ON ℓ-HEREDITARY RINGS

By

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CHAPTER I

INTRODUCTION

In his paper "On algebras close to hereditary algebras" [3] Bautista introduces the artin algebras Λ satisfying the *) condition. Following his definition we will say that a (perfect) ring R <u>satisfies the *) condition</u> if given any pair of indecomposable projective left R-modules P and Q and given any R-homomorphism $\psi: P \to Q$ then either $\psi = 0$ or ψ is a monomorphism. Bautista himself ([3] and [4]) has studied the artin algebras satisfying the *) condition in connection with their representation theory. Also, Martinez-Villa [14] has studied and characterized the algebras which are stably equivalent to artin algebras satisfying the *) condition.

Azumaya [2] and Morita [15] have proved that there is a (Morita) duality between the category of finitely generated left R-modules ${}_R\mathcal{FM}$ and the category of finitely generated right R-modules $\mathcal{FM}_R$ if and only if R is left artinian and the indecomposable injective left R-modules are finitely generated. Examples of artinian rings, whose indecomposable left and right injective modules are finitely generated and which do not have self-duality have not been known until very recently [17]. On the other hand, the list of the classes of rings which are known to have self-duality is not very long, and includes artin algebras, Q-F rings, some serial rings [10] and incidence rings over the division rings [11].

Azumaya calls a ring R <u>exact</u> if R is left artinian and has a composition series of (two sided) ideals

$$_R R_R = I_0 \geq I_1 \geq \dots \geq I_n = 0$$

such that for each $i = 1,\dots,n$ every left endomorphism of I_{i-1}/I_i is given by right multiplication of an element of R. He has conjectured that these rings have self-duality.

In this work we will be mainly concerned with a particular class of artinian rings satisfying the *) condition. It will follow from [6] that these rings are exact.

In Chapter II we will study the projective and injective modules over our rings and will give a characterization of the ring in terms of them. In Chapter III we will verify Azumaya's conjecture for the rings we are studying and will extend results in [9] and [11] by using ring theoretic tools.

The rest of Chapter I is devoted to fix the notation and to introduce the most basic notions. We will use [1] as our basic reference.

For a ring R and a left R module M, a submodule K of M will be called <u>essential in M</u>, abbreviated $K \trianglelefteq M$, if for every submodule $L \leq M$, $K \cap L = 0$ implies $L = 0$. Dually, a submodule K of M will be called <u>superfluous</u> in M, abbreviated $K << M$, if for every submodule $L \leq M$, $K + L = M$ implies $L = M$.

If $N \leq M$ is a submodule of M we will say that $N' \leq M$ is an <u>M-complement of</u> N if N' is maximal with respect to $N \cap N' = 0$. In such a case $N \oplus N' \trianglelefteq M$. [1, Prop. 5.21]. We will say that $T \leq M$ is an <u>M-supplement of</u> N if T is minimal with respect to $N + T = M$.

If $(M_i)_{i \in I}$ is a family of R-modules we will denote by $\pi_k : \bigoplus_{i \in I} M_i \to M_k$ the natural projection onto the kth summand, ι_N (or ι if the context is clear) will denote the natural inclusion map $N \hookrightarrow M$ for $N \leq M$. Similarly, η_N (or η if the context is clear) will denote the natural epimorphism $M \to M/N$ for $N \leq M$.

For a ring R, $J = J(R)$ will be the Jacobson radical of R. Also, a set $\{e_1, \ldots, e_n\}$ of idempotents of R will be called basic if it is pairwise orthogonal and $\{Re_1, \ldots, Re_n\}$ is a complete irredundant set of representatives of the primitive left R-modules.

Finally, we recall that if R is a left perfect ring, then R has a basic set of idempotents $\{e_1, \ldots, e_n\}$; it follows that $Re_1/Je_1, \ldots, Re_n/Je_n$ includes exactly one copy of each simple left R-module. With R left perfect, we also have

$$\text{Rad}\, M = JM << {}_RM \quad \text{and} \quad \text{Soc}(N) = \ell_N(J) \trianglelefteq N_R.$$

CHAPTER II

ℓ-HEREDITARY RINGS

A well known theorem of Cartan and Eilenberg states that a ring R is left hereditary if and only if submodules of projective left R-modules are projective if and only if quotient modules of injective left R-modules are injective [16]. An entirely analogous result holds for right R-modules. In this chapter we will establish a similar characterization for artinian rings which are sums of distributive modules and also satisfy the *) condition. The problem of left and right modules will also be addressed.

Although we are mainly concerned with rings with minimum condition, that is, with artinian rings, we will be stating some early results in a more general setting. The existence of projective covers as well as the need of the relations $\operatorname{Rad} M = JM \ll M$ (and $\operatorname{Soc} N = \ell_N(J) \underline{\Delta} N_R$) make perfect rings the natural objects of our study. So, let R be a left perfect ring, J its (Jacobson) radical and $\{e_1,\ldots,e_n\}$ a basic set of primitive idempotents. It is known that $\{Re_i\}_{i=1}^n$ constitutes an irredundant list of representatives of the indecomposable projective left R-modules and $\{Re_i/Je_i\}_{i=1}^n$ an irredundant set of representatives of the simple left R-module.

If ${}_RP$ is any indecomposable projective then P/JP is simple and JP is the unique maximal submodule of P. We will call <u>local</u> a module with this property, that is, a module with a unique maximal submodule.

<u>Proposition 2.1</u>

Let R be a left perfect ring. A non-zero module ${}_RM$ is local iff M is the homomorphic image of an indecomposable projective R-module.

Proof: $\Leftarrow$) Let ${}_RP$ be an indecomposable projective module

$\psi: P \to M$ an epimorphism.

Then $M \cong P/\mathrm{Ker}\,\psi$ and hence, $(P/\mathrm{Ker}\,\psi)/J(P/\mathrm{Ker}\,\psi) \cong M/JM$, that is

$P/JP \cong M/JM$.

Therefore, M/JM is simple and JM is maximal, that is, M is local.

$\Rightarrow$) If M is local then M/JM is simple. Let

$P: Re_i \to M/JM$

be a projective cover and let

$\eta: M \to M/JM$

denote the canonical projection. Then there exists a homomorphism

$h: Re_i \to M$

such that

$$\begin{array}{ccc} & & Re_i \\ & \overset{h}{\swarrow} & \downarrow p \\ M & \xrightarrow[\eta]{} & M/JM \end{array}$$

commutes. The fact that $JM \ll M$ and that p is onto implies that h is onto.

<u>Definition 2.2</u>

We will say that a left R-module N is <u>colocal</u> if it has a unique minimal submodule, or equivalently if its socle is simple.

It is clear that the indecomposable injective left R-modules are colocal.

Dual to proposition 2.1 we have

<u>Proposition 2.3</u>

${}_R M$ is colocal if and only if there exists a monomorphism $\psi: M \to E$ with E an indecomposable injective R-module.

Proof: $\Leftarrow$) Let $\psi: M \to E$ be any non-zero homomorphism, E an indecomposable injective. Then we have that $E = E(S)$ with S a simple R-module.

Clearly,

$$S \ \underline{\Delta}\ (M)\psi \ \underline{\Delta}\ E$$

that is $(M)\psi \cong M/\mathrm{Ker}\,\psi$ has S as its unique minimal submodule. In particular, if ψ is 1-1, M is colocal.

$\Rightarrow$) Let M be a colocal R module, S its unique simple submodule. Then the following diagram commutes

$$\begin{array}{ccccc} & & E(S) & & \\ & & \uparrow & \nwarrow^{\psi} & \\ 0 & \longrightarrow & S & \xrightarrow[\iota]{} & M \end{array}$$

Moreover, since $S = \mathrm{Soc}(M)\ \underline{\Delta}\ M$, it follows that ψ is 1-1.

We now start to examine rings which satisfy the *) condition.

<u>Definition 2.4</u>

Let R be an artinian ring. We will say that R is <u>left</u> <u>ℓ-hereditary</u> if, given any pair of indecomposable projective left R-modules P and Q and any non-zero map $\psi: P \to Q$, ψ is monic.

ℓ-hereditary rings are then artinian rings satisfying the *) condition.

The name "ℓ-hereditary" for these rings is partially justified in the following.

Proposition 2.5

Let R be a perfect ring. R satisfies the *) condition iff local submodules of indecomposable projective R-modules are projective.

Proof: Let ${}_RQ$ be an indecomposable projective with $K \leq Q$ local. Then K/JK is simple. Let $p: Re_i \to K/JK$ be a projective cover and $\eta: K \to K/JK$ the natural projection. Then there exists a homomorphism $h: Re_i \to K \leq Q$ such that $h\eta = p$. But p is onto and $JK \ll K$, hence h is onto. Also, since Re_i is an indecomposable projective, we have that h is a monomorphism and hence an isomorphism.

$\Leftarrow$) Let ${}_RP, {}_RQ$ be indecomposable projective R-modules, $\psi: P \to Q$ a non-zero homomorphism. Then $(P)\psi$ is local and hence $(P)\psi$ is projective. Therefore

$$P \xrightarrow{\psi} (P)\psi \longrightarrow 0$$

splits and we can write $P \cong \mathrm{Im}\,\psi \oplus \mathrm{Ker}\,\psi$. But $\mathrm{Ker}\,\psi < JP \ll P$. Therefore

$$P \cong \mathrm{Im}\,\psi$$

via ψ and ψ is 1-1.

Before we state our next result, notice the following: Suppose that M is a local module; then M/JM being simple implies that M/JM and hence M are indecomposable.

Similarly, a dual argument shows that colocal R-modules are also indecomposable.

We can extend proposition 2.5 as follows.

Proposition 2.6

Let R be a perfect ring that satisfies the *) condition and P a projective R module. Then, if $M \leq P$ is local, then M is projective.

Proof: We can put $P = P_1^{(A_1)} \oplus \ldots \oplus P_n^{(A_n)}$ where each P_i is an indecomposable projective module. Let $\pi_i : P \to P_i$ denote the natural projection onto P_i and let $M_i = M\pi_i \leq P_i$. We claim that M_i is local. If $N = JM$ is the unique maximal submodule of M, then $N_i = N\pi_i$ is maximal in M_i, for suppose that $N_i \subsetneq L_i \leq M_i$ then $N = (N_i)\pi_i^{-1} \subsetneq (L_i)\pi^{-1} \subsetneq (M_i)\pi_i^{-1} = M$. Moreover, N_i is the unique maximal submodule of M_i for if $N_i' \leq M_i$, $N_i' \neq N_i$ is maximal, $(N_i')\pi_i^{-1}$ is maximal in M, and $N_i'\pi_i^{-1} \neq N$.

By hypothesis then, M_i is projective and the sequence

$$0 \longrightarrow \operatorname{Ker} \pi_i \longrightarrow M \xrightarrow{\pi_i} M_1 \longrightarrow 0$$

splits, i.e., $M \cong M_i \oplus \operatorname{Ker} \pi_i$ which is a contradiction, for M is local. Therefore $M \cong M_i$ and M_i is projective.

It is known that an artinian left hereditary ring is right hereditary and vice versa [ ]. As the example below shows, this is no longer true for arbitrary left (right) hereditary rings.

Example:

Let

$$R = \{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a \in \mathbb{Z},\ b,c \in Q \}.$$

We may describe R in a more compact form and write

$$R = \begin{pmatrix} \mathbb{Z} & Q \\ & Q \end{pmatrix}.$$

The right ideals of R are

$$I_0^{(n)} = \begin{pmatrix} n\mathbb{Z} & Q \\ 0 & Q \end{pmatrix} \quad , \quad I_1^{(n)} = \begin{pmatrix} n\mathbb{Z} & Q \\ 0 & 0 \end{pmatrix} \ , \quad \text{with } n \geq 0,\ n \in \mathbb{Z}$$

$$I_2 = \begin{pmatrix} 0 & Q \\ 0 & 0 \end{pmatrix} \quad , \quad I_3 = \begin{pmatrix} 0 & Q \\ 0 & Q \end{pmatrix}$$

$$I_4 = \begin{pmatrix} 0 & 0 \\ 0 & Q \end{pmatrix} \quad , \quad I_5^{(\lambda)} = \left\{ \begin{pmatrix} 0 & \lambda q \\ & q \end{pmatrix} : q \in Q \right\} \ , \quad \lambda > 0,\ \lambda \in Q.$$

It is easily checked that $I_0^{(n)}$, $I_1^{(n)}$, I_2, I_3, I_4 and $I_5^{(\lambda)}$ are projective right R-modules, that is, R is right hereditary. However, the left ideal

$$L = \begin{pmatrix} 0 & {}_{\mathbb{Z}}Q \\ 0 & 0 \end{pmatrix}$$

a projective left R-module. To see this, define

$$\beta: R \to \mathbb{Z}$$

$$\begin{pmatrix} k & p \\ & q \end{pmatrix} \to k.$$

Clearly, β is a ring homomorphism. Then every left $\mathbb{Z}$-module M becomes a left R-module by "extension of the scalars", that is, by defining

$$rm = \beta(r)m = km \quad \text{where} \quad r = \begin{pmatrix} k & p \\ & q \end{pmatrix}.$$

Let

$$\alpha: {}_{\mathbb{Z}}Q \to {}_{\mathbb{Z}}R$$

$$p \to \begin{pmatrix} 0 & p \\ & 0 \end{pmatrix}$$

Then, if we define $(mr)\alpha = (\beta(r)m)\alpha = (km)\alpha$, α becomes an R-homomorphism.

Moreover, since α is clearly a monomorphism we get that ${}_R Q \cong {}_R L$ via α. (We remark that ${}_R Q$ is obtained by extension of scalars.) But ${}_{\mathbb{Z}}Q$ is not $\mathbb{Z}$-projective and hence not R-projective.

For rings satisfying the *) condition we have,

<u>Proposition 2.7</u> [19]

Let R be a left perfect ring. Then R satisfies the *) condition on the right iff R satisfies the *) condition on the left.

Proof: Assume R satisfies the *) condition on the right, let Re, Re' be two indecomposable projective left R modules and let f: Re → Re' be a non-zero homomorphism. If f is not a monomorphism, let p: P → Ker f be projective cover with $P = P_1^{(A_1)} \oplus \dots \oplus P_m^{(A_m)}$ and $P_i \cong Re''$ an indecomposable projective; that is, there is a primitive idempotent e'' such that

$$Re'' \xrightarrow{p} Re \xrightarrow{f} Re' \quad \text{and} \quad pf = 0.$$

In other words, if for every g: Re'' → Re where e'' ∈ R is a primitive idempotent gf ≠ 0, then f is a monomorphism.

Then let g: Re'' → Re by any homomorphism and apply $\mathrm{Hom}_R(-,R)$ to

$$Re'' \xrightarrow{g} Re \xrightarrow{f} Re'$$

Then

$$\begin{array}{ccccc} \mathrm{Hom}(Re',R) & \xrightarrow{f^*} & \mathrm{Hom}(Re,R) & \xrightarrow{g^*} & \mathrm{Hom}(Re'',R) \\ \cong \updownarrow & & \updownarrow \cong & & \updownarrow \cong \\ e'R & \xrightarrow{\hat{f}} & eR & \xrightarrow{\hat{g}} & e''R \end{array}$$

where the vertical arrows are the natural isomorphisms and $\hat{f}$ and $\hat{g}$ are the homomorphisms making the diagram commute. By hypothesis $\hat{g}$ is monic and hence $\hat{f}\hat{g} \neq 0$. Hence $f^*g^* = (fg)^* \neq 0$ and consequently $fg \neq 0$.

The remark below, although easy to establish will be very useful.

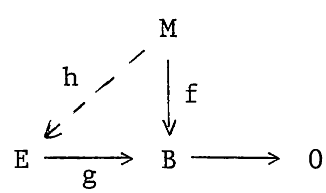
Remark

If Q is an R-module such that every colocal factor of Q is injective then every non-zero map $\phi: Q \to E$ into an indecomposable injective is an epimorphism. To see this, let $\phi: Q \to E$ be a non-zero map then $(Q)\phi \leq E$ and since E has a unique simple submodule so does $(Q)\phi \cong Q/\text{Ker}\,\phi$. Hence $(Q)\phi$ is injective and $(Q)\phi = Q$.

The next lemma is a projectivity test for local modules over semiprimary rings. Later in the sequel we will state an improved version under more restrictive conditions for the ring, which will be an essential tool in proving the main result of this chapter.

Lemma 2.8

Let R be a semiprimary ring and let ${}_RM$ be a local module. M is projective iff given the solid part of the diagram



with E an indecomposable injective R module, there exists a homomorphism $h: M \to E$ which makes the diagram commute.

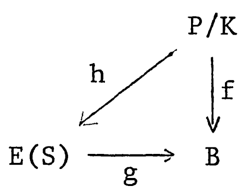
Proof: $\Leftarrow$) We may assume that M is a factor of an indecomposable projective module P with $\rho: P \to M$ the natural epimorphism. If $k = \text{Ker}\,\rho \neq 0$, let S be a simple submodule of P contained in $\text{Ker}\,\rho$. Pick k so that

$$S \cap J^k P = S, \quad S \cap J^{k+1} P = 0$$

Let $\eta: P \to P/J^{k+1}P$ be the natural epimorphism. Let $\iota: S \to E(S)$ be the injective envelope of S. Since η/S is a monomorphism, there exists a map

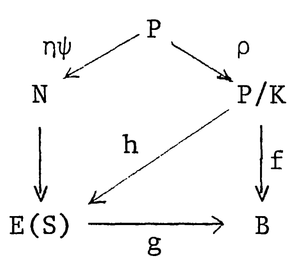
$$\psi: P/J^{k+1}P \to E(S) \quad \text{such that} \quad \iota = \eta\psi/S.$$

Let $N = \text{im}\eta\psi$ and let $B = E/JN$. Let $f: P/K \to B$ be given by $(x+K)f = (x)\eta\psi + JN$ and let $g: E(S) \to B$ be the natural epimorphism. Then by hypothesis, there exists $h: P/K \to E$ such that



commutes.

We claim that $\text{Im}h \subseteq N$. The diagram below commutes:



Let $P = Re$, $e \in R$ a primitive idempotent. We have that

$$\begin{aligned} (e)\eta\psi + JN &= e\eta\psi g = e\rho f \\ &= (e+K)f \\ &= (e+K)hg \\ &= (e+K)h + JN. \end{aligned}$$

Therefore, (*) $e\eta\psi - (e+K)h \in JN$ and hence $(e+K)h \in N$, for $e\eta\psi \in N$.

Therefore, $\operatorname{Im} h \leq N$ as desired.

Now, since P is projective, there exists an endomorphism α of P such that $\alpha\eta\psi = \rho h$. From (*) we have that

$$(P)(1_p - \alpha)\eta\psi = (P)(\eta\psi - \rho h) \leq JN, \quad \text{and hence}$$

$$(P)(1_p - \alpha) \leq JP.$$

Consequently,

$$(1_p - \alpha) \in J(\operatorname{End}({}_R P)).$$

Write $\alpha = 1_p + \beta$ with $\beta \in J(\operatorname{End}({}_R P))$. Let $0 \neq s \in S$. Then

$$\begin{aligned} 0 &= s\rho h = s\alpha\eta\psi \\ &= s\eta\psi + s\beta\eta\psi = s\iota + s\beta\eta\psi. \end{aligned}$$

But $\beta \in J\operatorname{End}({}_R P)$ implies that $s\beta \in J J^k P = J^{k+1} P$. Consequently, we have that $s\beta\eta = 0$ and hence $s\iota = 0$ which is a contradiction. We then conclude that $K = \operatorname{Ker} \rho = 0$.

Now we state the corresponding dual result. The following remark will be useful in proving it.

<u>Lemma 2.9</u>

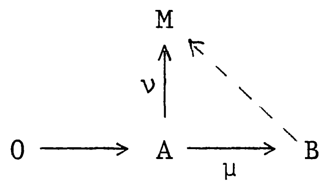
Let ${}_R M$, $z \in \operatorname{Soc}_k(M)$ and let $\beta \in J\operatorname{End}({}_R E)$. Then, $(z)\beta \in \operatorname{Soc}_{k-1}(M)$.

Proof: Let $r \in J^{k-1}$, $j \in J$. Then $jr \in J^k$ and $0 = (jr)z = j(rz)$, that is $rz \in \operatorname{Soc}(E)$. But $J(\operatorname{End}({}_R E)) = r_{\operatorname{End}({}_R E)}(\operatorname{Soc} E)$ and hence $0 = (rz)\beta = rz\beta$, i.e., $(z)\beta \in \operatorname{Soc}_{k-1}(E)$.

Now we prove the promised dual result

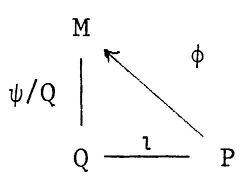
Lemma 2.10

Let R be a semiprimary ring and let $_RM$ be a colocal module. M is injective iff given the solid part of the diagram



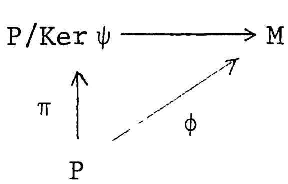
with B an indecomposable projective R module, there exists a homomorphism w: B → M which makes the diagram commute.

Proof: ⇐) Let M be an R-module satisfying the hypothesis, and assume M is not injective. Let T = Soc(M) and let E = E(M). Then E = E(T), and since T is simple, E is indecomposable. Let L be a maximal submodule of E containing M and let k be an integer, $0 \leq k \leq$ Loewy length (E) such that $\mathrm{Soc}_k(E) + L = E$ and $\mathrm{Soc}_{k-1}(E) + L = L$. Let p: P → E/L be the projective cover of E/L and let η: $\mathrm{Soc}_k(E) \to E/L$ be the canonical epimorphism. Then there exists a homomorphism ψ: $P \to \mathrm{Soc}_k(E)$ such that $\psi\eta = p$. Let Q = $(T)\psi^{-1}$; by assumption there exists ϕ: P → M such that the diagram



commutes.

Notice that $\mathrm{Ker}\,\psi = (0)\psi^{-1} \leq (T)\psi^{-1} = Q$; that is, $\mathrm{Ker}\,\psi \cap Q \leq \mathrm{Ker}\,\phi$. Hence, there exists h: $P/\mathrm{Ker}\,\psi \to M$ such that



commutes. (1)

Let $f\colon P/\mathrm{Ker}\,\psi \to \mathrm{Soc}_k(E)$ be the quotient map of ψ. Then, by (1) and since E is injective, there exists

$$\alpha \in \mathrm{End}({}_R E) \quad \text{such that} \quad h\alpha = f \tag{2}$$

Hence, $\pi h\alpha = \pi f$, so $\phi\alpha = \psi$ and $\iota\phi\alpha = \iota\psi$. That is,

$$\psi\alpha/Q = \psi/Q. \tag{3}$$

Let $t \in T$; then $t = x\psi$ for some $x \in Q$. Applying (2) gives $x\psi\alpha = t\alpha = x\psi = t$; that is, the restriction of α to T is the identity map. This implies that α is monic, for T is the unique simple submodule of E.

Moreover, since E is indecomposable, α is also epic and hence an isomorphism which fixes T elementwise. Let α' be the inverse isomorphism of α. Then α' also fixes T elementwise and $h = f\alpha'$. Hence

$$\phi = \psi\alpha'.$$

Let $\beta = \alpha' - 1_E$. Then $\mathrm{Ker}\,\beta$ contains T and consequently $\mathrm{Ker}\,\beta$ is essential in E. Then $\beta \in J(\mathrm{End}({}_R E))$ (see for example [1] 18.20).

We can write:

$$\phi = \psi(\beta + 1) = \psi\beta + \psi$$

$$\phi\eta = \psi\beta\eta + \psi\eta.$$

But $E\phi \leq M \leq L$, hence $\phi\eta = 0$, and we have

$$\psi\beta\eta + p = 0.$$

Let $x \in P$. Then $x\psi \in \mathrm{Soc}_k(E)$ and by Lemma 2.9, $p = 0$, which is a contradiction.

Now we start placing restrictions on our ring. Some definitions are in order.

Definition 2.11

Let R be an arbitrary ring.

a) A left R-module M is uniserial if the lattice of submodules $\mathcal{S}(M)$ of M is a chain, that is, for any submodules A and B of M either $A \subseteq B$ or $B \subseteq A$.

b) A left R-module M is distributive if the lattice of submodules $\mathcal{S}(M)$ of M is distributive, that is, for any submodules A, B, C of M we have $A \cap (B + C) = (A \cap B) + (A \cap C)$.

Proposition 2.12 ([5], [7], [9])

Let R be a semiperfect ring. The following are equivalent.

1) ${}_RM$ is distributive.

2) Every quotient module of M has at most one copy of every simple submodules in its socle.

3) For each primitive indecomposable projective P the set of submodules $\{\mathrm{Im}\gamma\colon \gamma \in \mathrm{Hom}(P,M)\}$ is linearly ordered.

3') For each simple left R-module T, the set of submodules $\{\mathrm{Ker}\,\gamma\colon \gamma \in \mathrm{Hom}(M,E(T)\}$ is linearly ordered.

4) For each primitive idempotent $e \in R$ the left eRe-module eM is uniserial.

4') For each simple left R-module T, the right $\mathrm{End}({}_RE(T))$-module $\mathrm{Hom}_R(M,E(T))$ is uniserial.

Definition 2.13

Let R be a left (right) perfect ring, $\{e_i\}_{i=1}^n$ a basic set of primitive idempotents of R. R is <u>left</u> (<u>right</u>) <u>semidistributive</u> if the left (right) R-modules Re_i (e_iR) are distributive.

A perfect ring R is <u>semidistributive</u> if it is both left and right semidistributive.

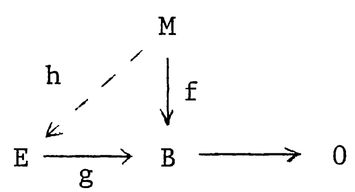
<u>Proposition 2.14</u> [8]

If R is an artinian semidistributive ring then the indecomposable injective R-modules are all distributive.

As promised earlier, a result similar to (2.8) is now proved.

<u>Lemma 2.15</u>

Let R be a semidistributive artinian ring. A local left R-module M is projective if and only if, given the solid part of the diagram



with E an indecomposable injective, B a colocal factor module of E and g the natural epimorphism, there exists a homomorphism h: $M \to E$ that completes the diagram commutatively.

Proof: $\Leftarrow$) We may assume that M is a factor of an indecomposable projective module P with ρ: $P \to M$ the natural epimorphism. If $K = \mathrm{Ker}\,\rho \neq 0$, let S be a simple submodule of P contained in $\mathrm{Ker}\,\rho$. Pick an integer k so that $S \cap J^kP = S$, $S \cap J^{k-1}P = 0$. Let η: $P \to P/J^{k+1}P$ be the natural epimorphism and let ι: $S \to E(S)$ be the injective envelope of S. Since η/S is a monomorphism, there exists a map ψ: $P/J^{k+1}P \to E(S)$ such that $\iota = \eta\psi/S$ Let $N = \mathrm{im}\,\eta\psi$ and let L/JN be a complement of N/JN in E/JN so that $(N+L)/L \cong N/(N\cap L) = N/JN$ is an essential simple submodule of $B = E/L$.

Then B is colocal (see [1] 5.21). Let $f\colon P/K \to B$ be given by $(x+K)f = x\eta\psi + L$, f is well-defined because $Kf \leq (JP)f = JN \leq L$. Let $g\colon E(S) \to B$ be the natural epimorphism. By hypothesis, there exists $h\colon P/K \to E(S)$ so that $f = hg$. Since R is semidistributive, E(S) is distributive, so the set of $\{Im\gamma\colon \gamma \in Hom(P,E)\}$ is linearly ordered under inclusion. Hence either $imh \leq N = im\eta\psi$ or $N \leq imh$. We claim that $imh = N$. First, imh is not strictly contained in N, for otherwise $imh \leq JN \leq L$, so that $0 = hg = f$, a contradiction. Hence $imh \geq N$. Since $imh + L = N + L$ we have

$$imh/JN = (imh \cap (N+L))/JN = N/JN \oplus (imh \cap L)/JN$$

by modularity. But imh/JN is local and hence indecomposable, so $imh \cap L = JN$ and $imh = N$ as claimed.

Now, since P is projective, there exists an endomorphism α of P such that $\alpha\psi\eta = \rho h$.

Since

$$JN \geq (P)(\eta\psi - \rho h) = (P)(1_p - \alpha)\eta\psi$$

we have that

$$(P)(1_p - \alpha) \leq JP.$$

Consequently,

$$1_p - \alpha \in J(End({}_R P)).$$

Let $\alpha = 1_p + \beta$ with $\beta \in JEnd({}_R P)$. Let $0 \neq s \in S$. Then

$$0 = s\rho h = s\alpha\eta\psi$$
$$= s\eta\psi + s\beta\eta\psi = s\iota + s\beta\eta\psi$$

Since $\beta \in J(\mathrm{End}({}_R P))$, $s\beta \in JJ^k P = J^{k+1}P$. Consequently we have that $s\beta\eta = 0$ and hence $s\iota = 0$ which is a contradiction. Therefore $K = 0$ and M is projective.

The condition of M being local in Lemma 2.15 is necessary as the next example shows.

Example:

Let D be an arbitrary division ring. Let

$$R = \left\{ \begin{pmatrix} a & x & y \\ & b & 0 \\ & & c \end{pmatrix} : \quad a,b,c,x,y \in D \right\}.$$

The ring structure of R is the one obtained by considering R as a subring of the ring of the 3×3 matrices over D. Let

$$M = \left\{ \begin{pmatrix} u \\ v \\ z \end{pmatrix} : u,v,z \in D \right\}.$$

Then M can be given an R-module structure by restriction of scalars. Let

$$P_1 = \left\{ \begin{pmatrix} u \\ 0 \\ 0 \end{pmatrix} : u \in D \right\}, \quad P_2 = \left\{ \begin{pmatrix} u \\ v \\ 0 \end{pmatrix} : u,v \in D \right\} \quad \text{and} \quad P_3 = \left\{ \begin{pmatrix} u \\ 0 \\ z \end{pmatrix} : u,z \in D \right\}.$$

Then $P_i \cong Re_i$, $i = 1,2,3$, where $e_{ij} \in R$ is the matrix with 1 in the (i,j) position, zero everywhere else and $e_i = e_{ii}$. The map $p\colon P_2 \oplus P_3 \to M$

$$\left(\begin{pmatrix} u \\ v \\ 0 \end{pmatrix}, \begin{pmatrix} u' \\ 0 \\ z \end{pmatrix} \right) \longrightarrow \begin{pmatrix} u - u' \\ -v \\ z \end{pmatrix}$$

is a projective cover and M is not projective. Also, P_2 and P_3 are maximal in M, so M is not local.

If ${}_RS$ is simple and ${}_RK$ is any colocal (and hence indecomposable) module we claim that the solid part of the diagram

$$\begin{array}{ccccc} & & M & & \\ & \swarrow & \downarrow & & \\ E(S) & \longrightarrow & K & \longrightarrow & 0 \end{array}$$

can be commutatively completed. Let $S_i \cong Re_i/Je_i$, $i = 1,2,3$, then $E(S_i) \cong M/L_i$ where

$$L_i = \sum_j \{Dm_j : e_{ij} = 0\} = \sum \{Rm_j : e_k Re_j = 0\} \qquad (*)$$

and $m_j = (\delta_{j,k})_k \in M$ [9]. Then $L_1 = 0$ so $E(S_1) = M$, and given

$$\begin{array}{ccc} & & M \\ & \overset{f'}{\swarrow} & \downarrow \phi \\ M & \underset{\eta}{\longrightarrow} & M/N \end{array}$$

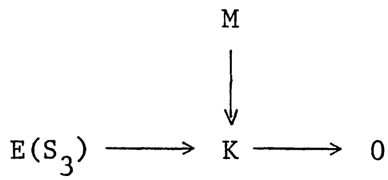
with M/N colocal, there exists a unique map $f'\colon M \to M \ni f'\pi = \psi$ (see [11] Lemma 3), $L_2 = Rm_3 + Rm_1 = P_3$ so $E(S_2) \cong M/P_3$ which is simple (isomorphic to S_2), that is $E(S_2)$ has no submodules other than the trivial ones from which we infer that

$$\begin{array}{ccccc} & & M & & \\ & & \downarrow & & \\ E(S_2) & \longrightarrow & K & \longrightarrow & 0 \end{array}$$

can always be commutatively completed. $L_3 = Rm_2 + Rm_1 = P_2$, then

$$E(S_3) \cong M/L_3 = M/P_2.$$

But $M/P_2 \cong S_3$ and the diagram



can (trivially) be completed.

<u>Proposition 2.16</u>

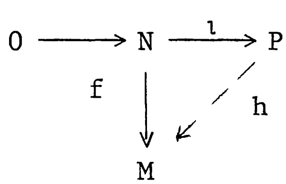
Let R be a perfect ring and let M be an R-module. If $N \leq M$ is a maximal submodule, then there exists a local submodule K of M such that $M = K + N$.

Proof: Let K be a supplement of N in M, that is, a submodule L which is minimal with respect to $N + L = M$. Then [12] $K \cap N \ll K$. Hence $K \cap N \leq JK$. We have $M/N = (K + N)/N \cong K/(K \cap N)$ which implies that $K/(K \cap N)$ is simple. Then the map $h: K/(K \cap N) \to K/JK$ defined by $(t + K \cap N)h = t + JK$ is an isomorphism. Hence K is local.

Dual to Lemma 2.15 we have

<u>Lemma 2.17</u>

Let R be a semidistributive semiprimary ring. A colocal left R-module M is injective if and only if given the solid part of the diagram

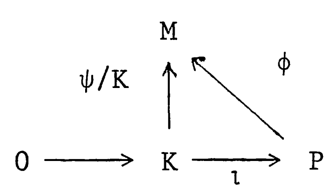


with P an indecomposable projective, N a local submodule of P and ι the natural inclusion, there exists a homomorphism $h\colon P \to M$ that completes the diagram commutatively.

Proof: $\Leftarrow$) Assume M is not injective. Let $T = \mathrm{Soc}(M)$, let $E = E(T)$ be the injective envelope of T. Then $E = E(M) = E(T)$ is an indecomposable injective.

Let ℓ denote the Loewy length of E, let $L \leq E$ be a maximal submodule of E containing M and let $k \leq \ell$ be an integer such that $\mathrm{Soc}_k(E) + L = E$, $\mathrm{Soc}_{k-1}(E) + L = L$. Let $\lambda\colon P \to E/L$ be the projective cover of E/L and let $\pi\colon E \to E/L$ be the natural epimorphism. Then there exists a homomorphism $\psi\colon P \to \mathrm{Soc}_k(E)$ such that $\psi\pi = \lambda$.

Let $Q = (T)\psi^{-1} \leq P$. Since $\mathrm{Ker}\,\psi \leq Q$, ψ/Q induces an isomorphism from $Q/\mathrm{Ker}\,\psi$ onto T, consequently, $\mathrm{Ker}\,\psi$ is maximal in Q, and there exists $K \leq Q$, K a local submodule, such that $K + \mathrm{Ker}\,\psi = Q$ (Prop. 2.16). By assumption, there exists a homomorphism $\phi\colon P \to M$ such that the diagram



(1)

commutes. This implies that

$$K \cap \mathrm{Ker}\,\phi = \mathrm{Ker}\,\psi \cap K. \qquad (*)$$

Since P is distributive and E is colocal we conclude ([18], Prop. 2.3) that either $\mathrm{Ker}\,\phi \leq \mathrm{Ker}\,\psi$ or $\mathrm{Ker}\,\psi \leq \mathrm{Ker}\,\phi$. We claim that $\mathrm{Ker}\,\psi = \mathrm{Ker}\,\phi$. First, if $\mathrm{Ker}\,\psi \leq \mathrm{Ker}\,\phi$, then, since $\mathrm{Ker}\,\psi$ is maximal in Q and $(P)\psi \cong P/\mathrm{Ker}\,\psi$ is colocal (Prop. 2.3) we see that

$$Q/\mathrm{Ker}\,\psi \leq \mathrm{Ker}\,\phi/\mathrm{Ker}\,\psi \leq P/\mathrm{Ker}\,\psi.$$

Hence, $\phi/Q = 0$. But $\phi/K = \psi/K$ and $\psi/K = 0$ which is a contradiction. Assume then, that $\operatorname{Ker}\phi \leq \operatorname{Ker}\psi$. From (*) and by modularity,

$$\operatorname{Ker}\phi = \operatorname{Ker}\phi + (\operatorname{Ker}\psi \cap K) = \operatorname{Ker}\psi \cap (\operatorname{Ker}\phi + K).$$

Hence,

$$Q/\operatorname{Ker}\phi = Q/(\operatorname{Ker}\psi \cap (\operatorname{Ker}\phi + K)) \cong Q/\operatorname{Ker}\psi \oplus Q/(\operatorname{Ker}\phi + K).$$

Since $Q/\operatorname{Ker}\phi$ is colocal, we conclude that $Q/(\operatorname{Ker}\phi + K) = 0$. But

$$Q/(\operatorname{Ker}\phi + K) = (\operatorname{Ker}\phi + \operatorname{Ker}\psi + K)/(\operatorname{Ker}\phi + K)$$

$$\cong \operatorname{Ker}\psi/((\operatorname{Ker}\phi + K) \cap \operatorname{Ker}\psi)$$

$$= \operatorname{Ker}\psi/\operatorname{Ker}\phi.$$

So, $\operatorname{Ker}\phi = \operatorname{Ker}\psi$ as claimed. Let $\bar{\psi}: P/\operatorname{Ker}\psi \to \operatorname{Soc}_k(E)$ be the monomorphism induced by ψ and let $\bar{\phi}: P/\operatorname{Ker}\phi \to M$ be the monomorphism induced by ϕ. Then there exists $\alpha \in \operatorname{End}({}_R E)$ such that $\bar{\psi}\alpha = \bar{\phi}$. It is then clear that α is a monomorphism; moreover, since E is indecomposable α is an isomorphism and $\psi\alpha = \phi$. Hence $\psi\alpha/K = \phi/K = \psi/K$.

Let $t \in T$, then $t = x\psi$ for $x \in K$

$$(t)\alpha = (x)\psi\alpha = (x)\psi = t.$$

Hence, $\alpha/T = 1_T$. Let $\beta = 1 - \alpha$, since $\operatorname{Ker}\beta \geq T$, $\operatorname{Ker}\beta \trianglelefteq E$ and $\beta \in J(\operatorname{End}_R E)$ we can write $\alpha = 1 + \beta$, $\beta \in J(\operatorname{End}({}_R E))$. $\bar{\psi}\alpha = \bar{\phi}$ implies $\bar{\psi}(1 + \beta) = \bar{\phi}$. But $\bar{\phi}\lambda = 0$. Hence $\bar{\psi}\lambda + \bar{\psi}\beta\lambda = 0$. But $x\bar{\psi}\beta \in \operatorname{Soc}_{k-1}(E)$ for $x \in P/\operatorname{Ker}\phi$. So $0 = x\bar{\psi}\beta\lambda$ and $\bar{\psi}\lambda = 0$ which is a contradiction.

Theorem 2.18

Let R be a semidistributive artinian ring. The following statements are equivalent.

1) R is left ℓ-hereditary.

2) Local submodules of (indecomposable) projective left R-modules are projective.

3) Colocal factor modules of (indecomposable) injective left R-modules are injective.

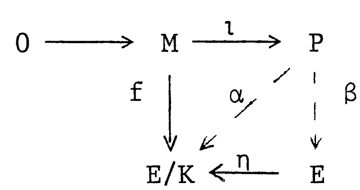
4) Nonzero maps between indecomposable injective left R-modules are epimorphisms.

Moreover, these statements are equivalent to those formed when left is replaced by right.

Proof:

1) $\Leftrightarrow$ 2) Propositions 2.5 and 2.6.

3) $\Rightarrow$ 2) Let E/K be a colocal factor module of an indecomposable injective left R-module E. Let P be an indecomposable projective left R-module with local submodule M. Consider the following diagram where ι: $M \to P$ is the inclusion map and η: $E \to E/K$ the natural epimorphism.



Since E/K is injective, then there exists a homomorphism α: $P \to E/K$ such that $\iota\alpha = f$. Since P is projective, there exists a homomorphism β: $P \to E$ such that $\beta\eta = \alpha$. Let $h = \iota\beta$; from Lemma 2.15, we see that M is projective.

2) $\Rightarrow$ 3) Let M be a local submodule of an indecomposable projective R-

module P. Let E be an indecomposable injective R-module, E/K a colocal factor of E. Consider the following diagram, where η: E $\to$ E/K is the natural epimorphism and ι: M $\to$ P the inclusion map.

$$\begin{array}{ccccc} E & \xrightarrow{\eta} & E/K & \longrightarrow & 0 \\ & & \uparrow f & & \\ P & \xleftarrow[\iota]{} & M & \longleftarrow & 0 \end{array}$$

By assumption M is projective and hence, there exists a homomorphism α: M $\to$ E such that $\alpha\eta = f$. The injectivity of E implies the existence of a homomorphism β: P $\to$ E such that $\iota\beta = \alpha$. Let $g = \beta\eta$. Then, by Lemma 2.17 we see that E/K is injective.

3) $\Rightarrow$ 4) Let $\tilde{Q}$ be an arbitrary injective R-module, Q an indecomposable injective R-module. Let ϕ: $\tilde{Q} \to Q$ be a nonzero homomorphism. Then, by assumption, $\tilde{Q}\phi \cong \tilde{Q}/\mathrm{Ker}\,\phi$ is injective and $Q \cong (Q)\phi \oplus L$, some R-module L. But Q being indecomposable implies that ϕ is onto.

4) $\Rightarrow$ 3) Let E be an indecomposable injective R-module, K a submodule such that E/K is colocal. Then Soc(E/K) is simple and E(E/K) is an indecomposable injective R-module. Let η: E $\to$ E/K be the natural epimorphism, ι: E/K $\to$ E(E/K) the natural inclusion. Let $\alpha = \eta\iota$. By assumption α is onto and hence so is ι. Then E/K is injective.

Finally, the dual result to Proposition 2.5 shows that 4) implies the non-parenthetical version of 3).

We close this chapter by showing some examples of semidistributive ℓ-hereditary rings.

1. Let D be a division ring and let $(X,\leq)$ be a finite ordered set. Define

$$R = \{ \sum_{i \leq j} d_{ij}\ x_{ij}, \qquad i,j \in X\}.$$

If we define

$$(\sum_{i \leq j} d_{ij}\ x_{ij})(\sum_{i \leq j} d'_{ij}\ x_{ij}) = \sum_{i \leq j} (\sum_{i \leq k \leq j} d_{ik}\ d'_{kj})x_{ij},$$

then R becomes a ring, called the _incidence ring of D over X_. This ring may be considered as a subring of the $|X| \times |X|$ upper triangular matrices. R is clearly an ℓ-hereditary semidistributive ring.

2. An example of an ℓ-hereditary semidistributive ring which is not an incidence ring is given below.

Let D be a division ring and let $\phi \in \mathrm{Aut}(D)$ be an automorphism which does not fix the center of D. Let

$$R_\phi = \left\{ \begin{pmatrix} a & 0 & x & m \\ & b & y & z \\ & & c & 0 \\ & & & d \end{pmatrix} : \ a,x,b,z,c,d \in D, \quad m \in {}_D M_D \right\}$$

with ${}_D M = {}_D D$ and the right D-multiplication in M is given by $m * d = m\phi(d)$. It is clear that R_ϕ is not an incidence ring [9].

CHAPTER III

MORITA DUALITY AND ℓ-HEREDITARY RINGS

In this chapter will establish that semidistributive ℓ-hereditary rings have self-duality. This will be accomplished by examining the quivers of these rings and by calculating their injective modules.

We begin the chapter by introducing the basic notions concerning (Morita) duality and by proving some necessary facts.

<u>Definition 3.1</u>

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. Let $H':\mathcal{C}\to\mathcal{D}$ and $H'':\mathcal{D}\to\mathcal{C}$ be two contravariant functors. We say that the pair (H',H'') is a <u>duality</u> between $\mathcal{C}$ and $\mathcal{D}$ if there exist natural isomorphisms such that $H''H' \cong 1_{\mathcal{C}}$ and $H'H'' \cong 1_{\mathcal{D}}$.

<u>Notation</u>. If R and S are rings, ${}_R\mathcal{M}$ and $\mathcal{M}_S$ will denote the <u>categories of left R-modules</u> and <u>right S-modules</u>, respectively. ${}_R\mathcal{FM}$ will denote the <u>category of finitely generated left R-modules</u>.

<u>Definition 3.2</u>

Let ${}_RU_S$ be a bimodule. The pair of contravariant additive functors $\mathrm{Hom}_R(-,{}_RU_S): {}_R\mathcal{M}\to\mathcal{M}_S$ and $\mathrm{Hom}_S(-,{}_RU_S): \mathcal{M}_S\to{}_R\mathcal{M}$ are called the U-duals.

We will sometimes denote $\mathrm{Hom}_R(M,U)$ by M^* and $\mathrm{Hom}_S(\mathrm{Hom}_R(M,U),U)$ by M^{**} if M is a left R-module. The same notation will be used for a right S-module N.

Definition 3.3

Let ${}_RU_S$ be a bimodule. We will say that a left R-module (or a right S-module) M is U-reflexive if the evaluation map σ_M: $M \to M^{**}$ defined by $(m^*)(m)\sigma_M = (m)m^*$, where $m \in M$, $m^* \in M^*$, is an isomorphism.

Theorem 3.4 [15]

Let R and S be rings and let ${}_R\mathcal{C}$ and $\mathcal{D}_S$ be full subcategories of ${}_R\mathcal{M}$ and $\mathcal{M}_S$ such that ${}_RR \in {}_R\mathcal{C}$ and $S_S \in \mathcal{D}_S$ and such that every module in ${}_R\mathcal{M}$ (respectively $\mathcal{M}_S$) isomorphic to one in ${}_R\mathcal{C}$ (respectively $\mathcal{D}_S$) is in ${}_R\mathcal{C}$ (respectively $\mathcal{D}_S$).

If H': ${}_R\mathcal{C} \to \mathcal{D}_S$ and H'': $\mathcal{D}_S \to {}_R\mathcal{C}$ is a duality between ${}_R\mathcal{C}$ and $\mathcal{D}_S$, then there exists a bimodule ${}_RU_S$ such that

1) ${}_RU \cong H''(S)$ and $U_S \cong H'(R)$,

2) there are natural isomorphisms

$$H' \cong \mathrm{Hom}_R(-,U) \quad \text{and } H'' \cong \mathrm{Hom}_S(-,U), \text{ and}$$

3) every $M \in {}_R\mathcal{C}$ and every $N \in \mathcal{D}_S$ is U-reflexive.

Definition 3.5

Let R and S be rings, ${}_RU_S$ a bimodule. We say that the duality given by the pair $\mathrm{Hom}_R(-,U)$ and $\mathrm{Hom}_S(-,U)$ is a Morita duality if

1) ${}_RR$ and S_S are U-reflexive, and

2) every submodule and every factor module of a U-reflexive module is U-reflexive.

Definition 3.6

An artinian ring R is said to have a (Morita) self-duality if there is a Morita duality D: ${}_R\mathcal{FM} \to \mathcal{FM}_R$, D': $\mathcal{FM}_R \to {}_R\mathcal{FM}$.

Theorem 3.7 ([2], [15])

Let R be an artinian ring. R has self-duality if and only if there exists an injective cogenerator ${}_R E$ of ${}_R\mathcal{FM}$ and a ring isomorphism $\phi: R \to \mathrm{End}({}_R E)$, which induces a right R-structure on E via $xr = x\phi(r)$, $x \in E$, $r \in R$, such that

$$D \cong \mathrm{Hom}_R(-, {}_R E) \quad \text{and} \quad D' \cong \mathrm{Hom}_R(-, E_R).$$

Definition 3.7

Let R be a ring with self-duality D. We will say that D is a weakly symmetric duality if $D(Re/Je) \cong eR/eJ$ for every primitive idempotent $e \in R$.

Theorem 3.8 [10]

Let R be an artinian ring. Then R has a weakly symmetric duality iff there is an injective cogenerator E of ${}_R\mathcal{FM}$ and a ring isomorphism $\phi: R \to \mathrm{End}({}_R E)$ such that $(E)(\phi e) \cong E(Re/Je)$ for every primitive idempotent $e \in R$. In particular $E = E(R/J)$; if R is basic, then E is the minimal injective cogenerator.

Proof: In view of (3.7) we just need to show that $D = \mathrm{Hom}(-, {}_R E_R)$ is weakly symmetric if and only if $Ee = (E)\phi(e) \cong E(Re/Je)$ for every primitive idempotent $e \in R$.

But D is a weakly symmetric duality iff

$$D(Re/Je) \cong eR/Je.$$

That is,

$$\text{iff} \quad [\mathrm{Hom}(Re/Je, E)]e \cong [D(Re/Je)]e \neq 0$$

iff $e\,\mathrm{Soc}(Ee) \neq 0$

iff $Re/Je \cong \mathrm{Soc}(Ee)$

iff $Ee \cong E(Re/Je)$ as needed.

Having established the basic necessary results we proceed toward our main goal. One of our basic techniques consists of analyzing the quivers of an artinian ring. A quiver is a finite set of points called vertices connected by arrows.

Given an artinian ring R and a basic set of primitive idempotents $\{e_1,\ldots,e_n\}$ of R we form <u>the (left) quiver $Q({}_RR)$ of R</u> as follows: the vertices are $v_1,\ldots,v_n$, one for each idempotent, with n_{ij} arrows from v_j to v_i iff the simple left R-module Re_i/Je_i appears exactly n_{ij} times as a direct summand of the semisimple left R-module Je_j/J^2e_j. (The <u>right quiver $Q(R_R)$ of R</u> is formed similarly, the vertices are $v'_1,\ldots,v'_n$, one for each idempotent, with n'_{ij} arrows from v'_j to v'_i iff e_iR/e_iJ appears exactly n'_{ij} times as a direct summand in the decomposition of the semisimple right R-module e_jJ/e_jJ^2.)

It is then clear that the quiver Q of an artinian ring is a multigraph.

We recall here some definitions and a few elementary facts from the theory of graphs that will be needed in the sequel (see [13]).

A <u>(finite) graph</u> G is defined to be an ordered pair (V,E) where V is a (finite) set and E is a binary relation in V. The elements in V are called the <u>vertices</u> and the ordered pairs in E are called the <u>edges</u> (or <u>arrows</u>) of the graph. If $v_i,v_j \in V$ are such that $a = (v_i,v_j) \in E$ then v_i is called the <u>initial vertex</u> and v_j is called the <u>terminal vertex</u>. A graph is said to be <u>directed</u> if directions are assigned to the edges.

We remark that in a directed graph the edge (v_i,v_j) is not the same as the edge (v_j,v_i). In a directed graph we will denote the edge (v_j,v_i) by $v_i \leftarrow v_j$.

Definition 3.9

a) In a directed graph, a path is a sequence of edges (arrows) $(a_{i_1},a_{i_2},\ldots,a_{i_k})$ such that the terminal vertex of a_{i_j} coincides with the initial vertex of $a_{i_{j+1}}$ for $1 \leq j \leq k-1$.

b) A path is simple if it does not use the same edge twice.

c) A path is elementary if it does not meet the same vertex twice.

d) A circuit (or closed path) is a path $(a_{i_1},a_{i_2},\ldots,a_{i_k})$ in which the terminal vertex of a_{i_k} coincides with the initial vertex of a_{i_1}.

e) A directed path in a directed graph Q is a path in which any two consecutive edges have the same direction, that is, if $v_{i_j}, v_{i_{j+1}}$ and $v_{i_{j+2}}$ are three consecutive vertices, then the directions assigned to the edges joining them are

$$v_{i_j} \xrightarrow{a_{i_j}} v_{i_{j+1}} \xrightarrow{a_{i_{j+1}}} v_{i_{j+2}} .$$

Similarly we define elementary circuits, simple circuits and directed circuits.

Definition 3.10

a) Two vertices v_i, v_j in a graph Q are said to be connected if there is a path in Q joining them.

b) A graph Q is connected if any two vertices in Q are connected.

c) A tree T is a connected graph which contains no circuits.

Definition 3.11

Let G be a graph with vertices $V = \{v_i\}_{i=1}^n$ and edges $E = \{a_i\}_{i=1}^n$. A graph G' with set of vertices V' and set of edges E' is a <u>subgraph</u> of G if $V' \subseteq V$ and $E' \subseteq E$.

<u>Definition 3.12</u>

a) A <u>tree of a graph</u> Q is a subgraph T of Q which is a tree.

b) A <u>spanning tree of a graph</u> Q is a tree of Q which contains all the vertices of Q.

<u>Proposition 3.13</u> [13]

a) Any two vertices in a tree are connected by a unique path.

b) A graph is connected iff it contains a spanning tree. Moreover, if the graph has n vertices, its spanning tree will contain $n-1$ edges.

For the quiver of an arbitrary artinian ring we state the following.

<u>Proposition 3.14</u> [9]

Let R be an artinian ring with (left) quiver Q. If Re_i/Je_i is (isomorphic to) a direct summand of $J^k e_j/J^{k+1}e_j$, then there is in Q a directed path $v_i \leftarrow v_{i_1} \leftarrow \dots \leftarrow v_{i_k} = v_j$ of length k from v_j to v_i. If in addition R is hereditary the converse is true.

Proof: Induct on k. By definition of a quiver the assertion is true for $k=0, 1$.

Let's now assume that Re_i/Je_i is (isomorphic to) a direct summand of $J^k e_j/J^{k+1}e_j$. Let

$$\bigoplus_{r=1}^{t} Re_{j_r} \xrightarrow{f} J^{k-1}e_j \longrightarrow 0 \qquad (*)$$

be a projective cover. By [1] Propositions 9.15, 9.19, 15.18 and by passing to the quotient, f induces an epimorphism

$$\bigoplus_{r=1}^{t} (Je_{j_r}/J^2e_{j_r}) \xrightarrow{\bar{f}} J^k e_j/J^{k+1}e_j \longrightarrow 0.$$

We have that R/J is a semisimple ring; consequently

$$\bigoplus_{r=1}^{t} (Je_{j_r}/J^2e_{j_r}) \quad \text{and} \quad J^k e_j/J^{k+1}e_j$$

are R/J semisimple modules and $\bar{f}$ is a splitting homomorphism (see [1] Prop. 4.3). Thus, there is r such that $Re_i/Je_i \cong Je_{j_r}/J^2e_{j_r}$, that is, there is an arrow from v_{j_r} to v_i.

From (*), and by the inductive hypothesis, we see there is a path of length k-1 from v_j to v_{j_r} We have thus obtained the desired path of length k from v_j to v_i.

Now, suppose that R is hereditary and let $v_i \leftarrow v_{i_{k-1}} \leftarrow \dots \leftarrow v_{i_p} = v_j$ be a directed path of length k from v_j to v_i. Assume that Re_{i_m}/Je_{i_m} is a direct summand of the semisimple module

$$J^m e_j/J^{m+1}e_j \quad \text{for } m < k$$

Since Re_j is projective, so is $J^m e_j$ and we can write

$$J^m e_j \cong Re_{i_m} \oplus M$$

for some left R-module M. We obtain that

$$J^{m+1}e_j/J^{m+2}e_j \cong Je_{i_m}/J^2e_{i_m} \oplus JM/J^2M.$$

The existence of an arrow $v_{i_{m+1}} \leftarrow v_{i_m}$ implies that $Re_{i_{m+1}}/Je_{i_{m+1}}$ is (isomorphic to) a direct summand of $Je_{i_m}/J^2e_{i_m}$. From this we conclude that $Re_{i_{m+1}}/Je_{i_{m+1}}$ is a direct summand of

$$Je_j^{m+1}/J^{m+2}e_j.$$

Proposition 3.15

Let R be an artinian ring with (left) quiver Q. Let $v_i = v_{i_k} \leftarrow v_{i_{k-1}} \leftarrow \ldots \leftarrow v_{i_1} \leftarrow v_{i_0} = v_j$ be a directed path of minimal length from v_j to v_i in Q. Then $e_iRe_j = e_iJ^ke_j$.

Proof: Let m be the least positive integer such that

$$e_iJ^ke_j \subsetneq e_iJ^{k-m}e_j \subseteq e_iRe_j.$$

Clearly,

$$e_iJ^{k-m}e_j/e_iJ^ke_j$$

is a nonzero left e_iRe_i-module and

$$J(e_iRe_i)(e_iJ^{k-m}e_j/e_iJ^ke_j) = e_iJe_i(e_iJ^{k-m}e_j/e_iJ^ke_j) = 0.$$

We then have ([1], Prop. 15.18) that $e_iJ^{k-m}e_j/e_iJ^ke_j$ is e_iRe_i-semisimple. Hence,

$$e_iJ^{k-m}e_j/e_iJ^ke_j \cong (e_iRe_i/e_iJe_i)^{(\lambda)}$$

for some λ, and

$$e_i(e_iJ^{k-m}e_j/e_iJ^ke_j) \neq 0.$$

Then

$$e_i(e_iJ^{k-m}e_j/e_iJ^{k-m+1}e_j) \neq 0 \quad \text{and}$$

$$e_i(J^{k-m}e_j/J^{k-m+1}e_j) \neq 0.$$

Consequently, Re_i/Je_i is (isomorphic to) a direct summand of $J^{k-m}e_j/J^{k-m+1}e_j$ and by Proposition 3.14 there is a path of length strictly less than k from v_j to v_i, contradicting our hypothesis.

For ℓ-hereditary rings we have

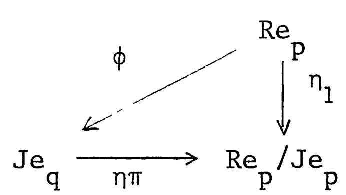
<u>Proposition 3.16</u>

Let R be an artinian ℓ-hereditary ring with quiver Q. Suppose there is a directed path (of length k) from v_j to v_i in Q, then $e_iRe_j \neq 0$.

<u>Proof</u>: We first claim that if there is an arrow $v_p \leftarrow v_q$, then there is a 1-1 map from $Re_p \to Re_q$. To see this, we have, by definition, that Re_p/Je_p is a direct summand of Je_q/J^2e_q.

Consider the composition $Je_q \xrightarrow{\eta} Je_q/J^2e_q \xrightarrow{\pi} Re_p/Je_p$ where η is the natural epimorphism and π the corresponding projection onto the direct summand.

Since Re_p is projective, there exists ϕ: $Re_p \to Je_q$, $\phi \neq 0$ such that the following diagram commutes.



with η_1 also the natural epimorphism. We may consider ϕ as a map into Re_q; since R is ℓ-hereditary ϕ is 1-1. If $v_i = v_{i_k} \leftarrow v_{i_{k-1}} \leftarrow \ldots \leftarrow v_{i_1} \leftarrow v_{i_0} = v_j$ is an oriented path we then get a sequence of maps

$$Re_i \xrightarrow{\phi_1} Re_{i_{k-1}} \xrightarrow{\phi_{i_{k-1}}} \cdots \longrightarrow Re_{i_1} \xrightarrow{\phi_{i_1}} Re_j$$

with each ϕ_{i_j} a monomorphism. Hence,

$$(Re_i)\phi_i\phi_{i_{k-1}} \cdots \phi_{i_1} \neq 0 \quad \text{and} \quad (e_iRe_i)\phi_i\phi_{i_{k-1}} \cdots \phi_{i_1} \neq 0.$$

But

$$(e_iRe_i)\phi_i\phi_{i_{k-1}} \cdots \phi_{i_1} \leq e_iRe_j,$$

so,

$$e_iRe_j \neq 0.$$

Proposition 3.17

If R is an artinian ℓ-hereditary ring, then the quiver Q of R has no directed circuits.

Proof: It is enough to show that there are no elementary circuits. Consider then a circuit

$$v_i = v_{i_k} \leftarrow v_{i_{k-1}} \leftarrow \cdots \leftarrow v_{i_1} \leftarrow v_{i_0} = v_i$$

where all the vertices other than $v_{i_k} = v_{i_0}$ are different. We thus get a sequence of monomorphisms

$$Re_{i_k} \xrightarrow{\phi_i} Re_{i_{k-1}} \xrightarrow{\phi_{i_{k-1}}} \cdots Re_{i_1} \xrightarrow{\phi_{i_1}} Re_i \xrightarrow{\phi_1} Re_{i_{k-1}}$$

$$\longrightarrow \cdots,$$

and then $J^{tk}e_i \neq 0$ which is a contradiction, for R is artinian.

Another simplification in the quiver of a ring occurs when we consider semidistributivity.

Proposition 3.18

If R is an artinian semidistributive ring, then its quiver Q is a graph, that is, for any pair of vertices v_i, v_j in Q there is at most one arrow from v_j to v_i.

Proof: If v_i and v_j are not connected by an arrow there is nothing to prove. We may then assume that $(Re_i/Je_i)^{(k)}$ is a direct summand of Je_j/J^2e_j. By hypothesis, R is semidistributive, that is, Re_j is distributive and hence so is Je_j. This implies [5] that $\mathrm{Soc}(Je_j/J^2e_j) = Je_j/J^2e_j$ is square free and hence $k = 1$.

If we combine Propositions 3.17 and 3.18 we see that the quiver Q of an artinian semidistributive ℓ-hereditary ring is a graph with no directed circuits. As a consequence of this fact we can partially order the set of vertices $\{v_1, \ldots, v_n\}$ of Q as follows.

Definition 3.19

Let R be an artinian semidistributive ℓ-hereditary ring with quiver Q. Let $\{v_1, \ldots, v_n\}$ be the vertices of Q. We will put $v_i \leq v_j$ if there is at least one directed path from v_j to v_i or $v_i = v_j$.

We will relabel the vertices $\{v_i\}$ of Q so that $v_i \leq v_j$ implies $i \leq j$. Notice also that under this condition v_1 is a minimal element and v_n is a maximal one.

We remark here that the quiver of an artinian ring R is connected if and only if R is an indecomposable ring ([1], Prop. 7.9).

In what follows we will assume, unless otherwise stated, that R is an artinian indecomposable semidistributive ℓ-hereditary ring with $\{e_1,$

$\ldots, e_n\}$ a basic set of primitive idempotents and with quiver Q.

As an immediate consequence of Proposition 3.17 we have that $e_1Je_1 = \ldots = e_iJe_i = \ldots = e_nJe_n = 0$. For, if $e_iJe_i \neq 0$ then $e_i(Je_i/J^2e_i) \neq 0$ which implies that there is an oriented path (of length one) from v_i to v_i.

This simple remark allows us to establish the following.

<u>Proposition 3.20</u>

e_iRe_i is a division ring for every $i = 1,\ldots,n$.

Proof: Since $e_iJe_i = 0$, we have

$$e_iRe_i = e_iRe_i/e_iJe_i = e_iRe_i/J(e_iRe_i)$$

$$\cong \mathrm{End}(Re_i)/J\mathrm{End}(Re_i)$$

$$\cong \mathrm{End}(Re_i/Je_i).$$

But Re_i/Je_i is simple and hence $\mathrm{End}(Re_i/Je_i)$ is a division ring.

It is our immediate task to derive from the quiver Q of R a few simple facts about R.

First, we notice that if there is an arrow $v_i \leftarrow v_j$ connecting v_j with v_i then $e_iJe_j \neq 0$ and we may consider the bimodule

$${}_{e_iRe_i}{e_iJe_j}_{e_jRe_j}.$$

Furthermore, since R is semidistributive

$$\dim_{e_iRe_i} e_iJe_j = \dim e_iJe_{j\,e_jRe_j} = 1,$$

for otherwise,

$$e_iRe_i \, e_iJe_j \cong (e_iRe_i)^{(k)}$$

which contradicts the fact that Re_i is distributive. We can then choose

$$e_{ij} \in {}_{e_iRe_i}\,{e_iJe_j}\,{}_{e_jRe_j}\,, \quad e_{ij} \neq 0$$

such that

$$e_iJe_j = e_iRe_ie_{ij} = e_{ij}e_jRe_j$$

and define

$$\sigma_{ij}\colon e_iRe_i \to e_jRe_j \quad \text{via} \quad ze_{ij} = e_{ij}(z)\sigma_{ij}.$$

Clearly σ_{ij} is an isomorphism, and by using the fact that R is indecomposable we conclude that $e_iRe_i \cong e_jRe_j$ for every i, j.

Applying Proposition 3.13 we can choose a spanning tree T of Q containing n-1 arrows. We will select T so that it contains all the arrows ending in v_1 (see [13]).

The existence of this tree will enable us to choose elements $e_{ij} \in R$ whenever $i \leq j$ independently of the path connecting v_j to v_i and will also make possible the construction of division ring D isomorphic to e_iRe_i which will embed into $\bigoplus_{i=1}^{n} e_iRe_i$ in a similar way as a ring A embeds into the main diagonal of the ring of n×n matrices over A

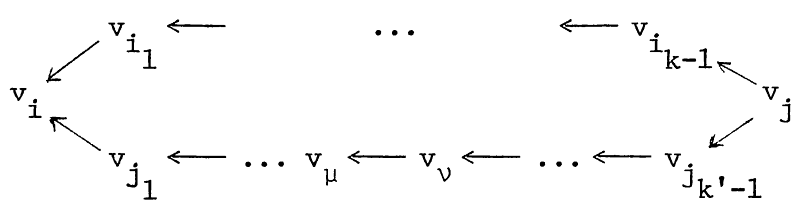
Let $v_i = v_{i_0} \leftarrow v_{i_1} \leftarrow \dots v_{i_{k-1}} \leftarrow v_{i_k} = v_j$ be a directed path from v_j to v_i which lies entirely in T. Define

$$e_{ij} = e_{ii}e_{i_1i_2}e_{i_2i_3} \cdots e_{i_{k-1}i_k}e_{jj} \in e_iJe_j$$

where $e_{kk} = e_k$ and $e_{i_ki_{k+1}}$ are defined as above for arrows $v_{i_k} \leftarrow v_{i_{k+1}}$ in T.

We will first define $e_{\mu\nu}$ for $v_\mu \leftarrow v_\nu$ that close two directed paths from one vertex to another. After this stage, we will add these arrows to the tree T and continue inductively until such additions are no longer possible.

Let $a_{\mu\nu} = v_\mu \leftarrow v_\nu$ be an arrow in Q-T; $v_\mu \leftarrow v_\nu$ will close necessarily a unique non-directed circuit in Q. We choose this arrow (if possible) so that it determines two directed paths



from v_j to v_i with $v_i \leftarrow v_{i_1} \leftarrow \ \dots \leftarrow v_j$ a directed path along T.

Let $\bar{e}_{\mu\nu} \in e_\mu J e_\nu$, $\bar{e}_{\mu\nu} \neq 0$ as before. That is,

$$e_\mu J e_\nu = e_\mu R e_\mu \bar{e}_{\mu\nu} = \bar{e}_{\mu\nu} e_\nu R e_\nu .$$

Then $e_{ii} e_{i_0 i_1} \dots e_{i_{k-1} i_k} e_{jj}$ and $e_{ii} e_{i_0 j_1} \dots \bar{e}_{\mu\nu} \dots e_{j_{k'-1} j_{k'}} e_{jj}$ are elements of $e_i J e_j$. Again, since R is semidistributive we have that

$$\dim_{e_i R e_i} e_i J e_j = \dim e_i J e_j{}_{e_j R e_j} = 1$$

and there exists $0 \neq t \in e_i R e_i$ such that

$$t e_{ii} e_{i_0 j_1} \dots \bar{e}_{\mu\nu} \dots e_{j_{k'-1} j_{k'}} e_{jj} = e_{ii} e_{i_0 i_1} \dots e_{i_{k-1} i_k} e_{jj} .$$

Let

$$e_{\mu\nu} = (t)\sigma_{i_0 j_1} \sigma_{j_1 j_2} \dots \sigma_{(\mu-1)\mu} \bar{e}_{\mu\nu}, \text{ then}$$

$$0 \neq e_{\mu\nu} \in e_\mu J e_\nu$$

$$e_{ii} e_{i_0 i_1} \cdots e_{i_{k-1} i_k} e_{jj} = e_{ii} e_{i_0 j_1} \cdots e_{\mu\nu} \cdots e_{j_{k'-1} j_{k'}} e_{jj} \qquad (1)$$

and

$$e_\mu J e_\nu = e_\mu R e_\mu e_{\mu\nu} = e_{\mu\nu} e_\nu R e_\nu .$$

Let $\beta_{\mu\nu}$: $e_\mu R e_\mu \to e_\nu R e_\nu$ be the isomorphism defined by $z e_{\mu\nu} = e_{\mu\nu}(z)\beta_{\mu\nu}$, $z \in e_\mu R e_\mu$. Let $\bar{\sigma}_{\mu\nu}$: $e_\mu R e_\mu \to e_\nu R e_\nu$ be the isomorphism determined by the (unique) (nondirected) path from v_ν to v_μ along T, that is, $\bar{\sigma}_{\mu\nu}$ is the isomorphism obtained by composition of the isomorphisms σ_{st} (and their inverses) determined by arrows $v_s \leftarrow v_t$ in T.

Then there exists $h_\nu^{\mu\nu} \in \mathrm{Aut}(e_\nu R e_\nu)$ such that the diagram below commutes.

$$\begin{array}{ccc} e_\mu R e_\mu & \xrightarrow{\beta_{\mu\nu}} & e_\nu R e_\nu \\ \downarrow 1 & & \downarrow h_\nu^{\mu\nu} \\ e_\mu R e_\mu & \xrightarrow[\bar{\sigma}_{\mu\nu}]{} & e_\nu R e_\nu \end{array}$$

i.e., $\beta_{\mu\nu} h_\nu^{\mu\nu} = \bar{\sigma}_{\mu\nu}$. Let $g_\mu^{\mu\nu} \in \mathrm{Aut}(e_\mu R e_\mu)$ such that

$$\begin{array}{ccc} e_\mu R e_\mu & \xrightarrow{\beta_{\mu\nu}} & e_\nu R e_\nu \\ g_\mu^{\mu\nu} \uparrow & & \uparrow 1 \\ e_\mu R e_\mu & \xrightarrow[\bar{\sigma}_{\mu\nu}]{} & e_\nu R e_\nu \end{array}$$

commutes. Then $\beta_{\mu\nu} h_\nu^{\mu\nu} = g_\mu^{\mu\nu} \beta_{\mu\nu}$ and $g_\mu^{\mu\nu} = \beta_{\mu\nu} h_\nu^{\mu\nu} \beta_{\mu\nu}^{-1}$. Define

$$\sigma_{\mu\nu} = g_\mu^{\mu\nu}\beta_{\mu\nu},$$

then we have that

$$\sigma_{\mu\nu} = \bar{\sigma}_{\mu\nu} \tag{2}$$

and

$$e_{\mu\nu}(z)\sigma_{\mu\nu} = e_{\mu\nu}(z)g_\mu^{\mu\nu}\beta_{\mu\nu} = (z)g_\mu^{\mu\nu}e_{\mu\nu}, \quad z \in e_\mu Re_\mu.$$

That is,

$$e_{\mu\nu}(z)\sigma_{\mu\nu} = (z)g_\mu^{\mu\nu}e_{\mu\nu}$$

or (3)

$$te_{\mu\nu} = e_{\mu\nu}(t)(g_\mu^{\mu\nu})^{-1}\sigma_{\mu\nu}$$

The commutative diagrams involved in the construction of $\sigma_{\mu\nu}$ from $\beta_{\mu\nu}$ for the directed path

$$v_i \leftarrow v_{j_1} \leftarrow v_{j_2} \quad \cdots \quad v_{\mu-1} \leftarrow v_\mu \leftarrow v_\nu \leftarrow \cdots \; v_{j_{k'-1}} \leftarrow v_j$$

are shown in Figure 1. By chasing this diagram we see that $h_t^{\mu\nu} = g_t^{\mu\nu}$, $i \leq t \leq j$. We will call the maps $h_t^{\mu\nu}$ the <u>twisting induced by the addition of the arrow</u> $v_\mu \leftarrow v_\nu$.

By (1) we can unambiguously define e_{ij} by

$$e_{ij} = e_{ii}e_{i_0j_1} \cdots e_{i_{k-1}i_k}e_{jj} = e_{ii}e_{j_0j_1} \cdots e_{j_{k'-1}j_k}e_{jj} \in e_iRe_j.$$

Similarly, from (3) we obtain

$$\sigma_{ij} = \sigma_{i_0i_1}\sigma_{i_1i_2} \cdots \sigma_{i_{k-1}j} = \sigma_{j_oj_1} \cdots \sigma_{j_{k'-1}j_{k'}}\sigma_{j_kj}. \tag{5}$$

Moreover, if $x \in e_iRe_i$, then

$$
\begin{array}{cccccccccccccc}
\cdots & e_{i_1}Re_{i_1} & \xrightarrow{\sigma^{-1}_{ii_1}} & e_iRe_i & \xrightarrow{\sigma_{ij_1}} & e_{j_1}Re_{j_1} & \xrightarrow{\sigma_{j_1j_2}} & \cdots \to e_{\mu-1}Re_{\mu-1} & \xrightarrow{\sigma_{\mu-1\mu}} & e_\mu Re_\mu & \xrightarrow{\bar{\sigma}_{\mu\nu}} & e_\nu Re_\nu & \to \cdots \to e_jRe_j & \to \cdots \to e_iRe_i \to \cdots \\
& \downarrow & & \downarrow g^{\mu\nu}_i & & \downarrow g^{\mu\nu}_{j_1} & & \downarrow g^{\mu\nu}_{\mu-1} & & \downarrow g^{\mu\nu}_\mu & & \downarrow 1 & \downarrow 1 & \downarrow 1 \\
\cdots & e_{i_1}Re_{i_1} & \xrightarrow{\sigma^{-1}_{ii_1}} & e_iRe_i & \longrightarrow & e_{j_1}Re_{j_1} & \longrightarrow & \cdots \to e_{\mu-1}Re_{\mu-1} & \longrightarrow & e_\mu Re_\mu & \xrightarrow{\beta_{\mu\nu}} & e_\nu Re_\nu & \to \cdots \to e_jRe_j & \to \cdots \to e_iRe_i \to \cdots \\
& \downarrow 1 & & \downarrow 1 & & \downarrow 1 & & \downarrow 1 & & \downarrow 1 & & \downarrow h^{\mu\nu}_\nu & \downarrow h^{\mu\nu}_j & \downarrow h^{\mu\nu}_i \\
\cdots & e_{i_1}Re_{i_1} & \xrightarrow{\sigma^{-1}_{ii_1}} & e_iRe_i & \longrightarrow & e_{j_1}Re_{j_1} & \longrightarrow & \cdots \to e_{\mu-1}Re_{\mu-1} & \longrightarrow & e_\mu Re_\mu & \xrightarrow{\bar{\sigma}_{\mu\nu}} & e_\nu Re_\nu & \to \cdots \to e_jRe_j & \to \cdots \to e_jRe_j \to \cdots
\end{array}
\qquad (4)
$$

Figure 1. Diagram of the Twisting Induced by the Addition of the Arrow $v_\mu \leftarrow v_\nu$

$$xe_{ij} = xe_{ii}e_{i_0j_1} \cdots e_{\mu\nu} \cdots e_{j_{k'-1}j_{k'}}e_{jj}$$

$$= e_{ii}e_{i_0j_1}(x)\sigma_{i_0j_1} \cdots e_{\mu\nu} \cdots e_{j_{k'-1}j_k}e_{jj}$$

$$= e_{ii}e_{i_0j_1} \cdots e_{\mu\nu}(x)\sigma_{i_0j_1} \cdots (g_\mu^{\mu\nu})^{-1}\sigma_{\mu\nu} \cdots e_{j_{k'-1}j_{k'}}e_{jj}$$

$$= e_{ii}e_{i_0j_1} \cdots e_{j_{k'-1}j_{k'}}(x)\sigma_{i_0j_1} \cdots (g_\mu^{\mu\nu})^{-1}\sigma_{\mu\nu} \cdots \sigma_{j_{k'-1}j_{k'}}.$$

Also,

$$xe_{ij} = xe_{ii}e_{i_0i_1} \cdots e_{i_{k-1}i_k}e_{jj}$$

$$= e_{ii}e_{i_0i_1} \cdots e_{i_{k-1}i_k}e_{jj}(x)\sigma_{i_0i_1}\sigma_{i_1i_2} \cdots \sigma_{i_{k-1}i_k}$$

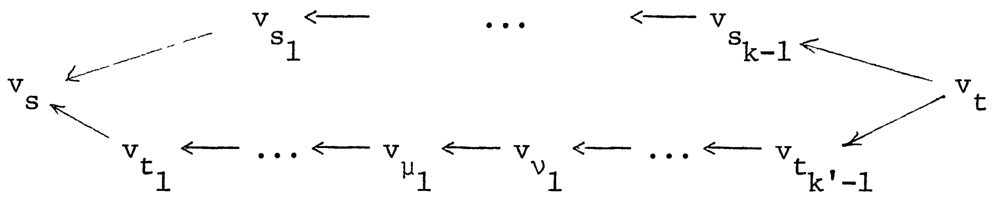
Hence

$$\sigma_{i_0i_1} \cdots (g_\mu^{\mu\nu})^{-1}\sigma_{\mu\nu} \cdots \sigma_{j_{k'-1}j_{k'}} = \sigma_{i_0i_1}\sigma_{i_1i_2} \cdots \sigma_{i_{k-1}i_k}.$$

Then (4) and the definition of $\sigma_{\mu\nu}$ imply that

$$g_\mu^{\mu\nu} = 1_{e_\mu Re_\mu}. \tag{6}$$

Again, let $v_{\mu_1} \leftarrow v_{\nu_1}$ be an arrow in Q-T, $v_{\mu_1} \leftarrow v_{\nu_1}$ different from $v_\mu \leftarrow v_\nu$. We choose $v_{\mu_1} \leftarrow v_{\nu_1}$ so that (if possible) it will determine two directed paths



from v_t to v_s.

We proceed exactly as in the previous case and we remark that $v_{\mu_1} \leftarrow v_{\nu_1}$ does not necessarily close a unique nondirected circuit in $T \cup \{v_\mu \leftarrow v_\nu\}$. However, an easy computation shows that, if $v_{\mu_1} \leftarrow v_{\nu_1}$ closes the circuit containing $v_\mu \leftarrow v_\nu$, the isomorphism $\bar{\sigma}_{\mu_1\nu_1} : e_{\mu_1} Re_{\mu_1} \to e_{\nu_1} Re_{\nu_1}$ is independent of the path along $T \cup \{v_\mu \leftarrow v_\nu\}$. We are now done with the first stage. We continue until this construction is no longer possible and collect all the arrows so obtained.

Let T_1 be the graph obtained by adding to T all the arrows $v_\mu \leftarrow v_\nu$, $v_{\mu_1} \leftarrow v_{\nu_1}$,.... Since Q is finite, T_1 exists (and might be equal to T). We remark that T_1 is a graph such that every (nondirected) circuit in T_1 contains two vertices joined by two (different) directed paths along T_1 and one of them along T.

Construct T_2 from T_1 as T_1 was obtained from T, that is, add to T_1 those arrows $v_\mu \leftarrow v_\nu$ in $Q-T_1$ which will close circuits composed of exactly two directed paths, one along T_1 and the other containing $v_\mu \leftarrow v_\nu$. Since Q is finite, this construction must end, say at T_r.

We remark that formulas (1), (2), (4), (5) and (6) remain valid for T_r.

We remark that T_r might not equal Q.

Let $v_\mu \leftarrow v_\nu \in Q - T_r$. This arrow cannot close a circuit with one directed path along T_r, that is, it cannot close a circuit in such a way that two directed paths are joining a pair of vertices in the circuit.

Let $0 \neq e_{\mu\nu} \in e_\mu J e_\nu$ as before, that is, such that

$$e_\mu R e_\mu e_{\mu\nu} = e_{\mu\nu} e_\nu R e_\nu$$

Let $\beta_{\mu\nu}: e_\mu R e_\mu \to e_\nu R e_\nu$ be the isomorphism defined by $z e_{\mu\nu} = e_{\mu\nu}(z)\beta_{\mu\nu}$, $z \in e_\mu R e_\mu$.

Let $\bar{\sigma}_{\mu\nu}: e_\mu R e_\mu \to e_\nu R e_\nu$ be the isomorphism determined by a nondirected path from v_μ to v_ν along T_r. It is clear by the construction of T_r that $\bar{\sigma}_{\mu\nu}$ is independent of the chosen path along T_r.

Then, there exists $h_\nu^{\mu\nu} \in \mathrm{Aut}(e_\nu R e_\nu)$, $g_\mu^{\mu\nu} \in \mathrm{Aut}(e_\mu R e_\mu)$ such that $\beta_{\mu\nu} h_\nu^{\mu\nu} = g_\mu^{\mu\nu} \beta_{\mu\nu}$.

Define $\sigma_{\mu\nu} = g_\mu^{\mu\nu} \beta_{\mu\nu}$, then $\sigma_{\mu\nu} = \bar{\sigma}_{\mu\nu}$ and

$$e_{\mu\nu}(z)\sigma_{\mu\nu} = (z) g_\mu^{\mu\nu} e_{\mu\nu}$$

or

$$t e_{\mu\nu} = e_{\mu\nu}(t) {g_\mu^{\mu\nu}}^{-1} \sigma_{\mu\nu}.$$

If $v_\mu \leftarrow v_\nu$, after inserted in T_r, determines a directed path from v_i to v_j in $T_r \cup \{v_\mu \leftarrow v_\nu\}$, that is if we have

$$v_i = v_{i_0} \leftarrow v_{i_1} \leftarrow \dots \leftarrow v_\mu \leftarrow v_\nu \leftarrow \dots v_{i_{k-1}} \leftarrow v_{i_k} = v_j$$

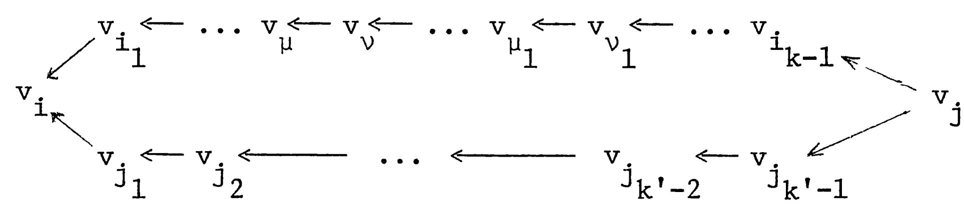
again, let $e_{\mu\nu} \in e_\mu J e_\nu$ be such that $e_\mu R e_\mu e_{\mu\nu} = e_{\mu\nu} e_\nu R e_\nu$.

Define $e_{ij} = e_{i_0 i_1} e_{i_1 i_2} \dots e_{\mu\nu} \dots e_{i_{k-1} i_k}$. Let

$$T_{r_1} = T_r \cup \{v_\mu \leftarrow v_\nu\}$$

and let $v_{\mu_1} \leftarrow v_{\nu_1} \in Q - T_{r_1}$ such that it closes a circuit containing two directed paths from one vertex to another one of which lies entirely in

T_{r_1}. Then $v_\mu \leftarrow v_\nu$ must necessarily be in such a path. If both $v_\mu \leftarrow v_\nu$ and $v_{\mu_1} \leftarrow v_{\nu_1}$ are along the same path we have; w.l.o.g.



with the lower path along T_{r_1}. Let $\bar{e}_{\mu_1\nu_1} \in e_{\mu_1} J e_{\nu_1}$ such that $e_{\mu_1} R e_{\mu_1} \bar{e}_{\mu_1\nu_1} = \bar{e}_{\mu_1\nu_1} e_{\nu_1} R e_{\nu_1}$. Then, there exists $t \in e_i R e_i$ such that

$$te_{ii}e_{i_0 i_1} \cdots e_{\mu\nu} \cdots \bar{e}_{\mu_1\nu_1} \cdots e_{i_{k-1}i_k} e_{jj}$$

$$= e_{ii}e_{i_0 j_1} \cdots e_{j_{k'-1}j_k} e_{jj}.$$

Let $e_{\mu_1\nu_1} = (t)\sigma_{i_0 i_1} \cdots (g_\mu^{\mu\nu})^{-1}\sigma_{\mu\nu} \cdots \sigma_{\mu_1 - 1\mu}\bar{e}_{\mu_1\nu_1}$, then

$$e_{ij} = e_{ii}e_{i_0 j_1} \cdots e_{jj} = e_{ii}e_{i_0 i_1} \cdots e_{\mu\nu} \cdots e_{\mu_1\nu_1} \cdots e_{jj} \qquad (7)$$

is unambiguously defined. Again, let $\beta_{\mu_1\nu_1} : e_{\mu_1} R e_{\mu_1} \to e_{\nu_1} R e_{\nu_1}$ be the isomorphism defined via

$$ze_{\mu_1\nu_1} = e_{\mu_1\nu_1}(z)\beta_{\mu_1\nu_1}.$$

We obtain, using the same procedure described before, the isomorphism

$$\sigma_{\mu_1\nu_1} : e_{\mu_1} R e_{\mu_1} \to e_{\nu_1} R e_{\nu_1}$$

defined by

$$\sigma_{\mu_1\nu_1} = g_{\mu_1}^{\mu_1\nu_1}\beta_{\mu_1\nu_1}$$

with

$$\beta_{\mu_1\nu_1}h_{\nu_1}^{\mu_1\nu_1} = g_{\mu_1}^{\mu_1\nu_1}\beta_{\mu_1\nu_1}$$

$$g_{\mu_1}^{\mu_1\nu_1} \in \mathrm{Aut}(e_{\mu_1}Re_{\mu_1})$$

and

$$h_{\nu_1}^{\mu_1\nu_1} \in \mathrm{Aut}(e_{\nu_1}Re_{\nu_1}).$$

We also obtain

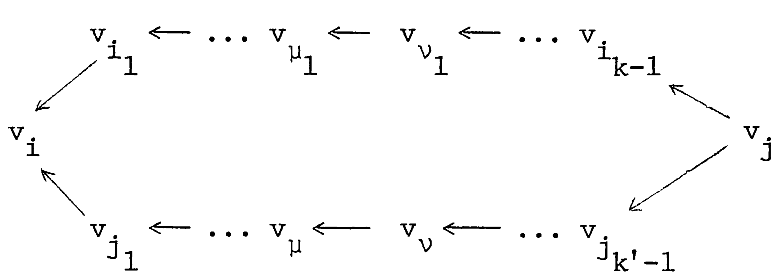
$$e_{\mu_1\nu_1}(z)\sigma_{\mu_1\nu_1} = (z)g_{\mu_1}^{\mu_1\nu_1}e_{\mu_1\nu_1}, \quad z \in e_{\mu_1}Re_{\mu_1}$$

and

$$\sigma_{i_0i_1} \cdots [g_{\mu}^{\mu\nu}]^{-1}\sigma_{\mu\nu} \cdots [g_{\mu_1}^{\mu_1\nu_1}]^{-1}\sigma_{\mu_1\nu_1} \cdots \sigma_{i_{k-1}i_k}$$

$$= \sigma_{i_0j_1}\sigma_{j_1j_2} \cdots \sigma_{j_{k'-1}j_{k'}} \qquad (8)$$

If $v_\mu \leftarrow v_\nu$ and $v_{\mu_1} \leftarrow v_{\nu_1}$ are not along the same path, that is, if



with the lower path along T_{r_1} we obtain

$$e_{\mu_1\nu_1} = (t)\sigma_{i_0i_1} \cdots \sigma_{\mu_1-1\mu}\bar{e}_{\mu_1\nu_1},$$

$$\sigma_{i_0i_1} \cdots [g_{\mu_1}^{\mu_1\nu_1}]^{-1}\sigma_{\mu_1\nu_1} \cdots \sigma_{i_{k-1}i_k} = \sigma_{i_0j_1} \cdots [g_{\mu}^{\mu\nu}]^{-1}\sigma_{\mu\nu}$$

$$\cdots \sigma_{j_{k'-1}j_{k'}}. \tag{9}$$

Let $T_{r_2} = T_{r_1} \cup \{v_{\mu_1} \leftarrow v_{\nu_1}\}$. We construct T_{r_3} from T_{r_2} as T_{r_2} was obtained from T_{r_1}. Since Q is finite the process must end, say at T_{r_k}. We consider $v_s \leftarrow v_t \in Q - T_{r_k}$ and obtain $T_{r_{k_1}}$ from T_{k_r} as T_{r_1} was obtained from T_r. Again, since Q is finite, we obtain Q and the total construction ends. We remark that formulas (8) and (9) adopt the form

$$\sigma_{i_0i_1} \cdots (g_{\mu_1}^{\mu_1\nu_1})^{-1}\sigma_{\mu_1\nu_1} \cdots (g_{\mu_2}^{\mu_2\nu_2})^{-1}\sigma_{\mu_2\nu_2} \cdots (g_{\mu_3}^{\mu_3\nu_3})^{-1}\sigma_{\mu_2\nu_2}$$

$$\cdots (g_{\mu_s}^{\mu_s\nu_s})^{-1}\sigma_{\mu_s\nu_s} \cdots \sigma_{i_{k-1}i_k}$$

$$= \sigma_{i_0i_1}\sigma_{j_1j_2} \cdots \sigma_{j_{k'-1}j_{k'}} \tag{10}$$

and

$$\sigma_{i_0i_1} \cdots [g_{\mu_1}^{\mu_1\nu_1}]^{-1}\sigma_{\mu_1\nu_1} \cdots [g_{\mu_2}^{\mu_2\nu_2}]^{-1}\sigma_{\mu_2\nu_2} \cdots [g_{\mu_s}^{\mu_s\nu_s}]^{-1} \cdots \sigma_{i_{k-1}i_k}$$

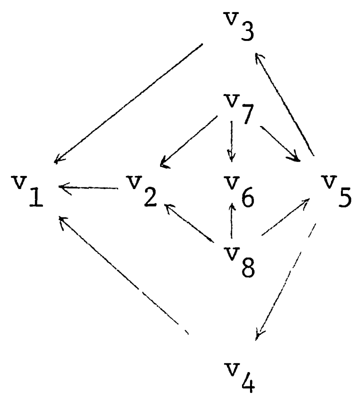
$$= \sigma_{i_0j_1} \cdots [g_{\alpha_1}^{\alpha_1\beta_1}]^{-1}\sigma_{\alpha_1\beta_1} \cdots [g_{\alpha_2}^{\alpha_2\beta_2}]^{-1}\sigma_{\alpha_2\beta_2} \cdots [g_{\alpha_t}^{\alpha_t\beta_t}]^{-1}\sigma_{\alpha_t\beta_t}$$

$$\cdots \sigma_{j_{k'-1}j_{k'}}. \tag{11}$$

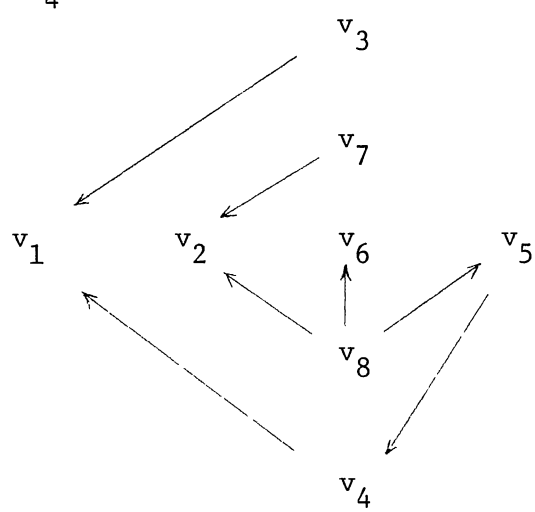
We also remark that the addition of every path $v_\mu \leftarrow v_\nu$ to Q induces a commutative diagram as in (4).

Example:

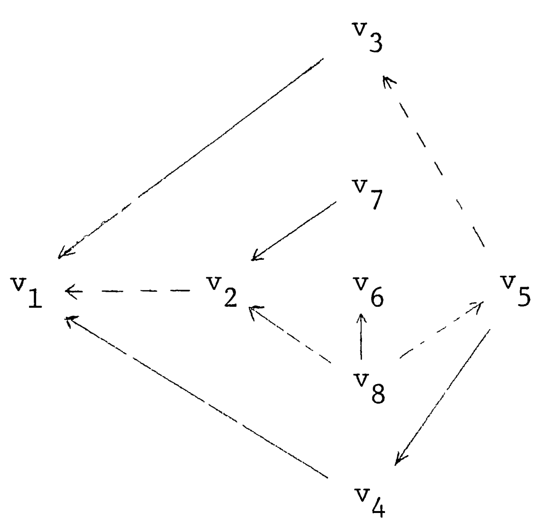
Let Q be as follows:



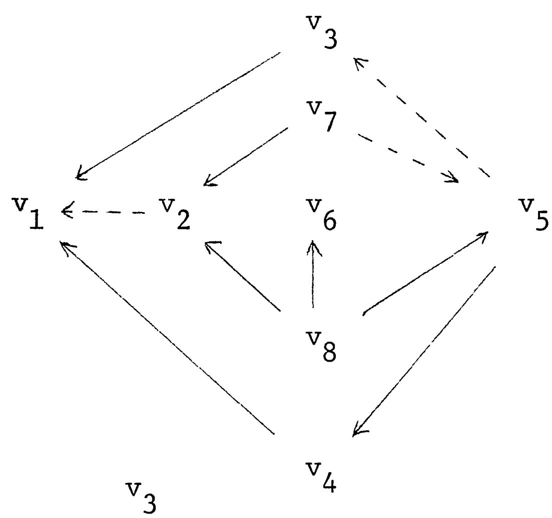
Let T be



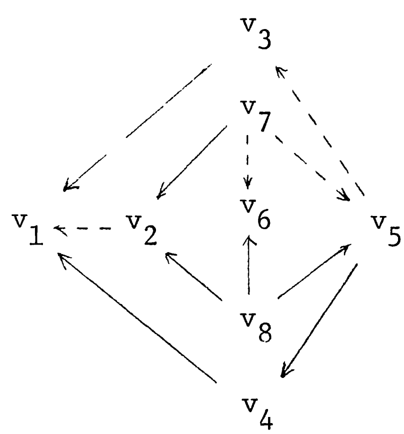
Then T_1 is



$T_2 = T_r$ is



and T_{r_1} is



Let $v_i \leftarrow v_j$ be an arrow in Q, let $\gamma_{ij} = \sigma_{ij}$ and let $\gamma_{ji} = \sigma_{ij}^{-1}$. If v_j is a vertex in Q, we can choose a (possible nondirected) path from v_1 to v_j along T: $v_1 = v_{i_0} \leftarrow \ldots v_{i_{k-1}} \leftarrow v_k = v_j$.

Define

$$\sigma_{ij} = \begin{cases} \gamma_{i_0 i_1}\gamma_{i_1 i_2} \cdots \gamma_{i_{k-1} i_k} & \text{if } j = 2,\ldots,n. \\ 1_{e_1 R e_1} & \text{if } j = 1 \end{cases}$$

and let

$$D = \{d \in \bigoplus_{i=1}^{n} e_i R e_i : d\pi_j = d\pi_1 \sigma_{1j}, \quad j = 1,\ldots,n\}$$

that is, $d \in D$ if an only if $d = \sum_{j=1}^{n} (x)\sigma_{1j}$, $x \in e_1 R e_1$.

It is then clear that D is isomorphic to $e_1 R e_1$ via

$$f\colon e_1Re_1 \to D$$

$$x \to \sum_{j=1}^{n} (x)\sigma_{1j}$$

and hence to e_iRe_i for $i = 2,\dots,n$. It is also clear that

1) $De_k = e_kRe_k$

2) $e_iRe_{ij} = e_iRe_ie_{ij} = e_iJe_j$, for $v_i \leftarrow v_j$

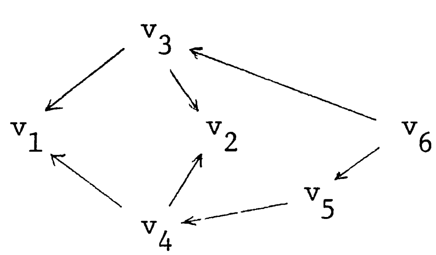
3) $De_{ij} = e_iJ^ke_j$, for $v_i \leq v_j$ via a path of length of k,

from which we conclude that

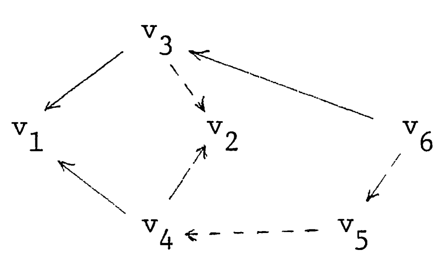
$$R = \sum_{i\leq j} De_{ij} = \sum_{i\leq j} e_iRe_j \tag{12}$$

Example: Let R be an artinian semidistributive ℓ-hereditary ring with $\{e_1,e_2,e_3,e_4,e_5,e_6\}$ a basic set of primitive idempotents. Let's assume the partial order generated by $1 \leq 3,4$; $2 \leq 3,4$; $3,4 \leq 6$; $4 \leq 5$.

The quiver Q of R can be pictured as



Let's choose the spanning tree T as the solid arrows below



Then T determines $e_{13}, e_{14}, e_{24}, e_{36}, e_{56}$ and $\sigma_{13}, \sigma_{14}, \sigma_{24}, \sigma_{36}, \sigma_{56}$ such that

$$ze_{13} = e_{13}(z)\sigma_{13}, \qquad ze_{14} = e_{14}(z)\sigma_{14}, \qquad ze_{24} = e_{24}(z)\sigma_{24}$$

$$ze_{36} = e_{36}(z)\sigma_{36}, \qquad ze_{56} = e_{56}(z)\sigma_{56},$$

for z in the correct e_iRe_i.

From $v_4 \leftarrow v_5$ we get $\bar{e}_{45} \in e_4Je_5$ and then

$$e_{13}e_{36} = te_{14}\bar{e}_{45}e_{56} = e_{14}((t)\sigma_{14}\bar{e}_{45})e_{56} \quad \text{for } t \in e_1Re_1.$$

Set $e_{45} = (t)\sigma_{14}\bar{e}_{45}$; then $e_{13}e_{36} = e_{14}e_{45}e_{56}$. From $e_{45} \in e_4Je_5$ we determine β_{45}: $e_4Re_4 \to e_5Re_5$ so that $ze_{45} = e_{45}(z)\beta_{45}$ by using the fact that

$$\dim_{e_4Re_4} e_4Je_5 = 1 = \dim e_4Je_5{}_{e_5Re_5}, \text{ for then,}$$

$$e_4Je_5 = e_4Re_4e_{45} = e_{45}e_5Re_5.$$

We then notice that $\sigma_{14}^{-1}\sigma_{13}\sigma_{36}\sigma_{65}^{-1}$: $e_4Re_4 \to e_5Re_5$ and put $\bar{\sigma}_{45} = \sigma_{14}^{-1}\sigma_{13}\sigma_{36}\sigma_{65}^{-1}$ and we can pick $h_5^{45} \in \mathrm{Aut}(e_5Re_5)$ so that $\beta_{45}h_5 = \bar{\sigma}_{45}$. Similarly, we pick $g_4^{45} \in \mathrm{Aut}(e_4Re_4)$ so that $g_4^{45}\beta_{45} = \bar{\sigma}_{45}$ and we define σ_{45} by

$$\sigma_{45} = g_4^{45}\beta_{45} = \bar{\sigma}_{45} \equiv \sigma_{14}^{-1}\sigma_{13}\sigma_{36}\sigma_{56}^{-1}$$

and get $\sigma_{13}\sigma_{36} = \sigma_{14}\sigma_{45}\sigma_{56}$ and we call this map

$$\sigma_{16}: e_1Re_1 \to e_6Re_6.$$

Then,

$$e_{45}(z)\sigma_{45} = e_{45}(z)g_4^{45}\beta_{45} = (z)g_4^{45}e_{45}, \qquad g_4^{45} \in \mathrm{Aut}(e_4Re_4)$$

Similarly

$$e_{23}(z)\sigma_{23} = (z)g_2^{23}e_{23}, \qquad g_2^{23} \in \mathrm{Aut}(e_2Re_2)$$

and

$$\sigma_{12} = \sigma_{13}\sigma_{23}^{-1} = \sigma_{14}\sigma_{24}^{-1}$$

Then e_{ij} and σ_{ij} are unambiguously defined. Let

$$D = \{d\colon d = \sum_{j=1}^{6} (x)\sigma_{1j}, \quad x \in e_1Re_1\}.$$

Then

$$e_1Re_1 \cong D$$

and 1) $De_k = e_kRe_k$,

2) $De_{ij} = e_iJ^ke_j$ for $v_i \leq v_j$ via a path of length k,

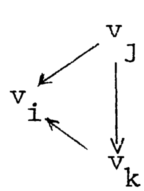
3) $R = \sum_{i \leq j} De_{ij}$.

We can say a little more about the quiver of an ℓ-hereditary semidistributive ring.

Proposition 3.21

Let R be a semidistributive ℓ-hereditary ring. Then the quiver Q of R contains no triangular circuits, that is, circuits which are formed with three edges.

Proof: Assume



is a triangular circuit in Q. Then

$$Je_k/J^2e_k \cong Re_i/Je_i \oplus Re_j/Je_j \oplus L$$

where L is a semisimple R-module.

Also,

$$Je_j/J^2e_j \cong Re_i/Je_i \oplus K$$

for some semisimple R-module K.

Let

$$Re_i/Je_i \oplus Re_j/Je_j \cong N/J^2e_k, \quad N \leq Je_k,$$

and let $p: Re_i \oplus Re_j \to N/J^2e_k$ be a projective cover.

Consider the following diagram

$$\begin{array}{ccc} & & Re_i \oplus Re_j \\ & & \downarrow p \\ N & \xrightarrow{\quad \eta \quad} & N/J^2e_k \end{array}$$

with η the natural projection. Then, there exists a nonzero homomorphism

$$\phi: Re_i \oplus Re_j \to N \text{ such that } \phi\eta = p.$$

Since $N \leq Je_k \leq Re_k$, ϕ/Re_i and ϕ/Re_j are either both monomorphisms or one of them is zero.

Let $\phi_i = \phi/Re_i$, $\phi_j = \phi/Re_j$. Since p is a projective cover, neither ϕ_i nor ϕ_j can be zero and consequently both must be monomorphisms, for

$N \leq Je_k \leq Re_k$. But this is a contradiction since Je_k is distributive.

The rather simple structure of the quiver of a semidistributive ℓ-hereditary ring has allowed us to construct the division ring D. It will also enable us to construct an R-module M whose existence fully characterizes those rings. Before that, we answer the following question: Given a directed graph G which contains no directed circuits, at most one edge between any two vertices and no triangular circuits. Is there a semidistributive ℓ-hereditary ring R whose quiver is G?

Proposition 3.22

Let G be a graph such that

1) There are no directed circuits in G.

2) There is at most one edge between any two vertices of G.

3) G contains no triangular circuits.

Then, there exists a semidistributive ℓ-hereditary ring R with quiver G.

Proof: Let $V = \{v_i\}_{i=1}^{n}$ be the set of vertices of G. Define, $v_i \leq v_j$ if and only if there exists a directed path in G from v_j to v_i. Thus, we induce a partial order in the set $X = \{1,\ldots,n\}$ via $i \leq j$ iff $v_i \leq v_j$. Let D be a division ring and let R be the incidence ring of D over X, that is,

$$R = \{ \sum_{i \leq j} d_{ij} x_{ij} : d_{ij} \in D,\ i,j \in X\}.$$

Then the quiver of R is G (see [9]).

Now, we proceed to the construction of M.

Theorem 3.23

Let R be an indecomposable semidistributive ring with $\{e_1,\ldots,e_n\}$

a basic set of primitive idempotents. Then, R is ℓ-hereditary if and only if there exists a left R-module M satisfying the following conditions:

1) For every $i = 1,\dots,n$ there exists a nonzero homomorphism $\alpha_i\colon Re_i \to M$.

2) If $\phi\colon Re_i \to M$, $i = 1,\dots,n$, is any nonzero homomorphism, then ϕ is a monomorphism.

Proof: Let ${}_RM$ satisfy conditions 1) and 2) in the statement and let $\psi\colon Re_i \to Re_j$ be a nonzero homomorphism. Then, there exists a monomorphism $\alpha_j\colon Re_j \to M$; by composing with ψ we obtain a nonzero map $\psi\alpha_j\colon Re_i \to M$ which by hypothesis is a monomorphism. Consequently, ψ is a monomorphism. Conversely, assume R is an ℓ-hereditary ring with quiver Q. For $i \le j$, that is, for $v_i \le v_j$, we define a map $\phi^i_j\colon Re_i \to Re_j$ via $z\phi^i_j = ze_{ij}$ where e_{ij} is the ring element in e_iRe_j determined, as before, by Q. It is then clear that ϕ^i_j is a well-defined monomorphism; moreover, if $i \le j \le k$ we easily obtain from the construction of the e_{ij}'s that $\phi^i_j\phi^j_k = \phi^i_k$. Hence, $\{Re_i,\phi^i_j\}$ is a directed system. We define $M = \varinjlim Re_i$, that is

$$M = (Re_1 \oplus \dots \oplus Re_n)/S$$

with

$$S = \langle\{x^j_i\phi^i_j\lambda_j - x^j_i\lambda_i : x^j_i \in Re_i,\ i \le j\}\rangle$$

and

$$\lambda_k\colon Re_k \to Re_1 \oplus \dots \oplus Re_n$$

the natural inclusion. We also define $\alpha_i: Re_i \to M$ via $(x)\alpha_i = (x)\lambda_i + S$. It is then clear that M satisfies 1); we claim it also satisfies 2). To see this, we will show that the maps α_i, $i = 1,\ldots,n$ are monomorphisms.

First, we notice that every $x_i \in Re_i$ can be written as

$$x_i = \sum_{k\leq i} x_{ki}, \quad x_{ki} \in De_{ki}, \ k \leq i.$$

Let $z \in Re_i$, $z \neq 0$ such that $(z)\alpha_i = 0$, then $z\lambda_i \in S$ and we can write

$$z\lambda_i = \sum_{\ell\leq j} [x_\ell^j \phi_j^\ell \lambda_j - x_\ell^j \lambda_\ell], \quad x_\ell^j \in Re_\ell. \tag{1}$$

Also, if $\pi_k: \bigoplus_{i=1}^{n} Re_i \to Re_k$, $k = 1,\ldots,n$, denotes the natural projections, we have, for fixed k,

$$z\lambda_i\pi_k = \sum_{\ell\leq j} [x_\ell^j \phi_j^\ell \lambda_j - x_\ell^j \lambda_\ell]\pi_k$$

$$= (\sum_{\ell<k} x_\ell^k \phi_k^\ell)\lambda_k\pi_k - (\sum_{k<\ell} x_k^\ell)\lambda_k\pi_k$$

$$= (\sum_{\ell<k} x_\ell^k \phi_k^\ell - \sum_{k<\ell} x_k^\ell)\lambda_k\pi_k$$

$$= \sum_{\ell<k} x_\ell^k \phi_k^\ell - \sum_{k<\ell} x_k^\ell, \quad x_j^s \in Re_j.$$

So,

$$z\lambda_i\pi_k = \sum_{\ell<k} x_\ell^k \phi_k^\ell - \sum_{k<\ell} x_k^\ell = \begin{cases} 0 & k \neq i \\ z & k = i \end{cases}. \tag{2}$$

Furthermore, if we write, as in the remark above,

$$z = \sum_{\mu\leq i} z_\mu, \ z_\mu = e_\mu z \in e_\mu Re_i, \ \mu \leq i$$

$$x_\ell^j = \sum_{t\leq\ell} x_{t\ell}^j, \ x_{t\ell}^j = e_t x_\ell^j \in e_t Re_\ell, \quad t \leq \ell.$$

From (2) we obtain

$$z\lambda_i\pi_k = \sum_{\mu\leq i}(z_\mu\lambda_i\pi_k) = \sum_{\ell<k}\sum_{j\leq\ell} x^k_{j,\ell}\phi^\ell_k - \sum_{k<\ell}\sum_{j\leq k} x^\ell_{j,k}$$

$$= \sum_{j\leq\ell}\sum_{\ell<k} x^k_{j,\ell}\phi^\ell_k - \sum_{j\leq k}\sum_{k<\ell} x^\ell_{j,k} = \begin{cases} 0 & k\neq i \\ z & k=i \end{cases}.$$

Hence, for fixed j, $j \leq k$ we get

$$\sum_{j\leq\ell<k} x^k_{j,\ell}\phi^\ell_k - \sum_{j\leq k<\ell} x^\ell_{j,k} = \begin{cases} 0 \in e_j Re_k & \text{if } k\neq i \\ z_j \in e_j Re_k & \text{if } k=i \end{cases}. \tag{3}$$

Next, we remark that if Re_n is the unique maximal element respect to the partial order $\leq$ induced by Q, then $\alpha_n : Re_n \to M$ is an isomorphism in which case there is nothing to prove. (See for example [16].)

Finally, we see that because of the relation $\alpha_i = \phi^i_j\alpha_j$ for $i \leq j$, it suffices to show that α_k is monomorphism for maximal indices k.

We will then assume that Re_m and Re_n are maximal with respect to $\leq$. So, let $z \in Re_m$ be such that $z\alpha_m = 0$, $z = \sum_{j\leq m} z_j$; then, $z_j\alpha_m = 0$, $j < m$.

Applying (3) we obtain

$$\sum_{j\leq\ell<k} x^k_{j,\ell}\phi^\ell_k - \sum_{j\leq k<\ell} x^\ell_{j,k} = 0 \in e_j Re_k \tag{3'}$$

for every k such that $j \leq k$, $k \neq m$. More explicitly, if $\ell_1, \ell_2, \ldots, \ell_{k_\ell}$ are the indexes such that there is a path of length 1 from v_{ℓ_s}, $s = 1,\ldots,k_\ell$, to v_j; $h_1, h_2, \ldots, h_{k_n}$ are the indexes for which there is a path of length 2 from v_{h_s}, $s = 1,\ldots,k_h$ to v_j; $i_1, i_2, \ldots, i_{k_i}$ the indexes for which there is a path of length 3 from v_{i_s}, $s = 1,\ldots,k_i$ to v_j, etc. We can write

$$-\sum_{j<\ell} x^\ell_{j,j} = 0 \in e_j Re_j \tag{4}$$

$$(x_{j,j}^{\ell_1})\phi_{\ell_1}^{j} - \sum_{\ell_1<\ell} x_{j,\ell_1}^{\ell} = 0 \in e_j Re_{\ell_1} \tag{5}$$

$$(x_{j,j}^{\ell_2})\phi_{\ell_2}^{j} - \sum_{\ell_2<\ell} x_{j,\ell_2}^{\ell} = 0 \in e_j Re_{\ell_2} \tag{6}$$

. . . .

$$(x_{j,j}^{\ell_{k_\ell}})\phi_{\ell_{k_\ell}}^{j} - \sum_{\ell_{k_\ell}<\ell} x_{j,\ell_{k_\ell}}^{\ell} = 0 \in e_j Re_{\ell_{k_\ell}} \tag{7}$$

$$\sum_{j\leq\ell<h_1} (x_{j,\ell}^{h_1})\phi_{h_1}^{\ell} - \sum_{j<h_1<\ell} x_{j,h_1}^{\ell} = 0 \in e_j Re_{h_1} \tag{8}$$

$$\sum_{j\leq\ell<h_2} (x_{j,\ell}^{h_2})\phi_{h_2}^{\ell} - \sum_{j<h_2<\ell} x_{j,h_2}^{\ell} = 0 \in e_j Re_{h_2} \tag{9}$$

. . . .

. . . .

$$\sum_{j\leq\ell<h_{k_h}} (x_{j,\ell}^{h_{k_h}})\phi_{h_{k_h}}^{\ell} - \sum_{j<h_{k_h}<\ell} x_{j,h_{k_h}}^{\ell} = 0 \in e_j Re_{h_{k_h}} \tag{10}$$

$$\sum_{j\leq\ell<i_1} (x_{j,\ell}^{i_1})\phi_{i_1}^{\ell} - \sum_{j<i_1<\ell} x_{j,i_1}^{\ell} = 0 \in e_j Re_{i_1} \tag{11}$$

$$\sum_{j\leq\ell<i_2} (x_{j,\ell}^{i_2})\phi_{i_2}^{\ell} - \sum_{i_2<\ell} x_{j,i_2}^{\ell} = 0 \in e_j Re_{i_2} \tag{12}$$

. . . .

$$\sum_{j\leq\ell<i_{k_i}} (x_{j,\ell}^{i_{k_i}})\phi_{i_{k_i}}^{\ell} - \sum_{i_{k_i}<\ell} x_{j,i_{k_i}}^{\ell} = 0 \in e_j Re_{i_{k_i}} \tag{13}$$

. . . .

$$\sum_{j<\ell<n} (x_{j,\ell}^{n})\phi_n^{\ell} = 0 \in e_j R e_n. \tag{14}$$

Now, we notice that ϕ_q^p/e_pRe_p: $e_pRe_p \to e_pRe_q$ is an isomorphism whenever $p \le q$. Hence there exists a unique $y_j^{\ell,\ell_1} \in e_jRe_j$, $\ell > \ell_1$ such that

$$x_{j,\ell_1}^{\ell} = (y_j^{\ell,\ell_1})\phi_{\ell_1}^{j}. \tag{*}$$

Introducing this in (5) we get

$$x_{jj}^{\ell_1} - \sum_{\ell_1<\ell} y_j^{\ell,\ell_1} = 0 \in e_jRe_j. \tag{5'}$$

By a similar argument, (6) and (7) will yield

$$x_{j,j}^{\ell_2} - \sum_{\ell_2<\ell} y_j^{\ell,\ell_2} = 0 \in e_jRe_j \tag{6'}$$

. . . .

$$x_{j,j}^{\ell_{k_\ell}} - \sum_{\ell_{k_\ell}<\ell} y_j^{\ell,\ell_{k_\ell}} = 0 \in e_jRe_j. \tag{7'}$$

From (8), (9) and (10) we obtain, after using $\phi_j^i\phi_k^j = \phi_k^i$,

$$x_{j,j}^{h_1} + y_j^{h_1,\ell_s} - \sum_{h_1<\ell} y_j^{\ell,h_1} = 0 \in e_jRe_j \tag{8'}$$

$$x_{j,j}^{h_2} + y_j^{h_2,\ell_s} - \sum_{h_2<\ell} y_j^{\ell,h_2} = 0 \in e_jRe_j \tag{9'}$$

. . . .

$$x_{j,j}^{h_{k_h}} + y_j^{h_{k_h},\ell_s} - \sum_{h_{k_h}<\ell} y_j^{\ell,h_{k_h}} = 0 \in e_jRe_j. \tag{10'}$$

From (11), (12) and (13) we obtain

$$x^{i_1}_{j,j} + y_j^{i_1,\ell_s} + y_j^{i_1,\ell_t} - \sum_{i_1<\ell} y_j^{\ell,i_1} = 0 \tag{11'}$$

$$x^{i_2}_{j,j} + y_j^{i_2,\ell_s} + y_j^{i_2,\ell_t} - \sum_{i_2<\ell} y_j^{\ell,i_2} = 0 \tag{12'}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$x^{i_{k_i}}_{j,j} + y_j^{i_{k_i},\ell_s} + y_j^{i_{k_i},\ell_t} - \sum_{i_{k_i}<\ell} y_j^{\ell,i_{k_i}} = 0. \tag{13'}$$

Finally from (14) we get

$$x^{n}_{j,j} + \sum_{j<\ell<n} y_j^{n,\ell} = 0. \tag{14'}$$

Adding (5') through (14') we get

$$- \sum_{\substack{\ell_1<\ell\leq m \\ \ell\neq n}} y_j^{\ell,\ell_1} - \sum_{\substack{\ell_2<\ell\leq m \\ \ell\neq n}} y_j^{\ell,\ell_2} - \cdots - \sum_{\substack{h_1<\ell\leq m \\ \ell\neq n}} y_j^{\ell,h_1} - \sum_{\substack{h_2<\ell\leq m \\ \ell\neq n}} y_j^{\ell,h_2}$$

$$- \cdots - \sum_{\substack{i_1<\ell\leq m \\ \ell\neq n}} y_j^{\ell,i_1} - \sum_{\substack{i_2<\ell\leq m \\ \ell\neq n}} y_j^{\ell,i_2} - \cdots - \sum_{\substack{\ell\leq m \\ \ell\neq n}} x^{\ell}_{j,j} = 0. \tag{15}$$

Applying ϕ^j_m to (15) and applying ϕ^k_m to (3') with $j < k < m$, $k \neq n$ we obtain,

$$- \sum_{\substack{\ell_1<\ell\leq m \\ \ell\neq n}} (x^{\ell}_{j,\ell_1})\phi_m^{\ell_1} - \sum_{\substack{\ell_2<\ell\leq m \\ \ell\neq n}} (x^{\ell}_{j,\ell_2})\phi_m^{\ell_2} - \cdots - \sum_{\substack{h_1<\ell\leq m \\ \ell\neq n}} (x^{\ell}_{j,h_1})\phi_m^{h_1} - \cdots$$

$$- \sum_{\substack{i_1<\ell\leq m \\ \ell\neq n}} (x^{\ell}_{j,i_1})\phi_m^{i_1} - \sum_{\substack{i_2<\ell\leq m \\ \ell\neq n}} (x^{\ell}_{j,i_2})\phi_m^{i_2} - \sum_{\substack{\ell\leq m \\ \ell\neq n}} x_{jj}\phi^j_\ell\phi^\ell_m = 0 \tag{16}$$

$$\sum_{j\leq \ell<k} x^k_{j,\ell}\phi^\ell_m - \Sigma x^\ell_{j,k}\phi^\ell_m = 0, \quad k \leq m, \; k \not< n. \tag{17}$$

Adding equations (16) and (17) gives

$$- \Sigma x^m_{j,\ell}\phi^\ell_m = 0.$$

Hence $z_j = \Sigma x^m_{j,\ell}\phi^\ell_m = 0$ as needed. This establishes that the maps α_j are monomorphisms.

If $m \in M$, then

$$m = \sum_{i=1}^{n} (m_i)\lambda_i + S$$

$$= \Sigma\, m_i(e_i)\alpha_i, \qquad m_i \in Re_i.$$

But

$$m_i = \sum_{k\leq i} e_k m_i = \sum_{k\leq i} m_{ki}, \qquad m_{ki} = e_k m_i \in e_k Re_i$$

and by (*) we can write $m_{ki} = x^i_k\phi^k_i$ for unique $x^i_k \in e_k Re_k$. Let

$$r_{kk} = m_{kk} - \sum_{i\geq k} x^i_k \in e_k Re_k, \qquad k = 1,\ldots,n$$

and let

$$r = \sum_{k=1}^{n} r_{kk}(e_k)\alpha_k.$$

Then

$$r = \sum_{k=1}^{n} r_{kk}(e_k)\alpha_k = \sum_k r_{kk}(e_k)\lambda_k + S = \sum_k m_k(e_k)\lambda_k + S = m.$$

Let

$$S_1^k = \sigma_{1,k}^{-1}(r_{kk}) \in e_1Re_1,$$

$$f\colon e_1Re_1 \xrightarrow{\cong} D$$

$$t \longrightarrow \sum_{j=1}^{n} (t)\sigma_{ij}$$

and $d_k = (S_{11}^k)f$, $k = 1,\ldots,n$. Then

$$r = \sum_k d_k(e_k)\alpha_k = m,$$

that is, we have proved that

$$\sum_{i=1}^{n} D(e_i)\alpha_i = M. \qquad (\Delta)$$

Now, let $g\colon Re_j \to M$ be a nonzero homomorphism; then, there exists $d_g \in D$ such that $(e_j)g = d_g(e_j)\alpha_j$ and

$$\begin{aligned}(re_j)\alpha_j = r(e_j)\alpha_j &= rd_g^{-1}d_g(e_j)\alpha_j \\ &= rd_g^{-1}(e_j)g \\ &= (rd_g^{-1}e_j)g, \qquad r \in R.\end{aligned}$$

That is, $\mathrm{Im}\alpha_j \leq \mathrm{Im}g$. But, since α_j is a monomorphism, $c(\mathrm{Im}\alpha_j) = c(Re_j)$. Hence, $c(\mathrm{Im}g) \leq c(\mathrm{Im}\alpha_j)$ and $\mathrm{Im}g = \mathrm{Im}\alpha_j$. Consequently,

$$c(\mathrm{Ker}\,g) = c(\mathrm{Ker}\,\alpha_j) = 0, \quad \text{so } \mathrm{Ker}\,g = 0.$$

If R is an incidence ring over a division ring K, the left R-module $M = \varinjlim Re_i$ which we introduced in Theorem 3.23 corresponds to the "last column vectors". As in this particular case, we can endow M with a right D-structure.

Let R be an ℓ-hereditary semidistributive ring with quiver Q. Let T be a tree in Q and let $M = \varinjlim Re_i$. We recall that for every vertex v_i in Q there exists a (possible non-directed) path from v_i to v_1 along T, σ_{1i} denotes the induced isomorphism from e_1Re_1 onto e_iRe_i. If $h \in \mathrm{Aut}(e_iRe_i)$ we will denote by $\tilde{h}$ the element in $\mathrm{Aut}(e_1Re_1)$ which make the following diagram commute.

$$\begin{array}{ccc} e_1Re_1 & \xrightarrow{\sigma_{1i}} & e_iRe_i \\ \tilde{h}\downarrow & & \downarrow h \\ e_1Re_1 & \xrightarrow[\sigma_{1i}]{} & e_iRe_i \end{array}$$

Let $i \leq j$, then there exists (at least one) a directed path

$$v_i \leftarrow v_{i_1} \cdots \leftarrow v_{i_{k-1}} \leftarrow v_j \quad \text{from } v_j \text{ to } v_i. \tag{*}$$

Let $v_{\mu_1} \leftarrow v_{\nu_1}, v_{\mu_2} \leftarrow v_{\nu_2} \cdots v_{\mu_t} \leftarrow v_{\nu_t}$ be the added arrows along the path (*), so that we have

$$v_i \leftarrow \cdots \leftarrow v_{\mu_1} \leftarrow v_{\nu_1} \leftarrow \cdots \leftarrow v_{\mu_2} \leftarrow v_{\nu_2} \leftarrow \cdots \leftarrow v_{\mu_t} \leftarrow v_{\nu_t} \leftarrow \cdots \leftarrow v_j.$$

Each $v_{\mu_m} \leftarrow v_{\nu_m}$ induces a twisting $h_{\mu_m}^{\mu_m\nu_m} \in \mathrm{Aut}(e_{\mu_m}Re_{\mu_m})$, $m = 1,\ldots,t$, and a commutative diagram as in (4).

$$
\begin{array}{ccccccccc}
e_iRe_i & \xrightarrow{\sigma_{ii_1}} & e_{i_1}Re_{i_1} & \rightarrow \cdots \rightarrow & e_{\mu_m}Re_{\mu_m} & \xrightarrow{\sigma_{\mu_m\nu_m}} & e_{\nu_m}Re_{\nu_m} & \rightarrow \cdots \rightarrow & e_jRe_j \\
\downarrow h_i^{\mu_m\nu_m} & & \downarrow h_{i_1}^{\mu_m\nu_m} & & \downarrow h_{\mu_m}^{\mu_m\nu_m} & & \downarrow 1 & & \downarrow 1 \\
e_iRe_i & \xrightarrow{\sigma_{ii_1}} & e_{i_1}Re_{i_1} & \rightarrow \cdots \rightarrow & e_{\mu_m}Re_{\mu_m} & \xrightarrow{\beta_{\mu_m\nu_m}} & e_{\nu_m}Re_{\nu_m} & \rightarrow \cdots \rightarrow & e_jRe_j \\
\downarrow 1 & & \downarrow 1 & & \downarrow 1 & & \downarrow h_{\nu_m}^{\mu_m\nu_m} & & \downarrow h_j^{\mu_m\nu_m} \\
e_iRe_i & \xrightarrow{\sigma_{ii_1}} & e_{i_1}Re_{i_1} & \rightarrow \cdots \rightarrow & e_{\mu_m}Re_{\mu_m} & \xrightarrow{\sigma_{\mu_m\nu_m}} & e_{\nu_m}Re_{\nu_m} & \rightarrow \cdots \rightarrow & e_jRe_j
\end{array}
$$

for $m = 1,2,\ldots,t$.

Chasing these diagrams we can write

$$\sigma_{ii_0} \cdots (h_{\mu_1}^{\mu_1\nu_1})^{-1}\sigma_{\mu_1\nu_1} \cdots (h_{\mu_2}^{\mu_2\nu_2})^{-1}\sigma_{\mu_2\nu_2} \cdots (h_{\mu_t}^{\mu_t\nu_t})^{-1}\sigma_{\mu_t\nu_t} \cdots \sigma_{i_{k-1}j}$$

$$= (h_i^{\mu_i\nu_i})^{-1}(h_i^{\mu_2\nu_2})^{-1} \cdots (h_i^{\mu_t\nu_t})^{-1}\sigma_{ii_0} \cdots \sigma_{\mu_1\nu_1} \cdots \sigma_{\mu_t\nu_t} \cdots \sigma_{i_{k-1}j}.$$

Let $f_{ij} = (h_i^{\mu_1\nu_1})^{-1}(h_i^{\mu_2\nu_2})^{-1} \cdots (h_i^{\mu_t\nu_t})^{-1} \in \mathrm{Aut}(e_iRe_i)$. By (11), f_{ij} is independent of the path from v_j to v_i. Let

$$\bar{m} = m + S = \sum_{i\leq j} d_{ij}(e_{ij})\alpha_j \in M \text{ and let } d = \sum_{k=1}^{n} (x)\sigma_{1k} \in D,\ x \in e_1Re_1.$$

Define

$$\bar{m}d = \sum_{i\leq j} d_{ij}(e_{ij}(x)(\tilde{f}_{ij})^{-1}\sigma_{1j})\alpha_j.$$

If $\bar{m}_1 = \bar{m}$, then

$$m - m_1 = \sum_{i\leq j} (x_i^j\phi_j^i\lambda_i - x_i^j\lambda_i) \in S.$$

Let $z_k = (m - m_1)\pi_k \in Re_k$,

$$z_k = \sum_{j\leq k} e_j z_k = \sum_{j\leq k} z_{j,k}, \quad z_{j,k} \in e_jRe_k.$$

Then

$$z_{j,k} = \sum_{j\leq\ell<k} x^{k}_{j,\ell}\phi^{\ell}_{k} - \sum_{j\leq k<\ell} x^{\ell}_{j,k}, \quad x^{j}_{i,\ell} \in e_i Re_\ell .$$

By chasing diagrams (4) and recalling formulas (3)

$$e_{\mu_m\nu_m}(z)\sigma_{\mu_m\nu_m} = (z)h^{\mu_m\nu_m}_{\mu_m} e_{\mu_m\nu_m}$$

$$m = 1,2,\ldots,t$$

$$te_{\mu_m\nu_m} = e_{\mu_m\nu_m}(t)(h^{\mu_m\nu_m}_{\mu_m})^{-1}\sigma_{\mu_m\nu_m}$$

we have that

$$x^{j}_{i,\ell}e_{\ell j}(x)(\tilde{f}_{ij})^{-1}\sigma_{1j} = x^{j}_{i,\ell}(x)\sigma_{1\ell}e_{\ell j}.$$

That is, $\bar{m}d$ is well defined.

Also, since each $e_k Je_\ell$ is a $e_k Re_k - e_\ell Re_\ell$ bimodule for $k \leq \ell$ and since α_j is an R-homomorphism, M has an $(R - D)$ bimodule structure.

We can now establish the following Corollaries.

Corollary 3.24

Let R be a semidistributive ℓ-hereditary ring, let $M = \varinjlim Re_i$ and let $N \leq M$. Then

$$N = \sum_j \{D(e_j)\alpha_j : (e_j)\alpha_j \in N\}.$$

Proof: Let $x \in N$, $x = \sum_{j=1}^{n} d_j(e_j)\alpha_j$, $d_j \in D$. If $d_k \neq 0$, we have

$$e_k d_k^{-1}x = e_k d_k^{-1}\left(\sum_{i=1}^{n} d_i(e_i)\alpha_i\right) = (e_k)\alpha_k.$$

That is, $e_k d_k^{-1}x = (e_k)\alpha_k$ and hence $(e_k)\alpha_k \in N$.

Corollary 3.25

Let R be a semidistributive ℓ-hereditary ring, $M = \varinjlim Re_i$. Then

1) Every nonzero homomorphism $g: M \to M$ is a monomorphism.

2) Every nonzero homomorphism $g: M \to M$ is an epimorphism.

Moreover, $\mathrm{End}_R(M)$ is a division ring isomorphic to D.

Proof:

1) Let $g: M \to M$, $g \neq 0$, define $g_i = \alpha_i g: Re_i \to M$. By Theorem 3.23 g_k is a monomorphism for every k.

Let $x \in \mathrm{Ker}\, g$, let $\{e_{i_k}\}_k$ be such that $(e_{i_k})\alpha_{i_k} \in \mathrm{Ker}\, g$. We can write

$$x = \sum_{k=1}^{t} r_{i_k i_k}(e_{i_k})\alpha_{i_k}, \qquad r_{i_k i_k} \in e_{i_k} Re_{i_k} \quad \text{(see page 63)}$$

Then

$$(x)g = \sum_{k=1}^{t} r_{i_k i_k}(e_{i_k})g_{i_k} = 0.$$

But M is distributive, hence $r_{i_k i_k}(e_{i_k})g_{i_k} = (r_{i_k i_k} e_{i_k})g_{i_k} = 0$; g_{i_k} being a monomorphism implies $x = 0$, that is, g is a monomorphism.

2) Let $g: M \to M$ be a nonzero homomorphism. Since M is a factor of R, it has finite length and hence $M = \mathrm{Ker}\, g^{\ell} \oplus \mathrm{Im}\, g^{\ell}$. By 1), g and consequently g^{ℓ} is a monomorphism.

Hence $M = \mathrm{Im}\, g^{\ell} \leq \mathrm{Im}\, g$. That is, g is an epimorphism.

We have established then that $\mathrm{End}({}_R M)$ is a division ring.

For $d \in D$, define $\psi_d: M \to M$ via $(m)\psi_d = md$. Then, since M is an R-D bimodule, $\psi_d \in \mathrm{End}({}_R M)$. The map $\psi: D \to \mathrm{End}_R(M)$ is a monomorphism, moreover, if $h \in \mathrm{End}({}_R M)$ then $h/(Re_i)\alpha_i \in \mathrm{End}((Re_i)\alpha_i)$. Hence

$$h/(Re_i)\alpha_i = \psi_d, \quad \text{for some } d \in D \text{ and } \quad h = \psi_d, \quad d \in D.$$

That is, ψ is an epimorphism and consequently an isomorphism.

Proposition 3.26

Let R be a semidistributive ℓ-hereditary ring, $M = \varinjlim Re_i$, $\alpha_i: Re_i \to M$ the canonical monomorphism. Let

$$L_k = \sum_j \{Re_j\alpha_j : j \not\geq k\}$$
$$= \sum_j \{De_j\alpha_j : j \not\geq k\}.$$

If $E_k = M/L_k$ then $Soc(E_k) \cong Re_k/Je_k$.

Proof: First notice that since M is distributive, so is E_k and hence $Soc(E_k)$ is square free [5]. Let $i > k$. Then,

$$0 \neq (e_k)\alpha_k + L_k = (e_k)\phi_i^k\alpha_i + L_k = (e_{ki})\alpha_i + L_k.$$
$$= e_{ki}(e_i)\alpha_i + L_k.$$

But $e_{ki} \in J$ and hence $J(e_i)\alpha_i + L_k \neq 0 \in E_k$. Because of $Soc(E_k) = r_{E_k}(J)$ we have

$$(e_i)\alpha_i + L_k \notin Soc(E_k) \cong Re_{k_i}/Je_{k_i} \oplus \dots \oplus Re_{k_\ell}/Je_{k_\ell}.$$

Hence, $i \neq k_1,\dots,k_\ell$. Then $e_i(Re_{k_j}/Je_{k_j}) = 0$ and $e_i Soc(E_k) = 0$. Let i be such that $i \not\geq k$. If $(e_i)\alpha_i \in Soc(E_k)$ then

$$Re_i/Je_i \cong Re_{k_j}/Je_{k_j}, \quad j = 1,\dots,\ell,$$

that is, $i = k_j$ and $(e_i)\alpha_i \notin L_k$ which is a contradiction. Hence,

$$(e_i)\alpha_i \notin Soc(E_k) \quad \text{for } i \not\geq k.$$

Because $Soc(E_k) \neq 0$, we conclude that $Soc(E_k) \cong Re_k/Je_k$.

Proposition 3.27

Let R, M, E_k be as in Proposition 3.26. Then E_k is isomorphic to $E(Re_k/Je_k)$, the injective envelope of Re_k/Je_k.

Proof: We have established that $Soc(E_k) \cong Re_k/Je_k$. Because R is semidistributive and ℓ-hereditary we obtain that

$$c(E(Re_k/Je_k)) = c(e_kR) \qquad [8].$$

But $c(e_kR) = \#\{j: k \leq j\} = n - \#\{j: j \not\geq k\}$

$$= c(M) - c(L_k) = c(E_k).$$

That is, $c(E(Re_k/Je_k)) = c(E_k)$. But $E(Soc(E_k)) = E(E_k) = E(Re_k/Je_k)$ from which it follows that

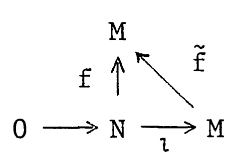
$$E_k = E(E_k) = E(Re_k/Je_k).$$

We now state a proposition which besides being interesting in its own right, will be useful in proving our main result.

Proposition 3.28

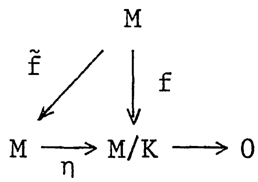
Let R be a semidistributive ℓ-hereditary ring, $M = \varinjlim Re_i$.

1) For every indecomposable submodule $N \leq M$ and every nonzero homomorphism $f: N \to M$, there exists a unique homomorphism $\tilde{f}: M \to M$ making the following diagram commute.



2) For every submodule $K \leq M$ such that M/K is a nonzero indecompos-

able factor of M and every nonzero homomorphism $f\colon M \to M/K$, there exists a unique homomorphism $\tilde{f}\colon M \to M$ such that the following diagram commutes.



Proof:

1) Let $N \leq M$ and let $\{e_{i_k}\}_k$ be the collection of idempotents of R such that $(e_{i_k})\alpha_{i_k} \in N$. Because $M = \sum_{i=1}^{n} D(e_i)\alpha_i$, we have

$$c({}_{e_jRe_j}e_jM) = 1 \text{ for every } j \text{ and then } N = eM \text{ for } e = \sum_k e_{i_k}.$$

Also, since N is indecomposable, the vertices $\{v_{i_k}\}_k$ form a connected subgraph of Q and $N = \varinjlim_k Re_{i_k}$. Moreover, since M is distributive, $Nf \leq N$. We conclude then, by applying Propositions 3.24 and 3.25 to ${}_{eRe}N$, that there exists $d \in D$ such that $nf = nd$, $n \in N$.

Define

$$\bar{\alpha}_i\colon Re_i \to M \quad \text{via} \quad z\bar{\alpha}_i = z\alpha_i d,\ i = 1,\ldots,n.$$

Then $z\bar{\alpha}_{i_k} = z\alpha_{i_k} f$. For $i \leq j$,

$$e_i\phi^i_j\bar{\alpha}_j = e_{ij}\bar{\alpha}_j = (e_{ij})\alpha_j d$$

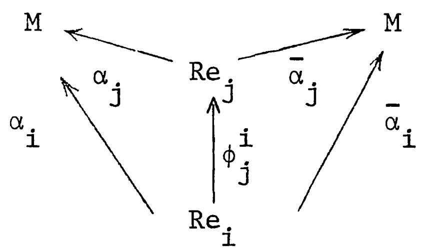
$$= (e_{ij}(x)\sigma_{1j}f_j^{-1})\alpha_j$$

$$= ((x)\sigma_{1i}e_{ij})\alpha_j$$

$$= (x\sigma_{1i}e_i)\phi^i_j\alpha_j = (x\sigma_{1i}e_i)\alpha_i$$

$$= (e_i(x)\sigma_{1i})\alpha_i = e_i\bar{\alpha}_i.$$

That is, $\phi^i_j\bar{\alpha}_j = \bar{\alpha}_i$ for $i \leq j$ and the following diagram commutes.

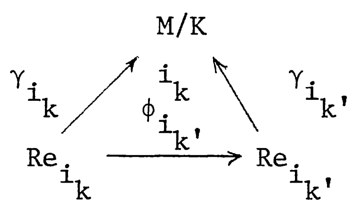


Hence, since $M = \varinjlim Re_i$, there exists a unique homomorphism $\tilde{f}: M \to M$ such that $\alpha_j\tilde{f} = \bar{\alpha}_j \ \forall j$. Clearly, $\tilde{f}$ extends f.

2) Let $K \leq M$ such that M/K is indecomposable and let $f: M \to M/K$ be an homomorphism. Define $(m+K)d = md + K$. Since $D \cong End(M)$, and since M is distributive, $Kd \leq K$ and M/K is a right D-module.

Let $\eta: M \to M/K$ be the natural epimorphism and let $\{e_{i_k}\}$ be the collection of idempotents such that $0 \neq (e_{i_k})\alpha_{i_k}\eta \in M/K$, that is, $(e_{i_k})\alpha_{i_k} \notin K$.

Let $\gamma_i = \alpha_i\eta$, then for $i_k \leq i_{k'}$



commutes

and $M/K = \varinjlim_k Re_{i_k}$.

Also, since $K \leq Ker\, f$ and M is distributive, f induces $f': M/K \to M/K$ so that $\eta f' = f$. Let $e = \sum_k e_{i_k}$, then $e(M/K)$ is an eRe-module and there exists $d \in D$ such that $(m+K)f' = md + K$.

Define $\tilde{f}: M \to M$, via $mf = md$. Then $m\tilde{f}\eta = m\tilde{f} + K = md + K = (m+K)f' = m\eta f' = mf$ for every $m \in M$. If $g: M \to M$ is such that $mg + K = m\tilde{f}$ for every $m \in M$ then $M(g - \tilde{f}) \leq K$, that is $Im(g - \tilde{f}) \neq M$ and by Corollary 3.25 $g = \tilde{f}$.

Now, we state and prove our main result.

Theorem 3.29

Let R be an indecomposable semidistributive ℓ-hereditary ring. Then R has a weakly symmetric duality.

Proof: By results in [1] we may assume R is basic. Let $\{e_1,\ldots,e_n\}$ be a basic set of primitive idempotents of R and let $E = E_1 \oplus \ldots \oplus E_n$ where

$$E_k = M/L_k = E(Re_k/Je_k), \quad M = \varinjlim Re_i, \text{ and } L_j \text{ is as in Proposition 3.26.}$$

Then E_k is an indecomposable injective R-module and, because R is basic, E is the minimal injective cogenerator in ${}_R M$.

Let $S = \mathrm{End}({}_R E)$ and let $w \in S$, then $w = \sum_{i,j} \psi^i_j$ with $\psi^i_j = \pi_i w \pi_j$, $\pi_k\colon E \to E_k$ the natural projection.

Since R is ℓ-hereditary, ψ^i_j is either zero or onto; moreover, if $i \not\leq j$ then $\psi^i_j = 0$ and consequently $w = \sum_{i \leq j} \psi^i_j$. We have that $\psi^i_j \neq 0$ if and only if $(e_j)\alpha_j + L_i \neq 0$. This implies that if $i \leq j$ then $0 \neq \phi^i_j \in \mathrm{Hom}(Re_i, Re_j)$. Consider the solid part of the following diagram

$$\begin{array}{ccccc}
Re_i & \longrightarrow & M & \xrightarrow{\eta_i} & E_i \\
\downarrow & & \downarrow & & \downarrow \psi^i_j \\
Re_j & \xrightarrow[\alpha_j]{} & M & \xrightarrow[\eta_j]{} & E_j
\end{array}$$

Then there exists $\delta\colon M \to M$ such that $\eta_i \psi^i_j = \delta \eta_j$. Hence, by Corollary 3.25 there exists a unique $d \in D$ such that $(m)\delta = md$. Define

$$\gamma^i_j\colon Re_i \to Re_j \quad \text{via} \quad (z)\gamma^i_j = ze_{ij}d = (z)\phi^i_j d.$$

Then the map $\theta\colon \psi^i_j \to \gamma^i_j$ is well defined and one-one.

Let $\theta: S \to R$, $\theta(\sum_{i \leq j} \psi_j^i) = \sum_{i \leq j} \gamma_j^i$. Then θ is a bijection.

Let $w' = \sum_{i \leq j} \psi_j'^i$, $\psi_j'^i = \pi_i w' \pi_j$. Then, if δ' and $\gamma_j'^i$ are the maps associated with $\psi_j'^i$ we have that

$$\delta + \delta' \quad \text{and} \quad \gamma_j^i + \gamma_j'^i \quad \text{correspond to} \quad \psi_j^i + \psi_j'^i.$$

Consequently, θ is additive.

Also, if $i \leq j \leq k$, $\gamma_j^i \gamma_k'^j$ and $\delta\delta'$ correspond to $\psi_j^i \psi_k'^j$, that is, θ is a ring isomorphism. Also, $\theta(\psi_i^i) = \gamma_i^i$ and R has a weakly symmetric duality (Theorem 3.8).

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