THE CAUCHY INTEGRAL THEOREM:

A HISTORICAL DEVELOPMENT

OF ITS PROOF

By

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TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER I

INTRODUCTION

The term "fundamental," when used to describe a mathematical theorem, implies the significance of the result in laying the foundation of a particular study. One result in complex analysis stands out as being truly fundamental to the theory. It is the Cauchy Integral Theorem, named for Augustin-Louis Cauchy who first published it. Its consequences and extensions are numerous and far-reaching, but a great deal of interest lies in the theorem itself. The rigorization which took place in complex analysis after the time of Cauchy's first proof and the development in the method of the proof are reasons why Cauchy's original proof is not found in a textbook on complex analysis of today. Several mathematicians since Cauchy have pondered the question as to how to improve upon the existing proof; consequently, the Cauchy Integral Theorem has undergone several changes in statement and in proof over the last 150 years. Yet it still remains the basic result in complex analysis it has always been.

The purpose of the thesis is to investigate the history of this theorem and the efforts of the mathematicians who sought to prove it as well as to develop a simple proof of the most general form of the theorem. The proofs are presented in a chronological fashion with the key differences in the methods of proof being noted. An effort has been made to retain the style of proof incorporated by these mathematicians;

however, where appropriate, more contemporary symbols and terminology have been employed.

The discussion is organized in the following way. Some remarks on terms essential to the theorem as well as Cauchy's and Riemann's proofs are discussed in Chapter II. In Chapter III, the first major development in the proof of the theorem is examined with Goursat's proof, and a remark is made concerning Moore's improvement of Goursat's proof. The proof by Pringsheim is presented in Chapter IV; this particular proof, or a version thereof, is the one often found in modern textbooks on complex analysis. The final stage in the development of the method of proof is given in Chapter V where the discussion is led up to the present time with Dixon's proof. A summary in Chapter VI concludes this thesis.

CHAPTER II

BASIC DEFINITIONS AND PROOFS

BY CAUCHY AND RIEMANN

Before the investigation into the history of the Cauchy Integral Theorem is begun, it is necessary to present several definitions essential to its understanding. Since the theorem deals with the integral of a complex function, it would be well to review this definition.

<u>Definition 2.1</u>: Let the path C be parametrized by C: $z = z(t)$, $a \leq t \leq b$, and let $z(t_k) = z_k$, $k = 1, 2, \ldots, n$, be points on C where $z(t_0) = z(a)$ and $z(t_{n+1}) = z(b)$. Then the definite integral along C of a function $f(z)$ defined on C has the definition

$$\int_C f(z)\, dz = \lim_{n\to\infty} \sum_{k=0}^{n} f(\xi_k)(z_{k+1} - z_k),$$

where ξ_k is any point on the arc $z_k z_{k+1}$ and $\max|z_{k+1} - z_k| \to 0$ as $n \to \infty$. It can be shown that this definite integral exists if the path C is rectifiable and the function $f(z)$ is continuous on C [10].

If one were to consider the integral $\int_a^b f(z)\, dz$, where a and b are complex numbers, then the path from a to b is a curve in the complex plane. Since there are many paths of integration from which to choose, one may question the very existence of the integral. Surely, for this integral to have any worthwhile meaning, it would be necessary for it to be independent of the path from a to b. This is what Cauchy's Theorem guarantees for analytic functions in certain special domains. Essentially, the theorem states that if a function f(z) is analytic in one of these special domains D and C is a closed curve lying in D, then $\int_C f(z)\, dz = 0$. Certainly, when verified, this result would be quite powerful; however, these special domains need careful defining. This is the purpose of the following discussion; it begins with an intensive look at an intuitive idea--the interior of a curve.

There are two approaches to the definition of the interior of a curve, both of which shall be presented here. The classic approach is through the Jordan Curve Theorem, a theorem simple in its statement, yet difficult in its proof. For many years the proof of this theorem plagued mathematicians. In fact, Jordan's actual argument was found insufficient, and later a valid proof was given by the American topologist Oswald Veblen [10]. The more contemporary approach to the definition of the interior of a curve is through the concept of the index of a curve; this method avoids the use of the Jordan Curve Theorem altogether.

The Jordan Curve Theorem states that a simple closed curve decomposes the complex plane into two disjoint domains: a bounded domain called the interior of the curve and an unbounded domain called the exterior of the curve. This is the classical definition of the interior of a curve.

A more contemporary approach to the definition of the interior of a curve is one which employs the integral of a particular function along a piecewise smooth closed curve lying in the complex plane. "Contemporary" is a relative term, as a discussion similar to that which follows, but which avoids the use of the integral, was given by Ames in 1905. Watson [17] gives a presentation patterned after Ames' argument.

Definition 2.2: Let C be a piecewise smooth closed curve lying in the complex plane $\mathbb{C}$. Let $\Omega = \mathbb{C} - C$. The index of C relative to the point $z \in \Omega$, denoted by $\text{Ind}_C(z)$, is given by

$$\text{Ind}_C(z) = \frac{1}{2\pi i} \int_C \frac{d\zeta}{\zeta - z}.$$

The index of a curve relative to a point tells how many times the curve winds around that point as well as the direction of the winding; hence, it is often referred to as the "winding number." For example, let C be the circle of radius r centered at the origin. Then C may be parametrized by C: $z(t) = re^{it}$, $0 \leq t \leq 2$. Then,

$$\text{Ind}_C(0) = \frac{1}{2\pi i} \int_C \frac{d\zeta}{\zeta} = \frac{1}{2\pi i} \int_0^{2\pi} \frac{rie^{it}}{re^{it}}\, dt = 1;$$

that is, the curve C winds around the origin once in the positive (counter-clockwise) direction. Note that for the curve C_0 parametrized by C_0: $z(t) = re^{it}$, $0 \leq t \leq 4\pi$, one obtains $\text{Ind}_{C_0}(0) = 2$; whereas for the curve C_1: $z(t) = re^{-it}$, $0 \leq t \leq 4\pi$, one finds $\text{Ind}_{C_1}(0) = -2$.

A component of $\mathbb{C} - C$ shall be a maximal open connected subset of $\mathbb{C} - C$. Since C is a compact subset of $\mathbb{C}$, there exists a real number $A > 0$ such that $C \subseteq \{z \mid |z| \leq A\}$ and $\mathbb{C} - C \supseteq \{z \mid |z| > A\}$. Thus, $\mathbb{C} - C$ has precisely one unbounded component.

Before indicating the relationship between the index of a curve relative to a point and the interior of a curve, the following theorem is necessary.

Theorem 2.1: The index of a curve C relative to a point $z \in \mathbb{C} - C$ is an integer-valued function on $\mathbb{C} - C$ which is constant on each component of $\mathbb{C} - C$ and which is zero in the unbounded component of $\mathbb{C} - C$.

Proof: The first part of the proof shall be to show that $\mathrm{Ind}_C(z)$ is an integer-valued function on $\mathbb{C} - C$. Let the piecewise smooth closed curve C be parametrized by C: $z = \zeta(t)$, $a \leq t \leq b$. Then

$$\mathrm{Ind}_C(z) = \frac{1}{2\pi i} \int_C \frac{d\zeta}{\zeta - z} = \frac{1}{2\pi i} \int_a^b \frac{\zeta'(s)}{\zeta(s) - z}\, ds.$$

Consider the function $\phi(t)$ defined by

$$\phi(t) = \int_a^t \frac{\zeta'(s)}{\zeta(s) - z}\, ds.$$

Note that $\phi(b) = 2\pi i\, \mathrm{Ind}_C(z)$ and that $e^{\phi(b)} = 1$ if and only if $\phi(b) = 2\pi i k$, k an integer. Setting $g(t) = e^{\phi(t)}$, $\mathrm{Ind}_C(z)$ can be proved to be integer-valued by showing $g(b) = 1$. Since $g(t) = e^{\phi(t)}$, then $g'(t) = e^{\phi(t)}\phi'(t)$. Hence,

$$\frac{g'(t)}{g(t)} = \phi'(t) = \frac{\zeta'(t)}{\zeta(t) - z}.$$

This implies that $g'(t)[\zeta(t) - z] - g(t)[\zeta'(t)] = 0$; whence

$$\frac{d}{dt}\left(\frac{g(t)}{\zeta(t) - z}\right) = 0.$$

But this implies that $\frac{g(t)}{\zeta(t) - z}$ is constant on finitely many intervals. Thus, $\frac{g(t)}{\zeta(t) - z}$ is constant for $a \leq t \leq b$. This implies that

$$\frac{g(t)}{\zeta(t) - z} = \frac{g(a)}{\zeta(a) - z}. \tag{1}$$

But $g(a) = e^{\phi(a)} = e^0 = 1$. Therefore,

$$\frac{g(a)}{\zeta(a) - z} = \frac{1}{\zeta(a) - z}. \tag{2}$$

In view of Equations (1) and (2), $g(t) = \frac{\zeta(t) - z}{\zeta(a) - z}$. Since C is a closed curve with $\zeta(b) = \zeta(a)$, $g(b) = \frac{\zeta(b) - z}{\zeta(a) - z} = 1$.

The second part of the proof is to show that $\text{Ind}_C(z)$ is constant in each component of $\mathbb{C} - C$. Let $f(z) = \text{Ind}_C(z)$. Choose z_0 in $\mathbb{C} - C$. Then there exists $\delta > 0$ such that $\{|z - z_0| \leq \delta\} \subseteq \mathbb{C} - C$. Let K be the circle $|z - z_0| = \delta$, and let $d = \text{dist}\,(C,K)$. Since K is compact and C is closed, d must be positive. Let $\varepsilon > 0$. Then

$$|f(z) - f(z_0)| = \left|\frac{1}{2\pi i}\int_C \frac{d\zeta}{\zeta - z} - \frac{1}{2\pi i}\int_C \frac{d\zeta}{\zeta - z_0}\right|$$

$$= \frac{1}{2\pi}\left|\int_C \left[\frac{1}{\zeta - z} - \frac{1}{\zeta - z_0}\right] d\zeta\right|$$

$$= \frac{|z - z_0|}{2\pi}\left|\int_C \frac{d\zeta}{(\zeta - z)(\zeta - z_0)}\right| .$$

Since for every z in the closed disk $|z - z_0| \leq \delta$ and $\zeta \in C$, $|\zeta - z| \geq d$ and $|\zeta - z_0| \geq d$, it follows that

$$|f(z) - f(z_0)| \leq \frac{|z - z_0|}{2\pi} \cdot \frac{\text{lgth } C}{d^2} .$$

Let $\delta_1 = \min\left\{\delta, \frac{\varepsilon 2\pi d^2}{\text{lgth } C}\right\}$. Then $|f(z) - f(z_0)| < \varepsilon$ whenever $|z - z_0| < \delta_1$. Thus $f(z)$ is continuous on each component of $\mathbb{C} - C$. Since each component is connected, its image under f must be connected. But $\text{Ind}_C(z)$ is integer-valued; consequently, f must be constant on each component.

The third and final part of the proof is to show $\text{Ind}_C(z) = 0$ in the unbounded component of $\mathbb{C} - C$. Recall that there exists $A > 0$ such that $C \subseteq \{|z| \leq A\}$. So for all $\zeta \in C$ and z in the unbounded component of $\mathbb{C} - C$, $|\zeta| \leq A$ and $|z| > A$. Thus, $|\zeta - z| \geq ||\zeta| - |z|| \geq |z| - A > 0$. Hence,

$$|\text{Ind}_C(z)| = \left|\frac{1}{2\pi i}\int_C \frac{d\zeta}{\zeta - z}\right| \leq \frac{1}{2\pi} \cdot \frac{\text{lgth } C}{|z| - A} .$$

Letting $z \to \infty$, it is clear that $\text{Ind}_C(z) = 0$.

It is now possible to define the interior of a curve.

Definition 2.3: A point z lies in the interior of a simple closed curve C if $Ind_C(z) \neq 0$.

In order for the Cauchy Integral Theorem to be valid, the domain in the hypothesis must have no "holes" in it. A precise definition of this condition which must be placed on the domain D was known in the formative years of complex analysis. A domain having this property was labeled "simply connected" (Figure 1). This term could be easily defined with the aid of the Jordan Curve Theorem; this classical definition is the following.

Definition 2.4: A simply connected domain D is a domain having the property that for every simple closed curve lying in D, the interior of the curve also lies in D.

Figure 1. Illustration of a Simply Connected Region and a Region Which is not Simply Connected

A more contemporary approach to the definition of simple connectivity is one which makes no use of the Jordan Curve Theorem. Rather, it hinges on the idea of being able to contract every closed curve lying in the domain down to a single point, all the while remaining in the domain. This definition involves homotopy and is explained in detail by the following discussion.

Definition 2.5: Let two closed curves γ_0: $z = z_0(s)$, $0 \leq s \leq 1$, and γ_1: $z = z_1(s)$, $0 \leq s \leq 1$, lie in a topological space X, and $z_0(0) = z_0(1) = z_1(0) = z_1(1)$. The curve γ_0 is said to be X-homotopic to γ_1, symbolically $\gamma_0 \sim \gamma_1$, if there exists a continuous mapping $H: [0,1] \times [0,1] \longrightarrow X$ such that $H(s,0) = z_0(s)$, $H(s,1) = z_1(s)$, and $H(0,t) = H(1,t)$ for all $s, t \in [0,1]$.

A constant mapping γ_1: $z = z_1(s)$, $0 \leq s \leq 1$, shall be defined by the equation $z_1(s) = a$ for every s, $0 \leq s \leq 1$, and for some $a \in X$. This leads to the following definition.

Definition 2.6: If γ_0 is X-homotopic to a constant mapping γ_1, then γ_0 is said to be null-homotopic in X.

Intuitively, if a closed curve is null-homotopic in a space, it can be collapsed to a single point while remaining in the space. This is the notion mentioned earlier as being the basis of simple connectivity; it is incorporated in the following definition.

Definition 2.7: If a topological space X is connected and if every closed curve in X is null-homotopic, then X is simply connected.

To see why the hypothesis of a simply connected domain is vital to the theorem, consider the following example.

Example: Let $f(z) = 1/z$ be defined in the domain D: $0 < |z| \leq 1$. Clearly, $f(z)$ is analytic in D. Let the closed curve C be the unit circle C: $z(t) = e^{it}$, $0 \leq t \leq 2\pi$. Then

$$\int_C f(z)\, dz = \int_0^{2\pi} \frac{ie^{it}}{e^{it}}\, dt = 2\pi i \neq 0.$$

The reason that the integral theorem fails in this case is that the domain D has a hole at $z = 0$ and C encloses that hole.

With these basic notions in hand it is possible to formulate the Cauchy Integral Theorem and to examine the developments in its proof. The theorem can now be stated as: If $f(z)$ is analytic in a simply connected domain D and if C is a simple closed rectifiable curve lying in D, then $\int_C f(z)\, dz = 0$.

A natural beginning in the study of the history of this theorem is with the man for whom it was named. However, other mathematicians besides Cauchy introduced some basic ideas about functions of a complex variable. Carl Friedrich Gauss and Siméon Denis Poisson made significant observations in the early nineteenth century concerning integrals of complex functions taken over paths in the complex plane, but neither man published a major paper on complex function theory. Because Cauchy did publish papers on this theory, he is credited with its founding.

But in his early works, Cauchy was reluctant to consider complex functions. He continually referred to these functions by considering their two real components as though that were their important feature. However, by 1822 he began to actually work with complex functions. In that year he wrote the paper which included his integral theorem for the special case when the closed curve was a rectangle. This result for rectangles was extended to more general paths in 1825 when Cauchy wrote the paper which is considered by many to be his most beautiful [5]. However, Kline suggests that Cauchy himself did not appreciate its worth; consequently, it was not published until 1874 [11].

In his 1825 memoir Cauchy claimed "If $f(x + iy)$ is finite and continuous for $x_0 \leq x \leq X$ and $y_0 \leq y \leq Y$, then the value of the integral $\int_{x_0 + iy_0}^{X + iY} f(z)\,dz$ is independent of the form of the functions $x = \phi(t)$ and $y = \psi(t)$." Using the method of calculus of variations, he considered $\phi(t) + \varepsilon u(t)$, $\psi(t) + \varepsilon v(t)$ as an alternate path and showed that the first variation of the integral with respect to ε vanishes. Note that no mention was made of the continuity of the derivative of $f(z)$ or even of its existence although Cauchy made use of both in his proof. Kline [10] suggests that a possible explanation for this is that Cauchy believed, as others of his time, that a continuous function was always differentiable and that its derivative was discontinuous only where the function itself was discontinuous. As this theorem is traced through history, the development of ideas more in keeping with present day views on continuity and differentiability will be seen.

In 1846, Cauchy published a paper in which he gave another proof of his integral theorem [4]. In this proof he made use of the notion of exact differentials, the definition of which follows.

Definition 2.8: $P\,dx + Q\,dy$ is an exact differential if there exists a function $U(x,y)$ such that $\frac{\partial U}{\partial x} = P$ and $\frac{\partial U}{\partial y} = Q$.

Cauchy also applied a result known as Green's Theorem, which is presented below. Whether Cauchy himself proved this result independently of George Green or actually utilized Green's work is not certain, for although Green's Theorem was published in a privately printed booklet in 1828, it was not published in a mathematical journal until 1850. So it is possible that Cauchy conceived of this result independently; however, there are indications that he was influenced by Green's work because he extended his integral theorem to areas on curved surfaces. Consequently, Green's Theorem shall be cited in the proof of Cauchy's Theorem. The original statement of Green's Theorem is uncertain; however, the following form shall be used in this thesis. The interested reader should refer to Apostol [2] for the proof of this form of Green's Theorem.

Theorem 2.2 (Green's Theorem): Let $P(x,y)$ and $Q(x,y)$ be continuous with continuous partials on a Jordan region R bounded by a rectifiable Jordan curve C. Then

$$\int_C P\,dx + Q\,dy = \int_R\int \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy,$$

where C is traversed in the positive sense.

The next theorem is Cauchy's Integral Theorem as presented in his 1846 paper.

Theorem 2.3: The line integral $\int_C P\,dx + Q\,dy$ defined in a domain D is zero for all closed paths C lying in D if and only if $P\,dx + Q\,dy$ is an exact differential.

Proof: Let C be a closed path lying in a domain D and let R be the region enclosed by C. If $P\,dx + Q\,dy$ is an exact differential, then there exists a function $U(x,y)$ such that $\frac{\partial U}{\partial x} = P$ and $\frac{\partial U}{\partial y} = Q$. Then

$$\frac{\partial P}{\partial y} = \frac{\partial^2 U}{\partial y \partial x} = \frac{\partial^2 U}{\partial x \partial y} = \frac{\partial Q}{\partial x} . \quad (3)$$

By Theorem 2.2,

$$\int_C P\,dx + Q\,dy = \int_R\int \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy,$$

but in view of Equation (3), the integrand of the right member is zero in R.

Conversely, suppose $\int_C P\,dx + Q\,dy = 0$. Fix a point (x_0,y_0) in R and join it to (x,y) by a polygonal path γ contained in R whose sides are parallel to the real and imaginary axes. Define a function by $U(x,y) = \int_\gamma P\,dx + Q\,dy$. Note that this function is well defined since the integral depends only on the endpoints. If γ is chosen so that the last segment is horizontal, then y can be kept constant while x varies without changing the other segments. On this last segment if x is chosen for parameter,

$$U(x,y) = \int^x P(x,y)\,dx + K,$$

where K is a constant and the lower limit of integration is irrelevant. From this it is clear that $\frac{\partial U}{\partial x} = P$. By a similar argument, choosing the last segment to be vertical, $\frac{\partial U}{\partial y} = Q$. Thus, P dx + Q dy is an exact differential.

Cauchy published many other papers besides those cited here; in fact, he wrote over 700 papers in mathematics. However, his work with the proof of his integral theorem appeared to end with his 1846 paper.

The next name of significance found in connection with the Cauchy Integral Theorem is that of Bernhard Riemann. During his brief life of 40 years, Riemann made many outstanding contributions to mathematics and in particular to complex analysis. In 1851, his inaugural dissertation at Göttingen included a proof that if a function $f(z) = u(x,y) + iv(x,y)$

is differentiable at a point z_0, then the equations, now known as the Cauchy-Riemann equations, must be satisfied at z_0 [15]. These equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} . \tag{4}$$

He also proved a near-converse to this theorem; that is, if $f(z) = u(x,y) + iv(x,y)$ is defined in a domain D and $u(x,y)$ and $v(x,y)$ have continuous partials that satisfy the Cauchy-Riemann Equations (4) at all points in D, then $f(z)$ is analytic in D. It is clear that the continuity of the partials in this result prevent it from being a strict converse of the previous theorem. Finally, by employing Equations (4) along with Green's Theorem, Riemann gave the following improved version of the Cauchy Integral Theorem.

Theorem 2.4: If $f(z)$ is analytic with a continuous derivative on and inside a closed contour C, then $\int_C f(z)\, dz = 0$.

Proof: Let $f(z) = u(x,y) + iv(x,y)$. By analyticity, $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$ for all points z inside or on C. Note that by equating the real and imaginary parts of the two expressions for $f'(z)$, the Cauchy-Riemann equations are obtained. If R is the closed region bounded by C, then by Theorem 2.2,

$$\begin{aligned} \int_C f(z)\, dz &= \int_C u\, dx - v\, dy + i \int_C v\, dx + u\, dy \\ &= \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\, dy + i \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\, dy \\ &= 0. \end{aligned}$$

Riemann's proof can be viewed as an improvement over Cauchy's in that it avoided the use of exact differentials and consequently applied to a wider variety of functions. However, the theorem was yet to be improved upon. In 1900, Edouard Goursat showed that the assumption of the continuity of the derivative could be lessened to simply the existence of it; his proof shall be the subject of the next chapter [8].

To conclude this chapter in the history of the Cauchy Integral Theorem, it is interesting to note that mathematicians before the time of Goursat's proof were by no means idle. The nineteenth century was filled with a great deal of mathematical activity in the area of complex analysis; however, most of the ideas and methods of the founders of the theory were pursued independently by the mathematicians of the time. Goursat's proof signified a return to the basics of the theory, and at the beginning of the twentieth century, unification of the theory began to take place.

CHAPTER III

GOURSAT'S PROOF

In 1900, Edouard Goursat proved Cauchy's Theorem without assuming the continuity of the derivative, a hypothesis which had heretofore remained unchallenged. In addition to eliminating this unnecessary hypothesis, Goursat also launched a totally new approach to the proof of this fundamental theorem, one in which the integral was worked with directly as opposed to being separated into real and imaginary parts. Goursat's basic idea was to divide up a region enclosed by a closed curve C through a process of subdivision which shall be described in his proof. This process would yield a finite number of square "regular" regions as well as some "irregular" regions near the curve C. By estimating the value of the integral of f(z) taken along the boundary of each regular and irregular region, Goursat was able to show that the sum of the values could be made arbitrarily small so that $\int_C f(z)\,dz$ must be zero. However, in order to insure the necessary estimates, Goursat found it necessary to prove a key lemma which now bears his name. Although Goursat's proof was good in that it eliminated an unnecessary hypothesis, it was not without its shortcomings. These weaknesses shall be discussed following his proof.

<u>Lemma 3.1</u>: If C is a closed curve, then $\int_C dz = 0$ and $\int_C z\,dz = 0$.

Proof: Let $z_1, z_2, \ldots, z_k$, where $z_1 = z_k$, be a partition of C. Then by definition of the integral,

$$\int_C dz = \lim_{k\to\infty} \sum_{j=1}^{k-1} (z_{j+1} - z_j) = \lim_{k\to\infty} (z_k - z_1) = 0.$$

For the second part of the proof, let $z_1, z_2, \ldots, z_k, z_1 = z_k$, be a partition of C. Let ξ_j be any point of the arc from z_j to z_{j+1}, including the endpoints. By definition of the integral,

$$\int_C z\, dz = \lim_{k\to\infty} \sum_{j=1}^{k-1} \xi_j (z_{j+1} - z_j).$$

If one first takes $\xi_j = z_j$ and then $\xi_j = z_{j+1}$, one finds

$$\int_C z\, dz = \lim_{k\to\infty} \sum_{j=1}^{k-1} \frac{z_j (z_{j+1} - z_j) + z_{j+1} (z_{j+1} - z_j)}{2}$$

$$= \lim_{k\to\infty} \sum_{j=1}^{k-1} \frac{z_{j+1}^2 - z_j^2}{2}$$

$$= \lim_{k\to\infty} \frac{z_k^2 - z_1^2}{2}$$

$$= 0.$$

Lemma 3.2: Suppose f(z) is defined on a domain D and C is any closed curve such that C and its interior lie in D. A cross-cut of the interior of C shall be a simple arc that lies in the interior of C except for its endpoints. If the region R bounded by C is divided into smaller parts $R_1, R_2, \ldots, R_n$ by cross-cuts and if $C_1, C_2, \ldots, C_n$ are the boundaries of $R_1, R_2, \ldots, R_n$, respectively, then

$$\int_C f(z)\, dz = \int_{C_1} f(z)\, dz + \int_{C_2} f(z)\, dz + \ldots + \int_{C_n} f(z)\, dz,$$

where the orientation of C_j, $j = 1, 2, \ldots, n$, is that induced by C.

<u>Proof</u>: It is clear that each cross-cut separates two adjoining regions; hence, it must be described twice in integration in opposite senses (Figure 2).

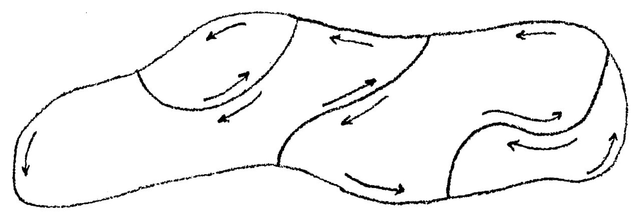


Figure 2. Cross-Cuts of the Interior of Closed Curve C

Since

$$\int_{C_j} f(z)\, dz = -\int_{-C_j} f(z)\, dz,$$

the sum

$$\sum_{j=1}^{n} \int_{C_j} f(z)\, dz$$

will result in only the integrals taken along the boundary curve C. This sum is the integral $\int_C f(z)\, dz$.

<u>Remark</u>: Watson [17] proves a theorem concerning orientation which he suggests Goursat should have used in the proof of Lemma 3.2.

A region R bounded by a closed curve C will be said to satisfy the Goursat Property with respect to the number ε if it is possible to find a point z' inside or on the curve C such that one always has

$$\left|f(z) - f(z') - (z - z')\, f'(z')\right| \leq \left|z - z'\right| \varepsilon$$

for all z on the curve C.

Lemma 3.3 (Goursat's Lemma): Let f(z) be differentiable on a closed contour C and on all points of a region R bounded by C; let ε be an arbitrary positive number. Then the region R can always be decomposed into sufficiently small subregions such that the contour of each of these subregions satisfies the Goursat Property relative to ε.

Proof: Let the region R be divided into smaller subregions by lines drawn parallel to the real and imaginary axes, the distance between adjacent parallels being constant and equal to l. If any of the smaller parts obtained by this division process satisfy the Goursat Property, then these subregions shall remain unchanged. All other parts are then divided into smaller ones by joining the midpoints of the opposite sides of the squares which form these subregions or enclose them. Now either this process will terminate, yielding a collection of regions all of which satisfy the Goursat Property and thereby proving the lemma, or it will not end. That is, one can always find a subdivision which does not satisfy the Goursat Property.

Considering the latter case, there must be at least one of the subregions obtained by the first division process which does not satisfy the Goursat Property. Call this subregion R_1. After the second subdivision, the part R_1 must contain at least one subregion R_2 which does not satisfy this property. Since it has been assumed that the division process does not terminate, a sequence of regions $\{R_n\}$ which are squares or portions of squares can be constructed such that

$\bar{R}_1 \supset \bar{R}_2 \supset \ldots \supset \bar{R}_n \supset \ldots .$ By construction,

$$\text{area } (R_n) \leq \left(\frac{l}{2^{n-1}}\right)^2$$

so that

$$\lim_{n\to\infty} \text{area } (R_n) \leq \lim_{n\to\infty} \frac{l^2}{4^{n-1}} = 0.$$

Therefore, there is one and only one point $z_0 \in \bigcap_{j=1}^{\infty} \bar{R}_j$; since $\bar{R} = R \cup C$ is closed, z_0 must lie either in R or on C. Because $f'(z_0)$ exists by hypothesis, there exists $\delta > 0$ such that

$$|f(z) - f(z_0) - (z - z_0)\ f'(z_0)| \leq |z - z_0|\ \varepsilon \tag{5}$$

whenever $|z - z_0| < \delta$. Since diam $(R_n) \leq \frac{l}{2^{n-1}} \sqrt{2}$, one can choose n sufficiently large so that the region R_n lies within the circle $|z - z_0| = \delta$. Then all points of the boundary of R_n satisfy Equation (5). But $z_0 \in \bar{R}_n$; that is, z_0 is an interior point or a boundary point of R_n. Thus R_n satisfies the Goursat Property with respect to ε, and a contradiction is obtained by supposing the lemma is not true.

Theorem 3.1 (Cauchy-Goursat Theorem): If f(z) is analytic in a region R bounded by the closed curve C and also on the curve, then $\int_C f(z)\, dz = 0$.

Proof: Divide the region R partly into smaller regular parts which shall be squares having their sides parallel to the real and imaginary axes and partly into irregular parts, which shall be portions of squares of which the remaining part lies beyond the boundary C. Some of the squares may be further divided into smaller squares by new parallels to the axes. By whatever manner of subdivision adopted, suppose there are

N regular parts and N' irregular parts; number the regular parts in any order from 1 to N and the irregular parts in any order from 1 to N'. Let l_j be the length of a side of the jth square and l'_k be the length of a side of the square to which the kth irregular part belongs. Let L be the length of C and P be the polygon formed by the outermost boundaries of the squares which contain points of C (Figure 3).

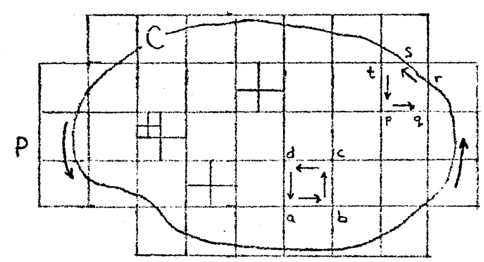


Figure 3. Subdivision of R

Let abdc be the jth square (Figure 3), let z_j be a point in the interior of the square or on one of its sides, and let z be any point on its boundary. Since f(z) is analytic on $\bar{R}$,

$$\frac{f(z) - f(z_j)}{z - z_j} = f'(z_j) + \varepsilon_j(z) \tag{6}$$

where $|\varepsilon_j(z)|$ is small, provided that l_j is small. Let pqrst be the kth irregular part (Figure 3), let z'_k be a point in the interior or on the perimeter of the part, and let z be any point on its perimeter. Then,

$$\frac{f(z) - f(z'_k)}{z - z'_k} = f'(z'_k) + \varepsilon'_k(z) \tag{7}$$

where $|\varepsilon_k'(z)|$ is small, provided that l_k' is small. Let $\varepsilon > 0$. By Lemma 3.3 the regular and irregular parts can be made so small that the absolute values of the quantities $\varepsilon_j(z)$ and $\varepsilon_k'(z)$ will be less than the positive number ε. Assuming this has been done, we proceed. From Equation (6) it follows that

$$f(z) = z\, f'(z_j) + f(z_j) - z_j\, f'(z_j) + (z - z_j)\, \varepsilon_j(z)$$

and

$$\int_{C_j} f(z)\, dz = f'(z_j) \int_{C_j} z\, dz + [f(z_j) - z_j\, f'(z_j)] \int_{C_j} dz$$
$$+ \int_{C_j} (z - z_j)\, \varepsilon_j(z)\, dz,$$

where the integrals are to be taken along the perimeter C_j of the jth square. By Lemma 3.1, this reduces to

$$\int_{C_j} f(z)\, dz = \int_{C_j} (z - z_j)\, \varepsilon_j(z)\, dz.$$

Hence,

$$\left|\int_{C_j} f(z)\, dz\right| \leq 4\sqrt{2}\, l_j^2 \varepsilon = 4\sqrt{2}\, w_j \varepsilon,$$

where w_j is the area of the jth regular part. By similar analysis, Equation (7) leads to

$$\left|\int_{C_k} f(z)\, dz\right| < \sqrt{2}\, l_k'\, \varepsilon (4\, l_k' + \text{arc } rs)$$
$$= 4\sqrt{2}\, w_k'\, \varepsilon + \sqrt{2}\, l_k'\, \varepsilon \text{arc } rs,$$

where w_k' denotes the area of the square which contains the kth irregular part. By Lemma 3.2 and the triangle inequality, one finds

$$\left|\int_C f(z)\,dz\right| < 4\sqrt{2}\;\varepsilon \sum_{j=1}^{N} w_j + \sqrt{2}\;\varepsilon \sum_{k=1}^{N'} (4w_k' + \sqrt{2}\; l_k'\; L)$$

$$< \varepsilon[4\sqrt{2}\,(\sum_{j=1}^{N} w_j + \sum_{k=1}^{N'} w_k') + \sqrt{2}\;\lambda\; L],$$

where λ is an upper bound for the sum of the sides l_k'. The sum

$$\sum_{j=1}^{N} w_j + \sum_{k=1}^{N'} w_k'$$

is less than or equal to the finite area of P so that because ε is arbitrary, $\int_C f(z)\,dz = 0$.

Published in the same volume of the Transactions of the American Mathematical Society as Goursat's Lemma was yet another proof of the Cauchy Integral Theorem. This proof, by Eliakim Hastings Moore [13], was essentially the same in spirit as Goursat's proof; however, by imposing a certain condition on the curve C, which Moore claimed that all the usual curves fulfilled, the necessity of introducing Goursat's Lemma was avoided. As far as the development in the technique of the proof of the integral theorem goes, little was actually added by Moore. However, he was careful to include in his paper the important notions of a simply connected region and a rectifiable curve.

This brings the discussion to the topic of the weaknesses in Goursat's proof. Goursat, as Cauchy and Riemann, used the idea of the function being analytic "on and inside a closed curve." Clearly, were this the case, the region bounded by the curve would be simply connected, but mathematical rigor requires a clearer description of this condition. Also these men made no provisions for the wildness of the curve. Goursat assumed the curve to be rectifiable when he let L be the length of C in his proof, but the hypothesis of his theorem does not warrant such an assumption. Moore should be credited for including the hypothesis of a rectifiable curve in his version of the theorem. Alfred Pringsheim [14] also used the assumption of a rectifiable curve, and his proof, which is a simpler version than Goursat's or Moore's, is the subject of the next chapter.

CHAPTER IV

THE PROOF BY PRINGSHEIM

Within a year after Moore had written and published his proof of the Cauchy Integral Theorem, Alfred Pringsheim wrote his version of the proof. A possible reason for Pringsheim's publication is that he was interested not so much in the theorem itself but rather in defense against a remark by Moore accusing Pringsheim of being "out of touch with the current notion of the general rectifiable curves" [13]. The resentment he must have felt from such a sharp criticism was reflected in Pringsheim's writing, and in his paper he pointed out that even Goursat's proof was not without its shortcomings.

Pringsheim's main criticism of Goursat's proof was that it lacked the rigor necessary for application to closed curves of a very general type. He went on to explain that Goursat's proof worked for curves which were piecewise monotone; that is, which could be decomposed into at most a finite number of subarcs, each of which is monotone increasing or monotone decreasing with respect to either the x-axis or the y-axis. For such a curve C, one can always choose $\lambda > 0$ sufficiently small so that no square of Goursat's decomposition with length of side less than or equal to λ can intersect C in more than two points (Figure 4). As previously stated, Pringsheim gave a proof which worked for all rectifiable curves, including ones which were not piecewise monotone. He also suggested that rather than decomposing the inside of a region bounded by the closed curve C, to instead prove the theorem first for triangles and then to use this result along with some limit considerations to approximate $\int_C f(z)\,dz$ by an integral of f(z) along an inscribed polygon of C. The following arguments are those of Pringsheim.

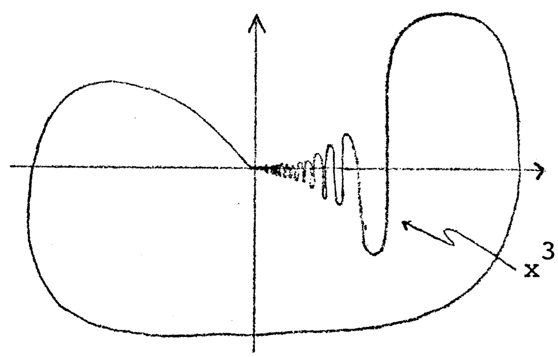


Figure 4. A Curve Which is not Piece-wise Monotone

Lemma 4.1: Suppose the function f(z) is analytic inside and on the triangle Δ. Then for each ε > 0, the triangle Δ can be decomposed into a finite number of similar triangles Δ_k such that there exists a point ζ_k either inside or on Δ_k such that

$$|f(z_k) - f(\zeta_k) - f'(\zeta_k)(z_k - \zeta_k)| \le \varepsilon |z_k - \zeta_k| \tag{8}$$

for any point z_k on the boundary of Δ_k.

Proof: Decompose the triangle Δ into four congruent triangles by joining the midpoints of the sides with straight line segments. If any of the triangles obtained by this division process satisfies Equation (8), it shall remain unchanged. If a triangle does not satisfy Equation

(8), it is decomposed further by joining the midpoints of its sides with straight line segments. By construction, the triangles formed by this decomposition are similar triangles (Figure 5). Either this process will end with a finite number of similar triangles all of which satisfy Equation (8), or it will not terminate.

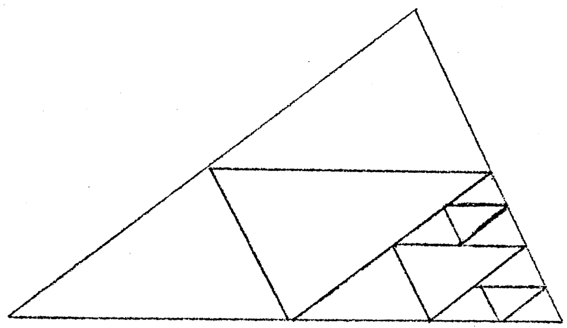


Figure 5. Decomposition of the Triangular Path Δ

Suppose the latter is the case. After the first subdivision, there must be at least one triangle, call it Δ_1, which does not satisfy Equation (8). After it is decomposed, there will be formed another triangle, Δ_2, which does not satisfy Equation (8). Since it has been assumed that the division process does not terminate, a sequence of triangles $\{\Delta_n\}$ can be constructed such that $\Delta \supset \Delta_1 \supset \Delta_2 \supset \ldots \supset \Delta_n \supset \ldots$. Also, the length of the sides of Δ_n are

$$\frac{a_1}{2^n}, \ \frac{a_2}{2^n}, \ \text{and} \ \frac{a_3}{2^n},$$

where a_1, a_2, a_3 are the lengths of the sides of Δ. There must exist a point ζ which lies inside or on all of the triangles in $\{\Delta_n\}$. Let $\varepsilon > 0$ be given. Since $f'(\zeta)$ exists by hypothesis, there exists $\delta > 0$ such that

$$|f(z) - f(\zeta) - (z - \zeta)\, f'(\zeta)| \le \varepsilon |z - \zeta|$$

whenever $|z - \zeta| < \delta$. Choose n sufficiently large so that $\frac{a_m}{2^n} < \delta$, $m = 1, 2, 3$. Then for z_k on the boundary of Δ_k, $k \ge n$,

$$|z_k - \zeta| \le \max_{m=1,2,3} \{\frac{a_m}{2^n}\}.$$

Hence, Δ_k satisfies Equation (8), and a contradiction has been obtained by supposing the lemma is not true.

Lemma 4.2: If the triangle Δ with perimeter $s = a_1 + a_2 + a_3$ has been decomposed as described in Lemma 4.1 into p similar triangles Δ_k with perimeters s_k, respectively, then

$$\sum_{k=1}^{p} s_k^2 = s^2.$$

Proof: First note that for any triangle subdivided into four smaller congruent triangles in the manner described in Lemma 4.1, the sum of the squares of the perimeters of these four triangles equals the square of the perimeter of the larger triangle which contains them. For suppose the triangle Δ' has been subdivided as previously described into Δ_1', Δ_2', Δ_3', Δ_4'. If the lengths of the sides of Δ' are a_1', a_2', a_3' and its perimeter is s', then the lengths of the sides of the four smaller triangles are $\frac{a_1'}{2}$, $\frac{a_2'}{2}$, $\frac{a_3'}{2}$; hence, the perimeter s_j' of Δ_j', $j = 1, 2, 3, 4$, is

$$s_j' = \frac{a_1'}{2} + \frac{a_2'}{2} + \frac{a_3'}{2} = \frac{s'}{2}.$$

Thus,

$$\sum_{j=1}^{4} (s_j')^2 = 4 \cdot (\frac{s'}{2})^2 = (s')^2.$$

By Lemma 4.1, the subdivision process must terminate, so there must be a smallest triangle in the collection of similar triangles. In fact, by construction, there must be at least four of these minimal triangles. Suppose these triangles were obtained after n subdivisions. Then the perimeter of each triangle equals $\frac{s}{2^n}$, and each set of four triangles is contained in a triangle obtained after n-1 subdivisions. Also, in each set the sum of the squares of the triangles' perimeters equals the square of the perimeter of that triangle which contains them by the previous argument; that is,

$$2^2 \cdot (\frac{s}{2^n})^2 = (\frac{s}{2^{n-1}})^2 . \qquad (9)$$

That triangle is one of four congruent triangles which are contained in a triangle obtained after n-2 subdivisions. Thus,

$$2^2 \cdot (\frac{s}{2^{n-1}})^2 = (\frac{s}{2^{n-2}})^2 .$$

In view of Equation (9), it follows that

$$(\frac{s}{2^{n-2}})^2 = (2^2) \cdot (\frac{s}{2^n})^2 .$$

That is, the sum of the squares of the perimeters of all triangles contained in a triangle from the (n-2)nd subdivision is equal to the square of that triangle's perimeter. In general, the sum of the squares of the perimeters of all triangles contained in a triangle from the (n-m)th subdivison, m = 1, 2, . . ., n, is equal to the square of the perimeter of that triangle which contains them; that is,

$$2^{2m} \left(\frac{s}{2^n}\right)^2 = \left(\frac{s}{2^{n-m}}\right)^2 .$$

Since the triangle Δ is the one obtained after n - m = 0 subdivisons, it follows that

$$\sum_{k=1}^{p} s_k^2 = 2^{2n} \left(\frac{s}{2^n}\right)^2 = s^2 .$$

Theorem 4.1: If f(z) is analytic inside and on a triangle Δ, then $\int_\Delta f(z)\, dz = 0$.

Proof: Choose $\varepsilon > 0$ arbitrarily small. By Lemma 4.1, Δ can be decomposed into p similar triangles Δ_k which satisfy Equation (8). Then

$$\int_\Delta f(z)\, dz = \sum_{k=1}^{p} \int_{\Delta_k} f(z)\, dz = \sum_{k=1}^{p} \int f(z_k)\, dz_k ,$$

where the integrals along Δ_k, k = 1, 2, . . ., p, are taken in the same sense as that along Δ. Note that

$$\int f(z_k)\, dz_k = \int [f(z_k) - f(\zeta_k) - (z_k - \zeta_k)\, f'(\zeta_k)]\, dz_k$$
$$+ [f(\zeta_k) - \zeta_k\, f'(\zeta_k)] \int dz_k + f'(\zeta_k) \int z_k\, dz_k .$$

The latter two integrals in the right member are zero by Lemma 3.1. Hence,

$$\left|\int f(z_k)\, dz_k\right| \leq \int |f(z_k) - f(\zeta_k) - (z_k - \zeta_k)\, f'(\zeta_k)|\,|dz_k|$$
$$\leq \varepsilon \int |z_k - \zeta_k|\,|dz_k| .$$

Denote the perimeter of Δ_k by s_k. Then for ζ_k inside or on Δ_k and z_k on Δ_k, $|z_k - \zeta_k| < \frac{1}{2} s_k$. Thus,

$$\left|\int f(z_k)\, dz_k\right| < \varepsilon \frac{1}{2} s_k^2.$$

By Lemma 4.2,

$$\left|\int_\Delta f(z)\, dz\right| < \frac{\varepsilon}{2} \sum_{k=1}^{p} s_k^2 = \frac{\varepsilon}{2} s^2.$$

Since ε is arbitrary, $\int_\Delta f(z)\, dz = 0$.

Pringsheim concluded his paper with the remark that Theorem 4.1 could be used to obtain Cauchy's Theorem for simple closed rectifiable curves; however, he chose not to include the argument for generalization in his paper. For completeness, the argument is essential; therefore, the following details have been furnished, although they were actually omitted by Pringsheim.

The first step in generalizing Theorem 4.1 is to show that it applies to arbitrary polygons. Let Π be any simple closed polygon whose interior and boundary lie in a domain D. Then Π may be triangulated; that is, there exist triangles $\Delta_1, \Delta_2, \ldots, \Delta_n$ such that the interiors of the triangles Δ_j and Δ_k have no point in common for $j \neq k$ and $\bigcup_{j=1}^{n} \Delta_j^* = \Pi^*$, where the asterisks denote the union of the interior and the boundary of the simple closed contour under consideration [10]. Thus,

$$\int_\Pi f(z)\, dz = \sum_{j=1}^{n} \int_{\Delta_j} f(z)\, dz = 0$$

so that the theorem holds for arbitrary polygons (Figure 6).

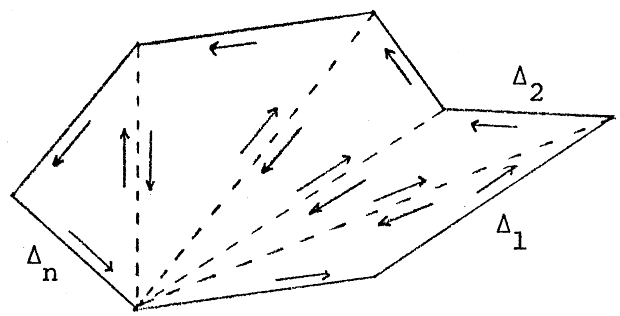


Figure 6. Triangulation of the Polygon Π

The second step is to show that the integral of f(z) along a closed curve C can be approximated by the integral along an inscribed polygon Π. Suppose that C is an arbitrary closed rectifiable curve such that C* is contained in the domain D. Since the complement of D is closed and C* is compact, the distance d from C* to the complement of D is positive. Choose ρ such that $0 < \rho < d$ and let

$$R = \{z \in \mathbb{C} \mid z \in C^* \text{ or dist } (C^*,z) \leq \rho\}.$$

Thus, $C^* \subset R \subset D$ (Figure 7).

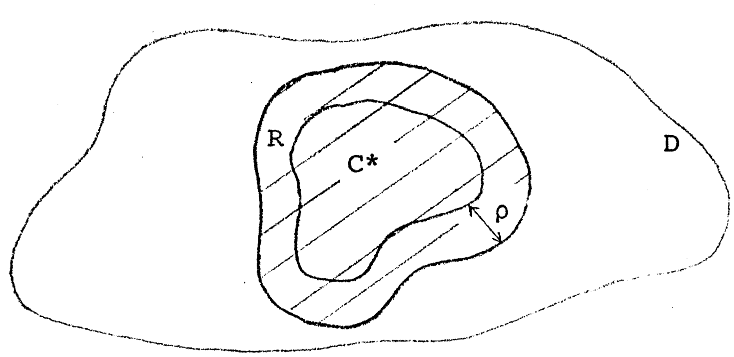


Figure 7. Relative Positions of the Sets C* and R Within Domain D

Let $\varepsilon > 0$ be given. Since f is uniformly continuous on R, there exists $\delta > 0$ such that $|f(z") - f(z')| < \varepsilon/2L$, where L is the length of C, whenever $|z" - z'| < \delta$. Let $z_1, z_2, \ldots, z_n$ be a partition of C. Choose the points z_j so close together, and hence n so large, that

(i) z_j precedes z_k on C if $j < k$;

(ii) $\left|\int_C f(z)\,dz - \sum_{k=1}^{n} f(z_k)(z_{k+1} - z_k)\right| < \frac{\varepsilon}{2}$;

(iii) $|z_k - z_{k+1}| < \frac{1}{2}\rho$;

(iv) $|z_k - z_{k+1}| < \delta$.

Clearly, a set of points can be constructed satisfying these conditions. Let $\Pi = [z_1, z_2, \ldots, z_n, z_1]$ and note that if Π is a simple closed polygon, then by the previous step,

$$\int_\Pi f(z)\,dz = 0. \tag{10}$$

But if Π should intersect itself, then it would be the union of a finite number of simple closed polygons plus, possibly, some double segments which are traversed twice in opposite directions when z describes Π. In such a case, Equation (10) would still be valid. Consider the expression

$$\int_\Pi f(z)\,dz - \sum_{k=1}^{n} f(z_k)(z_{k+1} - z_k).$$

Writing the finite sum as an integral along Π and taking absolute values, one finds

$$\left|\int_\Pi f(z)\,dz - \sum_{k=1}^{n} f(z_k)(z_{k+1} - z_k)\right|$$

$$= \left|\sum_{k=1}^{n} \int_{z_k}^{z_{k+1}} [f(z) - f(z_k)]\,dz\right| < \frac{\varepsilon}{2L} \cdot L = \frac{\varepsilon}{2}.$$

Note that the fact that the length of Π does not exceed L has been used here. Combining this result with the inequality in (ii), it follows that

$$\left|\int_C f(z)\,dz - \int_\Pi f(z)\,dz\right| \le \left|\int_C f(z)\,dz - \sum_{k=1}^{n} f(z_k)(z_{k+1} - z_k)\right|$$

$$+ \left|\int_\Pi f(z)\,dz - \sum_{k=1}^{n} f(z_k)(z_{k+1} - z_k)\right|$$

$$< \varepsilon.$$

In view of Equation (10), it is clear that $\left|\int_C f(z)\,dz\right| < \varepsilon$. Since ε was arbitrary, $\int_C f(z)\,dz = 0$.

A weakness in Pringsheim's proof is his omission of simple connectivity from his argument. Although Pringsheim, as Goursat before him, avoided the notion of simple connectivity by defining the function to be "analytic inside and on" the closed curve, the merits of his argument exceed the deficiencies. Pringsheim's proof is a straightforward version of the Cauchy Theorem for simple closed rectifiable curves, and he is credited with the first simple proof of the theorem for such curves. By the middle of the twentieth century, other developments in the proof of the Cauchy Integral Theorem had been made, but these proofs were topological in nature. The Pringsheim proof, because of its simplicity, is much more easily understood. Consequently, his proof, or a version thereof, can be found in many introductory texts on complex function theory.

CHAPTER V

DIXON'S PROOF

The 1971 proof of Cauchy's Integral Theorem by John D. Dixon will represent the final stage in the development of the proof of this theorem [6]. With it, the goal of a simple proof of a generalized version of the Cauchy Integral Theorem will be achieved. As a generalized theorem, Dixon's proof appears brief, but underlying his argument is a great deal of analysis, the development of which begins with the following definitions.

<u>Definition 5.1</u>: Let Ω be any open set of $\mathbb{C}$, and let $\gamma_1, \gamma_2, \ldots, \gamma_n$ be rectifable paths in Ω. A formal sum $\gamma = a_1\gamma_1 + a_2\gamma_2 + \ldots a_n\gamma_n$, where the a_j are integers, is called a chain. We define

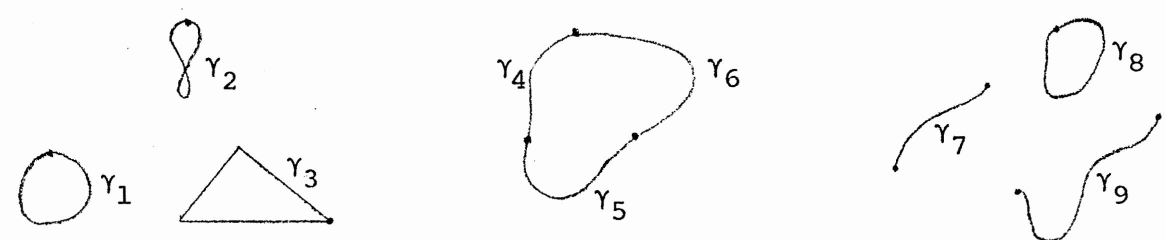
$$\int_\gamma f(z)\, dz = \sum_{j=1}^{n} a_j \int_{\gamma_j} f(z)\, dz.$$

<u>Definition 5.2</u>: Two chains $\gamma = a_1\gamma_1 + a_2\gamma_2 + \ldots + a_n\gamma_n$ and $\alpha = b_1\alpha_1 + b_2\alpha_2 + \ldots + b_m\alpha_m$ are said to be equal if

$$\int_\gamma f(z)\, dz = \int_\alpha f(z)\, dz$$

for every f continuous on $\gamma_1 \cup \gamma_2 \cup \ldots \cup \gamma_n \cup \alpha_1 \cup \alpha_2 \cup \ldots \cup \alpha_m$.

<u>Definition 5.3</u>: The chain Γ is called a cycle if each γ_j, $j = 1, 2, \ldots, n$, is a closed path (Figure 8).



$\gamma_1 + \gamma_2 + \gamma_3$ is a cycle

$\gamma_4 + \gamma_5 + \gamma_6$ is a chain but not a cycle

$\gamma_7 + \gamma_8 + \gamma_9$ is a chain

Figure 8. Chains and Cycles

Definition 5.4: The index of a cycle Γ with respect to a point a not on Γ is given by

$$\mathrm{Ind}_{\Gamma}(a) = \frac{1}{2\pi i}\int_{\Gamma}\frac{dw}{w-a}$$

Definition 5.5: A cycle Γ in Ω is said to be homologous to zero with respect to Ω if $\mathrm{Ind}_{\Gamma}(a) = 0$ for every $a \notin \Omega$. If Γ_1 and Γ_2 are cycles in Ω, then Γ_1 is homologous to Γ_2 with respect to Ω if and only if $\Gamma_1 - \Gamma_2$ is homologous to zero with respect to Ω. This is an equivalence relation.

It should be noted here that there is a direct correlation between homology and homotopy. Specifically, if Γ is null-homotopic in Ω, then Γ is homologous to zero in Ω. This fact will be demonstrated by the following discussion.

Lemma 5.1: If γ_0 and γ_1 are closed paths, if α is a complex number, and if $|z_1(s) - z_0(s)| < |\alpha - z_0(s)|$, $0 \leq s \leq 1$, then $\mathrm{Ind}_{\gamma_0}(a) = \mathrm{Ind}_{\gamma_1}(a)$.

Proof: Because of the strict inequality in the hypothesis, $\alpha \notin \gamma_0, \gamma_1$. Let

$$z(s) = \frac{z_1(s) - \alpha}{z_0(s) - \alpha}.$$

Then,

$$\frac{z'(s)}{z(s)} = \frac{z_1'(s)}{z_1(s) - \alpha} - \frac{z_0'(s)}{z_0(s) - \alpha}.$$

Also,

$$|1 - z(s)| = \frac{|z_0(s) - z_1(s)|}{|z_0(s) - \alpha|} < 1$$

for every s, $0 \leq s \leq 1$. Thus γ: $z = z(s)$, $0 \leq s \leq 1$, is a path lying in $|z - 1| < 1$. Hence $\text{Ind}_\gamma(0) = 0$. By definition,

$$\begin{aligned} \text{Ind}_\gamma(0) &= \frac{1}{2\pi i}\int_\gamma \frac{dz}{z} \\ &= \frac{1}{2\pi i}\int_0^1 \frac{z'(s)}{z(s)}\,ds \\ &= \frac{1}{2\pi i}\int_0^1 \frac{z_1'(s)}{z_1(s) - \alpha}\,ds - \frac{1}{2\pi i}\int_0^1 \frac{z_0'(s)}{z_0(s) - \alpha}\,ds \\ &= \text{Ind}_{\gamma_1}(\alpha) - \text{Ind}_{\gamma_0}(\alpha). \end{aligned}$$

Therefore, $\text{Ind}_{\gamma_0}(\alpha) = \text{Ind}_{\gamma_1}(\alpha)$.

<u>Theorem 5.1</u>: If Γ_0 and Γ_1 are Ω-homotopic closed paths and if $\alpha \notin \Omega$, then $\text{Ind}_{\Gamma_0}(\alpha) = \text{Ind}_{\Gamma_1}(\alpha)$.

<u>Proof</u>: Since $\Gamma_0 \sim \Gamma_1$ in Ω, there exists a continuous function H: $I \times I \to \Omega$, where $I = [0,1]$, such that $H(s,0) = \Gamma_0(s)$, $H(s,1) = \Gamma_1(s)$, $H(0,t) = H(1,t)$ for $s, t \in I$. Since $I \times I$ is compact, so is $H(I \times I)$. Therefore, there exists $\varepsilon > 0$ such that for $(s,t) \in I \times I$,

$$|\alpha - H(s,t)| > 2\varepsilon. \tag{11}$$

Since H is uniformly continuous on $I \times I$, there exists a positive integer n such that whenever $|s - s'| + |t' - t| < \frac{1}{n}$,

$$|H(s,t) - H(s',t')| < \varepsilon. \tag{12}$$

Define $\gamma_k(s)$ by the following equation.

$$\gamma_k(s) = H(\tfrac{j}{n}, \tfrac{k}{n})(ns + 1 - j) + H(\tfrac{j-1}{n}, \tfrac{k}{n})(j - ns),$$

$\frac{j-1}{n} \leq s \leq \frac{j}{n}$, $j = 1, 2, \ldots, n$. Then,

$$\begin{aligned}
|\gamma_k(s) - H(s, \tfrac{k}{n})| &= |H(\tfrac{j}{n}, \tfrac{k}{n})(ns + 1 - j) + H(\tfrac{j-1}{n}, \tfrac{k}{n})(j - ns) \\
&\quad - H(s, \tfrac{k}{n})\ [(ns + 1 - j) + (j - ns)]| \\
&\leq |H(\tfrac{j}{n}, \tfrac{k}{n}) - H(s, \tfrac{k}{n})|\ (ns + 1 - j) \\
&\quad + |H(\tfrac{j-1}{n}, \tfrac{k}{n}) - H(s, \tfrac{k}{n})|\ (j - ns) \\
&< \varepsilon(ns + 1 - j) + \varepsilon(j - ns) \quad \text{by (12)} \\
&= \varepsilon.
\end{aligned}$$

Hence,

$$|\gamma_k(s) - H(s, \tfrac{k}{n})| < \varepsilon,\ k = 0, 1, \ldots, n;\ 0 \leq s \leq 1. \tag{13}$$

Now,

$$\begin{aligned}
|\alpha - \gamma_0(s)| &\geq |\alpha - \Gamma_0(s)| - |\Gamma_0(s) - \gamma_0(s)| \\
&> 2\varepsilon - \varepsilon \text{ by (11) and (13)} \\
&= \varepsilon,
\end{aligned}$$

for all s, $0 \leq s \leq 1$. Similarly, $|\alpha - \gamma_k(s)| > \varepsilon$ for all s, $0 \leq s \leq 1$; $k = 0, 1, 2, \ldots, n$. By Equation (12),

$$|\gamma_{k-1}(s) - \gamma_k(s)| \leq \left|H\left(\frac{j}{n}, \frac{k-1}{n}\right) - H\left(\frac{j}{n}, \frac{k}{n}\right)\right| (ns + 1 - j)$$

$$+ \left|H\left(\frac{j-1}{n}, \frac{k-1}{n}\right) - H\left(\frac{j-1}{n}, \frac{k}{n}\right)\right| (j - ns)$$

$$< \varepsilon(ns + 1 - j) + \varepsilon(j - ns)$$

$$= \varepsilon.$$

Thus,

$$|\gamma_{k-1}(s) - \gamma_k(s)| < \varepsilon < |\alpha - \gamma_k(s)|, \; k = 0, 1, 2, \ldots, n.$$

By Lemma 5.1,

$$\text{Ind}_{\gamma_k}(\alpha) = \text{Ind}_{\gamma_{k-1}}(\alpha).$$

Also note that by Equation (13),

$$|\gamma_0(s) - \Gamma_0(s)| < \varepsilon$$

so that by Lemma 5.1,

$$\text{Ind}_{\gamma_0}(\alpha) = \text{Ind}_{\Gamma_0}(\alpha).$$

Similarly,

$$\text{Ind}_{\gamma_n}(\alpha) = \text{Ind}_{\Gamma_1}(\alpha).$$

Therefore,

$$\text{Ind}_{\Gamma_0}(\alpha) = \text{Ind}_{\Gamma_1}(\alpha).$$

Theorem 5.2: If Γ is null-homotopic in Ω, then Γ is homologous to zero in Ω.

Proof: Let $\Gamma = \Gamma_0 - \Gamma_1$, where Γ_0 and Γ_1 are Ω-homotopic closed paths. Let $a \notin \Omega$. Then, $\text{Ind}_\Gamma(a) = \text{Ind}_{\Gamma_0 - \Gamma_1}(a) = \text{Ind}_{\Gamma_0}(a) - \text{Ind}_{\Gamma_1}(a) = 0$

by Theorem 5.1. Hence Γ is homologous to 0.

The converse of this theorem is not true, as the following example illustrates.

Example: Let $\Omega = \mathbb{C} - \{P, Q\}$ and let A, B, C, D be the vertices of a rectangle enclosing P and Q as in Figure 9.

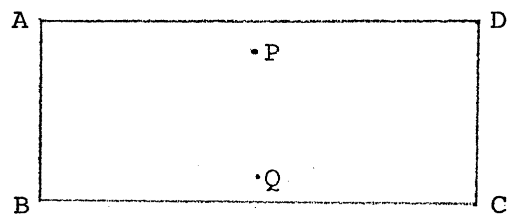


Figure 9. Rectangle ABCD

Let γ_0 be the path formed by connecting the vertices in the following order: A, D, B, C, D, A, C, B, A (Figure 10).

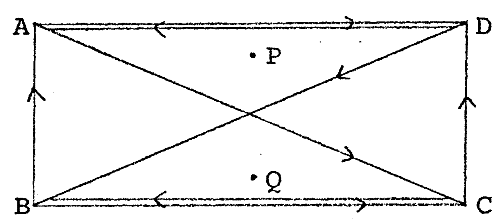


Figure 10. The Path γ_0

Notice that γ_0 is homologous to γ_1, which is formed by connecting the vertices in the following order: A, C, D, B, A (Figure 11).

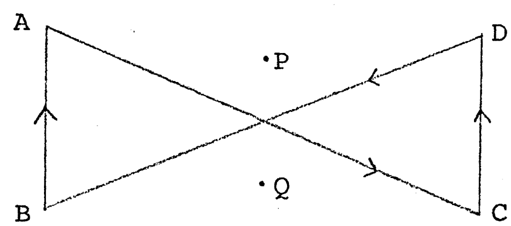


Figure 11. The Path γ_1

Since γ_1 is homologous to zero, γ_0 must be homologous to zero. But γ_0 is not null-homotopic since γ_0 cannot be contracted to a single point without passing through P or Q.

Actually, Emil Artin originally used the characterization of homology by vanishing winding numbers to prove the Cauchy Integral Theorem, and the reader is referred to Ahlfors for a proof closely patterned after Artin's [3] [1]. Dixon's proof differs from Artin's in that the latter used a topological argument whereas Dixon substituted the elementary properties of analytic functions that can be derived from the local version of Cauchy's Theorem for the topological considerations essential in Artin's proof. These properties which Dixon used as well as other essential results are listed in the immediately following theorems. Theorem 4.1 with a disk as the domain can be used as the local version of the Cauchy Theorem needed for their proof.

Theorem 5.3 (Fubini's Theorem): Let f(z) be continuous in a domain D and let Γ_1 and Γ_2 be cycles in D. Then

$$\int_{\Gamma_2} [\int_{\Gamma_1} f(z,w)\, dz]\, dw = \int_{\Gamma_1} [\int_{\Gamma_2} f(z,w)\, dw]\, dz.$$

Proof: See Sagan [16].

Theorem 5.4 (Morera's Theorem): Let f(z) be continuous in the disk D: $|z - a| < r$. If $\int_{\Delta} f(\zeta)\, d\zeta = 0$ for every triangular path Δ lying in D, then f(z) is analytic in D.

Proof: See Hille [10].

Theorem 5.5 (Liouville's Theorem): A bounded entire function must be a constant.

Proof: See Hille [10].

Theorem 5.6: Let Γ be a chain in $\mathbb{C}$. Let f(z) be continuous on Γ. Then

$$\phi(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(w)}{w - z}\, dw$$

is analytic on $\mathbb{C} - \Gamma$, and

$$\phi'(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(w)}{(w - z)^2}\, dw, \quad z \notin \Gamma.$$

Proof: See Hille [10].

For the actual proof of the Cauchy Integral Theorem by Dixon, a local version of the theorem is needed with the relaxed hypothesis that the function f(z) is analytic at every point in a domain D except at perhaps one point. This variation of the theorem is not difficult to establish as the following discussion indicates. The fact is used that if a function f(z) has a primitive, that is, an antiderivative, in D, then $\int_C f(z)\, dz = 0$ for every piecewise smooth closed curve C lying in D.

Theorem 5.7: Let f(z) be analytic in the disk D: $|z - a| < r$. Then $\int_C f(z)\, dz = 0$ for every piecewise smooth closed curve C lying in D.

Proof: It suffices to show that f(z) has a primitive in D. Let $F(z) = \int_a^z f(\zeta)\, d\zeta$, where the path of integration is the radial segment from a to z. Choose another point z + h lying in D. Then the points a, z, and z + h form the vertices of a triangle Δ, and

$$\int_\Delta f(\zeta)\, d\zeta = \int_z^{z+h} f(\zeta)\, d\zeta + \int_{z+h}^{a} f(\zeta)\, d\zeta + \int_a^z f(\zeta)\, d\zeta$$
$$= 0$$

by Theorem 4.1. Thus,

$$\int_z^{z+h} f(\zeta)\, d\zeta = -\int_{z+h}^{a} f(\zeta)\, d\zeta - \int_a^z f(\zeta)\, d\zeta$$
$$= F(z+h) - F(z),$$

from which one obtains

$$\frac{F(z+h) - F(z)}{h} - \frac{f(z)}{h} \int_z^{z+h} d\zeta = \frac{1}{h} \int_z^{z+h} [f(\zeta) - f(z)]\, d\zeta. \tag{14}$$

Taking absolute values of both sides of Equation (14) and applying the integral inequality,

$$\left|\frac{F(z+h) - F(z)}{h} - f(z)\right| < \max\left| f(\zeta) - f(z)\right|.$$

Since f(z) is continuous in D, we have for $\varepsilon > 0$ a $\delta > 0$ such that

$|f(\zeta) - f(z)| < \varepsilon$ whenever $|\zeta - z| < \delta$. Choose $|h|$ sufficiently small that $|\zeta - z| < \delta$. Then

$$\left|\frac{F(z+h) - F(z)}{h} - f(z)\right| < \varepsilon,$$

from which it may be concluded that $F'(z) = f(z)$.

With this result the necessary local version of the integral theorem may be proved as follows.

Theorem 5.8: Suppose $f(z)$ is analytic in $D - \{p\}$, $p \in D$, where D is the disk D: $|z - a| < r$. If $f(z)$ is continuous in D, then $\int_\Delta f(z)\,dz = 0$ for every closed triangular path Δ lying in D.

Proof: Let Δ be a closed triangular path in D. Three cases exist.

Case 1: $p \notin \Delta^*$. In this case, apply Theorem 4.1.

Case 2: p is a vertex of Δ. Denote the other vertices of Δ by a and b. Let x be any point on the segment from a to p; let y be a point on the segment from b to p. Let Δ_1 be the triangle with vertices p, x, y; let Δ_2 be the triangle with vertices a, y, x; let Δ_3 be the triangle with vertices a, b, y (Figure 12).

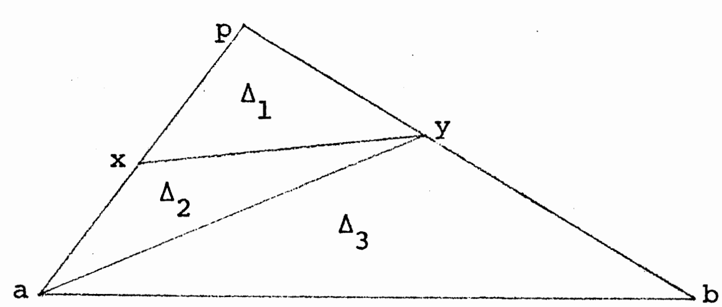


Figure 12. Decomposition of Triangle Δ With Point p a Vertex

Since

$$\int_{\Delta} f(z)\ dz = \int_{\Delta_1} f(z)\ dz + \int_{\Delta_2} f(z)\ dz + \int_{\Delta_3} f(z)\ dz,$$

and because by Theorem 4.1, the latter two integrals of the right member are zero, one has

$$\int_{\Delta} f(z)\ dz = \int_{\Delta_1} f(z)\ dz.$$

Applying the integral inequality,

$$\left|\int_{\Delta} f(z)\ dz\right| = \left|\int_{\Delta_1} f(z)\ dz\right| \leq \max_{z \in \Delta_1} |f(z)| \cdot \text{lgth } \Delta_1.$$

Since the points x and y were arbitrary, the perimeter of Δ_1 can be made arbitrarily small. Thus, $\int_{\Delta} f(z)\ dz = 0$.

<u>Case 3</u>: p is an interior point of Δ. In this case, connect the vertices of Δ to p forming three triangles with p as a vertex (Figure 13). Then apply Case 2 to each triangle.

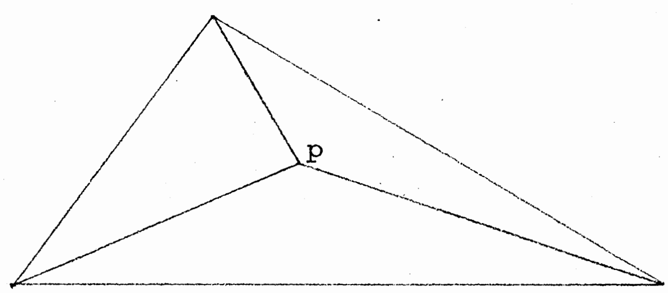


Figure 13. Decomposition of Triangle Δ With Point p an Interior Point

The following is John Dixon's proof of the Cauchy Integral Theorem.

Theorem 5.9: Suppose f(z) is analytic in an arbitrary open set Ω in $\mathbb{C}$. If Γ is a cycle which is homologous to zero with respect to Ω, then

$$\text{(i)} \quad f(z) \cdot \text{Ind}_{\Gamma}(z) = \frac{1}{2\pi i}\int_{\Gamma} \frac{f(w)}{w-z}\, dw, \text{ for all } z \in \Omega - \Gamma$$

and

$$\text{(ii)} \quad \int_{\Gamma} f(z)\, dz = 0.$$

Proof: (Part (i)) Define the function g on $\Omega \times \Omega$ by

$$g(z,w) = \begin{cases} \dfrac{f(w) - f(z)}{w-z}, & w \neq z \\ f'(z), & w = z \end{cases}$$

Since f is analytic in Ω, g(z,w) is continuous on $\Omega \times \Omega$. Let

$$h(z) = \frac{1}{2\pi i}\int_{\Gamma} g(z,w)\, dw, \; z \in \Omega.$$

Note that

$$\begin{aligned} h(z) &= \frac{1}{2\pi i}\int_{\Gamma} \frac{f(w) - f(z)}{w-z}\, dw \\ &= \frac{1}{2\pi i}\int_{\Gamma} \frac{f(w)}{w-z}\, dw - \frac{1}{2\pi i}\int_{\Gamma} \frac{f(z)}{w-z}\, dw \\ &= \frac{1}{2\pi i}\int_{\Gamma} \frac{f(w)}{w-z}\, dw - f(z) \cdot \text{Ind}_{\Gamma}(z). \end{aligned}$$

Thus, the first result will be established when it is shown that $h(z) = 0$ for all $z \in \Omega - \Gamma$.

First we show that h(z) is analytic on Ω. The continuity of g on Ω implies the uniform continuity of g on every compact subset of $\Omega \times \Omega$. Let $z \in \Omega$ and choose a sequence $\{z_n\} \subseteq \Omega$ which converges to z. Then the sequence $\{g(z_n,w)\}$ converges uniformly to $\{g(z,w)\}$ for all $w \in \Gamma$. Since

$$\lim_{n\to\infty} h(z_n) = \lim_{n\to\infty} \int_\Gamma g(z_n,w)\, dw = \int_\Gamma g(z,w)\, dw = h(z),$$

it is clear that $h(z)$ is continuous on Ω. Now let Δ be a triangular path in a disk $K \subseteq \Omega$. By Theorem 5.3,

$$\int_\Delta h(z)\, dz = \frac{1}{2\pi i} \int_\Delta [\int_\Gamma g(z,w)\, dw]\, dz = \frac{1}{2\pi i} \int_\Gamma [\int_\Delta g(z,w)\, dz]\, dw.$$

But by Theorem 5.8, $\int_\Delta g(z,w)\, dz = 0$ for fixed w in the disk K. Thus, by Theorem 5.4, $h(z)$ is analytic in K. Since this is true for any disk, $h(z)$ is analytic in Ω.

Next, $h(z)$ is extended to an entire function. Let Ω_1 be the set $\{z | \mathrm{Ind}_\Gamma(z) = 0\}$. Recall that by hypothesis, $\mathrm{Ind}_\Gamma(a) = 0$ for all $a \notin \Omega$. Hence, $\mathbb{C} - \Omega \subset \Omega_1$ and $\mathbb{C} = \Omega \cup \Omega_1$. Let $h_1(z)$ be defined by

$$h_1(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{w-z}\, dw, \quad z \in \Omega_1.$$

We claim that h_1 is an extension of h onto Ω_1. Suppose $z \in \Omega \cap \Omega_1$. Then

$$\begin{aligned} h(z) &= \frac{1}{2\pi i} \int_\Gamma g(z,w)\, dw \\ &= \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{w-z}\, dw - f(z) \cdot \frac{1}{2\pi i} \int_\Gamma \frac{dw}{w-z} \\ &= h_1(z) - f(z) \cdot \mathrm{Ind}_\Gamma(z). \end{aligned}$$

Since $z \in \Omega_1$, $\mathrm{Ind}_\Gamma(z) = 0$. Hence, $h(z) = h_1(z)$ for $z \in \Omega \cap \Omega_1$. Now extend h to an entire function by defining

$$\phi(z) = \begin{cases} h(z), & z \in \Omega \\ h_1(z), & z \in \Omega_1. \end{cases}$$

Since by Theorem 5.6, $h_1(z)$ is analytic on Ω_1, then $\phi(z)$ is an entire function.

Finally, we show $h(z) = 0$. By Theorem 2.1, the unbounded component of $\mathbb{C} - \Gamma$ is contained in Ω_1. Thus,

$$\lim_{z\to\infty} \phi(z) = \lim_{z\to\infty} h_1(z) = 0.$$

Hence, $\phi(z)$ is bounded. By Theorem 5.5, $\phi(z) \equiv 0$. Thus, $h(z) \equiv 0$.

(Part (ii)). Let $a \in \Omega - \Gamma$ and define $F(z) = (z - a)\, f(z)$. By Part (i) above,

$$\frac{1}{2\pi i} \int_\Gamma f(z)\, dz = \frac{1}{2\pi i} \int_\Gamma \frac{F(z)}{z-a}\, dz = F(a) \cdot \mathrm{Ind}_\Gamma(a).$$

Since $F(a) = (a - a)\, f(a) = 0$, $\int_\Gamma f(z)\, dz = 0$.

An immediate corollary to Theorem 5.9 is the following.

<u>Corollary</u>: If Γ_0 and Γ_1 are cycles in Ω such that $\mathrm{Ind}_{\Gamma_0}(a) = \mathrm{Ind}_{\Gamma_1}(a)$ for every $a \notin \Omega$, and $f(z)$ is analytic in Ω, then $\int_{\Gamma_0} f(z)\, dz = \int_{\Gamma_1} f(z)\, dz$.

<u>Proof</u>: Let $\mathrm{Ind}_{\Gamma_0}(a) = \mathrm{Ind}_{\Gamma_1}(a)$ for all $a \notin \Omega$, and let $\Gamma = \Gamma_0 - \Gamma_1$. Then, $\mathrm{Ind}_\Gamma(a) = \mathrm{Ind}_{\Gamma_0}(a) - \mathrm{Ind}_{\Gamma_1}(a) = 0$. Since $f(z)$ is analytic in Ω, by Theorem 5.9, $\int_\Gamma f(z)\, dz = 0$. This implies that $\int_{\Gamma_0} f(z)\, dz - \int_{\Gamma_1} f(z)\, dz = 0$, and the proof is complete.

The motivation for the inclusion of Dixon's proof in this historical development of Cauchy's Theorem is twofold. First, it better conforms to the analytical nature of the proofs presented in this thesis than would the more topological arguments developed in the middle of the twentieth century. Second, it achieves the goal of this thesis; that is, it provides a simple proof of the generalized form of the Cauchy Theorem.

Whereas Pringsheim generalized the theorem to include all rectifiable curves, Dixon generalized it to include all domains. His proof completes the final chapter in the history of the Cauchy Integral Theorem.

CHAPTER VI

SUMMARY AND CONCLUSIONS

Since 1825, when Cauchy first published his result, the Cauchy Integral Theorem has been improved many times by various mathematicians. Cauchy himself gave a second proof of his theorem in 1846 using the concept of exact differentials together with Green's Theorem. Following Cauchy, the next published version of the proof was that of Riemann. He broadened the scope of the theorem to include all continuous functions with continuous derivatives "on and inside" a simple closed curve. While many other significant developments were made during the remainder of the nineteenth century, the Cauchy Integral Theorem remained in this elementary stage of its evolvement.

Then in 1900, Goursat focused the attention of mathematicians back to the basics of the theory when he proved the Cauchy Theorem without the assumption of a continuous derivative. With this proof, a new method of approach was introduced, one of working directly with the integral rather than separating it into its two real components. Shortly after Goursat's proof was published, another mathematician, Moore, endeavored to rigorize the argument. Although he somewhat restricted the class of functions for which the theorem applied by imposing a special condition on them, he was careful to include the concepts of rectifiability for the curve and simple connectivity for the domain. Motivated by Moore's writing, Pringsheim published a proof of the theorem within the next year. His was a simple proof for simple closed rectifiable curves, the idea being to prove the theorem for triangles and then to apply this result together with some limit considerations to approximate the integral along a closed curve by one along an inscribed polygon.

For nearly half a century, the evolution of this proof once again stood at rest. Finally, some developments in the proof were made in the mid-twentieth century; however, these proofs were topological in nature. The last analytical development in the proof of the theorem came in 1971 with the proof by Dixon. By using the local version of the integral theorem, properties of analytic functions were developed, and using these properties, the Cauchy Integral Formula and ultimately, the Cauchy Integral Theorem were proved. The significance of Dixon's proof was that it broadened the scope of the theorem even further by including all domains in its hypothesis.

As stated in the introduction, the extensions and consequences of Cauchy's Theorem are quite numerous, too numerous to state all of them and of too great importance to single out one or two to mention. However, the omission of these results should not in any way affect the reader's appreciation of the theorem. In fact, it should promote the enjoyment of the theorem for its own sake, for like any work of art, the Cauchy Integral Theorem has a beauty and an elegance all its own. The careful development of the analysis together with the very nature of the result combine to make this theorem a monumental result, and this thesis serves as a tribute to all who helped to form it.

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