Extended length

$$s = \frac{Pix}{2w} [u_b \sqrt{1 + u_b^2} - u_0 \sqrt{1 + u_0^2} + \ln(u_b \sqrt{1 + u_b^2}) - \ln(u_0 \sqrt{1 + u_0^2})]$$

$$u_b = h/b + wb/2Pix$$

$$u_0 = h/b - wb/2Pix$$

2. Change in length

$$\Delta s = -(w^2b^3/12AEPix - Pixh^2/AEb)$$

3. Original length

$$s_0 = s - \Delta s$$

CABLE STIFFNESS

The stiffness of a loaded cable is not a constant function but rather varies extensively with the position of the cable. When the cable is slack the stiffness is small; as the cable becomes more taut the stiffness increases. It is therefore misleading to discuss a stiffness matrix for a cable. The forces and deformations of a cable can only be determined through iteration, applying equations for length and change in length to the known geometry of the structure.

Consider as an example two cables supporting a tower, as shown at right. The initial horizontal force, the weight of the cables, and the dimensions of the structure are known. Let $P = 6000$ lb, $E = 24 \times 10^6$ psi and $A = 0.3$ in².

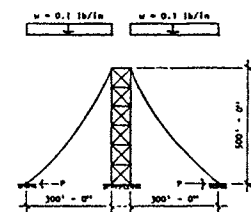
$$u_0 = h/b - wb/2Pix = 1.63667$$

$$u_b = h/b + wb/2Pix = 1.69667$$

$$u_0 \sqrt{1 + u_0^2} = 3.13911$$

$$u_b \sqrt{1 + u_b^2} = 3.34148$$

$$s = \left(\frac{6000}{2(0.1)} \right) (3.34148 - 3.13911 + \ln(3.34148) - \ln(3.13911)) = 7945.31 \text{ in}$$



$$\Delta s = (w^2 b^3 / 12 A E P_{ix} + P_{ix} h^2 / A E b) = 8.33423 \text{ in}$$

$$s_0 = s - \Delta s = 7936.98 \text{ in}$$

The x-force at the top of the tower is equal to the x-force at the lower end of the cables. The y-force at the top of the tower can be found from equilibrium. Consider only the cable on the left (see sketch, above right):

$$\Sigma M_A = 0 = -(6000 \text{ lb})(6000 \text{ in}) - (0.1 \text{ lb/in})(3600 \text{ in})^2 / 2 + P_{jy}(3600 \text{ in})$$

$$P_{jy} = 10180 \text{ lb}$$

The y-force on the right side of the tower can be calculated the same way, but it is obvious to the casual observer that it will be symmetric to that on the left side. The forces on the tower at the initially stated positions are shown on the sketch (below right).

Given the initial values of s , Δs and s_0 , the stiffness can be determined by moving the top of the tower one inch to the right and calculating the change in horizontal force, P_{ix} . For the cable on the left:

$$h = 6000 \text{ in}$$

$$b = 3601 \text{ in}$$

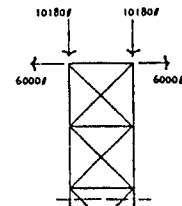
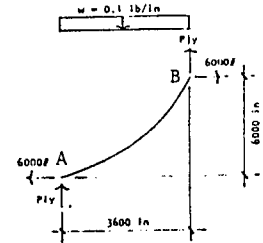
$$u_0 = 1.666203832 - 180.05 / P_{ix}$$

$$u_b = 1.666203832 + 180.05 / P_{ix}$$

Since $s - \Delta s - s_0 = 0$ and s_0 is a known constant, an equation solving algorithm such as the secant method (see Chapter One) can be used to find P_{ix} :

$$\left(\frac{P_{ix}}{0.2} \right) [u_b \sqrt{1 + u_b^2} - u_0 \sqrt{1 + u_0^2} + \ln(u_b \sqrt{1 + u_b^2}) - \ln(u_0 \sqrt{1 + u_0^2})] - (5.40450125 / P_{ix} + 0.001388503194 P_{ix}) - 7936.98 = 0$$

$$P_{ix} = 6693.35 \text{ lb}$$



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For the cable on the right:

$$h = 6000 \text{ in}$$

$$b = 3599 \text{ in}$$

$$u_0 = 1.667129758 - 179.95 / P_{ix}$$

$$u_b = 1.667129758 + 179.95 / P_{ix}$$

$$\left(\frac{P_{ix}}{0.2} \right) [u_b \sqrt{1 + u_b^2} - u_0 \sqrt{1 + u_0^2} + \ln(u_b \sqrt{1 + u_b^2}) - \ln(u_0 \sqrt{1 + u_0^2})] - (5.39550125 / P_{ix} + 0.001389274799 P_{ix}) - 7936.98 = 0$$

$$P_{ix} = 5308.86 \text{ lb}$$

These new values for P_{ix} are the new horizontal forces on the top of the tower resulting from a deflection of one inch to the right (see sketch above right).

The y-force at the top of the tower can be found from equilibrium. Again, consider the cable on the left (see sketch, middle right):

$$\Sigma M_A = 0 = -(6693.35 \text{ lb})(6000 \text{ in}) - (0.1 \text{ lb/in})(3601 \text{ in})^2 / 2 + P_{jy}(3601 \text{ in})$$

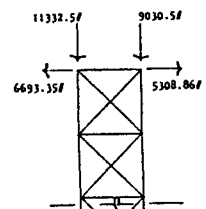
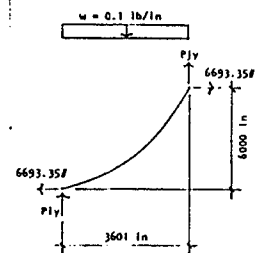
$$P_{jy} = 11332.53 \text{ lb}$$

Similarly for the cable on the right:

$$\Sigma M_A = 0 = -(5308.86 \text{ lb})(6000 \text{ in}) - (0.1 \text{ lb/in})(3599 \text{ in})^2 / 2 + P_{jy}(3599 \text{ in})$$

$$P_{jy} = 9030.51 \text{ lb}$$

And the new forces on the tower, after a one inch deflection to the right, are shown on the sketch (below right). Similar values can be calculated for a one inch vertical deflection, and a stiffness matrix can be formulated for the tower at that given position. It should be emphasized that the stiffness of a cable is always changing, and a new stiffness matrix must be calculated for each new position the cable assumes.



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After determining the change in force caused by one-inch horizontal and vertical deflections, the stiffness matrix for the cable at a single location can be determined:

Px_0 = initial horizontal force

Py_0 = initial vertical force

Px_1 = new horizontal force after 1" horizontal deflection

Py_1 = new vertical force after 1" horizontal deflection

Px_2 = new horizontal force after 1" vertical deflection

Py_2 = new vertical force after 1" vertical deflection

Calculate the changes in force from the values listed above:

$$\Delta Px_1 = Px_1 - Px_0$$

$$\Delta Py_1 = Py_1 - Py_0$$

$$\Delta Px_2 = Px_2 - Px_0$$

$$\Delta Py_2 = Py_2 - Py_0$$

The cable stiffness is the change in force divided by the deflection.

For a One-inch deflection, the stiffness is equal to the change in force:

$$k_{11} = \Delta Px_1$$

$$k_{12} = \Delta Py_1$$

$$k_{21} = \Delta Px_2$$

$$k_{22} = \Delta Py_2$$

The stiffness equation includes both the initial force at the joint and the change in force:

Px = final horizontal force

Py = final vertical force

Δx = actual horizontal deflection

Δy = actual vertical deflection

$$\begin{matrix} Px \\ Py \end{matrix} = \begin{matrix} Px_0 \\ Py_0 \end{matrix} + \begin{matrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{matrix} \begin{matrix} \Delta x \\ \Delta y \end{matrix}$$

EQUATION #4

$$Y = (X/0.2) * (D - C + \text{LOG}(D) - \text{LOG}(C)) - (5.40450125/X + 1.388503194E-3 * X) - 7936.98$$

$$A = 1.666203832 - 180.05/X$$

$$B = 1.666203832 + 180.05/X$$

$$C = A * \text{SQR}(1 + A^2)$$

$$D = B * \text{SQR}(1 + B^2)$$

$$\text{ROOT \#1} = 6693.352475$$

EQUATION #5

$$Y = (X/0.2) * (D - C + \text{LOG}(D) - \text{LOG}(C)) - (5.39550125/X + 1.389274799E-3 * X) - 7936.98$$

$$A = 1.667129758 - 179.95/X$$

$$B = 1.667129758 + 179.95/X$$

$$C = A * \text{SQR}(1 + A^2)$$

$$D = B * \text{SQR}(1 + B^2)$$

$$\text{ROOT \#1} = 5308.855425$$

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APPENDIX A

The following programs were written for the HP 41C calculator to solve three of the most common kinds of numerical problems: solution of a quadratic, solution of a cubic, and matrix operations.

'QUAD' - QUADRATIC SOLUTION

This program solves a general quadratic equation of the form

$$Ax^2 + Bx + C = 0$$

The program asks for values of A, C and B and stores the two roots in the x and y stack registers. If the roots are complex numbers, the word "IMAGINARY" appears in the display and the program stops. To see the complex roots, press "R/S". The real part of the root will be stored in the x-register and the imaginary part will be stored in the y-register.

This program uses only the four-register stack in all its calculations. The memory registers (00, 01, 02, 03, etc.) are not disturbed.

'CUBE' - CUBIC SOLUTION

This program solves a general cubic equation of the form

$$Ax^3 + Bx^2 + Cx + D = 0$$

The program asks for values of A, B, C and D and displays the three roots, one at a time, pausing after each number is displayed. The roots are stored in memory registers 01, 02 and 03.

If two of the roots are complex, their values will be displayed in two parts: first the real part followed by the letter "U", then the imaginary part followed by the letters "VI". Complex roots, which always occur in pairs, are stored in registers 02 and 03. Register 02 contains the real part and register 03 contains the imaginary part.

This program uses memory registers 01 to 05 and sets flag 01 if the roots include a complex pair. The radians mode is set automatically at the start of the program.

'MAT' - MATRIX OPERATIONS

This is a series of programs which, in combination with the Math Pack and Extended Functions Module, will store, recall, transpose and multiply square matrices and column vectors up to 12x12 in size. A distinction must be made here between "main memory", which is where most of the mathematical operation take place, and "extended memory", which is the storage memory contained in the Extended Functions Module.

To input a matrix into main memory:

```
XEQ "MATRIX"
Input size n (for n×n square matrix)
Input elements by rows
```

To input a column vector into main memory:

```
XEQ "MATRIX" *   *If size is the same as for a previous matrix, skip
Input size n *   these steps.
```

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```
XEQ "SIMEQ"
Input elements
```

To store a matrix in extended memory:

```
Input matrix as described above
XEQ "MSTO"
Input file name
```

To store a column in extended memory:

```
Input column as described above
XEQ "CSTO"
Input file name
```

To recall a matrix from extended memory:

```
XEQ "MRCL"
Input file name (must be a matrix, not a column)
Matrix is now in main memory. "MRCL" clears all main memory registers
prior to recalling matrix.
```

To recall a column from extended memory:

```
XEQ "CRCL"
Input file name (if file name for a matrix is given, the first row
will be recalled as a column)
Column is now in main memory. "CRCL" does not clear main memory. When
recalling both a matrix and a column, recall the matrix first.
```

To multiply two matrices: $[A][B] = [C]$

```
Matrix [A] must be in main memory ("MATRIX" or "MRCL")
Matrix [B] must be in extended memory ("MSTO")
XEQ "M*"
Input file name for matrix [B]
```

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The result ($\{C\}$) replaces $[A]$ in main memory

To multiply a matrix by a column: $[A]\{V\} = \{W\}$

Matrix $[A]$ must be in main memory ("MATRIX" or "MRCL")

Column $\{V\}$ must be in extended memory ("CSTO")

XEQ "M*"

Input file name for column $\{V\}$

The result ($\{W\}$) appears as a column in main memory. $[A]$ and $\{V\}$ are not disturbed.

To transpose a matrix: $[A]^T = [B]$

Matrix $[A]$ must be in extended memory ("MSTO")

XEQ "MTR"

Input file name for matrix $[A]$

The result appears in main memory, replacing whatever was there before.

To view a matrix or column in extended memory:

XEQ "MVIEW"

Input file name

To view without pausing, clear flag 21

Note: If file name has been given for a previous operation, it need not be reentered; just press "R/S" when asked for file name.

OTHER FUNCTIONS

To solve simultaneous equations:

Recall or input matrix

Recall or input column

XEQ "PVT"

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To take inverse or determinant:

Recall or input matrix

XEQ "INV" or XEQ "DET". The Math Pack does not save the inverse of a matrix in main memory; it only displays the terms.

To view a matrix or column in main memory

XEQ "VMAT" or "VCOL"

See Math Pack manual for further instructions.

QUAD

01 LBL "QUAD"	41 CHS
02 LBL 01	42 SQRT
03 2	43 X<>Y
04 "A=?"	44 /
05 PROMPT	45 X<>Y
06 *	46 LASTX
07 LASTX	47 /
08 "C=?"	48 RTN
09 PROMPT	49 GTO 01
10 *	50 END
11 4	
12 *	
13 CHS	
14 "B=?"	
15 PROMPT	
16 X+2	
17 LASTX	
18 CHS	
19 RDN	
20 +	
21 X<0?	
22 GTO 00	
23 SQRT	
24 R+	
25 X<>Y	
26 -	
27 LASTX	
28 R+	
29 +	
30 RDN	
31 X<>Y	
32 /	
33 R+	
34 LASTX	
35 /	
36 RTN	
37 GTO 01	
38 LBL 00	
39 "IMAGINARY"	
40 PROMPT	

125a

CUBE

01 LBL "CUBE"	41 /	81 STO 01	121 RCL 01
02 RAD	42 CHS	82 2	122 X+2
03 "A3=?"	43 STO 01	83 /	123 +
04 PROMPT	44 X+2	84 CHS	124 6
05 STO 04	45 RCL 04	85 STO 02	125 1/X
06 "A2=?"	46 3	86 RCL 03	126 Y+X
07 PROMPT	47 /	87 ST- 01	127 2
08 STO 03	48 3	88 ST- 02	128 *
09 "A1=?"	49 Y+X	89 RCL 04	129 STO 04
10 PROMPT	50 +	90 RCL 05	130 RCL 02
11 STO 02	51 STO 02	91 -	131 CHS
12 "A0=?"	52 X<=0?	92 3	132 SQRT
13 PROMPT	53 GTO 01	93 SQRT	133 RCL 01
14 STO 01	54 CF 01	94 *	134 /
15 RCL 04	55 SQRT	95 2	135 ATAN
16 ST/ 01	56 RCL 01	96 /	136 3
17 ST/ 02	57 +	97 STO 03	137 /
18 3	58 X<0?	98 LBL 02	138 STO 05
19 *	59 SF 01	99 "X1="	139 COS
20 ST/ 03	60 FS? 01	100 ARCL 01	140 *
21 RCL 02	61 CHS	101 PROMPT	141 RCL 03
22 RCL 03	62 3	102 "X2="	142 -
23 X+2	63 1/X	103 ARCL 02	143 STO 01
24 3	64 Y+X	104 "I"	144 RCL 05
25 *	65 FS? 01	105 PROMPT	145 PI
26 -	66 CHS	106 "+"	146 2
27 STO 04	67 STO 04	107 ARCL 03	147 *
28 RCL 03	68 RCL 01	108 "VI"	148 3
29 3	69 RCL 02	109 PROMPT	149 /
30 Y+X	70 SQRT	110 "X3="	150 +
31 2	71 -	111 ARCL 02	151 LASTX
32 *	72 FS? 01	112 "U"	152 X<>Y
33 RCL 02	73 CHS	113 PROMPT	153 +
34 RCL 03	74 3	114 "-"	154 LASTX
35 *	75 1/X	115 ARCL 03	155 COS
36 -	76 Y+X	116 "VI"	156 RCL 04
37 RCL 01	77 FS? 01	117 PROMPT	157 *
38 +	78 CHS	118 GTO 02	158 RCL 03
39 STO 05	79 STO 05	119 LBL 01	159 -
40 2	80 +	120 CHS	160 STO 02

125b

161 X<>Y
 162 COS
 163 RCL 04
 164 *
 165 RCL 03
 166 -
 167 STO 03
 168 LBL 03
 169 "X1="

170 ARCL 01
 171 PROMPT
 172 "X2="

173 ARCL 02
 174 PROMPT
 175 "X3="

176 ARCL 03
 177 PROMPT
 178 GTO 03
 179 END

125 c

MAT

01 LBL "MAT"
 02 LBL "MSTO"
 03 CF 00
 04 GTO 00
 05 LBL "CSTO"
 06 SF 00
 07 LBL 00
 08 XEQ 01
 09 FS?C 25
 10 GTO 00
 11 RCL 14
 12 FC? 00
 13 X+2
 14 1
 15 +
 16 CRFLD
 17 LBL 00
 18 CLX
 19 SEEKPT
 20 RCL 14
 21 SAVEX
 22 15.014
 23 FS? 00
 24 XEQ 04
 25 XEQ 02
 26 SAVERX
 27 GTO 06
 28 LBL "MRCL"
 29 CLRG
 30 CF 00
 31 CF 05
 32 XEQ 01
 33 STO 14
 34 15.014
 35 GTO 00
 36 LBL "CRCL"
 37 SF 00
 38 SF 05
 39 XEQ 01
 40 RCL 14

41 X≠Y?
 42 GTO 05
 43 XEQ 04
 44 LBL 00
 45 XEQ 02
 46 GETRX
 47 LBL 20
 48 6.013
 49 STO 00
 50 LBL 10
 51 CF IND 00
 52 ISG 00
 53 GTO 10
 54 SF 04
 55 GTO 06
 56 LBL "MVIEW"
 57 CF 00
 58 XEQ 01
 59 STO 01
 60 FLSIZE
 61 1
 62 -
 63 X=Y?
 64 SF 00
 65 1
 66 RCL 01
 67 XEQ 03
 68 STO 02
 69 STO 03
 70 RCLFLAG
 71 STO 00
 72 FIX 0
 73 CF 29
 74 "SIZE="

81 PROMPT
 82 LBL 11
 83 FIX 0
 84 CF 29
 85 "M"
 86 FS? 00
 87 "C"
 88 ARCL 02
 89 FS? 00
 90 GTO 00
 91 "F,"
 92 ARCL 03
 93 LBL 00
 94 "F="

121 1
 122 -
 123 X=Y?
 124 SF 01
 125 FS? 01
 126 SF 05
 127 RCLFLAG
 128 STO 00
 129 FIX 0
 130 CF 00
 131 CF 21
 132 CF 29
 133 XEQ 04
 134 STO 08
 135 SF 00
 136 1
 137 XEQ 02
 138 STO 01
 139 14.014
 140 +
 141 STO 06
 142 LBL 21
 143 1
 144 XEQ 02
 145 STO 02
 146 RCL 14
 147 X+2
 148 LASTX
 149 +
 150 14.9
 151 +
 152 STO 05
 153 RCL 14
 154 LBL 22
 155 "ROW"
 156 ARCL 01
 157 AVIEW
 158 FS? 01
 159 1
 160 1 E5

125d

161 /	201 RCL 00	241 X<>Y	281 +
162 .9	202 STOFLAG	242 LBL 07	282 FC? 00
163 +	203 GTO 06	243 ST* IND 00	283 RTN
164 STO 07	204 LBL 00	244 ISG 00	284 INT
165 LBL 12	205 FS? 01	245 GTO 07	285 1.001
166 RCL 06	206 GTO 22	246 GTO 06	286 *
167 STO 03	207 GTO 21	247 LBL 01	287 1 E-3
168 RCL 07	208 LBL "MTR"	248 CF 23	288 -
169 1	209 CF 00	249 SF 25	289 RTN
170 +	210 XEQ 01	250 "FILE NAME?"	290 LBL 05
171 STO 07	211 STO 14	251 AON	291 "NO"
172 STO 05	212 1 E5	252 PROMPT	292 PROMPT
173 ISG 05	213 /	253 AOFF	293 LBL 06
174 CLX	214 14.014	254 FC? 23	294 "DONE"
175 STO IND 05	215 +	255 CLA	295 PROMPT
176 LBL 13	216 XEQ 02	256 CLX	296 GTO 06
177 RCL 04	217 STO 00	257 SEEKPTA	297 END
178 SEEKPT	218 SF 00	258 FC? 25	
179 GETX	219 1	259 RTN	
180 RCL IND 03	220 XEQ 02	260 1	
181 *	221 STO 01	261 GETX	
182 ST+ IND 05	222 LBL 14	262 RTN	
183 ISG 04	223 RCL 01	263 LBL 02	
184 ISG 03	224 INT	264 RCL 14	
185 GTO 13	225 RCL 00	265 FC? 00	
186 FS? 01	226 +	266 X+2	
187 GTO 00	227 STO 02	267 LBL 03	
188 ISG 02	228 LBL 15	268 1 E3	
189 GTO 12	229 GETX	269 /	
190 RCL 08	230 STO IND 02	270 +	
191 REMOVE	231 ISG 02	271 RTN	
192 XEQ 02	232 GTO 15	272 LBL 04	
193 STO 08	233 ISG 01	273 1 E-6	
194 LBL 00	234 GTO 14	274 RCL 14	
195 RCL 14	235 GTO 20	275 +	
196 1.001	236 LBL "CON"	276 LASTX	
197 *	237 CF 00	277 *	
198 ST+ 06	238 15.014	278 LASTX	
199 ISG 01	239 XEQ 02	279 +	
200 GTO 00	240 STO 00	280 15.015	
	241 X<>Y		

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APPENDIX B

The following programs were written for the HP 9845 computer as applications of the basic methods presented herein. Since they are interactive programs--that is, they provide the user with instructions at each step--the guides for use that follow are intended to supplement, not replace, the instructions that appear on the computer screen.

The four available programs are briefly described below. More detailed guides for their use follow.

'EQN' - EQUATION SOLVER

This program plots, finds roots, and calculates y for a given value of x for any algebraic or transcendental equation of the form

$$y = f(x)$$

where $f(x)$ is any combination of mathematical operations that can be described to the computer via basic language program lines.

'POLY' - POLYNOMIAL SOLUTIONS

This program finds the roots of any polynomial up to one hundredth degree of the form

$$y = A_n x^n + A_{n-1} x^{n-1} + \dots + A_1 x + A_0$$

using Bairstow's method.

'MATRIX' - MATRIX OPERATIONS

This program solves systems of linear simultaneous equations using Gaussian reduction, Gauss-Jordan reduction, or Cholesky decomposition. It also includes provisions for partial or complete pivoting.

'STIFF' - STIFFNESS ANALYSIS

This program performs stiffness analysis, including shear stiffness, geometric stiffness, and the prediction of buckling, for three-dimensional moment-resisting frames. It allows up to 1000 joints, 2000 members and 15 load conditions for any one file, and allows an unlimited number of files to be stored and recalled by various users.

USER'S GUIDE TO 'EQN'

'EQN' solves a variety of algebraic and transcendental equations using either Newton's rule of tangents or the secant method. It also plots the equations within a region specified by the user and evaluates the equations for specific values of x.

When the program is started, a menu will appear at the top of the screen. Items from the menu are selected by pressing the keys at the upper right corner of the keyboard. The menu with corresponding key numbers is shown below.

k0	k1	k2	k3	k4	k5	k6
INPUT	PLOT	VAL	NEWT	SEC		

INPUT

Allows the user to input a new equation or edit a previously input equation. Up to nine different equations may be input and stored on the mass storage device.

"EQUATION #?": Choose equation number 1 to 9

"EDIT PREVIOUS EQUATION?": A "yes" response (1) will recall the equation stored previously under this number for editing. A "no" response (0) will clear the previous equation. After answering "yes" or "no" (1 or 0) use the keys to select degrees or radians (if appropriate) and press "CONT".

"DEFINITIONS?": A definition is an equation that defines a value to be used in a later equation. For example,

$$y = (1/2)(\sin(\pi x/2) - \cos(\pi x/2))$$

may be input as a definition:

$$A = \text{PI} * X / 2$$

followed by the equation:

$$Y = (1/2) * (\text{SIN}(A) - \text{COS}(A))$$

All definitions and equations must be written as basic language program lines and may only contain variables that have been previously defined (except, of course, the unknown x). Definitions use the letters A through H in order. Input the number of definitions (0 - 10) followed by the definitions themselves.

"FUNCTION?": The function is the equation to be solved, in terms of x and previous definitions. As with definitions, the function must be a basic language program line. The function statement must begin with "Y=". After the function has been input, the program will ask if it is correct. If not, answer "no" (0) to cancel the equation and definitions. If it is correct, answer "yes" (1) or just press "CONT" to return to the main program menu.

PLOT

Plots the value of a previously input equation on a Cartesian coordinate system. A new menu will appear on the screen:

k0	k1	k2	k3	k4	k5	k6
RETURN	NEWPLOT	OVERLAY				PRINT

"RETURN": Returns the program to the main menu. May be pressed at any time during a plotting operation--even when the menu is not on the screen.

"NEWPLOT": Clears the graphics screen in order to plot a new equation.

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"EQUATION #?": Input the number of a previously input equation (1 - 9). The equation and any definitions will appear on the screen.

"LIMITS": Input minimum and maximum values of x and y. These are the boundaries of the plot.

"NUMBER OF POINTS TO BE PLOTTED=?": The program plots a function as a series of straight lines from one point to the next. Input the number of points to determine the accuracy and resolution of the plot. The maximum resolution of the screen is about 500 points. Fewer will usually be sufficient.

When the program asks if the information is correct, input "no" (0) to return or just press "CONT" to go on. The plot will appear on the screen. Wait for the computer to finish drawing, then press "CONT" to return to the menu.

"OVERLAY": Plots another equation on top of the previous one. The limits will necessarily be the same. The procedure is the same as for a new plot.

"PRINT": Prints the current plot and overlays, along with the equations, on the thermal printer.

VAL

Calculates the y-values of the function for specific values of x. The value of the function at x will appear on the screen.

NEWT

Finds roots of an equation by Newton's rule of tangents.

"INPUT DERIVATIVE": The derivative of the function must be known to use Newton's rule of tangents. Input the derivative, including any

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definitions, as described under "INPUT".

"NUMBER OF SIGNIFICANT DIGITS=?": The program iterates until the root is found to a specified number of significant digits. Input from one to ten.

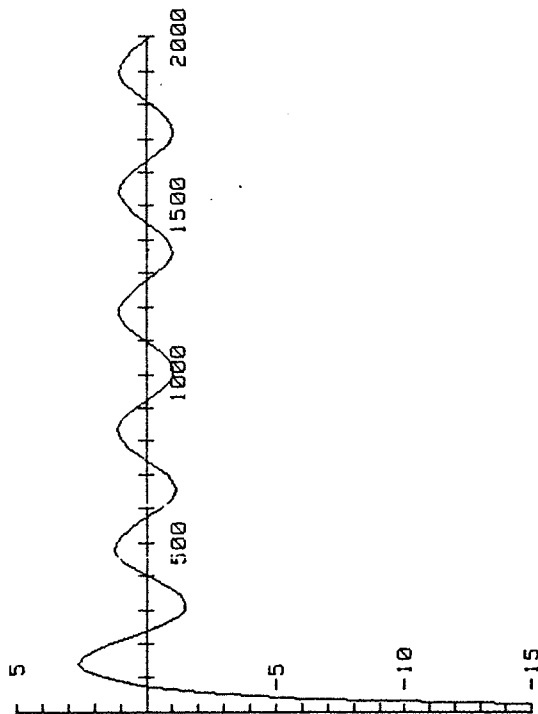
"GUESS=?": Input a value of x near where the root is expected to lie. The program will begin to iterate, displaying the iteration number and the value of x. When the root is found, it will appear on the screen along with the corresponding value of y (which should be close to 0).

SEC

Finds roots of an equation by the secant method. The procedure is the same as for Newton's rule of tangents, except that two guesses are required and the derivative is not.

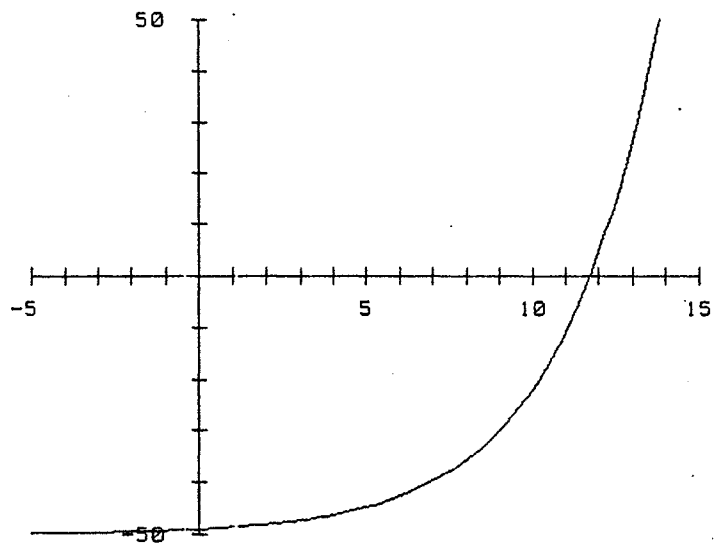
PRINT

At certain times, "PRINT" will appear at the right end of the menu. Pressing this key will print data from the most recent operation on the thermal printer. Examples of printouts are on the following pages.



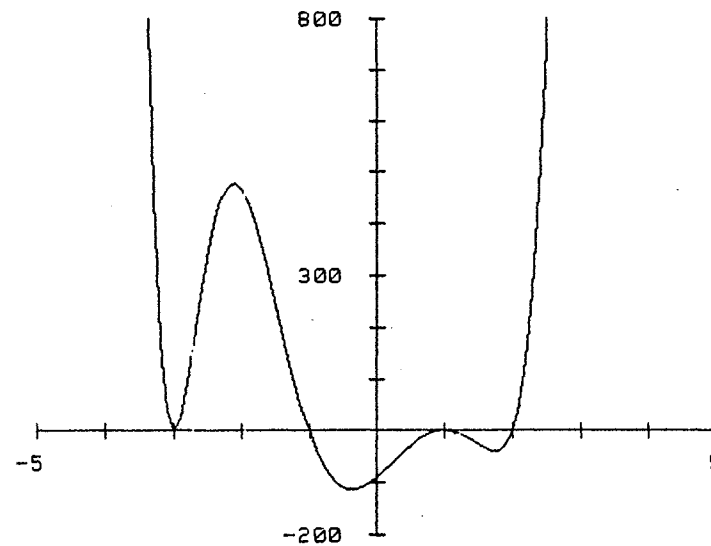
EQUATION #1
 $Y = \sin(X) - (360/X) * \cos(X)$

EQUATION #1
 $Y = \sin(X) - (360/X) * \cos(X)$
ROOT #1 = 1811.24153
ROOT #2 = 1632.43634
ROOT #3 = 1453.987216
ROOT #4 = 1275.758277
ROOT #5 = 1098.150383
ROOT #6 = 921.3422905
ROOT #7 = 745.767289
ROOT #8 = 401.8554014
ROOT #9 = 77.80458705



EQUATION #2
 $Y = \text{EXP}(X/3) - 50$

EQUATION #2
 $Y = \text{EXP}(X/3) - 50$
 $Y' = (1/3) * \text{EXP}(X/3)$
 ROOT #1 = 11.73606902



EQUATION #3
 $Y = X^8 + X^7 - 9X^6 + 13X^5 + 21X^4 - 125X^3 + 77X^2 + 111X - 90$

EQUATION #3
 $Y = X^8 + X^7 - 9X^6 + 13X^5 + 21X^4 - 125X^3 + 77X^2 + 111X - 90$
 $Y' = 8X^7 + 7X^6 - 54X^5 + 65X^4 + 84X^3 - 375X^2 + 154X + 111$
 ROOT #1 = 2
 ROOT #2 = 1.000001865
 ROOT #3 = 1.000000526
 ROOT #4 = .999999193
 ROOT #5 = -1
 ROOT #6 = -3.000001088

USER'S GUIDE TO 'POLY'

'POLY' uses Bairstow's method to find the roots of polynomials of the form

$$A_n x^n + A_{n-1} x^{n-1} + \dots + A_1 x + A_0$$

where n is the degree of the polynomial (1 - 100).

Input the degree, followed by the coefficients A_n to A_0 . The program will count down as it iterates, starting with $n/2$, which is the number of times it must extract a pair of roots. When all roots have been found, they will be displayed on the screen, and may be printed on the thermal printer by answering "yes" (1) to the question "Printout?". An example of a typical solution is shown below.

POLYNOMIAL SOLUTION

```
F(x) =  1 X^10
        +2 X^9
        +3 X^8
        +4 X^7
        +5 X^6
        +6 X^5
        +7 X^4
        +8 X^3
        +9 X^2
        +10 X

ROOTS:  .9217381477+.7963638244i
        .9217381477-.7963638244i
        .3102509091+1.242281903i
        .3102509091-1.242281903i
        0
        -.4656893762+1.229502343i
        -.4656893762-1.229502343i
        -1.097044088+.7569884336i
        -1.097044088-.7569884336i
        -1.338591186
```

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USER'S GUIDE TO 'MAT'

'MAT' solves systems of simultaneous equations using Gaussian elimination, Gauss-Jordan elimination, Cholesky condensation, or the computer's own inversion routine, either with or without pivoting. Its purpose is not only to give answers to systems of equations, but also to compare the speed and accuracy of various solution routines. The program has a menu of ten items, each of which is described below:

INPUT

Input matrix and constant vector(s). The order of the matrix may be from one to one hundred, and the number of constant vectors may be from one to ten. The program prompts for the matrix elements in row order, then prompts for the constant vector(s). Once input, the matrix may be solved or stored.

STORE

Stores the matrix in a file named by the user. The file name may contain from one to three characters. Once stored, the matrix and constant vector(s) will remain on the mass storage medium until the user erases the file (see "ERASE").

RECALL

Recalls a previously stored matrix and constant vector(s) for solution.

GAUSS

CHOLESKY

GAUSS-JORDAN

INVERT

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Solves the system of equations by the method named. The matrix and constants must be in the program's memory; this means that they must either have just been input or have just been recalled. Once solved, a matrix must be input again or recalled again to be solved again. Only the Gauss, Gauss-Jordan and Cholesky routines offer the option of partial or complete pivoting.

VIEW

Shows the original matrix, constant vector(s), and solution vector(s) on the screen after one of the solution schemes has been performed. Offers a copy on the thermal printer.

FILE CATALOG

Shows name, size and number of constant vectors in each matrix file on the mass storage device.

ERASE

Erases matrix files from the mass storage device.

A listing of the three solution schemes and the two pivoting schemes is on the following pages.

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ROUTINE FOR GAUSSIAN ELIMINATION

```

1  ! ROUTINE FOR GAUSSIAN ELIMINATION
2  ! N = Size of coefficient matrix
3  ! P = Number of constant vectors
4  ! A(N,N+P) = Augmented matrix
5  ! B(N,P) = Solution vector(s)
6  !
7  ! Row reduction
8  FOR I=1 TO N-1
9    FOR J=I+1 TO N
10     IF NOT A(J,I) THEN 16
11     X=A(J,I)/A(I,I)
12     A(J,I)=0
13     FOR K=I+1 TO N+P
14       A(J,K)=A(J,K)-A(I,K)*X
15     NEXT K
16   NEXT J
17 NEXT I
18 ! Back substitution
19 FOR I=1 TO P
20   FOR J=N TO 1 STEP -1
21     X=0
22     FOR K=N TO J+1 STEP -1
23       X=X+A(J,K)*B(K,I)
24     NEXT K
25     B(J,I)=(A(J,I+N)-X)/A(J,J)
26   NEXT J
27 NEXT I
28 END

```

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ROUTINE FOR GAUSS-JORDAN ELIMINATION

```

1 | ROUTINE FOR GAUSS-JORDAN ELIMINATION
2 | N = Size of coefficient matrix
3 | P = Number of constant vectors
4 | A(N,N+P) = Augmented matrix
5 | Solution is contained in augmented matrix, columns N+1 to N+P
6 |
7 | FOR I=1 TO N
8 |   DISP I
9 |   FOR J=I+1 TO N+P
10 |     A(I,J)=A(I,J)/A(I,I)
11 |   NEXT J
12 |   FOR J=1 TO N
13 |     IF I=J THEN 17
14 |     FOR K=I+1 TO N+P
15 |       A(J,K)=A(J,K)-A(I,K)*A(J,I)
16 |     NEXT K
17 |   NEXT J
18 | NEXT I
19 | END

```

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ROUTINE FOR CHOLESKY DECOMPOSITION

```

1 | ROUTINE FOR CHOLESKY DECOMPOSITION
2 | N = Size of coefficient matrix
3 | P = Number of constant vectors
4 | A(N,N+P) = Augmented matrix
5 | B(N,P) = Solution vector(s)
6 |
7 | Row reduction
8 | FOR I=2 TO N+P
9 |   A(I,I)=A(I,I)/A(1,I)
10 | NEXT I
11 | FOR I=2 TO N
12 |   X=0
13 |   FOR J=1 TO I-1
14 |     X=X+A(I,J)*A(J,I)
15 |   NEXT J
16 |   A(I,I)=A(I,I)-X
17 |   FOR J=I+1 TO N+P
18 |     X=0
19 |     FOR K=1 TO I-1
20 |       X=X+A(I,K)*A(K,J)
21 |     NEXT K
22 |     A(I,J)=(A(I,J)-X)/A(I,I)
23 |   NEXT J
24 |   FOR J=I+1 TO N
25 |     X=0
26 |     FOR K=1 TO I-1
27 |       X=X+A(J,K)*A(K,I)
28 |     NEXT K
29 |     A(J,I)=A(J,I)-X
30 |   NEXT J
31 | NEXT I
32 | Back substitution
33 | FOR I=1 TO P
34 |   FOR J=N TO 1 STEP -1
35 |     X=0
36 |     FOR K=N TO J+1 STEP -1
37 |       X=X+A(J,K)*B(K,I)
38 |     NEXT K
39 |     B(J,I)=A(J,I+N)-X
40 |   NEXT J
41 | NEXT I
42 | END

```

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SUB-ROUTINE FOR PARTIAL PIVOTING

```

1  ! SUB-ROUTINE FOR PARTIAL PIVOTING
2  ! N = Size of coefficient matrix
3  ! P = Number of constant vectors
4  ! A(N,N+P) = Augmented matrix
5  !
6  !   C=I
7  !   D=ABS(A(I,I))
8  !   FOR J=I+1 TO N
9  !     IF ABS(A(J,I))<=D THEN 12
10 !     C=J
11 !     D=ABS(A(J,I))
12 !   NEXT J
13 !   IF C=I THEN 19
14 !   FOR J=1 TO N+P
15 !     D=A(I,J)
16 !     A(I,J)=A(C,J)
17 !     A(C,J)=D
18 !   NEXT J
19 !   RETURN

```

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SUB-ROUTINE FOR COMPLETE PIVOTING

```

1  ! SUB-ROUTINE FOR COMPLETE PIVOTING
2  ! N = Size of coefficient matrix
3  ! P = Number of constant vectors
4  ! A(N,N+P) = Augmented matrix
5  ! B(N,P) = Solution vector(s)
6  ! B0(N) = Pointer vector for reordering solutions
7  !
8  !   C=I
9  !   D=I
10 !   E=ABS(A(I,I))
11 !   FOR J=I+1 TO N
12 !     FOR K=I+1 TO N
13 !       IF ABS(A(J,K))<=E THEN 17
14 !       C=J
15 !       D=K
16 !       E=ABS(A(J,K))
17 !     NEXT K
18 !   NEXT J
19 !   IF C=I THEN 25
20 !   FOR J=1 TO N+P
21 !     E=A(I,J)
22 !     A(I,J)=A(C,J)
23 !     A(C,J)=E
24 !   NEXT J
25 !   IF D=I THEN 34
26 !   FOR J=1 TO N
27 !     E=A(J,I)
28 !     A(J,I)=A(J,D)
29 !     A(J,D)=E
30 !   NEXT J
31 !   E=B0(I)
32 !   B0(I)=B0(D)
33 !   B0(D)=E
34 !   RETURN
35 !
36 ! SUB-ROUTINE TO REORDER SOLUTION VECTOR(S)
37 ! B(N,P) = Solution vector(s)
38 ! B0(N) = Pointer vector for reordering solutions
39 ! C(N,P) = Alternate solution vector(s)
40 !
41 !   FOR I=1 TO N
42 !     FOR J=1 TO P
43 !       C(B0(I),J)=B(I,J)
44 !     NEXT J
45 !   NEXT I
46 !   MAT B=C
47 !   RETURN

```

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"STIFF" performs stiffness analysis, including shear stiffness, geometric stiffness, and prediction of buckling, on three-dimensional moment-resisting frames. The program is divided into five different subprograms, each performing a specific function necessary to the solution of a frame analysis problem. The main program menu, which appears on the screen when the program is started, lists these five subprograms:

1. FILES
2. INPUT/EDIT
3. SOLVE
4. PRINT
5. PLOT

Upon selecting one of these options, the user will see a more specific menu for the particular subprogram. He may select options from that menu by inputting the corresponding number, or may return to the main program menu by pressing "CONT" with no input. Each of the five subprograms will be described in detail below.

The program has default values for most of the input requested. If, when asked a question, the user presses "CONT" without typing any number or character, the program will use the default value. In general, the default values are as follows:

INPUT REQUESTED	DEFAULT VALUE
file name	previous file
numerical data	0
'yes' or 'no'	yes
menu selection	return

Default values that do not follow these general guidelines are noted in the

program description and instructions that follow.

Often the question "PRINTOUT?" will appear on the screen. If the answer to the question is "yes" (1), the information requested will be printed on the computer's thermal printer. If "no" (0), the information will be printed on the screen. The default value for this question is always "no".

1. FILES

The program is structured so that each project has a separate set of files, designated by a one to three character name chosen by the user. These files are permanent unless they are erased intentionally, allowing several projects to be stored at the same time on the computer's mass storage device.

The menu for the "FILES" subprogram has three options:

1. FILE CATALOG
2. COPY FILE
3. ERASE FILE

Press "CONT" to return to the main program menu.

1. FILE CATALOG

Lists all projects currently in the program's filing system. Includes three character file name, title, number of joints, number of members, number of load conditions, and whether or not the project has been solved.

2. COPY FILE

Copies an existing file's input data to a new file (name must be from one to three characters). Output data is not copied.

3. ERASE FILE

Erases an existing file, including all input and output data. After being given the name of the file to be erased, the program will ask "Is this correct?" An answer of "yes" (1) or "no" (0) must be given; there is no default value for this question.

2. INPUT/EDIT

If given a new file name, the program will ask the user for all necessary information about the project: title, geometry, properties, supports and loads. If given an existing file name, the program will allow the user to change any previously input information.

INPUT (new file name)

Questions are listed in order of appearance on the screen.

"JOB TITLE AND COMMENTS":

Title and comments (one line) may be up to 60 characters long, and will be printed in the title block of all output pages.

"NUMBER OF JOINTS?"

"NUMBER OF MEMBERS?"

"NUMBER OF LOAD CONDITIONS?"

One to fifteen load conditions may be used. A unique solution is made for each load condition. The program will display "PLEASE WAIT" while it sets up the files for the project. From this point on all information will be permanently stored as soon as its input is complete. For example, once all joint coordinates are input, they will be stored in the program files. The computer can be turned off and re-started and input will resume with member properties.

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"SELECT UNITS FOR COORDINATES"

Select units of length using keys on the upper right side of the keyboard. These will be the units with which the joint coordinates will be input. Obviously, units of force are immaterial to joint coordinates.

"INPUT JOINT COORDINATES"

Type x, y and z coordinates for each joint, separating the numbers with commas. ONLY non-zero coordinates must be input; the default value is zero. The following examples will serve to show what need and need not be input:

COORDINATES	INPUT
x=0 y=0 z=0	nothing
x=5 y=0 z=0	5
x=0 y=5 z=0	0,5
x=0 y=0 z=5	0,0,5
x=5 y=5 z=5	5,5,5

If a coordinate is input incorrectly, continue to input all the other coordinates. When all coordinates are input, the program will ask "ALL COORDINATES CORRECT?" A "no" response (0) will allow the user to change any coordinates one at a time. After changes are made, just press "CONT" when asked for a joint number, and the program will go on to the next step.

"SELECT UNITS FOR PROPERTIES"

Select units for length and force.

"INPUT MEMBER PROPERTIES"

"MEMBER NUMBER(S)"

Input a list of member that all have the same properties. Member list uses commas and dashes. For example,

1,3,5,10-20,25,8,7

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is an acceptable member list. Numbers need not be in any particular order. After all properties are input for all members, press "CONT" without typing a member list to go on to the next step.

After the member list has been input, input the properties for those members as they are asked for. All properties must be input, whether or not they are appropriate for the particular analysis desired.

"INPUT MEMBER INCIDENCES"

Input the joint number for the i and j end of each member. If an incidence is input incorrectly, continue to input the rest of the member incidences. After all incidences are input, they may be changed selectively.

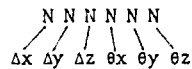
"INPUT SUPPORTS"

"JOINT NUMBER(S)"

Input joint list for a particular type of support, using the syntax of commas and dashes described earlier under "MEMBER PROPERTIES".

"SUPPORT CODE: ?"

Input support code: a six digit code describing x, y and z translations and rotations. For example, the code "111000" describes a pinned joint. The three ones mean that the joint is fixed from translating along the x, y and z axes, while the three zeroes mean that the joint may rotate about the x, y and z axes. The support code is structured as shown below, "1" meaning "fixed" and "0" meaning "free".



To change a support, input the joint number(s) and the new support code. The new code will supersede the old one. The default value for a support code is "111111"--that is, fixed.

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NOTE: Coplanar structures should be fixed from out-of-plane rotation and translation at all joints. Failure to do so will not only considerably increase the solution time but may result in a singular matrix. For example, a coplanar structure in the x-y plane should have all joints fixed against Δz , θ_x and θ_y (001110), plus any in-plane supports.

"SELECT UNITS FOR LOADS"

Select units for length and force.

"LOAD CONDITION #?"

"JOINT NUMBER(S): ?"

Input joint list as described under "SUPPORTS", followed by forces and moments as they are requested. Loads are additive; for example, a load $P_x = 100$ pounds at a particular joint, followed by a load $P_x = -25$ pounds at the same joint would result in a load of $P_x = 75$ pounds at that joint. Default value for loads is zero. After all loads are input, the program will return to the main program menu.

EDIT (existing file name)

The following menu will appear on the screen:

1. CHANGE TITLE
2. CHANGE UNITS
3. CHANGE JOINTS
4. CHANGE MEMBERS
5. CHANGE SUPPORTS
6. CHANGE LOADS

Descriptions of each menu item follow. To return to the main program menu, press "CONT".

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1. CHANGE TITLE

Edit project title and press "CONT". Title must not be longer than sixty characters.

2. CHANGE UNITS

Change units for coordinates, properties and loads. Changing units will not affect data that has already been input. New units are only for data that will be input after the units change.

3. CHANGE JOINTS

Change coordinates, add joints, or list joint coordinates. To add joints, input the number of joints to be added and their coordinates. Through this command and the command "ADD MEMBERS" (see below), the size and complexity of a structure may be increased for further analysis.

Note that joints may not be deleted, but any joint to which no members are connected is considered inactive and does not affect the solution. To list joint coordinates, input the joint numbers to be listed. For syntax, see "SUPPORTS." The default joint list is all joints.

4. CHANGE MEMBERS

Change member properties or incidences, add members, activate or inactivate members, or list member properties and incidences. To change member properties, all eight properties must be reinput. When a member is inactivated, it will not be included in the solution. Its properties and incidences, however, will remain in the program memory so that it can be reactivated for future analysis. Member incidences can only be changed when a member is active.

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5. CHANGE SUPPORTS

Change, add, delete or list support joints. To change supports, input the joint list and new support code, which will supersede the old support code.

6. CHANGE LOADS

Delete or add loads or load conditions; list loads. New loads are simply added on to previous loads. For example, a load of $P_x = 25$ pounds added to a joint with a previous load of $P_x = 75$ pounds will result in a total load of $P_x = 100$ pounds at that joint.

The maximum number of load conditions is 15; the minimum is one. If a project has only one load condition, it cannot be deleted.

3. SOLVE

This subprogram performs the stiffness analysis once a project has been input. The user has an option to include the effects of shear stiffness and geometric stiffness and to perform a buckling check. A buckling check necessarily requires that geometric stiffness be included.

Once the "SOLVE" subprogram has been started, the keyboard will be inactive. To stop the program after this point, the "RESET" button must be used. The program cannot then be continued from the point where it was stopped; it must be started over at the beginning.

Since the solution of a large structure takes a considerable amount of time, the program will display the steps it goes through in the solution. A geometric stiffness analysis, which must iterate for each load condition, will take longer to perform than an analysis that does not include geometric stiffness. Shear stiffness does not affect solution time.

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If a buckling check is performed, a warning will be printed on the thermal printer for each load condition under which the structure buckles.

Once the solution is complete, press "CONT" to return to the main program menu.

4. PRINT

This subprogram prints all input and output data either on the screen or on the thermal printer. The user can select which information he wishes to print, including deformations for only certain joints and forces for only certain members. The default value for the joint list in printing deformations is all joints; the default value for the member list in printing member forces is all members.

5. PLOT

This subprogram draws an isometric plot of the frame and its deformations either on the screen, the thermal printer or the external plotter. Inactive members are shown by dotted lines.

To plot deformations, input load condition and scale factor. The scale factor is a number by which the deformations are multiplied so that small deformations become visible.

The screen is not cleared between successive plots; to clear the screen select "2. CLEAR SCREEN."

After a plot has been drawn, press "CONT" to number joints or members or return to the subprogram menu.

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FILE STRUCTURE

Program files are grouped into three categories: input files, output files, and volatile files. Input files and output files are kept on the mass storage device for future reference until they are erased by the user. Volatile files are cleared as soon as they are no longer needed.

Every file has a six-character name. The first three characters are assigned by the program; the last three are the three-character file name chosen by the user. The three-character file name will be designated below as "---"

INPUT FILES

"s11---": Title, size of structure, units, coordinates, properties, and supports. Number of records varies; 256 bytes per record.

"s12---": Loads. One record per load condition; record size varies.

OUTPUT FILES

"s21---": Joint deformations. One record per joint; 48 bytes per record per load condition.

"s22---": Member forces and moments. One record per member; 96 bytes per record per load condition.

"s23---": Supplementary information: inactive joints, buckled load conditions. Number of records varies; 256 bytes per record.

VOLATILE FILES

"s31---" Joint stiffness matrix. One record per row; record size varies. This file is created and purged within the "SOLVE" program segment.

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EXAMPLE #1 - Bending and Shear Deformations

A cantilever beam with a point load at its free end was analyzed first with $1/A_V G = 0$ to demonstrate bending deformations, and then with $1/EI = 0$ to demonstrate shear deformations. The results match theoretical solutions of the problem.

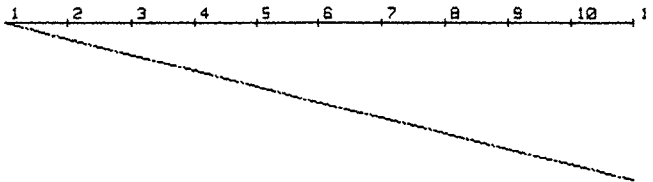
Bending deformation at free end by analysis: $U = 333.33$
by bending theory: $U = 333.33$

Shear deformation at free end by analysis: $U = 10.000$
by Castigliano's theorem: $U = 10.000$

STIFFNESS ANALYSIS
TEST BEAM: SHEAR DEFORMATION

12 DECEMBER 1985
FILE 000

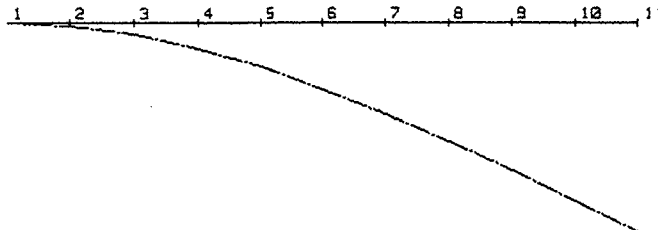
DEFORMATIONS
LOAD CONDITION 1
SCALE FACTOR: .25



STIFFNESS ANALYSIS
TEST BEAM: BENDING DEFORMATION

12 DECEMBER 1985
FILE 000

DEFORMATIONS
LOAD CONDITION 1
SCALE FACTOR: .01



STIFFNESS ANALYSIS
TEST BEAM

12 DECEMBER 1985
FILE 000

JOINT COORDINATES:

UNITS: INCHES

JOINT # 1	X= 0	Y= 0	Z= 0
JOINT # 2	X= 1	Y= 0	Z= 0
JOINT # 3	X= 2	Y= 0	Z= 0
JOINT # 4	X= 3	Y= 0	Z= 0
JOINT # 5	X= 4	Y= 0	Z= 0
JOINT # 6	X= 5	Y= 0	Z= 0
JOINT # 7	X= 6	Y= 0	Z= 0
JOINT # 8	X= 7	Y= 0	Z= 0
JOINT # 9	X= 8	Y= 0	Z= 0
JOINT # 10	X= 9	Y= 0	Z= 0
JOINT # 11	X= 10	Y= 0	Z= 0

MEMBER INCIDENCES:

MEMBER	JOINT I	JOINT J
1	1	2
2	2	3
3	3	4
4	4	5
5	5	6
6	6	7
7	7	8
8	8	9
9	9	10
10	10	11

MEMBER PROPERTIES:

UNITS: INCHES POUNDS

MEMBER	Ax	Ay	Az	Ix	Iy	Iz	E	G
1	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
3	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
4	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
5	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
6	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
7	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
8	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
9	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
10	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

STIFFNESS ANALYSIS
TEST BEAM

12 DECEMBER 1985
FILE 000

SUPPORTS:

JOINT #1	Px	Py	Pz	0x	0y	0z
JOINT #2	Pz	0x	0y			
JOINT #3	Pz	0x	0y			
JOINT #4	Pz	0x	0y			
JOINT #5	Pz	0x	0y			
JOINT #6	Pz	0x	0y			
JOINT #7	Pz	0x	0y			
JOINT #8	Pz	0x	0y			
JOINT #9	Pz	0x	0y			
JOINT #10	Pz	0x	0y			
JOINT #11	Pz	0x	0y			

LOADS:

UNITS: INCHES POUNDS

LOAD CONDITION #1
JOINT # 11 Px= 0 Py=-1 Pz= 0

JOINT DEFORMATIONS:

UNITS: INCHES

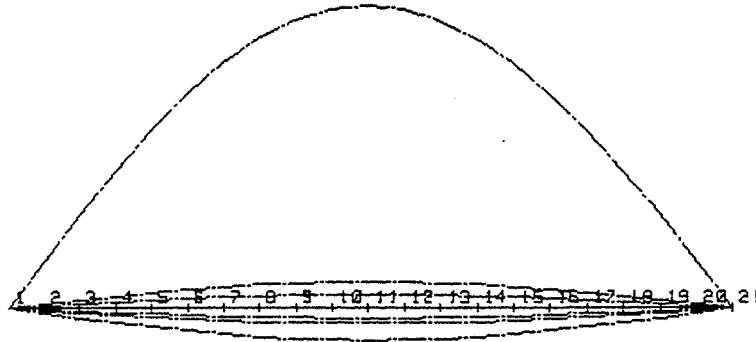
JOINT	LC	Ux	Uy	Uz	0x	0y	0z
1	1	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00
2	1	0.000E+00	-4.833E+00	0.000E+00	0.000E+00	0.000E+00	-9.500E+00
3	1	0.000E+00	-1.867E+01	0.000E+00	0.000E+00	0.000E+00	-1.800E+01
4	1	0.000E+00	-4.050E+01	0.000E+00	0.000E+00	0.000E+00	-2.550E+01
5	1	0.000E+00	-6.933E+01	0.000E+00	0.000E+00	0.000E+00	-3.200E+01
6	1	0.000E+00	-1.042E+02	0.000E+00	0.000E+00	0.000E+00	-3.750E+01
7	1	0.000E+00	-1.440E+02	0.000E+00	0.000E+00	0.000E+00	-4.200E+01
8	1	0.000E+00	-1.878E+02	0.000E+00	0.000E+00	0.000E+00	-4.550E+01
9	1	0.000E+00	-2.347E+02	0.000E+00	0.000E+00	0.000E+00	-4.800E+01
10	1	0.000E+00	-2.835E+02	0.000E+00	0.000E+00	0.000E+00	-4.950E+01
11	1	0.000E+00	-3.333E+02	0.000E+00	0.000E+00	0.000E+00	-5.000E+01

JOINT DEFORMATIONS: (SHEAR INCLUDED)

UNITS: INCHES

JOINT	LC	Ux	Uy	Uz	0x	0y	0z
1	1	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00
2	1	0.000E+00	-1.000E+00	0.000E+00	0.000E+00	0.000E+00	-9.500E-07
3	1	0.000E+00	-2.000E+00	0.000E+00	0.000E+00	0.000E+00	-1.800E-06
4	1	0.000E+00	-3.000E+00	0.000E+00	0.000E+00	0.000E+00	-2.550E-06
5	1	0.000E+00	-4.000E+00	0.000E+00	0.000E+00	0.000E+00	-3.200E-06
6	1	0.000E+00	-5.000E+00	0.000E+00	0.000E+00	0.000E+00	-3.750E-06
7	1	0.000E+00	-6.000E+00	0.000E+00	0.000E+00	0.000E+00	-4.200E-06
8	1	0.000E+00	-7.000E+00	0.000E+00	0.000E+00	0.000E+00	-4.550E-06
9	1	0.000E+00	-8.000E+00	0.000E+00	0.000E+00	0.000E+00	-4.800E-06
10	1	0.000E+00	-9.000E+00	0.000E+00	0.000E+00	0.000E+00	-4.950E-06
11	1	0.000E+00	-1.000E+01	0.000E+00	0.000E+00	0.000E+00	-5.000E-06

DEFORMATIONS
 LOAD CONDITION 4
 SCALE FACTOR: .2



EXAMPLE #2 - Buckling of a Pinned Column

A pinned column was loaded incrementally (by one-pound increments) until analysis showed buckling. The buckling load of 74023 pounds matches the Euler buckling load for a pinned column to at least four significant figures.

Buckling load by analysis: $P_{cr} = 74023$ pounds

by Euler's equation: $P_{cr} = \pi^2 EI/L^2 = 74022.03$ pounds

The shape of the buckled column also matches the theoretical sinusoidal mode shape $\sin(\pi x/L)$.

x	$\sin(\pi x/L)$	U_y/U_{max}
0	0	0
10	0.156	0.156
20	0.309	0.309
30	0.454	0.454
40	0.588	0.588
50	0.707	0.707
60	0.809	0.809
70	0.891	0.891
80	0.951	0.951
90	0.988	0.988
100	1.000	1.000

STIFFNESS ANALYSIS
BUCKLING OF PINNED COLUMN

12 DECEMBER 1985
FILE 001

JOINT COORDINATES:

UNITS: INCHES

JOINT # 1	X= 0	Y= 0	Z= 0
JOINT # 2	X= 10	Y= 0	Z= 0
JOINT # 3	X= 20	Y= 0	Z= 0
JOINT # 4	X= 30	Y= 0	Z= 0
JOINT # 5	X= 40	Y= 0	Z= 0
JOINT # 6	X= 50	Y= 0	Z= 0
JOINT # 7	X= 60	Y= 0	Z= 0
JOINT # 8	X= 70	Y= 0	Z= 0
JOINT # 9	X= 80	Y= 0	Z= 0
JOINT # 10	X= 90	Y= 0	Z= 0
JOINT # 11	X= 100	Y= 0	Z= 0
JOINT # 12	X= 110	Y= 0	Z= 0
JOINT # 13	X= 120	Y= 0	Z= 0
JOINT # 14	X= 130	Y= 0	Z= 0
JOINT # 15	X= 140	Y= 0	Z= 0
JOINT # 16	X= 150	Y= 0	Z= 0
JOINT # 17	X= 160	Y= 0	Z= 0
JOINT # 18	X= 170	Y= 0	Z= 0
JOINT # 19	X= 180	Y= 0	Z= 0
JOINT # 20	X= 190	Y= 0	Z= 0
JOINT # 21	X= 200	Y= 0	Z= 0

MEMBER INCIDENCES:

MEMBER JOINT I JOINT J

1	1	2
2	2	3
3	3	4
4	4	5
5	5	6
6	6	7
7	7	8
8	8	9
9	9	10
10	10	11
11	11	12
12	12	13
13	13	14
14	14	15
15	15	16
16	16	17
17	17	18
18	18	19
19	19	20
20	20	21

STIFFNESS ANALYSIS
BUCKLING OF PINNED COLUMN

12 DECEMBER 1985
FILE 001

MEMBER PROPERTIES:

UNITS: INCHES POUNDS

MEMBER	Ax	Ay	Az	Ix	Iy	Iz	E	G
1	1.00	1.00	1.00	10.0	10.0	10.0	30000000	10000000
2	1.00	1.00	1.00	10.0	10.0	10.0	30000000	10000000
3	1.00	1.00	1.00	10.0	10.0	10.0	30000000	10000000
4	1.00	1.00	1.00	10.0	10.0	10.0	30000000	10000000
5	1.00	1.00	1.00	10.0	10.0	10.0	30000000	10000000
6	1.00	1.00	1.00	10.0	10.0	10.0	30000000	10000000
7	1.00	1.00	1.00	10.0	10.0	10.0	30000000	10000000
8	1.00	1.00	1.00	10.0	10.0	10.0	30000000	10000000
9	1.00	1.00	1.00	10.0	10.0	10.0	30000000	10000000
10	1.00	1.00	1.00	10.0	10.0	10.0	30000000	10000000
11	1.00	1.00	1.00	10.0	10.0	10.0	30000000	10000000
12	1.00	1.00	1.00	10.0	10.0	10.0	30000000	10000000
13	1.00	1.00	1.00	10.0	10.0	10.0	30000000	10000000
14	1.00	1.00	1.00	10.0	10.0	10.0	30000000	10000000
15	1.00	1.00	1.00	10.0	10.0	10.0	30000000	10000000
16	1.00	1.00	1.00	10.0	10.0	10.0	30000000	10000000
17	1.00	1.00	1.00	10.0	10.0	10.0	30000000	10000000
18	1.00	1.00	1.00	10.0	10.0	10.0	30000000	10000000
19	1.00	1.00	1.00	10.0	10.0	10.0	30000000	10000000
20	1.00	1.00	1.00	10.0	10.0	10.0	30000000	10000000

SUPPORTS:

JOINT #1	Py	Pz	0x	0y
JOINT #2	Pz	0x	0y	
JOINT #3	Pz	0x	0y	
JOINT #4	Pz	0x	0y	
JOINT #5	Pz	0x	0y	
JOINT #6	Pz	0x	0y	
JOINT #7	Pz	0x	0y	
JOINT #8	Pz	0x	0y	
JOINT #9	Pz	0x	0y	
JOINT #10	Pz	0x	0y	
JOINT #11	Pz	0x	0y	
JOINT #12	Pz	0x	0y	
JOINT #13	Pz	0x	0y	
JOINT #14	Pz	0x	0y	
JOINT #15	Pz	0x	0y	
JOINT #16	Pz	0x	0y	
JOINT #17	Pz	0x	0y	
JOINT #18	Pz	0x	0y	
JOINT #19	Pz	0x	0y	
JOINT #20	Pz	0x	0y	
JOINT #21	Px	Py	Pz	0x 0y

STIFFNESS ANALYSIS
BUCKLING OF PINNED COLUMN

12 DECEMBER 1985
FILE 001

LOADS:

UNITS: INCHES POUNDS

LOAD CONDITION #	JOINT #	Px	Py	Pz
LOAD CONDITION #1	JOINT # 1	Px= 74020	Py= 0	Pz= 0
	JOINT # 11	Px= 0	Py= 1	Pz= 0
LOAD CONDITION #2	JOINT # 1	Px= 74021	Py= 0	Pz= 0
	JOINT # 11	Px= 0	Py= 1	Pz= 0
LOAD CONDITION #3	JOINT # 1	Px= 74022	Py= 0	Pz= 0
	JOINT # 11	Px= 0	Py= 1	Pz= 0
LOAD CONDITION #4	JOINT # 1	Px= 74023	Py= 0	Pz= 0
	JOINT # 11	Px= 0	Py= 1	Pz= 0
LOAD CONDITION #5	JOINT # 1	Px= 74024	Py= 0	Pz= 0
	JOINT # 11	Px= 0	Py= 1	Pz= 0
LOAD CONDITION #6	JOINT # 1	Px= 74025	Py= 0	Pz= 0
	JOINT # 11	Px= 0	Py= 1	Pz= 0

JOINT DEFORMATIONS: (GEOMETRIC STIFFNESS INCLUDED)

WARNING: STRUCTURE HAS BUCKLED UNDER LOAD CONDITION 4
WARNING: STRUCTURE HAS BUCKLED UNDER LOAD CONDITION 5
WARNING: STRUCTURE HAS BUCKLED UNDER LOAD CONDITION 6

UNITS: INCHES

JOINT	LC	Ux	Uy	Uz	0x	0y	0z
1	1	4.935E-01	0.000E+00	0.000E+00	0.000E+00	0.000E+00	3.038E-01
	2	4.935E-01	0.000E+00	0.000E+00	0.000E+00	0.000E+00	5.816E-01
	3	4.935E-01	0.000E+00	0.000E+00	0.000E+00	0.000E+00	6.656E+00
	4	4.935E-01	0.000E+00	0.000E+00	0.000E+00	0.000E+00	-7.027E-01
	5	4.935E-01	0.000E+00	0.000E+00	0.000E+00	0.000E+00	-3.343E-01
	6	4.935E-01	0.000E+00	0.000E+00	0.000E+00	0.000E+00	-2.191E-01
2	1	4.688E-01	3.026E+00	0.000E+00	0.000E+00	0.000E+00	3.001E-01
	2	4.688E-01	5.792E+00	0.000E+00	0.000E+00	0.000E+00	5.744E-01
	3	4.688E-01	6.629E+01	0.000E+00	0.000E+00	0.000E+00	6.574E+00
	4	4.688E-01	-6.998E+00	0.000E+00	0.000E+00	0.000E+00	-6.940E-01
	5	4.688E-01	-3.329E+00	0.000E+00	0.000E+00	0.000E+00	-3.302E-01
	6	4.688E-01	-2.182E+00	0.000E+00	0.000E+00	0.000E+00	-2.164E-01
3	1	4.441E-01	5.977E+00	0.000E+00	0.000E+00	0.000E+00	2.889E-01
	2	4.441E-01	1.144E+01	0.000E+00	0.000E+00	0.000E+00	5.531E-01
	3	4.441E-01	1.309E+02	0.000E+00	0.000E+00	0.000E+00	6.330E+00

STIFFNESS ANALYSIS
BUCKLING OF PINNED COLUMN

12 DECEMBER 1985
FILE 001

JOINT	LC	Ux	Uy	Uz	0x	0y	0z
3	4	4.441E-01	-1.382E+01	0.000E+00	0.000E+00	0.000E+00	-6.683E-01
	5	4.441E-01	-6.576E+00	0.000E+00	0.000E+00	0.000E+00	-3.179E-01
	6	4.442E-01	-4.310E+00	0.000E+00	0.000E+00	0.000E+00	-2.084E-01
4	1	4.194E-01	8.781E+00	0.000E+00	0.000E+00	0.000E+00	2.707E-01
	2	4.195E-01	1.681E+01	0.000E+00	0.000E+00	0.000E+00	5.182E-01
	3	4.195E-01	1.924E+02	0.000E+00	0.000E+00	0.000E+00	5.931E+00
	4	4.195E-01	-2.031E+01	0.000E+00	0.000E+00	0.000E+00	-6.261E-01
	5	4.195E-01	-9.661E+00	0.000E+00	0.000E+00	0.000E+00	-2.978E-01
	6	4.195E-01	-6.332E+00	0.000E+00	0.000E+00	0.000E+00	-1.952E-01
5	1	3.948E-01	1.137E+01	0.000E+00	0.000E+00	0.000E+00	2.458E-01
	2	3.948E-01	2.176E+01	0.000E+00	0.000E+00	0.000E+00	4.705E-01
	3	3.948E-01	2.491E+02	0.000E+00	0.000E+00	0.000E+00	5.385E+00
	4	3.948E-01	-2.629E+01	0.000E+00	0.000E+00	0.000E+00	-5.685E-01
	5	3.948E-01	-1.251E+01	0.000E+00	0.000E+00	0.000E+00	-2.704E-01
	6	3.948E-01	-8.198E+00	0.000E+00	0.000E+00	0.000E+00	-1.772E-01
6	1	3.701E-01	1.368E+01	0.000E+00	0.000E+00	0.000E+00	2.148E-01
	2	3.701E-01	2.618E+01	0.000E+00	0.000E+00	0.000E+00	4.112E-01
	3	3.701E-01	2.996E+02	0.000E+00	0.000E+00	0.000E+00	4.706E+00
	4	3.701E-01	-3.163E+01	0.000E+00	0.000E+00	0.000E+00	-4.969E-01
	5	3.701E-01	-1.505E+01	0.000E+00	0.000E+00	0.000E+00	-2.364E-01
	6	3.701E-01	-9.863E+00	0.000E+00	0.000E+00	0.000E+00	-1.549E-01
7	1	3.454E-01	1.565E+01	0.000E+00	0.000E+00	0.000E+00	1.786E-01
	2	3.454E-01	2.995E+01	0.000E+00	0.000E+00	0.000E+00	3.418E-01
	3	3.454E-01	3.428E+02	0.000E+00	0.000E+00	0.000E+00	3.912E+00
	4	3.454E-01	-3.619E+01	0.000E+00	0.000E+00	0.000E+00	-4.130E-01
	5	3.454E-01	-1.722E+01	0.000E+00	0.000E+00	0.000E+00	-1.965E-01
	6	3.455E-01	-1.128E+01	0.000E+00	0.000E+00	0.000E+00	-1.288E-01
8	1	3.208E-01	1.723E+01	0.000E+00	0.000E+00	0.000E+00	1.379E-01
	2	3.208E-01	3.299E+01	0.000E+00	0.000E+00	0.000E+00	2.640E-01
	3	3.208E-01	3.775E+02	0.000E+00	0.000E+00	0.000E+00	3.022E+00
	4	3.208E-01	-3.986E+01	0.000E+00	0.000E+00	0.000E+00	-3.190E-01
	5	3.208E-01	-1.896E+01	0.000E+00	0.000E+00	0.000E+00	-1.518E-01
	6	3.208E-01	-1.243E+01	0.000E+00	0.000E+00	0.000E+00	-9.946E-02
9	1	2.961E-01	1.839E+01	0.000E+00	0.000E+00	0.000E+00	9.388E-02
	2	2.961E-01	3.521E+01	0.000E+00	0.000E+00	0.000E+00	1.797E-01
	3	2.961E-01	4.030E+02	0.000E+00	0.000E+00	0.000E+00	2.057E+00
	4	2.961E-01	-4.255E+01	0.000E+00	0.000E+00	0.000E+00	-2.171E-01
	5	2.961E-01	-2.024E+01	0.000E+00	0.000E+00	0.000E+00	-1.033E-01
	6	2.961E-01	-1.327E+01	0.000E+00	0.000E+00	0.000E+00	-6.770E-02
10	1	2.714E-01	1.910E+01	0.000E+00	0.000E+00	0.000E+00	4.753E-02
	2	2.714E-01	3.657E+01	0.000E+00	0.000E+00	0.000E+00	9.098E-02
	3	2.714E-01	4.185E+02	0.000E+00	0.000E+00	0.000E+00	1.041E+00
	4	2.714E-01	-4.418E+01	0.000E+00	0.000E+00	0.000E+00	-1.099E-01
	5	2.714E-01	-2.102E+01	0.000E+00	0.000E+00	0.000E+00	-5.229E-02
	6	2.714E-01	-1.378E+01	0.000E+00	0.000E+00	0.000E+00	-3.427E-02
11	1	2.467E-01	1.934E+01	0.000E+00	0.000E+00	0.000E+00	2.554E-10
	2	2.467E-01	3.702E+01	0.000E+00	0.000E+00	0.000E+00	9.641E-10
	3	2.467E-01	4.237E+02	0.000E+00	0.000E+00	0.000E+00	9.685E-09
	4	2.467E-01	-4.473E+01	0.000E+00	0.000E+00	0.000E+00	-2.501E-09

EXAMPLE #3 - Effect of Shear on Buckling

A cantilever column with axial and lateral loads at the free end was analyzed first without shear stiffness and then with shear stiffness. The buckling load without shear stiffness (between 82 and 84 kips) approximates the Euler buckling load of 82.247 kips. The buckling load with shear stiffness included was reduced to between 76 and 78 kips.

The margin of difference for any one case will depend upon the relative values of shear area and shear modulus. For this case, the shear area was taken as 1/10 of the cross-sectional area, and the shear modulus was taken as 1/3 of the elastic modulus.

without shear: 82 kips < P_{cr} < 84 kips

with shear: 76 kips < P_{cr} < 78 kips

Euler's equation: $P_{cr} = 82.247$ kips

STIFFNESS ANALYSIS
EFFECT OF SHEAR ON BUCKLING

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JOINT COORDINATES:

UNITS: INCHES

JOINT # 1	X= 0	Y= 0	Z= 0
JOINT # 2	X= 10	Y= 0	Z= 0
JOINT # 3	X= 20	Y= 0	Z= 0
JOINT # 4	X= 30	Y= 0	Z= 0

MEMBER INCIDENCES:

MEMBER	JOINT I	JOINT J
1	1	2
2	2	3
3	3	4

MEMBER PROPERTIES:

UNITS:	INCHES			POUNDS				
MEMBER	Ax	Ay	Az	Ix	Iy	Iz	E	G
1	1.00	.10	.10	1.00	1.00	1.00	30000000	10000000
2	1.00	.10	.10	1.00	1.00	1.00	30000000	10000000
3	1.00	.10	.10	1.00	1.00	1.00	30000000	10000000

SUPPORTS:

JOINT #1	Pz	0x	0y
JOINT #2	Pz	0x	0y
JOINT #3	Pz	0x	0y
JOINT #4	Px	Py	Pz 0x 0y 0z

LOADS:

UNITS:	INCHES	POUNDS		
LOAD CONDITION #1				
JOINT # 1		Px= 76000	Py= 10	Pz= 0
LOAD CONDITION #2				
JOINT # 1		Px= 78000	Py= 10	Pz= 0
LOAD CONDITION #3				
JOINT # 1		Px= 80000	Py= 10	Pz= 0

LOAD CONDITION #4
JOINT # 1 Px= 82000 Py= 10 Pz= 0

LOAD CONDITION #5
JOINT # 1 Px= 84000 Py= 10 Pz= 0

JOINT DEFORMATIONS: (GEOMETRIC STIFFNESS INCLUDED)

WARNING: STRUCTURE HAS BUCKLED UNDER LOAD CONDITION 5

UNITS: INCHES

JOINT	LC	Ux	Uy	Uz	0x	0y	0z
1	1	7.600E-02	3.893E-02	0.000E+00	0.000E+00	0.000E+00	-2.030E-03
	2	7.800E-02	5.721E-02	0.000E+00	0.000E+00	0.000E+00	-2.987E-03
	3	8.000E-02	1.079E-01	0.000E+00	0.000E+00	0.000E+00	-5.642E-03
	4	8.200E-02	9.532E-01	0.000E+00	0.000E+00	0.000E+00	-4.990E-02
	5	8.400E-02	-1.393E-01	0.000E+00	0.000E+00	0.000E+00	7.304E-03
2	1	5.067E-02	1.953E-02	0.000E+00	0.000E+00	0.000E+00	-1.762E-03
	2	5.200E-02	2.867E-02	0.000E+00	0.000E+00	0.000E+00	-2.591E-03
	3	5.333E-02	5.401E-02	0.000E+00	0.000E+00	0.000E+00	-4.890E-03
	4	5.467E-02	4.767E-01	0.000E+00	0.000E+00	0.000E+00	-4.322E-02
	5	5.600E-02	-6.961E-02	0.000E+00	0.000E+00	0.000E+00	6.322E-03
3	1	2.533E-02	5.262E-03	0.000E+00	0.000E+00	0.000E+00	-1.025E-03
	2	2.600E-02	7.711E-03	0.000E+00	0.000E+00	0.000E+00	-1.503E-03
	3	2.667E-02	1.450E-02	0.000E+00	0.000E+00	0.000E+00	-2.830E-03
	4	2.733E-02	1.278E-01	0.000E+00	0.000E+00	0.000E+00	-2.496E-02
	5	2.800E-02	-1.862E-02	0.000E+00	0.000E+00	0.000E+00	3.643E-03
4	1	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00
	2	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00
	3	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00
	4	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00
	5	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00

MEMBER FORCES & MOMENTS: (GEOMETRIC STIFFNESS INCLUDED)

WARNING: STRUCTURE HAS BUCKLED UNDER LOAD CONDITION 5

UNITS: INCHES POUNDS

MEM	JT	LC	Px	Py	Pz	Mx	My	Mz
1	1	1	7.600E+04	1.000E+01	0.000E+00	0.000E+00	0.000E+00	1.000E-07
		2	7.800E+04	1.000E+01	0.000E+00	0.000E+00	0.000E+00	-1.000E-06
		3	8.000E+04	1.000E+01	0.000E+00	0.000E+00	0.000E+00	-4.000E-07
		4	8.200E+04	1.000E+01	0.000E+00	0.000E+00	0.000E+00	-3.000E-06
		5	8.400E+04	1.000E+01	0.000E+00	0.000E+00	0.000E+00	1.000E-07
1	2	1	-7.600E+04	-1.000E+01	0.000E+00	0.000E+00	0.000E+00	1.575E+03
		2	-7.800E+04	-1.000E+01	0.000E+00	0.000E+00	0.000E+00	2.326E+03
		3	-8.000E+04	-1.000E+01	0.000E+00	0.000E+00	0.000E+00	4.411E+03
		4	-8.200E+04	-1.000E+01	0.000E+00	0.000E+00	0.000E+00	3.918E+04
		5	-8.400E+04	-1.000E+01	0.000E+00	0.000E+00	0.000E+00	-5.758E+03
2	2	1	7.600E+04	1.000E+01	0.000E+00	0.000E+00	0.000E+00	-1.575E+03
		2	7.800E+04	1.000E+01	0.000E+00	0.000E+00	0.000E+00	-2.326E+03
		3	8.000E+04	1.000E+01	0.000E+00	0.000E+00	0.000E+00	-4.411E+03

STIFFNESS ANALYSIS
EFFECT OF SHEAR ON BUCKLING

12 DECEMBER 1985
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MEM	JT	LC	Px	Py	Pz	Mx	My	Mz
2	2	4	8.200E+04	1.000E+01	0.000E+00	0.000E+00	0.000E+00	-3.918E+04
		5	8.400E+04	1.000E+01	0.000E+00	0.000E+00	0.000E+00	5.758E+03
2	3	1	-7.600E+04	-1.000E+01	0.000E+00	0.000E+00	0.000E+00	2.759E+03
		2	-7.800E+04	-1.000E+01	0.000E+00	0.000E+00	0.000E+00	4.061E+03
		3	-8.000E+04	-1.000E+01	0.000E+00	0.000E+00	0.000E+00	7.672E+03
		4	-8.200E+04	-1.000E+01	0.000E+00	0.000E+00	0.000E+00	6.789E+04
		5	-8.400E+04	-1.000E+01	0.000E+00	0.000E+00	0.000E+00	-9.941E+03
3	3	1	7.600E+04	1.000E+01	0.000E+00	0.000E+00	0.000E+00	-2.759E+03
		2	7.800E+04	1.000E+01	0.000E+00	0.000E+00	0.000E+00	-4.061E+03
		3	8.000E+04	1.000E+01	0.000E+00	0.000E+00	0.000E+00	-7.672E+03
		4	8.200E+04	1.000E+01	0.000E+00	0.000E+00	0.000E+00	-6.789E+04
		5	8.400E+04	1.000E+01	0.000E+00	0.000E+00	0.000E+00	9.941E+03
3	4	1	-7.600E+04	-1.000E+01	0.000E+00	0.000E+00	0.000E+00	3.259E+03
		2	-7.800E+04	-1.000E+01	0.000E+00	0.000E+00	0.000E+00	4.762E+03
		3	-8.000E+04	-1.000E+01	0.000E+00	0.000E+00	0.000E+00	8.932E+03
		4	-8.200E+04	-1.000E+01	0.000E+00	0.000E+00	0.000E+00	7.846E+04
		5	-8.400E+04	-1.000E+01	0.000E+00	0.000E+00	0.000E+00	-1.141E+04

STIFFNESS ANALYSIS
EFFECT OF SHEAR ON BUCKLING

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JOINT DEFORMATIONS: (SHEAR AND GEOMETRIC STIFFNESS INCLUDED)

WARNING: STRUCTURE HAS BUCKLED UNDER LOAD CONDITION 2
WARNING: STRUCTURE HAS BUCKLED UNDER LOAD CONDITION 3
WARNING: STRUCTURE HAS BUCKLED UNDER LOAD CONDITION 4
WARNING: STRUCTURE HAS BUCKLED UNDER LOAD CONDITION 5

UNITS: INCHES

JOINT	LC	Ux	Uy	Uz	0x	0y	0z
1	1	7.600E-02	2.452E+00	0.000E+00	0.000E+00	0.000E+00	-1.186E-01
	2	7.800E-02	-1.282E-01	0.000E+00	0.000E+00	0.000E+00	6.213E-03
	3	8.000E-02	-6.243E-02	0.000E+00	0.000E+00	0.000E+00	3.031E-03
	4	8.200E-02	-4.123E-02	0.000E+00	0.000E+00	0.000E+00	2.005E-03
	5	8.400E-02	-3.076E-02	0.000E+00	0.000E+00	0.000E+00	1.499E-03
2	1	5.067E-02	1.226E+00	0.000E+00	0.000E+00	0.000E+00	-1.027E-01
	2	5.200E-02	-6.400E-02	0.000E+00	0.000E+00	0.000E+00	5.376E-03
	3	5.333E-02	-3.109E-02	0.000E+00	0.000E+00	0.000E+00	2.620E-03
	4	5.467E-02	-2.049E-02	0.000E+00	0.000E+00	0.000E+00	1.732E-03
	5	5.600E-02	-1.525E-02	0.000E+00	0.000E+00	0.000E+00	1.294E-03
3	1	2.533E-02	3.286E-01	0.000E+00	0.000E+00	0.000E+00	-5.929E-02
	2	2.600E-02	-1.706E-02	0.000E+00	0.000E+00	0.000E+00	3.096E-03
	3	2.667E-02	-8.243E-03	0.000E+00	0.000E+00	0.000E+00	1.505E-03
	4	2.733E-02	-5.402E-03	0.000E+00	0.000E+00	0.000E+00	9.924E-04
	5	2.800E-02	-3.999E-03	0.000E+00	0.000E+00	0.000E+00	7.393E-04
4	1	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00
	2	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00
	3	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00
	4	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00
	5	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00

MEMBER FORCES & MOMENTS: (SHEAR AND GEOMETRIC STIFFNESS INCLUDED)

WARNING: STRUCTURE HAS BUCKLED UNDER LOAD CONDITION 2
WARNING: STRUCTURE HAS BUCKLED UNDER LOAD CONDITION 3
WARNING: STRUCTURE HAS BUCKLED UNDER LOAD CONDITION 4
WARNING: STRUCTURE HAS BUCKLED UNDER LOAD CONDITION 5

UNITS: INCHES POUNDS

MEM	JT	LC	Px	Py	Pz	Mx	My	Mz
1	1	1	7.600E+04	1.000E+01	0.000E+00	0.000E+00	0.000E+00	1.000E-06
		2	7.800E+04	1.000E+01	0.000E+00	0.000E+00	0.000E+00	2.000E-07
		3	8.000E+04	1.000E+01	0.000E+00	0.000E+00	0.000E+00	2.000E-08
		4	8.200E+04	1.000E+01	0.000E+00	0.000E+00	0.000E+00	5.000E-08
		5	8.400E+04	1.000E+01	0.000E+00	0.000E+00	0.000E+00	-8.000E-08
1	2	1	-7.600E+04	-1.000E+01	0.000E+00	0.000E+00	0.000E+00	9.326E+04
		2	-7.800E+04	-1.000E+01	0.000E+00	0.000E+00	0.000E+00	-4.912E+03
		3	-8.000E+04	-1.000E+01	0.000E+00	0.000E+00	0.000E+00	-2.408E+03

EXAMPLE #4 - Three-dimensional Frame

A multistory frame with gravity and wind loads was analyzed to show the capabilities of the program. Diagonal bracing was made inactive on a second analysis to demonstrate its contribution to the stiffness of the structure.

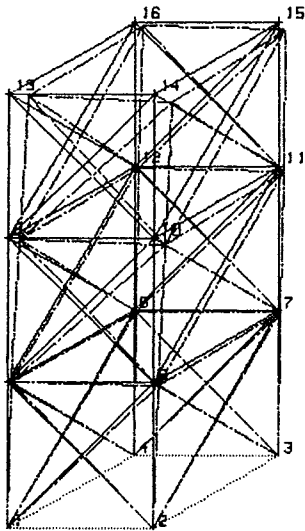
STIFFNESS ANALYSIS EFFECT OF SHEAR ON BUCKLING	12 DECEMBER 1985 FILE 002
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MEM	JT	LC	Px	Py	Pz	Mx	My	Mz
1	2	4	-8.200E+04	-1.000E+01	0.000E+00	0.000E+00	0.000E+00	-1.601E+03
		5	-8.400E+04	-1.000E+01	0.000E+00	0.000E+00	0.000E+00	-1.203E+03
2	2	1	7.600E+04	1.000E+01	0.000E+00	0.000E+00	0.000E+00	-9.326E+04
		2	7.800E+04	1.000E+01	0.000E+00	0.000E+00	0.000E+00	4.912E+03
		3	8.000E+04	1.000E+01	0.000E+00	0.000E+00	0.000E+00	2.408E+03
		4	8.200E+04	1.000E+01	0.000E+00	0.000E+00	0.000E+00	1.601E+03
		5	8.400E+04	1.000E+01	0.000E+00	0.000E+00	0.000E+00	1.203E+03
2	3	1	-7.600E+04	-1.000E+01	0.000E+00	0.000E+00	0.000E+00	1.616E+05
		2	-7.800E+04	-1.000E+01	0.000E+00	0.000E+00	0.000E+00	-8.473E+03
		3	-8.000E+04	-1.000E+01	0.000E+00	0.000E+00	0.000E+00	-4.135E+03
		4	-8.200E+04	-1.000E+01	0.000E+00	0.000E+00	0.000E+00	-2.738E+03
		5	-8.400E+04	-1.000E+01	0.000E+00	0.000E+00	0.000E+00	-2.048E+03
3	3	1	7.600E+04	1.000E+01	0.000E+00	0.000E+00	0.000E+00	-1.616E+05
		2	7.800E+04	1.000E+01	0.000E+00	0.000E+00	0.000E+00	8.473E+03
		3	8.000E+04	1.000E+01	0.000E+00	0.000E+00	0.000E+00	4.135E+03
		4	8.200E+04	1.000E+01	0.000E+00	0.000E+00	0.000E+00	2.738E+03
		5	8.400E+04	1.000E+01	0.000E+00	0.000E+00	0.000E+00	2.048E+03
3	4	1	-7.600E+04	-1.000E+01	0.000E+00	0.000E+00	0.000E+00	1.866E+05
		2	-7.800E+04	-1.000E+01	0.000E+00	0.000E+00	0.000E+00	-9.703E+03
		3	-8.000E+04	-1.000E+01	0.000E+00	0.000E+00	0.000E+00	-4.694E+03
		4	-8.200E+04	-1.000E+01	0.000E+00	0.000E+00	0.000E+00	-3.081E+03
		5	-8.400E+04	-1.000E+01	0.000E+00	0.000E+00	0.000E+00	-2.284E+03

STIFFNESS ANALYSIS
THREE DIMENSIONAL FRAME WITH X-BRACING

12 DECEMBER 1985
FILE 003

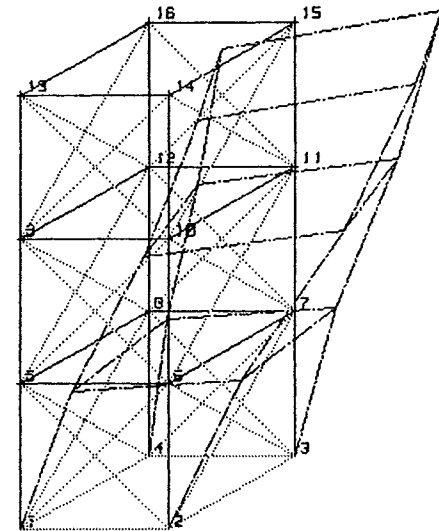
DEFORMATIONS
LOAD CONDITION 1
SCALE FACTOR: 100



STIFFNESS ANALYSIS
THREE DIMENSIONAL FRAME

12 DECEMBER 1985
FILE 003

DEFORMATIONS
LOAD CONDITION 1
SCALE FACTOR: 100



STIFFNESS ANALYSIS
THREE DIMENSIONAL FRAME

12 DECEMBER 1985
FILE 003

JOINT COORDINATES:

UNITS: FEET

JOINT # 1	X= 0	Y= 0	Z= 0
JOINT # 2	X= 10	Y= 0	Z= 0
JOINT # 3	X= 10	Y= 0	Z=-10
JOINT # 4	X= 0	Y= 0	Z=-10
JOINT # 5	X= 0	Y= 10	Z= 0
JOINT # 6	X= 10	Y= 10	Z= 0
JOINT # 7	X= 10	Y= 10	Z=-10
JOINT # 8	X= 0	Y= 10	Z=-10
JOINT # 9	X= 0	Y= 20	Z= 0
JOINT # 10	X= 10	Y= 20	Z= 0
JOINT # 11	X= 10	Y= 20	Z=-10
JOINT # 12	X= 0	Y= 20	Z=-10
JOINT # 13	X= 0	Y= 30	Z= 0
JOINT # 14	X= 10	Y= 30	Z= 0
JOINT # 15	X= 10	Y= 30	Z=-10
JOINT # 16	X= 0	Y= 30	Z=-10

MEMBER INCIDENCES:

MEMBER JOINT I JOINT J

1	1	2	(INACTIVE)
2	2	3	(INACTIVE)
3	3	4	(INACTIVE)
4	4	1	(INACTIVE)
5	5	6	
6	6	7	
7	7	8	
8	8	5	
9	9	10	
10	10	11	
11	11	12	
12	12	9	
13	13	14	
14	14	15	
15	15	16	
16	16	13	
17	1	5	
18	2	6	
19	3	7	
20	4	8	
21	5	9	
22	6	10	
23	7	11	
24	8	12	
25	9	13	
26	10	14	
27	11	15	
28	12	16	
29	1	6	(INACTIVE)
30	2	5	(INACTIVE)
31	2	7	(INACTIVE)

STIFFNESS ANALYSIS
THREE DIMENSIONAL FRAME

12 DECEMBER 1985
FILE 003

MEMBER	JOINT I	JOINT J	
32	3	6	(INACTIVE)
33	3	8	(INACTIVE)
34	4	7	(INACTIVE)
35	4	5	(INACTIVE)
36	1	8	(INACTIVE)
37	5	10	(INACTIVE)
38	6	9	(INACTIVE)
39	6	11	(INACTIVE)
40	7	10	(INACTIVE)
41	7	12	(INACTIVE)
42	8	11	(INACTIVE)
43	8	9	(INACTIVE)
44	5	12	(INACTIVE)
45	9	14	(INACTIVE)
46	10	13	(INACTIVE)
47	10	15	(INACTIVE)
48	11	14	(INACTIVE)
49	11	16	(INACTIVE)
50	12	15	(INACTIVE)
51	12	13	(INACTIVE)
52	9	16	(INACTIVE)

MEMBER PROPERTIES:

UNITS:	INCHES			POUNDS					
MEMBER	Rx	Ry	Rz	Ix	Iy	Iz	E	G	
1	2.00	1.50	1.50	50	25	25	29000000	12000000	
2	2.00	1.50	1.50	50	25	25	29000000	12000000	
3	2.00	1.50	1.50	50	25	25	29000000	12000000	
4	2.00	1.50	1.50	50	25	25	29000000	12000000	
5	2.00	1.50	1.50	50	25	25	29000000	12000000	
6	2.00	1.50	1.50	50	25	25	29000000	12000000	
7	2.00	1.50	1.50	50	25	25	29000000	12000000	
8	2.00	1.50	1.50	50	25	25	29000000	12000000	
9	2.00	1.50	1.50	50	25	25	29000000	12000000	
10	2.00	1.50	1.50	50	25	25	29000000	12000000	
11	2.00	1.50	1.50	50	25	25	29000000	12000000	
12	2.00	1.50	1.50	50	25	25	29000000	12000000	
13	2.00	1.50	1.50	50	25	25	29000000	12000000	
14	2.00	1.50	1.50	50	25	25	29000000	12000000	
15	2.00	1.50	1.50	50	25	25	29000000	12000000	
16	2.00	1.50	1.50	50	25	25	29000000	12000000	
17	2.00	1.50	1.50	100	50	50	29000000	12000000	
18	2.00	1.50	1.50	100	50	50	29000000	12000000	
19	2.00	1.50	1.50	100	50	50	29000000	12000000	
20	2.00	1.50	1.50	100	50	50	29000000	12000000	
21	2.00	1.50	1.50	100	50	50	29000000	12000000	
22	2.00	1.50	1.50	100	50	50	29000000	12000000	
23	2.00	1.50	1.50	100	50	50	29000000	12000000	
24	2.00	1.50	1.50	100	50	50	29000000	12000000	
25	2.00	1.50	1.50	100	50	50	29000000	12000000	
26	2.00	1.50	1.50	100	50	50	29000000	12000000	
27	2.00	1.50	1.50	100	50	50	29000000	12000000	

STIFFNESS ANALYSIS
THREE DIMENSIONAL FRAME WITH X-BRACING

12 DECEMBER 1985
FILE 003

JOINT DEFORMATIONS: (SHEAR INCLUDED)

UNITS: INCHES

JOINT	LC	Ux	Uy	Uz	0x	0y	0z
1	1	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00
2	1	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00
3	1	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00
4	1	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00
5	1	3.847E-02	-2.805E-03	3.131E-03	-1.707E-06	1.177E-04	-3.939E-04
6	1	4.217E-02	-2.881E-02	1.284E-03	2.672E-05	1.131E-04	-4.137E-04
7	1	1.437E-02	-2.230E-02	-5.312E-03	-5.802E-06	1.314E-04	-1.699E-04
8	1	9.879E-03	-9.433E-03	7.785E-04	-2.151E-05	1.360E-04	-1.438E-04
9	1	9.486E-02	-1.251E-02	3.364E-04	-5.446E-05	2.324E-04	-4.455E-04
10	1	9.768E-02	-5.164E-02	3.660E-03	3.499E-05	2.230E-04	-4.233E-04
11	1	3.590E-02	-4.225E-02	-1.493E-03	5.989E-05	2.327E-04	-1.904E-04
12	1	3.151E-02	-2.237E-02	-3.215E-03	-4.907E-05	2.421E-04	-2.057E-04
13	1	1.513E-01	-2.657E-02	-7.749E-03	-7.663E-05	3.081E-04	-4.177E-04
14	1	1.508E-01	-6.926E-02	8.853E-03	5.853E-05	2.927E-04	-3.993E-04
15	1	5.900E-02	-5.942E-02	6.443E-03	7.001E-05	2.954E-04	-2.019E-04
16	1	5.673E-02	-3.753E-02	-9.837E-03	-6.883E-05	3.107E-04	-2.124E-04

STIFFNESS ANALYSIS
THREE DIMENSIONAL FRAME

12 DECEMBER 1985
FILE 003

MEMBER	Ax	Ay	Az	Ix	Iy	Iz	E	G
28	2.00	1.50	1.50	100	50	50	29000000	12000000
29	1.00	.75	.75	2	1	1	29000000	12000000
30	1.00	.75	.75	2	1	1	29000000	12000000
31	1.00	.75	.75	2	1	1	29000000	12000000
32	1.00	.75	.75	2	1	1	29000000	12000000
33	1.00	.75	.75	2	1	1	29000000	12000000
34	1.00	.75	.75	2	1	1	29000000	12000000
35	1.00	.75	.75	2	1	1	29000000	12000000
36	1.00	.75	.75	2	1	1	29000000	12000000
37	1.00	.75	.75	2	1	1	29000000	12000000
38	1.00	.75	.75	2	1	1	29000000	12000000
39	1.00	.75	.75	2	1	1	29000000	12000000
40	1.00	.75	.75	2	1	1	29000000	12000000
41	1.00	.75	.75	2	1	1	29000000	12000000
42	1.00	.75	.75	2	1	1	29000000	12000000
43	1.00	.75	.75	2	1	1	29000000	12000000
44	1.00	.75	.75	2	1	1	29000000	12000000
45	1.00	.75	.75	2	1	1	29000000	12000000
46	1.00	.75	.75	2	1	1	29000000	12000000
47	1.00	.75	.75	2	1	1	29000000	12000000
48	1.00	.75	.75	2	1	1	29000000	12000000
49	1.00	.75	.75	2	1	1	29000000	12000000
50	1.00	.75	.75	2	1	1	29000000	12000000
51	1.00	.75	.75	2	1	1	29000000	12000000
52	1.00	.75	.75	2	1	1	29000000	12000000

SUPPORTS:

JOINT #	Px	Py	Pz	0x	0y	0z
JOINT #1	Px	Py	Pz	0x	0y	0z
JOINT #2	Px	Py	Pz	0x	0y	0z
JOINT #3	Px	Py	Pz	0x	0y	0z
JOINT #4	Px	Py	Pz	0x	0y	0z

LOADS:

UNITS: INCHES KIPS

LOAD CONDITION #1

JOINT # 5	Px= 1	Py= 0	Pz= 0
JOINT # 9	Px= 2	Py= 0	Pz= 0
JOINT # 13	Px= 3	Py=-10	Pz= 0
JOINT # 14	Px= 0	Py=-10	Pz= 0
JOINT # 15	Px= 0	Py=-10	Pz= 0
JOINT # 16	Px= 0	Py=-10	Pz= 0

JOINT DEFORMATIONS: (SHEAR INCLUDED)

UNITS:		INCHES						
JOINT	LC	Ux	Uy	Uz	0x	0y	0z	
1	1	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	
2	1	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	
3	1	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	
4	1	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	
5	1	5.058E-01	-1.033E-02	1.109E-01	9.197E-04	2.082E-03	-4.668E-03	
6	1	5.048E-01	-3.105E-02	-1.109E-01	-9.197E-04	2.076E-03	-4.658E-03	
7	1	2.398E-01	-3.012E-02	-1.109E-01	-9.197E-04	2.076E-03	-2.457E-03	
8	1	2.398E-01	-1.126E-02	1.109E-01	9.197E-04	2.082E-03	-2.457E-03	
9	1	1.221E+00	-2.496E-02	2.522E-01	7.464E-04	4.505E-03	-4.235E-03	
10	1	1.219E+00	-5.780E-02	-2.522E-01	-7.464E-04	4.493E-03	-4.228E-03	
11	1	6.158E-01	-5.633E-02	-2.522E-01	-7.464E-04	4.493E-03	-2.408E-03	
12	1	6.158E-01	-2.643E-02	2.522E-01	7.464E-04	4.505E-03	-2.408E-03	
13	1	1.734E+00	-4.346E-02	3.426E-01	3.773E-04	6.091E-03	-2.512E-03	
14	1	1.731E+00	-8.068E-02	-3.426E-01	-3.773E-04	6.073E-03	-2.506E-03	
15	1	9.038E-01	-7.902E-02	-3.426E-01	-3.773E-04	6.073E-03	-1.541E-03	
16	1	9.038E-01	-4.512E-02	3.426E-01	3.773E-04	6.091E-03	-1.542E-03	

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