SEQUENTIAL ESTIMATION FOR PARAMETERS

OF PARETO DISTRIBUTIONS

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CHAPTER I

INTRODUCTION AND SUMMARY

The Pareto distribution was first proposed by Vilfredo Pareto around 1897. This law as formulated by him dealt with the distribution of income in a population, and this is given by $N = Ax^{-a}$, where N is the number of persons having income $\geq x$, with A and a being positive parameters (a is known as Pareto's constant and also as a shape parameter. See Johnson and Kotz (1970), p. 233). We have considered the Pareto distribution in this study because of its wide applicability in the area of social sciences and especially in economics.

Over the years, several authors have rediscovered this distribution to provide a reasonably good fit for the distributions of firms, number of firms in various industries, sizes of cities, word frequencies and incomes. See, for example, Bhattacharya (1963), Krishnaji (1969), Mandelbrot (1960), Johnson (1958), Simon and Bonini (1958), Champernowne (1953), and Fisk (1961). This distribution has been particularly noted to fit very well in the upper tail of observed incomes.

In this study we consider a family of distributions having the density function

$$f(x;\theta,\sigma) = \frac{1}{\sigma}\,\theta^{\frac{1}{\sigma}}\,x^{-\frac{1}{\sigma}-1}, \qquad \ldots \quad (1.1)$$

where $x \geq \theta > 0$, $\sigma > 0$. This is known as the "Pareto distribution of the first kind", where θ and σ are known as the scale and shape parameters respectively. We assume that both θ and σ are unknown. Throughout this study, we write [u] for the largest integer smaller than u, and [a,b] for the closed intervals. We also write I(A) for the indicator function of the event A.

In this study, we consider two different problems in sequential estimation. These are one- and two-sample problems. Whenever necessary, the relevant literature has been reviewed, and in our derivations credits are given to the original authors accordingly. Our study is presented in four chapters.

In Chapter II, we consider a sequential procedure for estimating the scale parameter θ pointwise, such that it is asymptotically risk efficient, assuming a general loss function. The exact distribution of N, our stopping variable, is to be derived using Robbins' (1959) algorithm. Simulations on the computer have been carried out to study the behavior of our procedure for moderate sample sizes, and these results are reported in the form of several tables. Next, we consider a sequential procedure to construct a fixed-ratio confidence interval for θ, and we show that this procedure is "asymptotically consistent" and "asymptotically efficient" in the Chow-Robbins (1965) sense. We also propose a two-stage procedure and a modified two-stage procedure for the fixed-ratio confidence interval along the lines of Stein (1945, 1949) and Mukhopadhyay (1982a). As far as we know, the concepts of fixed-ratio confidence intervals have not been proposed earlier in related contexts.

Chapter III deals with the problem of estimating the shape parameter σ. We propose a purely sequential procedure for estimating the shape parameter assuming a loss function of the form of squared error plus linear cost. We show that the "regret" is bounded by exploiting the tools from Starr and Woodroofe (1972). Simulation studies have been carried out to examine the behavior of the "regret" for a few choices of moderate sample sizes.

In Chapter IV, we consider the problem of estimating the ratio of scale parameters of two Pareto distributions. We consider two cases separately:

1. the shape parameters are unknown but equal, and then
2. the shape parameters are unknown and unequal.

In either case, we propose several two-stage and purely sequential procedures to examine various asymptotic properties as we have done in other chapters.

Chapter V contains a summary of our findings, together with comments and some suggestions towards implementation of our procedures in practice.

CHAPTER II

ESTIMATION OF THE SCALE PARAMETER

2.1 Introduction

Several authors have considered different methods for estimating the parameters of the distribution in (1.1) when the sample size is fixed. Quandt (1966), for example, described some of these methods of estimation. Malik (1970) obtained the maximum likelihood estimators of the parameters. Kulldorff and Vannman (1973) estimated the scale and shape parameters through linear functions of order statistics. Aigner and Goldberger (1970) considered estimating the scale parameter in the Pareto distribution from grouped observations. Baxter (1980) derived the minimum variance unbiased estimators of the parameters.

A sequential procedure for estimating the scale parameter was considered only in Wang (1973). Wang's work has very little theoretical discussion, and also its mathematical and statistical analyses are at best incomplete. This chapter will fill some of the very important gaps present in Wang's (1973) research effort through a comprehensive study.

The present chapter deals with both the point and fixed-ratio confidence interval estimation problems for the scale parameter θ. In section 2.2, we consider the point estimation problem for θ. We propose a sequential procedure for estimating θ and show that our procedure is asymptotically risk efficient in the sense of Robbins (1959) and Starr

(1966). We also study the asymptotic behavior of the "regret" and more specifically show that the "regret" tends to zero at a very fast rate. It is worth noting that neither of these asymptotic results were considered in Wang's (1973) work. Next, in section 2.3, we compute the exact distribution of our stopping time N in order to evaluate the expected sample size (and its other characteristics) necessary to terminate sampling, and to obtain the corresponding exact values of the achieved risk and regret. We do so by using Robbins' (1959) algorithm. We also present a few descriptive statistics based on the exact distribution of N.

In section 2.4, we present some numerical results obtained through simulation studies for examining the moderate sample size behavior of our procedure considered in section 2.2.

In section 2.5, we address the problem of constructing a fixed-ratio confidence interval for θ. Here, we discuss both the sequential and two-stage procedures. We discuss the sequential procedure along the lines of Chow and Robbins (1965) and Mukhopadhyay (1974), while the two-stage procedure is developed along the lines of Mukhopadhyay (1980, 1982a), and Ghosh and Mukhopadhyay (1981).

Section 2.6 contains a few comments and suggestions related to the numerical results obtained from studying the moderate sample size performance of our procedures considered in section 2.5.

2.2 Point Estimation

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed (i.i.d.) random variables with the p.d.f. as in (1.1). Having observed $X_1, X_2, \ldots, X_n$, we consider the following transformation. Let $Y_i = \ell n(X_i)$,

$i = 1,2,\ldots,n$. This transformation gives the random variables $Y_1, Y_2, \ldots, Y_n$ which are i.i.d. with the p.d.f.

$$g(y;\ell n(\theta),\sigma) = \frac{1}{\sigma} \exp\{-(\frac{y - \ell n\,\theta}{\sigma})\}, \qquad \ldots \quad (2.1)$$

for $y > \ell n(\theta)$. Let $X_{n(1)} = \min(X_1, X_2, \ldots, X_n)$ and $Y_{n(1)} = \min(Y_1, Y_2, \ldots, Y_n)$. We propose to estimate θ by $X_{n(1)}$ which may be considered as equivalent to estimating $\ell n(\theta)$ by $Y_{n(1)}$. Consider the loss incurred in estimating θ by $X_{n(1)}$ to be

$$L_n^* = A\{\frac{X_{n(1)}}{\theta} - 1\}^s + cn^t, \qquad \ldots \quad (2.2)$$

A, s, c, t being all known positive numbers. Note that

$$\lim_{n\to\infty} \frac{\{\ell n(X_{n(1)}) - \ell n(\theta)\}}{(X_{n(1)} - \theta)} = \frac{1}{\theta}$$

with probability 1, which implies that

$$\{\ell n(X_{n(1)}) - \ell n(\theta)\}^s \doteq \{\frac{X_{n(1)}}{\theta} - 1\}^s.$$

Thus, our assumed loss (2.2) incurred in estimating θ by $X_{n(1)}$, can be approximated by

$$L_n = A\{Y_{n(1)} - \ell n(\theta)\}^s + cn^t, \qquad \ldots \quad (2.3)$$

where A, s, c, t are positive known constants. Wang (1973) considered a similar loss function with $s = 1$, $c = 1$, and $t = 1$ without giving any reasons towards that. Here, we have at least attempted to rationalize the loss function (2.3). Throughout section 2.2, we are going to work with the loss function (2.3). The risk associated with (2.3) is

$$E(L_n) = AE\{Y_{n(1)} - \ln(\theta)\}^s + cn^t.$$

Since $\frac{n(Y_{n(1)} - \ln\theta)}{\sigma}$ is distributed as an exponential random variable with mean one, we get

$$E(Y_{n(1)} - \ln\theta)^s = (\frac{\sigma}{n})^s \Gamma(s+1).$$

Therefore, the risk associated with (2.3) becomes

$$E(L_n) = A(\frac{\sigma}{n})^s \Gamma(s+1) + cn^t. \qquad \ldots \quad (2.4)$$

Now, treating n as a continuous variable, we differentiate (2.4) with respect to n and set the derivative equal to zero, obtaining the result that (2.4) is minimum when,

$$n = n^* = \{\frac{A\sigma^s s\Gamma(s+1)}{ct}\}^{\frac{1}{(s+t)}}. \qquad \ldots \quad (2.5)$$

Thus the corresponding minimum risk is given by

$$\xi(c) = E(L_{n^*}) = cn^{*t}(\frac{t+s}{s}). \qquad \ldots \quad (2.6)$$

Since n^* depends on the unknown σ, no fixed sample size procedure will solve our problem, which is to minimize the risk uniformly in σ. In section 2.2.1, we propose a purely sequential procedure as a solution.

2.2.1 Purely Sequential Procedure

Let $\hat{\sigma}_n = (n-1)^{-1} \sum_{i=1}^{n} (Y_i - Y_{n(1)})$, for $n \geq 2$. The stopping time $N = N(c)$ is defined as follows:

$$N = \inf\{n: n \geq m \geq 2,\ n \geq (\frac{A\hat{\sigma}_n^s\ s\Gamma(s+1)}{ct})^{\frac{1}{s+t}}\} \qquad \ldots \quad (2.7)$$

$$= \infty \qquad \text{if no such } n,$$

m being the starting sample size. When we stop, we estimate θ by $X_{N(1)}$. In the following theorem, we study some properties of N.

Theorem 2.1: For the sequential procedure (2.7), we have:

(i) N is well defined and non-increasing as a function of c,

(ii) $E(N) < \infty$,

(iii) $\lim_{c\to 0} (\frac{N}{n^*}) = 1$ a.s.,

(iv) $\lim_{c\to 0} N = \infty$ a.s., and $\lim_{c\to 0} E(N) = \infty$.

Proof:

Property (iii) can be verified by noting the following basic inequality:

$$\{\frac{As\Gamma(s+1)\hat{\sigma}_N^s}{ct}\}^{\frac{1}{s+t}} \le N \le m + \{\frac{As\Gamma(s+1)\hat{\sigma}_{N-1}^s}{ct}\}^{\frac{1}{s+t}}.$$

Parts (i), (ii) and (iv) are easy to verify along the lines of Chow and Robbins (1965) and Mukhopadhyay (1974). Further details are omitted.

The loss associated with (2.7) is given by

$$L_N = A\{Y_{N(1)} - \ln(\theta)\}^s + cN^t, \quad \ldots \quad (2.8)$$

and the corresponding achieved risk is

$$\bar{\xi}(c) = E(L_N) = AE\{Y_{N(1)} - \ln(\theta)\}^s + cE(N^t).$$

Now,

$$E(Y_{N(1)} - \ln\theta)^s = E\{E\{(Y_{N(1)} - \ln\theta)^s | N\}\}$$

$$= E\{\sum_{n=m}^{\infty} (Y_{n(1)} - \ln\theta)^s | N=n\}P(N=n)$$

$$= E\{\sum_{n=m}^{\infty} \{\frac{n^s(Y_{n(1)} - \ln\theta)^s}{\sigma^s} \frac{\sigma^s}{n^s} | N=n]\}P(N=n).$$

For every fixed $n \geq m$, by Basu's (1955) theorem, the events $\{N=n\}$ and $Y_{n(1)}$ are independent. Thus,

$$E(Y_{N(1)} - \ln\theta)^s = E\{\sum_{n=m}^{\infty} (\frac{n^s(Y_{n(1)} - \ln\theta)^s}{\sigma^s})\} \frac{\sigma^s}{n^s} P(N=n)$$

$$= \Gamma(s+1)\sigma^s \sum_{n=m}^{\infty} \frac{1}{n^s} P(N=n)$$

$$= \Gamma(s+1)\sigma^s E(\frac{1}{N^s}).$$

The risk associated with (2.8) is given by

$$\bar{\xi}(c) = E(L_N) = A\Gamma(s+1)\sigma^s E(\frac{1}{N^s}) + cE(N^t). \quad \ldots \quad (2.9)$$

We define the "risk efficiency" and "regret" of our procedure as

$$\eta(c) = \frac{E(L_N)}{E(L_{n*})} = \frac{\bar{\xi}(c)}{\xi(c)}, \quad \ldots \quad (2.10)$$

and

$$W(c) = E(L_N) - E(L_{n*}) = \bar{\xi}(c) - \xi(c)$$

$$= c\{\frac{tn*^{(s+t)}}{s} E(\frac{1}{N^s} - \frac{1}{n*^s}) + E(N^t - n*^t)\}, \quad \ldots \quad (2.11)$$

respectively, where $\xi(c)$ comes from (2.6). Now, we state and prove the following theorem.

Theorem 2.2: <u>For the procedure (2.7), we have</u>:

$$\lim_{c\to 0} \eta(c) = 1 \quad \underline{\text{if}}\ m > 1 + \frac{s^2}{(s+t)},$$

$$= 1+\gamma \quad \underline{\text{if}}\ m = 1 + \frac{s^2}{(s+t)},$$

$$= \infty \qquad \underline{\text{if}}\ m < 1 + \frac{s^2}{(s+t)},$$

<u>where</u> $\gamma(>0)$ <u>can be determined explicitly</u>.

<u>Remark 2.1</u>: The constant γ is actually $\delta(s)$ where $\delta(\cdot)$ is defined in the proof of Lemma 2.2.

To prove Theorem 2.2, we need the following Lemmas.

Lemma 2.1: <u>For the procedure (2.7), we have</u>:

$$\lim_{c\to 0} E\left(\frac{N}{n^*}\right)^w = 1, \ \underline{\text{for any fixed}}\ w(>0).$$

<u>Proof</u>: The proof of this Lemma will be complete if we show that $\liminf_{c\to 0} E\left(\frac{N}{n^*}\right)^w \geq 1$ and $\limsup_{c\to 0} E\left(\frac{N}{n^*}\right)^w \leq 1$. For the lim inf part, we use Fatou's Lemma and part (iii) of Theorem 2.1, and thus we note that

$$\liminf_{c\to 0} E\left(\frac{N}{n^*}\right)^w \geq E\{\liminf_{c\to 0}\left(\frac{N}{n^*}\right)^w\} = 1.$$

For the lim sup part, we proceed as in Starr (1966) and Mukhopadhyay (1974). Let $0 < \varepsilon < 1$ and $\beta = (1+\varepsilon)^{\frac{1}{w}} n^*$

$$E(N^w) = \sum_{n=m}^{\infty} n^w P(N=n) \leq (\beta+1)^w P(N \leq \beta+1) + T(\beta),$$

where $T(\beta) = \sum_{n\geq\beta+1} n^w P(N=n)$. Thus we have

$$E\left(\frac{N}{n^*}\right)^w \leq \left\{\frac{(\beta+1)}{n^*}\right\}^w P(N\leq \beta+1) + \frac{T(\beta)}{(n^*)^w}.$$

We will now prove that,

$T(\beta) < \lambda$, where λ is a constant independent of c.

Define $G_n = \dfrac{2(n-1)\hat{\sigma}_n}{\sigma}$,

$$h(n,c) = \{\frac{ctn^{s+w}}{As\Gamma(s+1)}\}^{\frac{1}{s}}\{\frac{2(n-1)}{\sigma}\}$$
$$= c^{\frac{1}{s}}\{\frac{tn^{s+w}}{As\Gamma(s+1)}\}^{\frac{1}{s}}\{\frac{2(n-1)}{\sigma}\}$$
$$= c^{\frac{1}{s}}h(n), \text{ say.}$$

For $n > m$, from (2.7) it follows that the event $\{N = n\} \subset \{G_{n-1} > h(n-1,c)\}$. Thus,

$$T(\beta) = \sum_{n \geq \beta+1} n^w P(N=n)$$

$$= \sum_{n \geq \beta} (n+1)^w P(N=n+1)$$

$$\leq \sum_{n \geq \beta} (n+1)^w P(G_n > h(n,c)).$$

The remainder of the proof follows along the lines of Mukhopadhyay's (1974) Lemma 2, with the modification that we substitute

$$\tau = \frac{(w+s)}{sw}, \qquad a(n) = \{\frac{tn^{s+w}}{As\Gamma(s+1)}\}^{\frac{1}{s}} 2(n-1)(1+\varepsilon)^{\tau},$$

in that proof. The following Lemma gives explicit conditions on m which allow us to study the extent of closeness between $E(N^{-w})$ and n^{*-w} as $c \to 0$, for every fixed $w > 0$.

Lemma 2.2: <u>For the procedure (2.7), we have</u>:

$$\lim_{c\to 0} E(\frac{n^*}{N})^w = 1 \qquad \underline{\text{if}} \quad m > 1 + \frac{w^2}{w+t},$$

$$= 1 + \delta(w,m) \quad \underline{\text{if}} \quad m = 1 + \frac{w^2}{w+t},$$

$$= \infty \qquad \underline{\text{if}} \quad m < 1 + \frac{w^2}{w+t}.$$

Proof:

Define

$$d(m,w) = \frac{1}{m^w}\frac{2^{1-m}}{(m-1)!},$$

$$b(w) = \{\frac{k\sigma^w}{t}\}^{\frac{w}{w+t}}, \quad \text{where } k = \{\frac{Aw\Gamma(w+1)}{c}\},$$

$$\delta(w) = \delta(m,w) = d(m,w)b(w)h^{m-1}(m),$$

$$\alpha = (1-\varepsilon)^{\frac{1}{w}} n^*,$$

$$\beta = (1+\varepsilon)^{\frac{1}{w}} n^*, \quad \text{where } 0 < \varepsilon < 1.$$

Our G_n is as defined in Lemma 2.1. The proof now follows from Theorem 3 of Starr (1966), noting that this $G_n \sim \chi^2_{2(n-1)}$, while Starr's G_n was $\chi^2_{(n-1)}$.

Proof of Theorem 2.2: From (2.9) and (2.10), we get

$$\eta(c) = (\frac{t}{s+t})E\{(\frac{n^*}{N})^s\} + (\frac{s}{s+t})E\{(\frac{N}{n^*})^t\}.$$

Thus Lemmas 2.1 and 2.2 with w replaced by t and s respectively prove Theorem 2.2. Here, we note that γ is the same as $\delta(s)$. The following result is a much stronger assertion than Theorem 2.2 for $s = t = 1$.

Theorem 2.3: For the procedure (2.7), we have:

$$W(c) = c + o(c) \quad \text{as } c \to 0 \quad \text{for } s = 1,\ t = 1.$$

Proof:

From (2.11), it is easily seen that,

$$W(c) = cE\{\frac{(N-n^*)^2}{N}\},$$

for $s = 1$, $t = 1$.

Let us now indicate a few steps in order to verify Theorem 2.3. Let us define a new stopping variable N' as

$$N' = \inf\{n:\ n \geq m \geq 2,\ \sum_{i=2}^{n} Z_i \leq \frac{c}{A}(1-\frac{1}{n})n^3\}, \qquad \ldots (2.12)$$

where $Z_2, Z_3, \ldots, Z_n$ are i.i.d. with the p.d.f. $\frac{1}{\sigma} e^{-\frac{z}{\sigma}} I(z>0)$. As in Swanepoel and van Wyk (1982), it can be shown that N and N' have exactly the same probability distribution. Note that (2.12) has the same form as Mukhopadhyay's (1982b) equation (2.2) and this also is of the same form as in Woodroofe's (1977) equation (1.1) with his $L_n = (1 - \frac{1}{n})$, $L_0 = -1$, $\alpha = 3$ and $c = \frac{c}{A}$. Following Ghosh and Mukhopadhyay (1975), it can be shown that,

$$\frac{(N-\lambda)}{\lambda^{\frac{1}{2}}} \xrightarrow{\mathscr{L}} N(0, \beta^2\tau^2\mu^{-2}) \quad \text{as } c \to 0,$$

where $\tau^2 = \sigma^2$, $\beta = \frac{1}{2}$, $\lambda = (\frac{A\sigma}{c})^{\frac{1}{2}}$. Also noting that $\frac{(N' - n^*)^2}{N'}$ is uniformly integrable in c, it can be shown that

$$E\{\frac{(N' - n^*)^2}{N'}\} = 1 + o(1),$$

and eventually this leads to the proof of Theorem 2.3.

Theorem 2.4: <u>For the procedure (2.7), we have</u>:

$$W(c) = O(c) \quad \underline{\text{as}}\ c \to 0$$

<u>if and only if</u> $m \geq s+1$ <u>where</u> $s \neq 1$ <u>and</u> $t = 1$.

<u>Proof</u>:

First note that from (2.11),

$$W(c) = c\{\frac{1}{s} n^{*s+1} E(\frac{1}{N^s} - \frac{1}{n^{*s}}) + E(N - n^*)\},$$

for $t = 1$. Now for $s > 0$,

$$\frac{1}{N^s} - \frac{1}{n^{*s}} = \frac{-s}{n^{*s+1}} (N - n^*) + \frac{(N - n^*)^2 (s+1)s}{2n_1^{s+2}}$$

where n_1 lies between n^* and N. Thus,

$$W(c) = \tfrac{1}{2}c(s+1)n^{*s+1} E\{\frac{(N-n^*)^2}{n_1^{s+2}}\}. \qquad \ldots (2.13)$$

Our (2.13) is of the same form as Mukhopadhyay's (1982b) equation (3.1). Hence our Theorem 2.4 follows from Mukhopadhyay's (1982b) Theorem 2.

Remark 2.2: Theorems 2.1 - 2.4 are all new, and they generalize and strengthen the structure of Wang's (1973) problem and give solid foundations.

In what follows, we review Robbins' (1959) algorithm as we have used this to obtain the exact probability distribution of the stopping time N.

2.3 The Exact Distribution of N and Application of Robbins' Algorithm

We need the probability distribution of N given by (2.7) to evaluate the expected sample size necessary to terminate sampling, and to obtain the corresponding values of the risk and regret. Basu (1971) and Wang (1973) carried out similar computations but their tabled values did differ for $s = 1$ and $t = 1$. We intend to see which of these tables are accurate, and then we will also give more elaborate tables.

For the procedure (2.7), the stopping time N is a random variable. In this section, we obtain the exact probability distribution of the stopping time N. From (2.12), we see that $V_i = \frac{2Z_i}{\sigma}$, i = 2,3,...,n are i.i.d. random variables with p.d.f. $e^{-v} I(v > 0)$. Let $S_{n-1} = \sum_{i=2}^{n} V_i$, $b_n = \sigma(\frac{n}{n*})^{\frac{s+t}{s}}$, $a_n = \frac{2(n-1)b_n}{\sigma}$. Then

$$P[N=n] = P[\hat{\sigma}_2 > b_2, \hat{\sigma}_3 > b_3, \ldots, \hat{\sigma}_{n-1} > b_{n-1}, \hat{\sigma}_n \leq b_n]$$

$$= P[S_1 > a_2, S_2 > a_3, \ldots, S_{n-2} > a_{n-1}, S_{n-1} \leq a_n].$$

Now, let us review Robbins' (1959) algorithm. Let

$$a_m = 2(m-1)(\frac{m}{n*})^{\frac{(s+t)}{s}}, \quad m = 1,2,\ldots.$$

Define $h_1(\cdot) = 1$, $c_1 = 1$. We compute recursively,

$$h_m(a_n) = \sum_{j=1}^{m-1} \frac{(a_n - a_m)^j}{j!} h_{m-j}(a_m),$$

$$c_m = e^{-a_m}\{\sum_{j=1}^{m-1} h_{m-j}(a_m)\}, \quad m = 2,3,\ldots$$
$$n = m+1, m+2, \ldots.$$

Then, according to the algorithm, we obtain $P[N = m+1] = p_m = c_m - c_{m+1}$, where $m = 1,2,3,\ldots$.

Using the algorithm described above, we have computed the exact probability distribution of our stopping time N for n* = 5(5)55 and these are shown in Tables I and IV. All computations were carried out on an IBM 3081D system with the help of FORTRAN language and WATFIV compiler.

Tables I - VI correspond to the sequential procedure (2.7). We fix A = 1, c = 1, t = 1. For Tables I - III, we fix s = 1, while in Tables IV - VI, we fix s = 2. The latter value of s is being used in order to make our tables comparable with those of Basu (1971) and Wang (1973). In Tables II and V, we present the expected sample size (and the standard deviation) necessary to terminate sampling, as well as the "risk" and "regret" associated with the sequential procedure (2.7). Tables III and VI contain the values of the mode, median and 99th percentiles of the stopping time N for $n^* = 5(5)55$.

2.3.1 Summary of Our Findings

In Tables I and IV, for each column, the minimum value of n such that $P[N=n] = 0$ always exceeds n^*. Also, the sum of the probabilities for each column gets closer to one and is exactly one for $n^* = 50$ and 55 in Table I.

In Tables II and V, we present the values of n^*, $E(N)$, $STD(N)$, $E(L_N)$, $E(L_{n^*})$, η, and W. In each table the expected sample size necessary to terminate sampling is a little smaller than the corresponding optimal sample size n^*. However, the "regret" W increases at a much faster rate when $s = 2$ in Table V than when $s = 1$ in Table II. This finding is similar to that of Basu (1971). On the other hand, for $s = 2$ in Table V the "risk" η increases at a slower rate; while for $s = 1$ in Table II the "risk" approaches one from the right as n^* gets larger. This latter behavior of the "risk" agrees with our results in Theorem 2.2.

Tables III and VI present purely descriptive statistics, the modes, medians, and 99th percentiles of the stopping time N. We notice that the modes and medians are all less than the corresponding n^* and 99th percentiles are also less than n^*, and except in few cases they are equal.

In conclusion, a comparison of our findings with those of Basu (1971) and Wang (1973) indicates that our tabled values are exactly

TABLE I

PROBABILITY DISTRIBUTION OF THE STOPPING TIME N FOR THE SEQUENTIAL PROCEDURE (2.7); s = 1

	n*					
n	5	10	15	20	25	30
2	0.27385090	0.07688367	0.03493088	0.01980132	0.01271844	0.00884950
3	0.22386220	0.03009123	0.00688118	0.00229257	0.00096178	0.00047016
4	0.28545930	0.03993976	0.00537056	0.00111115	0.00031310	0.00010872
5	0.18221980	0.07500321	0.00738603	0.00102705	0.00020039	0.00005060
6	0.03350292	0.14249620	0.01358932	0.00140291	0.00019974	0.00003749
7	0.00110062	0.21897380	0.02829325	0.00245583	0.00026965	0.00003952
8	0.00000383	0.22823720	0.05898809	0.00496560	0.00045174	0.00005186
9	0.00000000	0.13889620	0.11148260	0.01074648	0.00087214	0.00008476
10		0.04312814	0.17533910	0.02329248	0.00184417	0.00015581
11		0.00601257	0.21263430	0.04781681	0.00407541	0.00031728
12		0.00033129	0.18542210	0.08835870	0.00903308	0.00068980
13		0.00000633	0.10893980	0.14034930	0.01936120	0.00154585
14		0.00000004	0.04053971	0.18356250	0.03880322	0.00347865
15		0.00000000	0.00900119	0.18985920	0.07049710	0.00765383
16			0.00112450	0.14945010	0.11276320	0.01608896
17			0.00007455	0.08630115	0.15447640	0.03157747
18			0.00000247	0.03528415	0.17652600	0.05667663
19			0.00000004	0.00986649	0.16407190	0.09115732
20			0.00000000	0.00182393	0.12104850	0.12888160
21				0.00021552	0.06923288	0.15722940
22				0.00001574	0.02999893	0.16258440
23				0.00000069	0.00962773	0.14005790
24				0.00000002	0.00223844	0.09882236
25				0.00000000	0.00036884	0.05618071
26					0.00004215	0.02532133
27					0.00000327	0.00890577
28					0.00000017	0.00240629
29					0.00000001	0.00049184
30					0.00000000	0.00007489
31						0.00000837
32						0.00000068
33						0.00000004
34						0.00000000
	0.99999980	0.99999980	0.99999970	0.99999970	0.99999970	0.99999970

TABLE I (Continued)

n	n* 35	40	45	50	55
2	0.00650936	0.00498754	0.00394285	0.00319493	0.00264114
3	0.00025600	0.00015068	0.00009513	0.00006211	0.00004327
4	0.00004452	0.00002003	0.00000978	0.00000596	0.00000298
5	0.00001526	0.00000620	0.00000256	0.00000119	0.00000000
6	0.00000894	0.00000209	0.00000030	0.00000000	0.00000012
7	0.00000697	0.00000173	0.00000030	0.00000012	0.00000054
8	0.00000757	0.00000143	0.00000048	0.00000000	0.00000000
9	0.00001138	0.00000137	0.00000024	0.00000000	0.00000030
10	0.00001514	0.00000256	0.00000024	0.00000036	0.00000000
11	0.00002843	0.00000286	0.00000161	0.00000000	0.00000024
12	0.00005585	0.00000566	0.00000000	0.00000066	0.00000077
13	0.00011742	0.00000954	0.00000095	0.00000095	0.00000000
14	0.00025851	0.00002086	0.00000191	0.00000000	0.00000000
15	0.00058770	0.00004363	0.00000477	0.00000000	0.00000000
16	0.00133693	0.00009769	0.00000662	0.00000191	0.00000000
17	0.00299251	0.00022477	0.00001615	0.00000095	0.00000095
18	0.00649053	0.00051296	0.00003707	0.00000286	0.00000000
19	0.01337421	0.00116426	0.00008643	0.00000566	0.00000095
20	0.02581233	0.00258124	0.00019652	0.00001520	0.00000095
21	0.04590958	0.00549752	0.00045472	0.00003237	0.00000376
22	0.07416046	0.01113933	0.00101686	0.00007504	0.00000477
23	0.10727810	0.02119672	0.00222355	0.00017774	0.00001144
24	0.13706070	0.03743207	0.00465542	0.00039810	0.00003135
25	0.15263610	0.06068671	0.00929672	0.00088960	0.00006658
26	0.14624010	0.08931857	0.01749009	0.00190860	0.00015974
27	0.11902670	0.11809950	0.03068590	0.00394863	0.00035083
28	0.08128780	0.13887900	0.04989791	0.00778103	0.00077623
29	0.04601780	0.14374790	0.07446200	0.01446444	0.00163800
30	0.02133991	0.12970160	0.10111610	0.02529401	0.00334907

TABLE I (Continued)

n	n* 35	40	45	50	55
31	0.00801107	0.10102510	0.12392280	0.04121739	0.00651044
32	0.00240664	0.06729490	0.13603030	0.06216711	0.01201785
33	0.00057193	0.03797976	0.13263290	0.08622736	0.02092969
34	0.00010631	0.01799479	0.11402820	0.10921070	0.03415722
35	0.00001528	0.00709323	0.08575904	0.12549780	0.05199254
36	0.00000168	0.00230536	0.05601361	0.13001970	0.07337433
37	0.00000000	0.00061237	0.03153741	0.12065140	0.09542882
38		0.00013178	0.01519596	0.09967548	0.11379800
39		0.00002277	0.00622109	0.07285941	0.12372720
40		0.00000313	0.00214860	0.04683461	0.12204370
41		0.00000034	0.00062165	0.02632044	0.10862320
42		0.00000000	0.00000000	0.01285440	0.08680761
43				0.00542395	0.06196219
44				0.00196605	0.03931095
45				0.00060866	0.02205771
46				0.00016002	0.01089281
47				0.00003553	0.00471133
48				0.00000662	0.00177617
49				0.00000103	0.00058091
50				0.00000013	0.00016404
51				0.00000001	0.00003981
52				0.00000000	0.00000826
53					0.00000146
54					0.00000022
55					0.00000003
56					0.00000000
57					
58					
59					
60					
	0.99999950	0.99999950	0.99981480	1.00000000	1.00000000

TABLE II

OPTIMAL SAMPLE SIZE AND EXACT VALUES OF RISK AND EXPECTED SAMPLE SIZE AND SEQUENTIAL RISK FOR THE SEQUENTIAL PROCEDURE (2.7), BASED ON EXACT DISTRIBUTION OF N; s = 1

n*	E(N)	STD(N)	$E(L_N)$	$E(L_{n*})$	$\eta(c)$	W(c)
5	3.48098	1.1731	11.60838	10.00000	1.16084	1.60838
10	6.74405	2.1016	24.47794	20.00000	1.22390	4.47794
15	10.35928	2.5483	35.76460	29.99998	1.19215	5.76462
20	14.00189	2.7804	46.46208	39.99997	1.16155	6.46211
25	17.61467	2.9460	57.05821	49.99997	1.14116	7.05824
30	21.20158	3.0941	67.66019	59.99997	1.12767	7.66022
35	24.77229	3.2359	78.27480	69.99997	1.11821	8.27483
40	28.33327	3.3722	88.89522	79.99997	1.11119	8.89525
45	31.88025	3.5278	99.50078	89.99997	1.10556	9.50081
50	35.43875	3.6310	110.13890	99.99997	1.10139	10.13901
55	38.98660	3.7534	120.75950	110.00000	1.09781	10.75955

TABLE III

MODES, MEDIANS AND 99TH PERCENTILES OF THE STOPPING TIME N FOR THE SEQUENTIAL PROCEDURE (2.7); s = 1

n*	Mode	Median	99th Percentile
5	4	4	6
10	8	7	10
15	11	11	15
20	15	14	19
25	18	18	23
30	22	21	27
35	25	25	31
40	29	29	35
45	32	32	38
50	39	39	42
55	36	36	46

TABLE IV

PROBABILITY DISTRIBUTION OF THE STOPPING TIME N
FOR THE SEQUENTIAL PROCEDURE (2.7); s = 2

	n*					
n	5	10	15	20	25	30
2	0.39707600	0.16379820	0.09278238	0.06128699	0.04424614	0.03384072
3	0.23626040	0.07000726	0.02590603	0.01200461	0.00645983	0.00385219
4	0.21580180	0.08214337	0.02317846	0.00804734	0.00334197	0.00158668
5	0.11954360	0.11692790	0.03037995	0.00856525	0.00286490	0.00111169
6	0.02883972	0.15665190	0.04637241	0.01163560	0.00329328	0.00107521
7	0.00241946	0.17047940	0.07265526	0.01794815	0.00453305	0.00128812
8	0.00005819	0.13592850	0.10733590	0.02912933	0.00696403	0.00178123
9	0.00000034	0.07330382	0.14035850	0.04705364	0.01133984	0.00271606
10	0.00000000	0.02502473	0.15447080	0.07244831	0.01881373	0.00439298
11		0.00510550	0.13718980	0.10261480	0.03081834	0.00732094
12		0.00059093	0.09483415	0.12976980	0.04852527	0.01227254
13		0.00003696	0.04942061	0.14276180	0.07181162	0.02026278
14		0.00000119	0.01886581	0.13351760	0.09789699	0.03236932
15		0.00000002	0.00513823	0.10398720	0.12079500	0.04924268
16		0.00000000	0.00097433	0.06618160	0.13276010	0.07033283
17			0.00012570	0.03382964	0.12808410	0.09309739
18			0.00001079	0.01366573	0.10703080	0.11288380
19			0.00000060	0.00429712	0.07650310	0.12402050
20			0.00000002	0.00103688	0.04623634	0.12220630
21			0.00000000	0.00018939	0.02336875	0.10700700
22				0.00002585	0.00977564	0.08251977
23				0.00000260	0.00335160	0.05557708
24				0.00000019	0.00093295	0.03243293
25				0.00000001	0.00020897	0.01627674
26				0.00000000	0.00003734	0.00697388
27					0.00000528	0.00253356
28					0.00000059	0.00077528
29					0.00000005	0.00019854
30					0.00000000	0.00004229
31						0.00000745
32						0.00000108
33						0.00000013
34						0.00000001
35						0.00000000
	0.99999980	0.99999980	0.99999980	0.99999970	0.99999970	0.99999970

TABLE IV (Continued)

n	n* 35	40	45	50	55
2	0.02694970	0.02211255	0.01856494	0.01587272	0.01377285
3	0.00247443	0.00168055	0.00119281	0.00087577	0.00066203
4	0.00083345	0.00047302	0.00028527	0.00018138	0.00011992
5	0.00048548	0.00023341	0.00012076	0.00006652	0.00003839
6	0.00039840	0.00016373	0.00007379	0.00003594	0.00001854
7	0.00041348	0.00014836	0.00005829	0.00002503	0.00001132
8	0.00050783	0.00016177	0.00005639	0.00002098	0.00000876
9	0.00070405	0.00020081	0.00006354	0.00002289	0.00000858
10	0.00106299	0.00027889	0.00008035	0.00002533	0.00000846
11	0.00170493	0.00041908	0.00011021	0.00003088	0.00001031
12	0.00282788	0.00066215	0.00016528	0.00004411	0.00001222
13	0.00477076	0.00109208	0.00025868	0.00006557	0.00001788
14	0.00804287	0.00183898	0.00042188	0.00010121	0.00002629
15	0.01335067	0.00312525	0.00070804	0.00016397	0.00003946
16	0.02154994	0.00528491	0.00120413	0.00027263	0.00006390
17	0.03341442	0.00881469	0.00205249	0.00046355	0.00010520
18	0.04925740	0.01433563	0.00348562	0.00078988	0.00017852
19	0.06834817	0.02255136	0.00582713	0.00135553	0.00030333
20	0.08846492	0.03400415	0.00954282	0.00229919	0.00052100
21	0.10590600	0.04879600	0.01516891	0.00386024	0.00089586
22	0.11635720	0.06614697	0.02328157	0.00635588	0.00152153
23	0.11645970	0.08412874	0.03424817	0.01019138	0.00256455
24	0.10543530	0.09975833	0.04805845	0.01585418	0.00423372
25	0.08578026	0.10964080	0.06389117	0.02379370	0.00684816
26	0.06231683	0.11105790	0.08014137	0.03424978	0.01076931
27	0.04018480	0.10310040	0.09432983	0.04713172	0.01639235
28	0.02286732	0.08728588	0.10372780	0.06169373	0.02411973
29	0.01142139	0.06705570	0.10606710	0.07650173	0.03406936
30	0.00497995	0.04652470	0.10041300	0.08952075	0.04613221

TABLE IV (Continued)

	n*				
n	35	40	45	50	55
31	0.00188594	0.02902158	0.08766806	0.09848934	0.05957717
32	0.00061727	0.01620537	0.07030141	0.10148100	0.07318950
33	0.00017378	0.00806588	0.05158299	0.09760165	0.08523792
34	0.00004189	0.00356404	0.03450586	0.08731920	0.09381336
35	0.00000861	0.00139261	0.02096917	0.07244241	0.09728301
36	0.00000150	0.00047933	0.01153663	0.05555224	0.09477466
37	0.00000000	0.00014480	0.00572731	0.03925709	0.08649957
38		0.00003825	0.00255741	0.02549286	0.07376635
39		0.00000880	0.00102391	0.01516798	0.05861908
40		0.00000176	0.00036646	0.00824629	0.04330140
41		0.00000030	0.00011689	0.00408556	0.02965712
42		0.00000005	0.00003314	0.00183981	0.01879065
43		0.00000001	0.00000832	0.00075112	0.01098688
44		0.00000000	0.00000185	0.00027733	0.00591506
45			0.00000036	0.00009238	0.00292586
46			0.00000006	0.00002770	0.00132676
47			0.00000000	0.00000746	0.00055040
48				0.00000180	0.00020845
49				0.00000039	0.00007193
50				0.00000007	0.00002257
51				0.00000001	0.00000643
52				0.00000000	0.00000166
53					0.00000039
54					0.00000008
55					0.00000002
56					0.00000000
57					
58					
59					
60					
	0.99999940	0.99999950	0.99999970	0.99999950	0.99999950

TABLE V

OPTIMAL SAMPLE SIZE AND EXACT VALUES OF RISK AND EXPECTED SAMPLE SIZE AND SEQUENTIAL RISK FOR THE SEQUENTIAL PROCEDURE (2.7), BASED ON EXACT DISTRIBUTION OF N; s = 2

n*	E(N)	STD(N)	$E(L_N)$	$E(L_{n*})$	$\eta(c)$	W(c)
5	3.15430	1.1692	12.19435	7.50000	1.62591	4.69435
10	5.64525	2.3443	40.49281	15.00000	2.69952	25.49281
15	8.61003	3.2045	73.86827	22.49998	3.28304	51.36829
20	11.76799	3.7995	108.25280	29.99997	3.60844	78.25302
25	14.98837	4.2208	143.72090	37.49997	3.83256	106.22090
30	18.21750	4.5416	180.93060	44.99997	4.02068	135.93060
35	21.43750	4.8077	220.26100	52.49997	4.19545	167.76100
40	24.64505	5.0428	261.80760	59.99997	4.36346	201.80760
45	27.84134	5.2597	305.51830	67.49997	4.52620	238.01820
50	31.02881	5.4642	351.28170	74.99997	4.68376	276.28170
55	34.20956	5.6586	398.99160	82.50000	4.83626	316.49160

TABLE VI

MODES, MEDIANS AND 99TH PERCENTILES OF THE STOPPING TIME N FOR THE SEQUENTIAL PROCEDURE (2.7); s = 2

n*	Mode	Median	99th Percentile
5	2	3	6
10	7	6	10
15	10	9	14
20	13	12	18
25	16	16	22
30	19	19	26
35	23	22	29
40	26	25	33
45	29	28	36
50	32	32	40
55	35	35	44

the same as in Basu (1971) for $s = 1,2$ and $t = 1$ but these differ from Wang's (1973) computations for $s = 1$ and $t = 1$. Wang's (1973) "regret" W increases at a much faster rate than ours, because Wang's (1973) computations are possibly faulty.

2.4 Moderate Sample Size Behavior of the Sequential Procedure (2.7)

In this section we present simulation results carried out on an IBM 3081D computer system using the FORTRAN language and the WATFIV compiler.

For each row in Tables VII - IX, we repeat the sequential rule (2.7) 500 times. For a particular row, each time we first generate pseudo-random samples from a uniform (0,1) population.

We then transform the generated uniform random variable U to a two parameter negative exponential variable Y through the relationship $Y = \ell n(\theta) - \sigma \ell n(U)$. We fix $\ell n(\theta) = 1$, $\sigma = 1$, $s = 2$, $A = 1$, $t = 1$ and $n^* = 5,25,45,70,100(50)200$.

Tables VII - IX correspond to the sequential procedure (2.7) with $m = 3,5,10$. For each value of n^*, we start with m samples from the population with the p.d.f. $f(y;1,1)$ and we compute $\hat{\sigma}_m$. Next we check with the rule (2.7) to see whether we stop or observe the next sample. When we stop, we record the value $n(i)$ of N together with the observed value $e_{n(i)}$ of the smallest order statistic for the ith repetition in each row, $i = 1,\ldots,500$. Now, we write

$$\bar{e} = \frac{1}{J}\sum_{i=1}^{J} e_{n(i)},\quad s^2(\bar{e}) = \frac{1}{J(J-1)}\sum_{i=1}^{J}(e_{n(i)} - \bar{e})^2,$$

$$\bar{N} = \frac{1}{J}\sum_{i=1}^{J} n(i),\quad s^2(\bar{N}) = \frac{1}{J(J-1)}\sum_{i=1}^{J}(n(i) - \bar{N})^2,$$

where $J = 500$, $\bar{e} = \ell n(\hat{\theta})$. In Tables VII - IX, we report $s(\bar{e})$ and $s(\bar{N})$ as

TABLE VII

MODERATE SAMPLE SIZE BEHAVIOR OF THE SEQUENTIAL PROCEDURE (2.7): m = 3

n^*	$\bar{N}\times 10$	$SE(\bar{N})\times 10^2$	$\bar{e}\times 10^2$	$SE(\bar{e})\times 10^2$	$E(L_{n^*})\times 10^2$	$\hat{E}(L_N)\times 10^2$	$\hat{\eta}(c)\times 10^2$	$\hat{W}(c)\times 10^5$
5	43.7800	6.4236	123.4702	1.1216	24.0000	27.7123	115.4678	3712.2820
25	238.4198	18.7227	104.0596	0.2382	0.9600	1.1458	119.3543	185.8015
45	437.1396	22.7291	102.1880	0.1107	0.2983	0.3445	116.2791	48.2345
70	692.3198	28.9622	101.5946	0.0848	0.1224	0.1676	136.8363	45.1058
100	986.8997	29.4248	100.9708	0.0432	0.0600	0.0603	100.4932	0.2960
150	1491.1180	35.7451	100.6864	0.0310	0.0267	0.0267	100.0000	0.0000
200	1990.6580	44.7508	100.5068	0.0211	0.0150	0.0150	100.0000	0.0000

TABLE VIII

MODERATE SAMPLE SIZE BEHAVIOR OF THE SEQUENTIAL PROCEDURE (2.7): $m = 5$

n^*	$\bar{N}\times 10$	$SE(\bar{N})\times 10^2$	$\bar{e}\times 10^2$	$SE(\bar{e})\times 10^2$	$E(L_{n^*})\times 10^2$	$\hat{E}(L_N)\times 10^2$	$\hat{\eta}(c)\times 10^2$	$\hat{W}(c)\times 10^5$
5	53.2200	2.9907	119.7643	0.8524	24.0000	24.3409	101.4202	340.8606
25	239.7998	16.3275	104.3359	0.1905	0.9600	0.9985	104.0144	38.5385
45	443.0396	20.1728	102.3287	0.0991	0.2963	0.2997	101.1578	3.4305
70	688.0396	24.5978	101.3544	0.0623	0.1224	0.1233	100.7305	0.8945
100	992.7195	31.6508	101.0427	0.0460	0.0600	0.0603	100.5379	0.3228
150	1489.2780	36.2493	100.6605	0.0278	0.0627	0.0267	100.0000	0.0000
200	1990.6980	42.3714	100.5344	0.0238	0.0150	0.0150	100.0000	0.0000

TABLE IX

MODERATE SAMPLE SIZE BEHAVIOR OF THE SEQUENTIAL PROCEDURE (2.7): m = 10

n^*	$\bar{N}\times 10$	$SE(\bar{N})\times 10^2$	$\bar{e}\times 10^2$	$SE(\bar{e})\times 10^2$	$E(L_{n^*})\times 10^2$	$\hat{E}(L_N)\times 10^2$	$\hat{\eta}(c)\times 10^2$	$\hat{W}(c)\times 10^5$
5	100.0000	0.0000	110.1964	0.4648	24.0000	33.9999	141.6661	9999.8780
25	239.3597	17.3649	103.7890	0.1869	0.9600	0.9994	104.1087	39.4441
45	438.2197	20.7480	102.1967	0.0992	0.2963	0.3003	101.3638	4.0410
70	693.1396	25.5465	101.5355	0.0653	0.1224	0.1233	100.7290	0.8926
100	986.6597	29.4140	100.9700	0.0403	0.0600	0.0603	100.4963	0.2978
150	1491.9980	36.2298	100.6793	0.0304	0.0267	0.0267	100.0000	0.0000
200	1990.9780	43.1344	100.5027	0.0209	0.0150	0.0150	100.0000	0.0000

the corresponding standard errors (SE).

2.4.1 Summary of Numerical Findings

We notice that in Tables VII - IX, the estimated values $\bar{e}$ are very close to 1, as they are expected to be. Our numerical results for the sequential rule (2.7) agree reasonably well with the asymptotic results of Theorems 2.2 - 2.4 even for moderate values of n*. For example, estimated values $\bar{N}$ are very close to the corresponding values of n*, the optimal fixed-sample size required had σ been known. As expected, there is a considerable improvement in the performance of our procedure (2.7) when the starting sample size m increased from $m = 3$ to $m = 5$ or 10.

2.5 Fixed-Ratio Confidence Interval for θ

Suppose $X_1, X_2, \ldots, X_n$ are i.i.d. random variables having the p.d.f. as in (1.1). Here, we estimate θ by a confidence interval with a prescribed ratio and having a preassigned coverage probability, and along that line we propose to consider the interval

$$I_n = [\frac{\hat{\theta}_n}{d}, \hat{\theta}_n], \qquad \ldots \quad (2.14)$$

for θ, where $\hat{\theta}_n = X_{n(1)}$ and d(>1) being specified in advance. We require that $P\{\theta \in I_n\}$ is at least $(1-\alpha)$ or asymptotically (as $d \to 1+$) near $(1-\alpha)$. Here, $0 < \alpha < 1$ is also fixed in advance.

Fixed-ratio confidence interval problems have not been considered before. At the least, we have not found any such reference in this literature. However, fixed-width confidence interval problems for parameters of many other distributions have been considered. For example, we can cite Ray (1957), Starr (1966), Chow and Robbins (1965), Simons (1968), Khan (1969), Mukhopadhyay (1974). The "proportional closeness" criterion was considered in Nádas (1969).

To achieve a confidence coefficient at least $(1-\alpha)$ associated with the interval I_n, we require

$$P\{\theta \in I_n\} = P\{\frac{\hat{\theta}_n}{d} \leq \theta \leq \hat{\theta}_n\} \geq 1 - \alpha. \qquad \ldots (2.15)$$

From (2.14) we note that $\frac{\hat{\theta}_n}{\theta}$ is bounded below by 1 and also that n needs to be the smallest integer such that

$$n \geq \frac{\sigma \, \ell n(\frac{1}{\alpha})}{\ell n(d)} = \frac{\sigma \, \ell n(\frac{1}{\alpha})}{d^*} = C, \text{ say}, \qquad \ldots (2.16)$$

with $d^* = \ell n(d)$.

Notice that C, as given in (2.16), depends on σ which is actually unknown. In order to obtain the random sample size N in a very close proximity of C, we now propose a few purely sequential, two-stage and modified two-stage procedures.

2.5.1 Purely Sequential Procedure

Towards the end of achieving a suitable sequential procedure, the stopping time $N = N(d)$ is defined as

$$N = \inf\{n: n \geq m \geq 2,\; n \geq \frac{\hat{\sigma}_n \ell n(\frac{1}{\alpha})}{\ell n(d)}\}, \qquad \ldots (2.17)$$
$$= \infty \text{ if no such } n,$$

where $\hat{\sigma}_n = \frac{1}{n} \sum_{i=1}^{n} (Y_i - Y_{n(1)})$. When we stop, we propose the interval

$$I_N = [\frac{X_{N(1)}}{d}, X_{N(1)}].$$

for θ.

Remark 2.3: In estimating σ, we now take the divisor as n instead of $(n-1)$. This is just to get rid of botherations of working with ratios like $\frac{N}{(N-1)}$ in the technical proofs. Theorems 2.5 - 2.7 still hold with the earlier $\hat{\sigma}_n$.

Theorem 2.5: *The stopping time N from (2.17) satisfies the following properties:*

(i) N *is well defined, non-increasing as a function of* d *with* $E(N) < \infty$,

(ii) $\lim_{d\to 1+} N = \infty$ *a.s.*; $\lim_{d\to 1+} E(N) = \infty$,

(iii) $\lim_{d\to 1+} (\frac{N}{C}) = 1$ *a.s.*,

(iv) $\lim_{d\to 1+} P\{\theta \in I_N\} = 1-\alpha$, (*asymptotic consistency*)

(v) $\lim_{d\to 1+} E(\frac{N}{C}) = 1.$ (*asymptotic efficiency*)

Proof:

First note that as $d\to 1+$, $\ell n(d)\to 0$. The fact that N is well defined follows from the definition (2.17) of N. Now, we verify that $E(N) < \infty$. We get

$$E(N) - 1 = \sum_{n=1}^{\infty} P(N > n)$$

$$\leq 1 + \sum_{n=2}^{\infty} P\{n < \frac{\hat{\sigma}_n \ell n(\frac{1}{\alpha})}{\ell n(d)}\}.$$

But $2\sum_{i=1}^{n} (Y_i - Y_{n(1)}) \sim \sigma\chi^2_{2(n-1)}$. Thus,

$$E(N) - 2 \leq \sum_{n=2}^{\infty} \inf_{0<h<\frac{1}{2}} \exp\{-\frac{2d^*n^2h}{\sigma\ell n(\frac{1}{\alpha})}\}(1-2h)^{-(n-1)}$$

$$\leq \sum_{n=2}^{\infty} \exp\{-\frac{d^*n^2}{2\sigma\ \ell n(\frac{1}{\alpha})}\}2^{n-1},$$

where $h=\frac{1}{4}$. This infinite series is convergent and this leads to (i). Part (ii) is obvious. To verify (iii), we use the following basic inequality:

$$\frac{\hat{\sigma}_N\ \ell n(\frac{1}{\alpha})}{\ell n(d)} \leq N \leq m + \frac{\hat{\sigma}_{N-1}\ \ell n(\frac{1}{\alpha})}{\ell n(d)}\ .$$

Now, multiplying throughout by $\frac{\ell n(d)}{\ell n(\frac{1}{\alpha})}$ and taking limits yield the desired result. We turn now to prove (iv). We have

$$P(\theta \in I_N) = P\{0 < \ell n(\frac{\hat{\theta}_N}{\theta}) \leq \ell n(d)\}$$

$$= \sum_{n=m}^{\infty} P\{0 < \ell n(\frac{X_{n(1)}}{\theta}) \leq \ell n(d)\ |\ N=n\}P(N=n).$$

Using Basu's (1955) theorem, it can be shown that $I(N=n)$ and $X_{n(1)}$ are independent for every $n \geq m$. Here $I(\cdot)$ stands for the indicator function of $(\cdot)$. Hence,

$$P(\theta \in I_N) = \sum_{n=m}^{\infty} \{1 - \exp(-N\ \frac{\ell n(d)}{\sigma})\}P(N=n)$$

$$= E\{1 - \exp(-N\ \frac{\ell n(d)}{\sigma})\}. \qquad \ldots (2.18)$$

From part (iii) it follows that $\exp(-N\ \frac{\ell n(d)}{\sigma}) \to \alpha$ a.s.. Thus, utilizing (2.18) and the dominated convergence theorem, we can conclude (iv). For the proof of part (v), let us write

$$Y_n^* = \frac{\hat{\sigma}_n}{\sigma} = \frac{1}{n\sigma} \sum_{i=1}^{n} (Y_i - Y_{n(1)}),\ n = 2,3,\ldots.$$

Then

$$NY_N^* = \frac{1}{\sigma} \sum_{i=1}^{N} (Y_i - Y_{N(1)}) \le \frac{1}{\sigma} \sum_{i=1}^{N} (Y_i - \ln\theta) \quad \text{a.s..}$$

Using Wald's 1st equation, we get $E(NY_N^*) \le E(N)$, since $E(N) < \infty$. This gives $\frac{E(NY_N^*)}{E(N)} \le 1$. Thus, $\lim\limits_{d\to 1+} \sup \frac{E(NY_N^*)}{E(N)} \le 1$. Now, we can use Lemma 3 of Chow and Robbins (1965), having taken their $f(n) = n$, $g(n) = (\frac{n-1}{n})$, $c = t$ and $y_n = Y_n^*$ Hence we obtain

$$\lim_{d\to 1+} E(\frac{N}{C}) = 1.$$

2.5.2 Two-Stage Procedure

In order to propose a two-stage procedure, let us start with a sample of size $m(\ge 2)$. Now we define

$$N = \max\{m, [\frac{a_m \hat{\sigma}_m}{d^*}] + 1\}, \qquad \ldots (2.19)$$

where $d^* = \ln(d)$, and a_m is the upper $100\alpha\%$ point of the F-distribution with 2, $2(m-1)$ degrees of freedom. We write $[x]$ for the largest integer $< x$. When we stop, we propose the interval

$$I_N = [\frac{X_{N(1)}}{d}, X_{N(1)}],$$

for θ.

Theorem 2.6: <u>For the procedure in (2.19), we have:</u>

(i) $P\{\theta \in I_N\} \ge 1 - \alpha$,

(ii) $\frac{a_m \sigma}{d^*} \le E(N) \le m + \frac{a_m \sigma}{d^*}$,

(iii) $\lim_{d \to 1+} P\{\theta \in I_N\} = 1 - \alpha$,

(iv) $\lim_{d \to 1+} E(\frac{N}{C}) = \frac{a_m}{a} \quad (> 1)$.

Proof:

Parts (ii) and (iv) can be verified by using the basic inequality:

$$\frac{\ell n(\frac{1}{\alpha})\hat{\sigma}_m}{\ell n(d)} \le N \le m + \frac{\ell n(\frac{1}{\alpha})\hat{\sigma}_m}{\ell n(d)} \ .$$

We now verify that $a_m > a$. Using an approximation by Scheffé and Tukey (see Johnson and Kotz (1970), p. 84) we can write

$$a_m = a + \frac{a^2}{(m-1)} + O(\frac{1}{m^2}).$$

Now, dividing throughout by a, yields

$$\frac{a_m}{a} = 1 + \frac{a}{(m-1)} + O(\frac{1}{m^2})$$

$$> 1.$$

The proof of part (iii) follows in a similar way as in part (iv) of Theorem 2.5. To verify (i) first notice that

$$N \ge \frac{\ell n(\frac{1}{\alpha})\hat{\sigma}_m}{\ell n(d)} \ .$$

Now, we can write

$$P(\theta \in I_N) = E\{1 - \exp(- \frac{N\ \ell n(d)}{\sigma})\}$$

$$\geq E\{1 - \exp\{- \ell n(\frac{1}{\alpha})\frac{\hat{\sigma}_m}{\sigma}\}\}$$

$$= E(1-\alpha)$$

$$= 1-\alpha.$$

2.5.3 Modified Two-Stage Procedure

The two-stage procedure of this section is motivated by the works of Mukhopadhyay (1980, 1982a) and Ghosh and Mukhopadhyay (1981). We define,

$$m = \max\{2, [(\frac{a}{d^*})^{\frac{1}{(\gamma+1)}}] + 1\}, \qquad \ldots \quad (2.20)$$

where $\gamma > 0$ is fixed in advance. We will have more to say about γ while implementing this procedure for moderate sample sizes. Now, let

$$N = \max\{m, [\frac{a_m \hat{\sigma}_m}{d^*}] + 1\}. \qquad \ldots \quad (2.21)$$

If necessary, we extend $X_1,\ldots,X_m$ to $X_1,\ldots,X_N$ when we stop, and we propose the interval

$$I_N = [\frac{X_{N(1)}}{d}, X_{N(1)}],$$

for θ.

Theorem 2.7: <u>For the modified two-stage procedure (2.21), we have the following</u>:

(i) $P\{\theta \in I_N\} \geq 1-\alpha$,

(ii) $\lim_{d\to 1+} (\frac{N}{C}) = 1$ <u>a.s.</u>,

(iii) $\frac{(a_m \sigma)}{d^*} \leq E(N) \leq \frac{(a_m \sigma)}{d^*} + (\frac{a}{d^*})^{\frac{1}{(\gamma+1)}} + 4,$

(iv) $\lim_{d \to 1+} P\{\theta \in I_N\} = 1 - \alpha,$

(v) $\lim_{d \to 1+} E(\frac{N}{C}) = 1.$

Proof:

We start with the basic inequality:

$$\frac{a_m \hat{\sigma}_m}{d^*} \leq N \leq m + \frac{a_m \hat{\sigma}_m}{d^*} + 1$$

$$\leq N \leq 2 + [(\frac{a}{d^*})^{\frac{1}{1+\gamma}}] + 1 + [\frac{a_m \hat{\sigma}_m}{d^*}] + 1.$$

This implies

$$\frac{a_m}{d^*} E(\hat{\sigma}_m) \leq E(N) \leq \frac{a_m}{d^*} E(\hat{\sigma}_m) + (\frac{a}{d^*})^{\frac{1}{1+\gamma}} + 4.$$

This gives us part (iii). Parts (i), (ii) and (iv) can be verified by using the same basic inequality given above. Proof of part (v) is exactly the same as the proof of part (iv) in Theorem 2.6.

2.6 Moderate Sample Size Behavior of Our Procedures

In this section, we present numerical results in order to study moderate sample size performances of the procedures considered in sections 2.3 - 2.5. All computations were carried out on an IBM 3081D computer system, using the SAS (1982) version. For each row in Tables X - XIX, we repeat either the purely sequential, two-stage or modified two-stage procedure 500 times as the case may be. For each particular row, we generated pseudo-random samples Y from a negative exponential population and then transformed into Pareto variables X through $X = \exp(Y)$. We fixed $\ell n(\theta) = 1$, $\sigma = 1$.

Tables X - XII correspond to the sequential procedure (2.17) with $m = 3, 5, 10$; $\alpha = 0.05$ and $C = 10, 25, 50(50)200$. For each value of C, we start with m samples and compute $\hat{\sigma}_m$. We check with the rule (2.17) to see whether we stop or observe one more sample from the population. For each repetition we check whether $\ell n(\theta) = 1$ belongs to the actually constructed interval, and write $\hat{P}$ for the relative frequency of θ belonging to our actually constructed intervals out of 500 such intervals.

Tables XIII - XV present numerical results for the two-stage procedure (2.19). The first 5 columns are to be interpreted in exactly the same way as in the case of the first five columns in Tables X - XII. In addition, we also give $\frac{\bar{N}}{C}$ to compare with $\frac{a_m}{a}$.

In Tables XVI - XIX, we present the results for our modified two-stage procedure (2.21). We choose $\gamma = .01, .05, 0.1, 0.2$, for $\alpha = 0.05$. These tables contain values of C, $\bar{N}$, $SE(\bar{N})$, m, d, $\hat{P}$, $\frac{a_m}{a}$, $\frac{\bar{N}}{C}$.

2.6.1 Summary of Numerical Findings

In Tables X - XII, we notice that the estimated value $\hat{P}$ is very close to $(1 - \alpha)$. Also, the estimated values of $\bar{N}$ are very close to the corresponding values of C, the optimal fixed-sample size, had σ been known. These results substantiate that the purely sequential procedure (2.17) is nearly asymptotically efficient even for moderate C. As expected, there is improvement in the performance when the starting sample size m

TABLE X

MODERATE SAMPLE SIZE BEHAVIOR OF THE SEQUENTIAL PROCEDURE (2.17): $m = 3$, $\alpha = 0.05$

C	$\bar{N}$	SE($\bar{N}$)	d	$\hat{P}$
10	7.700	0.1662145	1.349283	0.816
25	22.350	0.3292047	1.127304	0.886
50	47.962	0.4414236	1.061746	0.904
100	99.329	0.4891234	1.030411	0.944
150	149.026	0.5354176	1.020172	0.956
200	198.464	0.6362539	1.015091	0.946

TABLE XI

MODERATE SAMPLE SIZE BEHAVIOR OF THE SEQUENTIAL PROCEDURE (2.17): $m = 5$, $\alpha = 0.05$

C	$\bar{N}$	SE($\bar{N}$)	d	$\hat{P}$
10	8.578	0.1368696	1.349283	0.892
25	23.196	0.2757449	1.127304	0.918
50	48.542	0.2893154	1.061746	0.908
100	99.510	0.4470565	1.030411	0.950
150	148.966	0.5585209	1.020172	0.960
200	198.436	0.6400618	1.015091	0.948

TABLE XII

MODERATE SAMPLE SIZE BEHAVIOR OF THE SEQUENTIAL PROCEDURE (2.17): $m = 10$, $\alpha = 0.05$

C	$\bar{N}$	SE($\bar{N}$)	d	$\hat{P}$
10	10.760	0.0647858	1.349283	0.952
25	23.600	0.2447362	1.127304	0.916
50	48.804	0.3397676	1.061746	0.924
100	99.412	0.4451256	1.030411	0.952
150	149.010	0.5555439	1.020172	0.946
200	198.386	0.6418239	1.015091	0.946

TABLE XIII

MODERATE SAMPLE SIZE BEHAVIOR OF THE TWO-STAGE PROCEDURE (2.19): $m = 3$, $\alpha = 0.05$

C	$\bar{N}$	$SE(\bar{N})$	d	$\hat{P}$	$\frac{a_m}{a}$	$\frac{\bar{N}}{C}$
10	24.022	0.7353	1.3493	0.962	2.3181	2.4022
15	34.550	1.0402	1.2211	0.946	2.3181	2.0303
25	58.652	1.8938	1.1273	0.936	2.3181	2.3461
50	109.584	3.2318	1.0617	0.946	2.3181	2.1917
100	229.870	6.9105	1.0304	0.954	2.3181	2.2987
150	350.602	10.9192	1.0202	0.946	2.3181	2.3373
200	489.864	15.0220	1.0151	0.960	2.3181	2.4493

TABLE XIV

MODERATE SAMPLE SIZE BEHAVIOR OF THE TWO-STAGE PROCEDURE (2.19): $m = 5$, $\alpha = 0.05$

C	$\bar{N}$	SE($\bar{N}$)	d	$\hat{P}$	$\frac{a_m}{a}$	$\frac{\bar{N}}{C}$
10	15.198	0.3366	1.3493	0.948	1.4884	1.5198
15	22.898	0.5006	1.2211	0.952	1.4884	1.5265
25	39.022	0.8391	1.1273	0.950	1.4884	1.5609
50	72.454	1.6436	1.0617	0.944	1.4884	1.4431
100	148.334	3.3462	1.0304	0.958	1.4884	1.4833
150	234.516	5.1022	1.0202	0.946	1.4884	1.5634
200	290.486	6.1678	1.0151	0.958	1.4884	1.4525

TABLE XV

MODERATE SAMPLE SIZE BEHAVIOR OF THE TWO-STAGE PROCEDURE (2.19): $m = 10$, $\alpha = 0.05$

C	$\bar{N}$	SE($\bar{N}$)	d	$\hat{P}$	$\frac{a_m}{a}$	$\frac{\bar{N}}{C}$
10	12.988	0.1500	1.3493	0.966	1.1865	1.2988
15	18.514	0.2528	1.2211	0.932	1.1865	1.2343
25	29.720	0.4392	1.1273	0.938	1.1865	1.1888
50	58.776	0.8677	1.0617	0.938	1.1865	1.1755
100	121.888	1.7668	1.0304	0.948	1.1865	1.2189
150	174.362	2.6317	1.0202	0.938	1.1865	1.1624
200	241.244	3.4373	1.0151	0.958	1.1865	1.2062

TABLE XVI

MODERATE SAMPLE SIZE BEHAVIOR OF THE MODIFIED TWO-STAGE PROCEDURE (2.21): $\alpha = 0.05$

γ	m	C	$\bar{N}$	SE($\bar{N}$)	d	$\hat{P}$	$\frac{a_m}{a}$	$\frac{\bar{N}}{C}$
0.01	10	10	12.988	0.1500	1.3493	0.966	1.1865	1.2988
	15	15	18.048	0.1607	1.2211	0.968	1.1150	1.2032
	25	25	28.342	0.1761	1.1273	0.940	1.0651	1.1337
	49	50	53.816	0.2530	1.0617	0.952	1.0319	1.0763
	96	100	103.394	0.3574	1.0304	0.938	1.0159	1.0340
	143	150	153.364	0.4515	1.0202	0.950	1.0106	1.0224
	190	200	204.408	0.5857	1.0151	0.944	1.0080	1.0220

TABLE XVII

MODERATE SAMPLE SIZE BEHAVIOR OF THE MODIFIED TWO-STAGE PROCEDURE (2.21): $\alpha = 0.05$

γ	m	C	$\bar{N}$	SE($\bar{N}$)	d	$\hat{P}$	$\frac{a_m}{a}$	$\frac{\bar{N}}{C}$
0.05	9	10	13.074	0.1805	1.3493	0.976	1.2130	1.3074
	14	15	18.124	0.1840	1.2211	0.958	1.1246	1.2083
	22	25	27.918	0.2338	1.1273	0.944	1.0748	1.1167
	42	50	52.712	0.3391	1.0617	0.942	1.0374	1.0542
	81	100	101.804	0.4922	1.0304	0.950	1.0190	1.0180
	119	150	151.910	0.6112	1.0202	0.940	1.0128	1.0127
	156	200	203.082	0.7317	1.0151	0.938	1.0097	1.0154

TABLE XVIII

MODERATE SAMPLE SIZE BEHAVIOR OF THE MODIFIED TWO-STAGE PROCEDURE (2.21): $\alpha = 0.05$

γ	m	C	$\bar{N}$	SE($\bar{N}$)	d	$\hat{P}$	$\frac{a_m}{a}$	$\frac{\bar{N}}{C}$
0.10	9	10	13.074	0.1805	1.3493	0.976	1.2130	1.3074
	12	15	17.998	0.2158	1.2211	0.954	1.1494	1.1999
	19	25	27.886	0.2576	1.1273	0.942	1.0880	1.1154
	36	50	52.654	0.3940	1.0440	0.934	1.0440	1.0531
	66	100	102.192	0.5657	1.0304	0.940	1.0234	1.0219
	96	150	153.024	0.6627	1.0202	0.958	1.0159	1.0202
	124	200	203.408	0.8103	1.0151	0.946	1.0123	1.0104

TABLE XIX

MODERATE SAMPLE SIZE BEHAVIOR OF THE MODIFIED TWO-STAGE PROCEDURE (2.20): $\alpha = 0.05$

γ	m	C	$\bar{N}$	$SE(\bar{N})$	d	$\hat{P}$	$\frac{a_m}{a}$	$\frac{\bar{N}}{C}$
0.20	7	10	13.622	0.2350	1.3493	0.960	1.2969	1.3622
	10	15	18.454	0.2493	1.2211	0.934	1.1865	1.2303
	15	25	28.714	0.3434	1.1273	0.938	1.1150	1.1486
	27	50	53.296	0.4793	1.0617	0.932	1.0599	1.0659
	47	100	103.818	0.6663	1.0304	0.942	1.0325	1.0382
	66	150	154.624	0.8347	1.0202	0.950	1.0234	1.0308
	83	200	204.228	1.0293	1.0151	0.946	1.0185	1.0211

increases. More specifically, $\hat{P}$ gets closer and closer to $(1-\alpha)$ as m increases.

Tables XIII - XV present results for the two-stage procedure (2.19). The last column in Tables XIII - XV correspond to part (v) of Theorem 2.6. Our results show that the values of $\frac{\bar{N}}{C}$ are close to the corresponding values of $\frac{a_m}{a}$. This is consistently the case in these tables.

Similar results from the modified two-stage procedure (2.21) are contained in Tables XVI - XIX. As we may expect, m increases as γ keeps on decreasing, thus our procedure performs better for smaller values of γ. For $\alpha = 0.05$, we present tables for $\gamma = 0.01, 0.05, 0.1, 0.2$. The results indicate that $\bar{N}$, $\hat{P}$ and $\frac{\bar{N}}{C}$ are very close to the true C, $(1-\alpha)$ and $\frac{a_m}{a}$ respectively. Quite impressively, $\frac{a_m}{a}$ converges to 1 as C gets larger and the estimated values of $\frac{\bar{N}}{C}$ do have the same feature. This result is very much on line with the asymptotic first-order efficiency property of the modified two-stage procedure. Another important feature coming out of this numerical study is that the pattern of results does indicate strongly that we choose $\gamma = 0.05$ in the absence of any other information.

CHAPTER III

ESTIMATION OF THE SHAPE PARAMETER

3.1 Introduction

In this chapter, our goal is to estimate σ, the shape parameter of the distribution given by (1.1). The problem of estimating the shape parameter has been addressed earlier in the fixed sample size case as cited before. While reviewing the literature for this study, we found that sequential estimation problems for the shape parameter have not been considered before. Therefore, it has become necessary to develop new tools for solving a problem of this nature, and our study has accomplished this task. In what follows, we address specifically the point estimation problem for the shape parameter σ.

In section 3.2, a formulation of the problem and some notations are presented. Section 3.3 deals with sequential estimation of the shape parameter. We assume a loss function of the form of squared error plus linear cost. The main result is presented in Theorem 3.1, where we derive the order of the "regret" associated with our proposed procedure.

In section 3.4, we study the moderate sample size performances of the procedure introduced in section 3.3.

3.2 Formulation of the Problem

Suppose we have $X_1, X_2, \ldots$ as i.i.d. random variables with the p.d.f.

given by (1.1). Let $Y_i = \ell n(X_i)$, $i = 1,2,\ldots$. Then $Y_1, Y_2, \ldots$ are i.i.d. random variables having the p.d.f. as in (2.1). Our goal is to estimate σ. The proposed estimator is $\hat{\sigma}_n$, where $\hat{\sigma}_n = \frac{1}{(n-1)} \sum_{i=1}^{n} (Y_i - Y_{n(1)})$ with $n \geq 2$. Suppose the loss incurred in estimating σ by $\hat{\sigma}_n$ is given by

$$L_n = A(\hat{\sigma}_n - \sigma)^2 + cn, \qquad \ldots (3.1)$$

A and c being known positive numbers. The associated risk is,

$$E(L_n) = AE(\hat{\sigma}_n - \sigma)^2 + cE(n), \qquad \ldots (3.2)$$

$$= \frac{A\sigma^2}{(n-1)} + cn \qquad \ldots (3.3)$$

With the usual techniques of calculus, we obtain the value of n which minimizes (3.3) to be

$$n^* = \left(\frac{A\sigma^2}{c}\right)^{\frac{1}{2}} + 1. \qquad \ldots (3.4)$$

Thus our optimal risk becomes,

$$E(L_{n^*}) = c(2n^* - 1). \qquad \ldots (3.5)$$

But n^* in (3.4) depends on σ which is unknown. In the next section, we propose a suitable sequential procedure for estimating σ by updating estimates of n^* at every stage.

3.3 Purely Sequential Procedure

Define the stopping variable $N^* = N^*(c)$ as follows:

$$N^* = \inf\{n:\ n \geq m \geq 2,\ n \geq \hat{\sigma}_n \left(\frac{A}{c}\right)^{\frac{1}{2}} + 1\}, \qquad \ldots (3.6)$$
$$= \infty \text{ if no such } n.$$

When we stop, we estimate σ by $\hat{\sigma}_{N^*}$. The associated loss function thus becomes

$$L_{N*} = A(\hat{\sigma}_{N*} - \sigma)^2 + cN^*. \qquad \ldots \quad (3.7)$$

Let $Z_2, Z_3, \ldots$ be a sequence of i.i.d. random variables with the p.d.f. $\frac{1}{\sigma} e^{-\frac{z}{\sigma}} I(z>0)$. Define $S_n = \sum_{i=2}^{n} Z_i$, $\bar{Z}_n = \frac{1}{(n-1)} S_n$ and a stopping variable N as

$$N = \inf\{n: \; n \geq m \geq 2, \; n \geq \bar{Z}_n (\tfrac{A}{c})^{\frac{1}{2}} + 1\},$$
$$= \infty \text{ if no such } n. \qquad \ldots \quad (3.8)$$

Along the lines of Swanepoel and van Wyk (1982), it can be shown that N* and N are identically distributed random variables. Now, let the loss associated with (3.8), be given by

$$L_N = A(\bar{Z}_N - \sigma)^2 + cN. \qquad \ldots \quad (3.9)$$

We now claim that $E(L_N) = E(L_{N*})$.

To ascertain this claim, we need the following two Lemmas.

Lemma 3.1:

$$E\{(\hat{\sigma}_{N*} - \sigma)^2 | N^* = n\} = E\{(\bar{Z}_N - \sigma)^2 | N = n\}.$$

This result follows from Lemma 3.2 which we state and prove below.

Lemma 3.2: <u>For each</u> $x > 0$,

$$P\{\frac{2(n-1)\hat{\sigma}_n}{\sigma} \leq x \mid N^*=n\} = P\{2 \sum_{i=2}^{n} \frac{Z_i}{\sigma} \leq x \mid N=n\}.$$

<u>Proof</u>:

To prove Lemma 3.2, it suffices to show that for each $x > 0$,

$$P\{\frac{2(n-1)\hat{\sigma}_n}{\sigma} \leq x, \; N^*=n\} = P\{2 \sum_{i=2}^{n} \frac{Z_i}{\sigma} \leq x, \; N=n\}, \qquad \ldots \quad (3.10)$$

for n = m,m+1,....

Consider the left hand side of (3.10).

$$P\{\frac{2(n-1)\hat\sigma_n}{\sigma} \le x,\ N^*=n\}$$

$$= P\{\hat\sigma_n \le \frac{x\sigma}{2(n-1)},\ \hat\sigma_i > (i-1)(\frac{c}{A})^{\frac{1}{2}},\ i=m,\dots,n-1,\ \hat\sigma_n \le (n-1)(\frac{c}{A})^{\frac{1}{2}}\}.$$

Define $d_n = \frac{x\sigma}{2(n-1)}$, $c_i = (i-1)(\frac{c}{A})^{\frac{1}{2}}$, for $i=m,\dots,n$. Let $S_n^* = \sum_{i=1}^{n}(Y_i - Y_{n(1)})$ and S_n be as defined previously. Thus,

$$P\{\hat\sigma_n \le \frac{x\sigma}{2(n-1)},\ \hat\sigma_i > (i-1)(\frac{c}{A})^{\frac{1}{2}},\ i=m,\dots,n-1,\ \hat\sigma_n \le (n-1)(\frac{c}{A})^{\frac{1}{2}}\}$$

$$= P\{\hat\sigma_n \le d_n,\ \hat\sigma_m > c_m,\ \hat\sigma_{m+1} > c_{m+1},\ \dots,\ \hat\sigma_n \le c_n\}$$

$$= P\{\hat\sigma_m > c_m,\ \hat\sigma_{m+1} > c_{m+1},\ \dots,\ \hat\sigma_n \le \min(d_n,c_n)\}$$

$$= P\{\frac{S_m^*}{m-1} > c_m,\ \frac{S_{m+1}^*}{m} > c_{m+1},\ \dots,\ \frac{S_{n-1}^*}{n-2} > c_{n-1},\ \frac{S_n^*}{n-1} \le \min(d_n,c_n)\}$$

$$= P\{\frac{S_m}{m-1} > c_m,\ \frac{S_{m+1}}{m} > c_{m+1},\ \dots,\ \frac{S_{n-1}}{n-2} > c_{n-1},\ \frac{S_n}{n-1} \le \min(d_n,c_n)\}$$

$$= P\{\frac{1}{n-1}\sum_{j=2}^{n} Z_j \le \frac{x\sigma}{2(n-1)},\ \frac{1}{i-1}\sum_{j=2}^{i} Z_j > (i-1)(\frac{c}{A})^{\frac{1}{2}},\ i=m,\dots,n-1,$$

$$\frac{1}{n-1}\sum_{j=2}^{n} Z_j \le (n-1)(\frac{c}{A})^{\frac{1}{2}}\}$$

$$= P\{2\sum_{j=2}^{n}\frac{Z_j}{\sigma} \le x,\ N=n\}$$

$$= \text{R.H.S. of (3.10).}$$

Thus, Lemma 3.2 leads to Lemma 3.1 which in turn proves our claim that $E(L_N) = E(L_{N^*})$. With $c=1$, our (3.9) is the same as that proposed by Starr and Woodroofe (1972). Thus our sequential risk becomes

$$E(L_N) = E\{A(\bar{Z}_N - \sigma)^2\} + cE(N). \qquad \ldots (3.11)$$

In part (b) of the following theorem, we show that the "regret" has the order $O(c)$ as $c \to 0$.

Theorem 3.1: <u>For the sequential procedure in (3.6), we have</u>:

(a) $\lim_{c \to 0} \eta(c) = 1$, where $\eta(c) = \dfrac{E(L_{N*})}{E(L_{n*})}$;

(b) $W(c) \leq O(c)$ as $c \to 0$, where $W(c) = E(L_{N*}) - E(L_{n*})$.

To prove Theorem 3.1, we require the following lemmas. In view of Lemma 3.1 we keep on working with N and $E(L_N)$ instead of $N*$ and $E(L_{N*})$.

Lemma 3.3:

$E(N) - (n* - 1) \leq O(1)$ <u>as</u> $c \to 0$.

Lemma 3.4:

$E(N^2) \leq \{(n* - 1) + m\}E(N)$.

Lemma 3.5: <u>Let</u> $p \geq m$ <u>be an integer</u>. <u>Then</u>

$$P\{p \leq N \leq \frac{(n* - 1)}{2}\} \leq O\{\frac{1}{(n* - 1)^p}\} \text{ as } c \to 0.$$

Lemma 3.6: <u>For</u> $k \geq 1$,

$$E\{(N - n* + 1)^{2k}\} = O\{(n* - 1)^k\} \text{ as } c \to 0.$$

Lemma 3.7: <u>Let</u> $e_i = \dfrac{Z_i}{\sigma}$, $i = 1, 2, \ldots$. <u>The</u> e_i<u>'s are independent, and exponentially distributed with mean unity</u>. <u>Let</u>

$$q_k = k(\bar{e}_k - 1) = e_1 + e_2 + \ldots + e_k - k, \quad k \geq 1.$$

<u>Then</u>

(i) $E(q_N^2) = (n^* - 1) + O(1)$,

(ii) $E(q_N^3) \geq O(n^* - 1)$,

(iii) $E(q_N^4) \leq O(n^* - 1)^2$,

as $c \to 0$.

Lemma 3.8:

$$(n^* - 1)^2 \, E\{(\bar{e}_N - 1)^2\} \leq (n^* - 1) + O(1).$$

To prove Lemmas 3.3 - 3.8, we use basically the tools from Starr and Woodroofe (1972). We omit the details. Now we return to the proof of Theorem 3.1.

Proof of Theorem 3.1: To prove part (a), we first assume part (b). From (3.11), we can write

$$E(L_N) = c\{(n^* - 1)^2 \, E(\frac{\bar{Z}_N}{\sigma} - 1)^2 + E(N)\}.$$

Thus,

$$\eta(c) = \frac{E(L_N)}{E(L_{n^*})} = \frac{c\{(n^* - 1)^2 \, E(\frac{\bar{Z}_N}{\sigma} - 1)^2 + E(N)\}}{c(2n^* - 1)},$$

which implies that

$$\eta(c) - 1 = \frac{W(c)}{2n^* - 1} = \frac{O(c)}{O(\sqrt{c})}$$

as $c \to 0$. Hence,

$$\lim_{c \to 0} \eta(c) = 1.$$

Now, in proving part (b), we first notice from (3.5) and (3.11) that

$$W(c) = c(n^* - 1)^2 E\{(\bar{e}_N - 1)^2\} - cn^* + cE(N) - cn^* + c$$

$$= c\{(n^* - 1)^2 E\{(\bar{e}_N - 1)^2\} - n^* + E(N) - n^* + 1\}$$

$$= c\{(n^* - 1)^2 E\{(\bar{e}_N - 1)^2\} - (n^* - 1) + E(N) - n^*\}.$$

From Lemma 3.3, we have $E(N) - n^* \leq O(1)$ and by Lemma 3.8, we get $(n^* - 1)^2 E\{(\bar{e}_N - 1)^2\} - (n^* - 1) \leq O(1)$. Thus,

$$(n^* - 1)^2 E\{(\bar{e}_N - 1)^2\} - (n^* - 1) + E(N) - n^* \leq O(1).$$

This completes the proof of Theorem 3.1.

3.4 Moderate Sample Size Behavior of the Sequential Procedure

We have studied extensively the moderate sample size behavior of the procedure (3.6) proposed in section 3.3. The results reported were carried out on an IBM 3081D computer system with the help of the FORTRAN language and the WATFIV compiler.

For each row in Tables XX - XXV, we repeat the sequential rule (3.6) 500 times. We fix $\ell n(\theta) = 1$, $\sigma = 1$ and consider starting sample sizes $m = 2(1)5(5)15$. For each value of m, we take $n^* = 10,25,50(50)200$. We start with m samples from the population $f(y;\ 1,1)$, for each row and compute $\hat{\sigma}_m$. Next we check with the rule in (3.6) to see whether we stop or observe one more sample. When we stop, we record the observed value n(i) of N* together with the value $G_{n(i)}$, the observed value of $\hat{\sigma}_{n(i)}$, for the ith repetition in each row, i=1,2,...,500. Let us write

$$\bar{G} = \frac{1}{J} \sum_{i=1}^{J} G_{n(i)}, \quad s^2(\bar{G}) = \frac{1}{J(J-1)} \sum_{i=1}^{J} (G_{n(i)} - \bar{G})^2,$$

$$\bar{N}^* = \frac{1}{J} \sum_{i=1}^{J} n(i), \quad s^2(\bar{N}^*) = \frac{1}{J(J-1)} \sum_{i=1}^{J} (n(i) - \bar{N}^*)^2,$$

where $J = 500$, $\bar{G} = \hat{\sigma}$. In the following tables we write $SE(\bar{N}^*)$ and $SE(\bar{G})$ for the standard errors $s(\bar{N}^*)$ and $s(\bar{G})$ respectively.

3.4.1 Summary of Numerical Findings

With the exception of the case where $m = 2$, the estimated values $\bar{N}^*$ of $E(N^*)$ are very close to the corresponding values of n^*, the optimal fixed-sample size. As m increases, we can see that the "risk-efficiency" and the "regret" both approach the right limit, namely, one and zero respectively. This is in agreement with the conclusions in Theorem 3.1. The estimated values of σ are very close to one, the fixed-value of the shape parameter. In the absence of any prior information, a starting sample size of at least three seems to be a good choice.

TABLE XX

MODERATE SAMPLE SIZE BEHAVIOR OF THE SEQUENTIAL PROCEDURE (3.6): m = 2

n^*	$c \times 10^3$	$\bar{N}^* \times 10$	$SE(\bar{N}^*) \times 10^2$	$\bar{G} \times 10^2$	$SE(\bar{G}) \times 10^2$	$\hat{E}(L_{N^*}) \times 10^2$	$E(L_{n^*}) \times 10^2$	$\hat{\eta}(c) \times 10^2$	$\hat{W}(c) \times 10$
10	12.346	74.5200	17.4684	73.7770	1.9810	20.3111	23.4568	86.5895	-0.3145
25	1.736	216.0198	36.2605	86.7266	1.5181	7.9170	8.5069	93.0653	-0.0589
50	0.416	463.0999	53.7442	92.9450	1.1401	3.9696	4.1233	96.2727	-0.0153
100	0.102	969.2197	75.5594	97.1621	0.7639	1.9990	2.0304	98.4532	-0.0031
150	0.045	1468.9580	93.7276	98.0681	0.6300	1.3328	1.3468	98.9619	-0.0014
200	0.025	1964.4780	107.6917	98.3248	0.5421	0.9986	1.0075	99.1097	-0.0009

TABLE XXI

MODERATE SAMPLE SIZE BEHAVIOR OF THE SEQUENTIAL PROCEDURE (3.6): m = 3

n^*	$c\times10^3$	$\bar{N}^*\times10$	$SE(\bar{N}^*)\times10^2$	$\bar{G}\times10^2$	$SE(\bar{G})\times10^2$	$\hat{E}(L_{N^*})\times10^2$	$E(L_{n^*})\times10^2$	$\hat{\eta}(c)\times10^2$	$\hat{W}(c)\times10$
10	12.346	81.6400	15.0219	81.5855	1.7117	21.1901	23.4568	90.3368	-0.2266
25	1.736	231.5199	28.5642	93.3170	1.1913	8.1861	8.5069	96.2286	-0.0320
50	0.416	478.2397	40.7906	96.0973	0.8334	4.0327	4.1233	97.8020	-0.0090
100	0.102	982.8799	54.3054	98.5208	0.5500	2.0129	2.0304	99.1397	-0.0017
150	0.045	1485.3180	64.3090	99.1795	0.4313	1.3402	1.3468	99.5397	-0.0006
200	0.025	1985.0570	61.7284	99.3760	0.3092	1.0038	1.0075	99.6255	-0.0003

TABLE XXII

MODERATE SAMPLE SIZE BEHAVIOR OF THE SEQUENTIAL PROCEDURE (3.6): m = 4

n^*	$c\times10^3$	$\bar{N}^*\times10$	$SE(\bar{N}^*)\times10^2$	$\bar{G}\times10^2$	$SE(\bar{G})\times10^2$	$\hat{E}(L_{N^*})\times10^2$	$E(L_{n^*})\times10^2$	$\hat{\eta}(c)\times10^2$	$\hat{W}(c)\times10$
10	12.346	85.1999	13.8580	85.3039	1.5903	21.6691	23.4568	92.3789	-0.1787
25	1.736	234.2999	26.4763	94.3826	1.1052	8.2344	8.5069	96.7959	-0.0272
50	0.416	480.2197	37.7408	96.4737	0.7732	4.0409	4.1233	98.0020	-0.0082
100	0.102	987.8799	46.4985	99.0299	0.4699	2.0180	2.0304	99.3909	-0.0012
150	0.045	1488.4380	54.6730	99.3977	0.3672	1.3416	1.3468	99.6134	-0.0005
200	0.025	1983.7370	63.7886	99.3992	0.3205	1.0034	1.0075	99.5925	-0.0004

TABLE XXIII

MODERATE SAMPLE SIZE BEHAVIOR OF THE SEQUENTIAL PROCEDURE (3.6): $m = 5$

n^*	$c\times10^3$	$\bar{N}^*\times10$	$SE(\bar{N}^*)\times10^2$	$\bar{G}\times10^2$	$SE(\bar{G})\times10^2$	$\hat{E}(L_{N^*})\times10^2$	$E(L_{n^*})\times10^2$	$\hat{\eta}(c)\times10^2$	$\hat{W}(c)\times10$
10	12.346	83.2000	12.9804	88.4005	1.5168	22.0148	23.4568	93.8526	-0.1442
25	1.736	234.3199	25.5645	0.2332	1.0569	8.2347	8.5069	96.7999	-0.0272
50	0.416	483.7598	34.1965	0.1096	0.6954	4.0556	4.1233	98.3596	-0.0067
100	0.102	980.3979	46.9622	0.0452	0.4746	2.0182	2.0304	99.3990	-0.0012
150	0.045	1488.8780	56.8837	0.0310	0.3817	1.3418	1.3468	99.6281	-0.0005
200	0.025	1984.7980	61.4445	0.0221	0.3091	1.0075	1.0037	99.6190	-0.0003

TABLE XXIV

MODERATE SAMPLE SIZE BEHAVIOR OF THE SEQUENTIAL PROCEDURE (3.6): m = 10

n^*	$c\times10^3$	$\bar{N}^*\times10$	$SE(\bar{N}^*)\times10^2$	$\bar{G}\times10^2$	$SE(\bar{G})\times10^2$	$\hat{E}(L_{N^*})\times10^2$	$E(L_{n^*})\times10^2$	$(c)\times10^2$	$\hat{W}(c)\times10^2$
10	12.346	108.1799	6.1977	97.1936	1.2785	24.4667	23.4568	104.3052	1.0099
25	1.736	235.2198	23.5796	94.7467	0.9768	8.2503	8.5069	96.9837	-0.2566
50	0.416	483.4998	34.2271	97.1273	0.6954	4.0546	4.1233	98.3333	-0.0687
100	0.102	989.2197	45.2643	99.2129	0.4567	2.0194	2.0304	99.4583	-0.0110
150	0.045	1488.8380	55.7146	99.4353	0.3743	1.3418	1.3468	99.6267	-0.0050
200	0.025	1984.5580	63.9614	99.3375	0.3221	1.0037	1.0075	99.6130	-0.0039

TABLE XXV

MODERATE SAMPLE SIZE BEHAVIOR OF THE SEQUENTIAL PROCEDURE (3.6): m = 15

n^*	$c\times10^3$	$\bar{N}^*\times10$	$SE(\bar{N}^*)\times10^2$	$\bar{G}\times10^2$	$SE(\bar{G})\times10^2$	$\hat{E}(L_{N^*})\times10^2$	$E(L_{n^*})\times10^2$	$\hat{\eta}(c)\times10^2$	$\hat{W}(c)\times10^2$
10	12.346	150.1799	0.5952	100.0913	1.1272	29.6518	23.4568	126.4105	6.1951
25	1.736	237.1998	22.6289	95.3924	0.9595	8.2847	8.5069	97.3877	-0.2222
50	0.416	481.5198	33.1364	96.7479	0.6762	4.0463	4.1233	98.1333	-0.0770
100	0.102	992.5598	45.1159	99.4893	0.4562	2.0228	2.0304	99.6261	-0.0076
150	0.045	1486.1780	55.0275	99.2427	0.3694	1.3406	1.3468	99.5378	-0.0062
200	0.025	1987.2780	61.9275	99.4714	0.3111	1.0043	1.0075	99.6812	-0.0032

CHAPTER IV

ESTIMATION OF THE RATIO OF SCALE PARAMETERS OF TWO PARETO DISTRIBUTIONS

4.1 Introduction and Formulation of the Problem

In this chapter, we consider the problem of constructing confidence interval for the ratio of scale parameters of two Pareto distributions through two-stage and sequential procedures. This problem has not been discussed earlier. We may mention that the distribution of the quotient of two Pareto variates has been derived only recently by Pederzoli and Rathie (1980). We consider below two separate cases.

In section 4.2, we consider the case where the shape parameters are equal but unknown, and propose both two-stage and sequential procedures to solve our two-sample problem.

The case where the shape parameters are unequal and unknown is considered in section 4.3. As in section 4.2, we consider both two-stage and sequential procedures. Now, let us turn to the formulation of the problem.

Let $U_1, U_2, \ldots$ be a sequence of i.i.d. random variables with the p.d.f. $f(u;\theta_1,\sigma_1)$ and $V_1, V_2, \ldots$ be i.i.d. random variables with the p.d.f. $f(v;\theta_2,\sigma_2)$ where $f(x;\theta,\sigma)$ is defined in (1.1). Let us assume that the U's and V's are independent.

After observing $U_1, U_2, \ldots, U_m$ from the first population and V_1, V_2

$\ldots, V_n$ from the second population, we consider the following transformations of the sets of random variables:

$X_i = \ell n(U_i)$, and $Y_j = \ell n(V_j)$

for $i = 1,\ldots,m$, $j = 1,\ldots,n$.

These transformations will give us random variables $X_1, X_2, \ldots, X_m$ as i.i.d. with the p.d.f. $g(x;\mu_1,\sigma_1)$ and also $Y_1, Y_2, \ldots, Y_n$ as i.i.d. with the p.d.f. $g(y;\mu_2,\sigma_2)$, where $g(t;\mu,\sigma)$ is defined as in (2.1), and the X's and Y's are independent, $\mu_i = \ell n(\theta_i)$, $i = 1,2$. The maximum likelihood estimators of μ_1 and μ_2 are respectively

$$X_{m(1)} = \min(X_1, X_2, \ldots, X_m), \quad \text{and} \quad Y_{n(1)} = \min(Y_1, Y_2, \ldots, Y_n).$$

Now for $m \geq 2$, $n \geq 2$, the usual unbiased estimators of σ_1 and σ_2 are respectively,

$$\tilde{U}_m = \frac{1}{(m-1)} \sum_{i=1}^{m} (X_i - X_{m(1)}) \quad \text{and}$$

$$\tilde{V}_n = \frac{1}{(n-1)} \sum_{j=1}^{n} (Y_j - Y_{n(1)}).$$

Let d (> 1) and $\alpha \in (0,1)$ be two given predetermined constants. The problem we consider is to construct a fixed-ratio confidence interval for $\frac{\theta_1}{\theta_2} = \delta$, say. We require the confidence interval to be constructed in such a way that the confidence coefficient is at least $(1 - \alpha)$. Now, we propose to consider the confidence interval

$$I_{m,n} = \left[\frac{U_{m(1)}}{dV_{n(1)}}, \frac{dU_{m(1)}}{V_{n(1)}}\right]$$

for δ. It can be shown that the problem cannot be solved by any fixed sample size procedure where σ_1 and σ_2 are completely unspecified (Lehmann

(1950)). Therefore we propose suitable two-stage and sequential procedures to solve the problem. The procedures proposed are along the lines of Stein (1945, 1949), Ghosh (1975), Ghosh and Mukhopadhyay (1980), Ghurye (1958), Ghurye and Robbins (1954), Mukhopadhyay (1980, 1982a) and Mukhopadhyay and Hamdy (1984a).

As a matter of convenience, let us consider the following transformation. Let $\ell n(\delta) = \delta^*$. It then follows that $\ell n(\frac{\theta_1}{\theta_2}) = \ell n(\theta_1) - \ell n(\theta_2) = \mu_1 - \mu_2 = \delta^*$. Let δ^* be estimated by $T_{m,n} = X_{m(1)} - Y_{n(1)}$. Next we propose the interval

$$I^*_{m,n} = [T_{m,n} \pm d^*]$$

for δ^*, with $d^* = \ell n(d)$. It can be shown that $P\{\delta \in I_{m,n}\}$ is exactly equal to $P\{\delta^* \in I^*_{m,n}\}$ which is given by

$$\frac{(\frac{\sigma_1}{m})\{1 - \exp(-\frac{md^*}{\sigma_1})\} + (\frac{\sigma_2}{n})\{1 - \exp(-\frac{nd^*}{\sigma_2})\}}{\frac{\sigma_1}{m} + \frac{\sigma_2}{n}} . \qquad \ldots (4.1)$$

Now, we require that

$$P\{\delta^* \in I^*_{m,n}\} \geq 1 - \alpha. \qquad \ldots (4.2)$$

The problem of minimizing the total sample size $(m + n)$ subject to the restriction that (4.1) is at least $(1 - \alpha)$ seems to be impossible to solve analytically (Mukhopadhyay and Hamdy (1984a)). However, if we choose $m \geq C = \frac{a\sigma_1}{d^*}$ and $n \geq D = \frac{a\sigma_2}{d^*}$ with $a = \ell n(\frac{1}{\alpha})$, the expression in (4.1) is indeed at least $(1 - \alpha)$.

4.2 Equal but Unknown Shape Parameters

In this case, we assume that $\sigma_1 = \sigma_2 = \sigma$ with σ unknown. It is natural to take $m = n$, since we see that in this case we have $C = D =$

$\frac{a\sigma}{d^*}$. Now, we propose the interval

$$I_m = [\frac{U_{m(1)}}{dV_{m(1)}}, \frac{dU_{m(1)}}{V_{m(1)}}]$$

for δ. This leads to the interval

$$I_m^* = [X_{m(1)} - Y_{m(1)} \pm d^*]$$

for δ^*. Also note that

$$P\{\delta \in I_m\} = 1 - \exp(-\frac{md^*}{d^*}). \qquad \ldots \quad (4.3)$$

4.2.1 Two-Stage Procedure

We start with k (≥ 2) samples from each population and we define

$W_k = \frac{1}{2}(\tilde{U}_k + \tilde{V}_k)$, and let

$$M = \max\{k\ ,\ [\frac{b_k W_k}{d^*}] + 1\}, \qquad \ldots \quad (4.4)$$

where b_k is the upper 100α% point of the F-distribution with 2 and (4k - 4) degrees of freedom, and [y] is the largest integer smaller than y. As in Stein (1945, 1949), and Mukhopadhyay and Hamdy (1984a), we propose the interval I_M for δ. In the following theorem, we study some properties of the two-stage procedure (4.4).

Theorem 4.1: For the two-stage procedure (4.4), we have:

(i) $P\{\delta \in I_M\} \geq 1 - \alpha$

(ii) $\frac{b_k\sigma}{d^*} \leq E(M) \leq \frac{b_k\sigma}{d^*} + k$,

(iii) $\lim_{d\to 1+} P\{\delta \in I_M\} = 1 - \alpha$,

(iv) $\lim_{d \to 1+} E(\frac{M}{C}) = \frac{b_k}{a} \quad (>1)$,

where $C = \frac{a\sigma}{d^*}$.

Proof:

The proof of part (ii) follows by merely noting the following basic inequality from (4.4),

$$\frac{b_k W_k}{d^*} \leq M \leq \frac{b_k W_k}{d^*} + k, \qquad \ldots \quad (4.5)$$

and then taking the expectation throughout. Part (iv) follows from part (ii) after dividing all sides by C and then taking the limits of all sides as $d \to 1+$. Now, we proceed to the proof of part (i). From the definition of M, we see that M depends only on W_k. Since W_k is independent of $(X_{i(1)}, Y_{i(1)})$, $i = k, k+1, \ldots$, thus the event "$M = m$" is independent of $(X_{m(1)}, Y_{m(1)})$ for all fixed integers $m = k, k+1, \ldots$. It can be shown that $P\{\delta \in I_M\}$ is equal to $P\{\delta^* \in I_M^*\}$ which is given by

$$E\{1 - \exp(-\frac{Md^*}{\sigma})\}$$

$$\geq E\{1 - \exp(-\frac{b_k W_k}{\sigma})\}, \qquad \ldots \quad (4.6)$$

where (4.6) follows from the lower bound in (4.5). Now, we can write (4.6) as

$$E\{P(0 < Q \leq \frac{b_k W_k}{\sigma}) | W_k\}, \qquad \ldots \quad (4.7)$$

where Q has the p.d.f. $f(q;0,1)$ and Q is chosen independently of W_k. Note that $2Q \sim \chi_2^2$, $\frac{4(k-1)W_k}{\sigma} \sim \chi_{4(k-1)}^2$ and they are independent. Therefore, $Q\sigma/W_k \sim F_{2,4(k-1)}$. Thus, (4.7) leads to

$$E\{P(0 < \frac{Q\sigma}{W_k} \leq b_k)\}. \qquad \ldots \quad (4.8)$$

So, by the choice of b_k, (4.7) can indeed be shown as $(1-\alpha)$. This proves part (i). To prove part (iii), first notice from (4.5) that $\lim_{d\to 1+} (Md) = b_k W_k$ a.s.. Thus, by the dominated convergence theorem, we can conclude that

$$\lim_{d\to 1+} \{\delta \in I_M\} = E\{1 - \exp(-\frac{b_k W_k}{\sigma})\}.$$

Now retracing the previous proof we can verify part (iii). This completes the proof of Theorem 4.1.

4.2.2 Purely Sequential Procedure

In the literature, property (iv) of Theorem 4.1 is referred to as "asymptotic inefficiency" as in Chow and Robbins (1965) or "first-order asymptotic inefficiency" as in Ghosh and Mukhopadhyay (1981) or Mukhopadhyay (1982a). Our goal is to propose a purely sequential procedure which achieves the property that $\lim_{d\to 1+} E(\frac{M}{C}) = 1$. Now, for every $m \geq 2$, let $W_m = \frac{1}{2}(\tilde{U}_m + \tilde{V}_m)$ and we define a stopping variable

$$M = \inf\{m: \; m \geq k, \; m \geq \frac{aW_k}{d^*}\}, \qquad \ldots \quad (4.9)$$

where k (≥ 2) is the starting sample size. As in Chow and Robbins (1965) and Mukhopadhyay (1974), it is clear that M is a bonafide stopping rule. The following results can easily be derived as in Mukhopadhyay (1974).

$$\text{(i)} \quad \lim_{d\to 1+} (\frac{M}{C}) = 1 \quad \underline{\text{a.s.}}, \quad \lim_{d\to 1+} E(\frac{M}{C}) = 1, \qquad \ldots \quad (4.10)$$

$$\text{(ii)} \quad \lim_{d\to 1+} P\{\delta \in I_M\} = 1 - \alpha. \qquad \ldots \quad (4.11)$$

Theorem 4.2: For the sequential procedure in (4.9), we have as $d \to 1+$ and for $k \geq 2$:

(i) $\frac{(M - C)}{C^{\frac{1}{2}}} \xrightarrow{\mathcal{L}} N(0,1)$,

(ii) $E(M) = C + \gamma + 0.5 + o(1)$,

(iii) $P\{\delta \in I_M\} = 1 - \alpha + \frac{d^*\alpha}{\sigma}(\gamma + 0.5 - 0.5a) + o(d^*)$,

where γ is a real number and can be determined by using the basic tools from Woodroofe (1977), and $d^* = \ell n(d)$.

Before proving Theorem 4.2, we will establish the following facts needed for the proof. Let us define a new stopping variable M' as

$$M' = \inf(m: \ m \geq k, \ \sum_{i=1}^{m-1} Z_i \leq \frac{m(m-1)d^*}{a}\} \qquad \ldots (4.12)$$

where $Z_1, Z_2, \ldots$ are i.i.d. r.v.'s with the p.d.f. $\frac{4z}{\sigma^2} \exp(\frac{2z}{\sigma}) I(z>0)$. Now, we state the following two lemmas.

Lemma 4.1: (Swanepoel and van Wyk (1982)). The stopping variable M from (4.9) and M' from (4.12) are identically distributed.

Lemma 4.2: (Woodroofe (1977)). Suppose $E(Z_1^r) < \infty$ for some $r \geq 2$ and let $M^* = \frac{(M' - C)}{C^{\frac{1}{2}}}$. If $0 < s < r$ and $k > \frac{s}{2}$ then $\{|M^*|^s\}$ is uniformly integrable.

From Lemma 4.1, we see that it is sufficient to prove (i) - (iii) of Theorem 4.2 for M being replaced by M'.

Proof of Theorem 4.2:

To prove (i), we appeal to the theorem of Ghosh and Mukhopadhyay (1975), and it follows that $\frac{(M' - C)}{C^{\frac{1}{2}}} \xrightarrow{\mathcal{L}} N(0,1)$ as $d \to 1+$. Parts (ii) and (iii) follow from Theorem 2 of Mukhopadhyay and Hamdy (1984a), but after noting that $d^* \to 0$ as $d \to 1+$.

4.3 Unequal and Unknown Shape Parameters

Let us now assume that σ_1, σ_2 are both unknown and unequal. We consider taking unequal sample sizes m and n from the U's and V's, respectively. We propose the interval

$$I_{m,n} = [\frac{U_{m(1)}}{dV_{n(1)}}, \frac{dU_{m(1)}}{V_{n(1)}}]$$

for δ. Now, $P\{\delta \in I_{m,n}\}$ is given by (4.1).

4.3.1 Two-Stage Procedure

We start with k (≥ 2) samples from each population, and we define

$$M = \max\{k\ ,\ [\frac{g_k \tilde{U}_k}{d^*}] + 1\}, \qquad \ldots (4.13)$$

$$N = \max\{k\ ,\ [\frac{g_k \tilde{V}_k}{d^*}] + 1\}, \qquad \ldots (4.14)$$

where g_k is a suitable constant that depends only on k and α. Now we propose the interval $I_{M,N}$ for δ. We have defined g_k properly in Theorem 4.3

Let us define R with the p.d.f. f(r;0,1) to be independent of $\tilde{U}_k$, $\tilde{V}_k$. We notice that $W' = 2R$, $S = \frac{2(k-1)\tilde{U}_k}{\sigma_1}$, $T = \frac{2(k-1)\tilde{V}_k}{\sigma_2}$ are all independent χ^2_2, χ^2_{2k-2} and χ^2_{2k-2} respectively. Let

$$G = \frac{\min(S,T)}{\{(k-1)W'\}}\ .$$

Theorem 4.3: <u>For the two-stage procedure (4.13) - (4.14), we have</u>

$$P\{\delta \in I_{M,N}\} \geq 1 - \alpha,$$

<u>where the constant</u> g_k <u>is determined to satisfy the condition</u>:

$$P(\frac{1}{g_k} < G < \infty) = 1 - \alpha.$$

<u>Proof</u>:

First, we notice that $P\{\delta \in I_{M,N}\}$ is equal to $P\{\delta^* \in I^*_{M,N}\}$. Now we proceed as in Mukhopadhyay and Hamdy (1984a). From their Theorem 3 and our (4.1), it is easily seen that

$$P\{\delta^* \in I_{M,N}\}$$

$$= E\left\{\frac{(\frac{\sigma_1}{M})(1 - \exp(-\frac{Md^*}{\sigma_1})) + (\frac{\sigma_2}{N})(1 - \exp(-\frac{Nd^*}{\sigma_2}))}{(\frac{\sigma_1}{M} + \frac{\sigma_2}{N})}\right\}. \quad \ldots \quad (4.15)$$

Once we notice that the expression inside the expectation in (4.15) is a convex combination of $1 - \exp(-\frac{Md^*}{\sigma_1})$ and $1 - \exp(-\frac{Nd^*}{\sigma_2})$, then it follows that

$$P\{\delta^* \in I^*_{M,N}\}$$

$$\geq E\{\min\{1 - \exp(-\frac{Md^*}{\sigma_1}), 1 - \exp(-\frac{Nd^*}{\sigma_2})\}\}$$

$$= E\{1 - \exp(-\min(\frac{Md^*}{\sigma_1}, \frac{Nd^*}{\sigma_2}))\}. \quad \ldots \quad (4.16)$$

From (4.13) and (4.14), we have $M \geq \frac{g_k \tilde{U}_k}{d^*}$ and $N \geq \frac{g_k \tilde{V}_k}{d^*}$. Thus, $\min(\frac{Md^*}{\sigma_1}, \frac{Nd^*}{\sigma_2}) \geq g_k \min(\frac{\tilde{U}_k}{\sigma_1}, \frac{\tilde{V}_k}{\sigma_2})$. Therefore, from (4.16) we obtain

$$P\{\delta \in I_{M,N}\}$$

$$= P\{\delta^* \in I^*_{M,N}\}$$

$$\geq E\{1 - \exp\{- g_k \min (\frac{\tilde{U}_k}{\sigma_1}, \frac{\tilde{V}_k}{\sigma_2})\}\}$$

$$= E P\{ 0 < R < g_k \min (\frac{\tilde{U}_k}{\sigma_1}, \frac{\tilde{V}_k}{\sigma_2}) | \tilde{U}_k, \tilde{V}_k\} \quad \ldots (4.17)$$

$$= P\{0 < \frac{W'}{\min(S,T)} < \frac{g_k}{(k-1)}\}$$

$$= P\{\frac{1}{g_k} < G < \infty\}. \quad \ldots (4.18)$$

Hence, to implement the two-stage procedures (4.13) - (4.14), we determine g_k in such a way that $P\{\frac{1}{g_k} < G < \infty\} = 1-\alpha$, where R and G are as defined earlier. This completes the proof.

For various values of α and k, the tables in Krishnaiah and Armitage (1964), Gupta and Sobel (1962), Guttman and Milton (1969), and Mukhopadhyay and Hamdy (1984b) will enable us to find g_k.

4.3.2 Purely Sequential Procedure

In this case we define two stopping variables M and N as follows:

$$M = \inf\{m: \ m \geq k, \ m \geq \frac{a\tilde{U}_m}{d^*}\}, \quad \ldots (4.19)$$

$$N = \inf\{n: \ n \geq k, \ n \geq \frac{a\tilde{V}_n}{d^*}\}, \quad \ldots (4.20)$$

where k (≥ 2) is the starting sample size. Along the lines of Mukhopadhyay (1974), it can be shown that M and N are bonafide stopping times. When we stop, we propose the interval $I_{M,N}$ for δ.

Theorem 4.4: <u>For the procedure in (4.19) - (4.20), we have as</u> $d \to 1+$:

(i) $P\{\delta \in I_{M,N}\} \to 1 - \alpha$,

(ii) $E(M+N) = C + D + 2\gamma + 1 + o(1)$, where γ is a real number and can be determined as mentioned in Theorem 4.2.

We can also show that the following theorem holds for our procedure in (4.19) - (4.20).

Theorem 4.5: For the procedure in (4.19) - (4.20), we have as $d \to 1+$:

$$P\{\delta \in I_{M,N}\} \geq (1-\alpha)^2 + (1-\alpha)H^*d^*(\frac{1}{\sigma_1} + \frac{1}{\sigma_2}) + o(d^{*2}),$$

where $H^* = 0.253 + \frac{1}{2}a$, $d^* = \ell n(d)$.

We omit the proofs of Theorems 4.4 and 4.5 since they follow along the same lines as in Mukhopadhyay and Hamdy (1984a) with obvious modifications.

CHAPTER V

CONCLUSIONS

In this study, we considered several different problems in sequential estimation. First, we addressed the problem of estimating the scale and shape parameter of a Pareto distribution. We considered a sequential procedure for estimating the scale parameter pointwise assuming a general loss function. It has been shown that our procedure is asymptotically risk efficient. The exact distribution of N, our stopping variable, has been derived using Robbins' (1959) algorithm. With the help of the exact distribution of N, we examine some of the exact characteristics. From the numerical studies of our sequential procedure for moderate sample sizes, we notice that our proposed procedure performs very well.

Next, we have constructed a fixed-ratio confidence interval through two-stage, modified two-stage and sequential procedures for θ. Our numerical studies indicate that two-stage procedures perform well for moderate sample sizes. Specifically, our coverage probabilities are seen to be very close to the prescribed goal.

In estimating the shape parameter, we considered a purely sequential procedure, assuming a loss function of the form of squared error plus linear cost. Theoretically, we have shown that our "regret" is $O(c)$. Extensive numerical studies indicate that our procedure performs very satisfactorily even for moderate sample sizes.

In the second set of problems, we considered estimating the ratio of the scale parameters of two Pareto distributions through several two-stage and sequential procedures. We separately examined two cases; when the shape parameters are unknown but equal, and when the shape parameters are unknown and unequal. Only asymptotic properties of our procedures were obtained.

Throughout this study, we have derived theoretical results which are either more general than those already available in the literature or they are simply new findings. We have built a theoretical foundation for dealing with sequential estimation problems for Pareto distributions.

For practical applications, we recommend the modified two-stage procedures (2.20) - (2.21), with the choice of $\gamma = 0.05$, and the sequential procedure (2.7) with $m = 5$ in estimating the scale parameter. In estimating σ, we recommend using the sequential procedure (3.6), and in the absence of prior information, a starting sample size of at least three is suggested. For the two-sample problems of Chapter IV the starting sample size k is recommended to be taken as 5 or 10.

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