SEQUENTIAL AND TWO-STAGE POINT ESTIMATION PROBLEMS FOR NEGATIVE EXPONENTIAL DISTRIBUTIONS

By

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Submitted to the Faculty of the Graduate College
of the Oklahoma State University
in partial fulfillment of the requirements
for the Degree of
DOCTOR OF PHILOSOPHY
July, 1984

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DISTRIBUTIONS

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ACKNOWLEDGMENTS

I wish to thank my advisor, Dr. Nitis Mukhopadhyay, for suggesting the topics included in this research. Also, and more importantly, I wish to thank him for holding before me the highest standards of discipline and excellence. They will be a continuing goal for me.

My thanks also go to the other members of my committee, Dr. Leroy Folks, Dr. William H. Stewart, and Dr. John P. Chandler, for giving their time and efforts on my behalf.

I appreciate also the understanding shown by Tom Popham and Dr. Bob Morrison in allowing me time off from work on occasion so that I could finish this dissertation.

I feel the remainder of those deserving thanks should appear roughly in the order determined by the degree of suffering my graduate study has inflicted upon them. Following this rule, my dear wife deserves the deepest appreciation. Her moral support has been a stabilizing influence during those inevitable moments of desperation when it seemed nothing was going right. Her support became more tangible when she typed this dissertation. My children, Holly and Geoffrey, deserve appreciation for foregoing a father much of the time. Even now, when my daughter doesn't want to be disturbed, she says she is doing her dissertation.

Thanks also to my parents and my wife's parents, who have all been a source of both moral and physical support.

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CHAPTER I

INTRODUCTION, NOTATIONS, REVIEW AND PRELIMINARY RESULTS

1.1 Introduction

We always assume that we may take any number of independent samples $X_1, X_2, \ldots$ from a negative exponential distribution with density

$$f(x;\mu,\sigma) = \sigma^{-1} e^{-\frac{x-\mu}{\sigma}} I(x>\mu), \tag{1.1.1}$$

where $I(\cdot)$ is the usual indicator function. The parameters μ in $(-\infty,\infty)$ and σ in $(0,\infty)$ are referred to as the location and scale respectively. We address two basic problems. First, the estimation of the location μ, and second the estimation of the mean, $\mu+\sigma$.

The negative exponential distribution is basic to data which represent a time of survival. This type of data occurs in medical research where the survival time is measured after treatment is rendered to patients (See Zelen (1966)). Another example is found in the area of quality control where frequently one wants to gather information regarding the lifespans of certain manufactured items. An example of a situation where the minimum time of survival is almost certainly positive is in the testing of materials for strength. One type of test requires stressing the material repeatedly until it breaks down in some sense. The stress applied at any one time is minimal. It is only the accumulated stress that causes this breakage.

1.2 Notations and Review

In all the work comprising this research, there are several things which are not subject to change from one chapter to another:

(i) It is always assumed that σ is unknown, and having recorded $X_1, X_2, \ldots, X_n$ we denote the i-th order statistic by $X_{n(i)}$, $i=1, \ldots, n$. We estimate σ by $\sigma_n = (n-1)^{-1} \sum_{i=1}^{n} (X_i - X_{n(1)})$ for $n \geq 2$. Note that $(n-1)\sigma_n/n$ is the maximum likelihood estimator for σ.

(ii) We always estimate μ by $X_{n(1)}$, where $X_{n(1)}$ is the smallest order statistic from a sample of size n.

(iii) Throughout, we conveniently ignore the fact that the "optimal" fixed sample size n^*, as derived mathematically had σ been known, may not be an integer. This is done for the purposes of brevity.

There are also several notations which are basic to our study.

(iv) We use the notations $[x]$ to mean the largest integer n such that $n<x$, and int (x) to mean the largest integer n such that $n \leq x$.

(v) We use the notation R for $E(L_N)$, where N is a suitable random sample size.

(vi) We use the notation n^* to mean the optimal fixed sample size if σ were known, and R_{n^*} to denote the risk, that is the expected loss, corresponding to a sample of size n^*.

To this date, the work done on the point estimation problem of the location parameter μ consists of a treatment of the sequential case in order to achieve the minimum risk associated with loss of the form $L_n = A(X_{n(1)} - \mu)^s + cn^t$ where A, s, c, and t are all assumed as known positive constants. This work is found in Mukhopadhyay (1974, 1982). Until now, no two-stage or modified two-stage procedures, along the lines of Mukhopadhyay (1980) and Ghosh and Mukhopadhyay (1981) had been proposed for the point estimation problems. In this case, $n=n^*$ that minimizes $E(L_n)$ is given by $n^* = \left(\frac{K\sigma^s}{ct}\right)^{\frac{1}{s+t}}$, where $K=As\Gamma(s+1)$. The main results referenced may be summarized as follows:

Let $N \equiv N(c)$ be the first positive integer $n \ge m$ such that

$$n \ge \left(\frac{K\sigma^s}{ct}\right)^{\frac{1}{s+t}}, \qquad (1.2.1)$$

m (≥ 2) being the starting sample size. We estimate μ by $X_{N(1)}$ when the rule (1.2.1) stops sampling. It has been shown in Mukhopadhyay (1974) that L_n and $I(N=n)$ are independent, where N is determined by (1.2.1). Using this fact, we see that

$$R = E(L_N) = AE(X_{N(1)} - \mu)^s + cE(N^t)$$

$$= A \sum_{n=m}^{\infty} E\{(X_{N(1)} - \mu)^s | N=n\} P(N=n) + cE(N^t)$$

$$= A \sum_{n=m}^{\infty} E\{(X_{n(1)} - \mu)^s | N=n\} P(N=n) + cE(N^t)$$

$$= \sum_{n=m}^{\infty} (K\sigma^s/s) n^{-s} P(N=n) + cE(N^t)$$

$$= (K\sigma^s/s) E(N^{-s}) + cE(N^t).$$

This result, namely,

$$R = (K\sigma^s/s) E(N^{-s}) + cE(N^t), \qquad (1.2.2)$$

is very important as it gives us a relatively easy way to compute R.

Now, we cite the following results.

Lemma 1.2.1: The stopping variable N in (1.2.1) is well-defined and nonincreasing in c with

(a) $E(N) < \infty$,

(b) $\lim_{c\to 0} N = \infty$ *a.s.*,

(c) $\lim_{c\to 0} E(N) = \infty$,

(d) $\lim_{c\to 0} N/n^* = 1$ *a.s.*.

where $n^* = \left(\frac{K\sigma^s}{ct}\right)^{\frac{1}{s+t}}$.

Theorem 1.2.1: $\lim_{c\to 0} R/R_{n^*} = 1$ *if* $m>1 + s^2/(t+s)$,

$= 1 + \gamma$ *if* $m=1 + s^2/(t+s)$,

$= +\infty$ *if* $m<1 + s^2/(t+s)$.

where γ *is a known positive constant.*

Theorem 1.2.2: *If* $s=t=1$, *then* $\lim_{c\to 0}(R-R_{n^*}) = c + o(c)$.

Theorem 1.2.3: *If* $t=1$, $s\neq 1$, *then* $\lim_{c\to 0}(R-R_{n^*}) = o(c)$ *if and only if* $m\geq s+1$.

Lemma 1.2.1 and Theorem 1.2.1 were proved in Mukhopadhyay (1974), where there is also an exact expression given for γ. Theorems 1.2.2 and 1.2.3 were proved in Mukhopadhyay (1982). All the comments and discussions given so far are pertinent for Chapter II.

The work of Chapter III combines methodologies found in Starr (1966), Simons (1968), Ghosh and Mukhopadhyay (1975), Woodroofe (1977), Lombard and Swanepoel (1978), and Mukhopadhyay (1982). We use these methods to try to achieve $R = E(L_N) \leq W$ in the two-stage and modified two-stage cases, or $R = E(L_N) \simeq W$ in the sequential case, for some specified W (>0), and loss $L_n = A(X_{n(1)}-\mu)^s$.

In Chapter IV, we deal with the estimation problems of the mean λ $(=\mu+\sigma)$ using the loss function $L_n = A(\overline{X}_n-\mu)^2+cn$. The work in this chapter follows the lines of development in Ghosh and Mukhopadhyay (1979) where in fact no specific distributional assumptions are made. The sequential procedure proposed there can be summarized as follows:

Define $N \equiv N(c)$ to be the smallest positive integer $n \geq m$ for which

$$n \geq b(s_n + n^{-\gamma}), \tag{1.2.3}$$

where $s_n^2 = (n-1)^{-1} \sum_{i=1}^{n} (X_i - \overline{X}_n)^2$, γ is a given positive constant, m (≥ 2) is the starting sample size and $b = (A/c)^{\frac{1}{2}}$. Then $\overline{X}_N$ is taken as the estimator for λ. Ghosh and Mukhopadhyay (1979) have proved the following result.

Theorem 1.2.4: With N as in (1.2.3) and no distributional assumptions other than $E|X_1|^8<\infty$ and $Var(X_1)>0$, then for $0<\gamma<\frac{1}{4}$, we have

$$\lim_{c\to 0} R/R_{n^*} = 1.$$

In our case, we assume the specific nature of the distribution of X_1, X_2, . . . , and a result analogous to Theorem 1.2.4 is proven. But there are some basic differences, the main one being the use of σ_n instead of s_n to estimate σ. Also, our corresponding stopping rule has a different form, which does not quite require the initial sample size to grow as $c\to 0$. However, this was the situation in Ghosh and Mukhopadhyay (1979).

1.3 Preliminary Results

Following are some results which will prove useful in later chapters.

Lemma 1.3.1: *If* N *is any stopping rule satisfying* $P(N<m) = 0$, $P(N=m) = P\{\chi^2(2(m-1)) \le 2m(m-1)/a\}$ *and for* $n>m$, $P(N=n) \le P\{\chi^2(2(n-1)) \le 2n(n-1)/a\}$, *then as* $a\to\infty$, *we have*

$$P\{N \le (1-\varepsilon)a\} = O_e(a^{-(m-1)}),$$

for every fixed ε *in* $(0,1)$, *and* m (≥ 2) *an integer*.

Proof: Let $U_n \sim \chi^2(2(n-1))$, $v = \text{int}((1-\varepsilon)a)$. Then, for every $h>0$ and n such that $m \le n$ and $n \le (1-\varepsilon)a$, we have

$$P\{U_n \le 2n(n-1)/a\} = P\{e^{-hU_n} \ge e^{-2hn(n-1)/a}\},$$

which implies

$$P\{U_n \le 2n(n-1)/a\} \le \inf_{0<h<\infty} e^{2hn(n-1)/a} \cdot E\{e^{-hU_n}\},$$

using Chebyshev's inequality. Thus,

$$P\{U_n \le 2n(n-1)/a\} \le \inf_{0<h<\infty} \{e^{2hn(n-1)/a} \cdot (1+2h)^{-(n-1)}\} \qquad (1.3.1)$$

Now, using elementary calculus to minimize the above with respect to h, we find that the h value required to obtain the infimum in the right hand side of (1.3.1) is given by $h = h^* = \frac{1}{2}(a/n-1)$. Thus, with $h = h^*$, (1.3.1) leads to

$$\begin{aligned} P\{U_n \le 2n(n-1)/a\} &\le \exp\{(2n(n-1)/a)(\tfrac{1}{2})(a/n-1)\} \times \\ &\quad \{1 + 2(\tfrac{1}{2})(a/n-1)\}^{-(n-1)} \\ &= e^{n-1-n(n-1)/a} \cdot (a/n)^{-(n-1)} \\ &= \{e^{(1-n/a)} \cdot (n/a)\}^{(n-1)}. \end{aligned}$$

Thus,

$$\begin{aligned} P\{N \le (1-\varepsilon)a\} &\le \sum_{n=m}^{v} P\{U_n \le 2n(n-1)/a\} \\ &\le P\{U_m \le 2m(m-1)/a\} \\ &\quad + (e/a)^m \sum_{n=m+1}^{v} \{(n/a)e^{(1-n/a)}\}^{n-m-1} n^m \end{aligned}$$

$= P_1 + P_2$, say.

Now, we note that for $n \le v$, there is an $r \equiv r(\varepsilon)$ with $0<r<1$, such that $(n/a)e^{(1-n/a)} \le r$. Thus,

$$P_2 \le (e/a)^m \sum_{n=m+1}^{v} r^{n-m-1} n^m$$

$$\le (e/a)^m \sum_{n=m+1}^{\infty} r^{n-m-1} n^m \qquad (1.3.2)$$

$$\le ka^{-m}, \qquad (1.3.3)$$

where k (>0) is some constant independent of a. We are able to go from (1.3.2) to (1.3.3) by using the ratio test for convergence.

Also, since $U_m \sim \chi^2(2(m-1))$ we have

$$k \int_0^{2m(m-1)/a} x^{m-2} e^{-n(n-1)/a} dx \le P_1 \le k \int_0^{2m(m-1)/a} x^{m-2} dx,$$

where k is a suitable positive constant independent of a. Evaluating these integrals yields $P_1 = O_e(a^{-(m-1)})$. (1.3.4)

Now, since we have assumed $P(N=m) = P_1$, equations (1.3.3) and (1.3.4) can be combined to give the desired result, if we also exploit the fact that $P\{N \le (1-\varepsilon)a\} \ge P\{N=m\} = O_e(a^{-(m-1)})$. The proof is now complete.

It is well known that $2(n-1)\sigma_n/\sigma \sim \chi^2(2(n-1))$. This fact can be easily used to prove the following lemma, which will be used repeatedly.

<u>Lemma 1.3.2</u>: $E(\sigma_n/\sigma)^w = (n-1)^{-w}\{\Gamma(n-1)\}^{-1}\Gamma(n-1+w)$, <u>for</u> $w>1-n$ <u>and</u> $n \ge 2$.

<u>Proof</u>: $E(\sigma_n/\sigma)^w = 2^{-w}(n-1)^{-w}E\{(2(n-1)\sigma_n/\sigma)^w\}$

$$= 2^{-w}(n-1)^{-w}(\tfrac{1}{2})^{-w}\{\Gamma(n-1)\}^{-1}\Gamma(n-1+w)$$

$$= (n-1)^{-w}\{\Gamma(n-1)\}^{-1}\Gamma(n-1+w).$$

Lemma 1.3.3 is a slightly generalized form of a result due to Chow and Robbins (1965). We cite it here for completeness.

Lemma 1.3.3: *Let* $Y_m, Y_{m+1}, \ldots$ *be any sequence of random variables such that* $Y_n>0$ *a.s.,* $\lim_{n\to\infty} Y_n=1$ *a.s., where* $m(\geq 1)$ *is a fixed integer. Let* $f(n)$ *be any sequence of positive constants such that* $\lim_{n\to\infty} f(n) = \infty$, *and* $\lim_{n\to\infty} \{f(n)/f(n-1)\} = 1$. For each t (>0) we let

$N\equiv N(t)$ be the first integer k m such that $Y_k \leq f(k)/t$.

Then N *is well-defined and nondecreasing as a function of* t,

$$\lim_{t\to\infty} N=\infty \text{ a.s.,} \quad \lim_{t\to\infty} E(N)=\infty,$$

and

$$\lim_{t\to\infty} f(N)/t=1 \text{ a.s..}$$

The following two lemmas summarize some simple properties of an exponential function.

Lemma 1.3.4: $\lim_{a\to\infty} a\{(1+b/a)^a e^{-b} - 1\} = -b^2/2$ *for every fixed* b *in* $\mathbb{R}$.

Proof: $\lim_{a\to\infty} a\{(1+b/a)^a e^{-b} - 1\} = \lim_{a\to\infty} \{(1+b/a)^a e^{-b} - 1\}(1/a)$

$$= \lim_{a\to\infty} e^{-b}(1+b/a)^a\{\ln(1+b/a) - b/(a+b)\}/(-1/a^2)$$

by L' Hospital's rule. Using L'Hospital's rule a second time, we obtain

$$\lim_{a\to\infty} \{\ln(1+b/a) - b/(a+b)\}/(-1/a^2)$$
$$= \lim_{a\to\infty} \{a(a+b)^{-1}(-b/a^2) + b/(a+b)^2\}/(2/a^3)$$
$$= (b/2)\lim_{a\to\infty} a^3\{1/(a+b)^2 - a^{-1}(a+b)^{-1}\}$$
$$= (-b^2/2)\lim_{a\to\infty} a^3\{a(a+b)^2\}^{-1}$$
$$= -b^2/2.$$

Thus,

$$\lim_{a\to\infty} a\{(1+b/a)^a e^{-b} - 1\}$$

$$= \lim_{a\to\infty} (1+b/a)^a e^{-b} \lim_{a\to\infty} \{\ln(1+b/a) - b/(a+b)\}/(-1/a^2)$$

$$= -b^2/2.$$

Now, the proof is complete.

<u>Lemma 1.3.5</u>: $\lim_{a\to\infty} a\{(1+b/a)^{b-\frac{1}{2}} - 1\} = b(b-\frac{1}{2})$ <u>for every fixed</u> b <u>in</u> $\mathbb{R}$.

<u>Proof</u>: Again with the aid of L'Hospital's rule, we find that

$$\lim_{a\to\infty} a\{(1+b/a)^{b-\frac{1}{2}} - 1\}$$

$$= \lim_{a\to\infty} \{(1+b/a)^{b-\frac{1}{2}} - 1\}/(1/a)$$

$$= \lim_{a\to\infty} (b-\tfrac{1}{2})(1+b/a)^{b-3/2}(-b/a^2)/(-1/a^2)$$

$$= b(b-\tfrac{1}{2}) \lim_{a\to\infty} (1+b/a)^{b-3/2}$$

$$= b(b-\tfrac{1}{2}).$$

Now, the proof is complete.

The following theorem can be deduced from Theorem 7 of Chow, Robbins and Teicher (1965). We cite it here for completeness.

<u>Theorem 1.3.1</u>: <u>If</u> $Z_1, Z_2, \ldots$ <u>are i.i.d. random variables with</u> $E(Z_1)=0$, $E(Z_1^2)=\sigma^2$, $E(Z_1^3)=\gamma$, $E(Z_1^4)=\beta<\infty$, <u>and</u> N <u>is a stopping variable with</u> $E(N^2)<\infty$, <u>then</u> $E(S_N^4)<\infty$, <u>and also</u>

$$E(S_N^4) = 6\sigma^2 E(NS_N^2) + 4\gamma E(NS_N) + \beta E(N) - 3\sigma^4 E\{N(N+1)\},$$

<u>where</u> $S_N = \sum_{i=1}^{N} Z_i$.

In chapter III we will need explicitly the following basic tools from Ghosh and Mukhopadhyay (1975), Woodroofe (1977) and Lombard and Swanepoel (1978).

Let $Z_1, Z_2, \ldots$ be a sequence of i.i.d. positive random variables. Let $L(n)$ be a positive, continuous function on $[0,\infty)$ having the form $L(n) = 1 + L_0 n^{-1} + o(n^{-1})$ as $n\to\infty$, where $-\infty<L_0<\infty$. Suppose a sequential sampling rule is now defined by letting

$N\equiv N(c)$ to be the first positive integer $n\geq m$ for which

$$S_n \leq cn^{\alpha}L(n), \tag{1.3.4}$$

where $S_n = \sum_{i=1}^{n} Z_i$, $m\geq 2$ is the initial sample size, α (>1) is a known constant, and c is a positive constant. Let $\beta = 1/(\alpha-1)$ and $\lambda = \mu^{\beta}c^{-\beta}$, where we write $\mu = E(Z_1)$ and $0 < \tau^2 = \mathrm{Var}(Z_1) < \infty$. Then from Ghosh and Mukhopadhyay (1975) the following theorem can be easily verified.

<u>Theorem 1.3.2</u>: *For the stopping variable* N *defined in (1.3.4) we have, as* $c\to 0$,

$$\lambda^{-\frac{1}{2}}\{N-\lambda\} \longrightarrow N(0,\beta^2\tau^2\mu^{-2}).$$

Suppose further that there exist positive constants B and a such that

$$F(x)\leq Bx^{a}, \text{ for all } x>0. \tag{1.3.5}$$

where F denotes the distribution function of Z_1, and that $E(Z_1^r) < \infty$, for some $r\geq 2$. Let $N^* = \lambda^{-\frac{1}{2}}(N-\lambda)$. Then, Woodroofe (1977) has shown the following results:

<u>Theorem 1.3.3</u>: *If* $0 < u < \min\{r,\frac{1}{2}(2a-1)r\}$ *and* $ma > \frac{1}{2}\beta u$, *then* $|N^*|^u$ is uniformly integrable in c.

<u>Theorem 1.3.4</u>: *If* $r(2\alpha-1) > 4$ *and* $ma > \beta$, *then*

$$E(N) = \lambda + \beta\mu^{-1}\nu - \beta L_0 - \tfrac{1}{2}\alpha\beta^2\tau^2\mu^{-2} + o(1) \text{ as } c\to 0,$$

where ν is a known constant.

An exact expression for computing the constant ν is also given in Woodroofe (1977).

The following theorem is due to Lombard and Swanepoel (1978).

Theorem 1.3.5: Let $S_n^* = \sum_{i=2}^{n} (n-i+1)\{X_{n(i)} - X_{n(i-1)}\} = (n-1)\sigma_n$. *Let* $S_n' = \sum_{i=1}^{n-1} Y_i$, *where* $Y_1, Y_2, \ldots, Y_{n-1}$, *are i.i.d. having the density* $f(y;0,\sigma) = \sigma^{-1}e^{-y/\sigma}I(y>0)$. *Then,* $\{S_n^*:n\geq 2\}$ *and* $\{S_n':n\geq 2\}$ *are identically distributed.*

Theorem 1.3.6: (a) $(m-1)^{-\rho}\Gamma(m-1+\rho)\{\Gamma(m-1)\}^{-1} < 1$ *if* $0<\rho<1$ *and* $m>1-\rho$,

(b) $(m-1)^{\rho}\Gamma(m-1-\rho)\{\Gamma(m-1)\}^{-1} > 1$ *if* $\rho>0$ and $m>1+\rho$.

Proof: to prove part (a), let $A=(m-1)^{-\rho}\Gamma(m-1+\rho)$ and $B=\Gamma(m-1)$. It will suffice to show that $A-B<0$.

$$A = (m-1)^{-\rho} \int_0^\infty x^{m-1+\rho-1}e^{-x}dx$$

$$= (m-1)^{-\rho} \int_0^\infty x^{\rho}\cdot x^{m-2}e^{-x}dx$$

$$= (m-1)^{-\rho}\Gamma(m-1)E(X^{\rho}),$$

where $X \sim \text{Gamma}(1,m-1)$. Since $0<\rho<1$, x^{ρ} is a concave function. Thus, using Jensen's inequality, we have

$$A-B = (m-1)^{-\rho}\Gamma(m-1)\{E(X^{\rho}) - (m-1)^{\rho}\}$$

$$< (m-1)^{-\rho}\Gamma(m-1)\{(E(X))^{\rho} - (m-1)^{\rho}\}$$

$$= 0,$$

and part (a) follows.

The proof of part (b) is similar, noting that $x^{-\rho}$ is a convex function for $\rho>0$.

CHAPTER II

POINT ESTIMATION OF THE LOCATION PARAMETER OF A NEGATIVE EXPONENTIAL DISTRIBUTION

2.1 Introduction

This chapter is intended to fill in the important gaps in the theory left by the absence of any two-stage procedures for the estimation of the location parameter. Here, we propose both two-stage and modified two-stage estimation procedures. In either case, the loss function is taken to be $L_n = A(X_{n(1)}-\mu)^s + cn^t$, where A, s, c, and t are all assumed as known positive constants. Noting that

$$\begin{aligned} R_n &= E(L_n) \\ &= AE(X_{n(1)}-\mu)^s + cn^t \\ &= (K\sigma^s/s)n^{-s} + cn^t, \end{aligned}$$

where $K = As\Gamma(s+1)$, we see that

$$\frac{dR_n}{dn} = -K\sigma^s n^{-(s+1)} + ctn^{t-1}.$$

Setting this equal to zero and solving for n yields $n^* = \left(\frac{K\sigma^s}{ct}\right)^{\frac{1}{s+t}}$.

Taking a second derivative shows that R_n has indeed a minimum for $n=n^*$. The corresponding minimum risk is given by $R_{n^*} = c(1+t/s)(n^*)^t$.

2.2 Two-Stage Procedure

Let the starting sample size $m \geq \text{int}(1+s^2/(s+t)) + 1$ be fixed.

Now we define our stopping rule as

$$N = \max\{m, \left[\left(\frac{K\sigma_m^s}{ct}\right)^{\frac{1}{s+t}}\right] + 1\}, \tag{2.2.1}$$

where m is the initial sample size. That is, we take $X_1, X_2, \ldots, X_m$ and compute $\left[\left(\frac{K\sigma_m^s}{ct}\right)^{\frac{1}{s+t}}\right] + 1$. N=m if $\left[\left(\frac{K\sigma_m^s}{ct}\right)^{\frac{1}{s+t}}\right] + 1 \le m$. If $\left[\left(\frac{K\sigma_m^s}{ct}\right)^{\frac{1}{s+t}}\right] + 1 > m$ then we take $\left[\left(\frac{K\sigma_m^s}{ct}\right)^{\frac{1}{s+t}}\right] + 1 - m$ additional samples. When N is determined, we estimate μ by $X_{N(1)}$.

<u>Theorem 2.2.1</u>: <u>For the sampling plan</u> N <u>defined in (2.2.1), we have</u>

(a) $P(N<\infty) = 1$,

(b) $E(N)<\infty$,

(c) $\lim_{c\to 0} N/n^* = (\sigma_m/\sigma)^{\frac{s}{s+t}}$ <u>a.s.</u>,

(d) $\lim_{c\to 0} |N-n^*| = \infty$ <u>a.s.</u>,

(e) $\lim_{c\to 0} E\{(N/n^*)^w\} = \{\Gamma(m-1)\}^{-1}(m-1)^{\frac{-sw}{s+t}}\Gamma(m-1+\frac{sw}{s+t})$ <u>for</u> (i) <u>every</u> $w>0$, <u>and</u> (ii) <u>every</u> $w<0$ <u>such that</u> $m > 1 - sw/(s+t)$,

(f) $\lim_{c\to 0} E(N-n^*) = -\infty$.

<u>Proof</u>: For part (a) note that

$$\begin{aligned} P(N<\infty) &= 1 - P(N=\infty) \\ &= 1 - \lim_{n\to\infty} P(N>n) \\ &= 1 - \lim_{n\to\infty} P(\max\{m, \left[\left(\frac{K\sigma_m^s}{ct}\right)^{\frac{1}{s+t}}\right] + 1\} > n) \end{aligned}$$

$$= 1,$$

since $\left[\left(\frac{K\sigma_m^s}{ct}\right)^{\frac{1}{s+t}}\right] + 1 < \infty$ a.s..

Next, from the definition of N, we get

$$\left(\frac{K\sigma_m^s}{ct}\right)^{\frac{1}{s+t}} \le N \le \left(\frac{K\sigma_m^s}{ct}\right)^{\frac{1}{s+t}} + m.$$

Since $E\left\{\left(\frac{K\sigma_m^s}{ct}\right)^{\frac{1}{s+t}}\right\}$ is finite, it now follows that $E(N)$ is finite, which is part (b).

Again we have,

$$\left(\frac{K\sigma_m^s}{ct}\right)^{\frac{1}{s+t}} \le N \le \left(\frac{K\sigma_m^s}{ct}\right)^{\frac{1}{s+t}} + m,$$

which implies

$$(\sigma_m/\sigma)^{\frac{s}{s+t}} \le N/n^* \le (\sigma_m/\sigma)^{\frac{s}{s+t}} + (m/n^*),$$

and thus we obtain

$$\lim_{c\to 0} (N/n^*) = (\sigma_m/\sigma)^{\frac{s}{s+t}} \text{ a.s.,}$$

and this is part (c).

To prove part (d), first we get from part (c),

$$\lim_{c\to 0} (N/n^*-1) = (\sigma_m/\sigma)^{\frac{s}{s+t}} - 1 \text{ a.s..}$$

This implies

$$\lim_{c\to 0} \{|N-n^*|/n^*\} = \left|(\sigma_m/\sigma)^{\frac{s}{s+t}} - 1\right| \text{ a.s.,}$$

which immediately leads to a proof of part (d), since $n^* \to \infty$ as $c \to 0$.

To verify part (e), we again start with

$$\left(\frac{K\sigma_m^s}{ct}\right)^{\frac{1}{s+t}} \le N \le \left(\frac{K\sigma_m^s}{ct}\right)^{\frac{1}{s+t}} + m,$$

and we obtain for every fixed $w > 0$,

$$\liminf_{c\to 0} E(\sigma_m/\sigma)^{\frac{sw}{s+t}} \le \liminf_{c\to 0} E(N/n^*)^w.$$

Now, using Lemma 1.3.2, we get

$$\liminf_{c\to 0} E(N/n^*)^w \ge \liminf_{c\to 0} \{\Gamma(m-1)\}^{-1}(m-1)^{\frac{-sw}{s+t}}\,\Gamma\{m-1+sw/(s+t)\}$$

$$= \{\Gamma(m-1)\}^{-1}(m-1)^{\frac{-sw}{s+t}}\,\Gamma\{m-1+sw/(s+t)\}.$$

Also, we have

$$\limsup_{c\to 0} E(N/n^*)^w \le \limsup_{c\to 0} E\{(\sigma_m/\sigma)^{\frac{s}{s+t}} + m/n^*\}^w.$$

But, $\{(\sigma_m/\sigma)^{\frac{s}{s+t}} + m/n^*\}^w$ is monotonic in c and converges to the integrable function $(\sigma_m/\sigma)^{\frac{sw}{s+t}}$ as $c \to 0$.

Thus, by the Monotone Convergence Theorem, we obtain

$$\limsup_{c\to 0} E\{(\sigma_m/\sigma)^{\frac{s}{s+t}} + m/n^*\}^w = E\{(\sigma_m/\sigma)^{\frac{sw}{s+t}}\}$$

$$= (m-1)^{\frac{-sw}{s+t}}\{\Gamma(m-1)\}^{-1}\,\Gamma\{m-1+sw/(s+t)\}.$$

Now, combining the lim inf and lim sup parts, we get part (e).

Next, using part (e) with w=1 and Theorem 1.3.6 part (a), we have

$$\lim E(N-n^*) = n^*\{(\Gamma(m-1))^{-1}(m-1)^{\frac{-s}{s+t}}\Gamma(m-1+s/(s+t)) - 1\}$$

$$= -\infty.$$

Hence, the proof of Theorem 2.2.1 is complete.

Theorem 2.2.2: For the sampling plan N defined in (2.2.1), we have,

$$\lim_{c\to 0} R/R_{n*} = (1+t/s)^{-1}\{(t/s)(m-1)^{\frac{s^2}{s+t}}\{\Gamma(m-1)\}^{-1}\Gamma(m-1-st/(s+t))$$

$$+ (m-1)^{\frac{-st}{s+t}}\{\Gamma(m-1)\}^{-1}\Gamma(m-1+s^2/(s+t))\}.$$

Proof: First we recall that

$$R_{n*} = c(1+t/s)n*^t$$

and from (1.2.2)

$$R = (K\sigma^s/s)E(N^{-s}) + cE(N^t).$$

Thus,

$$R/R_{n*} = (1+t/s)^{-1}\{(K\sigma^s/(sc))n*^{-t}E(N^{-s}) + E(N/n*)^t\}$$

$$= (1+t/s)^{-1}\{(t/s)E(N/n*)^{-s} + E(N/n*)^t\}.$$

Hence, using part (e) from Theorem 2.2.1 with $w=-s$ and $w=t$ completes the proof. One may note that the restriction on m needed for $w=-s$ in part (e) from Theorem 2.2.1, namely $m > 1 + s^2/(s+t)$, is already built into the definition of m given in (2.2.1).

Theorem 2.2.3: For the sampling plan N defined in (2.2.1), we have

$$\lim_{c\to 0} (R-R_{n*}) = O(c^{\frac{s}{s+t}}).$$

Proof: Again using

$$R_{n*} = c(1+t/s)n*^t$$

and

$$R = (K\sigma^s/s)E(N^{-s}) + cE(N^t),$$

we obtain

$$R - R_{n*} = (K\sigma^s/s)E(N^{-s}) + cE(N^t) - c(1+t/s)n*^t.$$

$$= (ct/s)n*^tE(N/n*)^{-s} + cE(N^t) - c(1+t/s)n*^t$$

$$= cn*^t\{(t/s)E(N/n*)^{-s} + E(N/n*)^t - (1+t/s)\}$$

$$= c^{1-\frac{t}{s+t}}(K\sigma^s/t)^{\frac{t}{s+t}}\{(t/s)E(N/n*)^{-s} + E(N/n*)^t - (1+t/s)\}$$

$$= 0(c^{\frac{s}{s+t}}),$$

noting that $E(N/n*)^t$ and $E(N/n*)^{-s}$ are finite by part (e) of Theorem 2.2.1 since $m > 1 + s^2/(s+t)$. Now the proof is complete.

2.3 Modified Two-Stage Procedure

Let $m = \max\{\text{int}(1+s^2/(s+t)) + 1, [(\frac{K}{ct})^{\frac{1}{s+t+\gamma}}] + 1\}$, where γ (>0) is a chosen constant. The problem of "good" choices for γ will be discussed later. For theoretical developments, we consider γ (>0) to be chosen and fixed in advance. Note that γ positive guarantees that although $\lim_{c\to 0} m=\infty$, $\lim_{c\to 0} (m/n*) = 0$. This is desirable from an intuitive point of view, and will be seen to be theoretically desirable in what follows. Then, we define a stopping rule by

$$N = \max\{m, \left[\left(\frac{K\sigma_m^s}{ct}\right)^{\frac{1}{s+t}}\right] + 1\}. \tag{2.3.1}$$

A few preliminary results are summarized below.

Theorem 2.3.1: For the sampling plan N defined in (2.3.1), we have

(a) $P(N<\infty) = 1$,

(b) $E(N)<\infty$,

(c) $\lim_{c\to 0} N/n* = 1$ a.s..

Proof: The proof of parts (a) and (b) are identical to the proofs of parts (a) and (b) in Theorem 2.2.1.

To prove (c), note that from the definition of N, we have

$$\left(\frac{K\sigma_m^s}{ct}\right)^{\frac{1}{s+t}} \le N \le \left(\frac{K\sigma_m^s}{ct}\right)^{\frac{1}{s+t}} + m,$$

which implies that

$$\lim_{c\to 0} (\sigma_m/\sigma)^{\frac{s}{s+t}} \le \lim_{c\to 0} N/n^* \le \lim_{c\to 0} (\sigma_m/\sigma)^{\frac{s}{s+t}} + m/n^*.$$

Thus, $\lim_{c\to 0} N/n^* = 1$ a.s., since $m\to\infty$ as $c\to 0$, which implies that $\sigma_m \to \sigma$ a.s. as $c\to 0$. Here we see the reason for introducing γ into the two-stage rule. The fact that $m\to\infty$ as $c\to 0$ is the crucial factor which makes the modified two-stage procedure perform better than the two-stage procedure.

Lemma 2.3.1: For b in $\mathbb{R}$ fixed, we have

$$\Gamma(a+b)a^{-b}\{\Gamma(a)\}^{-1} - 1 = 0(a^{-1}),$$

as $a\to\infty$.

Proof: By Sterlings series expansion, with

$$R_1 = \frac{1}{12} a^{-1} + \frac{1}{288} a^{-2} - \ . \ . \ . = 0(a^{-1}),$$

$$R_2 = \frac{1}{12} (a+b)^{-1} + \frac{1}{288} (a+b)^{-2} - \ . \ . \ . = 0(a^{-1}),$$

as $a\to\infty$, we now have

$$\lim_{a\to\infty} \{\Gamma(a+b)a^{-b}\{\Gamma(a)\}^{-1} - 1\}$$

$$= \lim_{a\to\infty} \{e^{-(a+b)}(a+b)^{a+b-\frac{1}{2}}(2\pi)^{\frac{1}{2}}(1+R_2)\{e^{-a}a^{a+b-\frac{1}{2}}(2\pi)^{\frac{1}{2}}(1+R_1)\}^{-1} - 1\}$$

$$= \lim_{a\to\infty} \{e^{-b}a^{a+b-\frac{1}{2}}(1+b/a)^{a}(1+b/a)^{b-\frac{1}{2}}(1+R_2)\{a^{a+b-\frac{1}{2}}(1+R_1)\}^{-1} - 1\}$$

$$= \lim_{a\to\infty} \{e^{-b}(1+b/a)^{a}\} \times$$

$$\lim_{a\to\infty} \{(1+b/a)^{b-\frac{1}{2}}(1+0(a^{-1}))(1+0(a^{-1}))^{-1} - 1\}. \qquad (2.3.2)$$

Applying Lemmas 1.3.4 and 1.3.5 to (2.3.2) yields

$$\lim_{a\to\infty} \{\Gamma(a+b)a^{-b}\{\Gamma(a)\}^{-1} - 1\} = \{(1+0(a^{-1}))(1+0(a^{-1}))(1+0(a^{-1}))/$$

$$(1+0(a^{-1}))\} - 1$$

$$= 0(a^{-1}),$$

This proves Lemma 2.3.1.

<u>Lemma 2.3.2</u>: $\lim_{c\to 0} E(N/n^*)^{w} = 1 + 0(c^{\eta})$ <u>for fixed</u> w <u>in</u> $\mathbb{R}$, <u>where</u>

$$\eta = \min\left\{\frac{1}{s+t+\gamma}, \frac{1}{s+t} - \frac{1}{s+t+\gamma}\right\}.$$

<u>Proof</u>: Suppose w>0. Since,

$$N \le \left(\frac{K\sigma_m^{s}}{ct}\right)^{\frac{1}{s+t}} + m,$$

we have

$$E(N/n^*)^{w} \le E\{n^{*-1}((\frac{K\sigma_m^{s}}{ct})^{\frac{1}{s+t}} + m)\}^{w}$$

$$= E\{(\sigma_m/\sigma)^{\frac{s}{s+t}} + (m/n^*)\}^{w}$$

$$\le E\{(\sigma_m/\sigma)^{\frac{ws}{s+t}}(1 + (m/n^*)(\sigma_m/\sigma)^{\frac{-s}{s+t}})^{[w]+1}\}$$

$$= E\{(\sigma_m/\sigma)^{\frac{ws}{s+t}} \sum_{n=0}^{[w]+1} \binom{[w]+1}{n}(m/n^*)^{n}(\sigma_m/\sigma)^{-\frac{ns}{s+t}}\}$$

$$= E\{(\sigma_m/\sigma)^{\frac{ws}{s+t}}(1 + \sum_{n=1}^{[w]+1} \binom{[w]+1}{n}(m/n^*)^{n}(\sigma_m/\sigma)^{-\frac{ns}{s+t}}\}$$

$$\le \Gamma(m-1+\tfrac{sw}{s+t})(m-1)^{-\frac{sw}{s+t}}\{\Gamma(m-1)\}^{-1}$$

$$+kc^{\frac{1}{s+t}-\frac{1}{s+t+\gamma}} \sum_{n=1}^{[w]+1} \binom{[w]+1}{n} E\{(\sigma_m/\sigma)^{\frac{s(w-n)}{s+t}}\}, \qquad (2.3.3)$$

for c small enough so that $m = \left[\left(\frac{K}{ct}\right)^{\frac{1}{s+t+\gamma}}\right] + 1$, and k a positive, generic constant independent of c. In the last step leading to (2.3.3) we note the relationships between m and c, n* and c, and the fact that by definition, $m \geq \text{int}(\frac{s}{s+t}+1) + 1$ guarantee the existence of $E\{(\sigma_m/\sigma)^{\frac{s(w-n)}{s+t}}\}$ for $1 \leq n \leq [w] + 1$. Applying Lemma 2.3.1 to the first part of (2.3.3) with a=m-1 and b = ws/(s+t), and again noting the relationship between m and c, yield,

$$E(N/n^*)^w \leq 1 + O(c^{\frac{1}{s+t+\gamma}})$$
$$+ kc^{\frac{1}{s+t}-\frac{1}{s+t+\gamma}} \sum_{n=1}^{[w]+1} \binom{[w]+1}{n} E\{(\sigma_m/\sigma)^{\frac{s(w-n)}{s+t}}\}, \qquad (2.3.4)$$

as $c \to 0$. Applying Lemma 1.3.2 to (2.3.4) gives,

$$E(N/n^*)^w \leq 1 + O(c^{\frac{1}{s+t+\gamma}}) + kc^{\frac{1}{s+t}-\frac{1}{s+t+\gamma}} \times$$
$$\sum_{n=1}^{[w]+1} \binom{[w]+1}{n} \Gamma(m-1+\frac{s(w-n)}{s+t})(m-1)^{-\frac{s(w-n)}{s+t}} \{\Gamma(m-1)\}^{-1}, \qquad (2.3.5)$$

as $c \to 0$. Lemma 2.3.1 implies the boundedness of $\sum_{n=1}^{[w]+1} \binom{[w]+1}{n} \Gamma(m-1+\frac{s(w-n)}{s+t})(m-1)^{-\frac{s(w-n)}{s+t}} \{\Gamma(m-1)\}^{-1}$ as $c \to 0$, which together with (2.3.5) give

$$\limsup_{c \to 0} E(N/n^*)^w \leq 1 + O(c^{\frac{1}{s+t+\gamma}}) + O(c^{\frac{1}{s+t}-\frac{1}{s+t+\gamma}})$$
$$= 1 + O(c^{\eta}),$$

where $\eta = \min\left\{\frac{1}{s+t+\gamma}, \frac{1}{s+t} - \frac{1}{s+t+\gamma}\right\}$.

Also, by definition of N,

$$N \geq \left(\frac{K\sigma_m^s}{ct}\right)^{\frac{1}{s+t}},$$

which implies that

$$(N/n^*)^w \geq (\sigma_m/\sigma)^{\frac{sw}{s+t}}.$$

Thus,

$$\liminf_{c\to 0} E(N/n^*)^w \geq \liminf_{c\to 0} E(\sigma_m/\sigma)^{\frac{sw}{s+t}}$$

$$= \liminf_{c\to 0} \Gamma(m-1+\frac{sw}{s+t})(m-1)^{\frac{-sw}{s+t}}\{\Gamma(m-1)\}^{-1}$$

$$= 1 + O\left(c^{\frac{1}{s+t+\gamma}}\right),$$

using Lemma 2.3.1. Putting together the lim sup and lim inf parts yield

$$\lim_{c\to 0} E(N/n^*)^w = 1 + O(c^{\eta}), \tag{2.3.6}$$

for positive w.

Let us now turn to the behaviors of the negative moment of N/n^*.

We have again from

$$N \geq \left(\frac{K\sigma_m^s}{ct}\right)^{\frac{1}{s+t}}$$

that

$$E(N/n^*)^{-w} \leq E\{n^{*-1}\left(\frac{K\sigma_m^s}{ct}\right)^{\frac{1}{s+t}}\}^{-w}$$

$$= E\{(\sigma_m/\sigma)^{\frac{-sw}{s+t}}\}$$

$$= \Gamma(m-1-sw/(s+t))\,(m-1)^{\frac{sw}{s+t}}\,\{\Gamma(m-1)\}^{-1},$$

for c small enough so that $m > 1 + sw/(s+t)$. Therefore,

$$\limsup_{c\to 0} E(N/n^*) \le 1 + O(c^{\frac{1}{s+t+\gamma}}),$$

using Lemma 2.3.1. Now,

$$E\{(N/n^*)^{-w}\} = E\{((N/n^*)^{w})^{-1}\}$$

$$\ge \{E(N/n^*)^{w}\}^{-1},$$

by Jensen's inequality. Thus, using (2.3.6), we have

$$\liminf_{c\to 0} E\{(N/n^*)^{-w}\} \ge \{1 + O(c^{\eta})\}^{-1}$$

$$= 1 + O(c^{\eta}),$$

Putting the lim sup and lim inf parts together yields

$$\lim_{c\to 0} E(N/n^*)^{-w} = 1 + O(c^{\eta}),$$

and the proof of Lemma 2.3.2 is complete.

<u>Theorem 2.3.2</u>: <u>For the sampling plan</u> N <u>defined in (2.3.1), we have</u>

$$\lim_{c\to 0} R/R_{n^*} = 1.$$

<u>Proof</u>: As in the proof of Theorem 2.2.2, we have

$$R/R_{n^*} = (1+t/s)^{-1}\{(t/s)E\{(N/n^*)^{-s}\} + E\{(N/n^*)^{t}\}\}$$

and Lemma 2.3.2 with $w=t$ and $w=-s$. Also note that the restriction put on m (by the definition) is needed in these verifications.

<u>Theorem 2.3.3</u>: <u>For the</u> N <u>defined in (2.3.1), we have</u>

$$\lim_{c\to 0} (R-R_{n^*}) = O(c^{\frac{s}{s+t}+\eta}),$$

where $\eta = \min\left\{\frac{1}{s+t+\gamma}, \frac{1}{s+t} - \frac{1}{s+t+\gamma}\right\}$.

<u>Proof</u>: As in the proof of Theorem 2.2.3,

$$\lim_{c\to 0} (R-R_{n^*}) = \lim_{c\to 0} c^{1-\frac{t}{s+t}} (K\sigma^s/t)^{\frac{t}{s+t}} \{(t/s)E\{(N/n^*)^{-s}\} + E\{(N/n^*)^t\}$$

$$- (1+t/s)\}$$

$$= (k\sigma^s/t)^{\frac{t}{s+t}} \lim_{c\to 0} c^{\frac{s}{s+t}} \{(t/s)(1+0(c^\eta))$$

$$+ 1 + 0(c^\eta) - (1+t/s)\},$$

using Lemma 2.3.2 with w=t and w=-s. Thus, we have

$$\lim_{c\to 0} (R-R_{n^*}) = 0\left(c^{\frac{s}{s+t} + \eta}\right),$$

and the theorem is proved.

Remark 2.3.1: We conjecture that possibly the order in Theorem 2.3.3 can be improved to $\lim_{c\to 0} (R-R_{n^*}) = 0\left(c^{\frac{s}{s+t} + \frac{1}{s+t+\gamma}}\right)$, but have not yet found a proof of this for general s and t.

2.4 Simulations

In order to study the moderate sample size behavior of our procedures, we ran simulations for various parameter values. In each of Tables I, II and III the following are fixed: A=50, σ=1, μ=10, s=2, and t=1. n* steps through the values 10, 15, 20, 25, 50 and 100. Then, c is determined by $n^* = \{K\sigma^s/(ct)\}^{\frac{1}{s+t}}$. Although not a topic of this dissertation the procedure of Mukhopadhyay (1974) is simulated here for purposes of comparison. We study the modified two-stage procedure with γ=0.2 and 0.3, and both the two-stage and sequential procedures are implemented with m=5 and 10.

The simulations were run on an IBM 3081 computer using the SAS language. The SAS function RANEXP was used to generate the pseudo random exponential deviates. RANEXP gives pseudo random deviates from a distribution with density $f(x;0,1)$ where $f(x;\mu,\sigma)$ is the same as in (1.1.1). These were transformed to pseudo random deviates from a distribution with density $f(x;10,1)$ by adding 10 to each value obtained from RANEXP. In this chapter as well as those that follow, we run 200 iterations for every row in each table. Here, the term iteration stands for the process of determining the random sample size N for the relevant procedure one time.

Table I gives the results for the two-stage procedure (2.2.1). For each value of m, we start with m samples from the population $f(x;10,1)$, and we compute σ_m. We then check to see whether to stop or take more samples. When we stop, we record the observed value n(i) of N together with the observed value $X_{n(i)(1)}$ of $X_{N(1)}$ for the i-th iteration in each row, i = 1, 2, . . . , 200. For each i we compute

$$L(i) = 50(X_{n(i)(1)}-10)^2 + cn(i).$$

We write,

$$\overline{N} = (200)^{-1} \sum_{i-1}^{200} n(i), \quad \text{s.e.}(\overline{N}) = \{(199)^{-1}(200)^{-1} \sum_{i=1}^{200} (n(i)-\overline{N})^2\}^{\frac{1}{2}}.$$

Similarly, we write

$$\overline{L} = (200)^{-1} \sum_{i=1}^{200} L(i), \quad \text{s.e.}(\overline{L}) = \{(199)^{-1}(200)^{-1} \sum_{i=1}^{200} (L(i)-\overline{L})^2\}^{\frac{1}{2}}.$$

These remarks are relevant to Tables II and III, also, with one exception. In the modified two-stage procedure we have the added task of computing m as defined in Section 2.3. We compute $\overline{N}$ and $\overline{L}$ for purposes of estimating E(N) and $R=E(L_N)$, respectively. The quantities R_{n^*},

TABLE I

MODERATE SAMPLE BEHAVIOR OF THE TWO-STAGE PROCEDURE (2.2.1)

n^*	m	c	$\bar{N}$	s.e.($\bar{N}$)	$\bar{L}$	s.e.($\bar{L}$)	R_{n^*}	$\bar{L}/R_{n^*}$	$\bar{L}-R_{n^*}$
10	5	0.2000	10.54	0.2418	3.4375	0.2541	3.0000	1.1458	0.4375
10	10	0.2000	11.17	0.1105	2.8571	0.0831	3.0000	0.9524	-.1429
15	5	0.0593	15.40	0.3642	1.5782	0.1297	1.3333	1.1837	0.2449
15	10	0.0593	15.24	0.2248	1.2653	0.0620	1.3333	0.9489	-.0681
20	5	0.0250	20.12	0.4645	0.8718	0.0575	0.7500	1.1624	0.1218
20	10	0.0250	20.25	0.3044	0.9524	0.1007	0.7500	1.2699	0.2024
25	5	0.0128	25.52	0.6228	0.5863	0.0430	0.4800	1.2215	0.1063
25	10	0.0128	24.73	0.4040	0.4765	0.0231	0.4800	0.9927	-.0035
50	5	0.0016	48.98	1.1387	0.1427	0.0116	0.1200	1.1889	0.0227
50	10	0.0016	50.07	0.7978	0.1252	0.0070	0.1200	1.0433	0.0052
100	5	0.0002	95.73	2.1920	0.0316	0.0019	0.0300	1.0521	0.0016
100	10	0.0002	100.63	1.5123	0.0357	0.0048	0.0300	1.1900	0.0057

TABLE II

MODERATE SAMPLE BEHAVIOR OF THE MODIFIED TWO-STAGE PROCEDURE (2.3.1)

n^*	γ	c	$\bar{N}$	s.e. $(\bar{N})$	$\bar{L}$	s.e. $(\bar{L})$	R_{n^*}	$\bar{L}/R_{n^*}$	$\bar{L}-R_{n^*}$
10	0.2	0.2000	11.04	0.1363	3.1402	0.1287	3.0000	1.0467	0.1402
10	0.3	0.2000	10.72	0.1330	3.2935	0.1674	3.0000	1.0978	0.2935
15	0.2	0.0593	15.71	0.1762	1.3867	0.0814	1.3333	1.0400	0.0533
15	0.3	0.0593	15.60	0.2097	1.3596	0.0744	1.3333	1.0197	0.0263
20	0.2	0.0250	20.63	0.2114	0.7157	0.0272	0.7500	0.9543	-.0343
20	0.3	0.0250	20.34	0.2167	0.7187	0.0323	0.7500	0.9582	-.0313
25	0.2	0.0128	25.55	0.2297	0.5032	0.0286	0.4800	1.0483	0.0232
25	0.3	0.0128	25.49	0.2712	0.5529	0.0291	0.4800	1.1518	0.0729
50	0.2	0.0016	50.49	0.3807	0.1165	0.0055	0.1200	0.9712	-.0035
50	0.3	0.0016	50.32	0.4140	0.1256	0.0060	0.1200	1.0464	0.0056
100	0.2	0.0002	100.64	0.5970	0.0308	0.0015	0.0300	1.0260	0.0008
100	0.3	0.0002	100.44	0.5884	0.0286	0.0014	0.0300	0.9547	-.0014

TABLE III

MODERATE SAMPLE BEHAVIOR OF THE SEQUENTIAL PROCEDURE (1.2.1)

n^*	m	c	$\bar{N}$	s.e.$(\bar{N})$	$\bar{L}$	s.e.$(\bar{L})$	R_{n^*}	$\bar{L}/R_{n^*}$	$\bar{L}-R_{n^*}$
10	5	0.2000	9.90	0.1759	3.1276	0.1377	3.0000	1.0425	0.1276
10	10	0.2000	10.76	0.0770	2.9354	0.1262	3.0000	0.9785	-.0646
15	5	0.0593	14.71	0.2423	1.4668	0.0925	1.3333	1.1001	0.1335
15	10	0.0593	14.93	0.1933	1.3155	0.0629	1.3333	0.9866	-.0178
20	5	0.0250	20.05	0.2428	0.7030	0.0277	0.7500	0.9373	-.0470
20	10	0.0250	19.96	0.2210	0.8205	0.0486	0.7500	1.0941	0.0705
25	5	0.0128	24.79	0.2485	0.4814	0.0209	0.4800	1.0029	0.0014
25	10	0.0128	25.17	0.2501	0.4739	0.0188	0.4800	0.9874	-.0061
50	5	0.0016	50.58	0.3219	0.1175	0.0049	0.1200	0.9794	-.0025
50	10	0.0016	49.97	0.3129	0.1090	0.0042	0.1200	0.9085	-.0110
100	5	0.0002	100.54	0.5359	0.0298	0.0013	0.0300	0.9943	-.0002
100	10	0.0002	100.86	0.4856	0.0279	0.0015	0.0300	0.9290	-.0021

$\bar{L}-R_{n^*}$ and $\bar{L}/R_{n^*}$ are also displayed to facilitate comparison of the simulations with the expected theoretical results.

The results of the simulations are encouraging. The average sample size ($\bar{N}$) can be seen to closely approximate n^*, regardless of which procedure is being used. Also, $\bar{L}=\hat{R}$ seems to be approaching zero in each case. Indeed, it is quite often the case that $\bar{L} - R_{n^*} < 0$. This may be explained by the fact that 200 iterations are too few to yield $\text{Var}(\bar{L})$ small enough relative to $R - R_{n^*}$ to draw crisp conclusions.

<u>Remark 2.4.1</u>: It should be noted that our simulations of the two-stage and modified two-stage procedures are based on stopping rules which are not exactly identical to (2.2.1) and (2.3.1). If $[\cdot]$ is replaced by $\text{int}(\cdot)$ in (2.2.1) and (2.3.1) then we have the rules used in the simulations. Recall that $[x]$ is the largest integer n such that $n<x$, while $\text{int}(x)$ is the largest integer n such that $n\leq x$. As an example, suppose that we have $m=5$ and $\left(\frac{K\sigma_m^s}{ct}\right)^{\frac{1}{s+t}} = 20.3$. Then both the simulations and the rules defined in the text would yield $N=21$. If, however, we had $\left(\frac{K\sigma_m^s}{ct}\right)^{\frac{1}{s+t}} = 20$, then the simulations would have $N=21$, while the rules defined in the text would have $N=20$. To see how minor this difference really is, note that theoretically $\left(\frac{K\sigma_m^s}{ct}\right)^{\frac{1}{s+t}}$ is an integer with probability zero, and in practice this happens only rarely.

<u>Remark 2.4.2</u>: Our choices of γ are based largely on simulations of other procedures, similar to those presented here, which have been carried out in the past. They seem to be good choices here as well.

CHAPTER III

POINT ESTIMATION OF THE LOCATION PARAMETER OF A NEGATIVE EXPONENTIAL DISTRIBUTION WITH BOUNDED RISK

3.1 Introduction

In this chapter we are still interested in estimating μ, but we consider the loss function $L_n = A(X_{n(1)}-\mu)^s$. With this loss function, $R_n = E(L_n)$ is strictly decreasing in n, and thus we cannot minimize the risk even for known σ. Instead, we consider a maximum "acceptable" risk, say W (>0), and attempt to devise procedures which have associated risk smaller than (or approximately equal to) W. For fixed W, it is easily shown that $n^* = \left(\frac{K\sigma^s}{sW}\right)^{1/s}$ is the minimum sample size required such that $R_{n^*} \le W$.

3.2 Two-Stage Procedure

Let $m \ge \text{int}(1+s) + 1$ be fixed. Define the two-stage rule by

$$N = \max\{m, \left[\left(\frac{Kh\sigma_m^s}{sW}\right)^{1/s}\right] + 1\}, \tag{3.2.1}$$

where $h = \{\Gamma(m-1)\}^{-1}(m-1)^s\Gamma(m-1-s)$. Note that the definition of m guarantees $m > s + 1$ which in turn implies that h is well-defined. When we stop sampling, we estimate μ by $X_{N(1)}$.

<u>Theorem 3.2.1</u>: <u>For the stopping variable</u> N <u>defined in (3.2.1), we have</u>

(a) $P(N<\infty) = 1,$

(b) $E(N)<\infty,$

(c) $\lim_{A\to\infty} N/n^* = h^{1/s}(\sigma_m/\sigma)$ <u>a.s.</u>,

(d) $\lim_{A\to\infty} |N-n^*| = \infty$ <u>a.s.</u>,

(e) $\lim_{A\to\infty} E(N/n^*) = \{\Gamma(m-1)\}^{-1/s}(m-1)\{\Gamma(m-1-s)\}^{1/s},$

(f) $\lim_{A\to\infty} E(N-n^*) = \infty,$

(g) $R = E(L_N) \leq W.$

<u>Proof</u>: For part (a) we note that

$$\begin{aligned} P(N<\infty) &= 1 - P(N=\infty) \\ &= 1 - \lim_{n\to\infty} P(N>n) \\ &= 1 - \lim_{n\to\infty} P\left(\max\left\{m, \left[\left(\frac{Kh\sigma_m^s}{sW}\right)^{1/s}\right] + 1\right\} > n\right) \\ &= 1, \end{aligned}$$

since $\left[\left(\frac{Kh\sigma_m^s}{sW}\right)^{1/s}\right] + 1 < \infty$ a.s..

Next, from the definition of N, we get

$$N \leq \left(\frac{Kh\sigma_m^s}{sW}\right)^{1/s} + m.$$

Since $E\left\{\left(\frac{Kh\sigma_m^s}{sW}\right)^{1/s}\right\}$ is finite, it now follows that E(N) is finite, which is part (b).

To see part (c), note that from the definition of N, we have,

$$\left(\frac{Kh\sigma_m^s}{sW}\right)^{1/s} \leq N \leq \left(\frac{Kh\sigma_m^s}{sW}\right)^{1/s} + m,$$

and thus

$$h^{1/s}(\sigma_m/\sigma) \le N/n^* \le h^{1/s}(\sigma_m/\sigma) + m/n^*,$$

which implies

$$\lim_{A\to\infty} N/n^* = h^{1/s}(\sigma_m/\sigma) \text{ a.s..}$$

To prove part (d), first we get from part (c),

$$\lim_{A\to\infty} (N/n^*-1) = h^{1/s}(\sigma_m/\sigma) - 1 \text{ a.s..}$$

This implies

$$\lim_{A\to\infty} \{|N-n^*|/n^*\} = |h^{1/s}(\sigma_m/\sigma) - 1| \text{ a.s.,}$$

which immediately leads to a proof of part (d) since $n^*\to\infty$ as $A\to\infty$.

To prove part (e) we recall from the proof of part (c) that,

$$h^{1/s}(\sigma_m/\sigma) \le N/n^* \le h^{1/s}(\sigma_m/\sigma) + m/n^*.$$

Thus,

$$\lim_{A\to\infty} E(N/n^*) = \lim_{A\to\infty} h^{1/s}E(\sigma_m)/\sigma$$

$$= \lim_{A\to\infty} h^{1/s}.$$

Since m does not depend on A, we have,

$$\lim_{A\to\infty} E(N/n^*) = h^{1/s}$$

$$= \{\Gamma(m-1)\}^{-1/s}(m-1)\{\Gamma(m-1-s)\}^{1/s}.$$

Next, Theorem 1.3.6 part (b) and the fact that $E(N/n^*)=h^{1/s}$, give

$$E(N-n^*) = n^*\{(\Gamma(m-1))^{-1/s}(m-1)\{\Gamma(m-1-s)\}^{1/s}$$

$$= \infty,$$

which is part (f).

Finally, since $N \ge \left(\frac{Kh\sigma_m^s}{sW}\right)^{1/s}$, we can write

$$N^{-s} \le \left(\frac{sW}{Kh\sigma_m^s}\right). \qquad (3.2.2)$$

Also, following the lines which led to (1.2.2) we have that

$$R = (K\sigma^{s}/s)E(N^{-s}). \tag{3.2.3}$$

Thus, combining (3.2.2) and (3.2.3), we have,

$$R \le \frac{K}{s} \cdot \frac{sW}{Kh} E\{(\sigma/\sigma_m)^{s}\}$$

$$= (W/h)E\{(\sigma_m/\sigma)^{-s}\}$$

$$= (W/h)(m-1)^{s}\{\Gamma(m-1)\}^{-1}\Gamma(m-1-s)$$

using Lemma 1.3.2, noting that $m \ge \text{int}(1+s) + 1$ in the definition of m guarantees the existence of $E\{(\sigma_m/\sigma)^{-s}\}$. But,

$$(m-1)^{s}\Gamma(m-1-s)\{\Gamma(m-1)\}^{-1} = h.$$

Thus,

$R \le W$, which is part (g).

3.3 Modified Two-Stage Procedure

Let $m = \max\{\text{int}(1+s) + 1, [\left(\frac{K}{sW}\right)^{\frac{1}{s+\gamma}}] + 1\}$, where we choose and fix $\gamma>0$. Define the new rule N by

$$N = \max\{m, \left[\left(\frac{Kh\sigma_m^{s}}{sW}\right)^{1/s}\right] + 1\}, \tag{3.3.1}$$

where h is the same as in the preceding section. When we stop sampling we again estimate μ by $X_{N(1)}$. The considerations involving the value of γ are the same as we discussed in section 2.3. Again we note that $\lim_{A\to\infty} m = \infty$ and that $\lim_{A\to\infty} m/n^* = 0$.

Theorem 3.3.1: *For the stopping variable* N *defined in (3.3.1), we have,*

(a) $P(N<\infty) = 1$,

(b) $E(N)<\infty$,

(c) $\lim_{A\to\infty} N/n^* = 1$ a.s.,

(d) $\lim_{A\to\infty} E(N/n^*) = 1$,

(e) $R = E(L_N) \le W$.

Proof: Parts (a) and (b) can be proved exactly as we did in Theorem 3.2.1 parts (a) and (b).

To prove (c) note that exactly as in the proof of Theorem 3.2.1 part (c), we have,

$$\lim_{A\to\infty} N/n^* = \lim_{A\to\infty} h^{1/s}(\sigma_m/\sigma) \text{ a.s..}$$

Recalling that $K = As\Gamma(s+1)$, we see that $m\to\infty$ as $A\to\infty$. Thus, Lemma 2.3.1 and the fact that $\sigma_m\to\sigma$ as $m\to\infty$ combine to give

$$\lim_{A\to\infty} N/n^* = 1 \text{ a.s.}$$

To prove part (d) note that as in the proof of Theorem 3.2.1 part (e), it follows that

$$\lim_{A\to\infty} E(N/n^*) = \lim_{A\to\infty} h^{1/s}.$$

Using Lemma 2.3.1, we see that $h^{1/s}\to 1$ as $m\to\infty$. Thus, the fact that $m\to\infty$ as $A\to\infty$ finishes the proof of (d).

Part (e) can be proved exactly as we did in Theorem 3.2.1 part (g).

3.4 Sequential Procedure

Let $N\equiv N(A)$ be the first integer $n\ge m$ for which

$$n \ge \left(\frac{K\sigma_n^s}{sW}\right)^{1/s}, \tag{3.4.1}$$

where m (≥ 2) is the initial sample size. When we stop sampling we again estimate μ by $X_{N(1)}$.

Theorem 3.4.1: *For the stopping variable N defined in (3.4.1), we have*

(a) $P(N<\infty) = 1$,

(b) $\lim_{A\to\infty} N/n^* = 1$ *a.s.*,

(c) $P\{N \le (1-\varepsilon)n^*\} = 0_e(n^{*-(m-1)})$,

as $A\to\infty$, *for any fixed* ε *in* $(0,1)$.

Proof: To prove (a) note that,

$$\begin{aligned} P(N<\infty) &= 1 - P(N=\infty) \\ &= 1 - \lim_{n\to\infty} P(N>n) \\ &\ge 1 - \lim_{n\to\infty} P\{n < \left(\frac{K\sigma_n^s}{sW}\right)^{1/s}\} \\ &= 1, \end{aligned}$$

since $\sigma_n \to \sigma$ in probability as $n\to\infty$.

Part (b) follows from Lemma 1.3.3, by letting $Y_n=\sigma_n/\sigma$, $f(n)=n$, and $t=n^*$.

Part (c) follows from Lemma 1.3.1 just merely noting that

$$\begin{aligned} P(N=m) &= P\{\left(\frac{K\sigma_m^s}{sW}\right)^{1/s} \le m\} \\ &= P\{\sigma_m \le \sigma m/n^*\} \\ &= P\{2(m-1)\sigma_m/\sigma \le 2m(m-1)/n^*\} \\ &= P\{\chi^2(2(m-1)) \le 2m(m-1)/n^*\} \end{aligned}$$

and, for $n>m$,

$$\begin{aligned} P(N=n) &\le P\{\left(\frac{K\sigma_n^s}{sW}\right)^{1/s} \le n\} \\ &= P\{\chi^2(2(n-1)) \le 2n(n-1)/n^*\}. \end{aligned}$$

We now consider some more sophisticated results regarding the properties of the risk. We will see in Theorem 3.4.2 that the asymptotic behavior of the risk can be characterized quite well. Before proceeding with Theorem 3.4.2, we give a Lemma which we will use in the proof of Theorem 3.4.2.

Lemma 3.4.1: If we expand N^{*-s} around n^* in a Taylor series, we obtain for $n^*>1$,

$$N^{*-s} = n^{*-s} - sn^{*-(s+1)}(N^*-n^*) + \tfrac{1}{2}s(s+1)Q^{-(s+2)}(N^*-n^*)^2 \tag{3.4.2}$$

where Q is a random variable with

$$Q^{-(s+2)} \le kn^{*-2},$$

and k is a constant independent of A, σ, and W.

Proof: Let $g(x) = x^{-s}$, then

$$g'(x) = -sx^{-(s+1)},$$

$$g''(x) = s(s+1)x^{-(s+2)}.$$

Everything but $Q^{-(s+2)} \le kn^{*-2}$ follows from this. To show that $Q^{-(s+2)} \le kn^{*-2}$, note that if

$$N^* \ge \tfrac{1}{2}n^*$$

then

$$\begin{aligned} Q^{-(s+2)} &\le 2^{s+2}n^{*-(s+2)} \\ &\le 2^{s+2}n^{*-2} \end{aligned} \tag{3.4.3}$$

since $n^*>1$, and Q is between N^* and n^*.

In the case $N^* < \tfrac{1}{2}n^*$ we solve (3.4.2) for $Q^{-(s+2)}$ and obtain,

$$\begin{aligned} Q^{-(s+2)} &= \{N^{*-s} - n^{*-s} + sn^{*-(s+1)}(N-n^*)\}\{\tfrac{1}{2}s(s+1)(N^*-n^*)^2\}^{-1} \\ &\le \{\tfrac{1}{2}s(s+1)(N^*-n^*)^2\}^{-1} \\ &\le \frac{8}{s(s+1)}\, n^{*-2}. \end{aligned} \tag{3.4.4}$$

Combining (3.4.3) and (3.4.4) completes the proof.

<u>Theorem 3.4.2</u>: <u>The risk associated with the stopping variable</u> N <u>defined in</u> (3.4.1) <u>satisfies the following</u>:

$$\lim_{A\to\infty} (R-W)$$

$$\begin{aligned} &= \infty && \text{if } m < s+1, \\ &= Ws^{s-1}/\Gamma(s) && \text{if } m = s+1, \\ &= O(A^{(s-m+1)/s}) && \text{if } s+1 < m < s+2 \text{ and } m \geq 3, \\ &= O(A^{-1/s}) && \text{if } m \geq s+2. \end{aligned} \tag{3.4.5}$$

<u>Proof</u>: The proof of the first two parts of (3.4.5) follows the lines of Starr (1966) noting that we are dealing with $\chi^2(2(n-1))$ rather than $\chi^2(n-1)$. For $0 < \varepsilon < 1$, let

$$u = \left[(1+\varepsilon)^{1/s} n^*\right],$$

$$v = \left[(1-\varepsilon)^{1/s} n^*\right],$$

$$\pi_1 = m^{-s} P(N=m),$$

$$\pi_2 = u^{-s} P\{m<N\leq u\}$$

$$\pi_3 = \sum_{n=m+1}^{v} n^{-s} P(N=n)$$

$$\pi_4 = (1-\varepsilon)^{-1} n^{*-s} P\{N \geq (1-\varepsilon)^{1/s} n^*\}.$$

Note that,

$$E(N^{-s}) = \sum_{n=m}^{\infty} n^{-s} P(N=n)$$

$$= m^{-s} P(N=m) + \sum_{n=m+1}^{\infty} n^{-s} P(N=n)$$

$$\geq \pi_1 + \sum_{n=m+1}^{u} u^{-s} P(N=n)$$

$$= \pi_1 + \pi_2.$$

Now,

$$\pi_1 = m^{-s}P\{\left(\frac{K\sigma_m^s}{sW}\right)^{1/s} \le m\}$$

$$= m^{-s}P\{\left(\frac{K}{sW}\right)^{1/s}\sigma(2(m-1))^{-1}2(m-1)\sigma_m/\sigma \le m\}$$

$$= m^{-s}P\{\chi^2(2(m-1)) \le \left(\frac{sW}{K}\right)^{1/s} 2m(m-1)/\sigma\}$$

$$= m^{-s}P\{\chi^2(2(m-1)) \le 2m(m-1)/n^*\}.$$

Define $2n(n-1)/n^* = \ell(n,n^*)$, $n=m, m+1, \ldots.$ Then,

$$\pi_1 = m^{-s}2^{-(m-1)}\{\Gamma(m-1)\}^{-1}\int_0^{\ell(n,n^*)} x^{m-2}e^{-x/2}dx$$

$$\ge m^{-s}2^{-(m-1)}\{\Gamma(m-1)\}^{-1}\int_0^{\ell(m,n^*)} x^{m-2}e^{-\ell(m,n^*)/2}dx$$

$$= m^{-s}2^{-(m-1)}\{\Gamma(m-1)\}^{-1}e^{-\ell(m,n^*)/2}\{\ell(m,n^*)\}^{-1}/(m-1)$$

$$= m^{-s}2^{-(m-1)}\{\Gamma(m)\}^{-1}e^{-\ell(m,n^*)/2}\{\ell(m,n^*)\}^{m-1}.$$

As $A\to\infty$, $n^*\to\infty$ which implies that $\ell(m,n^*) = 2m(m-1)/n^*\to 0$. Thus,

$$\liminf_{A\to\infty} n^{*s}E(N^{-s}) \ge m^{-s}2^{-(m-1)}\{\Gamma(m)\}^{-1}\liminf_{A\to\infty} n^{*s}\{2m(m-1)/n^*\}^{m-1}$$

$$+ (1+\varepsilon)^{-1}\liminf_{A\to\infty} P\{m<N\le u\}$$

$$= m^{m-1-s}(m-1)^{m-1}\{\Gamma(m)\}^{-1}\liminf_{A\to\infty} n^{*s-m+1}+1-\delta(\varepsilon)$$

where $\delta(\varepsilon)\to 0$ as $\varepsilon\to 0$.

Since ε was arbitrary, we now have

$$\liminf_{A\to\infty} n^{*s}E(N^{-s})$$

$$\ge \infty \quad \text{if } m < s+1,$$

$$\ge \frac{s^{s-1}}{\Gamma(s)} + 1 \text{ if } m = s+1,$$

$$\ge 1 \quad \text{if } m > s+1. \qquad (3.4.6)$$

In the above arguments, we use part (b) of Theorem 3.4.1 which states that $\lim_{A\to\infty} N/n^* = 1$ a.s. so that we may assert that

$P\{m<n\le u\}\to 1$ as $A\to\infty$.

Note also that

$$E(N^{-s}) = \sum_{n=m}^{\infty} n^{-s}P(N=n)$$

$$\le m^{-s}P(N=m) + \sum_{n=m+1}^{v} n^{-s}P(N=n) + (1-\varepsilon)^{-1}n^{*-s}P\{N\ge(1-\varepsilon)^{1/s}n^*\}$$

$$= \pi_1 + \pi_3 + \pi_4$$

Now,

$$\pi_1 = m^{-s}P(N=m)$$

$$= m^{-s}P\{\left(\frac{K}{sW}\right)^{1/s} 2(m-1)\sigma_m/\sigma \le 2m(m-1)/\sigma\}$$

$$= m^{-s}P\{\chi^2(2(m-1)) \le 2m(m-1)/n^*\}$$

$$= m^{-s}2^{-(m-1)}\{\Gamma(m-1)\}^{-1}\int_0^{\ell(m,n^*)} x^{m-2}e^{-x/2}dx$$

$$\le m^{-s}2^{-(m-1)}\{\Gamma(m-1)\}^{-1}\int_0^{\ell(m,n^*)} x^{m-2}dx$$

$$= m^{-s}2^{-(m-1)}\{\Gamma(m-1)\}^{-1}\{\ell(m,n^*)\}^{m-1}/(m-1)$$

$$= m^{-s}2^{-(m-1)}\{\Gamma(m)\}^{-1}\{\ell(m,n^*)\}^{m-1}$$

Similarly,

$$\pi_3 = \sum_{n=m+1}^{v} n^{-s}P\{\chi^2(2(n-1)) \le \ell(n,n^*)\}$$

$$\le (m+1)^{-s}O(n^{*-m})$$

using the techniques of Lemma 1.3.1. Thus,

$$\pi_3 = O(n^{*-m}).$$

This yields,

$$\limsup_{A\to\infty} E\{(N/n^*)^{-s}\} \le \limsup_{A\to\infty} n^{*s} m^{-s} 2^{-(m-1)} \{\Gamma(m)\}^{-1} \{\ell(m,n^*)\}^{m-1}$$

$$+ n^{*s} O(n^{*-m}) + (1-\varepsilon)^{-1} P\{N \ge (1-\varepsilon)^{1/s} n^*\}$$

$$= \limsup_{A\to\infty} m^{-s} 2^{-(m-1)} \{\Gamma(m)\}^{-1} \{2m(m-1)/n^*\}^{m-1} n^{*s}$$

$$+ O(n^{*s-m}) + 1 + \delta'. \qquad (3.4.7)$$

where $\delta' \equiv \delta'(\varepsilon) \to 0$ as $\varepsilon \to 0$. With some simplifications, (3.4.7) now leads to

$$\limsup_{A\to\infty} E\{(N/n^*)^{-s}\} \le m^{m-1-s} (m-1)^{m-2} \{\Gamma(m-1)\}^{-1} \limsup_{A\to\infty} n^{*s-m+1}$$

$$+ O(n^{*s-m}) + 1 + \delta(\varepsilon)'.$$

Since ε was arbitrary, we have

$$\limsup_{A\to\infty} E\{(N/n^*)^{-s}\}$$

$$\le 1 \qquad \text{if } m > s+1,$$

$$\le \frac{s^{s-1}}{\Gamma(s)} + 1 \text{ if } m = s+1, \qquad (3.4.8)$$

Now, we note that

$$R = (K\sigma^s/s) E(N^{-s})$$

$$= W n^{*s} E(N^{-s})$$

$$= W E\{(N/n^*)^{-s}\},$$

Thus,

$$R - W = W\{E\{(N/n^*)^{-s}\} - 1\}. \qquad (3.4.9)$$

Finally, (3.4.6) and (3.4.8) together with (3.4.9) yield the first two cases of the theorem.

The proof for the cases $s + 1 < m < s + 2$ and $m \ge s + 2$ follows the lines of derivation in Mukhopadhyay (1982). We diverge momentarily from direct efforts to finish proving Theorem 3.4.1. First we show that our stopping rule can be presented in the form discussed in

Woodroofe (1977) and Mukhopadhyay (1982). We stop sampling for the first integer n (≥m) such that

$$n \geq (K\sigma_n^s/(sW))^{1/s},$$

that is,

$$n(sW/K)^{1/s} \geq \sigma_n,$$

Using Theorem 1.3.5 we now note that a distributionally equivalent rule can be defined as the first integer n (≥m) such that

$$n(sW/K)^{1/s} \geq (n-1)^{-1}\sum_{i=1}^{n-1} Y_i, \qquad (3.4.10)$$

where $Y_1, Y_2, \ldots, Y_{n-1}$ are i.i.d. with density $f(y;0,\sigma)$, where $f(\cdot;\cdot,\cdot)$ is defined in (1.1.1). Denoting $\sum_{i=1}^{n-1} Y_i$ by S^*_{n-1}, (3.4.10) is equivalent to

$$S^*_{n-1} \leq n(n-1)(sW/K)^{1/s},$$

that is,

$$S^*_{n-1} \leq (sW/K)^{1/s}(n-1)^2(1+1/(n-1)).$$

Thus, we see that our rule is distributionally equivalent to N^*+1, where

$N^* \equiv N^*(A)$ is the first integer n (≥m-1) for which

$$S^*_n \leq (sW/K)^{1/s}n^2(1+1/n) \qquad (3.4.11)$$

which is of the form described in Woodroofe (1977) and reviewed in Section 1.3, with $L_n = 1 + 1/n$, $L_0=1$, $c = (sW/K)^{1/s}$, $\lambda=n^*$, $\alpha=2$, and $\beta=1$. Note that (1.3.5) is satisfied for $B > 1/\sigma$ and $a=1$, and that Y_1 has positive moments of all orders. Thus, the relevant condition for Theorem 1.3.3 is $m - 1 > \frac{1}{2}u$, and the relevant condition for Theorem 1.3.4 is $m - 1 > 1$. Since we will be needing Theorem 1.3.3 with u=2, both conditions are the same, that is, m≥3. In the case $s + 1 < m < s + 2$, we explicitly require $m \geq 3$. Since $s>0$, $m \geq 5 + 2$ guarantees that the condition $m \geq 3$ is satisfied implicitly.

We now return directly to the proof of Theorem 3.4.2. We note that

$$
\begin{aligned}
R &= (K\sigma^s/s)E(N^{-s}) \\
&= (K\sigma^s/s)E\{(N^*+1)^{-s}\} \\
&\leq (K\sigma^s/s)E(N^{*-s})
\end{aligned}
$$

Thus, using Lemma 3.4.1 and Theorem 1.3.4, we have,

$$
\begin{aligned}
\lim_{A\to\infty} (R-W) &\leq \lim_{A\to\infty} W\{1 - (s/n^*)E(N^*-n^*) \\
&\quad + \tfrac{1}{2}s(s+1)n^{*s}E\{Q^{-(s+2)}(N^*-n^*)^2\}\} - W \\
&= \lim_{A\to\infty} \{(-s/n^*)(\nu/\sigma-2+o(1)) + \tfrac{1}{2}Ws(s+1)n^{*s} \times \\
&\quad E\{Q^{-(s+2)}(N^*-n^*)^2\}\} \\
&= O_e(n^{*-1}) + \tfrac{1}{2}Ws(s+1)\lim_{A\to\infty} n^{*s}\ \{E\{Q^{-(s+2)}(N^*-n^*)^2 \times \\
&\quad I(N^*+1\leq(1-\varepsilon)n^*)\} + E\{Q^{-(s+2)}(N^*-n^*)^2 I(N^*+1>(1-\varepsilon)n^*)\}\}.
\end{aligned}
$$

We now note that for sufficiently large n^*, say $n^*>n_0$, $N^* + 1 > (1-\varepsilon)n^*$ will imply that $Q^{-(s+2)} \leq kn^{*-2}$ with k a constant independent of A. Using this fact along with the second part of Lemma 3.4.1, we obtain

$$
\begin{aligned}
\lim_{A\to\infty} (R-W) &\leq O_e(n^{*-1}) + \tfrac{1}{2}Ws(s+1)\lim_{A\to\infty} n^{*s}\{kP\{N \leq (1-\varepsilon)n^*\} \\
&\quad + kn^{*-(s+1)}E\{(N^*- n^*)^2/n^*\}\}.
\end{aligned}
$$

Now we use Theorem 3.4.1 part (c) and Theorem 1.3.3 to get

$$
\begin{aligned}
\lim_{A\to\infty} (R-W) &\leq O_e(n^{*-1}) + O_e(n^{*s-m+1}) + O_e(n^{*-1}) \\
&= O_e(n^{*-1}) + O_e(n^{*s-m+1}),
\end{aligned}
$$

and the last two parts of Theorem 3.4.2 follow upon noting the relationship between A and n^*.

We now state without proof the following lemma, which is essentially the same as Lemma 3.4.1.

<u>Lemma 3.4.2</u>: <u>If we expand</u> $(N^*+P)^{-s}$ <u>around</u> n^* <u>in a Taylor series, we obtain,</u>

$$(N^*+P)^{-s} = n^{*-s} - sn^{*-(s+1)}(N^*+P-n^*) + \frac{s(s+1)}{2}Q^{-(s+2)}(N+P-n^*)^2$$

<u>where</u> $Q^{-(s+2)} \le kn^{*-2}$ <u>and</u> k <u>is a constant independent</u> A, σ, <u>and</u> W.

The following result is proved in the spirit of the developments in Simons (1968) for a very different context.

<u>Theorem 3.4.4</u>: <u>There exists a non-negative integer</u> P, <u>independent of</u> A, σ, <u>and</u> W, <u>such that</u>

$$E(L_{N+P}) \le W,$$

<u>if</u> $m \ge s + 2$.

<u>Proof</u>: First note that the distribution of N, and hence that of N^* also, depend only on n^*. From Lemma 3.4.2 we know that

<u>Proof</u>: From Lemma 3.4.2 we know that

$$(N^*+P)^{-s} = n^{*-s} - sn^{*-(s+1)}(N^*+P-n^*) + \tfrac{1}{2}s(s+1)Q^{-(s+2)}(N^*+P-n^*)^2,$$

where Q is between $N^* + P$ and n^*. Thus,

$$\begin{aligned} E(L_{N^*+P}) = Wn^{*s}\{&n^{*-s} - sn^{*-(s+1)}E(N^*+P-n^*) \\ &+ \tfrac{1}{2}s(s+1)\{E\{Q^{-(s+2)}(N^*-n^*)^2\} + 2PE\{Q^{-(s+2)}(N^*-n^*)\} \\ &+ P^2E(Q^{-(s+2)})\}\}. \end{aligned} \quad (3.4.12$$

Noting Theorem 1.3.4 and the methods in Woodroofe (1977) for computing ν, (3.4.12) becomes

$$\begin{aligned} E(L_{N^*+P}) = W\{1 - sn^{*-1}(P+\beta+o(1)) + \tfrac{1}{2}s(s+1)n^{*s} \times \\ &\{E\{Q^{-(s+2)}(N^*-n^*)^2 I(N^*+1 \le (1-\varepsilon)n^*)\} \\ &+ E\{Q^{-(s+2)}(N^*-n^*)^2 I(N^*+1 > (1-\varepsilon)n^*)\} \\ &+ 2PE\{Q^{-(s+2)}(N^*-n^*)I(N^* \le n^*)\} \\ &+ 2PE\{Q^{-(s+2)}(N^*-n^*)I(N^* > n^*)\} \\ &+ P^2E\{Q^{-(s+2)}I(N^*+1 \le (1-\varepsilon)n^*\} \\ &+ P^2E\{Q^{-(s+2)}I(N^*+1 > (1-\varepsilon)n^*\}\}, \end{aligned} \quad (3.4.13)$$

as $n^*\to\infty$, for ε in (0,1) and $\beta\cong -1.253$. Making various simplifications, again noting that there exists n_0 such that $n^*>n_0$ and $N^*+1>(1-\varepsilon)n^*$ will imply that $Q^{-(s+2)}\le kn^{*-(s+2)}$, (3.4.13) becomes,

$$\begin{aligned}
E(L_{N^*+P}) \le W\{1 &- sn^{*-1}(P+\beta+o(1)) + kn^{*s}\times \\
&\{E\{Q^{-(s+2)}(N^*-n^*)^2 I(N^*+1\le(1-\varepsilon)n^*)\} \\
&+ kn^{*-(s+1)}E\{(N^*-n^*)^2/n^*\} \\
&+ 2Pn^{*-(s+3/2)}E\{|N^*-n^*|/n^{*\frac{1}{2}}\} \\
&+ P^2E\{Q^{-(s+2)}I(N^*+1\le(1-\varepsilon)n^*)\} \\
&+ kP^2n^{*-(s+2)}P(N^*+1>(1-\varepsilon)n^*)\}\} \\
\le W\{1 &- sn^{*-1}(P+\beta+o(1)) \\
&+ kn^{*s}E\{Q^{-(s+2)}(N^*-n^*)^2 I(N^*+1\le(1-\varepsilon)n^*)\} \\
&+ kn^{*-1}O(1) + 2PO(n^{*-3/2}) \\
&+ P^2n^{*s}E\{Q^{-(s+2)}I(N^*+1\le(1-\varepsilon)n^*)\} \\
&+ kP^2n^{*-2}\}
\end{aligned} \qquad (3.4.14)$$

as $n^*\to\infty$, using Theorem 1.3.3, where throughout k is a generic constant independent of A, σ, and W. Now, using Lemma 3.4.2 on the two terms in (3.4.14) which remain unevaluated, and noting the relationship between N and N^*, we have,

$$\begin{aligned}
E(L_{N^*+P}) \le W\{1 &- sn^{*-1}(P+\beta+o(1)) \\
&+ kn^{*s}E\{(N^*-n^*)^2n^{*-2}I(N^*+1\le(1-\varepsilon)n^*)\} \\
&+ O(n^{*-1}) + PO(n^{*-3/2}) \\
&+ kP^2n^{*s-2}E\{I(N^*+1\le(1-\varepsilon)n^*)\} + P^2O(n^{*-2})\} \\
\le W\{1 &- sn^{*-1}(P+\beta+o(1)) + kn^{*s}P\{N\le(1-\varepsilon)n^*\} \\
&+ O(n^{*-1}) + PO(n^{*-3/2}) + kP^2n^{*s-2}P\{N\le(1-\varepsilon)n^*\} \\
&+ P^2O(n^{*-2})\}.
\end{aligned} \qquad (3.4.15)$$

as $n^*\to\infty$. Now, using Theorem 3.4.1 part (c), (3.4.15) becomes,

$$E(L_{N^*+P}) \le W\{1 - sn^{*-1}(P+\beta+o(1)) + O(n^{*s-m+1}) + O(n^{*-1}) + PO(n^{*-3/2}) + P^2O(n^{*s-m-1}) + P^2O(n^{*-2})\}, \qquad (3.4.16)$$

as $n^*\to\infty$. Simplifying (3.4.16) further, with the assumption $m \ge s + 2$, yields,

$$E(L_{N^*+P}) \le W\{1 - sn^{*-1}(P+\beta+o(1)) + O(n^{*-1}) + PO(n^{*-3/2}) + P^2O(n^{*-2})\}$$

as $n^*\to\infty$. Thus, $E(L_{N^*+P}) \le W$ if

$$sn^{*-1}(P+\beta+o(1)) \ge O(n^{*-1}) + PO(n^{*-3/2}) + P^2O(n^{*-2}),$$

or,

$$P + \beta + o(1) \ge O(1) + PO(n^{*-\frac{1}{2}}) + P^2O(n^{*-1}),$$

or,

$$P + \beta \ge O(1) + PO(n^{*-\frac{1}{2}}) + P^2O(n^{*-1}), \qquad (3.4.17)$$

as $n^*\to\infty$. We note that our assumption $m \ge s + 2$ guarantees that $m\ge 3$ which satisfies the conditions of Theorems 1.3.3 and 1.3.4, and thus validates our earlier use of them.

Now, (3.4.17) is satisfied for all large n^*, say $n^*>n_0$, and for some large P, say $P\ge P_0$. If $n^*\le n$ then take $P\ge n_0$ and we will have $E(L_{N^*+P}) \le W$. Thus, for $P^* \ge \max\{P_0,P_1\}$ we have $E(L_{N^*+P^*}) \le W$. Noting that N has the same probability distribution as N^*+1, and that L_n is decreasing in n, we have $E(L_{N+P^*}) \le W$. Thus, choosing $P\ge P^*$ completes the proof.

3.5 Simulations

In each of our Tables IV, V, and VI we fix $W=1$, $s=2$, $\mu=10$, and $\sigma=1$, and let $n^*=10$, 15, 20, 25, 50, and 100. A is determined by

TABLE IV

MODERATE SAMPLE BEHAVIOR OF THE TWO-STAGE PROCEDURE (3.2.1): W=1

n*	m	A	$\bar{N}$	s.e.($\bar{N}$)	$\bar{L}$	s.e.($\bar{L}$)
10	5	50.0	15.58	0.5867	0.6253	0.1154
10	10	50.0	12.95	0.2227	0.5138	0.0902
15	5	112.5	24.69	0.8952	0.8220	0.1874
15	10	112.5	18.99	0.4276	0.7843	0.1327
20	5	200.0	33.65	1.2862	1.1055	0.4462
20	10	200.0	23.72	0.6279	1.0683	0.1790
25	5	312.5	41.70	1.4377	0.6741	0.1434
25	10	312.5	30.58	0.6936	1.1067	0.2041
50	5	1250.0	82.21	2.7408	0.9879	0.3135
50	10	1250.0	57.63	1.4245	1.2004	0.1907
100	5	5000.0	168.54	6.3185	0.6628	0.1478
100	10	5000.0	116.88	2.8700	0.9990	0.2559

TABLE V

MODERATE SAMPLE BEHAVIOR OF THE MODIFIED TWO-STAGE PROCEDURE (3.3.1): W=1

n^*	γ	A	$\bar{N}$	s.e. $(\bar{N})$	$\bar{L}$	s.e. $(\bar{L})$
10	0.2	50.0	13.82	0.3014	0.7011	0.1138
10	0.3	50.0	14.66	0.3810	0.4830	0.0822
15	0.2	112.5	18.76	0.3580	0.8846	0.1846
15	0.3	112.5	18.95	0.3609	0.9755	0.1622
20	0.2	200.0	23.74	0.3469	0.7733	0.0987
20	0.3	200.0	23.92	0.4306	0.9456	0.2398
25	0.2	312.5	28.78	0.4523	0.8955	0.1849
25	0.3	312.5	29.61	0.5306	1.1832	0.3332
50	0.2	1250.0	54.95	0.5596	0.8270	0.1384
50	0.3	1250.0	53.46	0.6490	0.7151	0.0944
100	0.2	5000.0	103.55	0.9807	1.1836	0.2248
100	0.3	5000.0	105.27	0.9498	0.8783	0.1160

TABLE VI

MODERATE SAMPLE BEHAVIOR OF THE SEQUENTIAL PROCEDURE (3.4.1): W=1

n*	m	A	$\bar{N}$	s.e.($\bar{N}$)	$\bar{L}$	s.e.($\bar{L}$)
10	5	50.0	9.82	0.2276	1.5308	0.2461
10	10	50.0	11.25	0.1311	1.0918	0.1631
15	5	112.5	13.79	0.3131	2.9320	0.6671
15	10	112.5	14.80	0.2609	1.4787	0.2066
20	5	200.0	19.45	0.3902	1.5649	0.3910
20	10	200.0	18.79	0.3528	1.5902	0.2816
25	5	312.5	23.76	0.4405	1.6107	0.3102
25	10	312.5	24.82	0.3833	1.1940	0.2116
50	5	1250.0	49.27	0.5678	0.9368	0.1298
50	10	1250.0	49.32	0.5178	1.1247	0.2149
100	5	5000.0	100.70	0.7347	0.8113	0.1092
100	10	5000.0	98.45	0.7460	1.2052	0.1729

$n^* = \left(\frac{K\sigma^s}{sW}\right)^{1/s}$. As in Chapter II, we let m=5 and 10 for the two-stage and sequential procedures, and γ=0.2 and 0.3 for the modified two-stage procedure. Once again 200 repetitions were executed in each case. $\bar{L}$, $\bar{N}$, s.e.($\bar{L}$) and s.e.($\bar{N}$) are exactly the same as they were in Section 2.4.

The results of the simulations are not what one would have hoped for. The results are particularly not very satisfying in the sequential case. There may be two factors influencing our numerical results. Again (even more than in Chapter II) the inadequacy of 200 repetitions is apparent. Also, the sequential procedure only guarantees that $\lim_{A\to\infty} R=W$, and not the exact result $R\leq W$. These considerations explain the possibility of obtaining a value such as $\bar{L}$=2.9320 when m=5 and n*=15.

In the two-stage and modified two-stage cases where we do have theoretically $R\leq W$, we have markedly improved moderate sample size behavior, but at the expense of greater average sample sizes. In particular, the modified two-stage procedure seems to behave well with only a few extra samples.

Remark 3.5.1: The content of Remarks 2.4.1 and 2.4.2 apply to the two-stage procedures of this chapter also.

CHAPTER IV

SEQUENTIAL POINT ESTIMATION OF THE MEAN OF A NEGATIVE EXPONENTIAL DISTRIBUTION

4.1 Introduction

We now turn our attention to the point estimation of the mean of the distribution, namely the parameter λ which equals $\mu + \sigma$. For brevity, only a sequential procedure is proposed. We first give some preliminary results. We will estimate λ by $\overline{X}_n$ and use the loss function $L_n = A(\overline{X}_n - \lambda)^2 + cn$, that is, the squared error loss plus the cost of sampling. We find that the optimal sample size to minimize $E(L_n)$ for fixed σ is $n^* = b\sigma$ where $b = (A/c)^{\frac{1}{2}}$. The minimum risk is given by $R_{n^*} = 2cn^*$.

A great deal of work has been done on this type of problem in various settings. For normal X's this problem was studied by Robbins (1959), Starr (1966) and Starr and Woodroofe (1969). Also, for Exponential X's, with location zero, this problem was studied by Starr and Woodroofe (1972). A purely nonparametric approach was introduced in Ghosh and Mukhopadhyay (1979). The present findings will fill some of the interesting gaps in this area.

4.2 Preliminary Results

Let $m \geq 2$ be given as the starting sample size. The following sequential procedure is proposed:

Let $N \equiv N(c)$ be the first integer $n \geq m$ for which

$$n \geq b\sigma_n. \tag{4.2.1}$$

When we stop, we propose to estimate λ by $\bar{X}_N$.

The preliminary results are summarized in:

<u>Lemma 4.2.1</u>: <u>For the sampling plan defined in (4.2.1), we have</u>

(a) $P(N<\infty) = 1$,

(b) N <u>is nonincreasing in</u> c; $\lim_{c\to 0} N=\infty$ <u>a.s.</u>,

(c) $\lim_{c\to 0} N/n^*=1$ <u>a.s.</u>,

(d) $\lim_{c\to 0} E(N/n^*)^w = 1$ <u>for every positive</u> w,

(e) <u>For every fixed</u> ε <u>in</u> $(0,1)$, $P\{N\leq(1-\varepsilon)n^*\} = 0_e(c^{\frac{m-1}{2}})$.

<u>Proof</u>: (a)
$$\begin{aligned} P(N<\infty) &= 1 - P(N=\infty) \\ &= 1 - \lim_{n\to\infty} P(N>n) \\ &\geq 1 - \lim_{n\to\infty} P(n<b\sigma_n) \\ &= 1, \end{aligned}$$

since $\sigma_n \to \sigma$ a.s. as $n\to\infty$.

The fact that N is nonincreasing in c is clear from the definition of N. The second part of part (b) as well as part (c) follow from Lemma 1.3.3 with $Y_n=\sigma_n/\sigma$, $f(n)=n$ and $t=n^*$. We must also note that $n^*\to\infty$ as $c\to 0$.

(d) From Fatou's lemma and part (c), we have

$$\liminf_{c\to 0} E(N/n^*)^w \geq E\{\liminf_{c\to 0} (N/n^*)^w\} = 1.$$

To prove that $\limsup_{c\to 0} E(N/n^*)^w \leq 1$, let $\varepsilon>0$ be given,

$u = [n^*(1+\varepsilon)] \geq m$. We now basically follow the lines of proofs given in Ghosh and Mukhopadhyay (1979). Note that, for $n^* \geq m/(1+\varepsilon)$,

$$E\{(N/n^*)^w\} = \sum_{j=m}^{\infty} (j/n^*)^w P(N=j)$$

$$\leq (1+\varepsilon)^w \sum_{j=m}^{u} P(N=j) + \sum_{j=u+1}^{\infty} (j/n^*)^w P(N=j)$$

$$= (1+\varepsilon)^w P\{N<(1+\varepsilon)n^*\} + n^{*-w} \sum_{j=u+1}^{\infty} j^w P(N=j)$$

Write $\pi_1 = (1+\varepsilon)^w P\{N<(1+\varepsilon)n^*\}$. With this notation we have,

$$E(N/n^*)^w \leq \pi_1 + n^{*-w} \sum_{j=u+1}^{\infty} j^w P(N=j)$$

$$= \pi_1 + n^{*-w} \sum_{j=u+1}^{\infty} j^w \{P(N\geq j) - P(N>j)\}$$

$$= \pi_1 + n^{*-w}\{(u+1)^w P(N>u) + \sum_{j=u+2}^{\infty} j^w P(N\geq j) - \sum_{j=u+1}^{\infty} j^w P(N>j)\}.$$

Write $\pi_2 = n^{*-w}(u+1)^w P(N>u)$. With this notation we now have,

$$E(N/n^*)^w \leq \pi_1 + \pi_2 + n^{*-w} \sum_{j=u+1}^{\infty} \{(j+1)^w - j^w\} P(N>j)$$

$$= \pi_1 + \pi_2 + n^{*-w} \sum_{j=u+1}^{\infty} \{j^w (1+1/j)^w - j^w\} P(N>j)$$

$$\leq \pi_1 + \pi_2 + n^{*-w} \sum_{j=u+1}^{\infty} \{j^w 2^w - j^w\} P(N>j)$$

$$= \pi_1 + \pi_2 + n^{*-w} (2^w - 1) \sum_{j=u+1}^{\infty} j^w P(N>j)$$

Now, $P(N>u) \to 0$ as $c \to 0$ since $N/n^* \to 1$ a.s. as $c \to 0$. Thus, recalling the meanings of π_1 and π_2, we have,

$$\limsup_{c\to 0} E\{(N/n^*)^w\} \leq (1+\varepsilon)^w + (2^w - 1) n^{*-w} \sum_{j=u+1}^{\infty} j^w P(N>j).$$

But, for $n \geq u + 1$,

$$P(N>n) \leq P(n<b\sigma_n)$$
$$= P\{2(n-1)\sigma_n/\sigma > 2n(n-1)/n^*\}$$
$$= P\{\chi^2(2(n-1)) > 2n(n-1)/n^*\}.$$

Let, $U_n \sim \chi^2(2(n-1))$ and $\ell(n,n^*) = 2n(n-1)/n^*$. Then,

$$P(N>n) \leq P\{U_n > \ell(n,n^*)\}$$
$$\leq \inf_{0<h<\frac{1}{2}} e^{-h\ell(n,n^*)} E(e^{hU_n})$$
$$= \inf_{0<h<\frac{1}{2}} e^{-h\ell(n,n^*)} (1-2h)^{-(n-1)}$$
$$= \{e^{(1-n/n^*)}(n/n^*)\}^{n-1}.$$

We now note that for $n \geq (1+\varepsilon)n^*$, there is a $p \equiv p(\varepsilon)$, with $0 < p < 1$, such that $e^{(1-n/n^*)}(n/n^*) \leq p$. Thus,

$$\sum_{n=u+1}^{\infty} n^w P(N>n) \leq \sum_{n=u+1}^{\infty} p^{n-1} n^w \leq k,$$

for some constant $k > 0$, independent of n^* (using the ratio test for convergence). Thus,

$$n^{*-w} \sum_{n=u+1}^{\infty} n^w P(N>n) \to 0 \text{ as } n^* \to \infty,$$

and it follows that

$$\limsup_{c\to 0} E(N/n^*)^w \leq (1+\varepsilon)^w.$$

Since $\varepsilon(>0)$ was arbitrary, we have

$$\limsup_{c\to 0} E(N/n^*)^w \leq 1,$$

and hence (d).

Part (e) follows from Lemma 1.3.1 and the facts that for $n>m$

$$P(N=n) \leq P(n>b\sigma_n)$$
$$= P\{2(n-1)\sigma_n/\sigma \leq 2n(n-1)/(b\sigma)\}$$

$$= P\{\chi^2(2(n-1)) \le 2n(n-1)/n^*\},$$

and

$$P(N=m) = P(m \ge b\sigma_m)$$

$$= P\{\chi^2(2(m-1)) \le 2m(m-1)/n^*\}.$$

4.3 The Main Result

Theorem 4.3.1: $\lim_{c\to 0} R/R_{n^*} = 1$ *if* $m \ge 6$.

Proof: First write

$$R/R_{n^*} = \tfrac{1}{2}E\{A(\overline{X}_N-\lambda)^2(cn^*)^{-1}\} + \tfrac{1}{2}E(N/n^*).$$

In view of Lemma 4.2.1 (d) with w=1, it suffices to show that

$$\lim_{c\to 0} E\{A(\overline{X}_N-\lambda)^2(cn^*)^{-1}\} = 1.$$

But,

$$\begin{aligned} E\{A(\overline{X}_N-\lambda)^2(cn^*)^{-1} &= E\{cb^2(\overline{X}_N-\lambda)^2(cn^*)^{-1}\} \\ &= E\{(b^2/n^*)(\overline{X}_N-\lambda)^2\} \\ &= E\{(n^*/\sigma^2)(\overline{X}_N-\lambda)^2\} \\ &= E\{(n^*/\sigma^2)(S_N-N\lambda)^2(n^{*-2}+N^{-2}-n^{*-2})\}, \end{aligned}$$

where $S_N = \sum_{i=1}^{N} X_i$. Since $E(N)<\infty$ (which follows from part (d) of Lemma 4.2.1 with w=1) and $E(X_1^2)<\infty$, using Wald's second equation we get $E(S_N-N\lambda)^2 = \sigma^2 E(N)$. Again using Lemma 4.2.1, part (d) with w=1, it remains to show that

$$E\{\{(S_N-N\lambda)^2/n^*\}\{(n^{*2}/N^2)-1\}\}\to 0 \text{ as } c\to 0.$$

From the theorem of Anscombe (1952) it follows that,

$$(S_N-N\lambda)^2/n^* \xrightarrow{\mathcal{L}} \sigma^2\chi^2(1)$$

as $c\to 0$. Also, as already noted earlier Wald's second equation yields

$$\lim_{c\to 0} E\{(S_N-N\lambda)^2/n^*\} = \lim_{c\to 0} \sigma^2 E(N/n^*)$$

$$= \sigma^2,$$

by Lemma 4.2.1 part (d) with w=1. Hence, $(S_N-N\lambda)^2/n^*$ is uniformly integrable in c. Now, given any ε in (0,1),

$$|(n^*/N)^2 - 1| I(|N-n^*| \le \varepsilon n^*) = \{|n^*-N||n^*+N|/N^2\} I(|N-n^*| \le \varepsilon n^*)$$
$$\le \{\varepsilon n^*(2+\varepsilon)n^*\}\{(1-\varepsilon)n^*\}^{-2} \times$$
$$I(|N-n^*| \le \varepsilon n^*)$$
$$\le \varepsilon(2+\varepsilon)(1-\varepsilon)^{-2}.$$

Therefore, $|(n^*/N)^2 - 1| I(|N-n^*| \le \varepsilon n^*)$ is bounded and hence $\{(S_N-N\lambda)^2/n^*\}|(n^*/N)^2 - 1| I(|N-n^*| \le \varepsilon n^*)$ is also uniformly integrable. Thus,

$$\lim_{c\to 0} E\{\{(S_N-N\lambda)^2/n^*\}|(n^*/N)^2 - 1| I(|N-n^*| \le \varepsilon n^*)\}$$
$$= E\{\lim_{c\to 0} \{\{(S_N-N\lambda)^2/n^*\}|(n^*/N)^2 - 1| I(|N-n^*| \le \varepsilon n^*)\}\}$$

$$= 0,$$

since $N/n^*\to 1$ a.s. as $c\to 0$ which implies that $\{(S_N-N\lambda)^2/n^*\}|(n^*/N)^2 - 1| I(|N-n^*| \le \varepsilon n^*) \to 0$ in probability as $c\to 0$.

Now, we consider

$$E\{\{(S_N-N\lambda)^2/n^*\}|(n^*/N)^2 - 1| I(N-n^*>\varepsilon n^*)\}.$$

On the set where $N - n^* > \varepsilon n^*$ we have $0 < (n^*/N)^2 < 1$, and this implies $|(n^*/N)^2 - 1| < 1$. Thus, once again we have $|(n^*/N)^2 - 1| I(N-n^*>\varepsilon n^*)$ to be bounded, and we thus conclude that $\{(S_N-N\lambda)^2/n^*\}|(n^*/N)^2 - 1| \times I(N-n^*>\varepsilon n^*)\}$ is also uniformly integrable. Thus, we have

$$E\{\{(S_N-N\lambda)^2/n^*\}|(n^*/N)^2 - 1| I(N-n^*>\varepsilon n^*)\} \to 0$$

as $c\to 0$, since $N/n^*\to 1$ a.s. as $c\to 0$, which again implies that

$$\{(S_N-N\lambda)^2/n^*\}|(n^*/N)^2 - 1| I(N-n^*>\varepsilon n^*) \to 0$$

in probability as $c\to 0$.

Now, consider

$$E\{\{(S_N-N\lambda)^2/n^*\}|(n^*/N)^2 - 1|I(N-n^*<-\varepsilon n^*)\}$$
$$= E\{\{(S_N-N\lambda)^2/n^*\}|(n^*/N)^2 - 1|I(N<(1-\varepsilon)n^*)\}$$
$$\le n^*E\{(S_N-N\lambda)^2 I(N<(1-\varepsilon)n^*)\}$$
$$\le n^*E\{(S_{N_u}-N_u\lambda)^2 I(N\le u)\}$$

where $N_u = \min(N,u)$, $u = [n^*(1-\varepsilon)]$. Using Schwarz's inequality, and Theorem 1.3.1 (noting that $EN^2<\infty$ by (d) of Lemma 4.2.1 with w=2) we get

$$n^*E\{(S_{N_u}-N_u\lambda)^2 I(N\le u)\} \le n^*\{E(S_{N_u}-N_u\lambda)^4 P(N\le u)\}^{\frac{1}{2}}$$
$$\le n^*\{6\sigma^2 E(N_u(S_{N_u}-N_u\lambda)^2) + 4\gamma E(N_u(S_{N_u}-N_u\lambda))$$
$$+ \beta E(N_u)\}^{\frac{1}{2}} P^{\frac{1}{2}}(N\le u)$$

where $\gamma = E(X_1-\lambda)^3$, $\beta = E(X_1-\lambda)^4$. Next, the fact that $N_u\le u$, Wald's second Lemma, Schwarz's inequality and (e) of Lemma 4.2.1 lead to

$$n^*E\{(S_{N_u}-N_u\lambda)^2 I(N\le u)\} \le n^*\{6\sigma^2 u^2\sigma^2 + 4\gamma u(E(S_{N_u}-N_u\lambda)^2)^{\frac{1}{2}}$$
$$+ \beta u\}^{\frac{1}{2}} P^{\frac{1}{2}}(N\le u)$$
$$\le kn^*\{6u^2\sigma^4 + 4\gamma\sigma u^{3/2} + \beta u\}^{\frac{1}{2}} c^{(m-1)/4}$$
$$\le kn^{*2} c^{(m-1)/4}$$
$$= kc^{\frac{m-5}{4}} \qquad (4.3.1)$$

for small c, say $c<c_0$, where k is a positive generic constant independent of c. Noting that if $m\ge 6$, $\lim kc^{(m-5)/4} = 0$ completes the proof.

4.4 Discussions

The condition "$m\ge 6$" in Theorem 4.3.1 may appear to be too restrictive. One may, however, note that m had to grow at a certain rate in the nonparametric version of Ghosh and Mukhopadhyay (1979). Still, we conjecture that it might be possible to relax this condition.

4.5 Simulations

For table VII we again fix $\mu=10$, $\sigma=1$ and $A=1$. Then, c is determined from $n^* = (A/c)^{\frac{1}{2}}\sigma$ and n^* cycles through the values 10, 15, 20, 25, 50 and 100. The quantities $\overline{N}$, $\overline{L}$, s.e.$(\overline{N})$, s.e.$(\overline{L})$, R_{n^*} and $\overline{L}/R_{n^*}$ are exactly as in Section 2.4. With the discussions of Section 4.4 in mind, we will again let m=5 and 10 in our simulation, the restriction "$m\geq 6$" in Theorem 4.3.1 notwithstanding.

Once again, with 200 replications, there is considerable uncertainty associated with the stability, but the results seem, in general, not to contradict the theoretical findings.

Remark 4.5.1: From Table VII, it seems that even for m=5, $\overline{L}/R_{n^*}$ is pretty close to 1. This seems to support our conjecture regarding the condition "$m\geq 6$" in Theorem 4.3.1. Thus, it seems that in the future, this condition should be investigated further as to how far it can be relaxed. Our attempts at this time have not been successful.

TABLE VII

MODERATE SAMPLE BEHAVIOR OF THE SEQUENTIAL PROCEDURE (4.2.1)

n^*	m	c	$\bar{N}$	s.e.(N)	$\bar{L}$	s.e.($\bar{L}$)	R_{n^*}	$\bar{L}/R_{n^*}$
10	5	0.0100	9.57	0.2232	0.2074	0.0091	0.2000	1.0369
10	10	0.0100	11.30	0.1311	0.1765	0.0061	0.2000	0.8827
15	5	0.0044	14.09	0.3063	0.1410	0.0074	0.1333	1.0575
15	10	0.0044	14.59	0.2522	0.1365	0.0062	0.1333	1.0241
20	5	0.0025	19.32	0.3529	0.1059	0.0064	0.1000	1.0591
20	10	0.0025	19.43	0.3469	0.1118	0.0053	0.1000	1.1183
25	5	0.0016	24.35	0.4257	0.0899	0.0056	0.0800	1.1237
25	10	0.0016	24.81	0.3381	0.0743	0.0035	0.0800	0.9289
50	5	0.0004	48.51	0.5801	0.0469	0.0029	0.0400	1.1722
50	10	0.0004	49.44	0.5656	0.0451	0.0040	0.0400	1.1270
100	5	0.0001	99.58	0.7558	0.0211	0.0011	0.0200	1.0542
100	10	0.0001	99.82	0.7271	0.0205	0.0011	0.0200	1.0256

CHAPTER V

SUMMARY

5.1 Objectives and Findings

The main objectives of this thesis were to fill in some important and interesting gaps in the two-stage and sequential theory for estimating parameters of the negative exponential distribution. Each time an estimation procedure was proposed, it was not only investigated from a theoretical point of view but was also simulated on the computer.

We consider two approaches to the estimation of the location. In the first approach, we propose two-stage and modified two-stage procedures to augment the existing sequential procedure. We showed in this context what one might expect. The theoretical properties of the sequential procedure are better than those of the modified two-stage procedure, which are in turn better than those of the two-stage procedure. However, all seem to be quite satisfactory in practice. With the second approach, which attempts to bound the risk, we propose all three types of procedure. For both the two-stage and modified two-stage procedures we achieve exactly the desired bound on the risk, while the sequential procedure only achieves the bound asymptotically. It appears however, from the simulations, that the exact achievement by the two-stage and modified two-stage procedures is at the expense of greater expected sample size.

The work of Chapter IV is directed toward estimation of the mean. Only a sequential procedure is proposed. This problem is not easy when attacked by conventional methods. Our procedure is based on the developments of an earlier nonparametric approach and gives a good start toward solving the problem.

5.2 Further Work

There is reason to believe that the conclusions of Theorem 2.3.3 can be strengthened. Also, there is evidence that the condition "m=6" in Theorem 4.3.1 is unnecessarily restrictive. Indeed, the work of Chapter IV should be considered a beginning, for there is quite likely some refinement possible in the results. In the future it may be possible to get a result regarding $\lim_{c\to 0} (R-R_{n^*})$ as well as a result similar to Theorem 4.3.1, but with less restrictive conditions.

Also, it would be profitable to carry out further simulations on a much larger (and more expensive) scale. This would stabilize the results and allow for sharper conclusions to be drawn regarding the moderate sample size behavior of all of the procedures.

Finnally, two general areas for possible future study come to mind. First, it may be that when σ is known, sequential methods could be devised which would do better at estimating μ or λ $(=\mu+\sigma)$ than the fixed sample size procedure with sample size n^*. In this case, n^* is not really "optimal", and we are emulating the wrong thing. Second, instead of using $X_{n(1)}$ to estimate μ, one could use the unbiased estimator $\mu-\sigma_n/n$. However, stopping rules based on this estimator would be more difficult to analyze theoretically than our stopping rules.

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