ISOMETRIES AND EPSILON-NEAR ISOMETRIES OF ANA-

LYTIC FUNCTION SPACES

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PREFACE

We deal in this thesis with the isometries and ε-near isometries (Definition 2.2) of analytic function spaces. The spaces we consider here are H^p spaces $(1 \leq p \leq \infty)$ and the disc algebra A (see 1.1 for definitions). We deal specifically with two questions: (1) Is every linear operator on H^p or A, which is "nearly" an isometry, close to an isometry? and (2) What are the isometries of the disc algebra A?

Question (1) will be formalized more precisely in 2.1. Those "nearly" isometries are the ones we call ε-near isometries. Question (1) was answered affirmatively for a number of well known Banach spaces. D. Amir and M. Cambern [2,8] worked out (independently) the case of ε-near isometries on C(K) onto C(S), the Banach spaces of continuous functions on K and S, respectively, where K and S are compact Hausdorff spaces. Y. Benyamini [4] dealt with the into case with K metrizable. He gave a counter example for a nonmetrizable K. D. Alspach [1] worked out the case of ε-near isometries on $L^p(\mu)$ into $L^p(\nu)$ where μ and ν are regular Borel measures and $(1 \leq p \neq 2 < \infty)$.

In Chapter II of this thesis we give examples to show that ε-near isometries on H^p spaces $(1 \leq p < \infty)$ and the disc algebra A are not always close to isometries on the same spaces. These results are interesting because it is known (see Boas [7]) that H^p spaces are isomorphic to L^p spaces $(1 \leq p < \infty)$. In other words, H^p spaces are "renormings" of L^p spaces. Now while ε-near isometries on L^p spaces are close to isometries, that is not the case with H^p spaces. This shows that the "renorming" process destroys this property. Of course, this was expected but no specific counter examples were given up to this point so the examples given in Chapter II serve that purpose. Question (2) is a natural question to ask about Banach spaces. Isometries of most of the well-known Banach spaces have beeh characterized. The onto isometries of C(K) spaces (for a compact Hausdorff K) were characterized by Banach and Stone. The into case was worked out by Holsztynski [17]. Isometries of $L^p(\mu)$ and H^p spaces were worked out by Lamperti [21] and Forelli [13], respectively. The onto isometries of the disc algebra A and H^∞ were determined by de-L-euw, Rudin and Wermer [22]. As for the disc algebra, partial results were obtained by McDonald [24] and Rochberg [31,32]. The question of characterizing the into isometries of the disc algebra was raised by Phelps [5].

In Chapter III of this thesis we give a complete characterization of the isometries of the disc algebra A.

Chapter I in this thesis will be devoted to setting up notation and listing some of the theorems, propositions, definitions, etc., that will be used in later chapters. Chapter II deals with the question of ε-near isometries (Definition 2.2) while Chapter III deals with the question of characterizing the isometries of the disc algebra A. Chapter IV contains some comments and open questions. In the Bibliography, references that were used in this thesis are listed as well as a few others which are of related interest. However, the listing does not exhaust all possible references related to this research.

A study like this could not have been completed without the good will and help of my major advisor, Professor John Wolfe, whose wise comments helped to clarify my thinking on many points in this study and who was always ready to come to the rescue when things seemed to come to a standstill. Special thankfulness from my heart goes to him for his patient guidance and assistance throughout my study at this institution. It has been a great privilege and pleasure to study under him, whose combination of fine scholarship and tutorship will be a guideline for my future career. A deeply felt gratitude goes to Professor D. Alspach for showing great interest and support in my work from the beginning till the end.

I acknowledge my obligation to Professor M. Keener and Yarlagada as my teachers and as members of my advisory committee for their time and effort on my behalf. Finally, special thanks goes to Barbara Newport for her long hours at the typewriter under the pressure of a deadline.

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CHAPTER I

PRELIMINARIES AND NOTATION

The notation in this thesis follows Rudin [34] and Hoffman [16]. D is used for the open unit disc in the complex plane, $\bar{D} = \{z: |z| \leq 1\}$ and $T = \{z: |z| = 1\}$. C(T), C($\bar{D}$) and C(K) are used to denote the supnorm Banach spaces of continuous complex-valued functions on T, $\bar{D}$, or a general compact Hausdorff space K, respectively. Lebesgue measure on T is denoted by m. Occasionally the letter T is also used to denote a linear operator where no confusion will occur. H(D) denotes the set of analytic functions on D.

1.1 H^p Spaces

Among the numerous definitions of H^p spaces we will give the ones that are going to be used throughout this thesis. Proofs of the equivalence of these definitions can be found in Hoffman [16] and Rudin [34].

If $0 < p \leq \infty$ we denote by H^p the class of analytic functions f in D for which $f_r(\theta) = f(re^{i\theta})$ are bounded in $L^p(m)$-norm as $r \to 1$. If $1 \leq p \leq \infty$, then H^p is a Banach space under the norm

$$||f|| = \lim_{r \to 1} ||f_r||p.$$

For $1 \leq p \leq \infty$ we shall identify H^p with the closed subspace of $L^p(T)$ consisting of functions f such that

$$\int_0^{2x} f(\theta)e^{in\theta}\, d\theta = 0, \qquad n = 1,2,3,\ldots.$$

For $1 \leq p \leq \infty$ we can also define H^p to be the class of functions f which have power series representations

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

such that $\lim_{r\to 1} f(re^{i\theta})$ exists for almost all θ and

$$\lim_{n\to\infty} \left[\frac{1}{2\pi} \int_0^{2\pi} \left|f(e^{i\theta}) - \sum_{k=1}^{n} a_k e^{ik\theta}\right|^p d\theta\right]^{1/p} = 0.$$

H^∞ is also defined to be the space of bounded analytic functions. The disc algebra is the space of functions continuous on $\bar{D}$, and analytic on D.

Functions in $L^p(T)$ $(1 \leq p \leq \infty)$ are actually equivalence classes where we identify two functions f_1, f_2 if the set

$$E = \{\theta: f_1(e^{i\theta}) \neq f_2(e^{i\theta})\}$$

has Lebesgue measure zero. The same identification is made with H^p spaces $(1 \leq p \leq \infty)$, i.e., the elements of an H^p space are actually classes of functions.

1.2 Some Important Lemmas and Theorems

We list here--for ease of reference--the most important lemmas and theorems that will be used throughout this thesis.

Lemma 1.1

The conjugate space of the disc algebra A (denoted by A*) is isometric to the space $M(T)|H_0^1$ where $M(T) = \{\mu: \mu$ is a regular Borel measure on $T\}$ ($M(T)$ is a representation of $C(T)^*$ with $||\mu||$ = total variation of μ on T) and

$$H_0^1 = \{f \in H^1: f(0) = 0\}.$$

Proof

The proof of this lemma can be found in many standard text books, e.g., Garnett [15].

Theorem 1.2

Let h be a non-negativ- Lebesque integrable function on the circle. Then there exists a function $f \in H^1$ such that $h = |f|$ if and only if $\log h$ is integrable. If $h \in L^\infty$, then $h = |g|$ for some $g \in H^\infty$ if and only if $\log h$ is integrable.

Proof

See Hoffman [16], p. 53.

Theorem 1.3 (de-Leeuw, Rudin and Wermer)

Let $T: H^\infty \to H^\infty$ be a bounded linear operator which is onto H^∞. Then T is an isometry if and only if

$$(Tf)(z) = \alpha f(\tau(z))$$

for all $f \in H^\infty$ and all $z \in D$, where τ is a complex member of modulus one and τ is a conformal map on $\bar{D}$. The result also holds if we replace H^∞ by the disc algebra everywhere.

Proof

See Hoffman [16], p. 147.

Theorem 1.4 (F. Forelli)

Let T: $H^p \to H^p$ be a bounded linear operator which is onto H^p $(1 \leq p < \infty,\ p \neq 2)$, then T is an isometry if and only if

$$(Tf)(z) = \alpha(\phi'(z))^{1/p} f(\phi(z))$$

for all $f \in H^p$ and all $z \in D$, where α is a modulus one constant, ϕ is a conformal map on $\bar{D}$ $(\phi(z) = \lambda\frac{z-a}{1-\bar{a}z}$ with $|\lambda| = 1$, $|a| < 1)$, and ϕ' is the complex derivative of ϕ $(\phi' = \lambda(1-|a|^2)/(1-\bar{a}z)^2)$.

Proof

See Forelli [13].

Note that the pth root of ϕ' is well defined on $\bar{D}$ since ϕ' has no zeros there and therefore can be represented as $\exp g(z)$ for some analytic function g on $\bar{D}$.

Definition 1.5

A boundary point β of a simply connected plane region Ω will be called a simple boundary point of Ω if β has the following property: To every sequence $\{\alpha_n\}$ in Ω such that $\alpha_n \to \beta$ as $n \to \infty$ there corresponds a curve γ with parameter interval [0,1] and a sequence $\{t_n\}$ such that $0 < t_1 < t_2 < \ldots$, $\lim_{n\to\infty} t_n = 1$, such that $\gamma(t_n) = \alpha_n$ $(n = 1,2,3,\ldots)$. In other words, there is a curve in Ω which passes through the points α_n and which ends at β

The last definition is only needed to state a form of the Riemann mapping theorem which will be used in Chapter II.

Theorem 1.6 (The Riemann Mapping Theorem)

If Ω is a bounded simply connected region in the plane and if every boundary point of Ω is simple, then every conformal mapping of Ω onto D extends to a homeomorphism of $\bar{\Omega}$ onto $\bar{D}$.

Proof

See Rudin [34], p. 281.

The following theorem, due to Rudin, will be used in Chapter III.

Theorem 1.7 (Rudin)

Let K be a closed set of Lebesque measure zero on the unit circle, and let F be any complex valued function in C(K). Then there exists a function in A whose restriction to K is F.

Proof

See Hoffman [16], p. 81.

CHAPTER II

OPERATORS APPROXIMABLE BY ISOMETRIES

2.0 Introduction

D. Amir and M. Cambern [2,8] proved independently that $C(K)$ and $C(S)$ are isomorphic with an isomorphism T satisfying $||T||\ ||T^{-1}|| < 2$, then K and S are homeomorphic. Hence $C(K)$ and $C(S)$ are isometric. Their proofs (especially in [8]) gives, in fact, more: If $||T|| \leq 1+\varepsilon$, $||T^{-1}|| \leq 1+\varepsilon$, and $||T||\ ||T^{-1}|| \leq 1+\varepsilon$ (with $0<\varepsilon<1$), then an isometry W can be chosen close to T, with $||T-W|| < 2\varepsilon$. In [4] Y. Benyamini showed that if K is a compact metric space and if $T: C(K) \to C(S)$ is an <u>onto</u> isomorphism satisfying $||f|| \leq ||Tf|| \leq (1+\varepsilon)||f||$, then there is an isometry W of $C(K)$ into $C(S)$ satisfying $||T-W|| \leq 3\varepsilon$. He also showed that the result is false if the assumption of metrizability of K is dropped.

In [1] D. Alspach proved a similar result for the case of L_p spaces. He showed that for each p, $1 \leq p \neq 2 < \infty$ there exists $\varepsilon > 0$ and a function $\tau: [0,\varepsilon) \to [0,\delta)$, $\lim_{s\to 0} \tau(s) = 0$, such that if T is an isomorphism of $L_p(\mu)$ into $L_p(\nu)$ such that $||f||_p \leq ||Tf||_p \leq (1+s)||f||_p$ for some $s \in [0,\varepsilon)$ then there exists an isometry $S: L_p(\mu) \to L_p(\nu)$ such that $||S-T|| < \tau(s)$. In this chapter we study the same problem for the case of the disc algebra and H^p spaces $1 \leq p < \infty$. The question we are trying to answer can be formulated as follows:

2.1 Statement of the Problem

Given a Banach space X and $0 < \varepsilon < \gamma < 1$. Is there a function $g: [0,\gamma) \to [0,\delta)$ with $\lim_{\varepsilon \to 0} g(\varepsilon) = 0$ such that if $T: X \to X$ is a linear isomorphism satisfying $(1-\varepsilon)||x|| \leq ||Tx|| \leq (1+\varepsilon)||x||$ we can find an isometry $W: X \to X$ such that $||T - W|| \leq g(\varepsilon)$?

The main results of this chapter will show that the answer to question 2.1 is negative in both the cases of the disc algebra A and the case of the H^p spaces $1 \leq p < \infty$. We begin by defining an ε-near isometry, a term which will be used throughout this chapter.

Definition 2.2

Let X, Y be Banach spaces, and let $T: X \to Y$ be an <u>into</u>-isomorphism such that $(1-\varepsilon)||x|| \leq ||Tx|| \leq (1+\varepsilon)||x||$ $(0 < \varepsilon < 1)$, then T is said to be an <u>ε-near isometry</u>.

We now state our main results for this chapter.

Theorem 2.3

Given $\varepsilon > 0$, there exists a linear operator $\tilde{S}$ on the disc algebra A <u>onto</u> itself satisfying

$$(1-\varepsilon)||f|| \leq ||\tilde{S}f|| \leq (1+\varepsilon)||f||$$

such that if $U: A \to A$ is an isometry and if $g \in A$, then

$$||\tilde{S} - gU|| \geq 1 - \varepsilon.$$

Theorem 2.4

Given $\varepsilon > 0$, there exists a linear operator S on H^p <u>onto</u> itself $(1 \leq p < \infty)$ satisfying

$$(1-\varepsilon)||f||_p \leq ||\tilde{S}f||_p \leq (1+\varepsilon)||f||_p$$

such that if T is an isometry on H^p, then

$$||\tilde{S} - T|| \geq 1-\varepsilon.$$

2.2 Proofs of the Results

Note that the statement of Theorem 2.3 is stronger than the statement of the problem 2.1. In fact Theorem 2.3 states that there exist ε-near isometries on A which are not even close to an isometry followed by a multiplication operator. The statement of the problem (2.1) follows from Theorem 2.3 if we take $g \equiv 1$. The reason for stating Theorem 2.3 as we have will be clear from the discussion after Theorem 2.6 below. We begin with a lemma.

<u>Lemma 2.6</u>

Let X, Y be Banach spaces. Let $T: X \to Y$ be an invertible operator and let $W: X \to Y$ be a bounded linear operator which does not have a dense range, then:

$$||T - W|| \geq \frac{1}{||T^{-1}||}.$$

<u>Proof</u>

Let $0 < \varepsilon < 1$. Let $Z = \overline{\text{Range of } W}$. Since Z is a closed proper subspace of Y, we can apply Riesz's Lemma to get an element $y_\varepsilon \in Y$ with $||y_\varepsilon|| = 1$ such that

$$||y_\varepsilon - z|| \geq 1-\varepsilon \quad \text{for all } z \in Z.$$

In particular $||y_\varepsilon - W(Ty^{-1})|| \geq 1-\varepsilon$. Thus

$$||1-WT^{-1}|| \geq ||(1-WT^{-1})y_\varepsilon|| = ||y_\varepsilon - W(T^{-1}y_\varepsilon)|| \geq 1-\varepsilon.$$

But $||1-WT^{-1}|| = ||(T-W)T^{-1}|| \leq ||T-W||\ ||T^{-1}||$, so

$$||T-W||\ ||T^{-1}|| \geq 1-\varepsilon$$

and since ε is arbitrary, we get

$$||T-W|| \geq \frac{1}{||T^{-1}||}.$$

For $g \in A$ the operator T_g: $A \to A$ defined by $T_g(f) = g \cdot f$ is called a <u>multiplication operator</u>. It can be easily shown that $||T_g|| = ||g||_\infty$.

<u>Theorem 2.7</u>

Given $0 < \varepsilon < 1$ there exists a function $g \in A$ such that T_g is an ε-near isometry of the disc algebra A and, if W is any isometry on A, then

$$||T_g - W|| \geq 1-\varepsilon.$$

<u>Proof</u>

Let g be a function in the disc algebra A such that g maps the unit disc $\bar{D}$ to the slit annular region $g(\bar{D})$ shown in Figure 1.

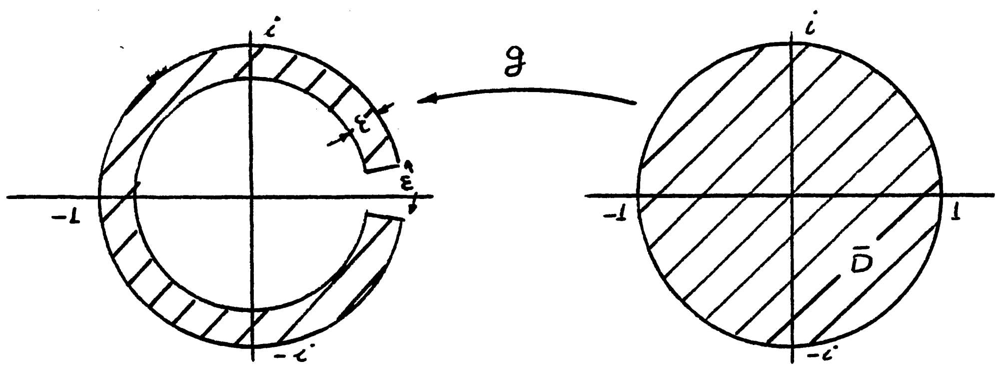

Figure 1. The Function g in Theorem 2.7

The region $g(\bar{D})$ is chosen as shown to satisfy the requirement that every point of the boundary of $g(\bar{D})$ is a simple boundary point (see definition 1.5) and therfore guaranteeing the existence of such a g (Theorem 1.6). It is easy to check that the multiplication operator Tg corresponding to g is an ε-near isometry. Since $|g(\xi)| \geq 1 - \varepsilon$ for all $\xi \in \bar{D}$, then $g^{-1} = \frac{1}{g}$ is in A. Thus $(Tg)^{-1} = Tg^{-1}$ exists and therefore Tg is onto. Also $||Tg||\ ||Tg^{-1}|| \leq \frac{1}{1-\varepsilon}$.

Now let W be an isometry on A. If W is into, then Lemma 2.6 implies

$$||Tg - W|| \geq \frac{1}{||Tg^{-1}||} \geq 1 - \varepsilon$$

and there is nothing to prove. If W is onto, then by Theorem 1.3 W has the form $Wf = \alpha f \circ \tau$ where $|\alpha| = 1$ and τ is a conformal map on $\bar{D}$.

<u>Claim</u>

If τ is not the identity map on $\bar{D}$, then $||Tg - W|| \geq 2 - \varepsilon$.

Proof of the Claim:

If not, then there exists a point $\xi_0 \in T$ such that $\tau(\xi_0) \neq \xi_0$. Let f be any function in A such that $f(\tau(\xi_0)) = \alpha^{-1}$ and $f(\xi_0) = -g^{-1}(\xi_0)(1-\varepsilon)$ and such that $||f|| = 1$ (a linear fractional transformation will do). Then

$$||Tg - W|| \geq ||Tgf - Wf|| = ||gf - \alpha f \circ \tau|| \geq |g(\xi_0)f(\xi_0) - \alpha f(\tau(\xi_0))|$$

$$= |g(\xi_0)(-g^{-1}(\xi_0))(1-\varepsilon) - \alpha\alpha^{-1}| = 2 - \varepsilon.$$

This proves the claim, and the theorem for $\tau \neq z$. On the other hand if $\tau \equiv z$, then

$$Wf = \alpha f.$$

But the choice of g implies that for each $\alpha \in T$ there exists a $\xi_\alpha \in T$ such $|g(\xi_\alpha) - \alpha| \geq 2 - \varepsilon$. Thus if $f \equiv 1$, we get

$$||Tg - W|| \geq ||Tgf - Wf|| = ||g - a|| \geq |g(\xi_\alpha) - \alpha| \geq 2 - \varepsilon > 1 - \varepsilon.$$

Q.E.D.

Some remarks on Theorem 2.7 are due

1. The quantifier $(1 - \varepsilon)$ is not the sharpest possible result. For example, if W is onto then the proof of the theorem shows that this estimate can be improved to $2 - \varepsilon$. It is suspected that the same estimate holds when W is into. In order to get such an estimate one would use the characterization of the into isometries of the disc algebra given in Chapter III of this thesis. However, the estimate we used in Theorem 2.7 is good enough for our purposes.

2. Instead of using the Riemann mapping theorem in the proof of Theorem 2.7 one could actually give an explicit formula for g with the aid of region transformation maps. The Riemann mapping theorem was preferred because such formulas are not easy to manipulate.

3. Theorem 2.7 shows that the multiplication operator as an ε-near isometry is not always close to an isometry. Thus it seems natural to try to approximate the ε-near isometry by operators of the form gW where $g \in A$ and W is an isometry on A since in this case the class of multiplication ε-near isometries will be approximated by operators of the same kind. However, Theorem 2.3 says that approximating ε-near isometries by operators of the form gW is still impossible.

Before proving Theorem 2.3 we need two lemmas.

<u>Lemma 2.8</u>

Let T_1 and T_2 be two operators defined on A into C(T) by

$$T_i f = \alpha_i f \circ \phi_i, \qquad i = 1,2$$

where α_i is a constant of modulus one and ϕ_i is a continuous unimodular function of T onto itself. If $\phi_1 \neq \phi_2$, then for any nonzero function $g \in A$

$$||T_1 - gT_2|| \geq 1 + \inf_{0 \leq \theta \leq 2\pi} |g(e^{i\theta})|.$$

In particular

$$||T_1 - T_2|| = 2.$$

Proof

Since $\phi_1 \neq \phi_2$, there exists a point $\xi_0 \in T$ such that

$$\phi_1(\xi_0) \neq \phi_2(\xi_0)$$

and by the continuity of ϕ_1 and ϕ_2, there exists an interval I_{ξ_0} such that $\phi_1(\xi) \neq \phi_2(\xi)$ for all $\xi \in I_{\xi_0}$. Now let $g \in A$ be a nonzero function. Pick a $\xi_1 \in I_{\xi_0}$ such that $g(\xi_1) \neq 0$. (If $g(I\xi_0) = \{0\}$, then $g \equiv 0$.) Choose $f \in A$ such that

(i) $||f|| = 1$

(ii) $\rho(\phi_1(\xi_1)) = \alpha_2$, and

(iii) $f(\phi_2(\xi_1)) = -\alpha_1/\mathrm{sgn}(g(\xi_1))$

where $\mathrm{sgn}(\xi) = \begin{cases} \xi/|\xi| & \xi \neq 0 \\ 0 & \xi = 0 \end{cases}$

Thus $||T_1 - gT_2|| \geq ||T_1 f - gT_2 f|| \geq |(T_1 f)(\xi_1) - g(\xi_1)(T_2 f)(\xi_1)$

$$= |\alpha_1 f(\phi_1(\xi_1)) - g(\xi_1)\alpha_2 f(\phi_2(\xi_1))| = |\alpha_1\alpha_2 + \alpha_1\alpha_2|g(\xi_1)||$$

$$= 1 + |g(\xi_1)| \geq 1 + \min_{0 \leq \theta \leq 2\pi} |g(e^{i\theta})|$$

Q.E.D.

<u>Lemma 2.9</u>

Let $0 < a < 1$. Let ϕ be the conformal map $\phi(\xi) = \frac{\xi - a}{1 - a\xi}$ for $\xi \in \bar{D}$. Let $F(\xi) = F(re^{i\theta}) = r^{1/2}\, e^{i\theta/2}$ for $0 \leq \theta < 2\pi$, a branch of the square root function. Then there exists a function ψ $C(T)$ with $T = \psi(T)$ such that $\psi^2(\xi) = \xi\phi(\xi)$ for all $\xi \in T$ and

$$||\psi - z|| = \sqrt{2 - 2\sqrt{1 - a^2}}.$$

<u>Proof</u>

Note first that according to the polar representation chosen for complex numbers in this lemma we have: for each $\xi \in T$

$$\arg \xi = \theta \qquad 0 \leq \theta < 2\pi$$

and

$$\arg \bar{\xi} = 2\pi - \theta = \arg \frac{1}{\xi}$$

Thus

$$\xi \frac{\xi - a}{1 - a\xi} = \xi\bar{\xi} \frac{\xi - a}{\xi - a} = e^{i\theta}\, e^{i(2\pi - \theta)} \frac{\xi - a}{\xi - a}$$

$$= \frac{\xi - a}{\xi - a} \quad \text{for all } \xi \in T.$$

Let $\arg(\xi - a) = \gamma$, then

$$\arg \frac{1}{(\bar{\xi} - a)} = \arg \frac{1}{\overline{(\xi - a)}} = 2\pi - \arg(\overline{\xi - a}) = 2\pi - (2\pi - \gamma) = \gamma.$$

Thus

$$\arg \frac{\xi - a}{\xi - a} = \begin{cases} 2\gamma & 0 \leq \gamma \leq \pi \\ 2\gamma - 2\pi & \pi \leq \gamma \leq 2\pi \end{cases}$$

Therefore

$$F\left(\xi \frac{\xi - a}{\bar{\xi} - a}\right) = \begin{cases} e^{i\gamma} & 0 \le \arg \xi < \pi \\ -e^{i\gamma} & \pi \le \arg \xi < 2\pi \end{cases} \qquad \text{(refer to Figure 2)}$$

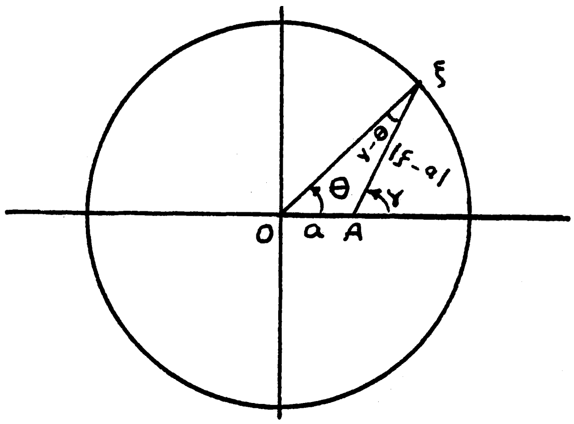

Figure 2. The Relation Between $|\xi - a|$, γ and θ

Define

$$\psi(\xi) = \begin{cases} F(\xi\phi(\xi)) & 0 \le \theta < \pi \\ -F(\xi\phi(\xi)) & \pi \le \theta < 2\pi \end{cases} \qquad \text{for } \xi \in T$$

Then $\psi(\xi) = e^{i\gamma}$ for $0 \le \theta < 2\pi$. Since γ depends continuously on θ (shown by Figure 2 or can be proved analytically) and since γ takes on every value in $[0,2\pi)$ as θ changes between 0 and 2π, it follows that ψ is continuous and onto. Clearly $\psi^2(\xi) = \xi\phi(\xi)$ as asserted. Now we consider $||\psi - z||_\infty$:

$$||\psi - z||_\infty = \sup_{0 \le \theta < 2\pi} |e^{i\gamma} - e^{i\theta}| = \sup_{0 \le \theta < 2\pi} |e^{i(\gamma-\theta)} - 1|.$$

Using Figure 3 it is clear that $|e^{i(\gamma-\theta)} - 1|$ is maximum when $(\gamma - \theta)$ is

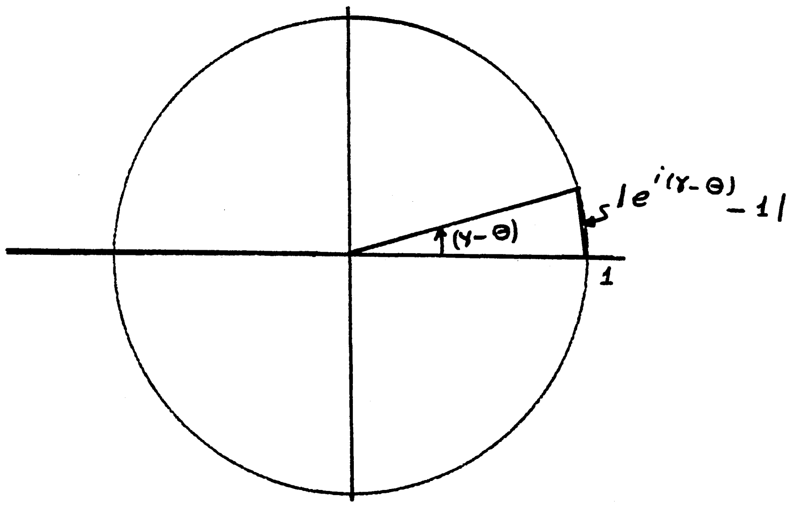

Figure 3. The Maximum of $\gamma - \theta$

maximum. Using the triangle $OA\xi$ in Figure 2 we get

$$\frac{\sin(\gamma - \theta)}{a} = \frac{\sin\gamma}{1}$$

Thus, $\sin(\gamma - \theta) = a\sin\gamma$. Since the maximum value of $\sin\gamma = 1$, we have that $(\gamma - \theta)$ is maximum when $\sin(\gamma - \theta) = a$. In this case, with the help of figure (3), we get

$$\max|e^{i(\gamma-\theta)} - 1|^2 = 1^2 + 1^2 - 2\sqrt{1-a^2} = 2 - 2\sqrt{1-a^2}.$$

Therefore

$$||z - \psi||_\infty = \sqrt{2 - 2\sqrt{1-a^2}}.$$

Note that ψ cannot be extended analytically to being the square root of an analytic function which has a simple zero at 0. Q.E.D.

The following corollary will be used in the proof of Theorem 2.4.

Corollary 2.10

If ψ and a are as defined in Lemma 2.9 and if, in addition, $a < \frac{1}{\sqrt{2}}$, then ψ has a continuous derivative with respect to θ, bounded away from zero on T.

Proof

Using the notation of the previous argument, since $\psi(\xi) = e^{i\gamma}$, and since $\sin(\gamma - \theta) = a \sin \gamma$, then

$$\frac{d\psi}{d\theta} = ie^{i\gamma} \frac{d\gamma}{d\theta}$$

and

$$\frac{d\gamma}{d\theta} = \frac{\cos(\gamma - \theta)}{\cos(\gamma - \theta) - a\cos\gamma}$$

Now since $\cos(\gamma - \theta) \geq \sqrt{1 - a^2}$, then

$$\cos(\gamma - \theta) - a\cos\gamma \geq \sqrt{1 - a^2} - a$$

But $\sqrt{1 - a^2} - a > 0$ since $a < \frac{1}{\sqrt{2}}$. Thus $\frac{d\psi}{d\theta}$ exists and is continuous on T.

Q.E.D.

Proof of Theorem 2.3

Given $0 < \varepsilon < 1$. Choose $a \in (0, \sqrt{\varepsilon^2 - \frac{\varepsilon^4}{4}})$. Let $\phi = \frac{z - a}{1 - az}$ (z is the identity function $z(\xi) = \xi$). Let ψ be the map defined in Lemma 2.9. Then

$$||\psi - z||_\infty^2 = 2(1 - \sqrt{1 - a^2}) < 2(1 - \sqrt{1 - \varepsilon^2 + \frac{\varepsilon^4}{4}}) = \varepsilon^2.$$

Therefore $||\psi - z||_\infty < \varepsilon$ and since $\psi^2 = z\phi$, ψ^2 extends to an analytic function on $\bar{D}$.

Let $\mathbb{P}$ be the algebra of complex polynomials on $\bar{D}$ viewed as a subalgebra of the disc algebra A. Define an operator $S: \mathbb{P} \to \mathbb{P}$ by

$$Sf = \sum_{k=0}^{[\frac{n}{2}]} a_{2k}\, \psi^{2k} + z \sum_{k=1}^{[\frac{n+1}{2}]} a_{2k-1}\, \psi^{2k-2} \qquad (*)$$

where $f = \sum_{k=1}^{n} a_k z^k \in \mathbb{P}$.

Claim 1:

S has the following properties:

(i) S is linear

(ii) S is an ε-near isometry

(iii) S is onto.

Proof of Claim 1:

(i) is easy to check. To prove (ii), let $p: A \to A$ be the projection

$$(pf)(\xi) = \frac{1}{2}(f(\xi) + f(-\xi)).$$

Then we can check that P is a norm one projection on A. Define $W: C(T) \to C(T)$ by

$$(Wf)(\xi) = f(\psi(\xi)) \text{ for all } f \in C(T) \text{ and } \xi \in T.$$

Then W is an isometry since ψ is onto (by Lemma 2.9). Now let $f \in \mathbb{P}$. then

$$\begin{aligned} ||Sf - Wf|| &= \left|\left| \sum_{k=0}^{[\frac{n}{2}]} a_{2k}\, \psi^{2k} + z \sum_{k=1}^{[\frac{n+1}{2}]} a_{2k-1}\, \psi^{2k-2} - \sum_{i=1}^{n} a_i\, \psi^{i} \right|\right| \\ &= \left|\left| z \sum_{k=1}^{[\frac{n+1}{2}]} a_{2k-1}\, \psi^{2k-2} - \sum_{k=1}^{[\frac{n+1}{2}]} a_{2k-1}\, \psi^{2k-1} \right|\right| \end{aligned}$$

$$= ||(z-\psi) \sum_{k=1}^{[\frac{n+1}{2}]} a_{2k-1} \psi^{2k-2}||$$

$$\leq ||z-\psi||\ ||(\sum_{k=1}^{[\frac{n+1}{2}]} a_{k2-1} z^{2k-2}) \circ \psi||$$

$$= ||z-\psi||\ ||\sum_{k=1}^{[\frac{n+1}{2}]} a_{2k-1} z^{2k-2}|| \quad \text{(since W is an}$$

isometry). And since $||z-\psi|| < \varepsilon$, we have

$$||Sf - Wf|| \leq \varepsilon || \sum_{k=1}^{[\frac{n+1}{2}]} a_{2k-1} z^{2k-2}|| = \varepsilon || \sum_{k=1}^{[\frac{n+1}{2}]} a_{2k-1} z^{2k-1}||.$$

The last equality holds because $|z(\xi)| = 1$ for all $\xi \in T$. Thus

$$||Sf - Wf|| \leq \varepsilon||(I-P)f|| \leq \varepsilon||f||.$$

Note that $(I-P)f(\xi) = \frac{1}{2}[f(\xi) - f(-\xi)]$ is also a norm one projection on A. Now since W is an isometry, it follows that

$$(1-\varepsilon)||f|| \leq ||Sf|| \leq (1+\varepsilon)||f|| \quad \text{for all } f \in A.$$

Thus S is an ε-near isometry on $\mathbb{P}$. This proves (ii).

To prove (iii), we show by induction that for each $k \geq 0$, z^k is in the range of S. In this proof the reader should recall that

$$\psi^2 = z\phi, \text{ and that } \phi = \frac{z-a}{1-az}.$$

<u>Claim 2</u>:

(a) $z^k \psi^2$ is in the range of S for $k \geq 0$

(b) z^k is in the range of S for $k \geq 0$.

Proof

For k = 0,1, we have

$$Sz^2 = \psi^2 \ , \ S1 = 1$$

and

$$Sz^3 = z\psi^2 \ , \ Sz = z.$$

For k = 2, we have

$$S(az + z^2 - az^3) = z^2 \text{ (which proves (b) for } k = 2)$$

and since $z^2 = az + \psi^2 - az\psi^2$, then

$$z^2\psi^2 = az\psi^2 + \psi^4 - az\psi^4 = S(az^3 + z^4 - az^5)$$

(which proves (a) for k = 2). Now suppose that the hypotheses are true for all $k \leq n$, and $n \geq 2$. Notice that

$$az^n + z^{n-1}\psi^2 - az^n\psi^2$$

$$= az^n + z^{n-1} z \frac{z-a}{1-az} - az^n z \frac{z-a}{1-az}$$

$$= az^n + \frac{a^{n+1} - az^n}{1-az} - \frac{az^{n+2} - a^2z^{n+1}}{1-az}$$

$$= az^n + z^{n+1}\frac{1-az}{1-az} - az^n\frac{1-az}{1-az} = z^{n+1}.$$

That is, for k = n+1, we have

$$z^k = az^{k-1} + z^{k-2}\psi^2 - az^{k-1}\psi^2 \qquad \text{——— (1)}$$

Therefore,

$$z^k\psi^2 = az^{k-1}\psi^2 + z^{k-2}\psi^4 - az^{k-1}\psi^4 \qquad \text{——— (2)}$$

Claim 3:

If $g \in \mathbb{P}$ is in the range of S, then S_0 is $g\psi^4$.

Proof

If g is in the range of S, then g = Sf for some $f \in \mathbb{P}$. Write

$$f = \sum_{\ell=0}^{m} a_\ell z^\ell$$

then

$$z^4 f = \sum_{\ell=0}^{m} a_\ell z^{\ell+4}$$

and

$$S(z^4 f) = \sum_{\ell=0}^{[\frac{m}{2}]} a_{2\ell} \psi^{2\ell+4} + z \sum_{\ell=0}^{[\frac{m+1}{2}]} a_{2\ell-1} \psi^{2\ell+2}$$

$$= \psi^4 (\sum_{\ell=0}^{[\frac{m}{2}]} a_{2\ell} \psi^{2\ell} + z \sum_{\ell=0}^{[\frac{m+1}{2}]} a_{2\ell-1} \psi^{2\ell-2})$$

$$= \psi^4 (Sf) = \psi^4 g.$$

This proves Claim 3.

Now using the induction hypothesis, the first term of the right-hand side of equation (2) is in the range of S by part (a) of the hypothesis. Part (b) of the induction hypothesis together with Claim 3 imply that the second and third terms of the right-hand side of equation (2) are in the range of S. Therefore, $z^k \psi^2$ is in the range of S for all $k \geq 0$. And thus part (b) of Claim 2 is proved. Part (a) follows by induction from part (b) together with equation (1). This proves Claim 2. In particular, Claim 2 shows that z^k is in the range of S, $k = 0,1,2\ldots.$ Therefore S is onto. This proves (iii) and completes the proof of Claim 1.

Since $\mathbb{P}$ is dense in A (in the sup norm), S extends to an ε-near isometry $\tilde{S}$ on the disc algebra A. $\tilde{S}f$ is given by formula (*) for every $f \in A$ (with n replaced by ∞). Since $\tilde{S}$ has closed range, the range of $\tilde{S}$ contains the closure of the range of S, i.e., contains the closure of $\mathbb{P}$ which is the disc algebra A. Thus $\tilde{S}$ is also onto.

Now let $U: A \to A$ be an isometry and let $g \in A$. If U is not onto, then so is gU. But Lemma 2.6 gives:

$$||\tilde{S} - gU|| \geq \frac{1}{||S^{-1}||} \geq 1 - \varepsilon.$$

On the other hand if U is onto, then U has the form:

$Uf = \alpha f \circ \tau$ where $|\alpha| = 1$ and τ is a conformal map on $\bar{D}$ (by Theorem 1.3). Since ψ cannot be extended analytically to $\bar{D}$ (being a square root of the analytic function $z\phi$ which has only a simple zero in D), then $\tau \neq \psi$. Therefore Lemma 2.8 gives:

$$||\tilde{S} - gU|| \geq ||gU - W|| - ||W - \tilde{S}|| \geq 1 - \varepsilon.$$

This completes the proof of Theorem 2.3. Q.E.D.

To prove Theorem 2.4, we need a Lemma similar to Lemma 2.7 in the H^p case.

<u>Lemma 2.11</u>

If T_1, T_2 are two operators defined on H^p space $(1 \leq p < \infty)$ by

$$T_i(f) = \alpha_i(\phi_i')^{1/p} f \circ \phi_i \qquad i = 1,2$$

where: α_i is a modulus one complex constant

$\phi_i: T \to T$ is an onto continuously differentiable function.

If $|\phi_i'|$ is bounded away from zero on T and if $\phi_1 \neq \phi_2$, then

$$||T_1 - T_2|| \geq 1.$$

Proof

Since $\phi_1 \neq \phi_2$, then there exists a point $\xi_0 \in T$ and an interval I_{ξ_0} such that

(i) $\phi_1(I_{\xi_0}) \cap \phi_2(I_{\xi_0}) = \phi$

(ii) either ϕ_1 or ϕ_2 is not constant on I_{ξ_0}.

(To get I_{ξ_0} let $\xi_1 = e^{i\theta_1}$ be any point in T such that $\phi_1(\xi_1) \neq \phi_2(\xi_1)$. Increase or decrease θ_1 till either ϕ_1 or ϕ_2 starts to change while $\phi_1(\xi) \neq \phi_2(\xi)$. Now choose ξ_0, I_{ξ_0} to satisfy (i), (ii).)

Assume without loss of generality that ϕ_1 is not constant on I_{ξ_0}. Since ϕ_1 is continuous, then $\phi_1(I_{\xi_0})$ is an interval I of positive Lebesgue measure ℓ. Let $0 < \delta < 1$. Since ϕ_i' is continuous and bounded away from zero, there exists a positive constant $M > 1$ such that

$$\frac{1}{M} \leq |\phi_i'| \leq M \qquad i = 1,2.$$

Choose an $\varepsilon > 0$ such that

$$\varepsilon < \frac{\delta}{2M^{2/p}}.$$

Let $g \in H^\infty$ be such that

$$|g(\xi)| = SX_I(\xi) + \varepsilon X_{I^c}(\xi)$$

where:

X_I is the characteristic function of the interval $I = \phi_1(I_{\xi_0})$

X_{I^c} is the characteristic function of the interval I^c, the complement of I.

and

$$S = \left(\frac{2\pi}{\ell}\right)^{1/p} \left(1 - \varepsilon^p\left(1 - \frac{\ell}{2\pi}\right)\right)^{1/p}.$$

We can check that $||g||_p = 1$. Note that such a choice of g is possible by Theorem 1.2. Now

$$||T_1g - T_2g||_p^p = \frac{1}{2\pi}\int_0^2 |\alpha_1(\phi_1')^{1/p}\ g \circ \phi_1 - \alpha_2(\phi_2')^{1/p}\ g \circ \phi_2|^p\ d\theta$$

$$\geq \frac{1}{2\pi}\int_0^{2\pi} ||\phi_1'|^{1/p}\ |g \circ \phi_1| - |\phi_2'|^{1/p}\ |g \circ \phi_2||^p\ d\theta$$

$$\geq \frac{1}{2\pi}\int_{I_{\xi_0}} ||\phi_1'|^{1/p}\ S - |\phi_2'|^{1/p}\ \varepsilon|^p\ d\theta.$$

Since $M^{-1} \leq |\phi_i'| \leq M$, we have

$$||\phi_1'|^{1/p}\ S - |\phi_2'|^{1/p}\ \varepsilon| \geq |\phi_1'|^{1/p}\ S - |\phi_2'|^{1/p}\ \varepsilon \geq |\phi_1'|^{1/p}\ S - M^{1/p}\ \varepsilon$$

$$\geq |\phi_1'|^{1/p}\ (S - M^{2/p}\ \varepsilon) \geq |\phi_1'|^{1/p}\ (S - \tfrac{\delta}{2}).$$

Thus

$$||T_1g - t_2g||_p^p \geq \frac{1}{2\pi}(S - \tfrac{\delta}{2}) \int_{I_{\xi_0}} |\phi_1'|^{1/p}\ d\theta = \frac{1}{2\pi}(S - \tfrac{\delta}{2}) \int_{\phi(I_{\xi_0})} d\theta$$

$$= \frac{\ell}{2\pi}(S - \tfrac{\delta}{2})^p.$$

So

$$||T_1g - T_2g||_p \geq \left(\frac{\ell}{2\pi}\right)^{1/p} \left[\left(\frac{2\pi}{\ell}\right)^{1/p} \left(1 - \varepsilon^p\left(1 - \frac{\ell}{2\pi}\right)\right)^{1/p} - \frac{\delta}{2}\right]$$

$$= \left(1 - \varepsilon^p\left(1 - \frac{\ell}{2\pi}\right)\right)^{1/p} - \frac{\delta}{2}\left(\frac{\ell}{2\pi}\right)^{1/p} \geq (1 - \varepsilon^p)^{1/p} - \frac{\delta}{2}$$

$$\geq 1 - \varepsilon - \frac{\delta}{2} \geq 1 - \delta.$$

Q.E.D.

Now we turn to the proof of Theorem 2.4

Proof of Theorem 2.4

Let $0 < \varepsilon < \frac{1}{4}$. Let ϕ and ψ be as defined in Lemma 2.9. Then

$$\frac{d\psi}{dz} - 1 = \frac{d\psi}{d\theta}\frac{d\theta}{dz} - 1 = \frac{1}{ie^{i\theta}}\, ie^{i\gamma}\, \frac{\cos(\gamma - \theta)}{\cos(\gamma - \theta) - a\sin\theta} - 1$$

(by the proof of Corollary 2.10). Using Figure 4 we can estimate $\left|\left|\frac{d\psi}{dz} - 1\right|\right|_\theta$ as follows

$$\frac{\cos(\gamma - \theta)}{\cos(\gamma - \theta) - a\sin\theta} \leq \frac{1}{\sqrt{1 - a^2} - a}.$$

Then

$$e^{i(\gamma-\theta)}\,\frac{\cos(\gamma - \theta)}{\cos(\gamma - \theta) - a\sin\theta}$$

is no farther from 1 than ℓ as shown in Figure 4, where $\cos\gamma_0 = \sqrt{1 - a^2}$. (This was proved in Lemma 2.9.) Here $\gamma_0 = \max(\gamma - \theta)$. Thus

$$\ell^2 = 1 + (\sqrt{1 - a^2} - a)^{-2} - 2\,\frac{\sqrt{1 - a^2}}{\sqrt{1 - a^2} - a}$$

$$= \frac{2a^2}{1 - 2a\sqrt{1 - a^2}}.$$

Therefore

$$||\psi' - 1||_\infty \leq \frac{\sqrt{2}\,a}{(1 - 2a\sqrt{1 - a^2})^{1/2}} \qquad \left(\text{where } \psi' = \frac{d\psi}{dz}\right)$$

Note that the right-hand side of the last inequality tends to zero as $a \to 0^+$. Thus we can choose a value a for which

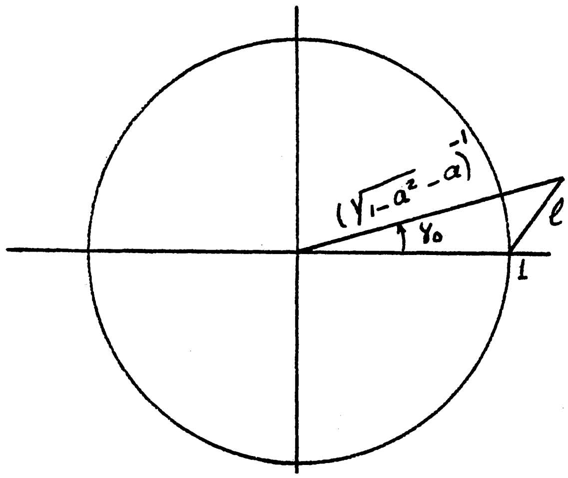

Figure 4. The Maximum of $|\frac{d}{dz} - 1|$

$$||\psi' - 1||_\infty < \frac{\varepsilon}{4} \quad \text{and} \quad ||\psi - z||_\infty < \frac{\varepsilon}{4}.$$

Since $\varepsilon < \frac{1}{4}$, then $1 < 2|\psi'|$ on T.

Now let $\mathbb{P}$ be the subspace of polynomials considered as a subspace of H^p. Define $S\colon \mathbb{P} \to \mathbb{P}$ by

$$S(f) = \sum_{k=0}^{[\frac{n}{2}]} a_{2k}\, \psi^{2k} + z \sum_{k=1}^{[\frac{n+1}{2}]} a_{2k-1}\, \psi^{2k-2} \qquad (*)$$

where

$$f = \sum_{k=0}^{n} a_k\, z^k \in \mathbb{P}.$$

We can easily check that S is linear.

Claim

(i) S is an ε-near isometry (i.e., $(1-\varepsilon)||f||_p \leq ||Sf||_p \leq (1+\varepsilon)\cdot ||f||_p$)

(ii) S is onto.

Proof

To prove (i) define W: $H^p \to L^p(T)$ by

$$(Wf)(\xi) = (\psi'(\xi))^{1/p} f(\psi(\xi)),$$

then W is an isometry. Now let $f \in \mathbb{P}$, then

$$||Wf - Sf||_p \le ||Wf - f \circ \psi||_p + ||f \circ \psi - Sf||_p$$

$$||Wf - f \circ \psi||_p^p = \frac{1}{2\pi} \int |f \circ \psi - Wf|^p \, d\theta$$

$$= \frac{1}{2\pi} \int |(\psi')^{1/p} - 1|^p \; |f \circ \psi|^p \, d\theta.$$

Since $p \ge 1$,

$$|(\psi')^{1/p} - 1| \le |\psi' - 1| < \frac{\varepsilon}{4}.$$

So

$$||Wf - f \circ \psi||_p^p \le \frac{1}{2\pi}\left(\frac{\varepsilon}{4}\right)^p \int |f \circ \psi|^p \, d\theta$$

and since $1 \le 2|\psi'|$, then

$$\frac{1}{2\pi} \int |f \circ \psi|^p \, d\theta \le \frac{1}{2\pi} \int 2|\psi'| \; |f \circ \psi|^p \, d\theta = 2||Wf||_p^p = 2||f||_p^p$$

(since W is an isometry). Therefore

$$||f \circ \psi - Wf||_p \le 2^{1/p} \frac{\varepsilon}{4}||f||_p < \frac{\varepsilon}{2}||f||_p,$$

also

$$||f \circ \psi - Sf||_p^p = \frac{1}{2\pi} \int |f \circ \psi - Sf|^p \, d\theta$$

$$= \frac{1}{2\pi} \int \left| \sum_{k=0}^{n} a_k \psi^k - \sum_{k=0}^{[\frac{n}{2}]} a_{2k} \psi^{2k} - z \sum_{k=0}^{[\frac{n+1}{2}]} a_{2k-1} \psi^{2k-2} \right|^p d\theta$$

$$= \frac{1}{2\pi} \int |z - \psi|^p \left| \sum_{k=0}^{[\frac{n+1}{2}]} a_{2k-1} \psi^{2k-2} \right|^p d\theta$$

$$\leq (\frac{\varepsilon}{4})^p \frac{1}{2\pi} \int \left| \sum_{k=0}^{[\frac{n+1}{2}]} a_{2k-1} \psi^{2k-1} \right|^p d\theta$$

(since $||\psi - z||_\infty < \frac{\varepsilon}{4}$ and $|\psi| = 1$ on T).

Define the operators A: $H^p \to H^p$ by

$$Af(\xi) = f(-\xi)$$

and C: $H^p \to H^p$ by

$$Af = f \circ \psi,$$

then

$$||(\frac{I - P}{2})f||_p = (\frac{1}{2\pi} \int |\tfrac{1}{2}(f(\xi) - f(-\xi))|^p \, d\theta)^{1/p}$$

$$\leq (\frac{1}{2\pi} \int |\tfrac{1}{2}f(\xi)|^p \, d\theta)^{1/p} + (\frac{1}{2\pi} \int |\tfrac{1}{2}f(-\xi)|^p \, d\theta)^{1/p}$$

$$= ||f||_p \text{ for all } f \in H^p$$

so that $||\frac{I - A}{2}|| = 1$. Also,

$$||Cf||_p^p = \frac{1}{2\pi} \int |f \circ \psi|^p \, d\theta \leq \frac{1}{2\pi} \int 2|\psi'| \; |f \circ \psi|^p \, d\theta = 2||f||^p$$

for $f \in H^p$ so that $||C|| \leq 2^{1/p}$.

Now

$$\left(\sum_{k=0}^{\frac{n+1}{2}} a_{2k-1}\,\psi^{2k-1}\right) = \left(C \circ \frac{I-A}{2}\right)(f).$$

Thus

$$\left|\left|\left(C \circ \frac{I-A}{2}\right)(f)\right|\right|_p \le ||C||\;\left|\left|\frac{I-A}{2}\right|\right|\;||f||_p \le 2^{1/p}\;||f||_p.$$

This means that

$$||f \circ \psi - Sf||_p \le \frac{\varepsilon}{4}\;\left|\left|\left(C \circ \frac{I-A}{2}(f)\right)\right|\right|_p \le 2^{1/p}\,\frac{\varepsilon}{4}\;||f||_p < \frac{\varepsilon}{2}\;||f||_p.$$

Therefore $||Sf - Wf||_p \le \varepsilon||f||_p$ and since W is an isometry, we get

$$(1-\varepsilon)||f||_p \le ||Sf||_p \le (1+\varepsilon)||f||_p \text{ for all } f \in \mathbb{P}.$$

This proves (i). The proof of (ii) is exactly the same as in the proof of Theorem 2.3. Thus the claim is proved.

Since the closure of $\mathbb{P}$ in L^p norm is H^p itself, then the operator $\tilde{S}$ extends to an ε-near isometry $\tilde{S}$ on H^p. And since $\tilde{S}$ has closed range, then $\tilde{S}$ is onto. $\tilde{S}f$ is given by formula (*) for every $f \in H^p$. Now let $T\colon H^p \to H^p$ be an isometry. If T is into then Lemma 2.5 implies that

$$||S-T|| \ge \frac{1}{||S^{-1}||} \ge 1-\varepsilon$$

and we are done. If T is onto, then T has the form

$$Tf = \alpha(\tau')^{1/p}\; f \circ \tau$$

where $|\alpha| = 1$ and τ is a conformal map (by Theorem 1.4). Since τ is analytic and ψ is not (Lemma 2.9), then $\tau \ne \psi$. And since τ' and ψ' satisfy the requirements of Lemma 2.11, then

$$||W - T|| \geq 1.$$

Thus

$$||S - T|| \geq ||W - T|| - ||S - W|| \geq 1 - \varepsilon.$$

Q.E.D.

CHAPTER III

INTO ISOMETRIES OF THE DISC ALGEBRA

3.0 Introduction

In his paper published in 1963 Frank Forelli [13] characterized the into isometries of H^p spaces for $1 \leq p \neq 2 < \infty$. In [5], p. 354, Phelps raised the question of characterizing the into isometries of the disc algebra. Rochberg [31] and McDonald [24] worked independently on that problem and obtained partial results. Their work amounts to describing into isometries of the type given in part (a) of Theorem 3.4 below. In this chapter we give a complete characterization of the into isometries of the disc algebra. The following two examples illustrate two different types of isometries of the disc algebra.

3.1 Examples

Example (1)

Let $\phi\colon \bar{D} \to \bar{D}$ be given by

$$\phi(\xi) = \xi^2$$

and let $\theta\colon \bar{D} \to \bar{D}$ be given by

$$\theta(\xi) = \xi.$$

Define a linear operator $U\colon A \to A$ by

$$(Uf)(\xi) = \theta(\xi)f(\phi(\xi)).$$

Then U is an into isometry. Indeed, for every $f \in A$, there is an $\xi_0 \in T$ such that $||f|| = |f(\xi_0)|$. Since ϕ is onto, there exists an $\xi_1 \in T$ such that $\phi(\xi_1) = \xi_0$. Therefore

$$||Uf|| = ||\theta,f \circ \phi|| = \sup_{\xi \in T} |\theta(\xi)f(\phi(\xi))| = \sup_{\xi \in T} |f(\phi(\xi))| =$$

$$|f(\phi(\xi_1)| = |f(\xi_0)| = ||f||.$$

Thus $||Uf|| = ||f||$ for all $f \in A$.

Example (2)

Let Δ be the cantor set of Lebesgue measure zero (embedded in T). There is a continuous surjection ϕ of Δ onto T (see Dugundji [10], p. 108). A theorem by Pelczynski [28] asserts the existence of a norm 1 linear operator E of $C(\Delta)$ into A such that $Ef = f$ on Δ. Define the linear operator $U: A \to A$ by

$$(Uf)(\xi) = E(f(\phi))(\xi).$$

Then U is an into isometry: $||Uf|| = ||E(f(\phi))|| = ||f \circ \phi|| = ||f||$ since ϕ is onto.

The question now is: are there other into isometries of the disc algebra? The results of this chapter will show that the answer is negative. In other words, the above two kinds are essentially the only ones, although the functions θ and ϕ defined in Example (1) will belong to larger classes of functions.

3.2 Statement of the Results

We begin with the following definitions.

Definition 3.1

Let B denote the subalgebra of H^∞ consisting of functions which are quotients of two functions with disc algebra, i.e.,

$$B = \{f \in H^\infty: f = h_1/h_2 \text{ with } h_1, h_2 \in A\}$$

Definition 3.2

Let B_1 be the subclass of B with the following properties:

(i) for each $f \in B$, there exists a closed subset $S_f \subset T$ such that $T \subset f(S_f)$

(ii) $|h_2(s)| = 1$ for all $s \in S_f$ and all representations $f = h_1/h_2$

(iii) $||f|| \leq 1$.

Definition 3.3

Let S be a subset of T of Lebesgue measure zero. For each $\theta, \psi \in C(S)$

$$Y(\theta,\psi) = \{\theta f(\psi): f \in A\}$$

Let $L[Y(\theta,\psi),A]$ denote the class of norm one linear extensions from $Y(\theta,\psi)$ to A, i.e., the class of operators $E: Y(\theta,\psi) \to A$ such that $||Ef|| = ||f||$ and $Ef = f$ on S for all $f \in Y(\theta,\psi)$.

With the above definitions we can now state the main theorem of this chapter.

Theorem 3.4

(i) Let $U: A \to A$ be an into isometry from the disc algebra A into itself, i.e., $||Uf||_\infty = ||f||_\infty$ for all $f \in A$, then either

(a) there exist norm one functions $\theta \in A$ and $\psi \in B$, such that

$$Uf = \theta f(\psi) \tag{1}$$

for each $f \in A$, or:

(b) there exists a closed subset $S \subset T$ with Lebesgue measure zero and functions $\theta, \psi \in C(S)$ where ϕ maps S onto T and θ is unimodular on S and there exists an $E \in L[Y(\theta,\psi),A]$ such that

$$Uf = E(\theta f(\psi)) \tag{2}$$

for each $f \in A$.

(ii) If U is the map defined by (1) (with $\psi \in B$, and $\theta = h_2$ where $\psi = \frac{h_1}{h_2}$) or the map defined by (2), then U is an isometry of A into itself.

3.3 Proof of the Results

The proof of Theorem 3.4 will follow from a string of lemmas and propositions which will be given now.

Lemma 3.5

$f \in B$ if and only if there exists a closed subset K of T with $m(K) = 0$ such that f is continuous outside of K.

Proof

Suppose $f = h_1/h_2$. Let $K = \{\xi \in T: h_2(\xi) = 0\}$. Then K is closed and has Lebesque measure zero (since otherwize $h_2 \equiv 0$). If $\xi \notin K$, then there exists an interval $I_\xi \subset T$ such that $I_\xi \cap K = \{\phi\}$ and $|h_2(s)| \geq M$ for some $M > 0$

and all $s \in I_\xi$. Since h_1 is continuous on T, it follows that f is continuous at ξ. Therefore f is continuous outside of K.

On the other hand, suppose f is continuous outside a set K of Lebesque measure zero. By Rudin's Theorem (Theorem 1.7) there exists a function $h_2 \in A$ such that h_2 vanishes on K. Since it is bounded ($f \in H^\infty$), then $h_2 f$ is continuous on T. Thus $h_2 f \in A$, i.e., $h_2 f = h_1$, or $f = h_1/h_2$. Therefore $f \in B$. Q.E.D.

In the following lemmas A* denotes the conjugate space of A, B_{A^*} denotes the unit ball of A*, and $\text{ext}(B_{A^*}) = \{\mu \in B_{A^*}: \mu \text{ is extreme}\}$. We note in passing that $\text{ext}(B_{A^*})$ is the set of measures of the form $\lambda\delta_\xi$ where $|\lambda| = 1$ and δ_ξ is the Dirac δ-measure associated with $\xi \in T$.

<u>Lemma 3.6</u>

Let U: A → A be an into isometry. Fix $\xi \in T$, then

$$(U^*)^{-1}(\delta_\xi) \cap \text{ext}(B_{A^*}) \neq \phi.$$

<u>Proof</u>

Define $K_\xi = (U^*)^{-1}(\delta_\xi) \cap B_{A^*}$. Since U* is onto, then K_ξ is not the empty set. Since K_ξ is W*-compact, and convex, then K_ξ contains and extreme point μ (the Krien-Millman Theorem). We claim that $\mu \in \text{ext}(B_{A^*})$. Indeed if μ is not an extreme point of B_{A^*}, then we can find a nonzero $\nu \in A^*$ such that $\mu + \nu \in B_{A^*}$ and $\mu - \nu \in B_{A^*}$, i.e.,

$$||\mu+\nu|| \leq 1 \quad \text{and} \quad ||\mu-\nu|| \leq 1, \tag{1}$$

then

$$||U^*(\mu \pm \nu)|| \leq ||U^*||\ ||\mu \pm \nu|| \leq ||\mu \pm \nu|| \leq 1.$$

But $U^*(\mu \pm \nu) = \delta_\xi \pm U^*(\nu)$. This means that $\delta_\xi + U^*(\nu)$ B_{A^*} and $\delta_\xi - U^*(\nu) \in B_{A^*}$. But since $\xi \in T$, then $\delta_\xi \in \text{ext}(B_{A^*})$. This implies that $U^*(\nu) = 0$. Thus $U^*(\mu + \nu) = \delta_\xi$ and $U^*(\mu - \nu) = \delta_\xi$. This together with inequalities (1) show that $\mu + \nu \in K_\xi$ and $\mu - \nu \in K_\xi$. But since μ is extreme in K_ξ it follows that $\nu = 0$, which is a contradition. Therefore $\mu \in \text{ext}(B_{A^*})$. (Note: $\mu = \lambda\delta_t$ for some $|\lambda| = 1$ and $t \in T$.) Q.E.D.

Note that the results in the above lemma can be generalized to function algebras following the same proof. We preferred to state it for the disc algebra in order to use it directly below. The following proposition is similar to that in Pelczynski [30].

Proposition 3.7

Let $U\colon A \to A$ be an into isometry, then there exists a closed subset $Q \subset T$, a continuous function $\varepsilon\colon Q \to T$ and a map ϕ of Q onto T such that

$$\varepsilon(q)(Ug)(q) = g(\phi(q)) \quad \text{for all } q \in Q.$$

Proof

In the following proof we identify T with its embedding in B_{A^*} (i.e., $t \mapsto \delta_t$) and we let $\lambda T = \{\lambda\delta_t : t \in T\}$ where $|\lambda| = 1$. For each $\lambda \in T$, let

$$Q_\lambda = (U^*)^{-1}(\lambda T) \cap T$$

and let

$$Q = \bigcup_{\lambda \in T} Q_\lambda. \tag{1}$$

Define $\varepsilon\colon Q \to T$ by

$$\varepsilon(q) = \lambda^{-1} \quad \text{for } q \in Q_\lambda,$$

and define $\phi: Q \to T$ by

$$\phi(q) = \varepsilon(q)U^*(\delta_q).$$

Then for $q \in Q$ and $g \in A$, we have

$$\varepsilon(q)(Ug)(q) = \varepsilon(q)[U^*(\delta_q)](g) = [\delta_{\sigma(q)}](g) = g(\phi(q)).$$

By Lemma 3.6, ϕ is onto. Thus we need to show that Q is closed and ε is continuous: Let F be a closed subset of T, then:

$$\varepsilon^{-1}(F) = \bigcup_{\lambda \in F} Q_{\lambda^{-1}} = \bigcup_{\lambda \in F} [(U^*)^{-1}(\lambda^{-1}T) \cap T] = (U^*)^{-1}[\bigcup_{\lambda \in F} (\lambda^{-1}T)] \cap T.$$

Since the map $(\lambda,\xi) \to \lambda^{-1}\xi$ is a homeomorphism, then $\bigcup_{\lambda \in F} (\lambda^{-1}T)$ is W*-closed. And since U* is W*-continuous, then $\varepsilon^{-1}(F)$ is closed and thus ε is continuous. Now since $Q = \varepsilon^{-1}(T)$, then Q is closed. Q.E.D.

Proof of Theorem 3.4

Let ε,ϕ be the maps associated with U in proposition 3.7 and let Q be the set associated with ε,ϕ. We have two cases:

Case (1): $m(Q) > 0$

Case (2) $m(Q) = 0$

Suppose Case (1) holds, then we have:

Proof of Part (a)

Claim (1)

For $f,g \in A$, and $n \geq 1$, we have;

(i) $(U1)(Ufg) = Uf\ Ug$

(ii) $(U1)^{n-1}\ U(f^n) = (Uf)^n$.

Proof

To prove (i) notice that the same identity holds on Q. Indeed, if $q \in Q$, then by proposition 3.7:

$$\varepsilon(q)(Ug)(q) = g(\phi(q)) \quad \text{for all } g \in A.$$

Then

$$\varepsilon(q)(U1)(q) = 1.$$

Therefore

$$(U1)(q) = \frac{1}{\varepsilon(q)} .$$

Thus for any $f \in A$,

$$(Uf)(q) = (U1)(q)f(\phi(q)).$$

And for $f,g \in A$, we have

$$(Ufg)(q) = (U1)(q)(fg)(\phi(q)) = (U1)(q)f(\phi(q))g(\phi(q))$$

so

$$(U1)(q)(Ufg)(q) = [(U1)(q)f(\phi(q))][(U1)(q)g(\phi(q))]$$

$$= (Uf)(q)(Ug)(q).$$

Now (i) follows since $m(Q) > 0$. Part (ii) of the claim follows from part (i) by taking $f \equiv g$ and using induction. Q.E.D.

Claim (2)

If $U1(\xi)$ has a zero of order n at a point $\xi_0 \in D$, then $(Uz^k)(\xi)$ has a zero of order at least n at ξ_0, $k = 1,2,3,\ldots$ (here z is the identity function $z(\xi) = \xi$).

Proof

By Claim (1), we have

$$(U1)Uz^{2k} = (Uz^k)^2,$$

therefore,

$$(U2^k)^2(\xi_0) = (U1)(\xi_0)(Uz^{2k})(\xi_0) = 0,$$

i.e., $(Uz^k)(\xi_0) = 0$ for all k. Since (U1) has a zero of order n at ξ_0, we have

$$\lim_{\xi\to\xi_0} \frac{(U1)^{n-1}(\xi_0)}{(\xi - \xi_0)^{n(n-1)}} = C \quad \text{where } 0 \leq |C| < \infty.$$

From (ii) of Claim (1) we have:

$$(U1)^{n-1}(\xi)(Uz^{kn})(\xi) = (Uz^k)^n(\xi).$$

Then

$$\lim_{\xi\to\xi_0} \frac{(U1)^{n-1}(\xi)(Uz^{kn})(\xi)}{(\xi - \xi_0)^{n(n-1)}} = \lim_{\xi\to\xi_0} \frac{(Uz^k)^n(\xi)}{(\xi - \xi_0)^{n(n-1)}}$$

$$= \lim_{\xi\to\xi_0} [\frac{uz^k}{(\xi - \xi_0)^{n-1}}]^n.$$

The left-hand side $= \lim_{\xi\to\xi_0} \frac{(U1)^{n-1}(\xi)}{(\xi - \xi_0)^{n(n-1)}} \lim_{\xi\to\xi_0} (Uz^{kn})(\xi)$

$$= C(Uz^{kn})(\xi_0) = 0.$$

Thus $\lim_{\xi \to \xi_0} \frac{Uz^k}{(\xi - \xi_0)^{n-1}} = 0.$

Therefore Uz^k has a zero of order at least n at ξ_0. Q.E.D.

Claim (3)

Define

$$\phi_1(\xi) = \frac{(Uz)(\xi)}{(U1)(\xi)}$$

then $\phi_1 \in H^\infty$ and $||\phi_1|| = 1$.

Proof

First note that Claim (2) implies ϕ_1 is analytic on D. To show that $||\phi_1|| \leq 1$, assume there is an $\xi \in D$ such that $|\phi_1(\xi)| > 1$. If $(U1)(\xi) \neq 0$, then

$$(Uz^n)(\xi) = (U1)(\xi)\phi_1^n(\xi), \qquad n = 1,2,3,\ldots$$

and since $||Uz^n|| = ||z^n|| = 1$, we have

$$|\phi_1^n(\xi)| = \left|\frac{Uz^n(\xi)}{U1(\xi)}\right| \leq \frac{1}{|U1(\xi)|} < \infty \quad \text{for all } n$$

which is a contradiction since $\lim_{n\to\infty} |\phi_1^n(\xi)| = \infty$. Now since $_1$ is analytic on D and the zeros of $U1(\xi)$ are isolated in D we must have that $|\phi_1(\xi)| \leq 1$ for all $\xi \in D$. Thus $\phi_1 \in H^\infty$ and $||\phi_1|| = 1$ (since $|\phi_1(\xi)| = 1$ for $\xi \in D$. Q.E.D.

Finally, to prove part (a) of Theorem 3.4 note that since

$$Uz(q) = U1(q)\phi(q) \quad \text{for } q \in Q$$

then $\phi_1 = \phi$ on Q. So for any $f \in A$

$$Uf(q) = U1(q)f(\phi(q)) = U1(q)f(\phi_1(q)) \quad \text{for all } q \in Q$$

and since $n(Q) > 0$, then

$$Uf = U1f(\phi_1) \text{ on } \bar{D}.$$

Thus, if we take $\theta = U1$, $\psi = \phi_1$, $h_1 = uz$ and $h_2 = U1$ it follows that $\psi \in B_1$ and $(Uf)(\xi) = \theta(\xi)f(\psi(\xi))$ for all $f \in A$ and $\xi \in \bar{D}$. This finishes Case (1).

Now suppose that Case (2) holds (i.e., $m(Q) = 0$). Then we have

Proof of Part (b)

Let $S = Q$, $\theta = \frac{1}{\varepsilon}$, and $\psi = \phi$ where Q, θ, ψ are as defined in proposition 3.7. Then we have:

$$(Ug)(q) = \theta(q)g(\phi(q)) \quad \text{for all } q \in S \text{ and } g \in A.$$

Let $Y(\theta,\psi)$ be as given by Definition 3.3. Note that if $g \in Y(\theta,\psi)$, then g has a unique representation as $g = \theta f(\psi)$ for a unique $f \in A$. Indeed, assume that $g = \theta f(\psi) = \theta h(\psi)$ for $f,h \in A$. Since

$$|\theta(s)| = \left|\frac{1}{\varepsilon(s)}\right| = 1 \text{ for each } s \in S,$$

then

$$f(\psi) = h(\psi)$$

and since ψ is onto, it follows that $f = h$ on T and therefore $f \equiv h$.

Now define a linear operator $E\colon Y(\theta,\psi) \to A$ by

$$E(\theta f(\psi)) = Uf.$$

It follows by the above discussion that E is well defined. We need to check that $||E|| = 1$. But this is true since ψ maps S onto T and $|\theta(s)| = 1$ for $s \in S$. Hence for any $f \in A$, $|f(\psi)|$ attains its maximum on S. Therefore given $f \in A$, let $\xi \in T$ be a point such that $|f(\xi)| = ||f||_\infty$. Let $g \in S$ be such that $\psi(q) = \xi$, then

$$|(Uf)(q)| = |(E(\theta f(\psi)))(q)| = |\theta f(\psi)(q)| = |\theta(q)|\ |f(\phi)(q)|$$

$$= |f(\xi)| = ||f|| = ||Uf||.$$

Hence if $g \in Y$, then $g = \theta f(\phi)$ for a unique $f \in A$. Then

$$||Eg|| = ||E\theta f(\psi)|| = ||Uf|| = |(\theta f(\psi))(q)| = |g(q)| < ||g||.$$

But since E is an extension operator, then $||Eg|| \geq ||g||$. Thus $||Eg|| = ||g||$. This shows that E has norm one. This finishes Case (2) and completes part (i) of Theorem 3.4.

<u>Proof of Part (ii) of Theorem 3.4</u>

Suppose a linear operator $U: A \to A$ is given by

$$Uf = \theta f(\psi).$$

In the case $\psi = h_1/h_2 \in B_1$ and $\theta = h_2$, we can show that $Uf \in A$ for every $f \in A$. It suffices to show that $Uz^n \in A$ for $n = 0,1,2,\dots.$ Let $K = \{s \in T: h_2(s) = 0\}$. Lemma 3.5 implies that ψ^n is continuous on $T - K$. Since $\psi^n \in H^\infty$, then ψ^n is bounded. Since $\theta = h_2$ then $\theta(K) = \{0\}$. Therefore $\theta\psi^n$ is continuous on T. Thus $\theta\psi^n = Uz^n \in A$ for $n = 0,1,2\dots.$ Now if $f \in A$, then there exists a point $\xi_0 \in T$ such that $|f(\xi_0)| = ||f||_\infty$. And since $\psi \in B_1$, there exists a point $\xi_1 \in T$ such that $\psi(\xi_1) = \xi_0$. Now

$$|Uf(\xi_1)| = |\theta(\xi_1)|\ |f(\phi(\xi_1))| = |\theta(\xi_1)|\ |f(\xi_0)| = ||f||_\infty$$

(since $|\theta(s)| = 1$ for $s \in S_\psi$). Therefore $||Uf||_\infty = ||f||_\infty$ and U is an isometry. In the case θ, ψ satisfy (b), then for each $f \in A$ we have

$$||Uf|| = ||E(\theta f(\psi))|| = ||\theta f(\psi)|| = \sup_{q \in S} |\theta(q) f(\psi(q))|$$

$$= \sup_{q \in S} |f(\psi(q))| = ||f||_\infty .$$

The last equality holds because ψ is onto. Therefore U is an isometry. This concludes the proof of Theorem 3.4.

CHAPTER IV

SOME COMMENTS AND FURTHER QUESTIONS

The operator $\tilde{S}$ defined in the proofs of Theorems 2.3 and 2.4 is a member of a class of ε-near isometries. $\tilde{S}$ is "almost" a composition operator with the function ψ which has the property that ψ^{2n} is analytic but ψ^{2n+1} is not. Similar ε-near isometries can be constructed using this periodicity feature of ψ with periods 3, 4, etc. For example ψ can be chosen such that ψ^{3n} is analytic while ψ^{3n+1} and ψ^{3n+2} are not, $n = 0, 1, 2, \ldots$. One can then define the operator

$$\tilde{S}f = \sum_{n=0}^{\infty} a_{3n}\,\psi^{3n} + z\sum_{n=0}^{\infty} a_{3n+1}\,\psi^{3n} + z^2\sum_{n=0}^{\infty} a_{3n+2}\,\psi^{3n}$$

for each holomorphic function f where we write $f = \sum_{n=0}^{\infty} a_n z^n$. With the proper choice of ψ, $\tilde{S}$ can be made to define an ε-near isometry. This leads us to the following question:

<u>Question (1)</u>

Are there any other ε-near isometries (aside from the perturbation of isometries)?

Although the ε-near isometries on L^∞ have not been studied yet, it is felt that a result similar to the results in Theorems 2.3 and 2.4 is true. We believe that the operator $\tilde{S}$ defined in Theorem 2.3 extends to an ε-near isometry of H^∞ onto itself. If that is the case, then Theorem 2.3

extends to H^{∞}. Thus we ask:

Question (2)

Are the same results true for H^{∞}?

Theorem 3.4 describes the isometries of the disc algebra A. The isometries of H^p spaces were described--as pointed out earlier--by Forelli [13] for $1 \leq p \neq 2 < \infty$. The onto isometries of H^{∞} were described by Dr. de Leeuw, Rudin and Wermer (see [22]). The problem of describing the into isometries of H^{∞} seems to still be open. Thus, we have:

Question (3)

What are the into isometries of H^{∞}?

We close with the following remark about "extending" linear extensions. The linear extension operator defined in part (b) of Theorem 3.4 is defined only on the subspace $Y \subset C(Q)$. Can this operator E be extended to an operator E on C(Q). More generally, one can put the question as follows:

Let Q be a closed subset of T of Lebesque measure zero, $\phi: Q \to T$ be an onto continuous map and $\psi \in C(Q)$ satisfy $|\psi(q)| = 1$ for each $q \in Q$. Let $Y(\psi,\psi) = \{\psi f(\phi): f \in A\}$. Then if $E: Y(\psi,\phi) \to A$ is a norm one linear extension operator, does there exist an extension operator $E': C(Q) \to A$ such that $E' = E$ on $Y(\psi,\phi)$?

The answer to the above question is negative. We construct here a counter example. The idea of the counter example is based on the fact that no linear extension operator $E: C(Q) \to A$ is such that $E1 = 1$. The proof of this fact is given in Michael and Pelczynski [25].

Proposition 4.1

If $T: A \to A$ is an isometry such that $T1 = 1$, and T is not multiplicative, then T is an isometry of the type (b) in Theorem 3.4.

Proof

This is an immediate consequence of Theorem 3.4 since all type (a) isometries which take 1 to 1 are multiplicative (being of the form $Tf = f(\psi)$). Q.E.D.

Now let Δ be the Cantor set of Lebesque measure zero, as a subset of T, let $\psi: \Delta \to T$ be continuous and onto. By Rudin's Theorem (see Hoffman [16] ψ extends to a function $h_1 \in A$ such that $||h_1|| = 1$ and $\psi(\xi) = h_1(\xi)$ for $\xi \in \Delta$. By adjoining a point to Δ and applying Rudin's Theorem again we can find $h_2 \in A$ of norm one such that $\psi(\xi) = h_2(\xi)$ and $h_1 \neq h_2$. Let $U: A \to A$ be the isometry: $(Uf)(\xi) = \frac{1}{2}(f(h_1(\xi)) + f(h_2(\xi)))$. It can be easily checked that U is not multiplicative and $U1 = 1$. It follows by proposition 4.1 that u is a type (b) isometry. Thus $Uf = E(\theta f(\psi))$, where E, θ, ψ satisfy the requirements of Theorem 3.4.

Since $U1 = 1$ and since E is a norm one extension operator it follows that $\theta = 1$. Thus $E: Y(1,\psi) \to A$ is an extension operator of norm one such that $E(1) = 1$. If this operator can be extended to an extension operator $E_1: C(\Delta) \to A$, then $E_1(1) = 1$ which is impossible. Therefore the extension operator E cannot be extended to all of $C(\Delta)$.

We note here that the above construction is due to Ryff and Forelli. (see Rochberg [31]).

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VITA 2

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