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THEORY OF RAREFACTION WAVES IN A PLASMA

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13. Plots of $p = 0$, $c^2 = 0$, and $d(N^2c^2)/dN = 0$ as functions of density n and temperature T for an isothermal expansion 69

LIST OF SYMBOLS

c	effective sound speed
C_+	a family of characteristic curves is the (x,t) plane
C_-	a second family of characteristic curves in the (x,t) plane
e	absolute value of the electron charge
E	electric field intensity
j	electric current density
k	Boltzmann's constant
m	mass of an electron
M^*	mass of a positive ion
M	mass of a neutral atom
n	electron number density
N_+	positive ion number density
N_o	neutral atom number density
N	total number density of heavy particles, $N = N_o + N_+$
p_-	electron partial pressure
p_+	positive ion partial pressure
p_o	neutral atom partial pressure
p	total pressure, $p = p_- + p_+ + p_o$
P_{-+}	total momentum transferred to the electrons per unit volume per unit time by collisions with the positive ions

P_{+-} total momentum transferred to the positive ions per unit volume per unit time by collisions with the electrons

P_{-o} total momentum transferred to the electrons per unit volume per unit time by collisions with the neutral atoms

P_{o-} total momentum transferred to the neutral atoms per unit volume per unit time by collisions with the electrons

P_{+o} total momentum transferred to the positive ions per unit volume per unit time by collisions with the neutral atoms

P_{o+} total momentum transferred to the neutral atoms per unit volume per unit time by collisions with the positive ions

t time variable

T_- electron temperature

T positive ion temperature

T_o neutral particle temperature

u electron flow velocity in the laboratory reference frame

U positive ion flow velocity in the laboratory reference frame

U_o neutral atom flow velocity in the laboratory reference frame

u^* effective plasma velocity in the laboratory reference frame - $u^* = U - (mj)/eMN$

U_R velocity of the rarefaction shock wave in the laboratory reference frame

U_S velocity of a compression shock wave in the laboratory reference frame

V_d electron drift velocity in the electric field

V effective plasma velocity in the shock wave reference frame - $V = u^* - U_R$

$\bar{v}$ random velocity of the electrons

$\bar{V}$ random velocity of the heavy particles

V_i ionization potential

x position variable

I_+ refers to the equation for the slope of C_+

I_- refers to the equation for the slope of C_-

II_+ refers to the equation relating the flow variables along C_+

II_- refers to the equation relating the flow variables along C_-

α degree of ionization, $\alpha = N_+/N$

δ ionization rate per unit volume

ϵ_o permitivity of vacuum

$\varkappa_i$ average fraction of its energy an electron loses to a positive ion in a collision

$\varkappa_o$ average fraction of its energy an electron loses to a neutral atom in a collision

$\varkappa$ effective fraction of its energy an electron loses in a collision

λ_o mean free path for electron-atom collisions

λ_i mean free path for electron-ion collisions

ν_o average collision frequency for electron-atom collisions

ν_i average collision frequency for electron-ion collisions

σ arc length along a characteristic curve

τ average collision time for an electron

φ $\varphi \equiv \varkappa T_-$

φ' $\varphi' \equiv \partial\varphi/\partial T_-$

φ'' $\varphi'' \equiv \partial^2\varphi/\partial T_-^2$

ψ undetermined multiplier

THEORY OF RAREFACTION WAVES IN A PLASMA

CHAPTER I

INTRODUCTION

In recent years considerable research, both experimental and theoretical has been done on various problems in plasma dynamics. This recent re-emphasis of fluid dynamics has been stimulated by various problems such as thermonuclear research, and research and development on new plasma devices. A considerable portion of the recent work in plasma dynamics has been directed toward the understanding of the nature and structure of compressional shock waves. Every shock wave is accompanied by a rarefaction process; consequently the nature of the rarefaction process in a plasma must also be understood if the complete plasma flow problem is to be solved.

The principal concern in this paper is the analysis of rarefaction waves in a strongly ionized plasma in which the electron temperature is much greater than the massive particle temperature. Principles of continuum plasma dynamics will be employed through the Eulerian formulation of the equations of motion.

Historical Background

Theory. The term "hydrodynamics" was first used by Bernoulli (1700-1783) as the name for the combined sciences of hydrostatics and hydraulics. The equations of motion for a perfect fluid were first derived by Euler (1) who is responsible for both the "Eulerian" and the "Lagrangian" formulations of the equations of hydrodynamics. Poisson (2) determined a simple wave solution of the differential equations of motion for an isothermal gas. Stokes (3) introduced the concept of balancing mass and momentum across a discontinuity surface in an isothermal gas marking the beginning of the theoretical analysis of shock waves. Riemann (4) developed a complete theory for simple isentropic waves by introducing a transformation of variables in the fluid dynamical equations. The technique for handling simple waves introduced by Riemann is the well known method of "Riemann invariants". Riemann also developed a theory for shock waves but incorrectly assumed that the shock transition was adiabatic and reversible. Hugoniot (5) showed that conservation of energy requires an increase in the specific entropy across a compressional shock. Rayleigh (6) pointed out that the entropy condition prevents the occurrence of a rarefaction shock in a perfect gas.

Thomson and Thomson (7) developed a phenomenological theory for small amplitude compressional waves in a plasma which was the first theoretical work on acoustical plasma waves. Recently, Fowler (8) has developed a theory for electron driven shock waves in which the plasma is treated as a three component fluid composed of neutral atoms, ions, and electrons.

Paxton and Fowler (9) have derived a theory for electron acoustic shock waves and have pointed out that electrical breakdown waves observed in the laboratory by Wheatstone (10), Thomson (11), Beams (12), and Loeb (13) and in lightning discharges by Schonland (14) are in fact electron acoustic shock waves.

Experiment. As just mentioned, the early experiments on electrical breakdown in gases were observations of hydrodynamical waves in a plasma. Strutt (15) demonstrated that the luminosity of an electrodeless discharge which extended beyond the exciting field resulted from a hydrodynamical motion of the plasma. Fowler, *et al*. (16, 17) found that the flow expansion observed by Strutt was composed of a succession of acoustical waves which could be made to reflect from obstacles placed in their path. Fowler, Paxton, and Hughes (18) observed a new class of shock waves in the electric shock tube and pointed out that the waves were driven by the electron gas. This marked the first realization of the importance of the electrons as a hydrodynamical fluid. A comprehensive treatment of the experiments and theory pertaining to the plasma flow in the electric shock tube has recently been given by Fowler (19).

Description of the Problem

The work presented in this paper has been motivated by the need for a theory of the rarefaction wave found in the "driver" section of the electric shock tube when the electron temperature is much greater than the massive particle temperature. This rarefaction of the plasma takes place during the early stages of the discharge before the massive particles have been appreciably heated (18) but after the electrical current has risen to a high value. Such a rarefaction will be named an "electrodiabatic" rarefaction.*

Fowler (8) has pointed out that the electrons in a plasma usually play a significant role in problems of plasma dynamics and that their fluid dynamical behavior must be included in the treatment of these problems.

The significance of the electrons can be demonstrated qualitatively by considering the relative effectiveness of electrons and heavy particles in transferring momentum to other heavy particles. For simplicity, let us assume that the heavy target particles are at rest and the incident electron and heavy particle have average velocities of v and $\overline{V}$ respectively. It is well known that the average transfer of momentum in a collision between the electron and a heavy particle is

$$\langle \Delta P_m \rangle = m\overline{v}$$

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*The word "electrodiabatic" means "electricity goes across"; hence an "electrodiabatic" rarefaction wave has a net electrical current flowing across it.

while the average transfer of momentum between two like, heavy particles in a collision is

$$\langle \Delta P_M \rangle = \frac{1}{2} M\overline{V} .$$

The collision frequency ν_- for the electron-heavy particle collisions is qualitatively related to the collision frequency ν_H for collisions between the heavy particles by

$$\nu_- = \frac{\overline{v}}{4\overline{V}} \nu_H ;$$

consequently the average momentum transfer per unit time per electron $\langle \Delta P_m \rangle \nu_-$ is related to that for the heavy particles $\langle \Delta P_M \rangle \nu_H$ by

$$\langle \Delta P_m \rangle \nu_- = \frac{1}{2} \frac{m\overline{v}^2}{M\overline{V}^2} \langle \Delta P_M \rangle \nu_H .$$

If the impact electrons and heavy particles are in thermal equilibrium ($m\overline{v}^2 = M\overline{V}^2$), it is observed that each electron transfers half as much momentum per unit time as does each heavy particle. Thus for a strongly ionized plasma the mechanical effects of the electrons are significant even when the electrons and heavy particles are in thermal equilibrium. If the electron temperature is much greater than the heavy particle temperature ($m\overline{v}^2 >> M\overline{V}^2$), which is the case considered in this paper, it is evident that the momentum transfer for the electrons far exceeds that of the heavy particles; thus the electrons will play the dominant role in the rarefaction process in a strongly ionized plasma with electron temperature much greater than the massive particle temperature.

As mentioned above, Rayleigh (6) observed that a rarefaction shock is an impossibility in a perfect gas due to the entropy restriction. In the case of a plasma that is subjected to an applied electric field, the possible existence of a rarefaction shock must be re-examined. In fact, it is shown in the following sections that under certain conditions a rarefaction shock exists in a plasma.

In Chapter II a general theory for waves in a plasma is developed from the basic equations of motion. The general criteria for simple waves and rarefaction shocks are derived, and general descriptions of each type of wave are given.

Electrodiabatic rarefaction waves are treated in detail in Chapter III by specializing the general theory of Chapter II to fit this case.

Chapter IV contains an application of the electrodiabatic wave theory to the case of a strongly ionized hydrogen plasma. It is shown that electrodiabatic rarefaction shocks indeed exist in the hydrogen plasma for certain values of the electron temperature and degree of ionization.

In Chapter V the electrodiabatic wave theory is applied to the rarefaction wave in the "driver" section of the electric shock tube. The relations connecting the shock velocity of the compressional shock produced in the electric shock tube to the known conditions in the driver section are formulated for both simple rarefaction waves and rarefaction shocks.

A very brief treatment of the effects of the plasma microfields on the structure of a rarefaction wave has been developed in Chapter VI.

CHAPTER II

THEORY OF ONE-DIMENSIONAL PLASMA FLOW

In the theory developed in this paper the plasma is treated as a continuum with the Eulerian form of the differential equations of motion used throughout the analysis. These plasmadynamical equations differ from the classical Eulerian equations of hydrodynamics only by the inclusion of electric and magnetic force terms. In the case of one-dimensional flow with no externally applied magnetic fields, the internally induced and the externally applied electric fields are the only new force terms in the plasma equations of motion.

Equations

The mechanical equations can be derived either by requiring continuity of mass and momentum in an infinitesimal volume of the plasma (20,21,22,23), or by taking moments of the Boltzmann equation (24). The individual equations of particle continuity of the three components of the plasma in the laboratory reference frame are

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x} (nu) = \delta \qquad (1)$$

$$\frac{\partial N_+}{\partial t} + \frac{\partial}{\partial x}(N_+U) = \delta \tag{2}$$

$$\frac{\partial N_o}{\partial t} + \frac{\partial}{\partial x}(N_oU_o) = -\delta \tag{3}$$

where δ is the ionization rate per unit volume. The momentum balance in the plasma gives

$$\frac{\partial(mnu)}{\partial t} + \frac{\partial}{\partial x}(mnu^2 + p_-) = -enE + P_{-+} + P_{-o} \tag{4}$$

$$\frac{\partial}{\partial t}(M^*N_+U) + \frac{\partial}{\partial x}(M^*N_+U^2 + p_+) = eN_+E + P_{+-} + P_{+o} \tag{5}$$

$$\frac{\partial}{\partial t}(MN_oU_o) + \frac{\partial}{\partial x}(MN_oU^2 + p_o) = P_{o-} + P_{o+} \tag{6}$$

where P_{-+}, P_{+-}, P_{-o}, P_{o-}, P_{+o}, and P_{o+} are the total momentum transfers to the particles denoted by the first subscript per unit volume per unit time by collisions with the particles denoted by the second subscript. Since the momentum is conserved in collisions between particles, $P_{-+} = -P_{+-}$; $P_{-o} = -P_{o-}$; and $P_{+o} = -P_{o+}$. A set of energy balance equations can also be obtained (8), but instead a slightly different approach will be used in order to characterize the plasma flow. The energy balance equations are actually relations which define the dependence of the temperature of the plasma particles on the other flow variables. In order to keep the presentation in this section as general as possible, instead of using the usual energy balance equations, a general functional dependence of the particle temperature on the flow variables is assumed, and in Chapter III this assumption will be justified for the specific problem of electrodiabatic wave motion in a plasma. The assumption regarding energy balance is contained in the following:

$$p = p(N) \,, \tag{7}$$

and

$$\alpha = \alpha(N) \,, \tag{8}$$

where p is the total plasma pressure ($p = p_- + p_+ + p_o$), N is the total number density of heavy particles ($N = N_+ + N_o$), and α is the degree of ionization of the plasma ($\alpha = N_+/N$).

The electrical current must also be conserved throughout the medium, and in the case where displacement currents are negligible, this principle is given by

$$j = e(N_+U - nu) = \text{constant} \,. \tag{9}$$

It is shown in Chapter III that the conservation of electrical current through the wave determines the energy balance of the electron gas.

If macroscopic charge neutrality is assumed throughout the plasma medium, one has

$$N_+ = n \,. \tag{10}$$

This assumption is completely valid in regions of constant state. In wave regions where the plasma is being accelerated in a given direction by internal pressure gradients, the electrons attempt to escape more quickly than the ions.

Escape of the electrons from this macroscopic region destroys charge neutrality and generates electrostatic forces which accelerate the positive ions in the direction of the expansion. The electrostatic force field generated by the charge separation is proportional to the pressure gradient in the plasma ($E \propto \partial p/\partial x$). Poisson's equation relates the deviation from charge neutrality in the plasma to the gradient of the electrostatic field by

$$\frac{\partial E}{\partial x} = \frac{e}{\epsilon_o} (N_+ - n) \ . \qquad (11)$$

When the equations of motion of the components of the plasma are combined to give the plasmadynamical equations for the plasma as a whole, terms emerge which are proportional to the net charge density $N_+ - n$; hence by equation (11) and the proportionality between the electric field and the pressure gradient, these terms are proportional to the second derivative of the pressure ($\partial^2 p/\partial x^2$). The charge separation term is of the same order as heat conduction and viscosity terms which have been neglected in the first order hydrodynamic theory; consequently charge separation, thermal conductivity, and viscosity determine the limiting wave profile for a wave which becomes discontinuous in the first order theory.

Because of the approximate equality of their masses, the positive ions and neutral atoms exchange momentum and energy quite rapidly. This rapid exchange of energy between the ions and neutrals keeps them in thermodynamic equilibrium;

hence one has

$$T_o = T \; , \tag{12}$$

and

$$U_o = U \; . \tag{13}$$

Adding equations (2) and (3) and using equation (13) along with the definition of the total massive particle density $N = N_+ + N_o$ gives

$$\frac{\partial}{\partial t} N + \frac{\partial}{\partial x} (NU) = 0 \; . \tag{14}$$

Adding equations (4), (5), and (6) and substituting equations (9), (10), (12), and (13) gives

$$\frac{\partial}{\partial t} (MNU) + \frac{\partial}{\partial x} \left(MNU^2 + \frac{mj^2}{e^2 \alpha N} - \frac{2mjU}{e} + p \right) = 0 \; . \tag{15}$$

Equations (14) and (15) are the general mechanical equations of a plasma. A transformation of the flow velocity variable U defined by

$$u^* \equiv U - \frac{m}{M} \frac{j}{eN} \tag{16}$$

reduces the equations of motion to the form:

$$\frac{\partial}{\partial t} N + \frac{\partial}{\partial x} (Nu^*) = 0 \; , \tag{17}$$

and

$$\frac{\partial}{\partial t} (MNu^*) + \frac{\partial}{\partial x} \left[MNu^{*2} + \frac{j^2}{e^2 N} \left(\frac{m}{\alpha} - \frac{m^2}{M} \right) + p \right] = 0 \; . \tag{18}$$

If we now impose the assumptions $\alpha = \alpha(N)$ and $p = p(N)$ and perform the differentiations, the equations of motion become

$$\frac{\partial N}{\partial t} + N \frac{\partial u^*}{\partial x} + u^* \frac{\partial N}{\partial x} = 0 \qquad (19)$$

and

$$MN \frac{\partial u^*}{\partial t} + MNu^* \frac{\partial u^*}{\partial x} + Mc^2 \frac{\partial N}{\partial x} = 0 \qquad (20)$$

where $c^2 \equiv \frac{1}{M} \frac{d}{dN} \left[\frac{j^2}{e^2 N} \left(\frac{m}{\alpha(N)} - \frac{m^2}{M} \right) + p(N) \right]$.

The quantity c is called the effective sound speed in the plasma. Again, it should be stated that the assumptions $\alpha = \alpha(N)$ and $p = p(N)$ place certain restrictions on the energy balance in the flow. Equations (19) and (20) completely determine the nature of the plasma flow and are observed to be identical in form to the classical hydrodynamics equations for isentropic flow.

If the quantity c^2 is negative ($c^2 < 0$), the equations are elliptic and no wave solutions exist. In the elliptic case the hodograph transformation transforms the two nonlinear equations into linear equations which are valid in the entire plasma region and which can be solved either analytically or numerically. When the equations of motion are elliptic the rarefaction process will be called an "extrusion". The "extrusion" rarefaction process will not be treated in detail in this paper.

If c^2 is positive ($c^2 > 0$), the plasmadynamical equations are hyperbolic and wave solutions exist. The analysis of the hyperbolic case is accomplished by making use of the notion of "characteristics".

Characteristics

In the mathematical theory of hyperbolic flow the dependent variables in the equations are in general differentiated in different directions.

If a linear combination of the equations is taken so that the dependent variables are all differentiated in the same direction, such a direction which in general depends on all variables both dependent and independent is called "characteristic". In the case of only two independent variables and two dependent variables the characteristic directions define two one-parameter families of "characteristic curves" or simply "characteristics".

Some important physical interpretations and theorems relative to the hyperbolic equations of motion are stated without proof:*

1. The characteristics in the (x,t) plane represent sound wave trajectories.
2. Fundamental Theorem: The flow in a region adjacent to a region of constant state is a simple wave.
3. In a simple wave zone, the characteristic curves of one family are straight lines.

These definitions and theorems are necessary in order to proceed with the analysis of plasma flow.

The characteristics of equations (19) and (20) are

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*For the proofs of these theorems see Courant (25).

determined by taking a linear combination of the equations:

$$\frac{\partial u^*}{\partial t} + (u^* + \psi N)\,\frac{\partial u^*}{\partial x} + \psi\,\frac{\partial N}{\partial t} + \left(\psi u^* + \frac{c^2}{N}\right)\frac{\partial N}{\partial x} = 0 \tag{21}$$

where ψ is a variable parameter. We now ask when this combination involves the derivatives of u^* and N in only one direction given by $\left(\frac{\partial x}{\partial \sigma}, \frac{\partial t}{\partial \sigma}\right)$. Evaluation of the coefficients in equation (21) shows that this condition is fulfilled by

$$\frac{\partial x}{\partial \sigma} = (u^* + \psi N)\,\frac{\partial t}{\partial \sigma} \tag{22}$$

and

$$\psi\,\frac{\partial}{\partial \sigma} = \left(\psi u^* + \frac{c^2}{N}\right)\frac{\partial}{\partial \sigma}\,. \tag{23}$$

Solving for ψ, one finds

$$\psi = \frac{c^2}{N^2} \quad \text{or} \quad \psi = \pm\,\frac{c}{N}\,. \tag{24}$$

With this result the two families of characteristic curves C_+ and C_- in the (x,t) plane are determined by the two directions

$$I_+: \qquad dx = (u^* + c)dt \tag{25}$$

and

$$I_-: \qquad dx = (u^* - c)dt\,. \tag{26}$$

In order to determine how the variables u^* and N vary along the characteristic curves, the values of ψ in equation (24) are substituted back into equation (21) giving

$$II_+: \qquad du^* = -\,\frac{c}{N}\,dN$$

and

$$\text{II}_-: \qquad du^* = \frac{c}{N}\,dN \; , \tag{28}$$

where II_+ and II_- are valid along C_+ and C_- respectively.

The equations I_+, I_-, II_+, and II_- are in some places referred to as the "characteristic equations" of the flow. Since $c = c(N)$, equations (27) and (28) can be integrated to give

$$u^* + \int \frac{cdN}{N} = \text{constant along } C_+ \tag{29}$$

and

$$u^* - \int \frac{cdN}{N} = \text{constant along } C_- \; . \tag{30}$$

Equations (29) and (30) are the usual "Riemann invariants".

Characterization of Rarefaction Waves

As mentioned earlier, a basic property of simple waves is that the characteristics of one kind are straight lines. We will consider here rarefaction waves in which the C_+ characteristics are straight lines. These waves are called "forward-facing" waves since particles enter the wave zone from a region with greater values of x. This is assured since these waves are propagated with a constant velocity $(u^* + c)$ which is always greater than the particle velocity u^* (c is the + root of c^2). Illustrations of "forward-facing" rarefaction waves are shown in Figures 1 and 2. The characteristic which separates the wave region from a region of constant state is called the "head" of the wave if the gas enters the wave across it or the "tail" of the wave if the

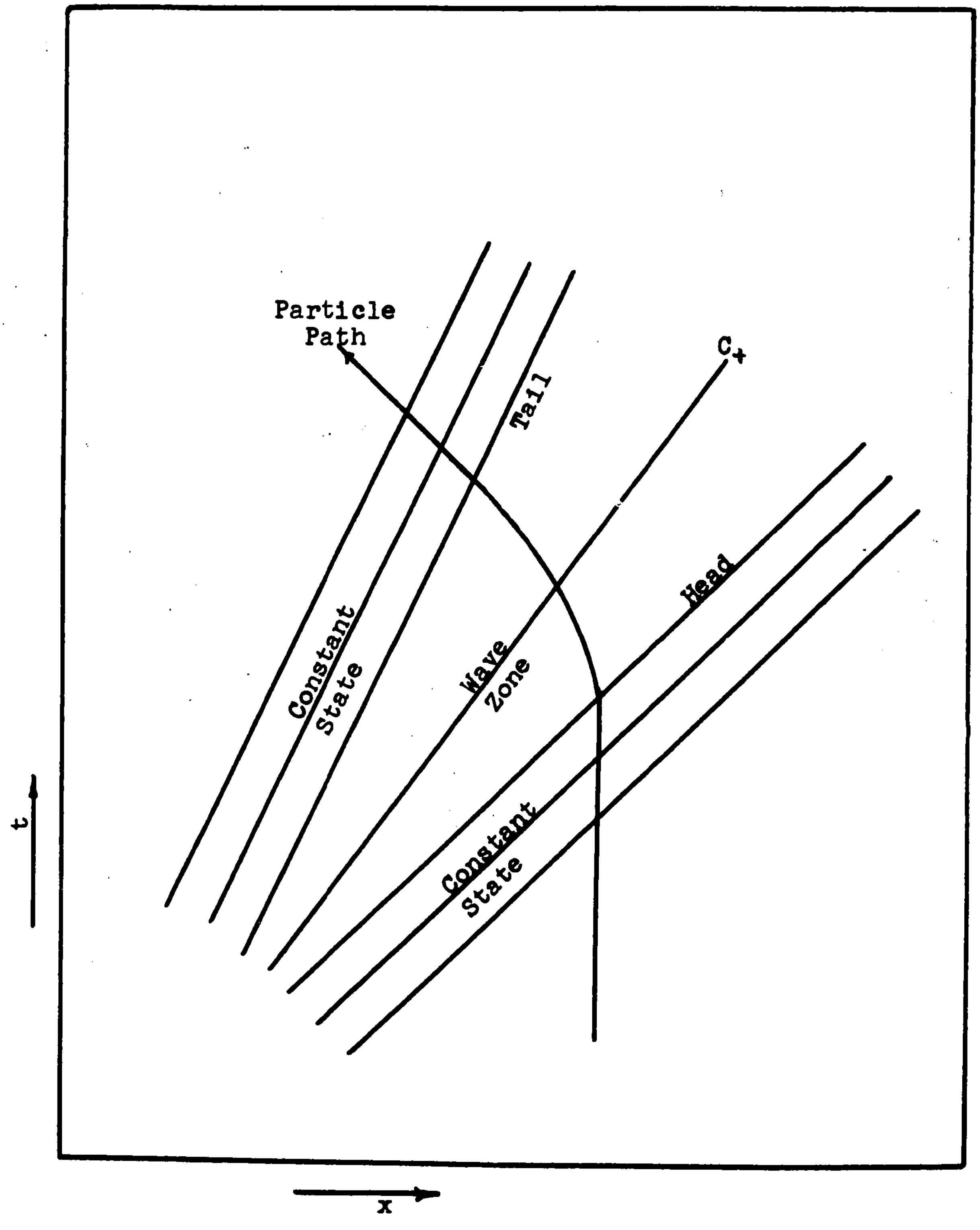

Figure 1. Forward-facing rarefaction wave with flattening wave profile - a simple wave at all times.

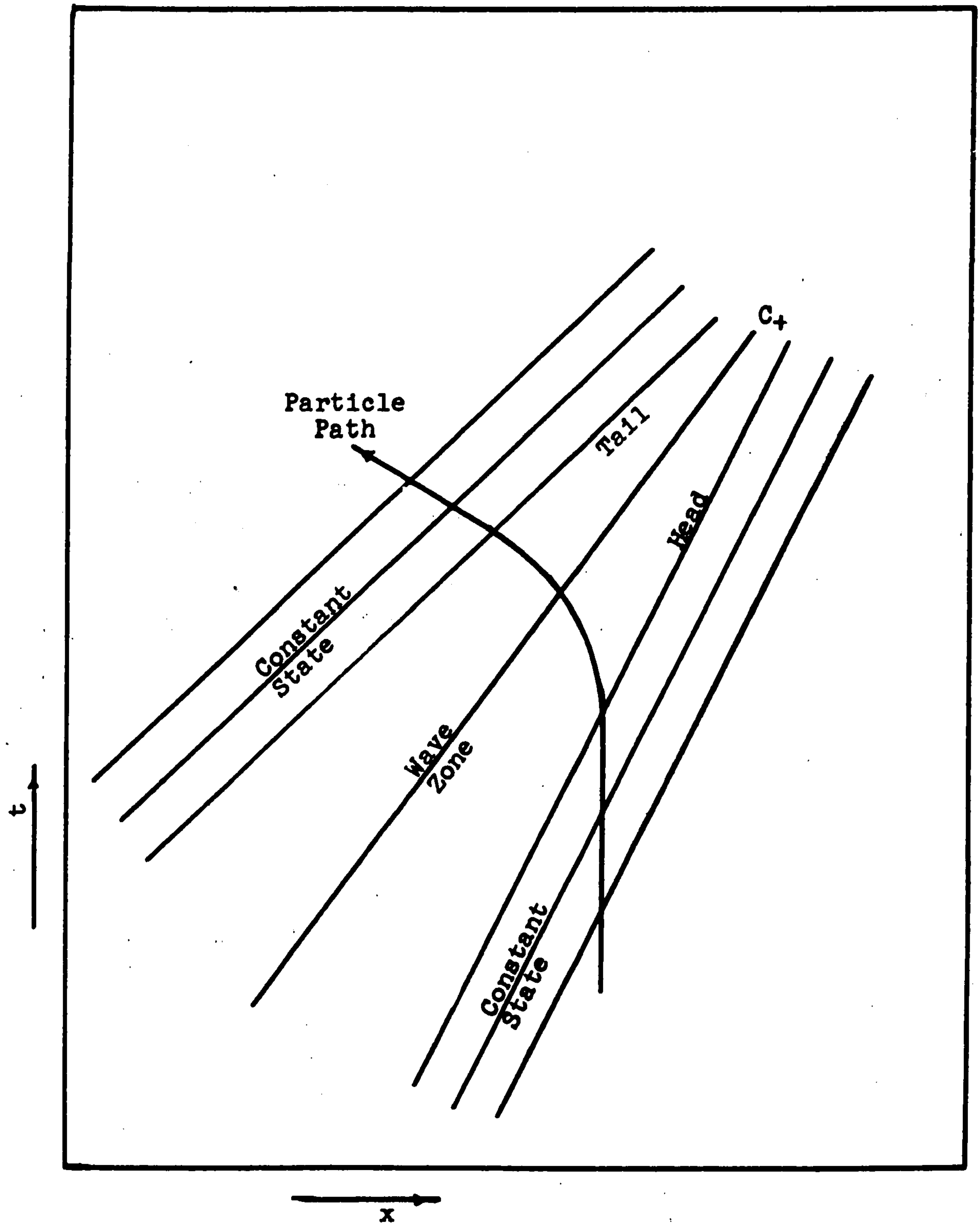

Figure 2. Forward-facing rarefaction wave with steepening wave profile - a rarefaction shock at the first intersection of C_+ characteristics.

gas leaves the wave across it.

Figures 1 and 2 are seen to represent quite different situations in that the characteristics are diverging in the former and converging in the latter. In the divergent case the illustration shows that the width of the wave increases as the wave propagates; consequently the rarefaction remains a simple wave. In the convergent case illustrated in Figure 2, the width of the wave decreases as the wave propagates. Eventually the characteristics intersect giving rise to an envelope on which multiple values of the propagation velocity exist. Beyond the point where the envelope forms, a unique continuation of a simple wave is mathematically impossible. This mathematical failure of the continuous plasma hydrodynamic theory corresponds physically to the production of a shock wave. Any flow in which the characteristics in the (x,t) plane converge as the wave propagates becomes a "shock" wave and must be analyzed by conservation of the flow through the wave front.

There are then mathematically two possibilities for a rarefaction wave in a plasma - a simple wave or a rarefaction shock.

Determining the nature of the rarefaction wave is equivalent to ascertaining how the slope of the straight C_+ characteristics changes across the wave. The slope of the C_+ characteristics is given by equation (25) to be $dx/dt = (u^* + c)$; hence it is necessary to determine how the quantity

$(u^* + c)$ changes through the wave zone. To determine $(u^* + c)$ on different C_+ characteristics, we examine the C_- characteristics which are called in this case the "cross-characteristics". For forward facing waves the C_- characteristics through the wave zone are curved lines in the (x,t) plane. The characteristic equation II_-, equation (28), gives a differential relationship which holds along the C_- characteristics, and since the C_- characteristics intersect the different straight C_+ characteristics in the wave zone, this relationship determines u^* as a function of the other variables on the different C_+ characteristics. Figure 3 illustrates these important points in the wave zone of a "forward facing" rarefaction wave.

A rarefaction wave is basically a wave across which the density decreases. In the case of the rarefaction of a plasma, the total number density N of the plasma particles decreases as the particles flow through the wave. With these remarks it follows that if, in the wave,

$$\frac{d(u^* + c)}{dN} < 0 \tag{31}$$

the C_+ characteristics will converge causing a rarefaction shock and if

$$\frac{d(u^* + c)}{dN} > 0 \tag{32}$$

the C_+ characteristics will diverge resulting in a simple rarefaction wave. By using the characteristic equation II_-, equation (28), we find

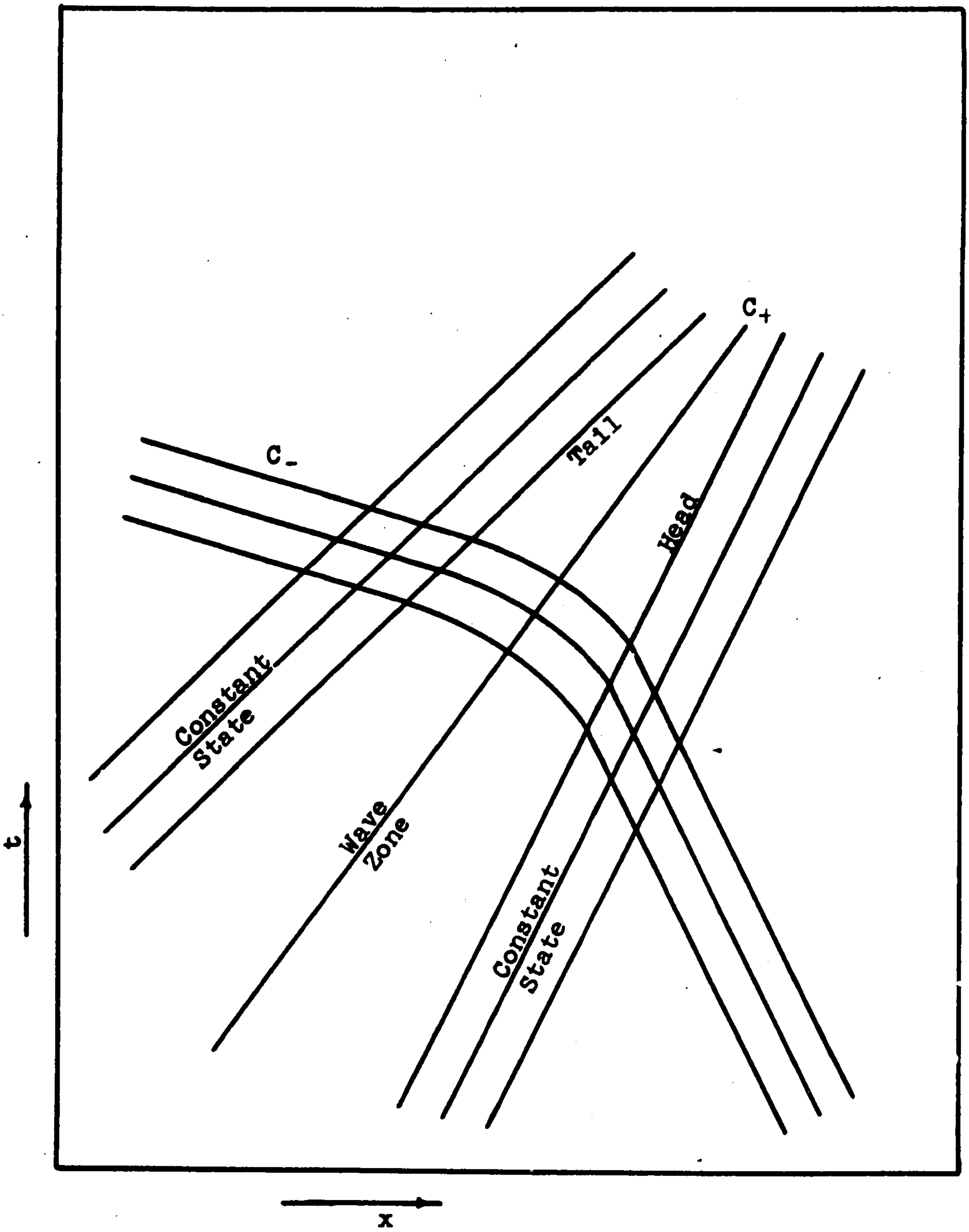

Figure 3. Cross-characteristics C_ in a steepening forward-facing rarefaction wave.

$$\frac{d(u^* + c)}{dN} = \frac{du^*}{dN} + \frac{dc}{dN} = \frac{c}{N} + \frac{dc}{dN} = \frac{1}{N}\frac{d(Nc)}{dN} \tag{33}$$

Thus the rarefaction wave is characterized by the quantity $d(Nc)/dN$ in that $d(Nc)/dN < 0$ implies a rarefaction shock while $d(Nc)/dN > 0$ implies a simple rarefaction wave.

General Comments on the Rarefaction Shock Wave

As mentioned in Chapter I, Rayleigh (6) pointed out that a rarefaction shock wave cannot occur in a perfect gas. This deduction was based on the fact that the specific entropy would decrease across a rarefaction shock which is prohibited by thermodynamics. Also, an analysis of isentropic waves in a perfect gas (25) shows that the nature of the waves is characterized by c^2 and $d(Nc)/dN$ in exactly the same manner as derived above for plasma waves. For a perfect gas c^2 and $d(Nc)/dN$ are always positive; hence the rarefaction wave will always be a simple wave.

On the other hand, in a plasma which is being "heated" by an electric field at the same time it is undergoing expansion, the change in the specific entropy of the plasma is not a useful criterion for determining the nature of the rarefaction wave. In this case there is no thermodynamic restriction on the existence of a rarefaction shock wave. Also, examination of the quantities c^2 and $d(Nc)/dN$ indicates that the conditions $c^2 > 0$ and $d(Nc)/dN < 0$ can exist in a plasma; therefore a rarefaction shock wave is a possible expansion process in a plasma.

Rarefaction Process in a Plasma

General. In the above analysis we have seen that there are three distinct possibilities for the rarefaction process in a current bearing plasma. These three possibilities are given by the following:

Extrusion: $c^2 < 0$

Simple Rarefaction Wave: $c^2 > 0$ and $\frac{d(Nc)}{dN} \geq 0$

Rarefaction Shock Wave: $c^2 > 0$ and $\frac{d(Nc)}{dN} < 0$.

Simple Rarefaction Wave. In the case of the simple rarefaction wave the states on each side of the "forward facing" wave are related by the Riemann invariant, equation (30). Thus across the wave we find

$$u_4^* - u_3^* = \int_{N_3}^{N_4} \frac{c}{N} \, dN \tag{34}$$

where for reasons which will become apparent later subscript 4 refers to the region in front of the rarefaction wave and subscript 3 refers to the region behind the rarefaction wave. Explicit determination of $c(N)$ will be dealt with in Chapter III.

Rarefaction Shock Wave. Relating the flow parameters across the rarefaction shock by integrating along the C_- characteristic is not possible since it has been shown that this analysis of the flow is not valid after the shock is formed. Instead, the so called "jump conditions" are derived from the following basic laws of physics:

1. Conservation of mass
2. Conservation of momentum
3. Conservation of energy
4. Conservation of current.

The jump conditions giving conservation of mass and momentum are derived by transforming equations (17) and (18) into a frame of reference attached to the shock wave. In this reference frame plasma particles come in one side at constant velocity and density and leave the other side at a different constant velocity and density; therefore the mass and momentum equations become steady state equations in the shock frame. These equations are given by

$$\frac{\partial}{\partial x}(NV) = 0 \tag{35}$$

and

$$\frac{\partial}{\partial x}\left[MNV^2 + \frac{j^2}{e^2N}\left(\frac{m}{\alpha} - \frac{m^2}{M}\right) + p\right] = 0 \tag{36}$$

where V is a velocity vector in the shock frame defined by V $\equiv u^* - U_R$. The rarefaction shock velocity in the laboratory frame is represented by U_R. Integration of equations (35) and (36) across the shock wave yields the jump conditions of mass and momentum balance:

$$N_3V_3 = N_4V_4 \tag{37}$$

$$MN_3V_3^{\ 2} + \frac{j^2}{e^2N_3}\left(\frac{m}{\alpha} - \frac{m^2}{M}\right) + p_3 = MN_4V_4^{\ 2} + \frac{j^2}{e^2N_4}\left(\frac{m}{\alpha} - \frac{m^2}{M}\right) + p_4 \tag{38}$$

The general jump condition for the current is given by

$$j_3 = j_4 = j \ . \tag{39}$$

The general jump condition for energy could also be written down here; however we will forego this and will consider only specific approximations to the energy equation in the next chapter.

CHAPTER III

ELECTRODIABATIC RAREFACTION WAVES

A rarefaction wave in a plasma with electron temperature much greater than massive particle temperature ($T_- >> T$) and with a time independent electric current driven by an external electric field normal to the wave has been named an "electrodiabatic" rarefaction wave. It has been pointed out in the previous chapter that conservation of the electric current through an electrodiabatic wave requires a definite energy balance in the electron gas. In this chapter the relation of the energy balance of the electrons to the nature of the electrodiabatic rarefaction wave is examined in detail.

Current Balance

Under the conditions just mentioned the conservation of energy loses its hydrodynamical importance since the electron gas, which is the energetic component of this plasma, is constantly gaining energy from the electromagnetic field by means of the Poynting vector and losing energy in collisions with the massive particles. This energy balance in the electron gas is controlled by the conservation of current; consequently current conservation is extremely important in determining the nature of the rarefaction wave. Since in the case under consideration the current is carried almost entirely by the electrons, one finds

$$j = - enV_d = \text{constant} \tag{40}$$

where V_d is the drift velocity of the electrons in the applied electric field. We must now determine the dependence of the electron drift velocity on the electron temperature and the degree of ionization in the plasma.

Electron Drift Velocity

It was just noted that to a very good approximation the electric current through the electrodiabatic wave is carried entirely by the electrons. In equation (40) we see that the current density is proportional to the product of the electron density and the electron drift velocity; consequently it is necessary to determine the functional dependence of the electron drift velocity V_d on the plasma variables in order to characterize the rarefaction wave.

In the following derivation average values (mean free path, average collision frequency, average thermal velocity, and etc.) are used from the beginning instead of integrating the microscopic processes over the velocity distribution. Even though the velocity distribution function is certainly warped in an applied electric field, the average value technique has been found to be quite useful in gaseous electronics in that the results obtained are quite simple and reasonably valid (26).

Let us assume the plasma is composed of electrons, neutral atoms, and one species of positive ions. In this case the electrons gain energy from the applied electric field and lose energy through collisions with both the neutral atoms and the positive ions. In the steady state of the heating process the energy loss rate of the electrons is equal to the rate the electrons receive energy from the field; therefore the functional dependence of the electron drift velocity on the plasma variables is determined by equating the electron's energy loss rate to its energy gain rate. The rate an electron gains energy from the applied electric field is

$$\text{Energy Gain Rate} = -\, eE_a V_d \tag{41}$$

while if $T_- \gg T$ the energy loss rate of the electrons is

$$\text{Energy Loss Rate} = \frac{m\overline{v}^2}{2}\left(\varkappa_1 \nu_1 + \varkappa_o \nu_o\right) \tag{42}$$

where E_a is the applied electric field, $\varkappa_1$ is the average fraction of its energy an electron loses to a positive ion during a collision, $\varkappa_o$ is the average fraction of its energy an electron loses to a neutral atom during a collision, ν_1 is the collision frequency for electron-ion collisions, and ν_o is the collision frequency for electron-atom collisions. Since the collisions tend to randomize the electron motion, the drift velocity is given approximately by

$$V_d = \frac{-eE_a}{m}\,\tau \tag{43}$$

where τ is the average collision time for an electron in the plasma. In terms of the "mean free paths" λ_o and λ_1, the average electron collision time is

$$\tau = \frac{1}{\frac{\bar{v}}{\lambda_o} + \frac{\bar{v}}{\lambda_1}} . \tag{44}$$

Equating the energy gain rate to the energy loss rate and substituting the relations for the drift velocity and the collision time given in equations (43) and (44) gives

$$V_d^{\,2} = \frac{\overline{v^2}}{2} \left(\frac{\varkappa_1\lambda_o + \varkappa_o\lambda_1}{\lambda_o + \lambda_1} \right) . \tag{45}$$

Substituting the electron temperature ($\overline{v^2} = 3kT_-/m$) into equation (45) gives

$$V_d^{\,2} = \frac{3kT_-}{2m} \left(\frac{\varkappa_1\lambda_o + \varkappa_o\lambda_1}{\lambda_o + \lambda_1} \right) . \tag{46}$$

For electrons with energies greater than a few electron volts, the electron-atom "mean free path" is

$$\lambda_o = \frac{A + BT_-}{N_o} \tag{47}$$

where A and B are parameters which depend on the atomic structure of the atom involved in the collision but are independent of the classical thermodynamic variables in the plasma. The electron-ion "mean free path" for singly charged ions is determined principally by coulombic scattering of the electron by the ions and is given by Spitzer (24):

$$\lambda_1 = \frac{DT_-^2}{N_+} \tag{48}$$

where $D \equiv (18\pi\epsilon_o^2 k^2)/(\ln \Lambda\, e^4)$ in MKS units. The dimensionless quantity $\ln \Lambda$ is a weakly dependent function of the electron temperature T_- and electron density n which takes into account the scattering of an electron by "distant" encounters due to the long range of the coulomb force.

Recalling that $N_o = (1 - \alpha)N$ and $N_+ = \alpha N$ and substituting equations (47) and (48) into equation (46) gives

$$V_d = \pm \left\{ \frac{3kT_-}{2m} \left[\frac{\varkappa_1 \alpha (A + BT_-) + \varkappa_o (1 - \alpha)\, DT_-^2}{\alpha (A + BT_-) + (1 - \alpha)\, DT_-^2} \right] \right\}^{\frac{1}{2}} . \tag{49}$$

The positive or negative root is to be chosen so that V_d and E_a have opposite sense.

In general $\varkappa_1$ and $\varkappa_o$ are functions of the electron temperature only - $\varkappa_1 = \varkappa_1(T_-)$ and $\varkappa_o = \varkappa_o(T_-)$ - and vary from a minimum of $(2m)/M^*$ and $(2m)/M$ respectively to a maximum of one half ($2m/M^* < \varkappa_1 < \frac{1}{2}$ and $2m/M < \varkappa_o < \frac{1}{2}$). The upper limit approaches one half because when the collision produces ionization the impact electron shares its excess energy with the ionized electron. However, if the ion has been completely stripped of its electrons as in the case of H^+, He^{++}, etc., the function $\varkappa_1$ is equal to $(2m)/M^*$ for all electron impact energies. The general temperature dependence of $\varkappa_o$ and $\varkappa_1$ ("non-stripped" ion) is illustrated in Figure 4.

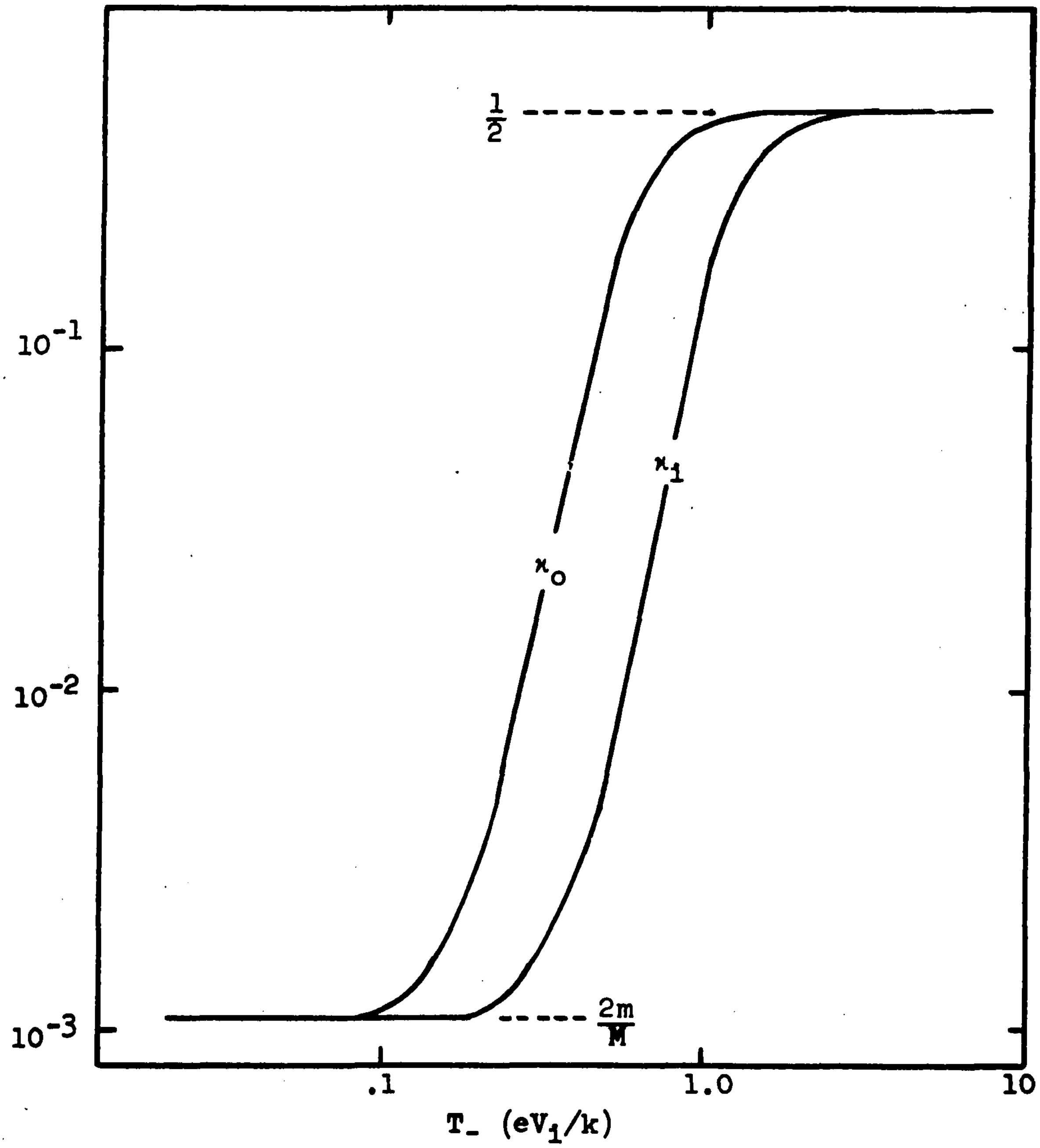

Figure 4. The average fractions $\varkappa_o$ and $\varkappa_1$ of electron energy lost in collisions with atoms and ions respectively as functions of the electron temperature T_- (qualitative dependence).

Equation (49) relates the electron drift velocity in the plasma to the electron temperature and degree of ionization. This equation plus the law of current conservation determines a functional relation between the total plasma density N, the electron temperature T_-, and the degree of ionization. The functional relation between T_-, N, and α along with an assumption on α (namely that α is approximately constant through the wave) is used to determine the effective sound speed c in the plasma as a function of the total plasma density. In the theory developed in Chapter II, the effective sound speed c(N) determines the nature of the rarefaction wave through the quantity d(Nc)/dN.

Characterization of Electrodiabatic Rarefactions

Substituting the drift velocity determined in equation (49) into the principle of current conservation which is given by equation (40) yields

$$j = \mp e\alpha N \left(\frac{3kT_-\varkappa}{2m} \right) = \text{constant} \tag{50}$$

or

$$\alpha^2 N^2 T_- \varkappa = \text{constant} \tag{50'}$$

where

$$\varkappa \equiv \frac{\varkappa_1 \alpha (A + BT_-) + \varkappa_0 (1 - \alpha) DT_-^2}{\alpha (A + BT_-) + (1 - \alpha) DT_-^2} . \tag{51}$$

The function $\varkappa = \varkappa(\alpha, T_-)$ defined by equation (51) is called the "effective" fraction of its energy an electron in a plasma loses per collision with a heavy plasma particle.

If α is constant through the wave, it can be treated as a parameter and equation (50) in principle can be solved for $T_- = T_-(N)$. Since in a strongly ionized plasma with the electron temperature much greater than the heavy particle temperature the total pressure is approximately equal to the electron partial pressure, we have

$$p \approx p_- = \alpha N k T_- = p(N) \ . \tag{52}$$

Equation (52) and the assumption that the degree of ionization is constant through the wave satisfy the assumptions that were made in developing the general theory in the preceding section; consequently electrodiabatic flow in a plasma with constant ionization through the wave can be analyzed using the general theory developed in Chapter II.

The effects of the electric microfields on the plasma equation of state have been neglected in equation (52) where the electron partial pressure is assumed to be produced by an "ideal" electron gas. The effects of the electric microfields on the nature of the plasma flow are examined quite briefly in Chapter VI for isentropic and isothermal expansions in a plasma in thermal equilibrium.

With the degree of ionization constant through the wave, the square of the effective sound speed given in equation (20) can be written

$$c^2 = \frac{-j^2}{Me^2N^2}\left(\frac{m}{\alpha} - \frac{m^2}{M}\right) + \frac{\alpha k}{M}\left(T_- + N\frac{\partial T_-}{\partial N}\right) \tag{53}$$

The conservation of current given by equation (50) is now used to calculate $\partial T_-/\partial N$. Let us define a new quantity φ by $\varphi(T_-) \equiv \varkappa(T_-)T_-$ which gives on substitution into equation (50)

$$N^2 \varphi(T_-) = \text{constant.} \tag{54}$$

Differentiating equation (54) with respect to N gives

$$\frac{\partial T_-}{\partial N} = - \frac{2\varphi}{N\varphi'} \tag{55}$$

where $\varphi' \equiv \partial\varphi/\partial T_-$. Substituting the values for the current density j and $\partial T_-/\partial N$ into equation (53) yields

$$c^2 = - \frac{3}{2}\left(1 - \frac{\alpha m}{M}\right)\frac{\alpha k}{M}\varkappa T_- + \frac{\alpha k}{M}\left(T_- - \frac{2\varphi}{\varphi'}\right) \tag{56}$$

or approximately

$$c^2 = \frac{\alpha k}{M}\left(T_- - \frac{2\varphi}{\varphi'} - \frac{3}{2}\right) \quad . \tag{56'}$$

Since the quantity $d(Nc)/dN$ which characterizes a rarefaction wave as either a shock or a simple wave is

$$\frac{d(Nc)}{dN} = \frac{1}{2Nc}\frac{d(N^2c^2)}{dN}$$

and since the quantity Nc is always positive, the quantity $d(N^2c^2)/dN$ can be used to characterize the electrodiabatic rarefaction process. Using equation (53) we find

$$\frac{d(N^2c^2)}{dN} = \frac{2\alpha kN}{M}\left(T_- - \frac{\varphi}{\varphi'} - \frac{2\varphi^2\varphi''}{\varphi'^3}\right) \tag{57}$$

where $\varphi'' \equiv \partial^2\varphi/\partial T_-^2$.

Equations (56) and (57) completely characterize the electrodiabatic flow process in the plasma. The different possible rarefaction processes predicted by the general theory for the electrodiabatic case are listed in the following:

$$\text{Extrusion:} \quad T_- - \frac{2\varphi}{\varphi'} - \frac{3}{2}\varphi < 0 \tag{58}$$

$$\text{Simple Rarefaction:} \quad T_- - \frac{2\varphi}{\varphi'} - \frac{3}{2}\varphi > 0$$

$$\text{and } T_- - \frac{\varphi}{\varphi'} - \frac{2\varphi^2\varphi''}{\varphi'^3} > 0 \tag{59}$$

$$\text{Rarefaction Shock:} \quad T_- - \frac{2\varphi}{\varphi'} - \frac{3}{2}\varphi > 0$$

$$\text{and } T_- - \frac{\varphi}{\varphi'} - \frac{2\varphi^2\varphi''}{\varphi'^3} < 0 \tag{60}$$

The quantities in inequalities (58), (59) and (60) determine the nature of the rarefaction process in terms of the function φ and its derivatives. In general, numerical determination of φ, φ' and φ'' is possible but would be quite involved.

In the case of a hydrogen plasma φ, φ', and φ'' can be determined as explicit functions of T_- and α. This analysis of a hydrogen plasma is developed in Chapter IV.

Simple Electrodiabatic Rarefaction Waves

The general theory relating the thermodynamic states on each side of the rarefaction wave which was developed in Chapter II is applied here to the electrodiabatic rarefaction. For an electrodiabatic rarefaction wave with effective sound speed given by equation (56'), equation (34) - a Riemann invariant across the wave - becomes

$$u_4^* - u_3^* = \int_{N_3}^{N_4} \frac{1}{N} \left[\frac{\alpha k}{M} \left(T_- - \frac{2\varphi}{\varphi'} - \frac{3}{2}\varphi \right) \right]^{\frac{1}{2}} dN \,. \tag{61}$$

Equations (54) and (55) are used to transform equation (61) to the following form:

$$u_4^* - u_3^* = -\int_{T_{-3}}^{T_{-4}} \frac{\varphi'}{2\varphi} \left[\frac{\alpha k}{M} \left(T_- - \frac{2\varphi}{\varphi'} - \frac{3}{2}\varphi \right) \right]^{\frac{1}{2}} dT_- \,. \tag{62}$$

Equation (62) relates the change in velocity u^* across the simple electrodiabatic rarefaction wave to a function of the thermodynamic variables on each side of the wave given by the integral on the right side of the equation. In Chapter IV the integral will be expressed as an integral over an explicit function of T_- for the case of an electrodiabatic wave in a hydrogen plasma.

Electrodiabatic Rarefaction Shocks

The three jump conditions, for conservation of mass, momentum, and current across an electrodiabatic rarefaction shock have been derived in Chapter II. With the functional dependence of the electron drift velocity developed in this chapter, the equation of current conservation - equation (39) - becomes

$$N_3^{\,2}\, \varphi_3 = N_4^{\,2}\, \varphi_4 \tag{63}$$

and the momentum conservation equation becomes

$$MN_3V_3^{\,2} + \left(\alpha - \frac{m\alpha^2}{M}\right)\frac{3}{2}\,kN_3\varphi_3 + \alpha N_3kT_{-3} = MN_4V_4^{\,2} +$$

$$\left(\alpha - \frac{m\alpha^2}{M}\right)\frac{3}{2}\,kN_4\varphi_4 + \alpha N_4kT_{-4}\ . \qquad (64)$$

The equation of mass conservation, equation (37), is recalled to be the following:

$$N_4V_4 = N_3V_3$$

Solving equations (37), (63) and (64) for the relative velocity ahead of the wave in terms of the states on each side of the wave, one finds

$$V_4^{\,2} = \frac{\alpha k}{M\left[1 - \left(\frac{\varphi_3}{\varphi_4}\right)^{\frac{1}{2}}\right]}\left\{\left(\frac{\varphi_4}{\varphi_3}\right)^{\frac{1}{2}}\left[\frac{3}{2}\left(1 - \frac{m\alpha}{M}\right)\varphi_3 + T_{-3}\right]\right.$$

$$\left. - \left[\frac{3}{2}\left(1 - \frac{m\alpha}{M}\right)\varphi_4 + T_{-4}\right]\right\} \qquad (65)$$

If φ is replaced by $\varkappa T_-$ and if $(m\alpha)/M$ is neglected, equation (65) can be rewritten approximately

$$V_4 = \pm\left\{\frac{\alpha k}{M\left[\left(\frac{\varkappa_3}{\varkappa_4}\right)^{\frac{1}{2}} - 1\right]}\left[\left(1 + \frac{3}{2}\,\varkappa_4\right)T_{-4}\right.\right.$$

$$\left.\left. - \left(\frac{\varkappa_4}{\varkappa_3}\right)^{\frac{1}{2}}\left(1 + \frac{3}{2}\,\varkappa_3\right)T_{-3}\right]\right\}^{\frac{1}{2}} \qquad (66)$$

The relative velocity behind the shock is given by

$$V_3 = \pm \left\{ \frac{\alpha k}{M\left[1 - \left(\frac{\kappa_4}{\kappa_3}\right)^{\frac{1}{2}}\right]} \left[\left(\frac{\kappa_3}{\kappa_4}\right)^{\frac{1}{2}} \left(1 + \frac{3}{2}\kappa_4\right) T_{-4} \right.\right.$$

$$\left.\left. - \left(1 + \frac{3}{2}\kappa_3\right) T_{-3} \right] \right\}^{\frac{1}{2}} . \qquad (67)$$

For a "forward facing" wave the negative value of the square root is the correct solution.

The velocity of the rarefaction shock in the laboratory frame is

$$U_R = u_4^* - V_4 , \qquad (68)$$

but by definition

$$u_4^* \equiv U_4 - \frac{mj}{MeN_4} \qquad (69)$$

which gives

$$U_R = U_4 - \frac{mj}{MeN_4} - V_4 . \qquad (70)$$

Since in the electrodiabatic rarefaction the velocity U_4 of the massive particles ahead of the wave is zero, one obtains

$$U_R = - \frac{mj}{MeN_4} + \left\{ \frac{\alpha k}{M\left[\left(\frac{\kappa_3}{\kappa_4}\right)^{\frac{1}{2}} - 1\right]} \left[\left(1 + \frac{3}{2}\kappa_4\right) T_{-4} \right.\right.$$

$$\left.\left. - \left(\frac{\kappa_4}{\kappa_3}\right) \left(1 + \frac{3}{2}\kappa_3\right) T_{-3} \right] \right\}^{\frac{1}{2}}. \qquad (71)$$

Equation (71) shows that the electrodiabatic rarefaction shock velocity depends on the polarity of the current j. It can be shown that the magnitude of the first term in equation (71) is smaller than that of the second term; consequently the rarefaction shock velocity is only slightly dependent on the polarity of the current j.

Degree of Ionization

Since the degree of ionization in the plasma has been assumed constant through the wave, this quantity enters the theory as a parameter. Even if this assumption is valid, there are other restrictions on the degree of ionization that must be fulfilled.

In the electrodiabatic rarefaction process the electrons are assumed to have a temperature much greater than the massive particle temperature ($T_- \gg T$). As discussed in Chapter II the rarefaction wave in a plasma accelerates the ions by the induced electric field resulting from the charge separation caused by the electrons escaping from a region of changing state faster than the ions. The ions in turn give momentum and energy to the neutral atoms by collisions. The ion-neutral atom collisions tend to randomize the energy of both the ions and neutrals; consequently, if the neutral atom density is too large the atoms and ions will actually be "heated" by the rarefaction wave. The result of this "heating" for $\alpha < \frac{1}{2}$ would be that the ion and atom partial pressures would increase through the rarefaction wave and would not be negligible. For this reason the theory developed here is applicable only for $\alpha > \frac{1}{2}$.

CHAPTER IV

APPLICATIONS OF THE ELECTRODIABATIC THEORY

It has been noted in the previous chapter that the nature of the electrodiabatic rarefaction wave in a strongly ionized plasma is determined by the conservation of the electric current which in turn requires a definite energy balance in the electron gas. The average energy per collision that an electron loses has been shown to be a very complex function of the electron temperature. In general this function is not known analytically; hence the analysis of these rarefaction waves must in general be handled numerically.

The unknown quantities in $\varkappa$ are $\varkappa_1$ and $\varkappa_0$. These functions can be calculated by considering all of the possible collision cross sections for each type of collision, but this approach would not permit an analytical evaluation of the nature of the electrodiabatic wave as a function of the electron temperature.

In the case of a partially ionized hydrogen plasma the requirement of the theory that the plasma have only three types of particles - electrons, neutral atoms, and one species of ions - is fulfilled. We assume that there are no hydrogen molecules or molecular ions in the plasma. Also, if the electron temperature in the hydrogen plasma undergoing rarefaction is of the order of the ionization energy, i.e. $kT_- \approx eV_i$, then $\varkappa_o \approx 1/2$ while since an electron can only lose energy to the hydrogen ion H^+ elastically $\varkappa_i = (2m)/M^*$ for all electron energies.

Characterization of the Waves

The theory developed in Chapters II and III is now applied to a hydrogen plasma in which the electron temperature is on the order of the ionization energy ($kT \approx eV_i$). Under the conditions which exist in the hydrogen plasma, the $\varkappa$ function is according to equation (51) the following explicit function of the electron temperature

$$\varkappa = \frac{\frac{2m}{M^*}\alpha(A + BT_-) + \frac{1}{2}(1 - \alpha)DT_-^2}{\alpha(A + BT_-) + (1 - \alpha)DT_-^2} . \tag{72}$$

The functions φ, φ', and φ'' ($\varphi \equiv \varkappa T_-$, $\varphi' \equiv \partial\varphi/\partial T_-$, $\varphi'' \equiv \partial^2\varphi/\partial T_-^2$ which determine the nature of the rarefaction by the inequalities (58), (59), and (60) can now be calculated from equation (72) as explicit functions of electron temperature.

Carrying out the differentiation gives

$$\varphi = \frac{A^*T_-}{B^*} , \tag{73}$$

$$\varphi' = \frac{B^*C^* - A^*D^*}{B^{*2}} , \tag{74}$$

$$\varphi'' = \frac{B^*(B^*E^* - A^*F^*) - 2D^*(B^*C^* - A^*D^*)}{B^{*3}\, T_-} . \tag{75}$$

The functions in equations (73), (74), and (75) are defined by the following:

$$A^* \equiv \frac{2m}{M^*} \alpha(A + BT_-) + \frac{1}{2} (1 - \alpha) DT_-^2 , \qquad (76)$$

$$B^* \equiv \alpha(A + BT_-) + (1 - \alpha) DT_-^2 , \qquad (77)$$

$$C^* \equiv \frac{2m}{M^*} \alpha(A + 2BT_-) + \frac{3}{2} (1 - \alpha) DT_-^2 , \qquad (78)$$

$$D^* \equiv \alpha BT_- + 2(1 - \alpha) DT_-^2 , \qquad (79)$$

$$E^* \equiv \frac{4m}{M^*} \alpha BT_- + 3(1 - \alpha) DT_-^2 , \qquad (80)$$

and

$$F^* \equiv 2(1 - \alpha) DT_-^2 . \qquad (81)$$

If equations (73), (74), and (75) are substituted into equations (56') and (57) for c^2 and $d(N^2c^2)/dN$, one obtains

$$c^2 = \frac{\alpha kT_-}{M} \left(1 - \frac{2A^*B^*}{B^*C^* - A^*D^*} - \frac{3}{2} \frac{A^*}{B^*}\right) , \qquad (82)$$

and

$$\frac{d(N^2c^2)}{dN} = \frac{2\alpha kNT_-}{M} \left\{1 - \frac{A^*B^*}{B^*C^* - A^*D^*} \right.$$

$$\left. - \frac{2A^{*2}B^*[B^*(B^*E^* - A^*F^*) - 2D^*(B^*C^* - A^*D^*)]}{(B^*C^* - A^*D^*)^3} \right\} \qquad (83)$$

According to the inequalities (58), (59), and (60) developed in the previous chapter, the nature of the electro-diabatic rarefaction process in a hydrogen plasma in terms of the above quantities depends on these inequalities:

$$\text{Extrusion:} \quad 1 - \frac{2A^*B^*}{B^*C^* - A^*D^*} - \frac{3}{2}\frac{A^*}{B^*} < 0. \tag{84}$$

$$\text{Wave Solution:} \quad 1 - \frac{2A^*B^*}{B^*C^* - A^*D^*} - \frac{3}{2}\frac{A^*}{B^*} > 0. \tag{85}$$

If the rarefaction is accomplished through a wave, the nature of the wave is given by

$$\text{Simple Rarefaction:} \quad 1 - \frac{A^*B^*}{B^*C^* - A^*D^*}$$
$$- \frac{2A^{*2}B^*[B^*(B^*E^* - A^*F^*) - 2D^*(B^*C^* - A^*D^*)]}{(B^*C^* - A^*D^*)^3} > 0. \tag{86}$$

$$\text{Rarefaction Shock:} \quad 1 - \frac{A^*B^*}{B^*C^* - A^*D^*}$$
$$- \frac{2A^{*2}B^*[B^*(B^*E^* - A^*F^*) - 2D^*(B^*C^* - A^*D^*)]}{(B^*C^* - A^*D^*)^3} < 0. \tag{87}$$

We now turn to the determination of the essential constants A, B and D. The mean free path for electrons in a gas of hydrogen atoms at 1 mm Hg pressure ($N_o = 3.5\ 10^{22}$ atoms/m^3) is plotted as a function of the electron temperature in Figure 5. The total collision cross section data used to plot this graph have been taken from Brackmann, et al. (27)

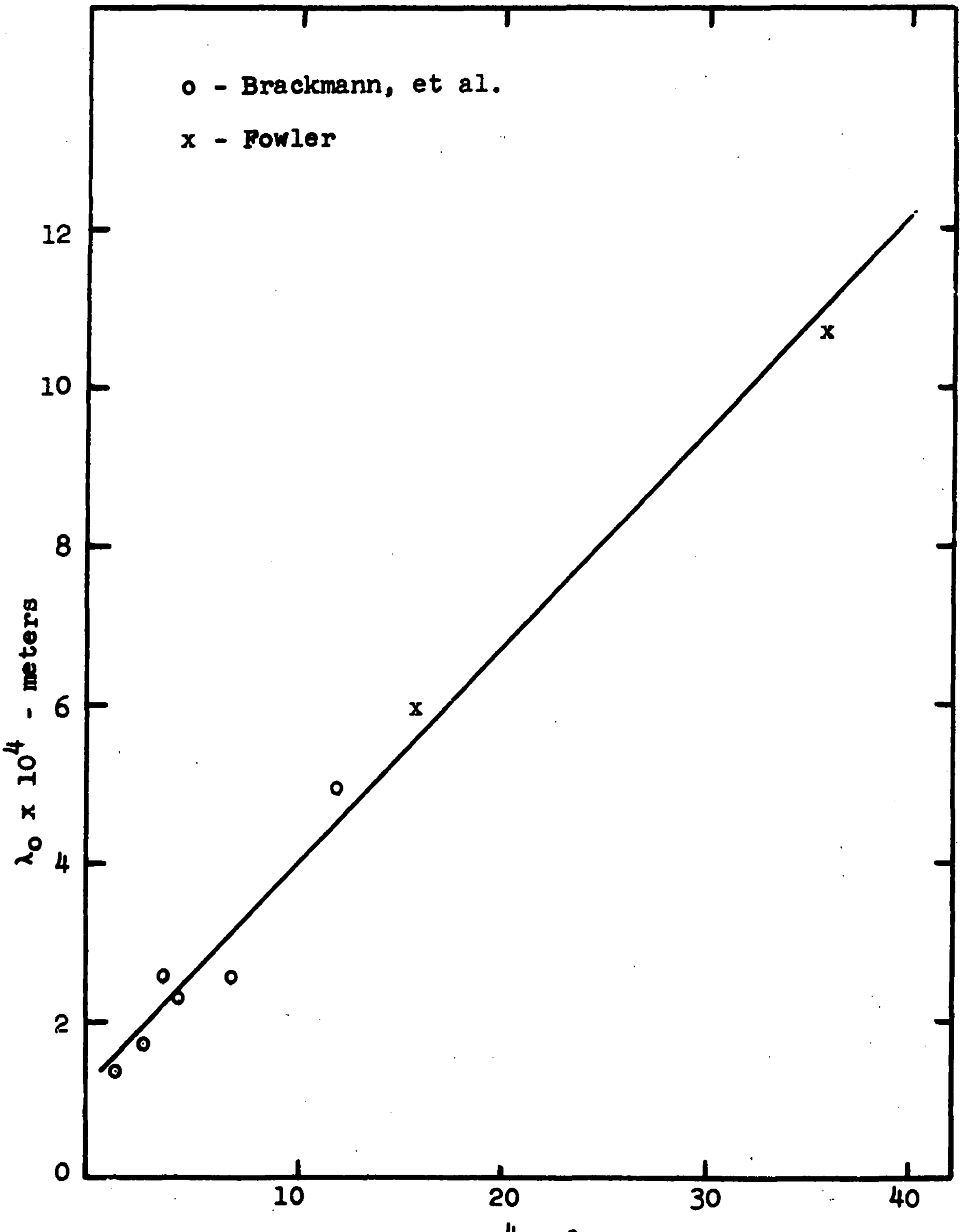

Figure 5. Electron-atom mean free path λ_o in a gas of H atoms as a function of electron temperature T_- (atom density = 3.54 10^{22} atoms/m^3).

for the low energy scattering while the high energy cross sections have been approximated from high energy cross sections in molecular hydrogen (28). The values for A and B are then obtained from the intercept and the slope of the curve by multiplying these numbers by the neutral particle density N_0 corresponding to 1 mm Hg pressure. This graphical determination of the constants A and B in atomic hydrogen gives

$$A = 5.3 \; 10^{18} \text{ per } m^2$$

and

$$B = 8.1 \; 10^{13} \text{ per } m^3 \text{ per } {}^\circ K \; .$$

The constant D in the equation for the electron-ion mean free path is calculated to be approximately

$$D = 2.1 \; 10^{8} \text{ per } m^2 \text{ per } {}^\circ K^2 \; .$$

The electron-ion mean free path λ_i is plotted in Figure 6 as a function of electron temperature for an ion density of $3.54 \; 10^{22}$ ions/m^3.

Calculations of c^2 and $d(N^2c^2)/dN$ for electron temperatures ranging from 10^4 to 10^6 °K and for degrees of ionization of $\alpha = 0.5$, 0.9, and 0.95 are plotted in Figures 7, 8, and 9 showing the dependence of the nature of the rarefaction process on the electron temperature and degree of ionization in the plasma. The illustrations show that all three rarefaction processes - extrusion, simple rarefaction, and rarefaction shock - can be expected in a given hydrogen plasma depending on the values of the electron temperature

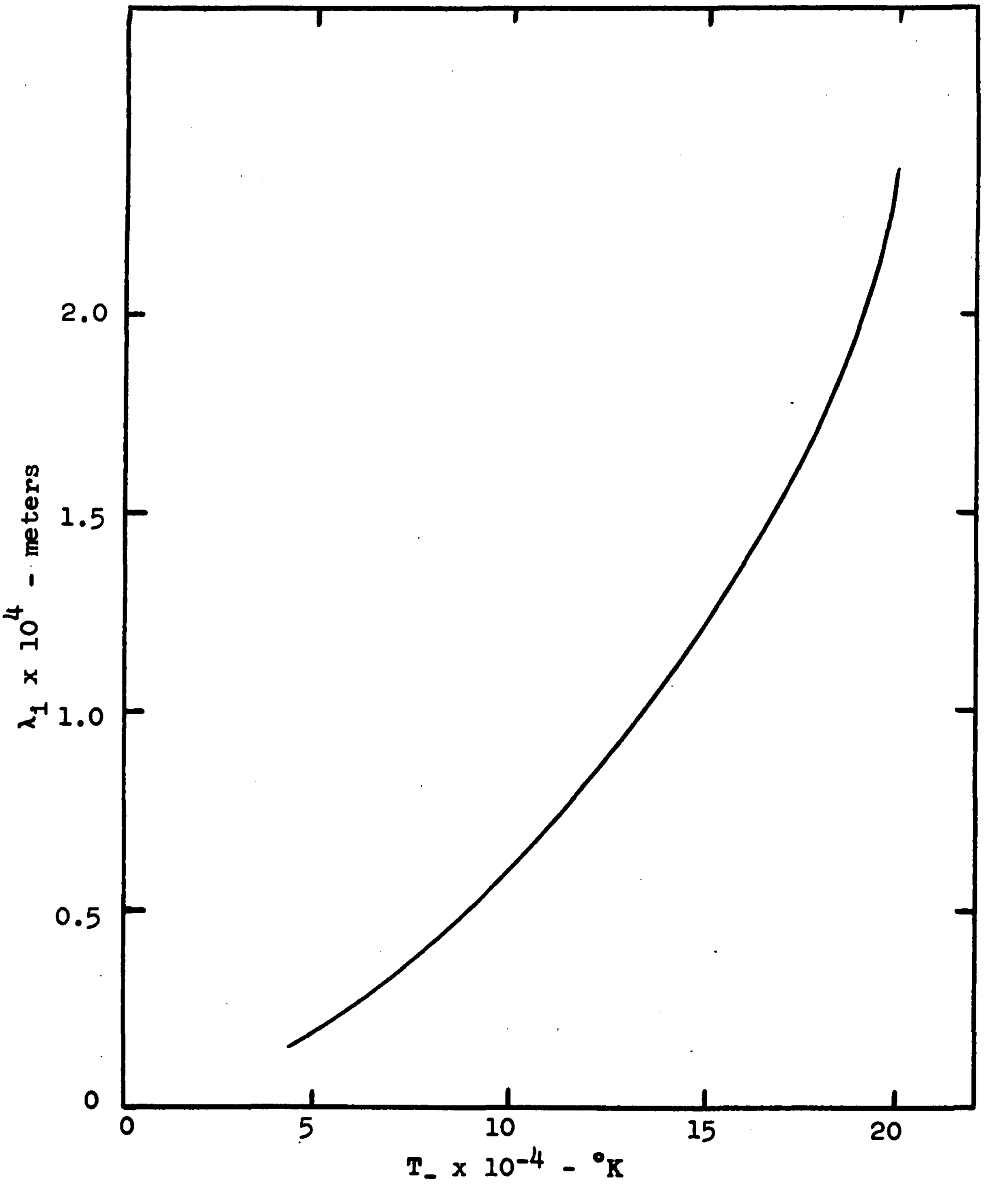

Figure 6. Electron-ion mean free path λ_1 in a gas of H^+ ions as a function of electron temperature T_- (ion density = $3.54\ 10^{22}$ ions/m^3).

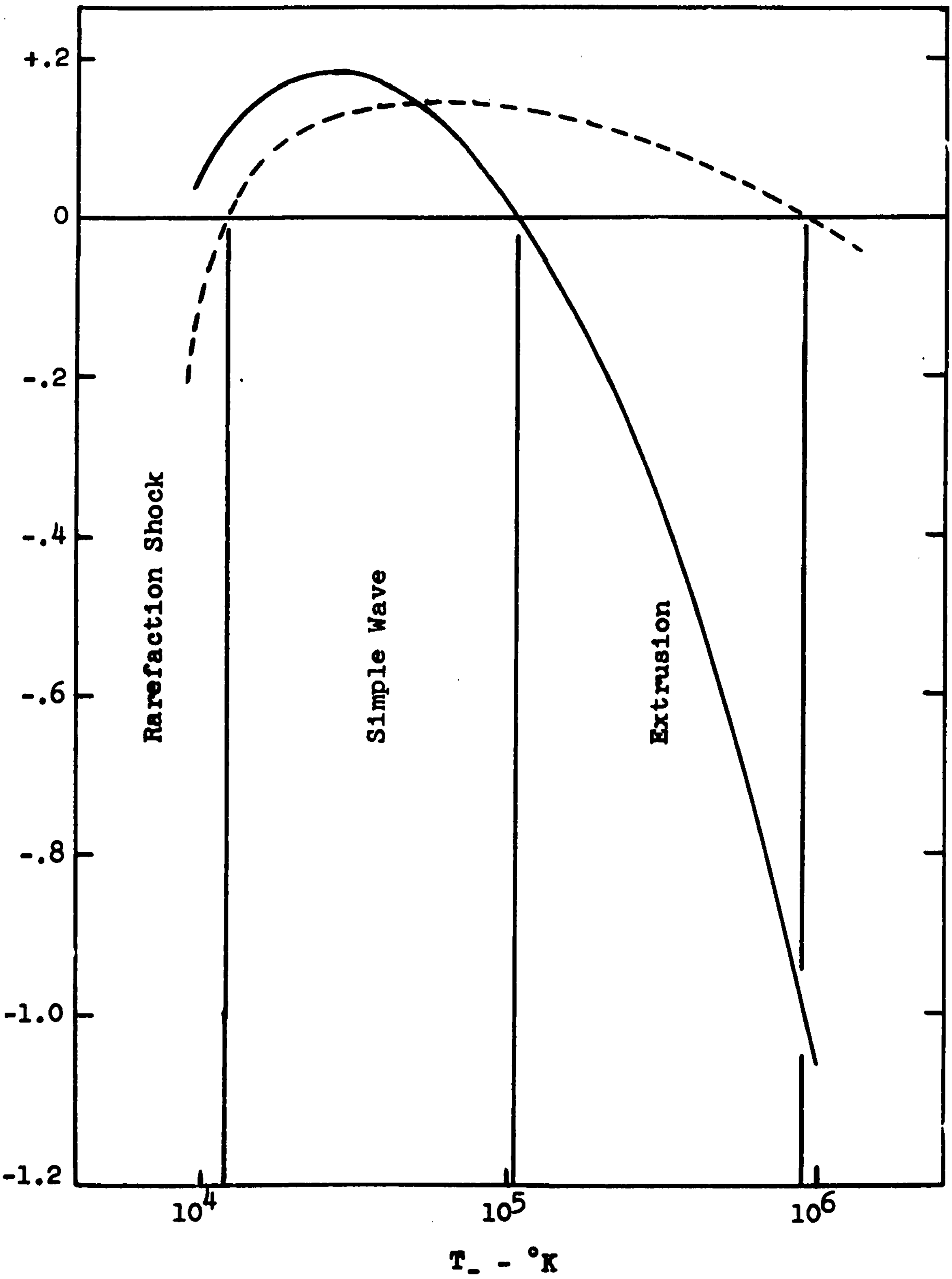

Figure 7. A plot of $Mc^2/\alpha kT_-$ and $M/2\alpha kT_-N \; d(N^2c^2)/dN$ as functions of electron temperature T_- for a degree of ionization $\alpha = .5$.

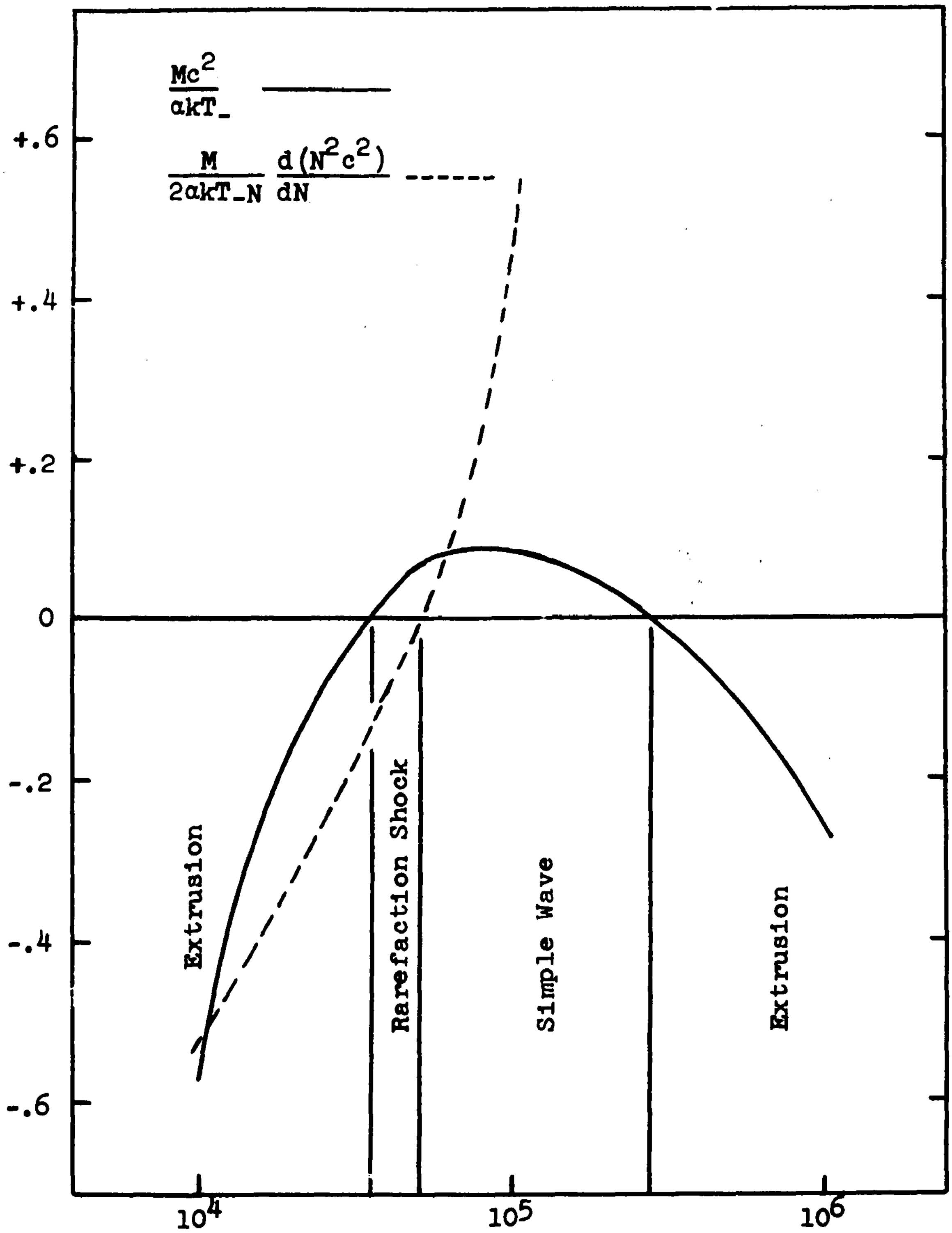

Figure 8. A plot of $Mc^2/\alpha kT_-$ and $M/2\alpha kT_-N \; d(N^2c^2)/dN$ as functions of electron temperature T_- for a degree of ionization $\alpha = .9$.

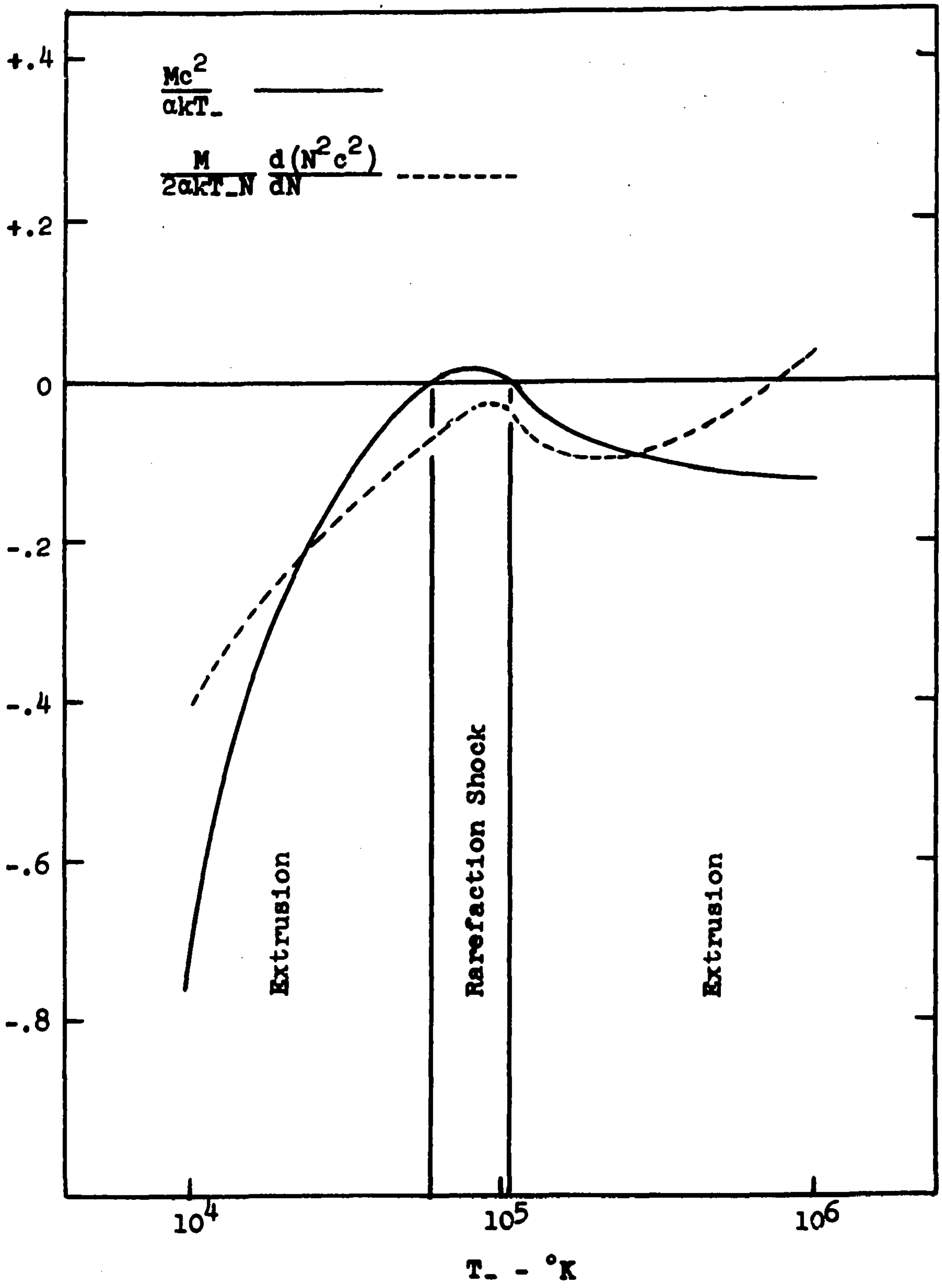

Figure 9. A plot of $Mc^2/\alpha kT_-$ and $M/2\alpha kT_-N \; d(N^2c^2)/dN$ as functions of electron temperature T_- for a degree of ionization $\alpha = .95$.

and degree of ionization that prevail.

Simple Rarefaction Wave

For the case of the simple electrodiabatic rarefaction wave in a hydrogen plasma, the functions derived in this chapter can be substituted into equation (62) to give the following relation between the flow variables on each side of the wave:

$$u_4^* - u_3^* = -\frac{1}{2}\left(\frac{\alpha k}{M}\right)^{\frac{1}{2}} \int_{T_{-3}}^{T_{-4}} \frac{B^*C^* - A^*D^*}{T_-^{\frac{1}{2}}\, B^*A^*} \left(1 - \frac{2B^*A^*}{B^*C^* - A^*D^*} - \frac{3A^*}{B^*}\right)^{\frac{1}{2}} dT_- \,. \tag{88}$$

No attempt to evaluate this integral has been made; however, since the integrand is a known function of the electron temperature, numerical integration can readily be performed.

Rarefaction Shock Wave

The velocity of the electrodiabatic rarefaction shock as a function of the electron temperature on each side of the wave is obtained by substituting the quantities derived above into equation (71). Performing this substitution gives

$$U_R = -\frac{mj}{MeN_4} + \left\{ \frac{\alpha k}{M\left[\left(\frac{A_3^*B_4^*}{B_3^*A_4^*}\right)^{\frac{1}{2}} - 1\right]} \left[\left(1 + \frac{3}{2}\frac{A_4^*}{B_4^*}\right)T_{-4} - \left(\frac{A_4^*B_3^*}{B_4^*A_3^*}\right)^{\frac{1}{2}}\left(1 + \frac{3}{2}\frac{A_3^*}{B_3^*}\right)T_{-3}\right]\right\}^{\frac{1}{2}}. \tag{89}$$

In Chapter V the quantities derived in this section are applied to the electrodiabatic rarefaction waves that are produced in the "driver section" of the electric shock tube.

CHAPTER V

ELECTRIC SHOCK TUBES

A simple electric shock tube is constructed from a long cylindrical tube in which the heating discharge takes place between a plane cathode placed in one end of the tube and a hollow ring anode placed a short distance from the cathode. The region between the two electrodes is called the "driver section" while the region beyond the hollow ring anode is referred to as the "expansion chamber". A large capacitor charged to a high voltage is quickly switched across the two electrodes in the driver section resulting in an electrical discharge through the gas. This electrical discharge creates a high temperature, high pressure plasma in the driver section which expands beyond the ring electrode causing a rarefaction wave to propagate through the plasma from the ring anode back to the plane cathode and causing a compressional shock to propagate up the tube. A schematic of the electric shock tube is shown in Figure 10.

The theory derived in the preceding sections is applied here to the rarefaction wave in the driver section of the electric shock tube. The development in this section will be limited to the formulation of the equations giving

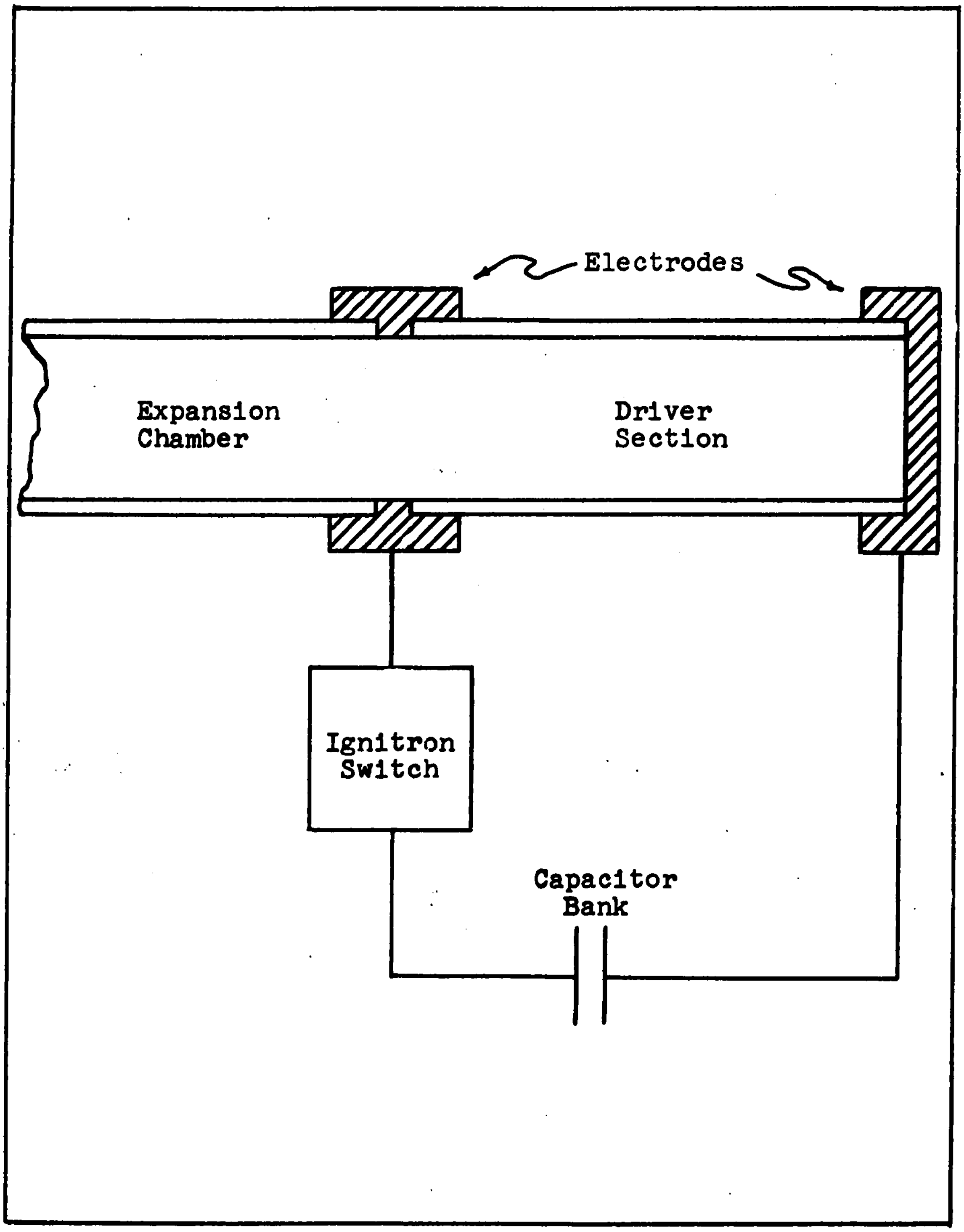

Figure 10. Schematic of a simple electric shock tube.

the functional dependence between the compressional shock velocity and the known conditions in the driver section of the tube.

General Description of the Flow

There are in general four distinct plasma regions in the simple one-dimensional electric shock tube. These four regions are separated by three different hydrodynamical waves - a compressional shock wave, a contact surface, and a rarefaction wave.

The first region, located in the expansion chamber, is composed of the as yet undisturbed stationary gas. The second region is composed of a partially ionized plasma which is moving with a constant velocity up the expansion chamber of the tube. All particles in region 2 have been produced from gas originally exterior to the driver section. Regions 1 and 2 are separated by a compressional shock wave which ionizes, heats, compresses and accelerates the gas passing through it.

The plasma in region 3 was generated from gas which was originally in the driver section of the tube. The constant flow velocity in region 3 is equal to the flow velocity in region 2; consequently there is no plasma flow through the wave separating the two regions. The wave separating regions 2 and 3 is called a "contact surface". The temperature and density change discontinuously across a contact surface, but the pressure remains constant across the wave. The plasma in region 3 can be pictured as a gaseous piston moving with a constant velocity into the expansion chamber.

Region 4 is composed of plasma in the driver section that has as yet not been disturbed by the hydrodynamical expansion of the plasma. The electrons in region 4 have a drift velocity in the applied electric field of the discharge; however, during the early stages of the discharge, the ions and neutrals are approximately static. Regions 3 and 4 are separated by an electrodiabatic rarefaction wave which has been shown in the preceding sections to be either a simple rarefaction wave or a rarefaction shock depending on the temperature and degree of ionization in the plasma. It should also be noted that the mass density in region 4 is equal to the known mass density in region 1.

When the expansion takes the form of an extrusion, region 4 disappears and region 3 extends instantaneously to the end electrode of the tube.

Figure 11 illustrates the wave profiles and the changes of the different thermodynamic variables across the three waves.

Fowler (8) has developed a theory for electron driven shock waves which gives the following relationship between the compressional shock velocity U_S, the electron temperature T_{-2}, and the degree of ionization α_2 behind the shock for the strong shock approximation ($p_2 >> p_1$):

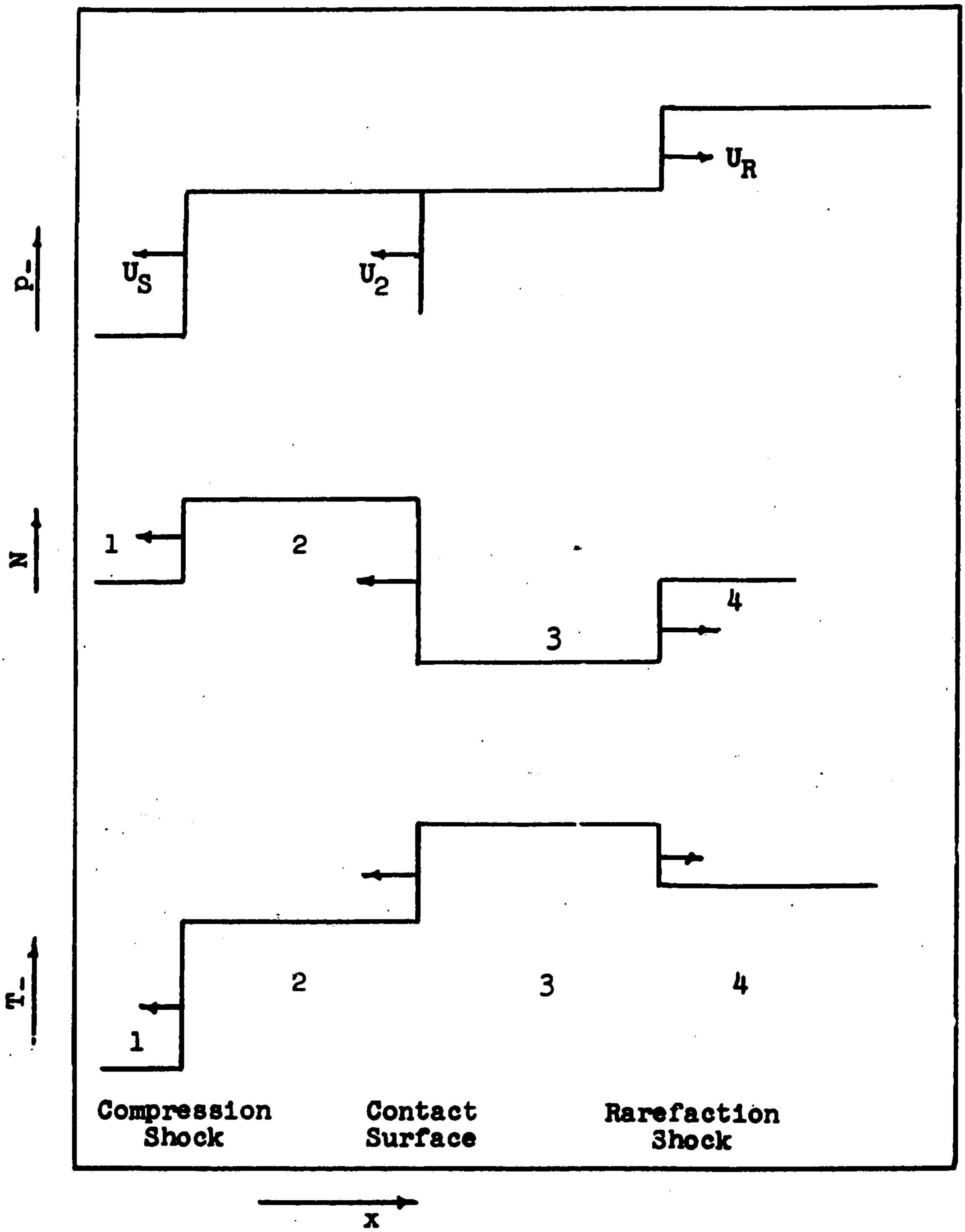

Figure 11. Schematic of the hydrodynamical waves in an electric shock tube showing the changes of the variables p_-, N, and T_- across the waves.

$$U_S = - \left[\frac{\alpha_2 \, kT_{-2} \left(4 + \frac{2eV_i}{kT_{-2}} \right)^2}{M \left(3 + \frac{2eV_i}{kT_{-2}} \right)} \right]^{\frac{1}{2}} . \tag{90}$$

The negative sign has been chosen for a shock traveling to the left as shown in Figure 11. The following equation for the plasma flow velocity U_2 and the total density N_2 behind the shock can also be determined from the above cited work:

$$U_2 = - \left[\frac{\alpha_2 \, kT_{-2} \left(3 + \frac{2eV_i}{kT_{-2}} \right)}{M} \right]^{\frac{1}{2}} . \tag{91}$$

and

$$N_2 = \left(4 + \frac{2eV_i}{kT_{-2}} \right) N_1 . \tag{92}$$

Combining equations (90) and (91) yields

$$U_2 = \frac{\left(3 + \frac{2eV_i}{kT_{-2}} \right)}{\left(4 + \frac{2eV_i}{kT_{-2}} \right)} U_S . \tag{93}$$

It has just been pointed out that $p_2 = p_3$ which implies

$$N_2 \, \alpha_2 \, kT_{-2} = N_3 \, \alpha_3 \, kT_{-3} . \tag{94}$$

If the theory for the electrodiabatic rarefaction process is now used to relate the variables in region 3 to the known variables in region 4, the velocity of the compressional shock wave is determined by the variables in region 4 of the plasma.

The density N_4 in region 4 is determined by the initial gas pressure in the tube. This leaves only the electron temperature and degree of ionization in region 4 to be determined. The so called "heating theory" (8,18,19) of the plasma determines the electron temperature and degree of ionization in region 4 in terms of the applied electric field and current. The heating theory of the electron gas is quite complex and as yet incomplete; however, in principle the electron temperature and degree of ionization in region 4 are known.

The velocity of the advancing compressional shock not only depends on the plasma density, applied electric field, and electric current in the driver section; but also depends on the nature of the electrodiabatic rarefaction wave.

We now assume that the expansion in the driver section of the electric shock tube is an electrodiabatic rarefaction wave with the degree of ionization constant across the wave. The assumption that the degree of ionization is constant across the electrodiabatic rarefaction wave is valid when the degree of ionization in region 4 is large ($\alpha_4 > \frac{1}{2}$). The theory developed in Chapter III is used to relate the plasma variables across the wave for the two cases - simple rarefaction wave and rarefaction shock.

Simple Rarefaction

If the electron temperature and degree of ionization satisfy the inequalities (59), the rarefaction is a simple wave across which equation (62) must be satisfied. With the definition $u^* \equiv U - (mj)/MeN$ and equation (50) for the current density, equation (62) is rewritten

$$U_4 - U_3 = \mp \frac{m}{M} \alpha_3 \left(\frac{3}{2} \frac{k \ \varphi_4}{m} \right)^{\frac{1}{2}} \pm \frac{m}{M} \alpha_3 \left(\frac{3}{2} \frac{k \ \varphi_3}{m} \right)^{\frac{1}{2}}$$
$$- \frac{1}{2} \left(\frac{\alpha_3 k}{m} \right)^{\frac{1}{2}} \int_{T_{-3}}^{T_{-4}} \frac{\varphi'}{\varphi} \left(T_- - \frac{2\varphi}{\varphi'} - \frac{3}{2} \varphi \right)^{\frac{1}{2}} dT_- \ , \qquad (95)$$

with the $\pm$ sign depending on the polarity of the applied electric field. The constant degree of ionization through the rarefaction is denoted by $\alpha_3 (\alpha_3 = \alpha_4)$. The flow velocity U_4 of the massive particles in the region ahead of the rarefaction wave is zero for the electrodiabatic rarefaction wave giving

$$- U_3 = \mp \frac{m}{M} \alpha_3 \left(\frac{3k}{2m} \right)^{\frac{1}{2}} (\varphi_4^{\frac{1}{2}} - \varphi_3^{\frac{1}{2}})$$
$$- \frac{1}{2} \left(\frac{\alpha_3 k}{M} \right)^{\frac{1}{2}} \int_{T_{-3}}^{T_{-4}} \frac{\varphi'}{\varphi} \left(T_- - \frac{2\varphi}{\varphi'} - \frac{3}{2} \varphi \right)^{\frac{1}{2}} dT_- \ . \qquad (96)$$

The current continuity given by equation (50') transforms equation (94) into

$$\alpha_2 T_{-2} = \frac{\alpha_3 N_4}{N_2} \left(\frac{\varphi_4}{\varphi_3} \right)^{\frac{1}{2}} (T_{-4} T_{-3})^{\frac{1}{2}} \ . \qquad (97)$$

If equation (92) and the fact $N_1 = N_4$ are substituted into equation (97), one finds

$$\left(4 + \frac{2eV_1}{kT_{-2}}\right) \alpha_2 T_{-2} = \alpha_3 \left(\frac{\varphi_4}{\varphi_3}\right)^{\frac{1}{2}} (T_{-4}T_{-3})^{\frac{1}{2}} \ . \qquad (98)$$

Equations (91) and (96) along with the fact $U_2 = U_3$ give the following:

$$\left[\frac{\alpha_2 kT_{-2}}{M}\left(3 + \frac{2eV_1}{kT_{-2}}\right)\right]^{\frac{1}{2}} = \mp \frac{m}{M}\alpha_3\left(\frac{3k}{2m}\right)^{\frac{1}{2}}(\varphi_4^{\frac{1}{2}} - \varphi_3^{\frac{1}{2}})$$

$$- \frac{1}{2}\left(\frac{\alpha_3 k}{M}\right)^{\frac{1}{2}} \int_{T_{-3}}^{T_{-4}} \frac{\varphi'}{\varphi}\left(T_- - \frac{2\varphi}{\varphi'} - \frac{3}{2}\varphi\right)^{\frac{1}{2}} dT_- \ . \qquad (99)$$

Equations (98) and (99) approximately determine the quantity $\alpha_2 T_{-2}$; hence these equations coupled with the appropriate heating theory give the velocity of the compressional shock in terms of the applied electric field, electrical current, and plasma density in the driver section.

Rarefaction Shocks

If the rarefaction wave in the driver section of the electric shock tube is a rarefaction shock, an analysis similar to that just completed relates the velocity of the compression shock to the known parameters in the driver section of the tube.

Equations (67) and (71) along with the definition $u^* \equiv U_R + V$ give

$$u_3^* = -\frac{mj}{MeN_4} + \left\{ \frac{\alpha_3 k}{M\left[\left(\frac{\varkappa_3}{\varkappa_4}\right)^{\frac{1}{2}} - 1\right]} \left[\left(1 + \frac{3}{2}\varkappa_4\right) T_{-4} \right.\right.$$

$$\left.\left. - \left(\frac{\varkappa_4}{\varkappa_3}\right)^{\frac{1}{2}} \left(1 + \frac{3}{2}\varkappa_3\right) T_{-3} \right] \right\}^{\frac{1}{2}}$$

$$- \left\{ \frac{\alpha_3 k}{M\left[1 - \left(\frac{\varkappa_4}{\varkappa_3}\right)^{\frac{1}{2}}\right]} \left[\left(\frac{\varkappa_3}{\varkappa_4}\right)^{\frac{1}{2}} \left(1 + \frac{3}{2}\varkappa_4\right) T_{-4} \right.\right.$$

$$\left.\left. - \left(1 + \frac{3}{2}\varkappa_3\right) T_{-3} \right] \right\}^{\frac{1}{2}} . \qquad (100)$$

The definition $u^* \equiv U - (mj)/MeN$ and equation (50) reduce equation (100) to

$$U_3 = \mp \frac{m}{M} \alpha_3 \left(\frac{3k}{2m}\right)^{\frac{1}{2}} \left[(\varkappa_3 T_{-3})^{\frac{1}{2}} - (\varkappa_4 T_{-4})^{\frac{1}{2}} \right]$$

$$+ \left\{ \frac{\alpha_3 k}{M\left[\left(\frac{\varkappa_3}{\varkappa_4}\right)^{\frac{1}{2}} - 1\right]} \left[\left(1 + \frac{3}{2}\varkappa_4\right) T_{-4} - \left(\frac{\varkappa_4}{\varkappa_3}\right)^{\frac{1}{2}} \left(1 \right.\right.\right.$$

$$\left.\left.\left. + \frac{3}{2}\varkappa_3\right) T_{-3} \right] \right\}^{\frac{1}{2}} - \left\{ \frac{\alpha_3 k}{M\left[1 - \left(\frac{\varkappa_4}{\varkappa_3}\right)^{\frac{1}{2}}\right]} \left[\left(\frac{\varkappa_3}{\varkappa_4}\right)^{\frac{1}{2}} \left(1 \right.\right.\right.$$

$$\left.\left.\left. + \frac{3}{2}\varkappa_4\right) T_{-4} - \left(1 + \frac{3}{2}\varkappa_3\right) T_{-3} \right] \right\}^{\frac{1}{2}} . \qquad (101)$$

If equation (91) along with $U_2 = U_3$ is substituted into equation (101), one has

$$- \left[\frac{\alpha_2 kT_{-2} \left(3 + \frac{2eV_1}{kT_{-2}} \right)}{M} \right]^{\frac{1}{2}} = \mp \frac{m}{M} \alpha_3 \left(\frac{3k}{2m} \right)^{\frac{1}{2}} \left[(\varkappa_3 T_{-3})^{\frac{1}{2}} \right.$$

$$\left. - (\varkappa_4 T_{-4})^{\frac{1}{2}} \right] + \left\{ \frac{\alpha_3 k}{M \left[\left(\frac{\varkappa_3}{\varkappa_4} \right)^{\frac{1}{2}} - 1 \right]} \left[\left(1 + \frac{3}{2} \varkappa_4 \right) T_{-4} \right. \right.$$

$$\left. \left. - \left(\frac{\varkappa_4}{\varkappa_3} \right)^{\frac{1}{2}} \left(1 + \frac{3}{2} \varkappa_3 \right) T_{-3} \right] \right\}^{\frac{1}{2}} - \left\{ \frac{\alpha_3 k}{M \left[1 - \left(\frac{\varkappa_4}{\varkappa_3} \right)^{\frac{1}{2}} \right]} \right.$$

$$\left. \times \left[\left(\frac{\varkappa_3}{\varkappa_4} \right) \left(1 + \frac{3}{2} \varkappa_4 \right) T_{-4} - \left(1 + \frac{3}{2} \varkappa_3 \right) T_{-3} \right] \right\}^{\frac{1}{2}} . \qquad (102)$$

Since the current continuity condition for the rarefaction shock is the same as for the simple rarefaction, equation (98) holds here also. Equations (98) and (102) determine $\alpha_2 T_{-2}$ in terms of $\alpha_3 T_{-4}$. As before, these quantities are determined by the heating theory. Thus the compression shock velocity given by equation (90) is determined by the conditions in region 4 of the plasma.

CHAPTER VI

ISENTROPIC AND ISOTHERMAL RAREFACTION WAVES

The effects of the plasma microfields on the nature of the rarefaction wave are considered very briefly here. For this derivation let us assume that there is no applied electric field; hence the current density is taken to be zero ($j = 0$). The plasma is also assumed to be completely ionized and in thermodynamic equilibrium at a temperature T.

Equations

The equations characterizing the plasma flow are the general equations derived in Chapter II. If the current density j is zero and the degree of ionization α is unity, the effective sound speed c defined in equation (20) reduces to the following:

$$c^2 = \frac{1}{M}\frac{\partial p}{\partial N} . \tag{103}$$

In order to determine the nature of the rarefaction wave it is necessary to obtain an expression for the plasma equation of state. For the case of the isentropic rarefaction the specific entropy must be expressed in terms of the plasma temperature and density.

The Debye-Huckel theory of electrolytes yields an equation of state for a plasma given by

$$p = 2NkT - \beta \frac{N^{3/2}}{T^{1/2}} \tag{104}$$

where

$$\beta \equiv \frac{e^3}{12\ \pi} \left(\frac{2}{\epsilon_o^3 k} \right)^{\frac{1}{2}}$$

The quantity β is given in MKS units. This theory also determines the specific entropy to be

$$s = k\ \ln \left(\frac{T^3}{n^2} \right) - \beta \frac{N^{1/2}}{T^{3/2}} + s_o \tag{105}$$

where s_o is a constant.

Theimer (29) from a different approach has derived equations for the plasma pressure p and specific entropy s in a plasma for the limiting case of very large plasma density. Theimer's equations for the plasma pressure and specific entropy as functions of plasma density and temperature are identical to the Debye-Huckel equations (104) and (105).

The nature of the plasma rarefaction process can be determined directly from equations (103), (104) and (105) for either an isentropic or isothermal expansion.

The rarefaction process is characterized as before by the following:

Extrusion: $c^2 < 0$

Simple Rarefaction Wave: $c^2 > 0$ $\qquad \frac{d(N^2c^2)}{dN} > 0$

Rarefaction Shock Wave: $c^2 > 0$ $\qquad \frac{d(N^2c^2)}{dN} < 0$

Isentropic Expansion

Combining equations (103) and (104) gives

$$c^2 = \frac{1}{M}\left[2kT - \frac{3}{2}\,\beta\,\frac{N^{1/2}}{T^{1/2}} + \left(2kN + \frac{1}{2}\,\beta\,\frac{N^{3/2}}{T^{3/2}}\right)\frac{\partial T}{\partial N}\right] . \tag{106}$$

For the case of isentropic flow the specific entropy s in equation (105) is constant. If equation (105) is differentiated with respect to the density, one finds

$$\frac{\partial T}{\partial N} = \frac{\dfrac{2k}{N} + \dfrac{\beta\ N^{-1/2}}{2\ T^{3/2}}}{\dfrac{3k}{T} + \dfrac{3\beta\ N^{1/2}}{2T^{5/2}}} \tag{107}$$

Thus the square of the effective sound velocity is

$$c^2 = \frac{T}{M}\left(\frac{10\ k^2 + \dfrac{\beta k N^{1/2}}{2T^{3/2}} - \dfrac{2\beta^2 N}{T^3}}{3k + \dfrac{3\beta N^{1/2}}{2T^{3/2}}}\right) . \tag{108}$$

Let us now examine $d(N^2c^2)/dN$. If equation (108) is multiplied by N^2 and then the product differentiated with respect to N, it is found that

$$\frac{d(N^2c^2)}{dN} = \frac{NT}{M\left(3k + \frac{3\beta N^{1/2}}{2\,T^{3/2}}\right)^3}\left(240k^4 + \frac{1029}{4}\frac{\beta k^3 N^{1/2}}{T^{3/2}}\right.$$

$$\left. + \frac{135}{4}\frac{\beta^2 k^2 N}{T^3} - \frac{303}{8}\frac{\beta^3 k N^{3/2}}{T^{9/2}} - \frac{21}{2}\frac{\beta^4 N^2}{T^6}\right) . \tag{109}$$

In order to illustrate the possible processes $p = 0$, $c^2 = 0$, and $d(N^2c^2)/dN = 0$ are plotted in Figure 12 with the arrow on the curves indicating the regions in which the respective quantities are positive. This illustration shows that the rarefaction shock wave is physically impossible because $d(N^2c^2)/dN < 0$ implies $p < 0$. Since $p > 0$ implies $c^2 > 0$, the rarefaction process in an isentropic plasma is a simple wave for all values of temperature and density.

Isothermal Expansion

The nature of the rarefaction process of the plasma described at the beginning of this chapter is now examined for the isothermal case. Since in this case the plasma temperature T is constant, equation (106) reduces to

$$c^2 = \frac{T}{M}\left(2k - \frac{3}{2}\,\beta\,\frac{N^{1/2}}{T^{3/2}}\right) \tag{110}$$

The quantity $d(N^2c^2)/dN$ is quite easily determined to be

$$\frac{d(N^2c^2)}{dN} = \frac{NT}{M}\left(4k - \frac{15}{4}\,\frac{\beta N^{1/2}}{T^{3/2}}\right) \tag{111}$$

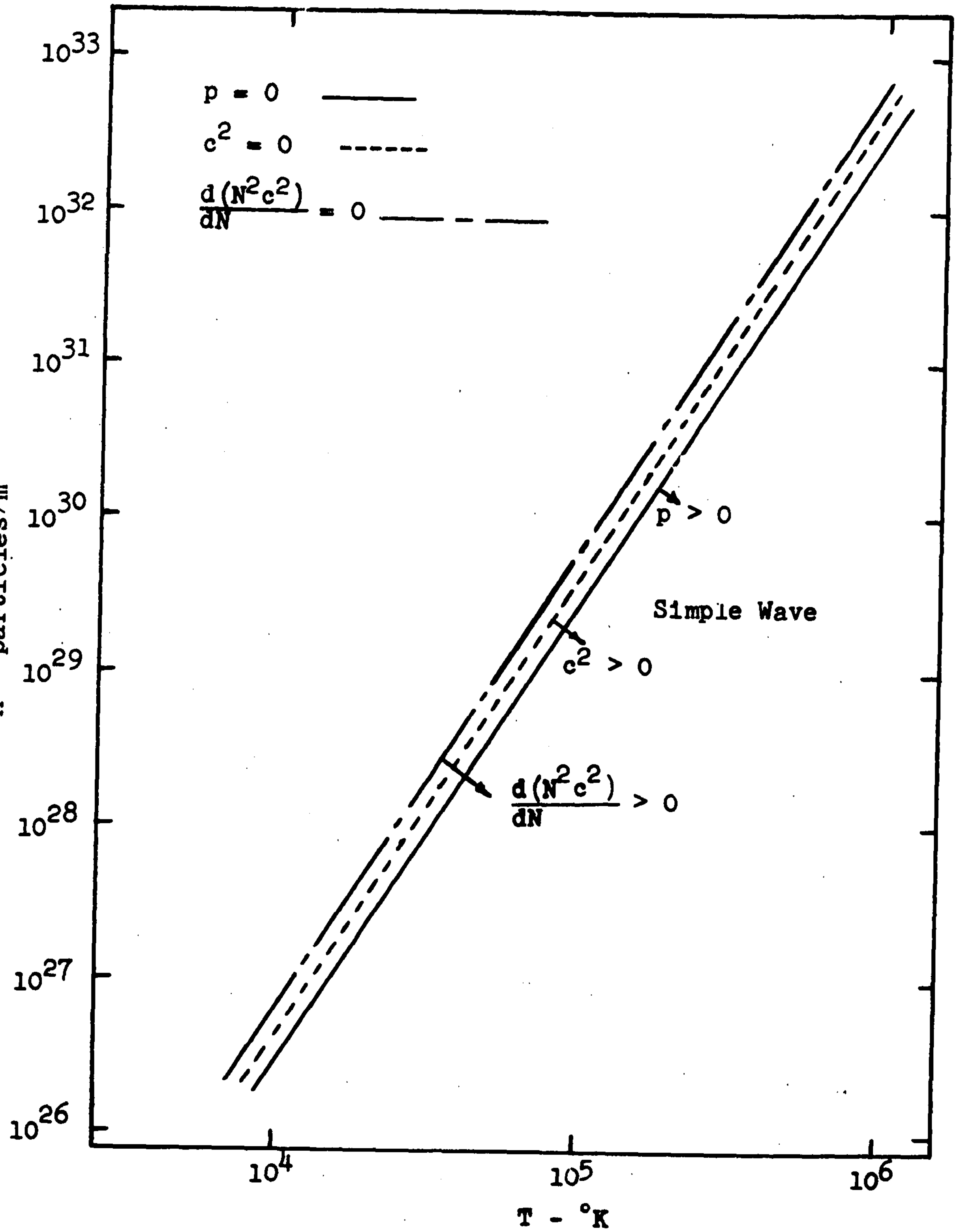

Figure 12. Plots of $p = 0$, $c^2 = 0$, and $d(N^2c^2)/dN = 0$ as functions of density n and temperature T for an isentropic expansion.

The relations defined by $p = 0$, $c^2 = 0$, and $d(N^2c^2)/dN = 0$ are plotted in Figure 13. This illustration shows that rarefaction shock waves are indeed possible for the isothermal expansion.

Examination of the numerical values for temperature and density indicates that for a reasonable value for the plasma temperature the density must be extremely large for the rarefaction to be a rarefaction shock. Even though these densities for a rarefaction shock far exceed those obtained in the electric shock tube, they are not unreasonable for a high density metal plasma.

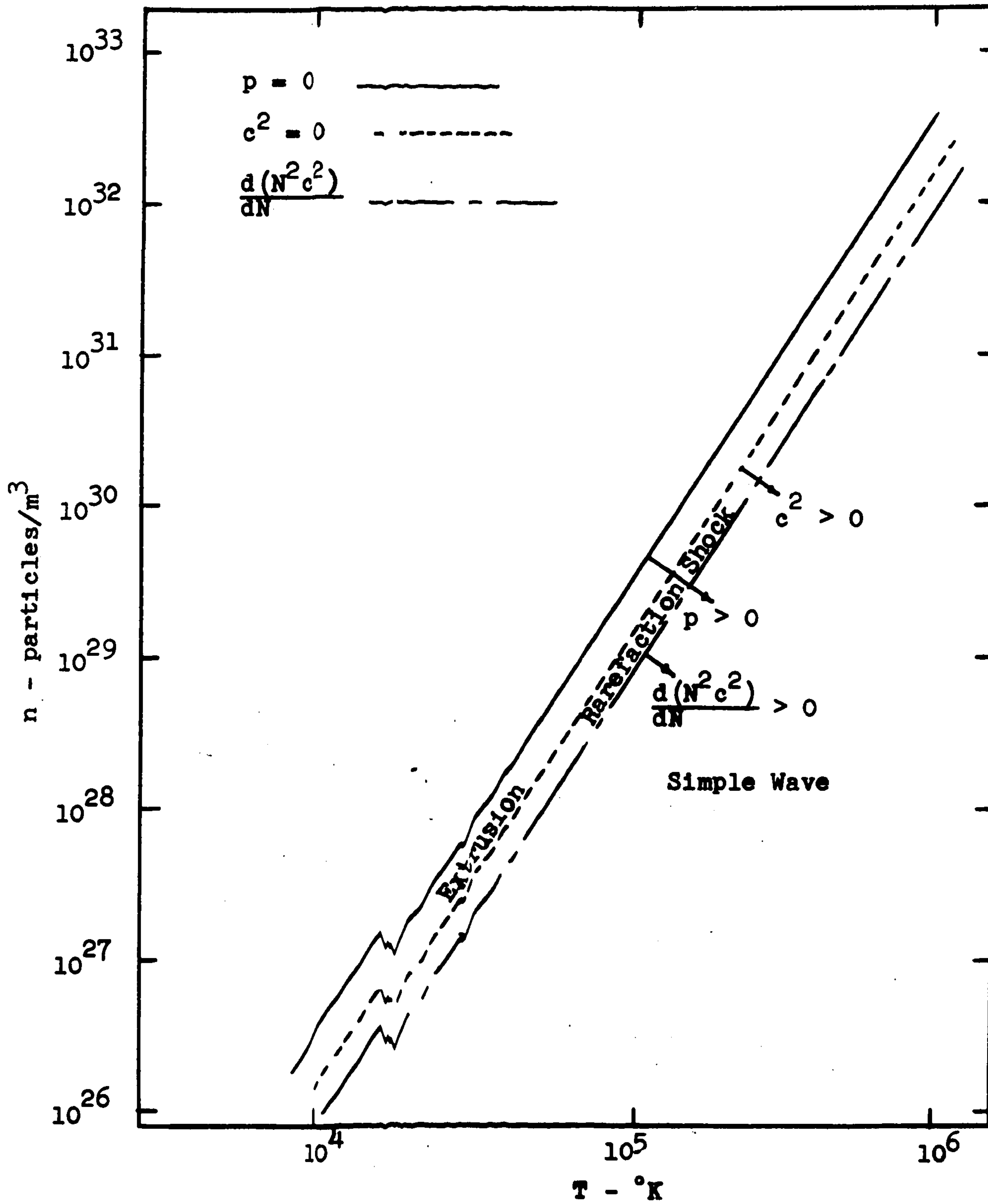

Figure 13. Plots of $p = 0$, $c^2 = 0$, and $d(N^2c^2)/dN = 0$ as functions of density n and temperature T for an isothermal expansion.

CHAPTER VII

CONCLUSIONS AND AN OBSERVATION

Conclusions

Perhaps the most significant result obtained in this work has been the demonstration of the existence under suitable conditions of rarefaction shock waves in a plasma. It has been shown that the existence of such a wave is the result of the energy balance in the electron gas of a plasma in which the electrons are the principal plasma-motive agent. Thus the characterization of the type of rarefaction process present in a plasma has been shown to be critically dependent on the microscopic interactions between the electrons and the massive particles in the plasma.

In general the nature of the rarefaction process has been shown to be characterized in the following way:

Extrusion: $c^2 < 0$.

Simple Rarefaction Wave: $c^2 > 0$, $\frac{d(N^2c^2)}{dN} > 0$.

Rarefaction Shock Wave: $c^2 > 0$, $\frac{d(N^2c^2)}{dN} < 0$.

The general functional dependence of c^2 and $d(N^2c^2)/dN$ on the electron temperature and degree of ionization in an electrodiabatic rarefaction have been shown to be given by the following:

$$c^2 = \frac{\alpha k}{M}\left(T_- - \frac{2\varphi}{\varphi'} - \frac{3}{2}\,\varphi\right)$$

$$\frac{d(N^2c^2)}{dN} = \frac{2\alpha kN}{M}\left(T_- - \frac{\varphi}{\varphi'} - \frac{2\varphi^2\varphi''}{\varphi'^3}\right)$$

where the quantity φ is proportional to the average energy an electron loses in a collision and is a function of the electron temperature and degree of ionization in the plasma.

The explicit dependence of φ or $\varkappa$ ($\varphi \equiv \varkappa T_-$) on electron temperature and degree of ionization has been derived for a strongly ionized hydrogen plasma. The result has been applied to the above criteria to determine the electron temperature regions for each type of rarefaction process - extrusion, simple wave, and rarefaction shock.

The theory for electrodiabatic rarefaction waves has been utilized in the general flow problem of the electric shock tube to determine the velocity of the compressional shock wave produced in terms of the known values of the electron temperature and degree of ionization in the "driver" plasma.

Observation

This work has been devoted principally to the rarefaction waves in a plasma, but a point should be made pertaining to compressional waves. If the conditions

$$c^2 > 0 \text{ and } \frac{d(N^2c^2)}{dN} < 0$$

are satisfied in a plasma undergoing compression, the compressional wave is propagated as a simple wave with its profile spreading out in time. Thus under the conditions that a rarefaction wave becomes a shock wave, a compressional wave remains a simple wave.

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