MEAN VALUE OF DIRICHLET SERIES

ASSOCIATED WITH CUSP FORMS

By

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CHAPTER 1

INTRODUCTION AND STATEMENT OF RESULTS

Dirichlet Series and the Riemann Zeta-function

We introduce the class of functions called Dirichlet series by outlining some standard facts about the Riemann zeta-function. The functions we are concerned with in this thesis, known as "Dirichlet series associated with cusp forms," share all of the key properties of the ζ-function, and historically the progress made on these functions has been attained by transferring the techniques which have been successfully applied to the ζ-function. We continue this process in this thesis. In this section we outline the relevant results for the Riemann ζ-function. In the next section we define our class of Dirichlet series and then describe the new results for these functions.

A *Dirichlet series* is a function which can be written in the form

$$L(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s},$$

where $s = \sigma + it$ is a complex variable and the series is absolutely convergent in some half-plane $\sigma > \sigma_0$. The classic example is the *Riemann zeta-function*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

the series being absolutely convergent for $\sigma > 1$.

Since we assume that our Dirichlet series converges absolutely in some half-plane, the series defines a holomorphic function there. The Dirichlet series studied in number theory have in common several properties which we illustrate by considering the Riemann zeta-function in some detail. Further details of the following material can be found in Titchmarsh's book [T]. First, $\zeta(s)$ has a meromorphic continuation to the whole s-plane with a single simple pole at $s = 1$. The function

$$\xi(s) = H(s)\zeta(s),$$

where

$$H(s) = \tfrac{1}{2}s(1-s)\pi^{-\frac{s}{2}}\Gamma(\tfrac{s}{2})$$

is entire and satisfies the *functional equation*

$$\xi(s) = \xi(1-s).$$

Finally, $\zeta(s)$ has the *Euler product* representation

$$\zeta(s) = \prod_p (1-p^{-s})^{-1},$$

where the product is over all primes and is absolutely convergent for $\sigma > 1$.

Of major significance in number theory is the location of the zeros of $\zeta(s)$, and the facts given above already give information about this. Since $\xi(s)$ is entire we see that $\zeta(s)$ is zero for $s = -2, -4, -6, \ldots$, these zeros being necessary to cancel the poles of the Γ-function. We refer to these as the *trivial zeros* of the ζ-function. From the Euler product representation we see that $\zeta(s)$ does not vanish for $\sigma > 1$, and so by the functional equation it also does not vanish for $\sigma < 0$, except for the trivial zeros. The remainder of the zeros, if any, are referred to as *nontrivial zeros*, and must lie in the region $0 \le \sigma \le 1$, which we refer to as the *critical strip*. In fact, the nontrivial zeros must lie in the open strip $0 < \sigma < 1$, as was shown by Hadamard and de la Valleé Poussin in the course of proving the prime number theorem. We denote a nontrivial zero of the ζ-function by $\rho = \beta + i\gamma$, and we define the zero counting function by

$$N(T) = \sum_{0<\gamma<T} 1,$$

where the sum is over the nontrivial zeros, and zeros are counted according to their multiplicity. Von Mangoldt proved that

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T).$$

The main conjecture about the location of the nontrivial zeros of the ζ-function is the Riemann Hypothesis, which is the conjecture that all of the nontrivial zeros of the zeta-function are of the form $\rho = \frac{1}{2} + i\gamma$. That is, all of the nontrivial zeros lie on the *critical line*, $\sigma = \frac{1}{2}$.

One step towards proving the Riemann Hypothesis is to establish lower bounds for the number of zeros of $\zeta(s)$ which are on the critical line. To this end, let

$$N_0(T) = \sum_{\substack{0<\gamma<T \\ \rho=\frac{1}{2}+i\gamma}} 1,$$

that is, $N_0(T)$ counts zeros in the critical line. The first result on $N_0(T)$ was given by Hardy in 1914. He showed that $N_0(T) \to \infty$ as $T \to \infty$. This was established by considering the integrals

$$\int_0^\infty \xi(\tfrac{1}{2} + it)\frac{t^n \cosh \alpha t}{t^2 + \frac{1}{4}}dt.$$

In 1921 Hardy and Littlewood improved this result to $N_0(T) > AT$ for some $A > 0$. The method used was different, but it involved considering integrals similar to the above. While this result shows that there are many zeros on the critical line, it falls short of showing that a positive proportion of the zeros are critical. Such a result was first obtained by Selberg in 1942, specifically, he proved that $N_0(T) > AT\log T$ for some (very small) $A > 0$. The key ingredient in his proof was that instead of considering an integral of $\zeta(\frac{1}{2}+it)$ he considered integrals of $\zeta(\frac{1}{2}+it)\phi(\frac{1}{2}+it)$, where $\phi(s)$ is an approximation to $1/\zeta(s)$. The function $\phi(s)$ is called a "mollifier" because, as an approximation to $1/\zeta(s)$, it helps dampen the wild behavior of $\zeta(s)$ near the critical line.

Mollifiers of the ζ-function also play a key role in Levinson's method of detecting simple zeros of $\zeta(s)$ on the critical line. This will be the subject of the remainder of this section. First we describe the mollifier $B_\zeta(s)$. We wish to approximate

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s},$$

where $\mu(n)$ is the Möbius function. Let

$$g_\alpha = \mu(\alpha)\, h\left(\frac{\log B/\alpha}{\log B}\right)$$

and

$$B_\zeta(s) = \sum_{\beta \le B} g_\beta \beta^{-s},$$

where $h(x) = \sum a_m x^m$ is a real polynomial with $h(0) = 0$ and $h(1) = 1$. Then $B_\zeta(s)$ is the necessary mollifier, and the most difficult step in applying Levinson's method is the evaluation of the following integral:

$$\int_1^T \zeta(\tfrac{1}{2}+u+it)\,\zeta(\tfrac{1}{2}+v-it)\,|B_\zeta(\tfrac{1}{2}+it)|^2 dt$$

$$\sim T\left(1 + \frac{1}{\theta}\int_0^1 T^{-y(u+v)}dy\,\frac{d}{d\alpha}\frac{d}{d\beta}B^{-\alpha u-\beta v}\int_0^1 h(x+\alpha)\,h(x+\beta)dx\right)\Bigg|_{\substack{\alpha=0\\ \beta=0}} \tag{0.1}$$

This formula is a special case of Theorem 2 in [C3], written in slightly different notation, and is valid for $0 < \theta < \frac{4}{7}$ and $|u| + |v| \ll 1/\log T$. We now describe the results obtainable from Levinson's method, and we outline the ideas behind the method.

Let $\tau_1^{(j)}$ denote the proportion of zeros of $\xi^{(j)}(s)$ which are simple and on the critical line. That is, if we let

$$N_1^{(j)}(T) = \#\{s = \tfrac{1}{2}+it \mid \xi^{(j)}(s) = 0,\ \xi^{(j+1)}(s) \neq 0,\ 0 < t < T\}$$

then

$$\tau_1^{(j)} = \liminf_{T\to\infty} \frac{N_1^{(j)}(T)}{N(T)}.$$

We now indicate how Levinson's method obtains lower bounds for $\tau_1^{(j)}$ from formula (0.1). One starts with

$$\eta(s) = i^n \left(\xi^{(n)}(s) + \frac{\lambda}{\log T} \xi^{(n+1)}(s) \right)$$

where λ is real. Since $\xi^{(n)}(\frac{1}{2}+it)$ is real when n is even and pure imaginary when n is odd, $i^n\xi^{(n)}(\frac{1}{2}+it) = Re\,\eta(\frac{1}{2}+it)$. Therefore one can count zeros of $\xi(\frac{1}{2}+it)$ by looking at the change in the argument of $\eta(\frac{1}{2}+it)$. Furthermore, if $Re\,\eta(\frac{1}{2}+it) = 0$ but $\eta(\frac{1}{2}+it) \neq 0$ then $\frac{1}{2}+it$ is a simple zero of $\xi^{(n)}(s)$. A zero of $\eta(s)$ on the $\frac{1}{2}$-line only contributes half as much change in argument as a zero to the right of $\sigma = \frac{1}{2}$, while Littlewood's lemma detects zeros with all zeros counted with equal weight. Thus we can arrange to find zeros which are both simple and on the critical line. Essentially, one replaces $\eta(s)$ by $H(s)V(s)$ where $Q(x) = (1+\lambda x)(1+2x)^n$ with λ real, and

$$V(s) = Q\left(\frac{1}{\log T}\frac{d}{ds}\right)\zeta(s).$$

This is sensible because

$$\frac{H^{(m)}}{H}(s) = \left(\frac{\log s/2\pi}{2}\right)^m \left(1 + O(t^{-1})\right). \tag{0.2}$$

Then Littlewood's lemma and the arithmetic-geometric mean inequality gives

$$N_1^{(n)}(T) \geq N(T) - \frac{T}{2(\frac{1}{2}-a)\pi}\log\left(\frac{1}{T}\int_0^T |VB(a+it)|^2\,dt\right) + O(T).$$

Putting $a = \frac{1}{2} - \frac{R}{\log T}$ and dividing by $\frac{1}{2\pi}T\log T$ we get

$$\tau_1^{(n)} \geq 1 - \frac{1}{R}\log\left(\frac{1}{T}\int_0^T |VB(\tfrac{1}{2} - \frac{R}{\log T} + it)|^2\,dt\right) + o(1).$$

We may differentiate the asymptotic formula in Theorem 1, so it is possible to evaluate the right hand side of the inequality above. Since $Q(\frac{d}{dx})e^{xy} = Q(y)e^{xy}$ we find that

(0.3)

$$\tau_1^{(n)} \geq 1 - \frac{1}{R}\log\left(1 + \frac{1}{\theta}\int_0^1\int_0^1 \frac{d}{d\alpha}\frac{d}{d\beta} Q_\zeta(y+\alpha\theta)Q_\zeta(y+\beta\theta)e^{R(2y+\alpha\theta+\beta\theta)}\, h(x+\alpha)h(x+\beta)\, dx dy\right)\Bigg|_{\substack{\alpha=0\\ \beta=0}}$$

To obtain lower bounds for $\tau_1^{(n)}$ one is given θ and then chooses the free parameters $h(x)$, R, and λ so as to make the right side of (0.3) as large as possible. Recall that T^θ is the length of our mollifying polynomial. Intuitively, one would expect that in order to get good results one would want to use the longest possible mollifier. We will investigate this dependence on θ. Expanding the term inside the logarithm in (0.3), ie., differentiating with respect to α and β, results in an expression of the form

$$1 + \theta X \int_0^1 h(x)^2\, dx + Y\big(h^2(1) - h^2(0)\big) + \frac{Z}{\theta}\int_0^1 h'(x)^2 dx. \tag{0.4}$$

Here X, Y, and Z each depend only on R, λ, and n. We wish to make this expression as small as possible. It is clear that when θ is very small then an increase in θ will result in a decrease in (0.4). Perhaps it is not so clear that this in fact holds even for large values of θ. The polynomial $h(x)$ is only restricted by the requirement $h(0) = 0$ and $h(1) = 1$. Clearly we can relax the polynomial condition on $h(x)$ and merely require it to be a smooth function. Thus we may choose $h(x)$ optimally by the calculus of variations, and this choice will depend continuously on R, λ, n, and θ. As θ becomes large the best choice of $h(x)$ will obviously have $\int_0^1 h^2(x)dx$ small. Of course the boundary conditions on $h(x)$ will then force $\int_0^1 h'(x)^2 dx$ to be large. It may not be immediately obvious that these competing factors result in an overall decrease in (0.4), but in Section 8, formula (8.1), the optimal expression is given and the reader will easily see that it is monotonically decreasing in θ (actually, (8.1) is a related expression, but the effect is the same). We also have a free choice in the parameters R and λ. The optimal choices for these is easily made once θ and n are fixed and $h(x)$ has been chosen.

To better illustrate the key role which θ plays we present a graph of θ vs. the right hand side of (0.3) when $n = 0$. This graph shows the lower bound we obtain for $\tau_1^{(0)}$ as a function of θ. For each value of θ the function $h(x)$ and the parameters R and λ were chosen optimally by a computer calculation. The decimal values given are truncations of the actual values.

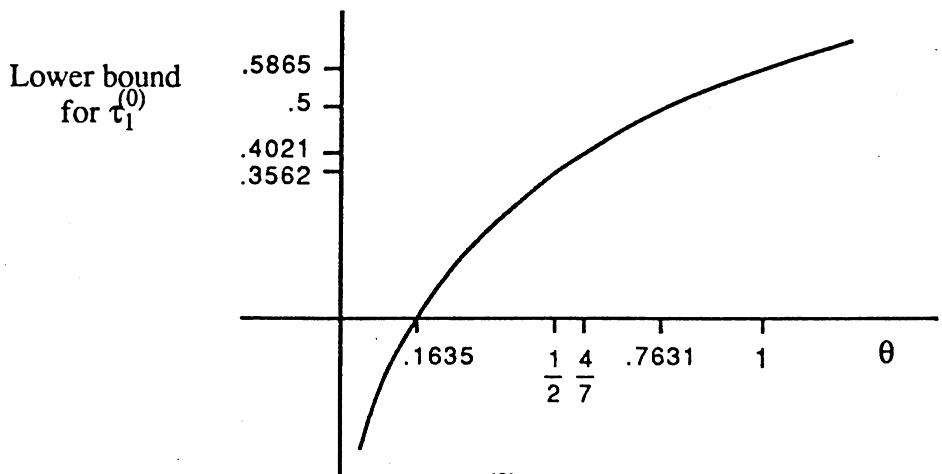

Figure 1. Lower Bound for $\tau_1^{(0)}$ as a Function of Mollifier Length

Originally Levinson [Lev] used $\theta = \frac{1}{2}$ with $h(x) = x$ and $\lambda = 1$ and chose R optimally to obtain $\tau_1^{(0)} > 0.3474$. As can be seen from the graph, this is not far from best possible. Small improvements on this were made in [C2] and [A]. These involved using $\theta = \frac{1}{2}$ and then making better choices for the other parameters. A significant improvement was made by Conrey [C3]. He showed that one may take $\theta = \frac{4}{7}$ in (0.1). This gives the best known lower bound for $\tau_1^{(0)}$. At this stage any improvement in our knowledge of $\tau_1^{(0)}$ using Levinson's method would require showing that one can take a yet larger value of θ. The bounds obtainable for $\tau_1^{(n)}$ using $\theta = \frac{4}{7}$ are as follows: $\tau_1^{(0)} > 0.40219$, $\tau_1^{(1)} > 0.7987$, $\tau_1^{(2)} > 0.9346$,

$\tau_1^{(3)} > 0.9673$, $\tau_1^{(4)} > 0.9800$, and $\tau_1^{(5)} > 0.9863$.

We mention one other mean value result for the ζ-function. Let

$$A(s) = \sum_{m \leq M} \frac{a(m)}{m^s}$$

with $a(m) \ll_\epsilon m^\epsilon$ for all $\epsilon > 0$, and define

$$I(T) = \int_1^T |\zeta(\tfrac{1}{2} + it)|^2 |A(\tfrac{1}{2} + it)|^2 dt.$$

Then a special case of Theorem 1 of [BCH-B] is the formula

$$I(T) \sim T \sum_{h,k \leq M} \frac{a(h)}{h} \frac{\overline{a(k)}}{k} (h,k) \log\left(\frac{T(h,k)^2}{hk}\right), \tag{0.5}$$

this being valid for $M \ll T^\theta$ with $\theta < \frac{1}{2}$.

The subject of this thesis is the extension of the above results, in particular (0.1), (0.3), and (0.5), to another class of Dirichlet series. The class of Dirichlet series, and the results, are described in the next section.

Cusp Forms and Dirichlet Series

We denote the upper half plane by $\mathcal{H} = \{z = x + iy \mid y > 0\}$, and $\Gamma(1) = SL(2, \mathbb{Z})$ denotes the full modular group. Let k be a positive even integer. Then a *holomorphic cusp form of weight k for* $\Gamma(1)$ is a holomorphic function F on $\mathcal{H}$ which satisfies

$$F\left(\frac{az+b}{cz+d}\right) = (cz+d)^k F(z)$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$, and which has a Fourier expansion of the form

$$F(z) = \sum_{n=1}^{\infty} f_n e(nz),$$

where $e(z) = e^{2\pi iz}$. The space of all such functions is denoted $S_k(\Gamma(1))$. The Hecke operators are maps $T_n : S_k(\Gamma(1)) \to S_k(\Gamma(1))$ given by

$$(T_nF)(z) = n^{k-1} \sum_{d|n} \sum_{b=0}^{d-1} F\left(\frac{nz+bd}{d^2}\right).$$

We fix $F(z) \in S_k(\Gamma(1))$ and further assume that $F(z)$ is an eigenfunction of all the Hecke operators and $f_1 = 1$. Hecke showed that this implies that f_n is multiplicative, that is, $f_a f_b = f_{ab}$ whenever $(a,b) = 1$. Define

$$f(n) = f_n n^{\frac{1-k}{2}}.$$

If p is a prime and $d(n)$ is the divisor function we have

$$|f(n)| \leq d(n) \tag{1.1}$$

and

$$f(p^\lambda) = f(p)f(p^{\lambda-1}) - f(p^{\lambda-2}). \tag{1.2}$$

The inequality above was proved by Deligne [Del]. The identity was established by Mordell [M] when f is the Ramanujan τ-function, and by Hecke [Hec1] in general. The *Dirichlet series associated to the cusp form* $F(z)$ is defined by,

$$L_F(s) = \sum_{n=1}^{\infty} f(n) n^{-s}.$$

This Dirichlet series is the subject of the remainder of the thesis.

As mentioned in the first section, $L_F(s)$ is in the class of Dirichlet series which have properties similar to the Riemann ζ-function. We now describe these properties in the case at hand. By (1.1) the series for $L_F(s)$ converges absolutely for $\sigma > 1$, and so $L_F(s)$ defines a holomorphic function in that half plane. The function $L_F(s)$ can be continued to an entire function, and the function

$$\xi_F(s) = H_F(s) L_F(s),$$

where $H_F(s) = (2\pi)^{-s}\Gamma(\frac{k-1}{2} + s)$, is also entire. We have a functional equation

$$\xi_F(s) = \xi_F(1-s),$$

and an Euler product

$$L_F(s) = \prod_p \left(1 - f(p)p^{-s} + p^{-2s}\right)^{-1}.$$

The shape of the Euler product comes directly from (1.2), and by (1.1) it converges absolutely for $\sigma > 1$. Thus all of the nontrivial zeros of $L_F(s)$ lie in the critical strip $0 \le \sigma \le 1$, and the analog of the Riemann Hypothesis for $L_F(s)$ is the conjecture that all of the nontrivial zeros lie on $\sigma = \frac{1}{2}$. In 1939 Rankin [R] showed that $L_F(s)$ does not vanish on the boundary of the critical strip, this result coming 45 years after the corresponding result for the Riemann ζ-function. To study the nontrivial zeros of $L_F(s)$ we adopt the same notation as used for the ζ-function. While the notation is formally the same, no confusion should result because for the rest of this paper we are concerned only with the Dirichlet series $L_F(s)$.

A nontrivial zero of $L_F(s)$ is denoted by $\rho = \beta + i\gamma$, and we have the zero counting function

$$N(T) = \sum_{0<\gamma<T} 1,$$

where the sum is over the nontrivial zeros, and zeros are counted according to their multiplicity. The same methods as used in the case of the Riemann ζ-function show that $N(T) \sim \frac{1}{\pi} T \log T$, so $L_F(s)$ has twice as many zeros as the ζ-function. Let

$$N_0(T) = \sum_{\substack{0<\gamma<T \\ \rho=\frac{1}{2}+i\gamma}} 1$$

be the counting function for zeros on the critical line. Lekkerkerker [Lek] showed in 1955 that $N_0(T) > AT$ for some $A > 0$, this result coming 34 years after the corresponding result for the ζ-function. Hafner [H] showed in 1983 that $N_0(T) > AT \log T$ for some $A > 0$, 41 years after Selberg showed the corresponding result for the ζ-function.

What has yet to be done is the extension of Levinson's method to $L_F(s)$. The main step in this is evaluating the analog of (0.1), and this is the main result of this thesis. First we need the appropriate mollifier of $L_F(s)$. With this in mind let T be a large parameter, $B = T^\theta$, and μ_f be the Dirichlet inverse of f, ie. for $\sigma > 1$ we have

$$\frac{1}{L_F(s)} = \sum_{n=0}^{\infty} \mu_f(n) n^{-s}.$$

From the Euler product representation of L_F we see that if p is a prime then $\mu_f(p) = -f(p)$ and $\mu_f(p^2) = 1$, and if $n \geq 3$ then $\mu_f(p^n) = 0$. Define

$$b_\alpha = \mu_f(\alpha)\, h\left(\frac{\log B/\alpha}{\log B}\right),$$

then the mollifier is

$$B(s) = \sum_{\beta \leq B} b_\beta \beta^{-s},$$

where $h(x) = \sum a_m x^m$ is a polynomial with $h(0) = 0$ and $h(1) = 1$. Then the analog of (0.1) is

Theorem 1. *If $0 < \theta < \frac{1}{6}$ and $|u| + |v| \ll 1/\log T$, then*

$$\int_1^T L_F(\tfrac{1}{2} + u + it)\, L_F(\tfrac{1}{2} + v - it)\, |B(\tfrac{1}{2} + it)|^2 dt$$

$$\sim T \left(1 + \frac{1}{\theta} \int_0^2 T^{-y(u+v)} dy \, \frac{d}{d\alpha} \frac{d}{d\beta} B^{-\alpha u - \beta v} \int_0^1 h(x + \alpha)\, h(x + \beta) dx \right)\Bigg|_{\substack{\alpha=0\\ \beta=0}}$$

uniformly in u and v.

Note that both sides of the above formula are holomorphic as functions of u and v. Since the error term is uniform in u and v Cauchy's theorem implies that the asymptotic formula holds after differentiating any number of times with respect to u and v, for $|u| + |v| \ll 1/\log T$.

Comparing Theorem 1 with (0.1), and recalling that $L_F(s)$ has twice as many zeros as the Riemann ζ-function (equivalently, L_F has twice as many Γ-factors in its functional equation), there is an obvious conjecture to be made about the shape of the mean square of LB where L has Λ times as many zeros as ζ and B is the appropriate mollifier of L. However, some additional conditions are needed. If

$$L(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$$

then define

$${}_2L(s) = \sum_{n=1}^{\infty} \frac{|a(n)|^2}{n^s}.$$

Both ${}_2\zeta(s)$ and ${}_2L_F(s)$ have *simple* poles at $s = 1$, and this fact plays a key part in the proofs of the asymptotic formulas. The function $\zeta^2(s)$, for example, does not have this property and we would not expect the mean square of $\zeta^2 B$ to look like the main term in Theorem 1. Conjecturally, if $L(s)$ is in the Selberg class (see [Sel2] or [CG2]) then the condition that ${}_2L(s)$ has a simple pole at $s = 1$ is equivalent to $L(s)$ being primitive, and so only for those functions is it reasonable to expect that the analogous asymptotic formula holds.

Now we illustrate some applications of Theorem 1. Let $\rho^{(j)} = \beta + i\gamma$ denote a zero of the j^{th} derivative $\xi_F^{(j)}(s)$, and denote its multiplicity by $m(\gamma)$. Define

$$N^{(j)}(T) = \sum_{\rho^{(j)}=\beta+i\gamma} 1 \qquad\qquad N(T) = N^{(0)}(T)$$

$$N_0^{(j)}(T) = \sum_{\rho^{(j)}=\frac{1}{2}+i\gamma} 1 \qquad\qquad N_1^{(j)}(T) = \sum_{\substack{\rho^{(j)}=\frac{1}{2}+i\gamma \\ m(\gamma)=1}} 1$$

$$M_r(T) = \sum_{\substack{\rho^{(0)}=\beta+i\gamma \\ m(\gamma)=r}} 1 \qquad\qquad M_{\le r}(T) = \sum_{\substack{\rho^{(0)}=\beta+i\gamma \\ m(\gamma)\le r}} 1$$

where all sums are over $0 < \gamma < T$, and zeros are counted according to their multiplicity. Let

$$\kappa_0^{(j)} = \liminf_{T\to\infty} \frac{N_0^{(j)}(T)}{N^{(j)}(T)}. \qquad\qquad \kappa_1^{(j)} = \liminf_{T\to\infty} \frac{N_1^{(j)}(T)}{N^{(j)}(T)}.$$

Thus, $\kappa_0^{(j)}$ is the proportion of zeros of $\xi_F^{(j)}(s)$ which are on the critical line, and $\kappa_1^{(j)}$ is the proportion which are simple and on the critical line. Lekkerkerker [Lek] has shown that $N^{(j)}(T) \sim \frac{1}{\pi}T\log T$ and $N_0^{(j)}(T) > AT$ as $T \to \infty$, and Hafner [Haf1] showed that $\kappa_0^{(0)} > 0$, but he did not obtain an explicit lower bound. It is immediate from Rolle's Theorem that $\kappa_0^{(j+1)} \geq \kappa_0^{(j)} \geq \kappa_1^{(j)}$. We have now the following

Corollary 1a. *With $\kappa_1^{(j)}$ as defined above we have $\kappa_1^{(1)} > 0.326$, $\kappa_1^{(2)} > 0.582$, $\kappa_1^{(3)} > 0.730$, $\kappa_1^{(4)} > 0.820$, $\kappa_1^{(5)} > 0.879$, and in general $\kappa_1^{(n)} = 1 + O(n^{-2})$.*

Corollary 1b. *For T sufficiently large, $M_{\leq 3}(T) > 0.165N(T)$, $M_{\leq 4}(T) > 0.325N(T)$, $M_{\leq 5}(T) > 0.460N(T)$, and $M_{\leq 6}(T) > 0.576N(T)$.*

In an unpublished work Conrey, Ghosh, and Gonek have used Montgomery's pair correlation method to show that if all of the nontrivial zeros of L_F are on $\sigma = \frac{1}{2}$ then

$$\sum_{0<\gamma<T} m_\gamma \leq \frac{13}{6}N(T),$$

where the sum is over the zeros of ξ_F. We have

$$2\sum_{\substack{0<\gamma<T\\ m_\gamma=1}} 1 \;+\; 2\sum_{\substack{0<\gamma<T\\ m_\gamma=2}} 1 \;\geq\; \sum_{0<\gamma<T} (3 - m_\gamma).$$

Thus,

$$\sum_{\substack{0<\gamma<T\\ m_\gamma=1,\,2}} 1 \;\geq\; \frac{5}{12}N(T).$$

That is, $M_{\leq 2}(T) > 0.416N(T)$. We also have $M_{\leq 3}(T) > 0.611N(T)$, $M_{\leq 4}(T) > 0.708N(T)$, $M_{\leq 5}(T) > 0.766N(T)$, and $M_{\leq 6}(T) > 0.805N(T)$. These results are conditional on the Riemann Hypothesis for L_F.

Corollary 1c. *Let $N^d(T)$ denote the number of distinct zeros of $\xi_F(s)$ in $0 < t < T$. Then if T is sufficiently large*

$$N^d(T) > 0.253N(T).$$

Corollary 2. *Let $\rho = \beta + i\gamma$ represent a zero of $L_F(s)$ with $\beta > \frac{1}{2}$. Then*

$$\sum_{0<\gamma<T} (\beta - \tfrac{1}{2}) < \big(0.204 + o(1)\big)T.$$

It is interesting to compare the above results to the corresponding statements for the Riemann ζ-function. Corollary 1a is just Levinson's method [Lev] applied to L_F. The only difference which occurs is for the Riemann ξ-function, $\xi(s) = H(s)\zeta(s)$ where

(1.3)
$$\frac{H^{(m)}}{H}(s) = \left(\frac{\log s/2\pi}{2}\right)^m \left(1 + O(t^{-1})\right)$$

while in our case $\xi_F(s) = H_F(s)L_F(s)$ where

(1.4)
$$\frac{H_F^{(m)}}{H_F}(s) = (\log s/2\pi)^m \left(1 + O(t^{-1})\right).$$

The factor of 2 difference in the above equations will change some of the constants in our formulas. The same analysis goes through as before and the result is that

(1.5)
$$\kappa_1^{(n)} \geq 1 - \frac{1}{2R}\log\left(1 + \frac{1}{\theta}\int_0^2\!\!\int_0^1 \frac{d}{d\alpha}\frac{d}{d\beta} Q(y+\alpha\theta)Q(y+\beta\theta)e^{R(2y+\alpha\theta+\beta\theta)}\, h(x+\alpha)h(x+\beta)\, dx dy\right)\Bigg|_{\substack{\alpha=0\\ \beta=0}}$$

In light of the discussion of Levinson's method in the previous section, one might think that since we may take $\theta = 0.166$ in (1.5) we could obtain a positive value for $\kappa_1^{(0)}$. However, the change of variable $y \mapsto 2y$ in (1.5) results in an expression identical to (0.3) with the substitutions

$$(\kappa_1^{(n)}, Q, \lambda, \theta, R) \leftrightarrow (\tau_1^{(n)}, Q_\zeta, \lambda/2, \theta/2, 2R).$$

In other words, it is "twice as hard" to get a result for L_F, and one would need $\theta > 0.327$ to obtain a positive value for $\kappa_1^{(0)}$ by this method. This seems far beyond reach of present technology. The correspondence θ(for L_F) $\leftrightarrow$ $\theta/2$(for ζ) is also apparent in the asymptotic formulas.

While our work does not show that a positive proportion of the zeros of ξ_F are simple and on the critical line, some results in this direction are already known. Hafner [Haf1] has shown that a positive proportion of the zeros of ξ_F have odd multiplicity and are on the critical line. This involved applying to ξ_F the method which Selberg [Sel1] used to show the corresponding result for the Riemann ξ-function. Conrey and Ghosh [CG1] showed that there exist arbitrarily large T such that for any $\epsilon > 0$, $L(s)$ has $\gg_\epsilon T^{\frac{1}{6}-\epsilon}$ simple zeros in the region $0 < t < T$. Here

$$L(s) = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s}$$

is the Dirichlet series formed with the Ramanujan τ-function. This result is stated only for the τ-function because the last step in their argument involves verifying that $L(s)$ has at least one nontrivial simple zero, and this is done in an ad hoc manner. Their proof uses Good's [G3] result $\xi_F(\frac{1}{2}+it) \ll t^{\frac{1}{3}+\epsilon}$ as $t \to \infty$, and an improvement in this estimate would yield an improvement in the simple zeros result.

We state one more result, namely, the analog of formula (0.5) Let

$$A(s) = \sum_{m \le M} \frac{a(m)}{m^s}$$

and define

$$I(T) = \int_1^T |L_F(\tfrac{1}{2}+it)|^2 |A(\tfrac{1}{2}+it)|^2 dt.$$

We have

Theorem 2. *Suppose $a(m) \ll m^\epsilon$ for any $\epsilon > 0$, and $M \ll T^\theta$ with $\theta < \frac{1}{6}$. Then*

$$I(T) \sim D_{-1} T \sum_{\alpha,\beta \le M} \frac{a(\alpha)}{\alpha} \frac{\overline{a(\beta)}}{\beta} (\alpha,\beta)\, \mathcal{F}\left(\frac{\alpha}{(\alpha,\beta)}\right) \mathcal{F}\left(\frac{\beta}{(\alpha,\beta)}\right) \log\left(\frac{T^2(\alpha,\beta)^2}{\alpha\beta}\right)$$

where $\mathcal{F}$ is the multiplicative function defined by

$$\mathcal{F}(p) = f(p)\frac{p}{p+1}$$

and

$$\mathcal{F}(p^n) = f(p)\mathcal{F}(p^{n-1}) - \mathcal{F}(p^{n-2})$$

for p prime, and D_{-1} is the residue at $s = 1$ of the function

$$D(s) = \sum_{n=1}^{\infty} \frac{f^2(n)}{n^s}.$$

In [BCH-B] it is conjectured that in the ζ-function case the asymptotic formula (0.5) should be valid for $\theta < 1$, and an example is given which shows that, at least for some choices of the $a(n)$, the formula fails to hold when $\theta > 1$. As can be seen from the graph previously given, the "$\theta = 1$" conjecture implies that more than 58.6% of the zeros of $\zeta(s)$ are simple and on the critical line. In view of the large sieve inequality for Dirichlet polynomials the $\theta = 1$ conjecture appears reasonable. Since the large sieve inequality is essentially best possible, the failure of the formula to hold in general for $\theta > 1$ is expected. The same reasoning would lead us to conjecture that the formula in Theorem 2 should hold for $\theta < 1$, and should not hold in general for $\theta > 1$.

There is the potential that for certain sequences $a(n)$ the formula may hold over a larger range, and if so, the case where $A(s)$ is a mollifier of the given Dirichlet series would be a likely candidate for this possibility. It may be that in the mollified case, that is, Theorem 1 or (0.1), the asymptotic formula remains valid for arbitrarily large values of θ. This would imply that almost all of the zeros of these primitive Dirichlet series are simple and on the critical line.

CHAPTER II

OUTLINE OF THE PROOF

We present the main ideas of the paper: Suppose Ψ_U is a smooth function with support in the interval $[1-1/U, 2+1/U]$ such that $\Psi_U(t) = 1$ for $1+1/U < t < 2-1/U$ and $\Psi_U^{(j)} \ll U^j$ for all j. Theorem 1 will follow directly from

Proposition 1. *If $B < T$, $|u| + |v| \ll 1/\log T$ and*

$$I(T; u, v) = \int_{-\infty}^{\infty} \Psi_U\left(\frac{t}{T}\right) L_F\left(\tfrac{1}{2} + u + it\right) L_F\left(\tfrac{1}{2} + v - it\right) |B(\tfrac{1}{2} + it)|^2 \, dt$$

then

$$I(T; u, v) = M(T; u, v) + O\big(TU^{-1} + U^{\frac{1}{2}}T^{\frac{1}{2}}B^{\frac{3}{2}} + T^{\frac{1}{2}}B^3 + T^{\frac{3}{4}}B^{\frac{3}{2}}\big)$$

where

$$M(T; u, v) \sim T \left(1 + \frac{\log T}{\log B} \int_0^2 T^{-y(u+v)} dy \, \frac{d}{d\alpha}\frac{d}{d\beta} B^{-\alpha u - \beta v} \int_0^1 h(x+\alpha)\, h(x+\beta)\, dx \right)\Bigg|_{\alpha=\beta=0}$$

The beginning of the proof of Proposition 1 is similar to [G3] where Good obtains an asymptotic representation for $\int_1^T |L|^2$ with the best known error term, and [Haf1] where Hafner bounds $\int_1^T |L|^2|B|^4$. Hafner did not need to obtain an asymptotic expression, nor did he need to use a long mollifying polynomial, while in our case both of these are critical. Nevertheless, our initial reductions closely follow Hafner's treatment, so we refer there for additional details. In Chapter 3 we begin the proof of Proposition 1. We start by squaring an approximate functional equation for LB. A trick of Good allows us to eliminate the cross terms. We are left with essentially three pieces. The terms far from the diagonal are disregarded by repeated integration by parts in a standard way. The terms on the diagonal provide the main term; this is done via Proposition 2. The near diagonal terms provide an error term; this is done by Proposition 3. Thus it remains to prove Propositions 2 and 3. In Chapter 4 we provide the background information and lemmas needed to prove the Propositions. In Chapter 5 we prove Proposition 2 using techniques of Conrey [C1] to extract an asymptotic expression. In Chapter 6 we prove Proposition 3. We transform the near diagonal terms into an expression involving Eisenstein series, Maass forms, and their Fourier coefficients. A lemma of Hafner [Haf2] along with theorems of Iwaniec [Iwa] and Deshouillers and Iwaniec [DI] are then used to bound this in a nontrivial way. In Chapter 7 we put the pieces together to finish the proof of Proposition 1, and so that of Theorem 1. The proof of Theorem 2 involves a very slight modification of the proof just given, and this is done in the last part of Chapter 7. In Chapter 8 we evaluate our formula for specific choices of h, u, v, and θ to obtain the corollaries.

CHAPTER III

INITIAL REDUCTION

This chapter contains the first part of the proof of Proposition 1.

We start with Good's [G1] approximate functional equation for L_F. Suppose $\varphi \in C_0^\infty(\mathbb{R})$ satisfies $\varphi(t) = \varphi(-t)$, and

$$\varphi(t) = \begin{cases} 0 & |t| \geq \frac{3}{2} \\ 1 & |t| \leq \frac{2}{3}. \end{cases} \tag{3.0}$$

For $g \in C_0^\infty(\mathbb{R})$ define $g_0(t) = 1 - g(1/t)$, and note that φ_0 satisfies (3.0). Then for $0 < \sigma < 1$ we have the approximate functional equation

$$L_F(s) = \mathcal{G}(s, x, \varphi, L_F) + X(s)\mathcal{G}(1-s, x^{-1}, \varphi_0, L_F), \tag{3.1}$$

where for $\epsilon > 0$,

$$\mathcal{G}(s, x, \varphi, L_F) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \varphi\left(\frac{2\pi n}{tx}\right) + \ O(x^{1-\sigma}|t|^{-\sigma+\epsilon})$$

and

$$X(s) = (2\pi)^{2s-1} \frac{\Gamma(\frac{k+1}{2} - s)}{\Gamma(\frac{k-1}{2} + s)}.$$

By multiplying (3.1) by $B(\frac{1}{2} + it)$ and replacing x by x/α we get

Lemma 1. *If* $|u| \ll 1/\log t$ *then for any* $\epsilon > 0$

$$L_F(\tfrac{1}{2}+u+it)\,B(\tfrac{1}{2}+it) = \sum_{n,\alpha} \frac{f(n)\,b_\alpha}{n^{\frac{1}{2}+u}\alpha^{\frac{1}{2}}}(n\alpha)^{-it}\varphi\Big(\frac{2\pi n\alpha}{tx}\Big)$$

$$+\,Q(u+it)\sum_{m,\beta}\frac{f(m)\,b_\beta}{m^{\frac{1}{2}-u}\beta^{\frac{1}{2}}}\Big(\frac{m}{\beta}\Big)^{it}\varphi_0\Big(\frac{2\pi m x}{t\beta}\Big)$$

$$+\,O\left(t^{-\frac{1}{2}+\epsilon}\sum_{\alpha\le B}\frac{b_\alpha}{\alpha^{\frac{1}{2}}}\left(\Big(\frac{x}{\alpha}\Big)^{\frac{1}{2}-u}t^{-u}+\Big(\frac{\alpha}{x}\Big)^{\frac{1}{2}+u}t^{u}\right)\right)$$

$$= A_1(t,u,\varphi)+Q(u+it)A_2(t,u,\varphi_0)+E_1(t;u,x),$$

where $Q(s) = X(\frac{1}{2}+s)$, *and* $E_1(t;u,x) \ll t^{-\frac{1}{2}+\epsilon}\left(x^{\frac{1}{2}}+Bx^{-\frac{1}{2}}\right)$.

Ultimately we will choose $x = B$. By Stirling's formula and the restriction on u,

$$Q(u+it) = e^{\frac{-i\pi}{2}(k-1)}\left(\frac{t}{2\pi e}\right)^{-2(u+it)}e^{-2u}\left(1+\frac{i}{t}(c_k+u^2)\right)+O\left(\frac{1}{t^2}\right), \tag{3.2}$$

where $c_k = \frac{k^2}{4}-\frac{k}{2}+\frac{1}{6}$.

From Lemma 1 and Cauchy's inequality we have

$$\begin{aligned} I(T;u,v) &= I_\varphi(T;u,v)+E_2(T;u,v,x) \\ &\quad + O\big(|E_2^{\frac{1}{2}}(T;u,u,x)I_\varphi^{\frac{1}{2}}(T;u,u)|+|E_2^{\frac{1}{2}}(T;v,v,x)I_\varphi^{\frac{1}{2}}(T;v,v)|\big), \end{aligned} \tag{3.3}$$

where for $\epsilon > 0$,

$$\begin{aligned} I_\varphi(T;u,v) = \int_{-\infty}^{\infty} \Psi_U\left(\frac{t}{T}\right)&\big(A_1(t,u,\varphi)+Q(u+it)A_2(t,u,\varphi_0)\big) \\ &\times\big(A_1(-t,v,\varphi)+Q(v-it)A_2(-t,v,\varphi_0)\big)\,dt \end{aligned} \tag{3.4}$$

and

$$E_2(T;u,v,x) \ll T^\epsilon(x+B+B^2x^{-1}).$$

We employ a trick of Good ([G2] p. 23) to break I_φ into manageable pieces. Write $\varphi = \varphi_1 + \varphi_2$ where φ_1 and φ_2 are smooth even functions satisfying $\varphi_1(t) = 0$ if $|t| > \frac{3}{4}$ and $\varphi_2(t) = 0$ if $|t| < \frac{2}{3}$. To make things look nicer, choose $\varphi,\ \varphi_1,\ \varphi_2$ so that $\varphi(t) = \varphi_0(t)$ and $\varphi_2(t) = \varphi_2(1/t)$, and note that $(\varphi + \varphi_2)_0 = \varphi - \varphi_2$. Now $\varphi + \varphi_2$ satisfies (3.0), so Lemma 1 gives

$$A_1(t,u,\varphi) + Q(u+it)A_2(t,u,\varphi) = A_1(t,u,\varphi+\varphi_2) + Q(u+it)A_2(t,u,\varphi-\varphi_2) + E_1(t;u,x).$$

Therefore,

$$Q(u+it)A_2(t,u,\varphi_2) = A_1(t,u,\varphi_2) + E_1(t;u,x). \tag{3.5}$$

The usefulness of (3.5) is this: when we multiply out the integrand in (3.4) we obtain cross terms, ie. terms containing exactly one Q as a factor, which shouldn't contribute anything to our main term. The usual trick of repeated integration by parts to show that these terms are small fails when t is close to 1. But this is exactly where φ_2 is supported, so that (3.5) allows us to exchange terms which contain one Q factor for terms which contain zero or two Q factors with the introduction of a reasonable error term.

Using (3.5) four times we get

$$I_\varphi(T;u,v) = \int_{-\infty}^{\infty} \Psi_U\left(\frac{t}{T}\right) \sum_{j=1}^{4} S_j(t;u,v)\,dt$$

where

$$S_1(t;u,v) = A_1(t,u,\varphi)A_1(-t,v,\varphi) + A_1(t,u,\varphi)A_1(-t,v,\varphi_2) + A_1(t,u,\varphi_2)A_1(-t,v,\varphi_1)$$

$$S_2(t;u,v) = Q(u+it)Q(v-it)\big(A_2(t,u,\varphi)A_2(-t,v,\varphi)$$

$$+ A_2(t,u,\varphi)A_2(-t,v,\varphi_2) + A_2(t,u,\varphi_2)A_2(-t,v,\varphi_1)\big)$$

$$S_3(t;u,v) = Q(u+it)\,A_2(t,u,\varphi_1)A_1(-t,v,\varphi_1) + Q(v-it)\,A_1(t,u,\varphi_1)A_2(-t,v,\varphi_1)$$

$$S_4(t;u,v) \ll |E_1(t;u,x)|\,(|A_1(t;v,\varphi_1)| + |Q(v+it)A_2(t;v,\varphi)|)$$

$$+ |E_1(t;v,x)|\,(|A_1(t;u,\varphi_1)| + |Q(u+it)A_2(t;u,\varphi)|)$$

We have expanded $I(T;u,v)$ into a large number of pieces, most of which we eliminate by

Lemma 2. *With the notation above, if $\epsilon > 0$ then*

$$I_\varphi(T;u,v) \ll T^{1+\epsilon}B$$

and

$$\int_{-\infty}^{\infty} \Psi_U\left(\frac{t}{T}\right)\big(S_3(t;u,v) + S_4(t;u,v)\big)\,dt = E_3(T;u,v,x),$$

where

$$E_3(T;u,v,x) \ll T^{\frac{1}{2}+\epsilon}B^{\frac{1}{2}}(x^{\frac{1}{2}} + Bx^{-\frac{1}{2}} + B^{-\frac{1}{2}}U^{\frac{1}{2}}x + B^{\frac{1}{2}}U^{\frac{1}{2}} + B^{\frac{3}{2}}U^{\frac{1}{2}}x^{-1}).$$

The details are in [Haf1] so we just give a sketch. The integral involving S_3 is small because the restriction of the support of φ_1 to the interval $[-\frac{3}{4},\frac{3}{4}]$ permits repeated integration by parts, as on page 140 of Hafner's paper. For I_φ and the integral involving S_4 we use the Cauchy-Schwarz inequality. Then integration by parts leads to an expression which we bound using familiar techniques; see page 139 of [Haf1], or [T], section 7.2.

The above Lemma, along with (3.3) and (3.4) brings us to

$$I(T;u,v) = \int_{-\infty}^{\infty} \Psi_U\left(\frac{t}{T}\right)\left(S_1(t;u,v) + S_2(t;u,v)\right) dt + O\left(T^{\frac{1}{2}+\epsilon} U^{\frac{1}{2}} E_2^{\frac{3}{2}}(T;u,v,x)\right).$$

The error term above is far from best possible, but we will see that it is not the limiting step in what we do.

To save on notation let $\Phi(x,y) = \varphi(x)\,\varphi(y) + \varphi(x)\,\varphi_2(y) + \varphi_2(x)\,\varphi_1(y)$. Note that the support of $\Phi(x,y)$ is contained in $[-\frac{3}{2},\frac{3}{2}] \times [-\frac{3}{2},\frac{3}{2}]$, and $\Phi(x,x)$ satisfies (3.0). We have

$$S_1(t;u,v) = \sum_{n,m,\alpha,\beta} \frac{f(n)f(m)b_\alpha \overline{b}_\beta}{(mn\alpha\beta)^{\frac{1}{2}} n^u m^v} \left(\frac{n\alpha}{m\beta}\right)^{-it} \Phi\left(\frac{2\pi n\alpha}{tx}, \frac{2\pi m\beta}{tx}\right)$$

and

$$S_2(t;u,v) = Q(u+it)Q(v-it) \sum_{n,m,\alpha,\beta} \frac{f(n)f(m)b_\alpha \overline{b}_\beta}{(mn\alpha\beta)^{\frac{1}{2}} n^{-u} m^{-v}} \left(\frac{n\beta}{m\alpha}\right)^{it} \Phi\left(\frac{2\pi nx}{t\alpha}, \frac{2\pi mx}{t\beta}\right).$$

And by (3.2)

$$Q(u+it)\,Q(v-it) = \left(\frac{t}{2\pi}\right)^{-2(u+v)} \left(1 + \frac{i}{t}(u^2 - v^2)\right) + O(t^{-2}).$$

Just as in the proof of Lemma 2, repeated integrations-by-part permit us to eliminate those terms in S_1 and S_2 which are "far from the diagonal," giving

$$I(T;u,v) = I_1(T;u,v) + I_2(T;u,v) + O\left(T^{\frac{1}{2}} U^{\frac{1}{2}} E_2^{\frac{3}{2}}(T;u,v,x)\right),$$

where for $\epsilon > 0$,

$$I_1(T;u,v) = \sum_{|\frac{m\beta}{n\alpha}-1|<T^{-1+\epsilon}} \frac{f(n)f(m)b_\alpha \overline{b}_\beta}{(mn\alpha\beta)^{\frac{1}{2}} n^u m^v} \int_{-\infty}^{\infty} \Psi_U\left(\frac{t}{T}\right)\left(\frac{n\alpha}{m\beta}\right)^{-it} \Phi\left(\frac{2\pi n\alpha}{tx}, \frac{2\pi m\beta}{tx}\right) dt$$

and

$$I_2(T;u,v) = \sum_{|\frac{n\beta}{m\alpha}-1|<T^{-1+\epsilon}} \frac{f(n)f(m)b_\alpha \overline{b}_\beta}{(mn\alpha\beta)^{\frac{1}{2}} n^{-u} m^{-v}}$$
$$\times \int_{-\infty}^{\infty} \Psi_U\left(\frac{t}{T}\right)\left(\frac{n\beta}{m\alpha}\right)^{it}\left(\frac{t}{2\pi}\right)^{-2(u+v)} \Phi\left(\frac{2\pi nx}{t\alpha}, \frac{2\pi mx}{t\beta}\right) dt.$$

The $1/t$ term from $Q(u+it)Q(v-it)$ does not appear in the expression for I_2; the restriction on the summation allowed us to incorporate it into the error term.

The main term in I_1 comes when $n\alpha = m\beta$; the rest will become an error term. In the error terms write $l = m\beta - n\alpha$, or $l = n\alpha - m\beta$, whichever happens to be positive, and eliminate m to get

$$I_1(T;u,v) = M_1(T;u,v) + R_1(T;u,v,x) + R_1(T;v,u,x)$$

where

$$M_1(T;u,v) = \sum_{\alpha,\beta} \frac{b_\alpha \overline{b}_\beta}{\alpha^{1+v}\beta^{-v}} \sum_n \frac{f(n)\, f\left(\frac{n\alpha}{\beta}\right)}{n^{1+u+v}} \int_{-\infty}^{\infty} \Psi_U\left(\frac{t}{T}\right) \Phi\left(\frac{2\pi n\alpha}{tx}\right) dt$$

and

$$R_1(T;u,v,x) = \sum_{\alpha,\beta} \frac{b_\alpha \overline{b}_\beta}{\alpha^{\frac{1}{2}}\beta^{-v}} \sum_n \sum_{1\le l<n\alpha T^{-1+\epsilon}} \frac{f(n)\, f\left(\frac{n\alpha+l}{\beta}\right)}{n^{\frac{1}{2}+u}\,(n\alpha+l)^{\frac{1}{2}+v}}$$
$$\times \int_{-\infty}^{\infty} \Psi_U\left(\frac{t}{T}\right) \left(\frac{n\alpha+l}{n\alpha}\right)^{it} \Phi\left(\frac{2\pi n\alpha}{tx}, \frac{2\pi(n\alpha+l)}{tx}\right) dt.$$

We have put $\Phi(z) = \Phi(z,z)$. Similarly,

$$I_2(T;u,v) = M_2(T;u,v) + R_2(T;u,v,x) + R_2(T;v,u,x),$$

where

$$M_2(T;u,v) = \sum_{\alpha,\beta} \frac{b_\alpha \overline{b}_\beta}{\alpha^{1-v}\beta^{v}} \sum_n \frac{f(n) f\left(\frac{n\alpha}{\beta}\right)}{n^{1-u-v}} \int_{-\infty}^{\infty} \Psi_U\left(\frac{t}{T}\right) \left(\frac{t}{2\pi}\right)^{-2(u+v)} \Phi\left(\frac{2\pi n x}{t\alpha}\right) dt$$

and

$$R_2(T;u,v,x) = \sum_{\alpha,\beta} \frac{b_\alpha \overline{b}_\beta}{\alpha^{\frac{1}{2}}\beta^{v}} \sum_n \sum_{1\le l<n\alpha T^{-1+\epsilon}} \frac{f(n) f\left(\frac{n\alpha+l}{\beta}\right)}{n^{\frac{1}{2}-u}(n\alpha+l)^{\frac{1}{2}-v}}$$
$$\times \int_{-\infty}^{\infty} \Psi_U\left(\frac{t}{T}\right) \left(\frac{n\alpha+l}{n\alpha}\right)^{-it} \left(\frac{t}{2\pi}\right)^{-2(u+v)} \Phi\left(\frac{2\pi n x}{t\beta}, \frac{2\pi(n\alpha+l)x}{t\alpha\beta}\right) dt.$$

All that remains is to get an asymptotic expression for M_1 and M_2 and to obtain bounds for R_1 and R_2. We evaluate M_j by Proposition 2, and we bound R_j via Proposition 3. After proving the Propositions we return to finish this proof.

CHAPTER IV

NOTATION AND LEMMAS

In this chapter we begin with terminology and results from the spectral theory of the Laplacian and Poincaré series. For more details on this material see Kubota's book [K]. After we have the necessary notation we present the lemmas needed in the sequel.

Let $\mathcal{H} = \{z = x + iy : y > 0\}$ be the complex upper half plane. The group $G = SL(2, \mathbb{R})$ acts on $\mathcal{H}$ by linear fractional transformation, and we equip $\mathcal{H}$ with the G-invariant measure $d\omega = y^{-2}dxdy$. Let $\Gamma \subset \Gamma(1) = SL(2, \mathbb{Z})/\{\pm I\}$ be a congruence subgroup of finite index $\mu(\Gamma) = [\Gamma(1), \Gamma]$. Let Λ be a set of inequivalent cusps for Γ, and set $h(\Gamma) = \#\Lambda$. For $\eta \in \Lambda$ let $\Gamma_\eta = \{\sigma \in \Gamma : \sigma\eta = \eta\}$, which is nontrivial by assumption. Note that

$$\Gamma_\infty = \left\{ \begin{pmatrix} 1 & nR \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}.$$

R is called the width of infinity for Γ. For each $\eta \in \Lambda$ choose $\sigma_\eta \in SL(2,\mathbb{R})$ such that $\sigma_\eta \infty = \eta$ and

$$\sigma_\eta^{-1}\Gamma_\eta\sigma_\eta = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}.$$

Note that $\sigma_\infty^{-1}x = x/R$. Of importance to us are the congruence subgroups

$$\Gamma_0(\beta, \alpha) = \left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in SL(2, \mathbb{Z}) : \beta|z, \ \alpha|y \right\}$$

and $\Gamma_0(N) = \Gamma_0(N, 1)$. We have

$$\mu(\Gamma_0(\beta, \alpha)) = \alpha\beta \prod_{p|\alpha\beta} \left(1 + \tfrac{1}{p}\right) \ll_\epsilon (\alpha\beta)^{1+\epsilon} \tag{4.1}$$

and

$$(4.2) \qquad h\big(\Gamma_0(\beta,\alpha)\big) = \sum_{d|\alpha\beta} \varphi\big((d, \tfrac{\alpha\beta}{d})\big) \ll_\epsilon (\alpha\beta)^\epsilon(\alpha,\beta).$$

Let $\mathcal{F}$ be a fundamental domain for the action of Γ; when convenient we will identify $\mathcal{F}$ with $\Gamma\backslash\mathcal{H}$. We are concerned with the spectral theory of the G-invariant Laplacian

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

acting on $L^2\big(\Gamma\backslash\mathcal{H}\big)$. There are two parts to this theory.

The discrete part of the spectrum is characterized by an orthonormal basis of Maass wave forms $e_j(z)$, $j = 0, 1, \ldots$ which are Γ-invariant and satisfy

$$\Delta e_j(z) = \lambda_j e_j(z),$$

and

$$\iint_{\mathcal{F}} e_i(z)\,\overline{e_j(z)}\,d\omega = \delta_{ij}.$$

We set $s_j = \frac{1}{2} + i\kappa_j$, so $\lambda_j = s_j(1-s_j) = \frac{1}{4} + \kappa_j^2$. It follows that

$$(4.3) \qquad e_j(\sigma_\eta z) = \sqrt{y} \sum_{m\neq 0} \rho_{j\eta}(m)\, K_{s_j - \frac{1}{2}}(2\pi|m|y)\, e(mx)$$

where K_ν is the modified Bessel function.

The continuous part of the spectrum is characterized by the Eisenstein series $E_\eta(z,s)$, where for $\eta \in \Lambda$ and $\sigma > 1$ we have

$$E_\eta(z,s) = \sum_{M\in\Gamma_\eta\backslash\Gamma} \Im^s(\sigma_\eta^{-1}Mz).$$

These satisfy $\Delta E_\eta(z,s) = s(1-s)E_\eta(z,s)$ and, as a function of z, $E_\eta(z,s)$ is Γ-invariant. We have the expansion

$$(4.4) \qquad E_\eta(\sigma_\kappa z, s) = \delta_{\eta\kappa} y^s + \varphi_{\eta\kappa}(s) y^{1-s} + \sqrt{y}\sum_{m\neq 0} \alpha_{m\eta\kappa}(s)\, K_{s-\frac{1}{2}}(2\pi|m|y)\, e(mx).$$

If $f \in L^2(\Gamma\backslash\mathcal{H})$ is bounded then we have the spectral decomposition

$$f(z) = \sum_{j\geq 0}\langle f, e_j\rangle e_j(z) + \sum_{\eta\in\Lambda}\frac{1}{4\pi i}\int_{(\frac{1}{2})}\langle f, E_\eta(\cdot,\xi)\rangle E_\eta(z,s)d\xi$$

where the Petersson inner product is defined by

$$\langle f, g\rangle_k = \iint_{\mathcal{F}} y^k f(z)\,\overline{g}(z)\,d\omega$$

whenever this integral exists. We put $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_0$.

Next we have non-holomorphic Poincaré series. For $m \geq 1$ and $\eta \in \Lambda$ put

$$U_{\eta,m}(z,s) = \sum_{\tau\in\Gamma_\eta\backslash\Gamma}\Im^s(\sigma_\eta^{-1}\tau z)\,e(m\sigma_\eta^{-1}\tau z).$$

As a function of z, $U_{\eta,m}(z,s)$ is Γ-invariant. Also,

$$(4.5)\; U_{\eta,m}(z,s) = \sqrt{\pi}\sum_{j>0}(4\pi m)^{\frac{1}{2}-s}\frac{\Gamma(s-s_j)\,\Gamma(s+s_j-1)}{\Gamma(s)}\,\rho_{j\eta}(m)\,e_j(z)$$
$$+\sqrt{\pi}\frac{1}{4\pi i}\int_{(\frac{1}{2})}(4\pi m)^{\frac{1}{2}-s}\frac{\Gamma(s-\xi)\,\Gamma(s+\xi-1)}{\Gamma(s)}\sum_{\kappa\in\Lambda}\alpha_{m\kappa\eta}(1-\xi)\,E_\kappa(z,\xi)\,d\xi.$$

The above formula is easily obtained from the definition of $U_{\eta,m}(z,s)$ and the spectral decomposition formula. One important use of Poincaré series is to exhibit Fourier coefficients of cusp forms. Suppose

$$H(z) = \sum_{n=1}^{\infty} h_n\,e(nz)$$

is a cusp form of weight k for $\Gamma(1)$ and

$$G(z) = \sum_{n=1}^{\infty} g_n\,e(nz/R)$$

is a cusp form of weight k for the congruence subgroup Γ, where R is the width of infinity for Γ. We have

Lemma 3. *In the notation above*

$$\langle U_{\infty,m}(\cdot,s),\ \overline{H}G\rangle_k = \frac{R^k\,\Gamma(k+s-1)}{(4\pi)^{k+s-1}}\sum_{n=1}^{\infty}\frac{h_n\,\overline{g}_{nR+m}}{(nR+m)^{k+s-1}}.$$

Proof. We compute:

$$\langle U_{\infty,m}(\cdot,s),\overline{H}G\rangle_k = \iint\limits_{\mathcal{F}} y^k H(z)\overline{G}(z)\sum_{\sigma\in\Gamma_\infty\backslash\Gamma}\Im^s\left(\frac{\sigma z}{R}\right)e\left(\frac{m\sigma z}{R}\right)d\omega$$

$$= R^{-s}\sum_{\sigma\in\Gamma_\infty\backslash\Gamma}\iint\limits_{\mathcal{F}} y^k H(z)\overline{G}(z)\Im^s(\sigma z)\,e\left(\frac{m\sigma z}{R}\right)d\omega$$

$$= R^{-s}\sum_{\sigma\in\Gamma_\infty\backslash\Gamma}\iint\limits_{\sigma\mathcal{F}}\Im^k(\sigma^{-1}z)H(\sigma^{-1}z)\overline{G}(\sigma^{-1}z)\,\Im^s(z)\,e\left(\frac{mz}{R}\right)d\omega$$

$$= R^{-s}\int_0^\infty\int_0^R y^{k+s-2}H(z)\overline{G}(z)\,e\left(\frac{mz}{R}\right)dx\,dy$$

$$= R^{-s}\int_0^\infty y^{k+s-2}\int_0^R\sum_{n,j}h_n\overline{g}_j\,e\left(\frac{x(nR-j+m)}{R}\right)e^{-2\pi y(nR+j+m)/R}\,dx\,dy$$

$$= R^{-s+1}\sum_{n=1}^{\infty}h_n\,\overline{g}_{nR+m}\int_0^\infty y^{k+s-2}\,e^{-4\pi y(n+m/R)}\,dy$$

$$= \frac{R^k\,\Gamma(k+s-1)}{(4\pi)^{k+s-1}}\sum_{n=1}^{\infty}\frac{h_n\,\overline{g}_{nR+m}}{(nR+m)^{k+s-1}},$$

as claimed.

We will apply Lemma 3 in the following case. With $F(z)$ as in Chapter 1 let $H(z) = F(z)$ and $G(z) = F(\beta z/\alpha)$. Then $G(z)$ is a cusp form for $\Gamma_0(\beta,\alpha)$. The Fourier coefficients of $G(z)$ satisfy

$$g_n = f_{n/\beta}$$

and

$$g(n) = \beta^{\frac{1-k}{2}} f(n/\beta),$$

so by Lemma 3

$$\sum_{n=1}^{\infty} \frac{f(n) f\left(\frac{n\alpha+l}{\beta}\right)}{n^{\frac{1-k}{2}}(n\alpha+l)^{\frac{k-1}{2}+s}} = \frac{\beta^{\frac{k-1}{2}}(4\pi)^{k+s-1}}{\alpha^k \Gamma(k+s-1)} \langle U_{\infty,l}(\cdot, s), \overline{F}G\rangle_k. \tag{4.6}$$

In the proof of Proposition 3 we will use the above formula along with (4.5) to express our error terms as a sum of Maass forms, Eisenstein series, and their Fourier coefficients. The next four lemmas are then used to bound the resulting expression.

Additional notation: The expression $\sum_{n\sim N}$ means that the sum is over $N \leq n < 2N$. If we are summing over the spectrum of Δ on $\Gamma\backslash\mathcal{H}$ we indicate the dependence on the group by writing $\sum^{\Gamma}$ or $\sum^{(N)}$ if $\Gamma = \Gamma_0(N)$. If $\mathbf{b} = (b_n)$ is a sequence of complex numbers then we write

$$||\mathbf{b}_N||_2 = \left(\sum_{n\sim N} |b_n|^2\right)^{\frac{1}{2}}.$$

Lemma 4. (Hafner [Haf2], Lemmas 3 and 4) *In the notation above, with* $\Gamma = \Gamma_0(\beta, \alpha)$, *we have*

$$|\langle e_j, \overline{F}G\rangle_k|^2 \ll \mu(\Gamma) h(\Gamma) \frac{s_j^{k+\epsilon}}{\cosh \pi i(s_j - \frac{1}{2})} \left(\frac{\alpha}{\beta}\right)^k$$

and

$$|\langle \sum_{\eta\in\lambda} E_\eta(\cdot, \tfrac{1}{2}+it), \overline{F}G\rangle_k|^2 \ll h(\Gamma) \frac{(1+|t|)^{k+\epsilon}}{\cosh \pi t} \left(\frac{\alpha}{\beta}\right)^k.$$

Lemma 5. (Deshouillers and Iwaniec [DI], Theorem 6) *Let* X, Q, N, *and* ϵ *be positive numbers, and* $\mathbf{a}$ *a sequence of complex numbers. Then one has*

$$\sum_{q\leq Q} \sum_{i\kappa_j>0}^{(q)} X^{4i\kappa_j} |\sum_{n\sim N} a_n \rho_{j\infty}(n)|^2 \ll_\epsilon (QN)^\epsilon (Q+N+NX) ||\mathbf{a}_N||_2^2.$$

Lemma 6. (Deshouillers and Iwaniec [DI], Theorem 2) *Suppose* $K > 1$, $N > \frac{1}{2}$, $\epsilon > 0$, *and* $\mathbf{a}$ *is a sequence of complex numbers. Then each of*

$$\sum_{i\kappa_j<K}^{(q)} \frac{1}{\cosh \pi i(s_j - \frac{1}{2})} |\sum_{n\sim N} a_n \rho_{j\infty}(n)|^2$$

and

$$\sum_{\eta\in\Lambda}^{(q)}\int_{-K}^{K}\frac{1}{\cosh\pi r}|\sum_{n\sim N}a_n n^{ir}\alpha_{l\eta\infty}(\tfrac{1}{2}+ir)|^2dr$$

is majorized by

$$(K^2+q^{-1}N^{1+\epsilon})||\mathbf{a}_N||_2^2.$$

Lemma 7. (Iwaniec [Iwa], Theorem 1) *In the notation above,*

$$\sum_{i\kappa_j>0}^{(N)}N^{4i\kappa_j}\ll N^{1+\epsilon}.$$

Lemmas 5, 6, and 7 are stated in the context of $\Gamma_0(N)$. Their proofs rely on Kuznetsov's trace formula, which relates Fourier coefficients of automorphic forms on a congruence group to Kloosterman sums defined over the group. If two congruence groups are conjugate then the sets of Kloosterman sums defined over them are exactly the same, so the proofs of the Lemmas, and so the Lemmas themselves, hold in the context of any group conjugate to $\Gamma_0(N)$. We will apply the Lemmas in the context of $\Gamma_0(\beta,\alpha)$, which is conjugate to $\Gamma_0(\beta\alpha)$.

Now we give the background needed for the proof of Proposition 2. We need to understand the function

$$D_{\alpha,\beta}(s)=\sum_n\frac{f(n)f\left(\frac{n\alpha}{\beta}\right)}{n^s}.$$

A simple calculation finds that

$$D_{\alpha,\beta}(s)=\left(\frac{\beta}{(\alpha,\beta)}\right)^{-s}D(s)\,P\left(\frac{\alpha}{(\alpha,\beta)},s\right)P\left(\frac{\beta}{(\alpha,\beta)},s\right),$$

where

$$D(s)=\sum_{j=1}^{\infty}\frac{f^2(j)}{j^s}=\prod_p\left(\sum_{j=1}^{\infty}\frac{f^2(p^j)}{p^{js}}\right)$$

and

$$P(M,s)=\left(\sum_{m|M^\infty}\frac{f(mM)f(m)}{m^s}\right)\cdot\left(\sum_{m|M^\infty}\frac{f^2(m)}{m^s}\right)^{-1}$$

$$= \prod_{p^r \| M} \left(\sum_{j=1}^{\infty} \frac{f(p^{j+r}) f(p^j)}{p^{js}} \right) \left(\sum_{j=1}^{\infty} \frac{f^2(p^j)}{p^{js}} \right)^{-1}.$$

The notation $m|M^{\infty}$ means that the prime divisors of m are divisors of M. We collect the important facts about D and P in

Lemma 8. *We have the following: $D(s)$ is regular for $\sigma > \frac{1}{2}$ except for a simple pole at $s = 1$ with residue D_{-1}, say. For $\frac{1}{2} < \sigma < 1$ we have $D(s) \ll |t|^{2-2\sigma+\epsilon}$ for $|t| > 1$, and $D(s) \gg \log^{-1}(2+t)$ for $1 - \sigma \ll \log^{-1} t$. For fixed s, $P(n,s)$ is a multiplicative function function of n, and if p is prime then*

$$P(p,s) = f(p) \frac{p^s}{p^s + 1},$$

and

$$P(p^n, s) = f(p)\, P(p^{n-1}, s) - P(p^{n-2}, s).$$

Consequently, $P(n,s) \ll n^{\epsilon}$ for $\sigma > \frac{1}{2}$.

The assertions about D are classical and are essentially contained in [R] and [Mn]. The assertions about P all follow directly from (1.1), (1.2) and the definition of P.

Finally, when we apply Propositions 2 and 3 the connection to the specific objects we have will be made through

Lemma 9. *Suppose that*

$$G(A,B,C) = \int_{-\infty}^{\infty} \Psi_U(t)\, t^A e^{iBt/C} \Phi\left(\frac{C}{t}\right) dt,$$

$$M(B,s) = \int_{0}^{\infty} \Phi(\xi) e^{iB/\xi} \xi^{s-1} d\xi,$$

and

$$M_U(X) = \int_{-\infty}^{\infty} \Psi_U(t) t^X dt.$$

Then

$$\int_0^\infty G(A,B,\xi)\xi^{s-1}d\xi = M(B,s)M_U(A+s).$$

In particular, by the Mellin inversion formula, for $c > 0$,

$$G(A,B,C) = \frac{1}{2\pi i}\int_{(c)} M(B,s)M_U(A+s)C^{-s}ds.$$

Furthermore, for real a, $M(a,s)$ *is regular in* $\sigma > \frac{1}{2}$, *except when* $a = 0$ *in which case it has a simple pole at* $s = 1$ *with residue 1.* $M_U(s)$ *is entire and both* $M(a,s)$ *and* $M_U(s)$ *decrease rapidly in any vertical strip.*

All of the statements in Lemma 9 are easily checked.

CHAPTER V

MAIN TERMS

With b_α and $P(n,s)$ as in the previous chapters we prove

Proposition 2. *If* $|w| \ll |v| + |u| \ll 1/\log B$ *and*

$$S(v,u,w) = \sum_{\alpha,\beta \le B} \frac{b_\alpha b_\beta (\alpha,\beta)^{1+w}}{\alpha^{1+v}\beta^{1+u}} P\left(\frac{\alpha}{(\alpha,\beta)}, 1+w\right) P\left(\frac{\beta}{(\alpha,\beta)}, 1+w\right),$$

then

$$S(v,u,w) \ll \frac{1}{\log B}$$

and

$$S(v,u,v+u) \sim \frac{1}{D_{-1}\log B} \frac{d}{d\alpha}\frac{d}{d\beta} B^{\alpha v + \beta u} \int_0^1 h(x+\alpha)\, h(x+\beta)\, dx \bigg|_{\alpha=\beta=0}$$

Proof: For convenience let $P_w(n) = P(n, 1+w)$. By the Möbius inversion formula

$$(\alpha,\beta)^{1+w} P_w\left(\frac{\alpha}{(\alpha,\beta)}\right) P_w\left(\frac{\beta}{(\alpha,\beta)}\right) = \sum_{d|(\alpha,\beta)} d^{1+w} \sum_{e|d} \frac{\mu(e)}{e^{1+w}} P_w\left(\frac{\alpha e}{d}\right) P_w\left(\frac{\beta e}{d}\right).$$

Thus,

$$S(v,u,w) = \sum_{d \le B} \frac{1}{d^{1+v+u-w}} \sum_{e|d} \frac{\mu(e)}{e^{1+w}} \sum_{\alpha \le B/d} \frac{b_{\alpha d} P_w(\alpha e)}{\alpha^{1+v}} \sum_{\beta \le B/d} \frac{b_{\beta d} P_w(\beta e)}{\beta^{1+u}}$$

$$= \sum_{d \le B} \frac{1}{d^{1+v+u-w}} \sum_{e|d} \frac{\mu(e)}{e^{1+w}} X(d,e,v,w)\, X(d,e,u,w),$$

say.

Recall that

$$b_n = \mu_f(n) \sum_{m=1}^{\infty} a_m \left(\frac{\log B/n}{\log B}\right)^m.$$

With $y = B/d$ we have

$$X(d,e,v,w) = \sum_{m=1}^{\infty} \frac{a_m}{\log^m B} \sum_{\alpha \le y} \frac{\mu_f(\alpha d) P_w(\alpha e)}{\alpha^{1+v}} \log^m(y/\alpha).$$

We will use Perron's formula to evaluate the inner sum of X. Let p_e (resp p_d) denote the order to which the prime p divides e (resp d), and recall that $e|d$ and P_w is multiplicative. Thus,

$$\sum_{\alpha=1}^{\infty} \frac{\mu_f(\alpha d) P_w(\alpha e)}{\alpha^s} = \prod_{p \nmid d} \sum_{j=0}^{\infty} \frac{\mu_f(p^j) P_w(p^j)}{p^{js}} \prod_{p|d} \sum_{j=0}^{\infty} \frac{\mu_f(p^{j+p_d}) P_w(p^{j+p_e})}{p^{js}}$$

$$= \prod_{p} \sum_{j=0}^{\infty} \frac{\mu_f(p^j) P_w(p^j)}{p^{js}} \prod_{p|d} \frac{\sum_{j=0}^{\infty} \frac{\mu_f(p^{j+p_d}) P_w(p^{j+p_e})}{p^{js}}}{\sum_{j=0}^{\infty} \frac{\mu_f(p^j) P_w(p^j)}{p^{js}}}$$

$$= \Omega_w(s) F_w(d,e,s),$$

say.

Perron's formula and the residue theorem give, for $c > 1 + |v|$,

$$X(d,e,v,w) = \sum_{m=1}^{\infty} \frac{a_m}{\log^m B} \frac{m!}{2\pi i} \int_{(c)} \frac{\Omega_w(s) F_w(d,e,s) y^{s-1-v}}{(s-1-v)^{m+1}} \, ds$$

$$= \sum_{m=1}^{\infty} \frac{a_m m!}{\log^m B} \operatorname*{Res}_{s=1+v} \left(\frac{\Omega_w(s) F_w(d,e,s) y^{s-1-v}}{(s-1-v)^{m+1}} \right)$$

$$+ \sum_{m=1}^{\infty} \frac{a_m}{\log^m B} \frac{m!}{2\pi i} \int_{C} \frac{\Omega_w(s) F_w(d,e,s) y^{s-1-v}}{(s-1-v)^{m+1}} \, ds,$$

where C is the piecewise linear path with vertices $[1 - i\infty, 1 - i\log^{10} y, 1 - b - i\log^{10} y, 1 - b + i\log^{10} y, 1 + i\log^{10} y, 1 + i\infty]$, and $b = \delta/\log\log y$ for some small δ. We require Ω_w to have properties similar to ζ^{-1} so that the methods of [C1] can be directly applied to bound the contribution of the integral above. Specifically, $\Omega_w(s)$ has a simple zero at $s = 1$ and $\Omega_w \ll \log(2 + |t|)$ on C. It is straightforward to check that $\Omega_w(s) = \Psi_w(s)/D(s)$ where

$$\Psi_0(s) = \prod_p \left(1 + \frac{(p^{4s} - p^{3s+1} - p^{3s} + p^{2s+1}) f^2(p)}{(p^{2s+1} + p^{2s})(p^{3s} + p^{2s} - p^{2s} f^2(p) + p^s f^2(p) - p^s - 1)} \right).$$

The product is absolutely convergent for $\sigma > \frac{1}{2}$, and $\Psi_0(1) = 1$. Thus $\Omega_w(s)$ inherits the desired properties from those of $D(s)$ mentioned in Lemma 8. Since $X(d,e,v,w)$ is part of the inner sum of $S(v,u,w)$, bounding the contribution of the integral is not trivial, but by virtue of the properties of $D(s)$ we obtain an acceptable error term, just as in [C1], pages 55-56. The main term comes from the residue of the pole at $s = 1+v$. Since $\Omega_w(1+v) \sim v\Omega'_w(1) \sim v/D_{-1}$ there are two main terms from the residue:

$$X(d,e,v,w) \sim \sum_{m=1}^{\infty} \frac{a_m \log^m y}{\log^m B}\left(\Omega_w(1+v)F_w(d,e,1+v) + \frac{1}{m\log y}\Omega'_w(1)F_w(d,e,1+v)\right)$$

$$\sim \frac{1}{D_{-1}}F_w(d,e,1+v)\left(v\,h\left(\frac{\log y}{\log B}\right) + \frac{1}{\log B}\,h'\left(\frac{\log y}{\log B}\right)\right).$$

We now have

$$S(v,u,w) = \frac{1}{D_{-1}^2}\sum_{d\le B}\frac{1}{d^{1+v+u-w}}\sum_{e|d}\frac{\mu(e)}{e^{1+w}}F_w(d,e,1+v)F_w(d,e,1+u)$$

$$\times\left(vu\,h\left(\frac{\log y}{\log B}\right)^2 + \frac{v+u}{\log B}\,hh'\left(\frac{\log y}{\log B}\right) + \frac{1}{\log^2 B}h'\left(\frac{\log y}{\log B}\right)^2\right).$$

To finish the calculation we must evaluate expressions of the form

$$\mathcal{S}_w^m(v,u) = \sum_{d\le B}\frac{1}{d^{1+v+u-w}}\sum_{e|d}\frac{\mu(e)}{e^{1+w}}F_w(d,e,1+v)F_w(d,e,1+u)\log^m(B/d).$$

First consider the case $m \ge 1$. If

$$H(v,u,w,s) = \sum_{d=1}^{\infty}\frac{1}{d^s}\sum_{e|d}\frac{\mu(e)}{e^{1+w}}F_w(d,e,1+v)F_w(d,e,1+u)$$

then for $c > 1 + |v| + |u| + |w|$ we have

$$\mathcal{S}_w^m(v,u) = \frac{m!}{2\pi i}\int_{(c)}\frac{H(v,u,w,s)B^{s-1-v-u+w}}{(s-1-v-u+w)^{m+1}}\,ds. \tag{5.1}$$

Again the main terms come from poles of the integrand. Much as before, $H(v,u,w,s) = D(s)\Psi(v,u,w,s)$ where $\Psi(v,u,w,1) \sim 1$ and Ψ has an Euler product representation converging absolutely for $\sigma > \frac{1}{2}$. Thus by Lemma 8, $H(s)$ has a simple pole at $s = 1$ with residue D_{-1}, and $H(s) \ll (1+|t|)^{\frac{1}{2}+\epsilon}$ for $\sigma > \frac{3}{4}$. We move the path of integration to the $\frac{3}{4}$-line. The resulting integral is easily seen to be bounded by $B^{-\frac{1}{4}}$, so the main term comes from the residues of the poles. In the case $w = v + u$ the integrand has a pole of order $m+2$, whence when $m \geq 1$ we have

$$\mathcal{S}^m_{v+u}(v,u) \sim \frac{D_{-1}}{m+1} \log^{m+1} B. \tag{5.2}$$

For $m = 0$ and $\epsilon > 0$ write

$$\mathcal{S}^0_{v+u}(v,u) = \frac{1}{2\pi i} \int_{1+\epsilon-iY}^{1+\epsilon+iY} \frac{H(v,u,v+u,s)B^{s-1}}{s-1}\, ds + O\left(\frac{B^{1+\epsilon}}{Y}\right).$$

The same analysis as above gives $\mathcal{S}^0_{v+u}(v,u) \sim D_{-1} \log B + O(B^{1+\epsilon}Y^{-1} + B^{\frac{3}{4}}Y^{\frac{1}{2}})$. By choosing $Y = B^{\frac{1}{4}}$ we see that (5.2) holds for $m = 0$ also. This is all we need:

$$S(v,u,v+u) \sim \frac{1}{D_{-1}} \left(vu \log B \int_0^1 h(x)^2\, dx + (v+u) \int_0^1 hh'(x)\, dx + \frac{1}{\log B} \int_0^1 h'(x)^2\, dx \right)$$

$$= \frac{1}{D_{-1} \log B} \frac{d}{d\alpha} \frac{d}{d\beta} B^{\alpha v + \beta u} \int_0^1 h(x+\alpha)\, h(x+\beta)\, dx \bigg|_{\alpha=\beta=0}$$

In the case of arbitrary w the above computation, slightly complicated by the fact that the integrand in (5.1) has two poles, will give an asymptotic expression. The estimate

$$\mathcal{S}^m_w(v,u) \ll \log^{m+1} B$$

is easily seen to hold and is sufficient for our purposes. This completes the proof of Proposition 2.

CHAPTER VI

ERROR TERMS

In this chapter we continue to use the notation of Chapter 4. We prove

Proposition 3. *Suppose* $W_{lY} \in C_0^\infty(\mathbb{R})$ *with* $W_{lY}(\xi) \ll \left((lY)^j + Z^j\right)\xi^{-j}$, *and* $|LY| + |Z| \ll X^\epsilon$ *for all* $\epsilon > 0$. *If* $L \ll Q \ll X$ *and* $\mathbf{b}_{\alpha,\beta}$ *is a sequence of complex numbers with* $b_{\alpha,\beta} \ll (\alpha\beta)^\epsilon$ *then*

$$\sum_{\alpha\beta\sim Q} b_{\alpha,\beta}\alpha^{\frac{k-1}{2}} \sum_{l\sim L}\sum_{n} \frac{f(n)\, f\left(\frac{n\alpha+l}{\beta}\right)}{n^{\frac{1-k}{2}}\,(n\alpha+l)^{\frac{k+1}{2}}} W_{lY}\left(\frac{n\alpha+l}{X}\right)$$

$$\ll X^{-\frac{1}{2}+\epsilon}L^{\frac{1}{2}}\left(Q^{\frac{3}{2}} + QL^{\frac{1}{2}} + X^{\frac{1}{4}}L^{\frac{1}{4}}Q^{\frac{1}{2}}\right).$$

Proof. Write $E(X,Y)$ for the expression in the proposition. By Mellin's formula

$$W_{lY}\left(\frac{n\alpha+l}{X}\right) = \frac{1}{2\pi i}\int_{(c)} \mathcal{W}_{lY}(s)\left(\frac{n\alpha+l}{X}\right)^{-s} ds \tag{6.1}$$

for $c > 1$, where

$$\mathcal{W}_{lY}(s) = \int_0^\infty W_{lY}(\xi)\xi^{s-1}d\xi.$$

Repeated integration by parts and the conditions on W_{lY} give that there exists a sequence $\mathbf{w}_{lY}$ such that for all $j \geq 0$

$$\mathcal{W}_{lY}(s) \ll_j w_{lY}^j s^{-j} \tag{6.2}$$

and $w_{lY} \ll X^\epsilon$.

By (6.1), (4.6), and (4.5) we have, for $c > 1$,

$$E(X,Y) = \sum_{\alpha\beta\sim Q} b_{\alpha,\beta}\alpha^{\frac{k-1}{2}} \sum_{l\sim L} \frac{1}{2\pi i}\int_{(c)} X^s \mathcal{W}_{lY}(s) \sum_n \frac{f(n)\, f\left(\frac{n\alpha+l}{\beta}\right)}{n^{\frac{1-k}{2}}\,(n\alpha+l)^{\frac{k+1}{2}+s}} ds$$

$$\ll X^{-1} \sum_{\alpha\beta\sim Q} \frac{b_{\alpha,\beta}}{\sqrt{\alpha\beta}} \left(\frac{\alpha}{\beta}\right)^{-\frac{k}{2}} \sum_{l\sim L}\int_{(c)} \frac{X^s\,(4\pi i)^{k-1+s}}{\Gamma(k-1+s)} \mathcal{W}_{lY}(s-1)\langle U_{\infty,l}(\cdot,s), \overline{F}G\rangle\, ds$$

$$\ll X^{-1} \sum_{\alpha\beta\sim Q} \frac{b_{\alpha,\beta}}{\sqrt{\alpha\beta}} \left(\frac{\alpha}{\beta}\right)^{-\frac{k}{2}} \int_{(c)} \frac{X^s}{\Gamma(s)\,\Gamma(k-1+s)}$$

$$\left\{\sum_{j\geq 0}^{(\beta,\alpha)} \Gamma(s-s_j)\,\Gamma(s+s_j-1)\langle e_j,\overline{F}G\rangle \sum_{l\sim L} \mathcal{W}_{lY}(s-1)\, l^{\frac{1}{2}-s}\rho_{j\infty}(l)\right.$$
$$\left.+\sum_{\eta\in\Lambda}\int_{(\frac{1}{2})} \Gamma(s-\xi)\,\Gamma(s+\xi-1)\langle E_\eta(\cdot,\xi),\overline{F}G\rangle \sum_{l\sim L}\mathcal{W}_{lY}(s-1)\, l^{\frac{1}{2}-s}\alpha_{l\eta\infty}(1-\xi)\, d\xi\right\} ds$$

The $\ll$ signs above just reflects our omission of various absolute constants. We move the path of integration left to the $(\frac{1}{2}+\epsilon)$-line, $\epsilon > 0$, encountering poles of the Γ-function at the exceptional eigenvalues of $\Gamma_0(\beta,\alpha)$. By the residue theorem

$$E(X,Y) \ll X^{-1} \sum_{\alpha\beta\sim Q} \frac{b_{\alpha,\beta}}{\sqrt{\alpha\beta}} \left(\frac{\alpha}{\beta}\right)^{-\frac{k}{2}} \left(Res(\alpha,\beta) + C(\alpha,\beta)\right)$$

where

$$Res(\alpha,\beta) = X^{\frac{1}{2}} \sum_{i\kappa_j>\epsilon}^{(\beta,\alpha)} \frac{X^{s_j-\frac{1}{2}}\Gamma(s_j-\frac{1}{2})}{\Gamma(k-1+s_j)} \langle e_j,\overline{F}G\rangle \sum_{l\sim L}\mathcal{W}_{lY}(s_j-1)\, l^{\frac{1}{2}-s_j}\rho_{j\infty}(l)$$

and

$$C(\alpha,\beta) = \int_{(\frac{1}{2}+\epsilon)} \frac{X^s}{\Gamma(s)\,\Gamma(k-1+s)} \left\{\sum_{j\geq 0}^{(\beta,\alpha)} \Gamma(s-s_j)\,\Gamma(s+s_j-1)\langle e_j,\overline{F}G\rangle \sum_{l\sim L}\mathcal{W}_{lY}(s-1)\, l^{\frac{1}{2}-s}\rho_{j\infty}(l)\right.$$
$$\left.+\frac{1}{4\pi i}\sum_{\eta\in\Lambda}\int_{(\frac{1}{2})} \Gamma(s-\xi)\,\Gamma(s+\xi-1)\langle E_\eta(\cdot,\xi),\overline{F}G\rangle \sum_{l\sim L}\mathcal{W}_{lY}(s-1)\, l^{\frac{1}{2}-s}\alpha_{l\eta\infty}(1-\xi)\, d\xi\right\} ds$$

$$= \int_{(\frac{1}{2}+\epsilon)} \frac{X^s}{\Gamma(s)\,\Gamma(k-1+s)} \left(E_r(\alpha,\beta,s) + E_c(\alpha,\beta,s)\right) ds,$$

say.

By the Cauchy-Schwarz inequality we have

$$X^{-1} \sum_{\alpha\beta\sim Q} \frac{b_{\alpha,\beta}}{\sqrt{\alpha\beta}} \left(\frac{\alpha}{\beta}\right)^{-\frac{k}{2}} Res(\alpha,\beta) \ll X^{-\frac{1}{2}} \left(V(X,Y)\, W(X,Y)\right)^{\frac{1}{2}}$$

where

$$V(X,Y) = \sum_{\alpha\beta\sim Q} \frac{\alpha^{-k-1}}{\beta^{-k+1}} \sum_{i\kappa_j>\epsilon}^{(\beta,\alpha)} Q^{4i\kappa_j} |\langle e_j, \overline{F}G\rangle|^2$$

and

$$W(X,Y) = \sum_{\alpha\beta\sim Q} \sum_{i\kappa_j>\epsilon}^{(\beta,\alpha)} \left(\frac{X}{Q^2}\right)^{2i\kappa_j} |\sum_{l\sim L} \mathcal{W}_{lY}(s_j-1) l^{-\kappa_j} \rho_{j\infty}(l)|^2.$$

By Lemma 4, (4.1) and (4.2), and Lemma 7 we have

$$V(X,Y) \ll \sum_{\alpha\beta\sim Q} (\alpha,\beta)(\alpha\beta)^{\epsilon} \sum_{i\kappa_j>0}^{(\beta,\alpha)} Q^{4i\kappa_j}$$

$$\ll Q^{2+\epsilon}.$$

And by (6.2) and Lemma 5

$$W(X,Y) \ll \sum_{\alpha\beta\sim Q} \sum_{i\kappa_j>0}^{(\beta,\alpha)} \left(\frac{X}{LQ^2}\right)^{2i\kappa_j} |\sum_{l\sim L} w_{lY} \rho_{j\infty}(l)|^2$$

$$\ll (QLX)^{\epsilon}(Q + L + X^{\frac{1}{2}} L^{\frac{1}{2}} Q^{-1}) ||\mathbf{w}_{LY}||_2^2$$

$$\ll X^{\epsilon}(Q + L + X^{\frac{1}{2}} L^{\frac{1}{2}} Q^{-1}) L.$$

Thus

$$X^{-1} \sum_{\alpha\beta\sim Q} \frac{b_{\alpha,\beta}}{\sqrt{\alpha\beta}} \left(\frac{\alpha}{\beta}\right)^{-\frac{k}{2}} Res(\alpha,\beta) \ll X^{-\frac{1}{2}+\epsilon} L^{\frac{1}{2}} \left(Q^{\frac{3}{2}} + QL^{\frac{1}{2}} + X^{\frac{1}{4}} L^{\frac{1}{4}} Q^{\frac{1}{2}}\right).$$

Next we bound the contribution of $C(\alpha,\beta)$. Suppose $s = \frac{1}{2} + \epsilon + it$, $\epsilon > 0$. We first have by Cauchy's inequality, (6.2), Lemmas 4 and 6, Stirling's formula, and the Selberg

Trace formula

$$E_r(\alpha,\beta,s) \ll \left(\sum_{j\geq 0}^{(\beta,\alpha)} |\Gamma(s-s_j)\,\Gamma(s+s_j-1)\langle e_j, \overline{F}G\rangle|^2 \kappa_j^4 \cosh \pi\kappa_j\right)^{\frac{1}{2}}$$

$$\times \left(\sum_{j\geq 0}^{(\beta,\alpha)} \frac{1}{\kappa_j^4 \cosh \pi\kappa_j} |\sum_{l\sim L} \mathcal{W}_{lY}(s-1) l^{\frac{1}{2}-s} \rho_{j\infty}(l)|^2\right)^{\frac{1}{2}}$$

$$\ll \left(\left(\frac{\alpha}{\beta}\right)^k (\alpha\beta)^{1+\epsilon}(\alpha,\beta) \sum_{j\geq 0}^{(\beta,\alpha)} \kappa_j^{k+4} |\Gamma(\epsilon+i(t-\kappa_j))\Gamma(\epsilon+i(t+\kappa_j))|^2\right)^{\frac{1}{2}}$$

$$\times \left(1+\frac{L^{1+\epsilon}}{\alpha\beta}\right)^{\frac{1}{2}} \|\mathcal{W}_{lY}(s-1)l^{\epsilon}\|_2$$

$$\ll \left(\frac{\alpha}{\beta}\right)^{\frac{k}{2}} (\alpha\beta)^{1+\epsilon}(\alpha,\beta)^{\frac{1}{2}} e^{-\pi|t|} ||w_{LY}^{k+4}||_2$$

$$\ll X^{\epsilon} L^{\frac{1}{2}} \left(\frac{\alpha}{\beta}\right)^{\frac{k}{2}} (\alpha\beta)^{1+\epsilon}(\alpha,\beta)^{\frac{1}{2}} e^{-\pi|t|}.$$

In the 2nd step we used the Selberg trace formula in the form $\sum_{\lambda_j \leq M}^{(\beta,\alpha)} 1 \ll (\alpha\beta)^{1+\epsilon} M$.

The same analysis gives

$$E_c(\alpha,\beta,s) \ll X^{\epsilon} L^{\frac{1}{2}} \left(\frac{\alpha}{\beta}\right)^{\frac{k}{2}} (\alpha\beta)^{\frac{1}{2}+\epsilon}(\alpha,\beta)^{\frac{1}{2}} e^{-\pi|t|}.$$

Thus

$$\sum_{\alpha\beta\sim Q} \frac{b_{\alpha,\beta}}{\sqrt{\alpha\beta}} \left(\frac{\alpha}{\beta}\right)^{-\frac{k}{2}} C(\alpha,\beta) \ll L^{\frac{1}{2}} \sum_{\alpha\beta\sim Q} b_{\alpha,\beta}(\alpha,\beta)^{\frac{1}{2}}(\alpha\beta)^{\frac{1}{2}+\epsilon} \int_{(\frac{1}{2}+\epsilon)} \frac{X^s e^{-\pi|t|}}{\Gamma(s)\Gamma(k-1+s)}\, ds$$

$$\ll X^{\frac{1}{2}+\epsilon} Q^{\frac{3}{2}} L^{\frac{1}{2}}$$

This completes the proof of Proposition 3.

CHAPTER VII

THE END OF THE PROOF

We use Proposition 2 to obtain asymptotic expressions for M_1 and M_2, and use Proposition 3 to bound R_1 and R_2, and so complete the proof of Proposition 1.

In the notation of Lemma 9,

$$M_1(T;u,v) = T\sum_{\alpha,\beta} \frac{b_\alpha \overline{b}_\beta}{\alpha^{1+v}\beta^{-v}} \sum_n \frac{f(n)f(\frac{n\alpha}{\beta})}{n^{1+u+v}}\, G\Big(0,0,\frac{2\pi n\alpha}{Tx}\Big)$$

and

$$M_2(T;u,v) = T\left(\frac{2\pi}{T}\right)^{2(u+v)} \sum_{\alpha,\beta} \frac{b_\alpha \overline{b}_\beta}{\alpha^{1-v}\beta^{v}} \sum_n \frac{f(n)f\big(\frac{n\alpha}{\beta}\big)}{n^{1-u-v}}\, G\Big(-2(u+v),0,\frac{2\pi nx}{T\beta}\Big).$$

Putting the above together with Lemma 9 and the calculation preceeding Lemma 8, and writing $w = u+v$, we get, for $c > 1+|w|$,

$$M_1(T;u,v) = \frac{2\pi}{x}\left(\frac{2\pi}{Tx}\right)^{w} \sum_{\alpha,\beta} \frac{b_\alpha \overline{b}_\beta}{\alpha^{-u}\beta^{-v}} \frac{1}{2\pi i}\int_{(c)} \left(\frac{Tx(\alpha,\beta)}{2\pi\alpha\beta}\right)^{s}$$
$$\times\, M(0,s-1-w)M_U(s-1-w)\, P\Big(\frac{\alpha}{(\alpha,\beta)},s\Big)\, P\Big(\frac{\beta}{(\alpha,\beta)},s\Big)\, D(s)\, ds$$

and

$$M_2(T;u,v) = 2\pi x\left(\frac{2\pi}{Tx}\right)^{w} \sum_{\alpha,\beta} \frac{b_\alpha \overline{b}_\beta}{\alpha^{1-v}\beta^{1-u}} \frac{1}{2\pi i}\int_{(c)} \left(\frac{T(\alpha,\beta)}{2\pi x}\right)^{s}$$
$$\times\, M(0,s-1+w)M_U(s-1-w)\, P\Big(\frac{\alpha}{(\alpha,\beta)},s\Big)\, P\Big(\frac{\beta}{(\alpha,\beta)},s\Big)\, D(s)\, ds$$

We move the path of integration to the $(\frac{1}{2}+\epsilon)$-line, $\epsilon > 0$. By Lemmas 8 and 9 the resulting integral is easily seen to be bounded by $T^{\frac{1}{2}+\epsilon}B^{\frac{1}{2}}$, so that the main term comes from the residues of poles of the integrand. Recall that $M(0,s)$ has a simple pole at $s=0$ with residue 1 and $D(s)$ has a simple pole at $s=1$ with residue D_{-1}. Thus,

$$\begin{aligned} M_1(T;u,v) \sim \frac{2\pi}{x}\left(\frac{2\pi}{Tx}\right)^w \sum_{\alpha,\beta} \frac{b_\alpha \bar{b}_\beta}{\alpha^{-u}\beta^{-v}} \\ \times \left\{ \frac{Tx(\alpha,\beta)}{2\pi\alpha\beta} M(0,-w) M_U(-w)\, P\Big(\frac{\alpha}{(\alpha,\beta)},1\Big)\, P\Big(\frac{\beta}{(\alpha,\beta)},1\Big)\, D_{-1} \right. \\ \left. + \left(\frac{Tx(\alpha,\beta)}{2\pi\alpha\beta}\right)^{1+w} M_U(0)\, P\Big(\frac{\alpha}{(\alpha,\beta)},1+w\Big)\, P\Big(\frac{\beta}{(\alpha,\beta)},1+w\Big)\, D(1+w) \right\} \end{aligned}$$

and

$$\begin{aligned} M_2(T;u,v) \sim 2\pi x\left(\frac{2\pi}{Tx}\right)^w \sum_{\alpha,\beta} \frac{b_\alpha \bar{b}_\beta}{\alpha^{1-v}\beta^{1-u}} \\ \times \left\{ \frac{T(\alpha,\beta)}{2\pi x} M(0,w) M_U(-w)\, P\Big(\frac{\alpha}{(\alpha,\beta)},1\Big)\, P\Big(\frac{\beta}{(\alpha,\beta)},1\Big)\, D_{-1} \right. \\ \left. + \left(\frac{T(\alpha,\beta)}{2\pi x}\right)^{1-w} M_U(-2w)\, P\Big(\frac{\alpha}{(\alpha,\beta)},1-w\Big)\, P\Big(\frac{\beta}{(\alpha,\beta)},1-w\Big)\, D(1-w) \right\}. \end{aligned}$$

Now

$$M_U(w) \sim M_U(0) = 1 + O(1/U),$$

and

$$D(1+w) \sim D_{-1}/w.$$

So in the notation of Proposition 2,

$$\begin{aligned} (M_1+M_2)(T;u,v) \sim \frac{D_{-1}T}{u+v}\left(S(v,u,u+v) - T^{-2(u+v)}S(-v,-u,-v-u)\right) \\ + T\,(Tx)^{-u-v}\, S(-u,-v,0)\big(M(0,-w)+M(0,w)\big) + O(T/U) \end{aligned}$$

$$\sim \frac{D_{-1}T}{u+v}\left(S(v,u,u+v) - T^{-2(u+v)}S(-v,-u,-v-u)\right) + O(T/U). \tag{7.1}$$

It is easy to see from Proposition 2 that

$$S(v,u,u+v)\sim S(-v,-u,-u-v)+\frac{u+v}{D_{-1}}.$$

Thus, if $U\gg\log T$,

$$\big(M_1+M_2\big)(T;u,v)\sim T\left(1+\frac{1-T^{-2(u+v)}}{u+v}D_{-1}S(-v,-u,-u-v)\right)$$

$$\sim T\left(1+\frac{\log T}{\log B}\int_0^2 T^{-y(u+v)}\,dy\,\frac{d}{d\alpha}\frac{d}{d\beta}B^{-\alpha u-\beta v}\int_0^1 h(x+\alpha)h(x+\beta)\,dx\right)\Bigg|_{\substack{\alpha=0\\ \beta=0}}$$

which is the main term in Proposition 1.

Finally, we use Proposition 3 to bound the error terms. In the notation of Lemma 9,

$$R_1(T;u,v,x)\ll T\sum_{\alpha,\beta}^{B}b_\alpha\bar{b}_\beta\alpha^{\frac{k-1}{2}}\sum_{1\le l<xT^\epsilon}\sum_n\frac{f(n)\,f\left(\frac{n\alpha+l}{\beta}\right)}{n^{\frac{1-k}{2}}\,(n\alpha+l)^{\frac{k+1}{2}}}\,G\left(0,\frac{l}{x},\frac{2\pi(n\alpha+l)}{x}\right)$$

and

$$R_2(T;u,v,x)\ll T\sum_{\alpha,\beta}^{B}b_\alpha\bar{b}_\beta\alpha^{\frac{k-1}{2}}\sum_{1\le l<\alpha\beta T^\epsilon/x}\sum_n\frac{f(n)\,f\left(\frac{n\alpha+l}{\beta}\right)}{n^{\frac{1-k}{2}}\,(n\alpha+l)^{\frac{k+1}{2}}}$$
$$\times\, G\left(-2(u+v),\frac{lx}{\alpha\beta},\frac{2\pi(n\alpha+l)x}{\alpha\beta}\right).$$

We write R_1 as a sum of terms, where in each term the sums are $l\sim 2^{-m}xT^\epsilon$ and $\alpha\beta\sim 2^{-n}B^2$. By partial integration,

$$G(*,l/x,\xi)\ll\left((l/x)^j+U^j\right)\xi^{-j},$$

so that if $U=\log T$ then Proposition 3 is applicable with $X=Tx$, $Y=x^{-1}$, $L=2^{-m}xT^\epsilon$, and $Q=2^{-n}B^2$ with

$$W_{lY}(\xi)=G(*,l/x,2\pi\xi).$$

We obtain

$$R_1(T;u,v,x)\ll T^{\frac{1}{2}+\epsilon}\big(B^3+T^{\frac{1}{4}}x^{\frac{1}{2}}B\big).$$

In almost the same way

$$R_2(T;u,v,x) \ll T^{\frac{1}{2}+\epsilon}\big(B^3 + T^{\frac{1}{4}}B^2x^{-\frac{1}{2}}\big).$$

The best choice is $x = B$. This finishes the proof of Proposition 1.

Since Ψ_U could be chosen to be either a majorant or minorant of the characteristic function of $[1,2]$, the formula in Proposition 1 holds for $\int_T^{2T} |L|^2|B|^2$, whence Theorem 1 follows by summing over intervals of the form $[2^{-m}T, 2^{-m+1}T]$. We observe that on the Selberg eigenvalue conjecture the terms involving $T^{\frac{1}{4}}$ in the bounds for R_1 and R_2 can be eliminated, but this does not result in an improvement in Theorem 1. It is possible that the first estimate in Lemma 4 can be reduced by a factor of $\mu(\Gamma)$, if so, Theorem 1 could be improved to $0 < \theta < \frac{1}{4}$.

To prove Theorem 2 we observe that in the proof just given only the application of Proposition 2 relied on information about b_α other than $b_\alpha \ll \alpha^\epsilon$. Thus, in the notation of Theorem 2, we have by (7.1)

$$I(T) \sim \frac{D_{-1}T}{u+v}\big(S(v,u,u+v) - T^{-(u+v)}S(-v,-u,-v-u)\big)$$

for $\theta < \frac{1}{6}$. Set $u = v = x$, take the limit as $x \to 0$, and note that $\mathcal{F}(n) = P(n,1)$ to finish the proof of Theorem 2.

CHAPTER VIII

PROOFS OF THE COROLLARIES

We evaluate (1.5) for specific values of the parameters to obtain Corollary 1a. As in [C3] the function $h(x)$ can be chosen optimally by the calculus of variations. The treatment there transfers to our situation almost without modification, so we do not repeat it. The result is that with $Q(x) = (1 + \lambda x)(1 + x)^n$, $w(y) = e^{Ry}Q(y)$, and $\alpha = \sqrt{C/A}$ where

$$A = \int_0^2 e^{2Ry} Q^2(y)\, dy$$

and

$$C = \theta^2 \int_0^2 e^{2Ry} \big(RQ(y) + Q'(y)\big)^2 dy$$

then

$$\kappa_1^{(n)} \geq 1 - \frac{1}{2R} \log \left(\frac{1}{2}\big(1 + w^2(2)\big) + \frac{A}{\theta} \alpha \coth \alpha \right). \tag{8.1}$$

Corollary 1a now follows by letting $\theta = \frac{1}{6} - \epsilon$, $\epsilon \to 0^+$, and making the choices in Table I.

TABLE I

CHOICES USED TO OBTAIN COROLLARY 1a

n	1	2	3	4	5
λ	0.53	0.52	0.51	0.51	0.50
R	1.12	1.03	0.99	0.95	0.92
$\kappa_1^{(n)}$	.3261	.5829	.7301	.8203	.8791

This method also gives a lower bound of $-.182$ for $\kappa_1^{(0)}$. This is vacuous, but it indicates how far we are from an actual result in this case. It is interesting to note that the pair correlation method gives a lower bound of $-.166$ for $\kappa_1^{(0)}$. The last statement in Corollary 1a is proven exactly as in [C1], page 73.

To obtain Corollary 1b we note that if ρ is a zero of ξ_F of order $m \geq n+2$ then ρ is a zero of order $m-n \geq 2m/(n+2) \geq 2$ for $\xi_F^{(n)}$. Thus,

$$N_1^{(n)}(T) \leq N(T) - \frac{2}{n+2}\big(N(T) - M_{\leq n+1}(T)\big),$$

which gives

$$M_{\leq n}(T) \geq \left(\frac{\kappa_1^{(n-1)}(n+1) - n + 1}{2}\right) N(T). \tag{8.2}$$

The bounds for $\kappa_1^{(j)}$ in Corollary 1a now give the result.

Next we obtain Corollary 1c. As a consequence of formula (A6) in the Appendix we have

$$N^d(T) \geq \left(\frac{\kappa_1^{(0)}}{2} + 2^{-j}\kappa_1^{(j)} + \sum_{n=1}^{j-1} 2^{-n-1}\kappa_1^{(n)}\right) N(T),$$

for T sufficiently large. Put $j = 5$ and use the bounds for $\kappa_1^{(n)}$ in Corollary 1a to obtain Corollary 1c.

Corollary 2 is slightly simpler. By Littlewood's lemma and the arithmetic-geometric mean inequality,

$$\begin{aligned}
\sum_{\substack{\beta > \frac{1}{2} \\ 0<\gamma<T \\ L_F(\beta+i\gamma)=0}} (\beta - \tfrac{1}{2}) &\leq \sum_{\substack{\beta > \frac{1}{2} \\ 0<\gamma<T \\ BL_F(\beta+i\gamma)=0}} (\beta - \tfrac{1}{2}) \\
&= \frac{1}{2\pi} \int_1^T \log |BL_F(\tfrac{1}{2} + it)|\, dt + O(\log T) \\
&\leq \frac{T}{4\pi} \log\left(\frac{1}{T}\int_1^T |BL_F(\tfrac{1}{2} + it)|^2\, dt\right) + O(\log T) \\
&\sim \frac{T}{4\pi} \log\left(1 + 2\frac{\log T}{\log B}\right).
\end{aligned}$$

Put $B = T^{\frac{1}{6}-\epsilon}$, $\epsilon \to 0^+$, to get the result.

We have not attempted to give an exhaustive list of the corollaries which can be obtained from our mean value theorem. Except for differences in numerical constants, results previously obtained for the Riemann ζ-function from the corresponding mean value theorem transfer over with little difficulty. For example, a zero-density estimate similar to that given by Jutila [J], but with a weaker exponent, can be obtained. It is also possible to use the methods in [LM] to rephrase Corollary 1a in terms of the proportion of zeros of $L_F^{(j)}$ to the left of the $\frac{1}{2}$-line.

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APPENDIX

THE COMBINATORICS OF DISTINCT ZEROS

In this appendix we discuss the combinatorics involved in extracting information from data on the zeros of the derivatives of a function. The discussion is specialized to the Riemann ξ-function, but most of our discussion is general. This Appendix is self-contained and only formula (A6) is used in the main body of this Thesis.

The Riemann ξ-function is defined by $\xi(s) = H(s)\zeta(s)$ where $H(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})$. The zeros of $\xi(s)$ and its derivatives are all located in the critical strip $0 < \sigma < 1$ and as $H(s)$ is regular and nonzero for $\sigma > 0$ the nontrivial zeros of $\zeta(s)$ exactly correspond to those of $\xi(s)$. Let $\rho^{(j)} = \beta + i\gamma$ denote a zero of the j^{th} derivative $\xi^{(j)}(s)$, and denote its multiplicity by $m(\gamma)$. Define

$$N^{(j)}(T) = \sum_{\rho^{(j)}=\beta+i\gamma} 1 \qquad N(T) = N^{(0)}(T)$$

$$N_s^{(j)}(T) = \sum_{\substack{\rho^{(j)}=\beta+i\gamma \\ m(\gamma)=1}} 1 \qquad N_{s,\frac{1}{2}}^{(j)}(T) = \sum_{\substack{\rho^{(j)}=\frac{1}{2}+i\gamma \\ m(\gamma)=1}} 1$$

$$M_r(T) = \sum_{\substack{\rho^{(0)}=\beta+i\gamma \\ m(\gamma)=r}} 1 \qquad M_{\le r}(T) = \sum_{\substack{\rho^{(0)}=\beta+i\gamma \\ m(\gamma)\le r}} 1$$

where all sums are over $0 < \gamma < T$, and zeros are counted according to their multiplicity. It is well known that $N^{(j)}(T) \sim \frac{1}{2\pi}T\log T$. Let

$$\alpha_j = \liminf_{T\to\infty} \frac{N_{s,\frac{1}{2}}^{(j)}(T)}{N^{(j)}(T)}. \qquad \beta_j = \liminf_{T\to\infty} \frac{N_s^{(j)}(T)}{N^{(j)}(T)}.$$

Thus, β_j is the proportion of zeros of $\xi^{(j)}(s)$ which are simple, and α_j is the proportion which are simple and on the critical line. The best currently available bounds are $\alpha_0 > 0.40219$,

$\alpha_1 > 0.79874$, $\alpha_2 > 0.93469$, $\alpha_3 > 0.9673$, $\alpha_4 > 0.98006$, and $\alpha_5 > 0.9863$. These bounds were obtained by combining Theorem 2 of [C3] with the methods of [C2]. Trivially, $\beta_j \geq \alpha_j$.

Let $N_d(T)$ be the number of distinct zeros of $\xi(s)$ in the region $0 < t < T$. That is,

$$N_d(T) = \sum_{n=1}^{\infty} \frac{M_n(T)}{n}.$$

We will use the bounds on β_j to obtain the following

Theorem. *For T sufficiently large*

$$N_d(T) > k\, N(T),$$

with $k = 0.63952\ldots$. Furthermore, given the bounds on β_j, this result is best possible.

We present two methods for determining lower bounds for $N_d(T)$. These methods employ combinatorial arguments involving the β_j. Our result is best possible in the sense that any improvement in the value of k in the Theorem would implicitly require an improvement in the lower bound for some β_j. We also note that the added information that α_j detects zeros on the critical line is not of any use in improving our result. To save on notation we adopt the convention that all inequalities contain an implicit $o(N(T))$ as $T \to \infty$. For example, $N_s^{(j)}(T) \geq \beta_j N(T)$ means that $N_s^{(j)}(T) \geq (\beta_j + o(1))\, N(T)$ as $T \to \infty$.

Our first method starts with the following inequality of Conrey, Ghosh, and Gonek [CGG]. A simple counting argument yields

$$(A1) \qquad N_d(T) \geq \sum_{r=1}^{R} \frac{M_{\leq r}(T)}{r(r+1)} + \frac{M_{\leq R+1}(T)}{R+1}.$$

To obtain lower bounds for $M_{\leq r}(T)$ we note that if ρ is a zero of $\xi(s)$ of order $m \geq n+2$ then ρ is a zero of order $m - n \geq 2m/(n+2) \geq 2$ for $\xi^{(n)}(s)$. Thus,

$$N_s^{(n)}(T) \leq N(T) - \frac{2}{n+2}\big(N(T) - M_{\leq n+1}(T)\big),$$

which gives

$$(A2) \qquad M_{\leq n}(T) \geq \left(\frac{\beta_{n-1}(n+1) - n + 1}{2} \right) N(T).$$

Using the bounds for α_j we find: $M_{\leq 1}(T) > 0.40219N(T)$, $M_{\leq 2}(T) > 0.69812N(T)$, $M_{\leq 3}(T) > 0.86938N(T)$, $M_{\leq 4}(T) > 0.91825N(T)$, $M_{\leq 5}(T) > 0.94019N(T)$, and finally $M_{\leq 6}(T) > 0.9520N(T)$. Inserting these bounds into inequality (A1) with $R = 5$ gives $N_d(T) > 0.62583N(T)$. We note that our lower bounds for $M_{\leq n}(T)$ are best possible in the sense that, for each n separately, equality could hold in (A2). However, it need not hold that (A2) is simultaneously sharp for all n, and this possibility imparts some weakness to the result. A lower bound for $N_d(T)$ was calculated in [CGG] in a spirit similar to the above computation, but it was mistakenly assumed that $M_{\leq n}(T) \geq \beta_{n-1}N(T)$, rendering their bound invalid.

Our second method eliminates the loss inherent in the first method. It is easy to show that

$$(A3) \qquad N_s^{(n)}(T) \leq \sum_{j=1}^{n+1} M_j(T) + n \sum_{j=n+2}^{\infty} \frac{M_j(T)}{j}.$$

Therefore

$$(A4) \qquad N_s^{(n)}(T) \leq nN_d(T) + n \sum_{j=1}^{n+1} \left(\frac{1}{n} - \frac{1}{j} \right) M_j(T).$$

Let I_n denote the inequality (A4). Then, in the obvious notation, a straightforward calculation finds that the inequality

$$I_J \ + \ \sum_{n=1}^{J-1} 2^{J-n-1} I_n$$

is equivalent to

$$(A5) \qquad (2^J - 1)N_d(T) + \sum_{n=1}^{J+1} \frac{M_n(T)}{n} \ \geq \ 2^{J-1} M_1(T) + N_s^{(J)}(T) + \sum_{n=1}^{J-1} 2^{J-n-1} N_s^{(n)}(T).$$

This implies

$$N_d(T) \geq 2^{-J}\left(2^{J-1}N_s^{(0)}(T) + N_s^{(J)}(T) + \sum_{n=1}^{J-1} 2^{J-n-1}N_s^{(n)}(T)\right)$$

$$\geq 2^{-J}\left(2^{J-1}\beta_0 + \beta_J + \sum_{n=1}^{J-1} 2^{J-n-1}\beta_n\right)N(T). \tag{A6}$$

Choose $J = 5$ and use the bounds for α_j to obtain the Theorem.

Finally, we show that our result is best possible. We will show that there exists a function which satisfies the bounds given for β_j and which has the number of distinct zeros as close to the bound given by (A6) as we wish. Suppose we have bounds of the form $\beta_j \geq \delta_j$, for $0 \leq j \leq J$, with $0 \leq \delta_j \leq 1$. Let $K \geq J + 2$ and suppose

$$M_1(T) = \delta_0 N(T),$$

$$M_K(T) = \tfrac{K}{K-J}(1 - \delta_J)N(T),$$

$$M_{J+1}(T) = \frac{J+1}{2}\left(\delta_J - \delta_{J-1} - \frac{1-\delta_J}{K-J}\right)N(T),$$

and for $2 \leq n \leq J$,

$$M_n(T) = \frac{n}{2}\left(\frac{3\delta_{n-1}}{2} - \delta_{n-2} - 2^{n-J-1}\delta_J - \frac{1-\delta_J}{2^{J-n+1}(K-J)} - \sum_{j=n}^{J-1} 2^{n-j-2}\delta_j\right)N(T)$$

and $M_j(T) = 0$ otherwise. Then $\sum_{j=1}^{\infty} M_j(T) = N(T)$ and for $0 \leq n \leq J$ we have

$$\sum_{j=1}^{n+1} M_j(T) + n\sum_{j=n+2}^{\infty} \frac{M_j(T)}{j} = \delta_n N(T), \tag{A7}$$

and

$$\sum_{n=1}^{\infty} \frac{M_n(T)}{n} = 2^{-J}\left(2^{J-1}\delta_0 + \delta_J + \sum_{n=1}^{J-1} 2^{J-n-1}\delta_n\right)N(T) + \frac{(1-\delta_J)2^{-J}}{K-J}N(T). \tag{A8}$$

If the values given for $M_j(T)$ could arise from an actual function then since the left side of (A8) is $N_d(T)$ we can find a function whose number of distinct zeros is as close to the bound given by (A6) as we like by choosing K sufficiently large. And by (A3) and (A7) the function will satisfy our bounds on β_j. The only thing remaining to be checked is that the values given for $M_j(T)$ are nonnegative when K is large. One can easily verify that this is the case for $J = 5$ and δ_j equal to our bounds for α_j, so our result is best possible. By computing further values of α_j, enabling us to take a larger value of J in (A6), we could improve the result slightly: this is due to a decrease in the loss in passing from (A5) to (A6). The bound $M_{\leq 6}(T) > 0.952N(T)$ implies that this improvement could increase the lower bound we obtained by at most $0.00021N(T)$.

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