FIXED-SIZE CONFIDENCE REGIONS FOR THE MEAN VECTOR OF A MULTINORMAL DISTRIBUTION

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CHAPTER I

INTRODUCTION AND REVIEW

1.1 Introduction

Let $\underset{\sim}{X}_1, \underset{\sim}{X}_2, \ldots, \underset{\sim}{X}_n, \ldots$ be a sequence of independent and identically distributed (i.i.d.) random variables with each $\underset{\sim}{X}$ being distributed as $N_p(\underset{\sim}{\mu}, \sigma^2 H)$, where $\underset{\sim}{\mu} \in \mathbb{R}^p$ is the unknown mean vector, $\sigma \in (0,\infty)$ is an unknown scale parameter, and H is a known p×p positive definite matrix.

The problem of constructing a fixed-size confidence region for $\underset{\sim}{\mu}$ is formulated as follows. Given two preassigned numbers $d \in (0,\infty)$ and $\alpha \in (0,1)$ and having recorded n (≥ 2) samples $\underset{\sim}{X}_1, \underset{\sim}{X}_2, \ldots, \underset{\sim}{X}_n$, we propose the following ellipsoidal confidence region for $\underset{\sim}{\mu}$:

$$R_n = \{\underset{\sim}{\omega} \in \mathbb{R}^p: \ (\bar{\underset{\sim}{X}}_n - \underset{\sim}{\omega})' \ H^{-1}(\bar{\underset{\sim}{X}}_n - \underset{\sim}{\omega}) \leq d^2\}, \qquad \ldots(1.1)$$

where $\bar{\underset{\sim}{X}}_n = \frac{1}{n} \sum_{i=1}^{n} \underset{\sim}{X}_i$. Let us use the notations $P(\cdot)$ and $E(\cdot)$ instead of $P_{\underset{\sim}{\mu},\sigma}(\cdot)$ and $E_{\underset{\sim}{\mu},\sigma}(\cdot)$, respectively, from this point onward. Now, the confidence coefficient associated with the region R_n is given by

$$\begin{aligned} P\{\underset{\sim}{\mu} \in R_n\} &= P\{(\bar{\underset{\sim}{X}}_n - \underset{\sim}{\mu})' \ H^{-1}(\bar{\underset{\sim}{X}}_n - \underset{\sim}{\mu}) \leq d^2\} \\ &= P\{(n/\sigma^2)(\bar{\underset{\sim}{X}}_n - \underset{\sim}{\mu})' \ H^{-1}(\bar{\underset{\sim}{X}}_n - \underset{\sim}{\mu}) \leq nd^2/\sigma^2\} \\ &= F(nd^2/\sigma^2), \end{aligned} \qquad \ldots(1.2)$$

where $F(u) = P\{U \leq u\}$ with U being distributed as χ^2 with p degrees of freedom. The region R_n is constructed in a way such that the length of its maximum diameter is at the most 2d. This fact is referred to as the region R_n being an ellipsoid of "fixed-size".

We also require that the confidence coefficient be at least $1-\alpha$, and thus the sample size n should be at least $a\sigma^2/d^2 = C$, say, where $F(a) = 1-\alpha$. This number "a" can easily be found from the chi-square tables. Our "C" is referred to as the optimal fixed sample size required had σ^2 been known. However, C is unknown since σ^2 is unknown, and thus no fixed-sample-size approach is feasible for our use.

For the sake of completeness, we now state definitions of some properties for any particular procedure giving rise to the stopping time, say, N.

Definitions:

(a) A procedure is called *consistent* in the Chow-Robbins (1965) sense if

$$P\{\underset{\sim}{\mu} \in R_{N(d)}\} \geq 1-\alpha, \qquad \ldots(1.3)$$

for all $\underset{\sim}{\mu} \in \mathbb{R}^p$ and $\sigma \in (0,\infty)$. The property (1.3) is also referred to as *exact consistency* in Mukhopadhyay (1982).

(b) A procedure is called *asymptotically consistent* in the Chow-Robbins (1965) sense if

$$\lim_{d\to 0} P\{\underset{\sim}{\mu} \in R_{N(d)}\} = 1-\alpha, \qquad \ldots(1.4)$$

for all $\underset{\sim}{\mu} \in \mathbb{R}^p$ and $\sigma \in (0,\infty)$.

(c) A procedure is called *asymptotically efficient* in the Chow-Robbins (1965) sense if

$$\lim_{d\to 0} E\{N(d)/C\} = 1, \qquad \ldots(1.5)$$

for all $\underset{\sim}{\mu} \in \mathbb{R}^p$ and $\sigma \in (0,\infty)$. The equation (1.5) is now referred to as *asymptotically first-order efficiency* property in Ghosh and Mukhopadhyay (1981).

(d) A procedure is called *asymptotically second-order efficient* in the Ghosh-Mukhopadhyay (1981) sense if

$$\lim_{d\to 0} E\{N(d) - C\} = k, \qquad \ldots(1.6)$$

for all $\underset{\sim}{\mu} \in \mathbb{R}^p$ and $\sigma \in (0,\infty)$, where k is a bounded constant. From this point onward, we will write N instead of N(d).

1.2 Review of Literature

We begin this literature review with the univariate normal theory of fixed-width interval estimation of the mean. The literature dealing with the corresponding multivariate normal theory for the mean vector is then considered next. Finally, we mention some of the work done on point estimation problems for the mean or the mean vector.

Stein (1945, 1949) developed a two-stage procedure for constructing a fixed-width confidence interval for the mean μ of a univariate normal distribution when the variance σ^2 is unknown. This procedure satisfies the properties (1.3) and (1.4), but it does not satisfy the property (1.5). See, for example, Chow and Robbins (1965) and Simons (1968).

Ray (1957) developed a purely sequential procedure to estimate the mean of a normal population with a confidence interval of preassigned width and confidence coefficient when the variance σ^2 is unknown. However, only the small sample approach was really discussed. More elaborate and thorough treatments came from Chow and Robbins (1965). This purely sequential procedure is known to satisfy the properties (1.4) and (1.5). The basic reason behind going through a sequential scheme was to achieve property (1.5). In achieving that goal, however, the sequential procedure lost the exact property of (1.3).

Recently, Mukhopadhyay (1980) proposed a two-stage procedure (this is now called the "modified two-stage procedure") for constructing a fixed-width confidence interval for the mean μ of a normal distribution when the variance σ^2 is unknown. This procedure has all the properties of (1.3), (1.4), and (1.5).

A natural question then arises. If the *asymptotic efficiency* property (1.5) can also be achieved by suitably modifying Stein's (1945, 1949) two-stage procedure, then exactly in what sense is the purely sequential procedure superior? Ghosh and Mukhopadhyay (1981) settled this issue by introducing a concept known as the *second-order efficiency* property. The sequential procedure satisfies property (1.6), whereas the modified two-stage procedure satisfies only the weaker property, namely (1.5).

Mukhopadhyay (1982) also showed that a fixed-width confidence interval for the mean of a univariate population can be constructed in a fairly reasonable way so as to achieve *exact consistency* even without the normality assumption. In Stein's construction, normality assumption is not crucial, and this was replaced by independence of some estimators of a pivotal nature. Modified two-stage procedures were also proposed along the lines of Mukhopadhyay (1980), and they were shown to be asymptotically *first-order efficient*.

Woodroofe (1977) obtained the second-order approximations of the expected sample size and the risk associated with sequential procedures of the Ray-Chow-Robbins type. Woodroofe (1977) considered both point and interval estimation of the mean of a normal distribution when the variance is unknown.

Hall (1981) studied a three-stage procedure for constructing a fixed width confidence interval for the mean μ of a univariate normal distribution when σ^2 is unknown. If a third stage was appended to Stein's two-stage procedure, it lost its exactness (property (1.3)) but it became strongly competitive with the Ray-Chow-Robbins procedure from the efficiency point of view (properties (1.4) and (1.5)). Hall (1981) considered the asymptotic theory of triple sampling as it pertained to the estimation of the mean of a univariate normal distribution. He obtained various limit theorems and expansions, and his results showed in turn that a suitable triple sampling procedure actually combines the simplicity of Stein's double sampling techniques with that of the Ray-Chow-Robbins sequential procedure.

In the area of multivariate sequential estimation, Chatterjee (1959, 1960) extended the works of Stein (1945, 1949) for developing suitable two-stage procedures in the multivariate normal case with unknown mean vector $\underset{\sim}{\mu}$ and completely unknown positive definite dispersion matrix Σ. It was demonstrated how that procedure could be used to obtain a fixed-size ellipsoidal confidence region for $\underset{\sim}{\mu}$.

Srivastava (1967) extended Chow and Robbins' (1965) sequential procedure to construct ellipsoidal or spherical confidence regions with preassigned confidence coefficients for (i) the linear regression parameters and (ii) the mean vector of a multivariate population. No assumptions regarding the population distribution were made; and as a result, all results were asymptotic in nature.

Mukhopadhyay (1979) considered the construction of fixed-size simultaneous confidence region for $\underset{\sim}{\mu}$ and Σ, the unknown mean vector and dispersion matrix of a multinormal population. Results analogous to those of Chow and Robbins (1965) and Srivastava (1967) were obtained.

Mukhopadhyay (1981) considered the problem of simultaneously estimating the mean and variance of a normal distribution. A semicircular region $\Phi_n = \{(a,b)': b > 0\}$ of radius $d > 0$ and with approximately a preassigned coverage probability $(1-\alpha)$ was proposed to solve the problem through a sequential procedure.

We must mention that there are parallel results for minimum risk point estimation of the mean vector $\underset{\sim}{\mu}$ of a multinormal population when Σ is completely unknown and positive definite. One is referred to Ghosh, Sinha, and Mukhopadhyay (1976) and Sinha and Mukhopadhyay (1976). Wang (1981) also considered the same point estimation problem when $\Sigma = \sigma^2 A$ where σ^2 is unknown and A is a known $p \times p$ positive definite matrix. The corresponding results for the univariate situation were introduced in Robbins (1959), and a brief review is available in Ghosh and Mukhopadhyay (1976). Even though we borrowed tools from the papers on related point estimation problems, we do not explicitly study herein any point estimation problem.

In the present study, we develop procedures and results along the lines of the recent contributions of Woodroofe (1977, 1982), Mukhopadhyay (1980, 1982), and Hall (1981). Some partial results along the lines of Ghosh and Mukhopadhyay (1981), and Mukhopadhyay (1982) in the direction of "first-" and "second-order efficiencies" will also be discussed. Some of the tools necessary had already been developed and utilized for a few related problems studied in Woodroofe (1977, 1982), Ghosh and Mukhopadhyay (1981), Hall (1981), and Mukhopadhyay and Hamdy (1983). We will also report some thorough computational studies conducted to make comparisons among the different procedures proposed in the present study.

The basic idea of our plan of attack is very simple. Because C is unknown, somehow we must estimate C using a suitable positive integer valued random variable N, say. Once we determine this random sample size N, we then propose the same confidence region R_N for $\underset{\sim}{\mu}$ as in (1.1) based on N samples, namely $\underset{\sim}{X}_1, \underset{\sim}{X}_2, \ldots, \underset{\sim}{X}_N$. Naturally, the characteristics of any sort of "goodness" of having this region R_N will undoubtedly depend on the "closeness" between N and C.

In Chapter II, we propose a two-stage procedure along the lines of Stein (1945, 1949), Chatterjee (1959, 1960), and Mukhopadhyay (1982).

Chapter III deals with a modified two-stage procedure to obtain "asymptotic efficiency". This procedure was motivated by the results of Mukhopadhyay (1980, 1982).

Chapter IV is devoted to the purely sequential procedure where we take one sample at a time after we start, to decide the stopping stage. We derive second-order expansions for E(N) and $P\{\underset{\sim}{\mu} \in R_N\}$ using the nonlinear renewal theory of Woodroofe (1977, 1982).

In Chapter V, we introduce a three-stage procedure. The motivation behind this procedure is as follows. After starting the experiment with m (≥ 2) samples, we estimate a fraction rC of the optimal fixed sample size C by, say, M. Then, depending on the size of M, we decide whether to obtain all the remaining samples of size N - M at the third stage. In this way, we attempt to avoid the problems of overestimating C that tend to occur while using the two-stage and the modified two-stage procedures discussed in Chapters II and III. The numerical results on simulations for all of these competitive procedures, as well as various remarks and comments on each are presented in Chapter VI.

In what follows, we write $[x]$ for the largest integer $< x$. Let us now introduce some of the preliminary information. Suppose that we have a random sample $\underset{\sim}{X}_1, \underset{\sim}{X}_2, \ldots, \underset{\sim}{X}_n$ of size n (≥ 2) from the $N_p(\underset{\sim}{\mu}, \sigma^2 H)$ population. Let $H = BB'$ where B is a known $p \times p$ matrix having full rank. Then, we let $\underset{\sim}{Y}_j = B^{-1}\underset{\sim}{X}_j$ where we write $\underset{\sim}{Y}_j = (Y_{1j}, Y_{2j}, \ldots, Y_{pj})'$, $j = 1, 2, \ldots, n$. So, $\underset{\sim}{Y}_1, \underset{\sim}{Y}_2, \ldots, \underset{\sim}{Y}_n$ are all i.i.d. $N_p(\underset{\sim}{\nu}, \sigma^2 I)$, where $\underset{\sim}{\nu} = B^{-1}\underset{\sim}{\mu}$. Also, we write $\bar{Y}_{in} = n^{-1}\sum_{j=1}^{n} Y_{ij}$, $i = 1, 2, \ldots, p$ and $S_n^2 = \{p(n-1)\}^{-1}\sum_{i=1}^{p}\sum_{j=1}^{n}(Y_{ij} - \bar{Y}_{in})^2$. It can be easily seen that S_n^2 is invariant with respect to all possible choices of the matrix B, and that indeed $S_n^2 = \{p(n-1)\}^{-1}\sum_{j=1}^{n}(\underset{\sim}{X}_j - \bar{\underset{\sim}{X}}_n)'H^{-1}(\underset{\sim}{X}_j - \bar{\underset{\sim}{X}}_n)$. For computational purposes, one can use the latter expression. However, in order to use the tools from Woodroofe (1977) and Hall (1981), we will need the expression of S_n^2 in terms of the corrected sums of squares of the Y's. We propose to estimate $\underset{\sim}{\mu}$ and σ^2 by $\bar{\underset{\sim}{X}}_n = n^{-1}\sum_{i=1}^{n}\underset{\sim}{X}_i$ and S_n^2, respectively. Now, we will consider the elliptic confidence region R_n for $\underset{\sim}{\mu}$ defined in (1.1). Notice that the particular structure we have makes it impossible to use the methods developed in Chatterjee (1959, 1960) or Srivastava (1967). Yet this structure is very common in statistical analyses. Suppose, for a simple example, we have four treatments characterized by observable independent random variables Z_1, Z_2, Z_3, and Z_4 where $Z_i \sim N(\mu_i, \sigma^2)$, $i = 1, \ldots, 4$. We now define two contrasts $\theta_1 = \mu_1 - \mu_2$ and $\theta_2 = \mu_1 + \mu_4 - 2\mu_3$ with natural estimators $X_1 = Z_1 - Z_2$ and $X_2 = Z_1 + Z_4 - 2Z_3$, respectively. Now $(X_1, X_2)'$ has the bivariate normal distribution with mean vector $= (\theta_1, \theta_2)'$ and dispersion matrix $= \sigma^2 H$ where $H = \begin{pmatrix} 2 & 1 \\ 1 & 6 \end{pmatrix}$. While estimating the contrast vector

$(\theta_1, \theta_2)'$, we may very well ask for a confidence region of the type R_n with prescribed accuracy. Naturally, in most of these types of applications, the present formulation and its solutions are the reasonable ones to use.

We write $f(x) = \{2^{\frac{1}{2}p}\Gamma(\frac{1}{2}p)\}^{-1} x^{\frac{1}{2}p-1} \exp(-\frac{1}{2}x) I\ (x > 0)$, with $I(\cdot)$ being the indicator of $(\cdot)$. Thus, $F(u) = \int_0^u f(x)dx$ for $u > 0$. Now, we turn to the introduction of the specific statistical procedures in separate chapters one after the other.

CHAPTER II

A TWO-STAGE PROCEDURE

We start the experiment with m (≥ 2) random samples $\underset{\sim}{X}_1, \underset{\sim}{X}_2, \ldots, \underset{\sim}{X}_m$. We propose the following two-stage procedure in a manner similar to that defined in Stein (1945, 1949), Chatterjee (1959, 1960), and Mukhopadhyay (1982). We let $b = pb'$ where b' is the upper $100\alpha\%$ point of a $F(p, p(m-1))$ distribution, and define

$$N = \max\{m, [bS_m^2/d^2] + 1\}, \qquad \ldots(2.1)$$

where N is a positive integer valued random variable denoting the stopping time. If $N = m$, we stop sampling at the starting stage. Otherwise, we sample the difference $N - m$ at the second stage. This way, we will have $\underset{\sim}{X}_1, \underset{\sim}{X}_2, \ldots, \underset{\sim}{X}_N$ as our random samples. We then compute $\bar{\underset{\sim}{X}}_N$ and propose the region R_N as defined in (1.1). The following theorem lists some of the properties of the two-stage procedure (2.1).

Theorem 1: For the two stage procedure (2.1) we have:

i) $P\{\underset{\sim}{\mu} \in R_N\} \geq 1 - \alpha$ for all $\underset{\sim}{\mu} \in \mathbb{R}^p$ and $\sigma \in (0,\infty)$;

ii) $\lim_{d\to 0} E(N/C) = b/a$, where $C = a\sigma^2/d^2$;

iii) $\lim_{d\to 0} \{\frac{1}{2}V(N)\ p(m-1)\ (b\sigma^2/d^2)^{-2}\} = 1$; and

iv) $\lim_{d\to 0} P\{\underset{\sim}{\mu} \in R_N\} = 1 - \alpha$ for all $\underset{\sim}{\mu} \in \mathbb{R}^p$ and $\sigma \in (0,\infty)$.

Proof: To prove part (i), notice that

$$P\{\underset{\sim}{\mu} \in R_N\} = \sum_{m \le n < \infty} P\{\underset{\sim}{\mu} \in R_N \, , N = n\}$$

$$= \sum_{m \le n < \infty} P\{\underset{\sim}{\mu} \in R_n \, , N = n\}.$$

The event $\{N = n\}$ depends only on S_m^2, and the event $\{\underset{\sim}{\mu} \in R_n\}$ depends only on $\bar{\underset{\sim}{X}}_n$ for every fixed $n \ge m$. Using Helmert's orthogonal transformation, we see that $\bar{\underset{\sim}{X}}_n$ is independent of S_m^2 for every $n \ge m$, and thus we can write

$$P\{\underset{\sim}{\mu} \in R_N\} = \sum_{m \le n < \infty} P\{\underset{\sim}{\mu} \in R_n\} \; P\{N = n\}$$

$$= \sum_{m \le n < \infty} F(nd^2/\sigma^2) \; P\{N = n\}$$

$$= E\{F(Nd^2/\sigma^2)\}. \qquad \ldots(2.2)$$

However, we have $N \ge bS_m^2/d^2$ which implies that $Nd^2/\sigma^2 \ge bS_m^2/\sigma^2$. Thus,

$$P\{\underset{\sim}{\mu} \in R_N\} \ge E\{F(bS_m^2/d^2)\}.$$

Let $U \sim \chi^2(p)$ and be independent of S_m^2. Then,

$$E\{F(bS_m^2/\sigma^2)\} = E\{P\{U \le bS_m^2/\sigma^2 \mid S_m^2\}\}$$

$$= P\{U/(S_m^2/\sigma^2) \le pb' \mid S_m^2\}$$

$$= P\{F(p, p(m-1)) \le b'\}$$

$$= 1 - \alpha,$$

by the choice of b'.

The proof of part (ii) is trivial once we notice the basic inequality,

$$bS_m^2/d^2 \leq N \leq m + bS_m^2/d^2, \qquad \ldots(2.3)$$

and then we divide by C and take expectations throughout.

To prove part (iii), we again use the inequality (2.3); and we obtain

$$(bS_m^2/d^2)^2 \leq N^2 \leq (m + bS_m^2/d^2)^2,$$

and this leads to

$$E(bS_m^2/d^2)^2 \leq E(N^2) \leq m^2 + (2mb/d^2)E(S_m^2) + (b/d^2)^2E(S_m^2)^2.$$

However, $p(m-1)S_m^2/\sigma^2 \sim \chi^2(p(m-1))$ and so we have $E(S_m^2) = \sigma^2$ and

$$E(S_m^2)^2 = \{2/(p(m-1)) + 1\}\sigma^4.$$

Therefore, we can write

$$(b\sigma^2/d^2)^2(2/(p(m-1)) + 1) \leq E(N^2) \leq m^2 + 2m(b/d^2)\sigma^2$$

$$+ (b\sigma^2/d^2)^2\{2/(p(m-1)) + 1\}. \qquad \ldots(2.4)$$

Now, part (ii) gives us

$$(b\sigma^2/d^2)^2 \leq \{E(N)\}^2 \leq (m + (b\sigma^2/d^2))^2. \qquad \ldots(2.5)$$

Combining (2.4) and (2.5), we get

$$(b\sigma^2/d^2)^2(2/p(m-1)) - (m^2 + 2mb\sigma^2/d^2) \leq \mathrm{Var}(N)$$

$$\leq m^2 + 2mb\sigma^2/d^2 + (b\sigma^2/d^2)^2\{2/(p(m-1))\},$$

which implies that

$$1 - (m^2 + 2mb\sigma^2/d^2)\{(b\sigma^2/d^2)(2/(p(m-1)))\}^{-1}$$

$$\leq \mathrm{Var}(N)\{(b\sigma^2/d^2)^2\{2/(p(m-1))\}^{-1}$$

$$\leq 1 + (m^2 + 2mb\sigma^2/d^2)\{(b\sigma^2/d^2)(2/(p(m-1)))\}^{-1}.$$

Part (iii) follows immediately by taking limits as $d \to 0$.

To prove part (iv), we take the limit as $d \to 0$ in (2.2) and apply the dominated convergence theorem to write

$$\lim_{d\to 0} P\{\underset{\sim}{\mu} \in R_N\} = \lim_{d\to 0} E\{F(Nd^2/\sigma^2)\}$$

$$= E\{F(\lim_{d\to 0} Nd^2/\sigma^2)\},$$

From the inequality (2.3), it follows that

$$\lim_{d\to 0} Nd^2/\sigma^2 = bS_m^2/\sigma^2 \text{ w.p. } 1,$$

and thus we have

$$\lim_{d\to 0} P\{\underset{\sim}{\mu} \in R_N\} = E\{F(bS_m^2/d^2)\}.$$

This was shown earlier to be equal to $1-\alpha$. This completes the proof of Theorem 1.

Remark 1: If we take $p = 1$, Stein's (1945, 1949) results will follow as special cases of our Theorem 1. In part (ii), we have the limiting ratio b/a which is almost always numerically found to be greater than one. However, this naturally depends on the values of p, m, and α. The reader is referred to Corollary 4.2 in Ghosh (1973). The part (iv) tells us that the procedure (2.1) is "asymptotically consistent", while part (i) shows that the property of "exact consistency" holds. One is referred to

(1.3) and (1.4). Now, in order for the limiting ratio in part (ii) of Theorem 1 to be unity, we consider next a modified version of the two-stage procedure.

CHAPTER III

A MODIFIED TWO-STAGE PROCEDURE

Motivated by the results of Mukhopadhyay (1980, 1982) and of Ghosh and Mukhopadhyay (1981), we first choose and fix a real number $\gamma \in (0,\infty)$ and let

$$m = \max\{2, [(a/d^2)^{1/(1+\gamma)}] + 1\}.$$

Then, with the starting sample size m determined this way, we define

$$N = \max\{m, [bS_m^2/d^2] + 1\}, \qquad \ldots(3.1)$$

where N, the stopping time, is a positive integer valued random variable. The number b remains the same as in (2.1). Again if $N = m$, we stop sampling at the starting stage itself. Otherwise, we sample the difference $N - m$. We compute $\bar{\underset{\sim}{X}}_N$ and propose the confidence region R_N for $\underset{\sim}{\mu}$. The main point to observe here is that $m \to \infty$ as $d \to 0$, but $m/C \to 0$ as $d \to 0$. Thus, $b/a \to 1$ as $d \to 0$. Some properties of the modified two-stage procedure (3.1) are listed in the following theorem.

<u>Theorem 2</u>: <u>For the modified two-stage procedure</u> (3.1) <u>we have</u>:

i) $P\{\underset{\sim}{\mu} \in R_N\} \geq 1 - \alpha$, <u>for all</u> $\underset{\sim}{\mu} \in \mathbb{R}^p$ <u>and</u> $\sigma \in (0,\infty)$;

ii) $\lim_{d\to 0} E(N/C) = 1$, <u>where</u> $C = a\sigma^2/d^2$;

iii) $\lim_{d\to 0} \{\tfrac{1}{2}V(N)p(m-1)(b\sigma^2/d^2)^{-2}\} = 1$; <u>and</u>

iv $\lim_{d\to 0} P\{\underset{\sim}{\mu} \in R_N\} = 1 - \alpha$ <u>for all</u> $\underset{\sim}{\mu} \in \mathbb{R}^p$ <u>and</u> $\sigma \in (0,\infty)$.

Proof: Parts (i), (iii), and (iv) follow along the same lines as those of the two-stage procedure discussed in Chapter II. To prove part (ii), we we consider the new basic inequality

$$bS_m^2/d^2 \leq N \leq (a/d^2)^{1/(1+\gamma)} + bS_m^2/d^2 + 4.$$

Taking expectations on all sides and dividing by C now yield

$$b/a \leq E(N)/C \leq \sigma^{-2}(d^2/a)^{\gamma/(1+\gamma)} + b/a + 4/C.$$

After taking limits as $d \to 0$, we then conclude that

$$E(N/C) \to 1 \quad \text{as} \quad d \to 0.$$

This completes the proof of our Theorem 2.

Remark 2: If we fix $p = 1$, Mukhopadhyay's (1980) results will follow as a special case of our Theorem 2.

Remark 3: It is particularly important to note part (ii) which shows that the modified two-stage procedure satisfies equation (1.5). Parts (i) and (iv) verify that (1.3) and (1.4), respectively, are satisfied by the procedure in (3.1). One major advantage here is that in order to conclude $\lim_{d\to 0} E(N/C) = 1$, one does not have to go through a purely sequential procedure. Utilizing the remark 4 from Mukhopadhyay (1982), one can easily conclude that $\liminf_{d\to 0} E(N - C) = +\infty$ for our modified two-stage procedure (3.1) with $p = 1$ or 2. We conjecture that this is true for all integers $p \geq 1$. With a view toward removing this type of undesirable property, we now resort to a purely sequential scheme along the lines of Ray (1957) and of Chow and Robbins (1965).

CHAPTER IV

A PURELY SEQUENTIAL PROCEDURE

We start the experiment with m (≥ 2) random samples $\underset{\sim}{X}_1, \underset{\sim}{X}_2, \ldots, \underset{\sim}{X}_m$. After that we take one sample at a time and define N to be the first integer such that $n \geq aS_n^2/d^2$. ...(4.1)

Once again the stopping time N is a positive integer valued random variable with $P(N<\infty) = 1$ for all $\underset{\sim}{\mu} \in \mathbb{R}^p$ and $\sigma \in (0,\infty)$. After sampling is stopped, we have N random samples $\underset{\sim}{X}_1, \underset{\sim}{X}_2, \ldots, \underset{\sim}{X}_N$ in hand. We compute $\bar{\underset{\sim}{X}}_N$ and propose the confidence region R_N for $\underset{\sim}{\mu}$ as in (1.1). Some of the properties of the purely sequential procedure (4.1) are listed in Theorems 3 and 4.

<u>Theorem 3</u>: <u>For the purely sequential procedure</u> (4.1), <u>we have</u>:

i) $P\{\underset{\sim}{\mu} \in R_N\} = E\{F(Nd^2/\sigma^2)\}$;

ii) $\lim_{d\to 0} P\{\underset{\sim}{\mu} \in R_N\} = 1-\alpha$ <u>for all</u> $\underset{\sim}{\mu} \in \mathbb{R}^p$ <u>and</u> $\sigma \in (0,\infty)$;

iii) $N/C \to 1$ <u>w.p. 1 as</u> $d \to 0$, <u>for all</u> $\underset{\sim}{\mu} \in \mathbb{R}^p$ <u>and</u> $\sigma \in (0,\infty)$;

iv) $E(N) \leq C + O(1)$;

v) $\lim_{d\to 0} E(N/C) = 1$; <u>and</u>

vi) $(2C/p)^{-1/2}(N-C) \xrightarrow{L} N(0,1)$ <u>as</u> $d \to 0$.

<u>Proof</u>: To prove part (i) first notice that the event "$N=n$" and $\bar{\underset{\sim}{X}}_n$ are independent for every fixed $n \geq m$. Now,

$$P\{\underset{\sim}{\mu} \in R_N\} = \sum_{m\leq n<\infty} P\{\underset{\sim}{\mu} \in R_N, N=n\}$$

$$= \sum_{m\leq n<\infty} F(nd^2/\sigma^2)P\{N=n\}$$

$$= E\{F(Nd^2/\sigma^2)\},$$

which is part (i). To prove parts (ii) and (iii), we first note that $aS_N^2/d^2 \leq N$, since we stopped at the N*th* stage. Also, $aS_{N-1}^2/d^2 > N-1$ for $N > m$, since we did not stop at the $(N-1)$*th* stage. Thus, we can easily write the following inequality:

$$aS_N^2/d^2 \leq N \leq aS_{N-1}^2/d^2 + m.$$

Dividing throughout by C and then taking limits as $d \to 0$ we obtain $\lim_{d\to 0}(N/C) = 1$ w.p. 1 which proves part (iii). In other words, we can also write $Nd^2/\sigma^2 \to a$ w.p. 1 as $d \to 0$. Hence, using the dominated convergence theorem, we obtain

$$\lim_{d\to 0} P\{\underset{\sim}{\mu} \in R_N\} = E\{F(\lim_{d\to 0} Nd^2/\sigma^2)\} = E\{F(a)\} = 1-\alpha,$$

by the choice of "a". This proves part (ii). To prove part (iv), we recall that

$$S_n^2 = \{p(n-1)\}^{-1} \sum_{i=1}^{p} \sum_{j=1}^{n} (Y_{ij} - \bar{Y}_{in})^2,$$

and

$$N-1 \leq (m-1) + (a/d^2)S_{N-1}^2$$

$$= (m-1) + (a/d^2)\{p(N-2)\}^{-1} \sum_{i=1}^{p} \sum_{j=1}^{N-1} (Y_{ij} - \bar{Y}_{i,N-1})^2$$

$$\leq (m-1) + (a/d^2)\{N-2\}^{-1} \sum_{i=1}^{p} \sum_{j=1}^{N} (Y_{ij} - \mu_i)^2/p.$$

This implies that

$$(N-1)(N-2) \leq (m-1)N + (a/d^2) \sum_{i=1}^{p} \sum_{j=1}^{N} (Y_{ij} - \mu_i)^2/p.$$

By using the facts that $(N-1)(N-2) \geq N^2 - 3N$ and that $E(N^2) \geq \{E(N)\}^2$, and by combining them with Wald's first equation, we get

$$\{E(N)\}^2 - 3E(N) \leq (m-1)E(N) + (a/d^2)\sigma^2 E(N),$$

assuming $E(N) < \infty$. Then, dividing throughout by E(N) yields

$$E(N) \leq C + m + 2.$$

In case E(N) can possibly be infinity, we can use the techniques of truncation by defining $N_k = \min(k,N)$ for $k = 1,2,3,\ldots$. We can immediately see from the preceding proof that $E(N_k) \leq C + m + 2$ since $E(N_k) < \infty$. Now, the monotone convergence theorem will lead to part (iv), since $N_k \uparrow N$ w.p. 1 as $k \to \infty$.

To prove part (v), we utilize part (iv) to write

$$E(N/C) \leq \{(m+2)/C\} + 1,$$

and thus

$$\limsup_{d\to 0} E(N/C) \leq 1.$$

Also, Fatou's Lemma implies that $\liminf_{d\to 0} E(N/C) \geq E\{\liminf_{d\to 0} (N/C)\}$. But, part (iii) implies that $\liminf_{d\to 0} (N/C) = 1$ w.p. 1 and hence we obtain

$$\liminf_{d\to 0} E(N/C) = \limsup_{d\to 0} E(N/C) = 1,$$

which proves part (v). The part (vi) follows from the main theorem of Ghosh and Mukhopadhyay (1975). Here, we give a sketch of the proof. From the stopping time N defined in (4.1), we can equivalently write

$$N = \inf\{n \geq m\ (\geq 2):\ nd^2 \geq a(n-1)^{-1} \sum_{i=1}^{n-1} (Y_i/p)\},$$

where $Y_1, Y_2, \ldots$ are i.i.d. $\sigma^2\chi^2(p)$ random variables. Since S_n^2 has the form of a sample mean, the part (iii) and Anscombe's (1952) theorem will now immediately give

$$\frac{N^{\frac{1}{2}}(S_N^2 - \sigma^2)}{(2/p)^{\frac{1}{2}}\sigma^2} \xrightarrow{L} N(0,1) \quad \text{as } d \to 0;$$

and

$$\frac{N^{\frac{1}{2}}(S_{N-1}^2 - \sigma^2)}{(2/p)^{\frac{1}{2}}\sigma^2} \xrightarrow{L} N(0,1) \quad \text{as } d \to 0.$$

Now, applying Theorem 3 in Ghosh and Mukhopadhyay (1979) with $\nu = 1/d^2$, $\psi_\nu = a/d^2$, and $T_n = S_n^2$ we can conclude part (vi). This completes the proof of Theorem 3.

Before we state and prove the next theorem, let us discuss basic notation from the nonlinear renewal theoretic results of Woodroofe (1977). By using Helmert's orthogonal transformation, the sequential procedure (4.1) can be equivalently written as

$$N = \inf\{n \geq m:\ \sum_{j=1}^{n-1} V_j^* \leq n^2(1-n^{-1})p/C\}, \qquad \ldots(4.2)$$

where $V_1^*, V_2^*, \ldots$ are i.i.d. $\chi^2(p)$ random variables. The condition (2.5) in Woodroofe (1977) is easily shown to be satisfied. Also, one can readily see that (4.2) has the same form as Woodroofe's (1977) equation (1.1) with his $\alpha = 2$, $\beta = 1$, $c = p/C$, $\mu = p$, $\tau^2 = 2p$, $\lambda = C$, $a = \frac{1}{2}p$, and starting sample size (m-1). Let $x^+ = \max(0,x)$ for $x \in \mathbb{R}$. Now, if we write

$$f(x) = \{\Gamma(a)b^a\}^{-1} x^{a-1} e^{-bx}\, I\ (x > 0)$$

and then use Woodroofe's (1977) example (on page 986) with his $\mu=E(X)=a/b$, $\tau^2 = V(X) = a/b^2$, and $S_n = \frac{1}{2}\sum_{i=1}^{n} V_i^*$, we can easily define

$$\nu = \nu_\alpha(a,b) = \beta(2a/b)^{-1}\{(\alpha-1)^2(a^2/b^2) + (a/b^2)\} - \sum_{n=1}^{\infty} n^{-1}E\{(S_n - na\alpha b^{-1})^+\}$$

$$= \beta(2ab)^{-1}\{(\alpha-1)^2 a^2 + a\} - b^{-1}\sum_{n=1}^{\infty} n^{-1}E\{(bS_n - na\alpha)^+\}$$

$$= b^{-1}\{\beta(2a)^{-1}((\alpha-1)^2 a^2 + a) - \sum_{n=1}^{\infty} n^{-1}E\{(bS_n - na\alpha)^+\}\}.$$

Therefore, we get

$$\nu_\alpha(a,1) = \beta(2a)^{-1}\{(\alpha-1)^2 a^2 + a\} - \sum_{n=1}^{\infty} n^{-1}E\{(S_n - n\alpha a)^+\},$$

and it is easily seen that

$$\nu = \nu_\alpha(a,b) = b^{-1}\nu_\alpha(a,1).$$

So, if $X \sim \chi^2(p)$, then $a = \frac{1}{2}p$ and $b = \frac{1}{2}$; and thus,

$$\nu = \nu_\alpha(\tfrac{1}{2}p, \tfrac{1}{2}) = 2\nu_\alpha(\tfrac{1}{2}p, 1)$$

$$= 1 + \tfrac{1}{2}p - 2h(p), \qquad \ldots(4.3)$$

where $h(p) = \sum_{n=1}^{\infty} n^{-1}k(n,p)$, with $k(n,p) = E\{(\frac{1}{2}\sum_{i=1}^{n} V_i^* - np)^+\}$. Since the V_i^*'s are i.i.d. $\chi^2(p)$ random variables, it can be easily seen that $T = \frac{1}{2}\sum_{i=1}^{n} V_i^*$ is a gamma $(\frac{1}{2}np, 1)$ random variable. Therefore, $E(T)=\frac{1}{2}np=V(T)$. We can now write,

$$E\{(T - np)^+\} = \int_{np}^{\infty} (t - np)\, f(t)\, dt$$

$$= \{\Gamma(\tfrac{1}{2}np)\}^{-1}\{\int_{np}^{\infty} t^{\frac{1}{2}np}\, e^{-t}\, dt - np\int_{np}^{\infty} t^{\frac{1}{2}np-1}\, e^{-t}\, dt\}.$$

Let

$$I_1 = \int_{np}^{\infty} t^{\frac{1}{2}np} e^{-t} \, dt , \quad I_2 = \int_{np}^{\infty} t^{\frac{1}{2}np} e^{-t} \, dt , \quad \text{and}$$

$$G(u;v) = \int_{v}^{\infty} y^{u-1} e^{-y} \, dy \quad \text{for } u > 0, \ v > 0.$$

Using integration by parts we get,

$$I_1 = (np)^{\frac{1}{2}np} e^{-np} + \tfrac{1}{2}npG(\tfrac{1}{2}np;np).$$

Also,

$$I_2 = G(\tfrac{1}{2}np;np).$$

Therefore, we can conclude that

$$E\{(T-np)^+\} = \{\Gamma(\tfrac{1}{2}np)\}^{-1}\{(np)^{\frac{1}{2}np} e^{-np} - \tfrac{1}{2}npG(\tfrac{1}{2}np;np)\}.$$

Thus, ν can be computed to any desired level of accuracy by utilizing a table of incomplete gamma functions. The tables from Pearson (1922) and Harter (1964) are helpful for this purpose. Let us define

$$\eta = 1.5 - p^{-1}(1+2h(p)). \qquad \ldots(4.4)$$

Theorem 4: *For the sequential procedure* (4.1) *we have:*

i) $E(N) = C + \eta + o(1)$ *as* $d \to 0$, *if* $m > 1 + 2p^{-1}$; *and*

ii) $P\{\underset{\sim}{\mu} \in R_N\} = (1-\alpha) + d^2\sigma^{-2}\{\eta f(a) + ap^{-1}F''(a)\} + o(d^2)$
as $d \to 0$, if (a) $m \geq 7$ *for* $p = 1$, (b) $m \geq 3$ *for* $p = 2$ or 3; *and*
(c) $m \geq 1 + 2p^{-1}$ *for* $p \geq 4$,

where the number η *is defined in* (4.4).

Proof: Part (i) of Theorem 4 follows directly from Theorem 2.4 of

Woodroofe (1977) with the number η coming from (4.4). From part (i) of Theorem 3, we have

$$P\{\underset{\sim}{\mu} \in R_N\} = E\{F(Nd^2/\sigma^2)\}, \qquad \ldots(4.5)$$

where N, the stopping time, comes from (4.1) or equivalently from (4.2). Using the Taylor's expansion, we obtain from (4.5)

$$E\{F(Nd^2/\sigma^2)\} = (1-\alpha) + (d^2/\sigma^2)f(a)E(N-C) + \tfrac{1}{2}(ad^2/\sigma^2)E\{N^{*2}F''(W)\}, \qquad \ldots(4.6)$$

where W is a suitable random variable between a and Nd^2/σ^2, and $N^* = C^{-\frac{1}{2}}(N-C)$. It is also clear that $W \to a$ in probability as $d \to 0$. Now, let $h(x;p) = e^{-x/2}\, x^{(p/2)-1}$. Then, h(x;p) attains its maximum at $x=x^*(p)=p-2$ for every fixed $p > 2$. Also, we can write for $x > 0$:

$$F''(x) = -\{2^{\frac{1}{2}p+1}\Gamma(\tfrac{1}{2}p)\}^{-1}h(x;p) + (\tfrac{1}{2}p-1)\{2^{\frac{1}{2}p}\Gamma(\tfrac{1}{2}p)\}^{-1}h(x;p-2).$$

We now consider the separate cases for p, namely $p > 4$, $p = 1,2,3$, and 4.

<u>Case 1</u>: Let $p > 4$. Then

$$|N^{*2}F''(W)| \le N^{*2}\{|k_1h(p-2;p)| + |k_2h(p-4;p-2)|\}$$

where

$$k_1 = \{2^{\frac{1}{2}p+1}\Gamma(\tfrac{1}{2}p)\}^{-1} \text{ and } k_2 = \{\tfrac{1}{2}p-1\}\{2^{\frac{1}{2}p}\Gamma(\tfrac{1}{2}p)\}^{-1}.$$

Notice that the two terms inside the brackets are bounded above by positive constants. Also, Woodroofe's (1977) Theorem 2.3 implies that N^{*2} is uniformly integrable if $m > 1 + 2p^{-1}$. Thus, $N^{*2}F''(W)$ is also uniformly integrable. Now, from part (vi) of Theorem 3, it follows that

$\frac{1}{2}pN^{*2} \xrightarrow{L} \chi^2(1)$ as $d \to 0$. Since $W \to a$ in probability as $d \to 0$, $\frac{1}{2}pF''(W)N^{*2} \xrightarrow{L} F''(a)\chi^2(1)$ as $d \to 0$. Hence, we obtain $E\{N^{*2}F''(W)\} = (2/p)F''(a) + o(1)$ as $d \to 0$. Thus, (4.6) immediately leads to part (ii).

<u>Case 2</u>: Let $p = 4$. Then,

$$\begin{aligned} |N^{*2}F''(W)| &= N^{*2}|-\{2^3\Gamma(2)\}^{-1}h(W;4) + \{2^2\Gamma(2)\}^{-1}h(W;2)| \\ &\leq N^{*2}|-\tfrac{1}{8}h(2;4) + \tfrac{1}{4}h(0;2)| \\ &\leq N^{*2}\{|\tfrac{1}{8}h(2;4)| + \tfrac{1}{4}\} \end{aligned}$$

where the quantity inside the brackets is a bounded positive constant. Therefore, $N^{*2}F''(W)$ is again uniformly integrable if $m > 1 + 2p^{-1}$; and we obtain the same result as in Case 1, after we use (4.6).

<u>Case 3</u>: Let $p = 3$. Then,

$$\begin{aligned} |N^{*2}F''(W)| &= N^{*2}|-\{2^{5/2}\Gamma(3/2)\}^{-1}e^{-W/2}W^{\frac{1}{2}} + \tfrac{1}{2}\{2^{3/2}\Gamma(3/2)\}^{-1}e^{-W/2}W^{-\frac{1}{2}}| \\ &= N^{*2}|-\{2^{5/2}\Gamma(3/2)\}^{-1}e^{-W/2}W^{\frac{1}{2}} + \{2^{5/2}\Gamma(3/2)\}^{-1}e^{-W/2}W^{-\frac{1}{2}}| \\ &= |Z|, \text{ say.} \end{aligned}$$

Let A be the event that $N > \frac{1}{2}C$. Write $Z = ZI(A) + ZI(A')$ which implies

$$\lim_{d\to 0} E\{Z\} = \lim_{d\to 0} E\{ZI(A)\} + \lim_{d\to 0} E\{ZI(A')\}, \text{ if the limits exist.}$$

Now,

$$\begin{aligned} |ZI(A)| &= N^{*2}|-\{2^{5/2}\Gamma(3/2)\}^{-1}e^{-W/2}W^{\frac{1}{2}} + \{2^{5/2}\Gamma(3/2)\}^{-1}e^{-W/2}W^{-\frac{1}{2}}|I(N > \tfrac{1}{2}C) \\ &\leq N^{*2}|\{2^{5/2}\Gamma(3/2)\}^{-1}e^{-W/2}W^{\frac{1}{2}}|I(N > \tfrac{1}{2}C) + \end{aligned}$$

$$N^{*2}|\{2^{5/2}\Gamma(3/2)\}^{-1}e^{-W/2}W^{-\frac{1}{2}}|I(N > \tfrac{1}{2}C).$$

Since W is between a and Nd^2/σ^2 and A is the set where $N > \frac{1}{2}C$, this implies $Nd^2/\sigma^2 > \frac{1}{2}a$. Thus, $W > \frac{1}{2}a$ on the set A, and we obtain

$$\begin{aligned}|ZI(A)| &\leq \{2^{5/2}\Gamma(3/2)\}^{-1}N^{*2}\{e^{-W/2}W^{\frac{1}{2}} + e^{-W/2}W^{-\frac{1}{2}}I(N > \tfrac{1}{2}C)\}\\ &\leq \{2^{5/2}\Gamma(3/2)\}^{-1}N^{*2}\{e^{-\frac{1}{2}} + (\tfrac{1}{2}a)^{-\frac{1}{2}}\}.\end{aligned}$$

Hence, $|ZI(A)|$ is uniformly integrable if $m > 1 + 2p^{-1}$. Also, $I(A) \to 1$ in probability as $d \to 0$. Thus, we have

$$E\{ZI(A)\} = (2/p)F''(a) + o(1) \text{ as } d \to 0.$$

On the other hand, we know that $N \leq \frac{1}{2}C$ on the set A' and thus,

$$\begin{aligned}E\{ZI(A')\} &= \int_{A'} N^{*2}F''(W)dP\\ &\leq -\{2^{5/2}\Gamma(3/2)\}^{-1}\int_{A'} N^{*2}e^{-W/2}W^{\frac{1}{2}}\,dP +\\ &\quad \{2^{5/2}\Gamma(3/2)\}^{-1}\int_{A'} N^{*2}e^{-W/2}W^{-\frac{1}{2}}\,dP.\end{aligned}$$

Again, since W is between a and Nd^2/σ^2 and $N \leq \frac{1}{2}C$ on the set A', $Nd^2/\sigma^2 \leq \frac{1}{2}a$. Thus $W < a$ and $W > Nd^2/\sigma^2$ implies $W^{-\frac{1}{2}} < (Nd^2/\sigma^2)^{-\frac{1}{2}}$. Therefore,

$$\begin{aligned}E\{|ZI(A')|\} &\leq a^{\frac{1}{2}}\{2^{5/2}\Gamma(3/2)\}^{-1}\int_{A'} C(1-\tfrac{N}{C})^2\,dP +\\ &\quad \{2^{5/2}\Gamma(3/2)\}^{-1}\int_{A'} C(1-\tfrac{N}{C})^2(Nd^2/\sigma^2)^{-\frac{1}{2}}\,dP\\ &\leq a^{\frac{1}{2}}C\{2^{5/2}\Gamma(3/2)\}^{-1}\int_{A'} dP +\\ &\quad a^{-\frac{1}{2}}C\{2^{5/2}\Gamma(3/2)\}^{-1}\int_{A'} (C/N)^{\frac{1}{2}}\,dP\\ &\leq \{2^{5/2}\Gamma(3/2)\}^{-1}\{a^{\frac{1}{2}}CP(N \leq \tfrac{1}{2}C) + a^{-\frac{1}{2}}C^{3/2}P(N \leq \tfrac{1}{2}C)\}.\end{aligned}$$

From Lemma 2.3 of Woodroofe (1977), we have for $0 < \gamma < 1$,

$$P(N \leq \tfrac{1}{2}C) = O(C^{-3(m-1)/2}) + O(C^{-\frac{1}{2}r\gamma}),$$

as $d \to 0$ where $E(V_1^{*r}) < \infty$ with $r \geq 2$. Thus, one can readily see that for $m > 2$, $\lim_{d \to 0} E\{ZI(A')\} = 0$. This leads us to part (ii) for p=3, since now we can write $E(Z) = (2/p)F''(a) + o(1)$ as $d \to 0$, and we of course utilize (4.6) as well.

Case 4: Let $p = 2$. Then,

$$|N^{*2}F''(W)| = N^{*2}|-\tfrac{1}{4}e^{-\frac{1}{2}W}| \leq \tfrac{1}{4}N^{*2}.$$

Since, N^{*2} is uniformly integrable for $m > 1 + 2p^{-1}$. so is $|N^{*2}F''(W)|$. Using the same arguments as before, we can write

$$E\{N^{*2}F''(W)\} = (2/p)F''(a) + o(1) \quad \text{as } d \to 0.$$

Case 5: Let $p = 1$. Then,

$$|N^{*2}F''(W)| = N^{*2}|\{2^{3/2}\Gamma(\tfrac{1}{2})\}^{-1}\{e^{-W/2}W^{-\frac{1}{2}} + e^{-W/2}W^{-3/2}\}|.$$

Again, let A denote the event that $N > \tfrac{1}{2}C$. Then,

$$\begin{aligned} |ZI(A)| &\leq N^{*2}|\{2^{3/2}\Gamma(\tfrac{1}{2})\}^{-1}e^{-W/2}W^{-\frac{1}{2}}|\ I\ (N > \tfrac{1}{2}C) + \\ &\qquad N^{*2}|\{2^{3/2}\Gamma(\tfrac{1}{2})\}^{-1}e^{-W/2}W^{-3/2}|\ I\ (N > \tfrac{1}{2}C) \\ &\leq N^{*2}|\{2^{3/2}\Gamma(\tfrac{1}{2})\}^{-1}(a/2)^{-\frac{1}{2}}| + N^{*2}|\{2^{3/2}\Gamma(\tfrac{1}{2})\}^{-1}(a/2)^{-3/2}| \\ &= N^{*2}\{|\{2^{3/2}\Gamma(\tfrac{1}{2})\}^{-1}(a/2)^{-\frac{1}{2}}| + |\{2^{3/2}\Gamma(\tfrac{1}{2})\}^{-1}(a/2)^{-3/2}|\} \end{aligned}$$

where the quantities inside the absolute values are bounded positive constants. Hence, $|ZI(A)|$ is uniformly integrable if $m > 3$, which in turn implies that $E\{ZI(A)\} = (2/p)F''(a) + o(1)$ as $d \to 0$. Again, with $b^* = \{2^{3/2}\Gamma(\frac{1}{2})\}^{-1}$, we can write

$$E\{|ZI(A')|\} = \int_{A'} |Z| \, dP$$

$$\leq \int_{A'} N^{*2} |b^* e^{-W/2} W^{-\frac{1}{2}}| \, dP + \int_{A'} N^{*2} |b^* e^{-W/2} W^{-3/2}| \, dP$$

$$\leq b^* \int_{A'} N^{*2} W^{-\frac{1}{2}} \, dP + b^* \int_{A'} N^{*2} W^{-3/2} \, dP.$$

Also, $W^{-\frac{1}{2}} \leq (Nd^2/\sigma^2)^{-\frac{1}{2}}$ on the set A', and so we obtain

$$E\{|ZI(A')|\} \leq b^* \int_{A'} N^{*2} (Nd^2/\sigma^2)^{-\frac{1}{2}} \, dP + b^* \int_{A'} N^{*2} (Nd^2/\sigma^2)^{-3/2} \, dP$$

$$= b^* a^{-\frac{1}{2}} \int_{A'} C(1 - \frac{N}{C})^2 \; (C/N)^{\frac{1}{2}} \, dP +$$

$$b^* a^{-3/2} \int_{A'} C(1 - \frac{N}{C})^2 \; (C/N)^{3/2} \, dP$$

$$\leq b^* a^{-\frac{1}{2}} C^{3/2} P(N \leq \tfrac{1}{2}C) + b^* a^{-3/2} C^{5/2} P(N \leq \tfrac{1}{2}C).$$

In order to make $C^{3/2} P(N \leq \frac{1}{2}C)$ and $C^{5/2} P(N \leq \frac{1}{2}C)$ both converge to zero as $d \to 0$, the same basic techniques used at the end of Case 3 would lead us to the sufficient condition that $\frac{5}{2} - \frac{1}{2}(m - 1) < 0$, that is we need $m > 6$. Earlier, we found the condition $m > 3$. Thus, for $m \geq 7$, we have $\lim_{d \to 0} E\{ZI(A')\} = 0$. Hence, for $m \geq 7$, we have part (ii), since

$$E\{N^{*2} F''(W)\} = (2/p)F''(a) + o(1)$$

as $d \to 0$.

This completes the proof of Theorem 4.

Remark 4: The part (i) of our Theorem 4 shows that the purely sequential procedure (4.1) is indeed "asymptotically second-order efficient" in the Ghosh-Mukhopadhyay (1981) sense, since we have here $\lim_{d \to 0} E(N - C) = \eta$. One is also referred to remark 3.

4.1 Rate of Convergence for the Distribution of N

Let us write $\tilde{N} = (2C/p)^{-\frac{1}{2}}(N - C)$ where N comes from (4.1). Let

$G(x;d) = P(\tilde{N} \leq x)$ and $\Phi(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{x} \exp(-\frac{1}{2}t^2)\,dt$ for $x \in (-\infty,\infty)$.

The part (vi) of Theorem 3 shows that $G(x;d) \to \Phi(x)$ as $d \to 0$, for each $x \in (-\infty,\infty)$. The following theorem studies the rate of convergence for this result.

Theorem 5: *For the sequential procedure* (4.1), *we have as* $d \to 0$:

$$\sup_{-\infty<x<\infty} |G(x;d) - \Phi(x)| = O(d^{\frac{1}{2}-\gamma}),$$

for every fixed $\gamma \in (0,\frac{1}{2})$.

Proof: The stopping time N defined in (4.1) can be equivalently written as follows:

$$N = \inf\{n \geq m\ (\geq 2): n^{*2}(1+n^{*-1}) \geq (a/d^2) \sum_{i=1}^{n^*} (Y_i/p)\}, \qquad \ldots(4.7)$$

where $n^* = n-1$ and $Y_1, Y_2, \ldots$ are i.i.d. $\sigma^2\chi^2(p)$ random variables.

Notice that N defined in (4.7) has exactly the same form as that of Ghosh's (1980) representation with his $\alpha = 2$, $L(n) = (1+n^{*-1})$, $\psi_\nu = a/d^2$ and $T_n = \sum_{i=1}^{n^*}(Y_i/p)$, where the Y_i's are i.i.d. gamma random variables with parameters $\frac{1}{2}p$ and $\frac{1}{2}\sigma^{-2}$ both being positive. Now Ghosh's (1980) Theorem 2 implies

$$P\{|NC^{-1} - 1| > C^{\frac{1}{2}+\gamma}\} = O(C^{-\frac{1}{4}+\frac{1}{2}\gamma}), \quad \text{as } d \to 0. \qquad \ldots(4.8)$$

Now, let us write $G_i = Y_i/p$ with $E(G_1) = \sigma^2$ and $V(G_1) = (2/p)\sigma^4$. Let $N^* = N-1$. Then, $N^{*2}(S_N^2 - \sigma^2) = \sum_{i=1}^{N^*} (G_i - \sigma^2)$ implies that

$$\left(\frac{\tfrac{1}{2}p}{N^*}\right)^{\frac{1}{2}} \sigma^{-2} \sum_{i=1}^{N^*} (G_i - \sigma^2) = (\tfrac{1}{2}pN^*)^{\frac{1}{2}} \sigma^{-2}(S_N^2 - \sigma^2).$$

Thus, the main results from Landers and Rogge (1976) and Ghosh's (1980) equation (1.3) imply

$$\sup_{-\infty<x<\infty} |P\{(\tfrac{1}{2}pN^*)^{\frac{1}{2}}(S_N^2 - \sigma^2) \le x\sigma^2\} - \Phi(x)| = O(C^{-\frac{1}{4}+\frac{1}{2}\gamma}).$$

However, we also have

$$P\{|N^{*\frac{1}{2}}N^{-\frac{1}{2}} - 1| > C^{-\frac{1}{4}+\frac{1}{2}\gamma}\}$$

$$= P\{|(N-1)^{\frac{1}{2}} N^{-\frac{1}{2}} - 1| > C^{-\frac{1}{4}+\frac{1}{2}\gamma}\}$$

$$= P\{|(N-1)^{\frac{1}{2}} N^{-\frac{1}{2}} - 1||(N-1)^{\frac{1}{2}} N^{-\frac{1}{2}} - 1| > C^{-\frac{1}{2}+\gamma}\}$$

$$\le P\{|(N-1)^{\frac{1}{2}} N^{-\frac{1}{2}} - 1||(N-1)^{\frac{1}{2}} N^{-\frac{1}{2}} + 1| > C^{-\frac{1}{2}+\gamma}\}$$

$$= P\{|(N-1)N^{-1} - 1| > C^{-\frac{1}{2}+\gamma}\}$$

$$= P\{N^{-1} > C^{-\frac{1}{2}+\gamma}\}$$

$$= P\{CN^{-1} > C^{\frac{1}{2}+\gamma}\}$$

$$\le P\{|CN^{-1} - 1| > C^{\frac{1}{2}+\gamma} - 1\}$$

$$\le P\{|CN^{-1} - 1| > kC^{\frac{1}{2}+\gamma}\},$$

for sufficiently small d, and for some $k \in (0,1)$. Now, we get

$$P\{|NC^{-1} - 1| > kC^{\frac{1}{2}+\gamma}\}$$

$$\le P\{|NC^{-1} - 1| > kC^{-\frac{1}{2}+\gamma}\}$$

$$= O(C^{-\frac{1}{4}+\frac{1}{2}\gamma}).$$

Hence, we can write

$$\sup_{-\infty<x<\infty} |P\{(\tfrac{1}{2}pN^*)^{\frac{1}{2}}(S_N^2 - \sigma^2) \leq N^{*\frac{1}{2}} N^{-\frac{1}{2}} x\sigma^2\} - \Phi(x)|$$

$$= \sup_{-\infty<x<\infty} |P\{(\tfrac{1}{2}pN)^{\frac{1}{2}}(S_N^2 - \sigma^2) \leq x\sigma^2\} - \Phi(x)|$$

$$= O(C^{-\frac{1}{4}+\frac{1}{2}\gamma}), \qquad \ldots(4.9)$$

as $d \to 0$, after using Lemma 10 from Landers and Rogge (1976). In a similar manner, we can show that

$$\sup_{-\infty<x<\infty} |P\{(\tfrac{1}{2}pN)^{\frac{1}{2}}(S_{N-1}^2 - \sigma^2) \leq x\sigma^2\} - \Phi(x)| = O(C^{-\frac{1}{4}+\frac{1}{2}\gamma}), \qquad \ldots(4.9)$$

as $d \to 0$.

Then, combining (4.8), (4.9), (4.10), and Ghosh's (1980) Theorem 3 we conclude that

$$\sup_{-\infty<x<\infty} |P\{(2C/p)^{-\frac{1}{2}}(N - C) \leq x\} - \Phi(x)|$$

$$= \sup_{-\infty<x<\infty} |P\{\tilde{N} \leq x\} - \Phi(x)|$$

$$= O(d^{\frac{1}{2}-\gamma}),$$

as $d \to 0$. This proves Theorem 5.

CHAPTER V

A THREE-STAGE PROCEDURE

Motivated by the results of Hall (1981), we now propose the following three-stage procedure.

We start the experiment with m (≥ 2) random samples $\underset{\sim}{X}_1, \underset{\sim}{X}_2, \ldots, \underset{\sim}{X}_m$. We fix a real number $r \in (0,1)$. We define

$$M = \max\{m\,,\ [raS_m^2/d^2] + 1\}, \qquad \ldots(5.1)$$

and take new samples, if needed, to form $\underset{\sim}{X}_1, \underset{\sim}{X}_2, \ldots, \underset{\sim}{X}_M$. We let

$$N = \max\{M\,,\ [aS_M^2/d^2] + 1\}, \qquad \ldots(5.2)$$

and take new samples, if needed, to form $\underset{\sim}{X}_1, \underset{\sim}{X}_2, \ldots, \underset{\sim}{X}_N$. Once we determine N, we propose the confidence region R_N for $\underset{\sim}{\mu}$ as indicated in (1.1). The following theorems study expansions of E(N) and $P\{\underset{\sim}{\mu} \in R_N\}$ as $d \to 0$.

Using the representations analogous to those in (4.2), we can easily rewrite (5.1) - (5.2) in the following equivalent fashion:

$$M = \max\{m\,,\ [ra\bar{U}_m/d^2] + 1\}, \qquad \ldots(5.3)$$

and

$$N = \max\{M\,,\ [a\bar{U}_M/d^2] + 1\}, \qquad \ldots(5.4)$$

where $\bar{U}_k = (k-1)^{-1} \sum_{i=1}^{k-1} (U_i/p)$, $k = m, m+1, \ldots,$ the U_i's being i.i.d. $\sigma^2\chi^2(p)$.

Theorem 6: Let $C = \lambda\sigma^2$ with $\lambda = a/d^2$. Then, for the three-stage procedure (5.1) - (5.2), we have as $d \to 0$:

i) $E(N) = C + \frac{1}{2} - 2(pr)^{-1} + o(1)$;

ii) $V(N) = 2(pr)^{-1}C + o(\lambda)$; and

iii) $E\{|N - E(N)|^3\} = o(\lambda^2)$.

Proof: We follow very closely the developments in Hall (1981). We indicate only some of the basic steps assuming $\sigma^2 = 1$. Using (4.1) of Hall (1981), we get

$$\begin{aligned} \lambda E(\bar{U}_M) &= \lambda - r^{-1}V(p^{-1}U_1) + o(1) \\ &= \lambda - 2(pr)^{-1} + o(1). \end{aligned} \qquad \ldots(5.5)$$

Also, $E\{\lambda\bar{U}_M - [\lambda\bar{U}_M]\} = \frac{1}{2} + o(1)$, and this can be justified along the lines of Hall (1981). Let $T = [\lambda\bar{U}_M] + 1$. Then, Hall's (1981) equation (4.2) will lead to

$$\begin{aligned} E(N) &= E(T) + o(1) \\ &= 1 + E(\lambda\bar{U}_M) - E\{\lambda\bar{U}_M - [\lambda\bar{U}_M]\} + o(1) \\ &= \lambda + \tfrac{1}{2} - 2(pr)^{-1} + o(1), \end{aligned} \qquad \ldots(5.6)$$

where the last step was obtained by using (5.5). Again, by using (4.3) and (4.4) from Hall (1981), we obtain

$$\begin{aligned} V(N) &= V(T) + o(1) \\ &= r^{-1}\lambda V(p^{-1}U_1) + o(1) \\ &= 2(pr)^{-1}\lambda + o(\lambda). \end{aligned} \qquad \ldots(5.7)$$

In (5.6) and (5.7), replacing λ by $\lambda\sigma^2$ we obtain parts (i) and (ii) of

Theorem 6. We omit the proof of part (iii) as it can be tackled along the similar lines of Hall (1981). This completes the proof of Theorem 6.

We now modify the three-stage procedure (5.1) - (5.2) slightly so as to be able to conclude that the resulting coverage probability turns out as $(1-\alpha)+o(d^2)$. Towards that end, we define:

$$m_1 = \{f(a)\}^{-1}(\{2(pr)^{-1} - \tfrac{1}{2}\}f(a) - a(pr)^{-1}F''(a)),$$

$$M = \max\{m\,,\, [raS_m^2/d^2]+1\}, \text{ and} \qquad \ldots(5.8)$$

$$N^* = \max\{M\,,\, [aS_M^2 d^{-2} + m_1]+1\}. \qquad \ldots(5.9)$$

We extend the starting samples $\underset{\sim}{X}_1, \underset{\sim}{X}_2, \ldots, \underset{\sim}{X}_m$ to $\underset{\sim}{X}_1, \underset{\sim}{X}_2, \ldots, \underset{\sim}{X}_{N^*}$, and propose the corresponding region R_{N^*} for $\underset{\sim}{\mu}$.

<u>Theorem 7</u>: <u>For the modified three-stage procedure</u> (5.8) - (5.9), <u>we have as</u> $d \to 0$:

i) $P\{\underset{\sim}{\mu} \in R_{N^*}\} = (1-\alpha)+o(d^2)$, <u>and</u>

ii) $E(N^*) = C - aF''(a)\{prF'(a)\}^{-1} + o(1)$.

<u>Proof</u>: We first verify part (i). In fact, we start working with (M,N) from (5.1) - (5.2); and at the end show that N must be modified to N* to conclude part (i).

We still have

$$P\{\underset{\sim}{\mu} \in R_N\} = E\{F(Nd^2/\sigma^2)\} = E\{F(\ell N)\},$$

where $\ell = d^2/\sigma^2$. Now, we can write

$$E\{F(\ell N)\} = F(\ell E(N)) + \tfrac{1}{2}\ell^2 E\{(N-E(N))^2\}F''(\ell E(N)) + r_1(d),$$

say, where

$$|r_1(d)| \leq k\ell^3 E\{|N - E(N)|^3\} = o(d^2),$$

by part (iii) of Theorem 6. Here k (> 0) is a generic constant independent of d. Again, we have

$$F(\ell E(N)) = F(a) + \{\ell E(N) - a\}F'(a) + r_2(d), \qquad \ldots(5.10)$$

say, where

$$r_2(d) = \tfrac{1}{2}(\ell E(N) - a)^2 F''(z),$$

for a suitable positive number z.

Let us now use $\lambda = \lambda(d) = a(1+\varepsilon)/d^2$; and with this choice, $|r_2(d)| = o(d^2 + |\varepsilon|)$. Also, we have from part (i) of Theorem 6,

$$\begin{aligned} \ell E(N) - a &= \ell(\lambda\sigma^2 + \tfrac{1}{2} - 2(pr)^{-1} + o(1)) - a \\ &= a\varepsilon + d^2\sigma^{-2}(\tfrac{1}{2} - 2(pr)^{-1}) + o(d^2). \qquad \ldots(5.11) \end{aligned}$$

Thus, combining (5.10) and (5.11), we obtain

$$F(\ell E(N)) = F(a) + F'(a)\{a\varepsilon + d^2\sigma^{-2}(\tfrac{1}{2} - 2(pr)^{-1})\} + o(d^2) + o(d^2 + |\varepsilon|). \qquad \ldots(5.12)$$

Again, we have from part (ii) of Theorem 6,

$$\tfrac{1}{2}\ell E\{(N - E(N))^2\} = (pr)^{-1} a d^2 \sigma^{-2} + o(d^2). \qquad \ldots(5.13)$$

By combining (5.12) and (5.13), we get

$$\begin{aligned} E\{F(\ell N)\} &= (1-\alpha) + F'(a)\{a\varepsilon + d^2\sigma^{-2}(\tfrac{1}{2} - 2(pr)^{-1})\} + o(d^2) \\ &\quad + \{ad^2\sigma^{-2}(pr)^{-1} + o(d^2)\}F''(a) + o(d^2 + |\varepsilon|) \\ &= (1-\alpha) + \{a\varepsilon F'(a) + d^2\sigma^{-2}(\tfrac{1}{2} - 2(pr)^{-1})F'(a) \end{aligned}$$

$$+ ad^2\sigma^{-2}(pr)^{-1}F''(a)\} + o(d^2) + o(d^2 + |\varepsilon|). \qquad \ldots(5.14)$$

To make the second term from the left in (5.14) vanish, we choose ε such that

$$C\varepsilon = \{(2(pr)^{-1} - \tfrac{1}{2})F'(a) - a(pr)^{-1}F''(a)\}/F'(a)$$

which leads to

$$\lambda\sigma^2 = C + a\varepsilon\sigma^2 d^{-2}$$

$$= C + \{F'(a)\}^{-1}\{(2(pr)^{-1} - \tfrac{1}{2})F'(a) - a(pr)^{-1}F''(a)\}$$

$$= C + m_1.$$

Now, we can immediately see from (5.14) that

$$P\{\underset{\sim}{\mu} \in R_{N^*}\} = (1 - \alpha) + o(d^2),$$

as $d \to 0$. This proves part (i).

For part (ii), simply notice from part (i) of Theorem 6 that

$$E(N^*) = C + \tfrac{1}{2} - 2(pr)^{-1} + (2(pr)^{-1} - \tfrac{1}{2} - aF''(a)\{prF'(a)\}^{-1}) + o(1)$$

$$= C - aF''(a)\{prF'(a)\}^{-1} + o(1).$$

This completes the proof of Theorem 7.

CHAPTER VI

MODERATE SAMPLE SIZE PERFORMANCES OF THE VARIOUS PROCEDURES

In this chapter, we present numerical results obtained through simulations using FORTRAN programs on an IBM 3081 D computer system with WATFIV Compiler. Our subsections 6.1, 6.2, 6.3, and 6.4 contain respectively the numerical results of simulation studies for the two-stage procedure of Chapter II, the modified two-stage procedure of Chapter III, the purely sequential procedure of Chapter IV and the three-stage procedure of Chapter V.

Let us now explain the way we carry out the simulations. In any particular table we use a particular "rule" to determine the sample size N. We utilize the subroutine called GGNML from IMSL (1982) to generate samples from $N_p(\underset{\sim}{0}, I)$ with $p = 1,2,3$ depending on the situation, i.e., $\underset{\sim}{\mu} = \underset{\sim}{0}$ and $H = I$ with $\sigma^2 = 1$. A particular "rule" is replicated R times, the jth replicate giving rise to observed values of N, and $\bar{X}_{iN}$ as, say, n(j) and $\bar{X}_{in(j)}$ respectively. Then, we estimate E(N) and μ_i by $\bar{n} = R^{-1} \sum_{j=1}^{R} n(j)$ and $\hat{\mu}_i = R^{-1} \sum_{j=1}^{R} \bar{X}_{in(j)}$ respectively. We also compute the corresponding standard errors

$$S(\bar{n}) = \{(R^2 - R)^{-1} \sum_{j=1}^{R} (n(j) - \bar{n})^2\}^{\frac{1}{2}},$$

$$S(\hat{\mu}_i) = \{(R^2 - R)^{-1} \sum_{j=1}^{R} (\bar{X}_{in(j)} - \hat{\mu}_i)^2\}^{\frac{1}{2}}.$$

We consider $i=1,\ldots,p$ and $j=1,\ldots,R$. While using a particular rule, we also estimate the coverage probability of the region R_N by, say, C.P. where

$$\text{C.P.} = \text{relative frequency of } \sum_{i=1}^{p} \bar{X}^2_{in(j)} \leq d^2$$

among all the replicates for $j = 1,2,\ldots,R$. Here, we are considering 95% confidence regions only, that is, we keep $\alpha = .05$ fixed and d is computed using the realtionship $d = (a/C)^{\frac{1}{2}}$.

6.1 Moderate Sample Size Performances of the Two-Stage Procedure

We use the "rule" as being the two-stage procedure of Chapter II. We give results for $p = 2$ and 3, $m = 5$ and 10, $C = 10$, 15, 20, 25, 50, 100, with $R = 300$. The results for $p = 1$ can be found in Starr (1966). The Table I summarizes our findings.

Remark 5: From Table I, we notice that $\bar{n}$ is always somewhat larger than C, however, almost always the coverage exceeds the target $(1 - \alpha) = .95$. The amount of oversampling reduces when we go from $p = 2$ to $p = 3$, and this is because of the increment in the degree of freedom of the estimate of σ^2. The results also get better as m increases, and this is generally expected. We suggest that m be taken as 5 or 10 in the absence of any further knowledge. The values of $S(\bar{n})$ are quite stable, so $\bar{n}$ can be taken as good estimators of E(N).

6.2 Moderate Sample Size Performances of the Modified Two-Stage Procedure

Here, we use the "rule" as being the modified two-stage procedure of Chapter III. We naturally have to choose γ (> 0) suitably. We first fix

TABLE I

TWO-STAGE PROCEDURE (2.1) WITH R = 300

p	m	C	d	$\bar{n}$	$S(\bar{n})$	$\hat{\mu}_1$	$\hat{\mu}_2$	$\hat{\mu}_3$	C.P.
2	5	10	0.774	15.27	0.44	-0.005	-0.001	.	0.953
		15	0.632	23.64	0.73	0.011	-0.004	.	0.967
		20	0.547	29.52	0.81	-0.011	0.008	.	0.953
		25	0.489	39.58	1.16	-0.018	0.017	.	0.950
		50	0.346	75.26	2.11	0.011	-0.007	.	0.963
		100	0.245	158.15	4.87	0.009	-0.009	.	0.950
2	10	10	0.774	12.59	0.18	0.004	0.016	.	0.970
		15	0.632	18.19	0.34	-0.012	-0.007	.	0.963
		20	0.547	24.57	0.46	-0.008	-0.008	.	0.957
		25	0.489	29.41	0.56	0.016	0.012	.	0.960
		50	0.346	60.28	1.19	-0.002	-0.001	.	0.953
		100	0.245	119.68	2.18	0.013	-0.003	.	0.970
3	5	10	0.884	13.87	0.31	-0.034	-0.015	-0.006	0.950
		15	0.722	20.90	0.49	-0.010	-0.002	0.016	0.960
		20	0.625	27.10	0.66	0.031	-0.003	-0.008	0.933
		25	0.559	34.09	0.81	-0.016	-0.004	0.023	0.953
		50	0.395	69.01	1.73	0.005	0.012	-0.012	0.957
		100	0.280	134.36	3.02	-0.002	0.002	-0.001	0.950
3	10	10	0.884	12.49	0.15	-0.001	-0.013	-0.022	0.987
		15	0.722	17.78	0.27	0.007	0.000	0.017	0.973
		20	0.625	23.73	0.38	0.000	-0.018	0.005	0.967
		25	0.559	28.93	0.46	-0.002	0.019	-0.020	0.953
		50	0.395	56.47	0.96	0.010	0.004	-0.002	0.947
		100	0.280	112.37	1.68	-0.009	0.001	-0.003	0.930

$p = 1$, and $C = 24$, 43, 61, 76, 96, 125, 171, 246, 384 as in Hall (1981). We select $\gamma = .1$, .3, .5, .7 and 1.0 with $R = 300$. The results are most promising for $\gamma = .3$. In Table II, we summarize our findings for $p = 1$ with $R = 300$, except that for $\gamma = .3$ in the third block, we consider $R = 1000$. So, in Table II, the third block is comparable with Hall's (1981) findings. This modified two-stage procedure performs almost as good as or better than Hall's (1981) three-stage scheme.

The Table III still uses the rule (3.1), but for $p = 2$, 3 and $\gamma = .1$ .3, $C = 10$, 15, 20, 25, 50, 100, and $R = 300$. Here, $\gamma = .1$ or .3 seems to be the right choice.

Remark 6: From our Tables II and III, we notice that the modified two-stage procedure (3.1) performs very satisfactorily for $\gamma = .3$. The values of $\bar{n}$ are very close to C, and C.P. is also very much on the target. We recommend using the procedure (3.1) in practice with $\gamma = .3$ in the absence of any further knowledge.

6.3 Moderate Sample Size Performances of the Purely Sequential Procedure

Here, we use the "rule" as being the purely sequential procedure of Chapter IV. Just to show the stability of the generator we are using, we provide with Table IV for $p = 1$, $C = 10$, 15, 20, 25, 50, 100 and $m = 5$, 10 with $R = 300$. One can compare our Table IV with the corresponding findings in Starr (1966). Naturally, for increasing m, the procedure performs better. Also, $\bar{n}$ and C.P. are close to C and $(1 - \alpha)$ respectively.

In the case $p = 2$ and 3, Table V represents the results of simulation

TABLE II

MODIFIED TWO-STAGE PROCEDURE (3.1)
WITH $p = 1$, $R = 300$

γ	C	d	$\bar{n}$	$S(\bar{n})$	$\hat{\mu}_1$	$S(\hat{\mu}_1)$	C.P.
0.1	24	0.400	28.480	0.550	0.034	0.011	0.953
	43	0.299	48.180	0.706	-0.009	0.009	0.937
	61	0.251	64.467	0.843	-0.011	0.007	0.957
	76	0.225	81.620	0.890	0.010	0.006	0.950
	96	0.200	97.423	0.948	-0.002	0.006	0.947
	125	0.175	126.633	1.171	-0.003	0.005	0.950
	171	0.150	177.197	1.379	-0.001	0.004	0.957
	246	0.125	251.277	1.812	0.009	0.003	0.977
	384	0.100	383.463	2.154	0.002	0.003	0.953
0.3	24	0.400	31.317	0.757	0.027	0.012	0.957
	43	0.299	49.590	0.994	-0.002	0.009	0.940
	61	0.251	68.627	1.178	-0.011	0.007	0.947
	76	0.225	86.540	1.303	0.007	0.007	0.947
	96	0.200	101.237	1.416	0.001	0.006	0.947
	125	0.175	126.147	1.517	-0.006	0.005	0.940
	171	0.150	181.243	2.100	-0.001	0.004	0.967
	246	0.125	256.660	2.663	0.009	0.003	0.967
	384	0.100	406.406	3.133	0.003	0.003	0.947
0.3 *	24	0.400	30.677	0.410	-0.002	0.006	0.951
	43	0.299	50.557	0.518	0.003	0.005	0.960
	61	0.251	66.800	0.632	-0.005	0.004	0.941
	76	0.225	83.884	0.720	0.002	0.004	0.960
	96	0.200	102.356	0.799	0.007	0.003	0.953
	125	0.175	130.439	0.896	0.003	0.003	0.953
	171	0.150	182.273	1.137	-0.003	0.002	0.966
	246	0.125	258.830	1.391	-0.000	0.002	0.952
	384	0.100	396.600	1.749	-0.000	0.002	0.948
0.5	24	0.400	34.023	1.088	0.028	0.012	0.937
	43	0.299	55.227	1.235	-0.008	0.008	0.957
	61	0.251	75.083	1.550	-0.005	0.007	0.967
	76	0.225	87.230	1.685	0.003	0.006	0.960
	96	0.200	110.090	1.891	-0.002	0.006	0.960
	125	0.175	134.060	2.105	-0.005	0.005	0.933
	171	0.150	184.903	2.819	0.003	0.004	0.977
	246	0.125	271.537	3.642	0.007	0.004	0.953
	384	0.100	412.063	4.149	0.003	0.003	0.940
0.7	24	0.400	38.610	1.311	0.016	0.012	0.943
	43	0.299	58.973	1.553	-0.017	0.008	0.943
	61	0.251	77.180	1.844	-0.004	0.007	0.940
	76	0.225	94.097	2.282	0.003	0.006	0.957
	96	0.200	112.703	2.680	0.002	0.006	0.923
	125	0.175	140.097	2.873	-0.003	0.006	0.933
	171	0.150	195.887	3.716	0.003	0.004	0.960
	246	0.125	265.717	4.497	0.010	0.004	0.953
	384	0.100	410.400	5.582	0.002	0.003	0.950
1.0	24	0.400	52.703	1.992	0.007	0.011	0.950
	43	0.299	66.233	2.230	-0.014	0.008	0.967
	61	0.251	85.457	2.571	0.005	0.007	0.970
	76	0.225	103.010	2.907	-0.000	0.007	0.940
	96	0.200	128.040	3.713	-0.002	0.006	0.957
	125	0.175	152.737	3.667	-0.004	0.005	0.957
	171	0.150	205.000	4.713	0.003	0.004	0.947
	246	0.125	284.213	5.843	0.007	0.004	0.927
	384	0.100	454.386	8.556	0.000	0.003	0.947

*This block is based on 1000 replications for ease of comparisons with Hall's (1981) table.

TABLE III

MODIFIED TWO-STAGE PROCEDURE (3.1) WITH R = 300

γ	p	C	d	$\bar{n}$	$S(\bar{n})$	$\hat{\mu}_1$	$\hat{\mu}_2$	$\hat{\mu}_3$	C.P.
0.1	2	10	0.774	12.84	0.22	0.024	0.041	.	0.953
		15	0.632	18.12	0.26	0.008	0.038	.	0.953
		20	0.547	22.74	0.29	0.003	0.001	.	0.947
		25	0.489	27.33	0.37	-0.001	0.011	.	0.977
		50	0.346	53.24	0.55	0.004	0.003	.	0.940
		100	0.245	103.95	0.76	-0.006	0.004	.	0.963
0.3	2	10	0.774	14.25	0.32	0.016	0.003	.	0.957
		15	0.632	18.19	0.38	0.003	-0.030	.	0.960
		20	0.547	23.96	0.47	0.002	0.002	.	0.957
		25	0.489	28.93	0.46	-0.003	0.009	.	0.970
		50	0.346	56.46	0.71	-0.009	-0.006	.	0.953
		100	0.245	108.84	1.11	0.003	0.000	.	0.950
0.1	3	10	0.884	12.18	0.17	-0.002	0.013	-0.006	0.977
		15	0.722	16.77	0.20	-0.002	0.001	-0.022	0.973
		20	0.625	22.09	0.25	0.006	-0.010	-0.011	0.957
		25	0.559	27.56	0.30	0.019	-0.007	0.003	0.963
		50	0.395	52.10	0.42	-0.007	-0.009	-0.003	0.957
		100	0.280	106.66	0.62	0.004	0.004	0.000	0.953
0.3	3	10	0.884	13.33	0.25	-0.010	0.013	0.007	0.973
		15	0.722	17.66	0.31	0.010	-0.011	0.008	0.953
		20	0.625	22.54	0.33	0.021	-0.003	-0.005	0.943
		25	0.559	28.22	0.40	0.003	0.005	-0.003	0.957
		50	0.395	55.56	0.61	0.012	0.003	-0.010	0.953
		100	0.280	106.10	0.84	-0.012	0.013	-0.001	0.957

TABLE IV

SEQUENTIAL PROCEDURE (4.1) WITH p = 1, R = 300

m	C	d	$\bar{n}$	$S(\bar{n})$	$\hat{\mu}_1$	$S(\hat{\mu}_1)$	C.P.
5	10	0.620	8.513	0.205	-0.019	37.839	0.893
	15	0.506	11.973	0.326	0.023	37.728	0.843
	20	0.438	16.227	0.438	-0.001	24.100	0.897
	25	0.392	20.557	0.528	-0.003	21.823	0.873
	50	0.277	45.723	0.750	0.001	8.214	0.933
	100	0.196	97.113	0.833	0.001	3.271	0.933
10	10	0.620	11.017	0.120	0.007	32.031	0.940
	15	0.506	14.400	0.247	-0.001	24.451	0.923
	20	0.438	17.820	0.351	0.002	18.073	0.927
	25	0.392	22.643	0.464	0.015	15.753	0.923
	50	0.277	46.880	0.611	-0.000	7.586	0.927
	100	0.196	98.397	0.850	-0.014	3.048	0.950

TABLE V

SEQUENTIAL PROCEDURE (4.1) WITH R = 300

p	m	C	d	$\bar{n}$	$S(\bar{n})$	$\hat{\mu}_1$	$\hat{\mu}_2$	$\hat{\mu}_3$	C.P.
2	5	10	0.774	8.57	0.18	-0.003	0.021	.	0.903
		15	0.632	12.87	0.27	0.005	-0.027	.	0.887
		20	0.547	17.73	0.31	-0.015	-0.015	.	0.870
		25	0.489	23.18	0.36	-0.003	0.018	.	0.903
		50	0.346	48.19	0.48	0.024	-0.006	.	0.943
		100	0.245	99.36	0.59	-0.001	-0.005	.	0.937
2	10	10	0.774	10.91	0.09	-0.011	0.007	.	0.957
		15	0.632	13.97	0.20	-0.000	-0.001	.	0.940
		20	0.547	18.16	0.29	-0.019	0.000	.	0.927
		25	0.489	23.47	0.33	-0.005	-0.001	.	0.910
		50	0.346	48.56	0.41	0.000	-0.022	.	0.920
		100	0.245	98.41	0.63	-0.001	0.000	.	0.943
3	5	10	0.884	8.73	0.15	-0.020	0.019	0.035	0.920
		15	0.722	13.58	0.23	-0.002	0.014	-0.001	0.923
		20	0.625	18.79	0.26	-0.029	-0.020	0.019	0.947
		25	0.559	24.20	0.30	0.024	-0.000	-0.011	0.900
		50	0.395	49.40	0.37	-0.004	0.000	0.011	0.923
		100	0.280	98.73	0.45	-0.004	-0.010	0.009	0.947
3	10	10	0.884	10.70	0.07	-0.003	-0.005	0.021	0.933
		15	0.722	14.03	0.18	0.000	0.013	-0.013	0.920
		20	0.625	18.88	0.23	0.023	0.024	-0.008	0.903
		25	0.559	23.99	0.26	0.009	-0.026	0.020	0.903
		50	0.395	49.40	0.34	0.004	-0.006	0.008	0.953
		100	0.280	98.35	0.50	-0.005	0.016	-0.006	0.957

studies with $R = 300$, $m = 5$, 10 and $C = 10$, 15, 20, 25, 50, 100. The entries of this table are similar to those of Table III except for the first column, where we now have the values of p. The conclusions from this table are similar to those discussed when $p = 1$ in Table IV. We recommend using the sequential procedure (4.1) with $m = 5$ or 10 in practice.

6.4 Moderate Sample Size Performances of the Three-Stage Procedure

In this section, we use the "rule" as being the three-stage procedure of Chapter V. For $p = 1$, some numerical studies have been reported in Hall (1981). We consider $p = 2$, 3 and $C = 10$, 15, 20, 25, 50, 100 with $r = .25$, .50, .75, $R = 300$, and $m = 5$, 10.

The values of m_1 as needed to implement (5.8) - (5.9) with $\alpha = .05$ are given in the fifth column of Table VI for $r = .5$. However, using the third and fourth columns of Table VI, one can find m_1 for any value of $r \in (0,1)$. We may point out that our m_1 for $p = 1$ is the same as that of Hall (1981).

While carrying out simulations with $r = .75$, we noticed some instability in the achieved coverage, namely C.P., with no detectable change in the estimates of the average sample sizes. Also, $S(\bar{n})$ was somewhat higher than our usual expectations in some instances. On the other hand, the average sample sizes seemed to increase for $r = .25$. The results for $r = .5$ seemed to be most stable, and these are reported in Table VII.

We compute

$$\bar{m} = R^{-1} \sum_{j=1}^{R} m(j), \quad \bar{n}^* = R^{-1} \sum_{j=1}^{R} n^*(j),$$

TABLE VI

VALUES OF m_1 AS NEEDED IN (5.8) - (5.9)

p	a	F'(a)	F"(a)	m_1
1	3.84	0.0298	-0.0188	8.339997
2	5.99	0.0250	-0.0125	4.494996
3	7.82	0.0224	-0.0097	3.106661
4	9.49	0.0206	-0.0081	2.372498
5	11.07	0.0193	-0.0070	1.913997
6	12.59	0.0183	-0.0062	1.598331
7	14.07	0.0174	-0.0056	1.367141
8	15.51	0.0167	-0.0051	1.188749
9	16.92	0.0160	-0.0047	1.046664
10	18.31	0.0155	-0.0044	0.930999
11	19.68	0.0150	-0.0041	0.834544
12	21.03	0.0145	-0.0038	0.752500
13	22.36	0.0142	-0.0036	0.681538
14	23.69	0.0138	-0.0034	0.620714
15	24.99	0.0135	-0.0032	0.566001
16	26.30	0.0131	-0.0031	0.518749
17	27.59	0.0128	-0.0029	0.475882
18	28.87	0.0126	-0.0028	0.437222
19	30.14	0.0123	-0.0027	0.402105
20	31.41	0.0121	-0.0026	0.370501
21	32.67	0.0119	-0.0025	0.341428
22	33.92	0.0117	-0.0024	0.314545
23	35.17	0.0115	-0.0023	0.290000
24	36.42	0.0113	-0.0022	0.267499
25	37.65	0.0111	-0.0022	0.245999
26	38.89	0.0110	-0.0021	0.226538
27	40.11	0.0108	-0.0020	0.207778
28	41.34	0.0106	-0.0020	0.190714
29	42.56	0.0105	-0.0019	0.174482
30	43.77	0.0104	-0.0019	0.159000

TABLE VII

THREE-STAGE PROCEDURE (5.8) - (5.9)
WITH R = 300, r = .5

p	m	C	d	$\bar{m}$	$S(\bar{m})$	$\bar{n}^*$	$S(\bar{n}^*)$	$\hat{\mu}_1$	$\hat{\mu}_2$	$\hat{\mu}_3$	C.P.
2	5	10	0.774	6.23	0.12	14.10	0.21	0.013	0.016	.	0.980
		15	0.632	7.94	0.20	17.61	0.30	0.015	-0.000	.	0.973
		20	0.547	10.37	0.26	21.99	0.38	0.025	0.003	.	0.957
		25	0.489	13.16	0.34	28.53	0.48	0.000	-0.010	.	0.963
		50	0.346	25.04	0.69	52.68	0.67	0.005	0.025	.	0.933
		100	0.245	50.63	1.45	103.88	0.96	-0.004	-0.001	.	0.953
2	10	10	0.774	10.01	0.01	15.13	0.19	0.008	-0.004	.	0.983
		15	0.632	10.45	0.06	20.19	0.26	-0.009	0.005	.	0.980
		20	0.547	11.54	0.14	24.27	0.30	0.014	0.006	.	0.967
		25	0.489	13.04	0.20	27.98	0.40	-0.006	-0.010	.	0.963
		50	0.346	24.47	0.46	51.65	0.65	-0.004	0.004	.	0.953
		100	0.245	50.56	0.91	103.49	0.84	-0.009	-0.002	.	0.950
3	5	10	0.884	5.94	0.08	12.64	0.19	-0.005	0.004	0.013	0.963
		15	0.722	8.25	0.16	17.62	0.28	0.009	0.022	0.011	0.957
		20	0.625	10.43	0.23	22.08	0.34	-0.019	-0.012	0.007	0.950
		25	0.559	13.35	0.32	26.89	0.39	-0.015	-0.011	-0.007	0.923
		50	0.395	25.03	0.59	51.50	0.57	0.010	0.008	0.006	0.923
		100	0.280	51.47	1.22	101.37	0.78	0.006	-0.015	0.000	0.960
3	10	10	0.884	10.00	0.00	13.58	0.15	-0.003	0.001	0.015	0.983
		15	0.722	10.17	0.03	18.24	0.21	0.026	-0.005	-0.002	0.967
		20	0.625	11.44	0.12	23.17	0.28	-0.003	0.021	-0.008	0.967
		25	0.559	12.80	0.16	26.54	0.33	-0.004	0.009	0.010	0.933
		50	0.395	25.38	0.38	52.24	0.53	-0.005	0.004	0.005	0.940
		100	0.280	50.83	0.78	102.92	0.68	0.005	-0.001	-0.002	0.957

$$S(\bar{m}) = \{(R^2 - R)^{-1} \sum_{j=1}^{R} (m(j) - \bar{m})^2\}^{1/2}, \text{ and}$$

$$S(\bar{n}^*) = \{(R^2 - R)^{-1} \sum_{j=1}^{R} (n^*(j) - \bar{n}^*)^2\}^{1/2}, \text{ as well.}$$

We may note that $\bar{n}^*$ and C.P. are very close to C and $(1-\alpha)$, respectively. In the absence of any further knowledge, we suggest using the three-stage procedure (5.8) - (5.9) with $r = .5$ and $m = 5$ or 10.

Remark 7: In a particular application, if all our procedures can possibly be implemented, we will suggest using the modified two-stage or the three-stage procedure, simply because these will be less time-consuming. However, the sequential procedure will give the best theoretical results if it can be implemented. The main point to note is that the three-stage procedure can be almost as good. The final recommendation should also consider the structure and design of the particular applied problem. Overall, the choice seems to be between the three-stage and the purely sequential procedures. We must also stress that we have $P\{\mu \in R_N\}$ to be at least $(1-\alpha)$ with $\lim_{d\to 0} E(N/C) = 1$ for the modified two-stage procedure of Chapter III. However, the coverage becomes only asymptotically $(1-\alpha)$ for the three-stage and sequential procedures. So, depending on the goal, the modified two-stage procedure can be just as appealing and possibly be more practical to use because in this case one does not need to go to the third stage for sampling.

Remark 8: For all the problems discussed in Chapters II - VI, it will indeed be very interesting to study various effects of considering the James-Stein (1961) estimators instead of the more conventional ones like

$\bar{\underset{\sim}{X}}_N$ in defining the region R_N. Both numerical and theoretical developments would be very challenging. This particular area has just started to grow only recently. One is referred to Ghosh and Sen (1983).

CHAPTER VII

CONCLUSIONS

In this study we have presented two-stage, modified two-stage, purely sequential, and three-stage procedures to construct "fixed-size" elliptic confidence regions for estimating the mean vector of a p-dimensional normal distribution when the dispersion matrix is of the form $\sigma^2 H$ where $\sigma \in (0,\infty)$ and H is a p×p known positive definite matrix.

For the univariate case, namely when $p = 1$, we find that some of the well-known existing procedures follow as special cases of those presented in this study. As examples, the reader is referred to Stein's (1945, 1949) two-stage procedure, Mukhopadhyay's (1980) modified two-stage procedure, and Chow and Robbins' (1965) purely sequential procedure. We also report extensive simulation studies to put various competitive procedures in proper perspective. The proposed two-stage and modified two-stage procedures guarantee the coverage probability to be at least $(1-\alpha)$. For the purely sequential and three-stage procedures, the coverage probability is shown to be asymptotically $(1-\alpha)$. However, since the purely sequential procedure does not violate the sufficiency principle, it is expected to give the best theoretical results in terms of having the asymptotic second-order efficiency property. On the other hand, the three-stage procedure combines the simplicity of a two-stage procedure with some of the merits of a purely sequential procedure. In terms of the average sample size and the achieved coverage probability, the modified two-stage procedure can be almost as good as the three-stage procedure. Also, the modified two-stage procedure is less time-consuming in terms of implementation. Our findings might possibly limit the usefulness of the three-stage procedure in some applications. The final choice among those procedures should depend on the goals and types of results one would expect to have in a particular context. Various second-order expansions are derived when the purely sequential and three-stage procedures are proposed. The rate of convergence to normality for the distribution of N is also given for the purely sequential case. Results from Mukhopadhyay (1974, 1980), Ghosh and Mukhopadhyay (1975, 1981), Srivastava (1967), Woodroofe (1977), and Hall (1981) have proved to be extremely important and useful for the theoretical ground work in this present study. For practical implementation we recommend using the modified two-stage or the three stage-procedure, simply because these will be less time consuming than the purely sequential one when implemented. For more detailed recommendations, the reader is also referred to Remark 7.

Also, we can easily examine which values of the estimated coverage probability (C.P.) from our tables are consistent with the target coverage probability (T.C.P.), namely, .95. To be more specific, for any particular row in any of the tables, suppose we wish to test the null hypothesis H_0: T.C.P. = .95 against the alternative H_1: T.C.P. $\neq$ .95. We can now compute the magnitude of $(\text{C.P.} - .95)\{(.95)(.05)/300\}^{-\frac{1}{2}}$, and then reject H_0 at the 5% level if it is larger than 1.96. We have checked most of the achieved values of C.P. and failed to reject H_0, that is, the achieved C.P. values are consistent with .95.

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