ON QUASI-VARIETIES OF LATTICE ORDERED GROUPS GENERATED BY CYCLIC EXTENSIONS

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PREFACE

A variety of ℓ-groups is an equational class consisting precisely of those ℓ-groups which satisfy a given set of equations. Alternatively, varieties are those classes of ℓ-groups closed with respect to ℓ-subgroups, cardinal products, and ℓ-homomorphic images. A quasi-variety of ℓ-groups is a class of ℓ-groups closed with respect to ℓ-subgroups, cardinal products, and ultraproducts. These can also be described as the classes which are determined by given sets of implications. Thus, a quasi-variety of ℓ-groups is a weaker algebraic structure than a variety of ℓ-groups.

In Chapter I we give a brief review of the theory of partially ordered groups, wreath products, and notations that are necessary for the discussion that follows. Chapter II is devoted primarily to considering special types of ℓ-groups called cyclic extensions. In this chapter we also give several examples which are the most general types of cyclic extensions. Several theorems are proved which describe when two cyclic extensions can be compared.

In the third chapter we study representable ℓ-groups. An ℓ-group is representable iff it satisfies the equation

$$(x \wedge y)^2 = x^2 \wedge y^2 .$$

Let $\mathcal{A}$ be the variety of abelian ℓ-groups, M^+ and M^- be the Medvedev varieties, and $\mathcal{R}$ be the variety of representable ℓ-groups. In this chapter we first construct two infinite chains of distinct quasi-varieties between $\mathcal{A}$ and $\mathcal{R} \cap \mathcal{A}^2$ using properties of M^-. Similarly, infinite chains of distinct quasi-varieties of representable ℓ-groups can be constructed using properties of M^+.

Chapter IV contains a study of nonrepresentable, normal valued ℓ-groups. An ℓ-group G is normal valued iff for every $a,b \in G$ $a^{-1}b^{-1}ab << |a|v|b|$. Let G_n, S_n, H_o, and L_n be the Scrimger ℓ-group, the variety generated by it, the quasi-variety generated by it, and the variety of ℓ-groups defined by $[x^n,y^n] = 1$, respectively. We know that if n is prime S_n covers $\mathcal{A}$ and is contained in L_n in the lattice of varieties. First, we find a set of implications satisfied by H_o and then construct an ℓ-group that fails those implications but contains G_n. Next, we generalize this construction to get an infinite chain of distinct nonrepresentable ℓ-groups between $\mathcal{A}$ and $L_n \cap \mathcal{A}^2$, and hence an infinite chain of distinct quasi-varieties of nonrepresentable ℓ-groups between $\mathcal{A}$ and $L_n \cap \mathcal{A}^2$.

Chapter V contains a study of three important properties of classes of ℓ-groups: the subalgebra property, the divisibility property, and the amalgamation property. It is devoted to proving that every quasi-variety of ℓ-groups which is an element of one of the chains constructed in either Chapters III or IV fails the amalgamation property. Also, relations between these properties are given and used to establish whehter or not a quasi-variety of ℓ-groups satisfies or fails the amalgamation property.

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CHAPTER I

INTRODUCTORY CONCEPTS AND NOTATION

In this chapter we review many of the basic definitions and notations necessary for our study of quasi-varieties of ℓ-groups. A more complete discussion can be found in Bigard [2].

By Z and R we will mean the sets of integers and real numbers, respectively. Unless otherwise specified, these sets will have their usual operations and order.

A partially ordered set (poset) is a pair $(A, \leq)$ where A is a set and $\leq$ is a relation on A which is reflexive, antisymmetric, and transitive. The set $(A, \leq)$ is called a chain if $\leq$ defines a total order on A (i.e., if $a,b \in A$ either $a \leq b$ or $b \leq a$). If A, B are partially ordered sets, then we will use the following notation. By A-B we mean the set $\{a \in A | a \notin B\}$. Let A be a poset and H a subset of A. The set H has an upper bound (lower bound) if and only if there exists an element $x \in A$ such that $a \leq x$ $(x \leq a)$ for every element $a \in H$. Now, $(A, \leq)$ is a lattice if $(A, \leq)$ is a poset such that for every two elements a and b in P the least upper bound and the greatest lower bound of the set $\{a,b\}$ exist. As for notations the least upper bound of $\{a,b\}$ is denoted by $a \vee b$ (a join b), and the greatest lower bound of $\{a,b\}$ by $a \wedge b$ (a meet b).

The lexicographic product of A and B is the usual direct product $A \vec{\times} B$ ordered by the relation $(a_1,b_1) \leq (a_2,b_2)$ if and only if $a_1 < a_2$ or $a_1 = a_2$ and $b_1 < b_2$. $A \vec{\times} B$ is ordered analogously. Now, if (A, ., $\leq$ is a group, then A^{-1} will be $\{a^{-1} | a \in A\}$. Also, $a_1 << a_2$ for $a_1, a_2 \in A$ means $a_1^n \leq a_2$ for all $n \in Z$.

A <u>partially ordered group</u> is a group $(G, \cdot)$ together with a partial order $\leq$ such that $a \leq b$ implies $ac \leq bc$ and $ca \leq cb$ for all $a, b, c \in G$. If the order is a lattice order, then G is called an <u>ℓ-group</u>, and if the order is total, then G is called a totally ordered group.

If X is any ordered group, where 0 is the identity element, then X^+ will denote $\{x \in X | x \geq 0\}$. X^+ is called the <u>positive cone</u> of X. This set completely determines the partial order of the set structure. For X^+ to be the positive cone of a partially ordered group, it is necessary and sufficient that $X^+ \wedge (X^+)^{-1} = \{0\}$, $X^+ \cdot X^+ \leq X^+$ and $x X^+ x^{-1} \leq X^+$ for all $x \in X$.

Let X be an ordered set and $\Upsilon: X \to X$, Υ is an <u>o-permutation</u> of X if and only if Υ is a bijective map where if $x \leq y$ then $\Upsilon(x) \leq \Upsilon(y)$. An <u>o-homomorphism</u> is a homomorphism ϕ: $X \to Y$ between partially ordered groups such that $x_1 \leq x_2$ implies $\phi(x_1) \leq \phi(x_2)$. An <u>o-monomorphism</u> is an injective o-homomorphism while an <u>o-epimorphism</u> ϕ: $X \to Y$ is a surjective o-homomorphism such that $\phi(X^+) = Y^+$. An <u>o-isomorphism</u> is a bijection ϕ such that both ϕ and ϕ^{-1} are o-epimorphisms.

An <u>ℓ-homomorphism</u> is a homomorphism ϕ: $X \to Y$ such that $\phi(x_1 \wedge x_2) = \phi(x_1) \wedge \phi(x_2)$ and $\phi(x_1 \vee x_2) = \phi(x_1) \vee \phi(x_2)$ for all $x_1, x_2 \in X$. As in the preceding paragraph, we define <u>ℓ-monomorphism</u>, <u>ℓ-epimorphism</u>, and <u>ℓ-isomorphism</u>.

A subset S of a partially ordered group X is called convex if $0 \leq x \leq s$ and $s \in S$ imply $x \in S$. An <u>o-ideal</u> I of X is a kernel of some o-epimorphism, and if X is lattice ordered, the kernel of an ℓ-epimorphism is called an <u>ℓ-ideal</u>. For ℓ-groups, the ℓ-ideals are the convex normal ℓ-subgroups (i.e., subgroups that are also sublattices).

If G is an ℓ-group and C a normal convex ℓ-subgroup of G, then $G(C) = \{xC | x\varepsilon G\}$ is the set of <u>left cosets</u>. If C is a maximal convex ℓ-subgroup with respect to not containing a where $a\varepsilon G$, then C is a <u>value of a</u> in G. Now C is <u>prime</u> if whenever $C = A \cap B$ where A and B are convex ℓ-subgroups of G then A=C or B=C. In this case let u be the <u>canonical map</u> from G into Aut G(C) (i.e., automorphisms of the set G(C)) defined by: for every g, $h \varepsilon G$, $(u(g))(hC) = ghC$. For u(G) to <u>operate transitively</u> in G(C), means for every x and $y \varepsilon G(C)$, there exists $\phi \varepsilon u(G)$ such that $\phi(x) = y$. An ℓ-group G is <u>transitive</u> if there exists a totally ordered set T and an ℓ-monomorphism $u:G \to \text{Aut } T$, such that u(G) operates transitively in T. Holland has proved that every ℓ-group can be embedded in $\text{Aut}(\Lambda)$ for some totally ordered set Λ where the cardinal of Λ is smaller than or equal to the cardinal of G. Hence, every ℓ-group is isomorphic to a subdirect product of transitive groups (see later in the chapter for the definition of subdirect product).

If X is lattice ordered and $x \varepsilon X$, then $x^+ = xvo$ is called the <u>positive part</u> of x, $x^- = (x^{-1})vo$ is called the <u>negative part</u> of x, and $|x| = xvx^{-1}$ is called the <u>absolute value</u> of x. Note that $x = x^+(x^-)^{-1}$ and $|x| = x^+ \cdot x^-$. Also, if x and y are elements of x, $y^x = x^{-1}yx$ is called the <u>conjugate</u> of y by x.

If x and y are ordered, algebraic structures, then X + Y will denote the usual <u>direct sum</u> without regard to order. If we write X + Y, we mean the <u>cardinal sum</u> which is the usual direct sum but with order determined by $(X + Y)^+ = \{(x,y) \varepsilon X + Y \mid x\varepsilon X^+, y\varepsilon Y^+\}$. If X and Y are two posets, then $XxY = \{(a,b)$ such that $a\varepsilon X$ and $b\varepsilon Y\}$ with the ordering $\leq$ defined by $(a,b) \leq (c,d)$ if and only if $a \leq b$ and $c \leq d$. $(XxY, \leq)$ is called the <u>direct product</u> of X and Y. Also, if X and Y are ℓ-groups, then the direct product is the *cardinal product* of X and Y.

Note that all types of products and sums can be defined for any collection $(X_\alpha)_{\alpha \varepsilon A}$ of ordered sets.

A subset S of a direct product $\pi_{\alpha \varepsilon A} S_\alpha$ is called a *subdirect product* of the S_α if the projection on each S_α is a surjection. An object S is said to be *subdirectly irreducible* if whenever it is a subdirect product of a collection $(S_\alpha \mid \alpha \varepsilon A)$, then at least one of the projections is an isomorphism. Let C be a class of objects, then every object in C is a subdirect product of subdirectly irreducible objects in C if and only if C is an equational class. This is also equivalent to the condition that C is closed under the formation of products, substructures, and quotients. An equational class C is called a *variety* of objects. In particular, we denote by $\underline{A}$, the variety of *abelian ℓ-groups*, $\underline{R}$ the variety of *representable ℓ-groups* where every ℓ-group G satisfies the equation $x^{+} \wedge (y^{-1}.x^{-1}.y) = 1_G$ for any x,y in the ℓ-group G and where 1_G is the identity element in G, N the variety of *normal valued ℓ-groups* and L the variety of *all ℓ-groups*. We know that $A \lneq R \lneq N \lneq L$.

Let I be an index set with a minimal element z, F a subset of I such that F is stable with respect to finite intersection, z does not belong to F, and x ε F and t ≥ x implies t ε F. Then F is a *filter* of I. It is an *ultrafilter* if it is a maximal filter. If C is a class of ℓ-groups, $(G_\alpha \mid \alpha \varepsilon I)$ a collection of elements of C, and U an ultrafilter over I, then an ultraproduct is $\pi_{\alpha \varepsilon I} G_\alpha / U$. Now, if C is closed under the formation of products, substructures, and ultraproducts, then C is called a *quasi-variety* of objects. And every object in C is a direct product of ultraproducts (i.e., $\pi_{j \varepsilon I} \pi_{i \varepsilon I_\alpha} G_i / U_\alpha$ where I, I_α are index sets, U_α an ultrafilter over I_α and G_i an object in C for every i).

Let A and B be groups, we denote by $A^{[B]}$, the group of all functions from B into A with the usual multiplication and by $A^{(B)}$ the subgroup of $A^{[B]}$ consisting of all such functions with finite support. For each $f \in A^{[B]}$ and $b \in B$ we define a function $f(b)$ by $f(b)(b') = f(bb')$. Consider the set product $BxA^{[B]}$ with multiplication given by $(b,f)\ (b',f') = (bb',\ f(bb')\cdot f(b))$, then $BxA^{[B]}$ is called the big wreath product of the group A by the group B and is denoted by A Wr B, while the subgroup $BxA^{(B)}$ of $BxA^{[B]}$ is their small wreath product and is denoted by A wr B.

CHAPTER II

CYCLIC EXTENSIONS

In this chapter special types of ℓ-groups are studied along with methods of constructing them. In later chapters their importance will be seen in generating quasi-varieties and in determining whether or not a class of ℓ-groups fails the amalgamation property.

Definition 2.1 Let $(G,.)$ be an ℓ-group, α an ℓ-automorphism of G, and

$$G(\alpha) = G\times\langle\alpha\rangle = \{(g,\alpha^n) \mid g\varepsilon G,\ n\varepsilon Z\} .$$

$G(\alpha)$ is called the cyclic extension of G by α.

$G(\alpha)$ carries a natural group structure if we define multiplication as follows:

$$(g,\alpha^n)\cdot(h,\alpha^m) = (g\cdot\alpha^n(h),\ \alpha^{n+m}) .$$

Note that $(g,\alpha^n)^{-1} = (\alpha^{-n}(g^{-1}),\alpha^{-n})$, and the identity in $G(\alpha)$ is $(1_G,\ \alpha^0)$.

Also, define an order $\leq$ on $G(\alpha)$ by $(g,\alpha^n) \leq (h,\ \alpha^m)$ iff $n < m$ or $n = m$ and $g \leq h$. This defines a lattice order on $G(\alpha)$. In fact,

$$(g,\alpha^n) \wedge (h,\alpha^m) = \begin{cases} (g,\alpha^n) & \text{if } n<m \\ (h,\alpha^m) & \text{if } n>m \\ (g\wedge h,\alpha^n) & \text{if } n=m \end{cases}$$

and

$$(g,\alpha^n) \vee (h,\alpha^m) = \begin{cases} (g,\alpha^n) & \text{if } n>m \\ (h,\alpha^m) & \text{if } n<m \\ (g\vee h,\alpha^n) & \text{if } n=m \end{cases} .$$

Further, $G(\alpha)$ is an ℓ-group. We need to verify only that if $a,b,c \in G(\alpha)$ where $a\leq b$, then $ac \leq bc$ and $ca \leq cb$. Since a,b,c are in $G(\alpha)$ write $a = (g,\alpha^n)$, $b = (h,\alpha^m)$, and $c = (f,\alpha^k)$. Now $a\leq b$ implies that $n<m$ or $n=m$ and $g\leq h$. If $n<m$, then $n+k < m+k$ and $ac = (g,\alpha^n)(f,\alpha^k) = (g\cdot\alpha^n(f),\ \alpha^{n+m}) \leq (h\cdot\alpha^m(f),\ \alpha^{m+k}) = (h,\alpha^m)\cdot(f,\alpha^k) = bc$. Otherwise, if $n=m$, then $g \leq h$ implies $g\cdot\alpha^n(f) = g\cdot\alpha^m(f) \leq h\cdot\alpha^m(f)$ and $ac \leq bc$. We can verify in a similar way that $ca \leq cb$.

One remark is that G can be embedded in $G(\alpha)$, and is isomorphic to $G(\alpha)$ whenever α is the identity automorphism. To see this just identify each element g of G with the element (g,α^0) of $G(\alpha)$.

It is also important to observe that $G(\alpha)$ is abelian iff G is abelian and α is the identity map. Clearly, if $G(\alpha)$ is abelian, then G is abelian since $G \leq G(\alpha)$. Suppose α is not the identity automorphism. Then there exists an element g of G such that $\alpha(g) \neq g$. Let $x = (g,\alpha^0)$, and $y = (1_G,\alpha)$. The two elements x,y of $G(\alpha)$ do not commute, since $x\cdot y = (g,\alpha^0)(1_G,\alpha) = (g\cdot\alpha^0(1_G),\alpha) = (g\cdot 1_G,\alpha) = (g,\alpha)$, and $y\cdot x = (1_G,\alpha)(g,\alpha^0) = (1_G\cdot\alpha(g),\alpha) = (\alpha(g),\alpha)$.

Cyclic extensions of ℓ-groups have been used in the literature to generate varieties and to study the amalgamation and divisible embedding properties. The following example was used by Feil [4] to create uncountably many different varieties.

Example 2.2. Let $G = (R,+)$, t any number inside the interval $[0,1]$, and α_t an automorphism of G defined by the following: For every r in G, $\alpha_t(r) = (\frac{t}{t+1})r$. Consider $G(\alpha_t)$ to be the cyclic extension of the real numbers. Let U_t be the variety generated by $G(\alpha_t)$. The variety U_t strictly contains $\mathcal{A}$ (variety of abelian ℓ-groups) and is contained in $\mathcal{R}$ (variety of representable ℓ-groups) since U_t is totally ordered.

In [4] Feil established the following result.

Theorem 2.3. The varieties U_t $(0 < t < 1)$ constructed in the above example are distinct and form an uncountable chain between $\mathcal{A}$ and $\mathcal{R}$.

In order to prove this theorem, first he proved that if $0 \leq t \leq \frac{p}{q} \leq 1$ where p,q are in Z_+ then $G(\alpha_t)$ satisfies the equation

$$|[x,|[x,y]|]|^p \geq |[x,y]|^q \ .$$

Further, if $0 \leq \frac{p}{q} \leq t \leq 1$, then $G(\alpha_t)$ fails the equation for p and q.

More generally, cyclic extensions can also be constructed by using not just permutations over the totally ordered group itself, but also permutations over any totally ordered set. Therefore, the following examples arise.

Example 2.4. (i) Let G be an ℓ-group, I a totally ordered set, and $\bar{\alpha}$ an o-permutation of I. Let $\bar{G} = \bigoplus_{i\in I} G_i$, where $G_i \simeq G$. Define an automorphism α: $\bar{G} \to \bar{G}$ by $\alpha(\bar{g}) = \bar{h}$ where $\bar{h}_i = \bar{g}_{\bar{\alpha}(i)}$. Consider $\bar{G}(\alpha)$ the cyclic extension of $\bar{G}$ by α. If G is any group, then $\bar{G}(\alpha)$ is generated by the set of elements $\{e, e_i \mid i \in I\}$ where

$$e = (\bar{0},\alpha)$$

$$e_i = (\bar{a},\alpha^0) \text{ where } \bar{a}_j = \begin{cases} 1_G & \text{if } j=i \\ 0 & \text{if } j\neq i \end{cases} \quad \text{and}$$

1_G is the identity element of G.

Multiplication in $\bar{G}(\alpha)$ is given by

$$(\bar{g},\alpha^n)(\bar{h},\alpha^m) = (\bar{g} + \alpha^n(\bar{h}),\alpha^{n+m}) = (\bar{k},\alpha^{n+m})$$

where $\bar{k}_i = \bar{g}_i + \bar{h}_{\bar{\alpha}^n(i)}$.

(ii) Let $G = \bigoplus_Z Z$ and $\alpha:G \to G$ be defined as follows: for an element $\bar{a}$ of G, write $\bar{a} = (a_i)_{i\varepsilon Z}$ where $a_i \varepsilon Z$ and $a_i \neq 0$ for only finitely many indexes in Z, $\alpha(\bar{a}) = \bar{b}$ where $\bar{b} = (b_i)_{i\varepsilon Z}$, $b_i \varepsilon Z$ and $b_i = a_{i+1}$. Consider the cyclic extension, $G(\alpha)$, of G by α. Note that $\alpha^n(\bar{a}) = \bar{c} = (c_i)_{i\varepsilon Z}$ where $c_i = a_{i+n}$. Multiplication is defined by

$$(\bar{g},\alpha^n)(\bar{h},\alpha^m) = (\bar{g} + \alpha^n(\bar{h}),\alpha^{n+m}) = (\bar{k},\alpha^{n+m})$$

where $k_i = g_i + h_{i+n}$.

Now, let ZwrZ (i.e. the small wreath product of Z and Z), be the set $\{(\bar{g},n)$ where $\bar{g} = (g_i)_{i\varepsilon Z}$ where $g_i \neq 0$ for finitely many i's, $g_i \varepsilon Z$, and $n\varepsilon Z\}$. Multiplication in ZwrZ is defined by

$$(\bar{g},n)(\bar{h},m) = (\bar{k},n+m) \text{ where } \bar{k}_i = \bar{g}_i + \bar{h}_{i+n} .$$

Note that $G(\alpha)$ is isomorphic to ZwrZ. If ϕ is the map from $G(\alpha)$ into ZwrZ defined by $\phi((\bar{g},\alpha^n)) = (\bar{g},n)$, then ϕ is a well-defined isomorphism. First, let us show that ϕ is a homomorphism. Let $(\bar{g},\alpha^n)$ and $(\bar{h},\alpha^m)$ be two arbitrary elements of $G(\alpha)$. Then, $\phi((\bar{g},\alpha^n)\cdot(\bar{h},\alpha^m)) = \phi((\bar{g} + \alpha^n(\bar{h}),$

$\alpha^{n+m})) = (\bar{g} + \alpha^n(\bar{h}),n+m) = (\bar{k},n+m)$ where $k_i = g_i + h_{i+n}$. But $(\bar{g},n)(\bar{h},m) = (\bar{k},n+m)$. Therefore, $\phi((\bar{g},\alpha^n)\cdot(\bar{h},\alpha^m)) = \phi((\bar{g},\alpha^n))\ \phi((\bar{h},\alpha^m))$. The map ϕ is clearly a bijection.

(iii) Let G in (ii) be furnished with the following order: $(\bar{a},\alpha^n)\geq 0$ if and only if $n>0$ or $n=0$ and $a_i\geq 0$ where is is the maximum index with respect to $a_i\neq 0$. Then G is an ℓ-group and is isomorphic to $W^- = Z\vec{w}rZ$, where W^- is ZwrZ with a similar order as G, (i.e. $(\bar{g},n) \geq 0$, if and only $n>0$ or $n=0$ and $g_i \geq 0$ where is is the maximum index with respect to $g_i\neq 0$). Thus, ϕ defined in (ii) is an ℓ-isomorphism. Hence, W^- is a cyclic extension.

Now, if we change the order on G, and define it as follows: $(\bar{a},\alpha^n) \geq 0$ if and only if $n \geq 0$ or $n = 0$ and $a_i \geq 0$ where is is the minimum index with respect to $a_i\neq 0$, then $W^+ = Z\overleftarrow{w}rZ$, which is ZwrZ with similar order as G, (i.e., $(\bar{g},n) \geq 0$ if and only if $n \geq 0$ or $n=0$ and $g_i \geq 0$ where i is the minimum index with respect to $g_i \neq 0$). Again, ϕ in (ii) is an ℓ-homomorphism, and W^+ is a cyclic extension.

Let M^+ and M^- be the varieties generated by W^+ and W^- respectively, Medvedev [13] has established that M^+ and M^- are both covers of A.

(iv) Let $G = \prod_0^{n-1} Z$, $\bar{\alpha}$ a permutation of the set $\{0,1,2,\ldots,n-1\}$ defined by $\bar{\alpha}(i) = (i+1)$ modulo(n). Define an automorphism $\alpha:G \to G$ by, $\alpha(g) = h$ where $h_i = g_{\bar{\alpha}(i)}$ $i = 0,\ldots,n-1$. Consider $G(\alpha)$ the cyclic extension of G by α. Multiplication in $G(\alpha)$ is defined by $(g_0,\ldots,g_{n-1};\alpha^m)$ $(h_0,\ldots,h_{n-1};\alpha^p) = (g_0+h_{\alpha^m(0)},\ldots,g_i+h_{\alpha^m(i)},\ldots,g_{n-1}+h_{\alpha^m(n-1)};\alpha^{m+p}) = (g_0+h_{0+m},\ldots,g_i+h_{i+m},\ldots,g_{n-1} + h_{n-1+m};\alpha^{m+p})$ where all subscripts are read modulo n.

Now, the Scrimger ℓ-group G_n of order n is defined by Scrimger [21] as follows:

$G_n = \{(a_0,..,a_{n-1};n)$ where $n \varepsilon Z$, and $a_i \varepsilon Z$ for every $i = 0,..,n-1\}$, furnished with the following order and multiplication. An element $(a_0,..,a_{n-1};m) \geq 0$ if and only if $m>0$ or $m=0$ and $a_i \geq 0$ for every $i=0,..,n-1$, and $(a_0,..,a_{n-1};m)(b_0,..,b_{n-1};p) = (a_0+b_{0+m},..,a_{n-1} + b_{n-1+p};m+p)$ where all subscripts are read modulo n.

Next, define an order over $G(\alpha)$ as follows: $(g_0,..,g_{n-1}:\alpha^m) \geq (0,0)$ if and only if $m>0$ or $m=0$ and $g_i \geq 0$ for every $i=0,..,n-1$. Then $G(\alpha)$ is an ℓ-group and is isomorphic to G_n. Let $\phi:G(\alpha) \to G_n$ be a map defined by $\phi((g_0,..,g_{n-1};\alpha^m)) = (g_0,..,g_{n-1};m)$. Easily, we can prove as in (ii) that ϕ is an ℓ-isomorphism.

Note that if n is a prime number the variety S_n generated by G_n covers A (Scrimger [21]).

These types of cyclic extensions will play an important role in constructing countably infinite different quasi-varieties in Chapters III and IV. Therefore, it is necessary to learn how to compare cyclic extensions.

<u>Theorem 2.5</u>: Let Λ_1, Λ_2 be two totally ordered sets, $G = \bigoplus_{\Lambda_1} Z$, $G' = \bigoplus_{\Lambda_2} Z$, and $\bar{\alpha}_1$, $\bar{\alpha}_2$ be o-permutations of Λ_1 and Λ_2, respectively. Consider $G(\alpha_1)$ and $G'(\alpha_2)$. If there exists an injective map γ_o from Λ_1 into Λ_2 such that $\bar{\alpha}_2 o \gamma_o = \gamma_o o \bar{\alpha}_1$ then $G(\alpha_1) \leq G'(\alpha_2)$.

<u>Proof</u>: Suppose there exists such a γ_o so that the diagram on the next page commutes. Let ϕ be the map from $G(\alpha_1)$ into $G'(\alpha_2)$ defined by $\phi((g,\alpha_1^k)) = (h,\alpha_2^k)$ where $h_{\gamma o(i)} = g_i$ and $h_j = 0$ otherwise. The map ϕ is well defined since γ_o is injective. Also ϕ is an ℓ-homomorphism. Let

h_1, h_2 be elements of $G(\alpha_1)$. We need to show that $\phi(h_1 \cdot h_2) = \phi(h_1)\cdot\phi(h_2)$. First, $h_1 = (g,\alpha_1^n)$, $h_2 = (h,\alpha_1^m)$, and $h_1 \cdot h_2 = (g,\alpha_1^n)(h,\alpha_1^m) = (k,\alpha_1^{n+m})$ where $k_i = g_i + h_{\bar{\alpha}_1^n(i)}$. Now $\phi(h_1 \cdot h_2) = (t,\alpha_2^{n+m})$ where $t_{\gamma_o(i)} = g_i + h_{\bar{\alpha}_1^n(i)}$ and $t_j = o$ otherwise. But $\phi(h_1) = (\ell, \alpha_2^n)$, $\phi(h_2) = (p,\alpha_2^m)$ where $\ell_{\gamma_o(i)} = g_i$, $p_{\gamma_o(i)} = h_i$ and $\ell_j = p_j = o$ otherwise. Now $\phi(h_1)\cdot\phi(h_2) = (\ell,\alpha_2^n)\cdot(p,\alpha_2^m) = (q,\alpha_2^{n+m})$ where $q_{\gamma_o(i)} = \ell_{\gamma_o(i)} + p_{\bar{\alpha}_2^n(\gamma_o(i))}$. Note that $\bar{\alpha}_2^n o \gamma_o = \gamma_o o \bar{\alpha}_1^n$ since $\bar{\alpha}_2 o \gamma_o = \gamma_o o \bar{\alpha}_1$. Then $q_{\gamma_o(i)} = \ell_{\gamma_o(i)} + p_{\gamma_o(\bar{\alpha}_1^n(i))} = g_i + h_{\bar{\alpha}_1^n(i)} = t_{\gamma_o(i)}$.

$$\begin{tikzcd} \Lambda_1 \arrow[r, "\gamma_o"] \arrow[d, "\bar{\alpha}_1"'] & \Lambda_2 \arrow[d, "\bar{\alpha}_2"] \\ \Lambda_1 \arrow[r, "\gamma_o"'] & \Lambda_2 \end{tikzcd}$$

So, $\phi(h_1)\phi(h_2) = \phi(h_1 \cdot h_2)$ and ϕ is a homomorphism. The map ϕ is clearly injective. Since $G(\alpha_1)$ and $G'(\alpha_2)$ are totally ordered, then ϕ is an ℓ-monomorphism.

The above theorem remains true if γ_o is found so that $\bar{\alpha}_2 o \gamma_o = \gamma_o o \bar{\alpha}_1^k$ for some k in Z^+.

However, the converse is not true. The following example is an illustration of this fact.

Example 2.6: Let $\Lambda_1 = \Lambda_2 = Z$ in the theorem. Then $G = G' = \bigoplus_Z Z$. Let $\bar{\alpha}_2$ be the o-permutation of Z that maps n to n+1, and $\bar{\alpha}_1$ the identity on Z. There is no γ_o from Z to Z that makes the diagram commute in the theorem. Suppose there exists γ_o such that $\bar{\alpha}_2 o \gamma_o = \gamma_o o \bar{\alpha}_1$, and let n be any element of Z. Now $\bar{\alpha}_2 o \gamma_o(n) = \bar{\alpha}_2(\gamma_o(n)) = \gamma_o(n)+1$, and $\gamma_o o \bar{\alpha}_1(n) = \gamma_o(\bar{\alpha}_1(n)) = \gamma_o(n)$ $\gamma_o(n)$. If $\bar{\alpha}_2 o \gamma_o = \gamma_o o \bar{\alpha}_1$, then $\gamma_o(n) = \gamma_o(n)+1$ and γ_o is the constant zero function and, therefore, not injective.

Note that $G(\alpha_1) \simeq G \leq G(\alpha_2)$, but there exists no injective mapping between Λ_1 and Λ_2.

Note that Theorem 2.7 remains valid if the cyclic extension over the totally ordered group is taken by adjoining an order-preserving automorphism of the group itself.

Corollary 2.7: Let G_1, G_2 be two totally ordered groups and α_1 and α_2 two o-automorphisms of G_1 and G_2, respectively. Suppose there exists an o-monomorphism γ_o between G_1 and G_2 such that $\gamma_o o \alpha_1 = \alpha_2^{\lambda} o \gamma_o$ for some γ in Z. Then $G_1(\alpha_1)$ is ℓ-isomorphic to an ℓ-subgroup of $G_2(\alpha_2)$.

Proof: Suppose there exists such γ_o. Define the map ϕ from $G_1(\alpha_1)$ into $G_2(\alpha_2)$ by

$$\phi((g, \alpha_1^k)) = (\gamma_o(g), \alpha_2^{\lambda k}) .$$

Clearly ϕ is a well-defined injective map (since γ_o is injective we need to prove that ϕ is an ℓ-homomorphism. Consider two elements (g, α_1^n) and (h, α_1^m) of $G_1(\alpha_1)$. Now

$$\phi((g,\alpha_1^n)(h,\alpha_1^m) = \phi(g\ \alpha_1^n(h),\ \alpha_1^{n+m})$$

$$= (\gamma_o(g\ \alpha_1^n(h)),\ \alpha_2^{\lambda(n+m)})$$

$$= (\gamma_o(g)\ \gamma_o(\alpha_1^n(h)),\ \alpha_2^{\lambda n+\lambda m})\ .$$

But $\gamma_o o\alpha_1^n = \alpha_2^{\lambda n} o\gamma_o$ because $\gamma_o o\alpha_1 = \alpha_2^{\lambda} o\gamma_o$. Then

$$\phi((g,\alpha_1^n)(h,\alpha_1^m)) = (\gamma_o(g)\ \alpha_2^{\lambda n}(\gamma_o(h)),\ \alpha_2^{\lambda n+\lambda m})$$

$$= (\gamma_o(g),\ \alpha_2^{\lambda n})(\gamma_o(h),\ \alpha_2^{\lambda m})$$

$$= \phi((g,\ \alpha_1^n))\cdot\phi((h,\alpha_1^m))\ .$$

Hence, ϕ is a homomorphism. Finally, since $G_1(\alpha_1)$ and $G_2(\alpha_2)$ are both totally ordered, ϕ is an ℓ-monomorphism.

To conclude this chapter, we prove a theorem that clarifies for which special cyclic extensions of totally-ordered groups the converse of theorem 2.7 is true.

<u>Theorem 2.8</u>: Let G be a totally ordered group and α_1 and α_2 two ordered preserving automorphisms of G. Consider $G(\alpha_1)$ and $G(\alpha_2)$. There exists an ℓ-monomorphism between $G(\alpha_1)$ and $G(\alpha_2)$ that fixes G iff there exists an o-monomorphism γ_o of G such that $\gamma_o o\alpha_1 = \alpha_2^{\lambda} o\gamma_o$ for some λ in Z.

<u>Proof</u>: (1) Suppose there exists a monomorphism γ_o of G such that $\gamma_o o\alpha_1 = \alpha_2^{\lambda} o\gamma_o$. Define the map ϕ from $G(\alpha_1)$ into $G(\alpha_2)$ by

$$\phi((g,\alpha_1^k)) = (\gamma_o(g),\ \alpha_2^{\lambda k})\ .$$

Clearly ϕ is a well-defined injection since γ_o is. It remains to prove that ϕ is an ℓ-homomorphism. First, let us prove that ϕ is a homomorphism. Consider two elements (g,α_1^n) and (h,α_1^m) of $G(\alpha_1)$. We have

$$\begin{aligned}\phi((g,\alpha_1^n)\cdot(h,\alpha_1^m)) &= \phi((g\ \alpha_1^n(h),\ \alpha_1^{n+m}))\\ &= (\gamma_o(g\ \alpha_1^n(h)),\ \alpha_2^{\lambda(n+m)})\\ &= (\gamma_o(g)\cdot\gamma_o(\alpha_1^n(h)),\ \alpha_2^{\lambda n+\lambda m}).\end{aligned}$$

But $\gamma_o o\alpha_1^n = \alpha_2^{\lambda n} o\gamma_o$ because $\gamma_o o\alpha_1 = \alpha_2^{\lambda} o\gamma_o$. So

$$\begin{aligned}\phi((g,\alpha_1^n)(h,\alpha_1^m)) &= (\gamma_o(g)\ \alpha_2^{\lambda n}(\gamma_o(h)),\ \alpha_2^{\lambda n+\lambda m})\\ &= (\gamma_o(g),\ \alpha_2^{\lambda n})(\gamma_o(h),\ \alpha_2^{\lambda m})\\ &= \phi((g,\ \alpha_1^n))\cdot\phi((h,\ \alpha_1^m)).\end{aligned}$$

Hence ϕ is a homomorphism and since $G(\alpha_1)$ and $G(\alpha_2)$ are totally ordered, clearly ϕ is an ℓ-homomorphism.

(2) Suppose there exists an ℓ-monomorphism ϕ between $G(\alpha_1)$ and $G(\alpha_2)$ that fixes G (i.e. $\phi(g,\alpha_1^0) = (g',\alpha_2^0)$). The map ϕ is an automorphism since $G(\alpha_1)$ and $G(\alpha_2)$ have the same cardinality. In fact $|G(\alpha_1)| = |G||Z| = |G(\alpha_2)|$. Define γ_o from G into itself by $\gamma_o(g) = g'$ where $\phi((g,\alpha_1^0)) = (g',\ \alpha_2^0)$. The map γ_o is an o-automorphism of G. Since ϕ is an o-automorphism γ_o is an o-automorphism of G. It remains to prove that $\gamma_o o\alpha_1 = \alpha_2^{\lambda} o\gamma_o$ for some λ in Z. Let λ be such that $\phi((1,\alpha_1)) = (1,\alpha_2^{\lambda})$. Now $\gamma_o o\alpha_1(g) = \gamma_o(\alpha_1(g)) = h$ where $\phi((\alpha_1(g),\alpha_1^0)) = (h,\alpha_2^0)$. But

$$\begin{aligned}
\phi((\alpha_1(g),\alpha_1^0)) &= \phi((1,\alpha_1)(g,\alpha_1^0)(1,\alpha_1^{-1})) \\
&= \phi((1,\alpha_1))\ \phi((g,\alpha_1^0))\ \phi((1,\alpha_1^{-1})) \\
&= (1,\alpha_2^\lambda)(g',\alpha_2^0)(1,\alpha_2^{-\lambda}) \\
&= (1\cdot\alpha_2^\lambda(g'),\ \alpha_2^\lambda)(1,\alpha_2^{-\lambda}) \\
&= (\alpha_2^\lambda(g'),\ \alpha_2^0).
\end{aligned}$$

Hence $h = \alpha_2^\lambda(g') = \alpha_2^\lambda(\gamma_o(g))$ and $\gamma_o o \alpha_1(g) = \alpha_2^\lambda o \gamma_o(g)$. Since g was chosen arbitrarily, $\gamma_o o\alpha_1 = \alpha_2^\lambda o \gamma_o$.

CHAPTER III

REPRESENTABLE ℓ-GROUPS

A variety of ℓ-groups is a class of ℓ-groups which is equationally defined. A quasi-variety is a class of ℓ-groups defined by a set of one or more implication. An equation or an implication may involve not only the group operations but also the lattice operation v and $\wedge$. The existence of an uncountable number of ℓ-group varieties was first shown by Kopytov and Medvedev. However, they use the fact that there exists an uncountable number of group varieties to show their result. The approach used in this chapter is constructive and does not rely on previous results in varieties or quasi-varieties. It will establish the existence of an infinite countable chain of quasi varieties, each of which contains the abelian variety and is contained in $\mathcal{R} \cap \mathcal{A}^2$. The ℓ-groups used to generate these quasi-varieties are totally ordered.

Definition 3.1: Let $(G_i | i \varepsilon I)$ be a family of ℓ-groups. Then H is a subdirect product of the G_i's iff H is an ℓ-subgroup of $\prod_I G_i$ such that every projection of H on a G_i is surjective. It is clear that an ℓ-group G is isomorphic to a subdirect product of the G_i's iff for every i in I, there exists an ℓ-epimorphism $u_i : G \to G_i$ such that $\bigcap_{i \varepsilon I} \text{Ker } u_i$ is reduced to the identity element.

Example 3.2: Let G be an arbitrary ℓ-group, $(M_i | i \varepsilon I)$ the family of minimal prime ℓ-subgroups, and $G(M_i)$ the left cosets of M_i. For every $i \varepsilon I$, let u_i be the canonical surjection from G onto $G(M_i)$ (i.e. $u_i(g) = gM_i$), and let $G_i = u_i(G)$. Then, the intersection of the

u_i-kernels is equal to the intersection of all the M_i's, and is therefore reduced to the identity element. Hence, G is a subdirect product of the G_i's. Note that the G_i's are transitive groups. Therefore, every ℓ-group is isomorphic to a subdirect product of transitive groups, and if the ℓ-group is commutative the G_i's can be chosen to be totally ordered (Bigard [2]).

The latter property is of course not valid for an arbitrary ℓ-group. However, the case where the G_i's are totally ordered is of special significance.

Definition 3.3: Let G be an ℓ-group. G is representable if G is isomorphic to a subdirect product of totally ordered groups. Equivalently, G is representable iff there exist a family of totally ordered groups $(G_i | i \in I)$ together with ℓ-epimorphisms $(u_i : G \to G_i | i \in I)$. Such that every G_i is totally ordered and $\cap \mathrm{Ker}\, u_i$ is reduced to the identity element.

Example 3.4: The following are examples of representable ℓ-groups. Other examples can be constructed from these by taking the product of such ℓ-groups, and quotients by ℓ-ideals.

(i) Any totally ordered group G is representable.

(ii) Let A, B be two totally ordered groups and AwrB be the small wreath product of A and B. AwrB can be totally ordered in two distinct ways: First, $(a,b) \leq (a',b')$ iff $b < b'$ or $b = b'$ and $a_i < a_i'$ where i is the smallest index such that $a_i \neq a_i'$. Secondly, $(a,b) \leq (a',b')$ iff $b < b'$ or $b = b'$ and $a_i < a_i'$ where i is the largest index such that $a_i \neq a_i'$.

(iii) In (ii) let A and B be both equal to Z. Since Z is totally ordered by the usual order ZwrZ is representable. Note that, W^+ and W^- defined in 2.4(iii), are obtained by defining two distinct total orders on ZwrZ.

The big wreath product AWrB of totally ordered groups does not admit any total order compatible with the group operation (see Neumann [14].

There are numerous characterizations of representable ℓ-groups. The two most useful ones are the following: An ℓ-group is representable iff G satisfies one of the following relations:

(i) $x \wedge y^{-1} \cdot x^{-1} \cdot y \leq e$,

(ii) $(x \wedge y)^2 = x^2 \wedge y^2$.

These two identities characterizing the representable ℓ-groups will permit us to prove whether an ℓ-group is representable or not. As we will notice in these examples.

<u>Example 3.5</u>: i) Let $G = \{f | f:[0,1] \to \mathbb{R}$ is a continuous function$\}$. Define an order on G as follows: $f \leq g$ iff $f(x) \leq g(x)$ for every $x \in [0,1]$. The operation in G is the usual componentwise addition of functions. Consider two elements f, g of G we need to prove that $f \wedge (g^{-1} + f^{-1} + g) \leq 0$. Note that $f^{-1} = -f$ and $g^{-1} = -g$, so $g^{-1} + f^{-1} + g = -g - f + g = -f$ and $f \wedge (g^{-1} + f^{-1} + g) = f \wedge -f \leq 0$. Note also that G satisfies (ii) (i.e. $(x \wedge y)^2 = x^2 \wedge y^2$). In another way G is isomorphic to $\prod_{[0,1]} R$ where R is the set of real numbers and the $\prod_{[0,1]} R$ is totally ordered, hence G is representable.

(ii) Let K = Z and G = ZxZ and τ a homomorphism between K and the group of automorphisms of G such that $\tau(1)(m,n) = (n,m)$. Let H = KxG furnished with an internal multiplication defined by $(a,x)\ (b,y) = (a+b,\ x+y^a)$ where $y^a = \tau(a)(y)$. It can be easily verified that H is a group. Let $(a,x) \geq e$ if $a > e$ or $a = e$ and $x \geq e$. Then, H is an ℓ-group in which:

$$(a,x) \vee e = \begin{cases} (a,x) & \text{if } a>e \\ (e,x\vee e) & \text{if } a=e \\ e & \text{if } a<e \end{cases}$$

where e is the identity. Let $x = (1,(0,0))$ and $y = (1,(1,-1))$. Then $x\wedge y = (1,(0,-1))$ and $(x\wedge y)^2 = (2,(-1,-1))$, but $x^2\wedge y^2 = x^2 = y^2 = (2,(0,0))$. Hence H is not representable.

Another property of ℓ-groups which is close to the representable conditions deals only with the group structure.

Definition 3.6: A group G is said to be an R-group if whenever $x^n=y^n$ for $n\in Z^+-\{0\}$ we have $x=y$. Hence, any totally ordered group is an R-group. But representable ℓ-groups are always isomorphic to a subdirect product of totally ordered groups. Therefore, any representable ℓ-group is an R-group. However, an R-group is not always representable.

Example 3.7: i) An R-group which may not be representable is ZwrZ. Since ZwrZ can be totally ordered it must be an R-group. However, there are lattice orders on ZwrZ which are not representable. For example the Scrimger ℓ-groups G_n for $n \in Z^+-\{0,1\}$, are not representable (see 2.4(iv)).

(iii) Neumann's [14] proof that AWrB is not orderable actually showed first that it is not an R-group.

The small wreath product of Z and Z with its two orders, W^+ and W^-, is of special significance in generating varieties and quasi-varieties of ℓ-groups. We first concentrate on W^+.

Let $\alpha = (\bar{a},0)$ where

$$a_i = \begin{cases} 1 & \text{if } i=0 \\ 0 & \text{if } i\neq 0 \end{cases} \quad \text{and } \beta = (0,1) .$$

Note that α and β are positive and they generate W^+. Also, for $\delta \in W^+$ of the form $\delta = (c, k)$ we have $\delta^{-1} = (c', -k)$ where $c_i' = -c_{i-k}$. Consider now $\delta = (c,k)$, $\bar{\gamma} = (a,n)$ in W^+. Then

$$\begin{aligned}
\gamma^\delta = \delta^{-1}\gamma\delta &= (c',-k)(a,n)(c,k) \\
&= (d, -k+n)(c,k) \quad \text{where } d_i = c_i' + a_{i-k} \\
&= (e, -k+n+k) \\
&= (e,n) \quad \text{with } e_i = d_i + c_{i-k+n} .
\end{aligned}$$

Hence, $e_i = c_i' + a_{i-k} + c_{i-k+n}$, but $c_i' = -c_{i-k}$ and $e_i = -c_{i-k} + a_{i-k} + c_{i-k+n}$. If $n=0$ then $\gamma^\delta = (e,0)$ where $e_i = a_{i-k}$; i.e., conjugation by $\delta = (c,k)$ when $\gamma = (a,0)$ shifts a k places to the right making γ^δ smaller than γ. Since α is equal to $(a,0)$ and β is equal to $(0,1)$ then α^β will be α with the 1 shifted to the 1-component in a. Thus, W^+ is characterized by the property that it has two generators α and β with $\beta >> \alpha >> \alpha^\beta$.

Similarly, we can define W^- or $Z\overleftarrow{wr}Z$. The multiplication is exactly as in W^+, but the order is changed. Here, α and β defined above generate W^- with $\beta >> \alpha^\beta >> \alpha$.

Let M^+ and M^- be the varieties generated by W^+ and W^-, respectively.

It is a known fact that if G is a totally ordered group generated by x and y and either $x >> y >> y^x$ or $y >> y^x >> x$ and the set $\{y^{x^n} | n \in Z\}$ is an independent set then G is isomorphic to W^+ or W^-, respectively (Medvedev [13]). The varieties M^+ and M^- strictly contain $\mathcal{A}$ and are contained in $\mathcal{R}$.

The following theorem of Medvedev [13] demonstrates the significance of the wreath product in generating varieties of ℓ-groups.

Theorem 3.8: M^+ and M^- are covers of $\mathcal{A}$ in the lattice of varieties of ℓ-groups.

In order to prove the theorem, Medvedev used the following lemma:

Lemma 3.9: Let G be an ℓ-group. If G contains two elements u and v such that either $u^v \ll u \ll v$ or $u \ll u^v \ll v$ and the set $\{u^{v^n} \mid n \in Z\}$ is independent, then G contains either a copy of W^+ or W^-.

We know that quasi-varieties are weaker algebraic structures than varieties. We know that $\mathcal{A} \leq M^+ \leq \mathcal{R} \cap \mathcal{A}^2$. We will construct an infinite number of different quasi-varieties between $\mathcal{A}$ and $\mathcal{R} \cap \mathcal{A}^2$.

First, we establish the analog of Medvedev's theorem for quasi-varieties. For notational purposes let Q_o^+ be the quasi-variety generated by W^+ and Q_o^- the quasi-variety generated by W^-.

Theorem 3.10: The quasi-varieties Q_o^+ and Q_o^- cover $\mathcal{A}$ in the lattice of quasi-varieties of ℓ-groups.

Proof: Let Q be a quasi-variety properly containing $\mathcal{A}$ and contained in Q_o^+. Then, there exists G a non-abelian ℓ-group G in Q. Since G belongs to Q_o^+, G is isomorphic to a subgroup of H where $H = \prod_{\alpha \in I}(\prod_{I'} W^+/U_\alpha)$ for which I, I' are index sets and each U_α is an ultrafilter over I'. There exist x and y elements of G such that $[x,y] \neq 1$. We consider G as a subgroup of H so x and y can be viewed as elements of H. So $x = (t_\alpha)_{\alpha \in I}$ and $y = (s_\alpha)_{\alpha \in I}$, for which $t_\alpha = (t_i^\alpha | U)_{i \in I'}$ and $s_\alpha = (s_i^\alpha | U)_{i \in I'}$ with t_i^α, s_i^α in W^+. Let $u = |[x,y]|$ and $v = |x||y|$. Now, for some $\alpha \in I$ $[t_\alpha, s_\alpha] \neq 1$ since $[x,y] \neq 1$. Fix such an α, then for every i in I' $|[t_i^\alpha, s_i^\alpha]| \ll |t_i^\alpha||s_i^\alpha|$, since t_i^α and s_i^α are elements of W^+. Thus $|[t_\alpha, s_\alpha]| \ll |t_\alpha||s_\alpha|$. Also, there exists $J \leq I'$ such that $|[t_i^\alpha, s_i^\alpha]| \neq 1$ for every $i \in J$. Therefore, we may assume that $u^v \ll u \ll v$ and the set $\{u^{v^n} | n \in Z\}$ is an independent set. Let K be the ℓ-subgroup of G generated by u and v then by lemma 3.9 K is isomorphic to W^+. Thus G contains a copy of W^+ and $Q_o^+ \leq Q$. But $Q \leq Q_o^+$ so $Q = Q_o^+$. Similarly, we can prove that Q_o^- covers A.

The preceding theorem does not indicate whether or not Q_o^- is identical to M^- and whether or not Q_o^+ is identical to M^+. We concentrate on comparing Q_o^- and M^- since the case for Q_o^+ and M^- is similar. Before doing so we need to learn more about the quasi-variety Q_o^-. In general quasi-varieties can be defined by a set of one or more implications. It is not always easy to find the implications that define a given quasi-variety. But, it is possible, however, to find a set of implications that are satisfied by all elements of the quasi-variety. Note that if two quasi-varieties are equal, then all of the implications that are true for one of them will have to be true for the other. Since our goal in this chapter is to construct a chain of quasi-varieties, it is important to determine some implications satisfied by each quasi-variety constructed. We consider first some properties of W^-.

<u>Proposition 3.11</u>: If a,b,c are positive elements of W^- with $b \ll a \ll c$ and $a \ll a^c$, then $b \ll b^c$.

<u>Proof</u>: Clearly, the proposition is true if $a = b = c = (0,0)$. Assume that a,b,c are positive elements of W^-. Then, there exist n, m, $k \in Z^+$ and $x,y,z \in \bigoplus_Z Z$ such that $a = (x,n)$, $b = (y,m)$ and $c = (z,k)$. Two possibilities arise.

First, if $k = 0$ then $n=m=0$ since $b \ll a \ll c$. Now, $c^{-1} = (z',0)$ with $z_i' = -z_i$ and so

$$\begin{aligned} a^c = c^{-1}ac &= (z',0)(x,0)(z,0) \\ &= (h,0)(z,0) \text{ where } h_i = z_i' + x_i = -z_i + x_i, \\ &= (t,0) \text{ where } t_i = h_i + z_i = -z_i + x_i + z_i = x_i\ . \end{aligned}$$

Hence, $a^c = a$ and for a to be infinitely smaller than a^c, a must be $(0,0)$. But if $a = (0,0)$ then $b = (0,0)$ since $(0,0) \leq b^n \leq a$ for every n in Z. Thus, $b << b^c$ since $b = b^c = (0,0)$.

Secondly, if $k > 0$ then $n=m=0$. Otherwise, if $n > 0$ then there exists $\lambda > 0$ such that $n\lambda > k$ contrary to $a << c$. Consider $a^\lambda = (x', n\lambda)$ where $x_i' = x_i + (\lambda-1)x_{i+n}$. Therefore, $a^\lambda > c$ since $n\lambda > k$. This is a contradiction to $a << c$. So $n=0$. But if $n=0$ and $b << a$ then $m=0$. Now, $b = (y,0)$, $b^n = (y^n,0)$ where $y_i^n = ny_i$, and

$$\begin{aligned} b^c &= c^{-1}bc = (z',-k)(y,0)(z,k) \text{ where } z_i' = -z_{i-k}. \text{ So,} \\ &= (h,-k)(z,k) \text{ where } h_i = z_i' + y_{i-k} = -z_{i-k} + y_{i-k}, \\ &= (t,0) \text{ where } t_i = h_i + z_{i-k} = -z_{i-k} + y_{i-k} \\ &\quad + z_{i-k} = y_{i-k} \end{aligned}$$

Note that $b^c - b^n = (t,0) - (y^n,0) = (s,0)$ where $s_i = b_i - y_i^n$. Hence, $s_i = y_{i-k} - ny_i$. Let λ be such that $y_\lambda > 0$ and λ is maximum with respect to $y_i \neq 0$. Then $s_{\lambda+k} = y_{\lambda+k-k} - ny_{\lambda+k} = y_\lambda - ny_{\lambda+k}$. But $y_{\lambda+k} = 0$ by the maximality of λ, and $s_{\lambda+k} = y_\lambda - 0 = y_\lambda > 0$. Now, $s_i = 0$ if $i > \lambda+k$. Since $s_i = s_{\lambda+k+j} = y_{\lambda+j} - ny_{\lambda+k+j} = 0-0 = 0$. Hence, $b^c - b^n \geq 0$ and $b^n \leq b^c$. Since n was chosen arbitrarily in Z^+, $b << b^c$.

The condition in this proposition can be stated in an equivalent way by using an infinite set of implications.

<u>Proposition 3.12</u>: The ℓ-group W^- satisfies the property "If a,b,c are positive elements of W^- with $b << a << c$ and $a << a^c$ then $b << b^c$," iff it satisfies the following implications:

(I1) $b \leq a \leq c$ and $a \leq a^c$ implies $b \leq b^c$

(I2) $b^2 \leq a \leq a^2 \leq c$ and $a^2 \leq a^c$ implies $b^2 \leq b^c$

⋮

(In) $b^n \leq a \leq a^n \leq c$ and $a^n \leq a^c$ implies $b^n \leq b^c$

⋮

<u>Proof</u>: Clearly (I1), (I2),..., and (In),...,imply that the condition holds. Conversely, choose a,b,c three positive elements of W^- such that a,b,c satisfy the property, and write $a = (x,n)$, $b = (y,m)$, and $c = (z,k)$ as before. If $k = 0$, then $n=m=0$, and for $a << a^c$ a must be $(0,0)$ (see Proof 3.11). Hence, $b = a = (0,0)$ since $b \leq a$ and (I1), (I2),...,(In) are satisfied. Now if $k > 0$ and $b << a << c$, then $a = (x,0)$ and $b = (y,0)$ (see Proof of 3.11). Thus, $b^n \leq b^c$ for every $n \geq 1$. Hence, (I1),...,(In) are satisfied.

Note that for every $\lambda \geq 2$, $(I\lambda)$ is satisfied by W^-. Fix $\lambda \in Z^+ - \{0,1\}$, and let a,b,c be positive elements of W^-. Write $a = (x,n)$, $b = (y,m)$, and $c = (z,k)$ as above. If $k=0$, then $n=m=0$ since $b^\lambda \leq a \leq a^\lambda \leq c$. So $a = a^c$ and $a^\lambda \leq a^c = a$ giving $a^\lambda = a$. Thus $a = (0,0)$ and $b = (0,0)$ since $b^\lambda \leq a$. Hence, $b^\lambda = b^c = (0,0)$ and $b^\lambda \leq b^c$. Otherwise, if $k > 0$ then $a^\lambda \leq a^c$ forces n to be zero. Suppose $n > 0$. Then $a^c = (p,n)$ where $p_i = -z_{i-k} + x_{i-k} + z_{i-k+n}$ and $a^\lambda = (p',\lambda n)$ where $p'_i = \lambda x_i$. Thus, $a^c < a^\lambda$ and we have a contradiction.

Then $n=0$ and $m=0$ since $b^\lambda \leq a$. Thus $b^\lambda \leq b^c$. Therefore (In) is true in W^- hence true in Q_o^- for every $n \geq 2$. The implication (I1) is true only if the condition holds.

Note that the implication $b \leq a \leq c$ and $a \leq a^c$ implies $b \leq b^c$ is not always true in W^-. For this we will give the following example:

<u>Example 3.13</u>: Let a,b,c be elements of W^- such that $a = (x,n) = c$ and $b = (y,n)$ where $x_i = 0$ for every i, $y_i = 0$ for every $i \neq k$, and $y_k = -1$ for a fixed positive element k of Z. Now, $b \leq a = c$ and $a = a^c$. But $b^c = c^{-1}bc = (x,-n)(y,n)(x,n) = (h,n)$ where $h_i = x_i + y_{i-n} + x_i = y_{i-n}$.

Thus, $h_i = 0$ for every $i \neq k+n$ and $h_{k+n} = y_{k+n-n} = -1$. So $b^c < b$ since $b^c - b = (h,n) - (y,n) = (h,n) + (y',-n)$ where $y_i' = -y_{i-n}$. Thus, $y_i' = 0$ for every $i \neq k+n$ and $y_{k-n}' = -y_{k+n-n} = -y_k = 1$. Now $b^c - b = (t,0)$ where $t_i = h_i + y_{i+n}'$. Hence, $t_i = 0$ for every $i \neq k$ or $n+k$, $t_k = h_k + y_{k+n}' = 0 + 1 = 1$ and $t_{k+n} = h_{k+n} + y_{k+n+n}' = h_{k+n} + y_{k+2n}' = -1 + 0 = -1$. Therefore, $b^c - b < (0,0)$ since $h_{k+n} = -1 < 0$ and $k+n$ is the maximum index for which $h_{k+n} \neq 0$. Hence, $b^c < b$ and (I1) fails in W^-. Therefore it fails in Q_o^-.

In the above set of implications the idea was to force b to be in the form $(y,0)$. But in W^- the commutator of two arbitrary elements is always in the form $(\ell,0)$. Based on this we can form a new infinite sequence of implications, such that whenever they are all satisfied they lead to the following result:

Proposition 3.14: If a,b,c are positive elements of W^- with $b^{c^{-1}} \leq b \leq a$ and $a \ll a^c$ then $[b,c] \ll [b,c]^c$.

Proof: First, $b^{c^{-1}} \leq b$ is equivalent to $[b,c] \geq (0,0)$. Choose a,b,c three positive elements of W^- and write $a = (x,n)$, $b = (y,m)$, and $c = (z,k)$ as before. If $k = 0$, then $n = 0 = m$ since $a \ll a^c$ and $b \leq a$. Hence, $[b,c] = (0,0) = [b,c]^c$ and $[b,c] \ll [b,c]^c$. Now, if $k > 0$, then $n = m = 0$ (see proof 3.10) and $[b,c] = (\ell,0)$. Hence, $[b,c] \ll [b,c]^c$ (see proof 3.10).

The condition in this proposition can be stated in an equivalent way by using an infinite set of implications.

Proposition 3.15: The ℓ-group W^- satisfies the property "$b^{c^{-1}} \leq b \leq a$ and $a \ll a^c$ imply $[b,c] \ll [b,c]^c$ for positive elements", iff it satisfies the following implications:

(I'1) $b^{c^{-1}} \leq b \leq a \leq a^c$ then $[b,c] \leq [b,c]^c$

(I'2) $b^{c^{-1}} \leq b \leq a \leq a^2 \leq a^c$ then $[b,c]^2 \leq [b,c]^c$

⋮

(I'n) $b^{c^{-1}} \leq b \leq a \leq a^n \leq a^c$ then $[b,c]^n \leq [b,c]^c$.

⋮

Proof: Clearly, (I'1), (I'2),...(I'n) imply that the condition holds. It suffices to prove that for every j, (I'j) is satisfied by W^-. Therefore (I'j) is true in Q_o^-. Let λ be an arbitrary index in $Z^+-\{0\}$ and a,b,c be positive elements of W^-. Then $b^{c^{-1}} \leq b$ implies that $[b,c] \geq (0,0)$, and $a^\lambda \leq a^c$ will imply that $c = (y,n)$ with $n > 0$ and $a = (x,0)$ if $\lambda \geq 2$. Now, $[b,c] = (h,0)$ and since $c = (y,n)$ with $n > 0$ then $[b,c]^c = (t,0)$ with $t_i = h_{i-n}$ and thus $[b,c]^\lambda \leq [b,c]^c$. If $\lambda=1$ and $b^{c^{-1}} \leq b$ then $[b,c] \geq (0,0)$. Now, if $c = (y,0)$ then $[b,c]^c = [b,c]$. Otherwise, $c = (y,n)$ with $n>0$ and $[b,c] < [b,c]^c$. In all cases (I'λ) is true.

Propositions 3.11 and 3.14 give a characterization of W^- and provide a way of determining whether an ℓ-group is different than W^-. Generally, to show an ℓ-group is not in Q_o^- it suffices to find elements in the ℓ-group that do not satisfy one of (I1), (I2),...(In), or one of (I'1), (I'2),...,(I'n) as we will see in the following example:

Example 3.16: Let Λ_1 be the set $Z_1 \cup \{t_1\} \cup Z_2$ where $Z_1 = Z_2 = Z$ totally ordered in the following way: For every $n, m \in \Lambda_1$ let $n \leq t_1 \leq m$ whenever $n \in Z_1$ and $m \in Z_2$. Let $\bar{\alpha}$ be an order preserving permutation of Λ defined by:

$$\bar{\alpha}(n) = \begin{cases} n+1 & \text{if } n \in Z_1 \cup Z_2 \\ n & \text{if } n = t_1 \end{cases}$$

Let $G = \bigoplus_{\Lambda_1} Z$ and α be an automorphism of G defined by $\alpha(g) = g'$ where

$g_i' = g_{\bar{\alpha}(i)}$. Consider $G(\alpha)$, the cyclic extension of G by α. Define an order over $G(\alpha)$ by $(g,\alpha^n) \geq (0,0)$ iff $n > 0$ or $n = 0$ and $g_i \geq 0$ where i is the maximum index with $g_i \neq 0$. $G(\alpha)$ is totally ordered and fails (I1), (I2),...,(In)... . To see this let a,b,c be elements of G such that $a = (x,\alpha^0)$ with $x_i = 0$ for $i \leq t_1$, $b = (y,\alpha^0)$ with $y_i = 0$ for $i \neq t_1$ and $y_{t_1} > 0$ $c = (z,\alpha^1)$ where $z_i = 0$ for every i. Clearly $b << a << c$ and $a << a^c$. But $b^c = b$ since $b^c = (h,\alpha^0)$ where $h_i = z_i + y_{\alpha^{-1}(i)} + z_{\alpha(i)} = b_{\alpha^{-1}(i)}$. Hence, $h_i = 0$ for every $i \neq t_1$ and $h_{t_1} = y_{\alpha^{-1}(t_1)} = y_{t_1}$. Thus, $b^n > b^c$ for every $n \geq 2$. Hence, (I2), (I3)...,(In) fail to be true in $G(\alpha)$.

We later apply the same technique of Example 3.16 to construct the countable infinite chain of different quasi-varieties containing Q_o^- and contained in $L_n \cap A^2$. But, first we need to prove that Q_o^- is different than M^-.

<u>Theorem 3.17</u>: The quasi-variety Q_o^- generated by W^- is strictly contained in the variety M^- generated by W^-.

<u>Proof</u>: Clearly $Q_o^- \leq M^-$.

In order to prove that $Q_o^- \lneq M^-$, it suffices to find an ℓ-group in M^- that does not belong to Q_o^-. Fix $\lambda \varepsilon Z^+ - \{0\}$ and let H be the ℓ-subgroup generated by all elements in W^- of the form $(g,0)$ where $g_i = 0$ if $i > \lambda$. Let $G = W^-/H$. Then G belongs to M^- since M^- is closed under the formation of quotients. Choose $n \varepsilon Z^+ - \{0\}$ where $-n + \lambda \leq -3$, $c = (z,n)$ where $z_i = 0$ for $i \neq \lambda$ and $z_\lambda = 1$, $b = (y,0)$ where $y_i = 0$ for every $i \neq -n + \lambda$ and $y_{-n+\lambda} = 1$, and $a = (x,0)$ where $x_i = 0$ for every $i \neq -n + \lambda + 1$ and $x_{-n+\lambda+1} = 1$. Now $b^{-1} = (y',0)$ where $y_i' = 0$ for every $i \neq -n + \lambda$

and $y'_{-n+\lambda} = -1$, $c^{-1} = (z', -n)$ where $z'_i = 0$ for every $i \neq \lambda+n$ and $z'_{\lambda+n} = -1$. Notice, $b \leq a \leq c$ and $a \leq a^c$. Now

$$[b,c] = b^{-1}c^{-1}bc = (y',0)(z',-n)(y,0)(z,n)$$

$$= (h',-n)(y,0)(z,n)$$

where $h'_i = y'_i + z'_i$.

Thus, $h'_i = 0$ for every $i \neq -n+\lambda$ and $i \neq \lambda+n$, $h'_{-n+\lambda} = y'_{-n+\lambda} = -1$, and $h'_{n+\lambda} = z'_{\lambda+n} = -1$. Then

$$[b,c] = (h',-n)(y,0)(z,n) = (h'',-n)(z,n)$$

where $h''_i = h'_i + y_{i-n}$.

Hence $h''_i = 0$ for every $i \neq -n+\lambda$, λ, $\lambda+n$ and we have:

$$h''_{-n+\lambda} = h'_{-n+\lambda} + y_{-2n+\lambda} = h'_{-n+\lambda} = -1,$$

$$h''_{\lambda} = h'_{\lambda} + y_{-n+\lambda} = 0 + 1 = 1$$

and

$$h''_{n+\lambda} = h'_{n+\lambda} + y_{n+\lambda-n} = h'_{n+\lambda} + y_{\lambda} = h'_{n+\lambda} = -1 .$$

So $\quad [b,c] = (h'',-n)(z,n) = (h,0)$ where $h_i = h''_i + z_{i-n}$.

Thus, $h_i = 0$ for every $i \neq -n+\lambda$ or λ and we have:

$$h_{-n+\lambda} = h''_{-n+\lambda} + z_{-n+\lambda-n} = h''_{-n+\lambda} + z_{-2n+\lambda} = h''_{-n+\lambda} + 0 = -1,$$

$$h_{\lambda} = h''_{\lambda} + Z_{\lambda-n} = h''_{\lambda} + 0 = 1,$$

and $h_{\lambda+n} = h''_{\lambda+n} + z_{\lambda+n-n} = h''_{\lambda+n} + z_{\lambda} = (-1) + (1) = 0$. Hence, $[b,c] \geq (0,0)$. Now,

$$[b,c]^c = (z',-n)(h,0)(z,n)$$

$$= (t',-n)(z,n) \text{ where } t'_i = z'_i + h_{i-n} .$$

Hence, $t'_i = 0$ for every $i \neq \lambda$ and we have:

$$t'_{\lambda} = z'_{\lambda} + h_{\lambda-n} = 0+(-1) = -1$$

$$t'_{\lambda+n} = z'_{\lambda+n} + h_{\lambda+n-n} = z'_{\lambda+n} + h_{\lambda} = (-1) + (1) = 0 .$$

Thus, $[b,c]^c = (t',-n)(z,n) = (t,0)$ where $t_i = t'_i + z_{i-n}$.

Hence, $b_i = 0$ for every $i \neq \lambda$ or $\lambda+n$ and we have:

$$t_{\lambda} = t'_{\lambda} + z_{\lambda-n} = (-1) + 0 = -1$$

and $$t_{\lambda+n} = t'_{\lambda+n} + z_{\lambda+n-n} = t'_{\lambda+n} + z_{\lambda} = 0 + 1 = 1 .$$

Notice that $[b,c] \leq [b,c]^c$ in W^-. But in G, $\overline{[b,c]^c} = \overline{(p,0)}$ where $p_i = 0$ for every $i \neq \lambda$ and $p_{\lambda} = -1$. Thus $\overline{[b,c]^c} \leq \overline{(0,0)}$. Also, $\overline{[b,c]} > \overline{(0,0)}$ since $[b,c] = (h,0)$ with $h_{\lambda} = 1 > 0$ therefore $\overline{[b,c]^c} < \overline{[b,c]}$ and (I'1) fails in G. Notice, for every $n \in Z^+-\{0\}$ we have $b^n \leq a \leq a^n \leq c$ and $a^n \leq a^c$ but $\overline{[b,c]^c}^{\,n} < \overline{[b,c]}$ in G. Thus, (I'1),...and (I'n)... are not satisfied in G. Hence, G cannot be in Q_o^- and $Q_o^- \lneq M^-$.

The above theorem provides an example where the implications (I'1), (I'2) and (I'n) fail to be true.

By introducing Example 3.16, we know of the existence of ℓ-groups different than W^- and therefore the existence of quasi-varieties different than Q_o^-. It is important to first construct the ℓ-groups and then take the quasi-varieties generated by them. In order to establish a nested sequence of ℓ-groups, first we need the following definitions and notations.

Let $\Lambda_n = Z_1 \cup \{t_1, t_2, \ldots, t_n\} \cup Z_2$ where $Z_1 = Z_2 = Z$. The index set Λ_n can be totally ordered by $\leq$ if $\leq$ is defined by: If n, m are elements of Z_1, Z_2 respectively, then $n < t_i$ and $t_i < m$ for every $i = 1, \ldots, n$. Also, $t_i < t_j$ iff $i < j$. Let $\bar{\alpha}_n$ be an o-permutation of Λ_n such that

$$\begin{cases} \bar{\alpha}_n(z) = z+1 & \text{if } z \in Z_1 \cup Z_2 \\ \bar{\alpha}_n(t_i) = t_i & \text{for } i = 1,..,n \end{cases}$$

Let $G = \bigoplus_{\Lambda_n} Z$ and α_n an o-automorphism of G defined by $\alpha_n(g) = g'$ where $g_i' = g_{\bar{\alpha}_n(i)}$. Consider $W_n^- = G(\bar{\alpha}_n)$ the cyclic extension of G by α_n with the same order as W^-, and let Q_n^- be the quasi-variety generated by W_n^-. For every $n \in Z^+ - \{0\}$ W_n^- is an ℓ-group [Example 2.4(i), Chapter I]. The family of ℓ-groups $(W_i^-)_{i \in Z^+ - \{0\}}$ form a nested sequence of ℓ-groups. And if they are pairwise different then we have an infinite chain of quasi-varieties each of which contains Q_o^- and is contained in $L_n \cap A^2$. Hence, it is necessary to prove that the Q_n^-'s are pairwise different. Therefore, we need to find one or more implications satisfied by each Q_n^- and make sure that the implications will fail in Q_k^- if $k > n$.

<u>Lemma 3.18</u>: If $a, b, c, d_1, d_2, \ldots, d_n$ are positive elements of W_n^- with

$b \ll d_1 \ll d_2 \ll \ldots \ll d_n \ll a \ll c$, $a \ll a^c$ and $d_i = d_i^c$ for every $i = 1,\ldots,n$, then $b \ll b^c$.

<u>Proof</u>: Clearly, the Lemma is true if $a = b = c = d_1 = d_n = (0, \alpha_n^0)$. Let $a,b,c, d_1,\ldots,d_n$ be positive elements of W_n^-. Write $a = (x,\alpha_n^t)$, $b = (y,\alpha_n^m)$, $c = (z,\alpha_n^k)$, $d_1 = (x^1, \alpha_n^{k_1})$, $d_2 = (x^2, \alpha_n^{k_2}),\ldots,$and $d_n = (x^n, \alpha_n^{k_n})$. First, if $k = 0$ then $n = m = k_1 = \ldots = k_n = \ldots = 0$. Then, $a = a^c$ and for $a \ll a^c$, a must be $(0, \alpha_n^0)$. Hence, $d_1 = d_2 = \ldots = d_n = b = (0, \alpha_n^0)$ since $b \ll d_1 \ll \ldots \ll d_n \ll a$. Next, if $k > 0$, then $t = 0$. Suppose $t > 0$ then there exists λ such that $\lambda t > k$. So, $a^\lambda = (x', \alpha_n^{\lambda t}) > c = (z, \alpha_n^k)$ since $\lambda t > k > 0$. This is a contradiction with $a \ll c$. Thus we must have $t = 0$ and $m = k_1 = k_2 = \ldots = k_n = 0$ since $b \ll d_1 \ll d_2 \ll \ldots \ll d_n \ll a$. There exists $\eta \in Z_1 \cup Z_2$ for which $x_\eta > 0$. We know that $a = (x,\alpha^0) > (0,\alpha^0)$. Then there exists $x_i > 0$ where i is the maximum index with $x_i \neq 0$. If $i \in Z_1 \cup Z_2$, then choose $\eta = i$. Suppose there exists $j = 1,\ldots,n$ such that $i = t_j$. Then $a^c = c^{-1}ac = (z', \alpha^{-k})(x, \alpha^0)(z, \alpha^k) = (h, \alpha^0)$ where $h_i = z_i' + x_{\alpha^{-k}(i)} + z_{\alpha^{-k}(i)}$. But $z_i' = -z_{\alpha^{-k}(i)}$, hence $h_i = x_{\alpha^{-k}(i)}$. So $h_{t_j} = x_{t_j} > 0$, $h_{t_j} < 2x_{t_j}$, and $a^c < a^2$. We have a contradiction to $a \ll a^c$. Therefore, $i \neq t_j$ for every $j = 1,\ldots,n$. In addition, η is the maximal index for which $x_\eta \neq 0$ and $x_\eta > 0$. Now, for every $i = 1,\ldots,n$, $d_i = (x^i,\alpha^0)$ where $x_j^i = 0$ for $j \in Z_1 \cup Z_2$. Suppose there exists $j \in Z_1 \cup Z_2$ such that $x_j^i \neq 0$. Now, $d_j^c = (z', \alpha_n^{-k})(x^i, \alpha_n^0)(z, \alpha_n^k) = (p, \alpha_n^0)$ where

$p_\ell = z'_\ell + x^i_{\alpha_n^{-k}(\ell)} + z_{\alpha_n^{-k}(\ell)} = p_\ell = -z_{\alpha_n^{-k}(\ell)} + z_{\alpha_n^{-k}(\ell)} + x^i_{\alpha_n^{-k}(\ell)}$. Hence,

$p_{j+k} = x^i_{\alpha_n^{-k}(j+k)} = x^i_j$ and $p_{t_i} = x_{\alpha_n^{-k}(t_i)} = x_{t_i}$. Thus, $d_j^c \neq d_j$

since $d_j^c = (p, \alpha_n^0)$ where $p_j \neq x_j^i$ and $p_{j+k} \neq x^i_{j+k}$. Also, $x^i_j = 0$ for every

$j \in Z_1 \cup Z_2$. Let γ_i be the maximal index for which $x^i_{\gamma_i} > 0$. There

exists such γ_i since $d_i > (0, \alpha_n^0)$. Now, if η belongs to Z_1 and $d_1 <<$

$d_2 << \ldots << d_n << a$, then $d_1 = d_2 = \ldots = d_n = (0, \alpha_n^0)$. Hence,

$b = (0, \alpha_n^0)$ and the lemma is proved. Otherwise, if $\eta \in Z_2$, then $\gamma_1 <$

$\gamma_2 < \ldots < \gamma_n$. Suppose that $d_i > (0, \alpha_n^0)$ for every $i = 1,\ldots,n$. Then,

$\gamma_i = t_i$ and $b = (y, \alpha_n^0)$ where $y_i \neq 0$ if i belongs to Z_1 and $y_i = 0$,

otherwise. So

$$b^c = c^{-1}bc = (z', \alpha_n^{-k})(y, \alpha_n^0)(z, \alpha_n^k) = (s, \alpha_n^0)$$

where

$$s_i = z'_i + y_{\alpha_n^{-k}(i)} + z_{\alpha_n^{-k}(i)} = -z_{\alpha_n^{-k}(i)} + y_{\alpha_n^{-k}(i)} + z_{\alpha_n^{-k}(i)} .$$

Then $s_i = y_{\alpha_n^{-k}(i)} = y_{i-k}$, and $b^n = (s', \alpha_n^0)$ where $s'_i = ny_i$. Note

that $b^n \leq b^c$ for every $n \geq 1$. Otherwise, if there exists $j \geq 1$ such that

$d_j = (0, \alpha_n^0)$. Then $d_i = (0, \alpha_n^0)$ for every $i < j$. Since $b << d_j$, we

have $b = (0, \alpha_n^0)$ and $b^n = (0, \alpha_n^0) = b^c$.

The condition in the above Lemma determines a set of implications that are simultaneously satisfied by Q_n^-. Let $a, b, c, d_1, d_2, \ldots, d_n$ be positive elements of Q_n^-, and consider the implications:

(T_1^n) If $b \le d_1 \le d_2 \le \dots \le d_n \le a \le c$, $a \le a^c$ and $d_i = d_i^c$ for $i = 1,\dots,n$ then $b \le b^c$.

(T_2^n) If $b^2 \le d_1 \le d_1^2 \le d_2 \le d_2^2 \le d_3 \dots \le d_{n-1} \le d_{n-1}^2 \le d_n \le d_n^2 \le a \le a^2 \le c$, $a^2 \le a^c$ and $d_i = d_i^c$ for $i = 1,\dots,n$ then $b^2 \le b^c$

$\vdots$

(T_k^n) If $b^k \le d_1 \le d_1^k \le d_2 \le d_2^k \le \dots \le d_n \le d_n^k \le a \le a^k \le c$, $a^k \le a^c$ and $d_i = d_i^c$ for $i = 1,\dots,n$ then $b^k \le b^c$.

$\vdots$

Proposition 3.19: The condition "$b \ll d_1 \ll \dots \ll d_n \ll a \ll c$, $a \ll a^c$ and $d_i = d_i^c$ for every $i = 1,\dots,n$ then $b \ll b^c$ for positive elements of the ℓ-group W_n^-", is satisfied iff the implications $(T_1^n),\dots,(T_k^n),\dots,$ are satisfied.

Proof: Clearly, (T_1^n), $(T_2^n),\dots,(T_k^n)$ imply that the condition holds. It suffices to prove (T_λ^n) for $\lambda \ge 1$. Choose $a,b,c,d_1\dots,d_n$ positive elements of W_n^- that satisfy the condition. Write $a = (x,\alpha_n^t)$, $b = (y,\alpha_n^m)$, $c = (z,\alpha_n^k)$, $d_1 = (x^1,\alpha_n^{k_1})$, $d_2 = (x^2,\alpha_n^{k_2}),\dots,$and $d_n = (x^n,\alpha_n^{k_n})$. First, if $k=0$, the $n=m=k_1 = k_2 = \dots = k_n = 0$. Then, $a = a^c$ and for $a \ll a^c$, a must be $(0,\alpha_n^0)$. Hence, $d_1 = d_2 = \dots = d_n = b = (0,\alpha_n^0)$ since $b \ll d_1 \ll \dots \ll d_n \ll a$ and $b^n = (0,\alpha_n^0) = b^c$. Therefore, (T_λ^n) is satisfied for all $\lambda \ge 1$. Next, if $k > 0$ then $t = 0$ (see Proof 3.18) and $m = k_1 = k_2 = \dots = k_n = 0$. Also, for every $i = 1,\dots,n$, $d_i = (x^i,\alpha_n^0)$ with $x_j^i = 0$ if $j \ne t_i$ (see Proof 3.18), and $b = (y,\alpha_n^0)$ where $y_i = 0$ if $i \notin Z_1$. Hence, $b^\lambda \le b^c$ for every $\lambda \ge 1$, and $(T_1^n),(T_2^n),\dots,(T_\lambda^n),\dots,$are satisfied. Note that, for every $\lambda \ge 2$ (T_λ^n) is satisfied by Q_n^-. In this case $a^\lambda \le a^c$ implies $a = (x,\alpha_n^0)$, $c = (z,\alpha_n^k)$ with $k > 0$ (see 3.18) and $b = (y,\alpha_n^0)$ where $y_i = 0$ if $i \notin Z_1$. Hence, $b^\lambda \le b^c$.

It is important that every Q_n^- contains Q_o^- and that any pair W_n^-, W_k^- will generate two different quasi-varieties. Therefore, the following lemmas:

<u>Lemma 3.20</u>: For every n, Q_n^- strictly contains Q_o^-.

<u>Proof</u>: Recall that W^- can be regarded as a cyclic extension of a totally ordered ℓ-group $G_o = \bigoplus_Z Z$ by an automorphism α of G_o (see 2.3(iii)). If we let $\Lambda_o = Z$, $\bar{\alpha}$ the o-permutation of Z mapping i to i+1, and $\bar{\alpha}_n$ the o-permutation of Λ_n as defined above. Then there exists $\gamma_o : \Lambda_o \to \Lambda_n$ such that the diagram below commutes. That is, $\bar{\alpha}_n o \gamma_o = \gamma_o o \bar{\alpha}_o$. Define γ_o as follows: $\gamma_o(i) = i$ where i is in Z_1.

$$\begin{tikzcd}
\Lambda_o \arrow[r, "\gamma_o"] \arrow[d, "\bar{\alpha}_o"'] & \Lambda_n \arrow[d, "\bar{\alpha}_n"] \\
\Lambda_o \arrow[r, "\gamma_o"'] & \Lambda_n
\end{tikzcd}$$

Then, $\bar{\alpha}_n(\gamma_o(i)) = \bar{\alpha}_n(i) = i+1$ where i+1 belongs to Z_1. Now, $\gamma_o(\bar{\alpha}_o(i)) = \gamma_o(i+1) = i+1$ where i+1 belongs to Z_1. Hence, $\bar{\alpha}_n o \gamma_o = \gamma_o o \bar{\alpha}_o$ and by Theorem 2.7 $G(\alpha_o) \leq G(\alpha_n)$. Thus, $Q_o^- \leq Q_n^-$.

In order to prove the strict inclusion, it suffices to find positive elements a,b,c of W_n^- that do not satisfy the condition in Proposition 3.11.

Let $a = (x,0)$, $b = (y,0)$, $c = (z,n)$ where $x_i = 0$ for $i \leq t_n$ and $x_i > 0$ otherwise, $y_i = 0$ for $i \neq t_n$ and $y_i = 1$ if $i = t_n$, $z_i = 0$ for every i, and $n > 0$. Notice $b \ll a \ll c$ and $a \ll a^c$ but $b = b^c$. Hence, $b^n > b^c$ for every $n > 1$. Thus, the condition in Proposition 3.11 fails to be true in W_n^-. By Proposition 3.11 we have $Q_o^- \lneq Q_n^-$.

<u>Lemma 3.21</u>: For every n $Q_n^- \leq R \cap A^2$.

Proof: Clearly W_n^- is a totally ordered ℓ-group. Thus $Q_n^- \leq R$. Next, we need to prove that $W_n^- \in A^2$. For this it suffices to find an ℓ-ideal $H \leq W_n^-$ such that H and W_n^-/H are in A. Let $H = \{(g,0) | (g,0) \in W_n^-\}$. Clearly, H is an abelian ℓ-ideal of W_n^-. Also, $W_n^-/H \simeq Z$. Then, W_n^-/H and $H \in A^2$. Hence, $Q_n^- \leq R \cap A^2$.

The above lemma indicates that every Q_n^- is contained in $R \cap A^2$. If we can prove that the Q_n^-'s are pairwise different and form a nested chain, then infinitely many of them are strictly included in $R \cap A^2$.

Lemma 3.22: For every $n \geq 1$, Q_n^- strictly contains Q_{n-1}^-.

Proof: Let Λ_n, Λ_{n-1} be the two index sets used in defining W_n^- and W_{n-1}^-, respectively, and let $\bar{\alpha}_n$, $\bar{\alpha}_{n-1}$ be the o-permutations of Λ_n and Λ_{n-1}, respectively, as before. There exists γ_o, an injective map between Λ_{n-1} and Λ_n, such that $\bar{\alpha}_n o \gamma_o = \gamma_o o \bar{\alpha}_{n-1}$. Define γ_o by $\gamma_o(i) = i$. It can be easily verified that $\bar{\alpha}_n o \gamma_o = \gamma_o o \bar{\alpha}_{n-1}$. Thus $W_{n-1}^- \leq W_n^-$ and $Q_{n-1}^- \leq Q_n^-$ by 2.5. In order to prove the strict inclusion it suffices to find positive elements b, d_1, ..., d_{n-1}, a and c of W_n^- such that the hypothesis in Lemma 3.18 is true for $n-1$ but the conclusion of the lemma fails. Let b, $d_1,\ldots,d_{n-1}$, a, c be such that $b = (y,0)$, $d_1 = (t^1,0),\ldots,d_{n-1} = (t^{n-1},0)$, $a = (x,0)$, $c = (z,n)$ where the following are true:

$y_i = 0$ for $i \neq t$, and $y_i = 1$ otherwise,

$t_i^j = 0$ for $i \neq t_{j+1}$ and $t_i^j = 1$ otherwise,

$x_i = 0$ for $i \leq t_n$ and $x_i > 0$ otherwise, and

$z_i = 0$ for every i and $n > 0$.

Note that $b << d_1 << d_2 << \cdots << d_{n-1} << a << c$ and $a << a^c$ but $b = b^c$. Hence, $b^n > b^c$ for every $n > 1$ and Lemma 3.15 fails to be true.

Finally, combining Lemmas 3.21 and 3.22 we have the following important result.

Theorem 3.23: There is an infinite chain of distinct quasi-varieties between $\mathcal{A}$ and $\mathcal{R} \cap \mathcal{A}^2$.

Proof: Clearly the $(Q_n^- | n \in Z^+)$ forms a countably infinite chain of different quasi-varieties between $\mathcal{A}$ and $\mathcal{R} \cap \mathcal{A}^2$.

By using techniques similar to those in constructing the W_n^-'s and a different set of implications for W^- we are able to construct another infinite chain of quasi-varieties between $\mathcal{A}$ and $\mathcal{R} \cap \mathcal{A}^2$.

Proposition 3.24: Let a,b,c be positive elements of W^-. If $b << a << c$, $b = b^c$ and $a << a^c$, then $b = (0,0)$.

Proof: Clearly, the proposition is true if $a = b = c = d = (0,0)$. Let a,b,c,d be elements of W^- such that $a = (x,n)$, $b = (y,m)$, $c = (z,k)$. First, if $k = 0$ then $n = m = 0$ since $b << a << c$. Now, since $n = 0$, $a^c = a$ and for a to be infinitely smaller than a^c, a must be equal to $(0,0)$. But, if $a = (0,0)$, then $b = (0,0)$. Second, if $k > 0$, then $n = 0 = m$ since $b << a << c$. Now, $b = (0,0)$ since $b = b^c$. Suppose $b > (0,0)$ then there exists an index λ maximal with respect to $y_\lambda > 0$. Now, $b^c = (y',0)$ where $y_i' = -z_{i-k} + y_{i-k} + z_{i-k} = y_{i-k}$. Thus $y'_{\lambda+k} = y_\lambda > (0,0)$ and $y_j' = 0$ for $j > \lambda+k$. Hence, $b^c > b$. Thus b cannot be bigger than $(0,0)$. But by the choice of b, b must be equal to $(0,0)$ and the proposition is proved.

We next construct an example of an ℓ-group where the condition of Proposition 3.24 is not true.

Example 3.25: Let $N_1 = \{t_1\} \cup Z$ and $G = \bigoplus_{N_1} Z$. The index set N_1 is totally ordered as follows: If i,j are elements of N_1 and if $i,j \in Z$, then let $i \le j$ in N_1 iff $i \le j$ in the natural order of Z. Otherwise, if $i = t_1$, then $i \le j$. Let $\bar{\gamma}_1$ be an o-permutation of Λ_1 defined by $\bar{\gamma}_1(i) = i+1$ if $i \ne t_1$ and $\bar{\gamma}_1(t_1) = t_1$. Now define an automorphism γ_1

of G by $\gamma_1(g) = g'$ where $g'_i = g_{\bar{\gamma}_1(i)}$. Let M_1^- be the cyclic extension of G by γ_1, and P_1^- the quasi-variety generated by M_1^-. Consider a,b,c to be three special positive elements of M_1^-. Fix $n > 0$ let $c = (0,n)$, $a = (x,0)$, and $b = (y,0)$ where $x_i = 0$ if $i = t$, and $x_i > 0$ otherwise, $y_i = 0$ if $i \neq t_1$ and $y_i > 0$ otherwise. Notice that $b << a << c$, $b = b^c$, and $a << a^c$, but $b \neq (0,0)$.

The ideas of Example 3.25 can be generalized to find other ℓ-groups which fail the condition of Proposition 3.24.

Let $N_n = \{t_1, t_2, \ldots, t_n\} \cup Z$ be totally ordered by: $t_i \leq t_j \leq k$ iff $i \leq j$ and $k \in Z$, and $i \leq j$ iff $i,j \in Z$ and $i \leq j$ in the order of Z. Let $G = \bigoplus_{N_n} Z$ and let $\bar{\gamma}_n$ be an o-permutation of Λ_n such that $\bar{\gamma}_n(i) = i+1$ iff $i \neq t_j$ for $j = 1,\ldots,n$ and $\bar{\gamma}_n(i) = i$ otherwise. Let γ_n be the o-automorphism of G defined by $\gamma_n(g) = g'$ where $g'_i = g_{\bar{\gamma}_n(i)}$. Let M_n^- be the cyclic extension of G by γ_n and P_n^- the quasi-variety generated by M_n^-.

As for the Q_n^-'s we will prove that the $(P_n^- | n \in Z^+ - \{0\})$ forms a countably infinite family of different quasi-varieties between A and $\mathcal{R} \cap A^2$. We will first give a characterization of M_n^-.

Lemma 3.26: Let $a,b,c,d_1,\ldots,d_n$ be positive elements of M_n^- such that $b << d_1 << d_2 << \ldots << d_n << a << c$, $d_i = d_i^c$ for every $i = 1,\ldots,n$ and $a << a^c$. Then $b = (0,\gamma_n^0)$.

Proof: Clearly the lemma is true if $c = (0,\gamma_n^0)$. Suppose that $c = (z,\gamma_n^k)$, $b = (y,\gamma_n^m)$, $a = (x,\gamma_n^p)$ and $d_i = (t^i,\gamma_n^{p_i})$. First, if $k = 0$, then $k = m = p = p_i = 0$ for every $i = 1,\ldots,n$. Thus, $a = a^c$ and for a to be infinitely smaller than a^c, it must be equal to $(0,\gamma_n^0)$. Hence, $d_i = b = (0,\gamma_n^0)$ for every i. Secondly, if $k > 0$, then $p = m = p_i = 0$ since $b << d_i << a << c$ for every i. Clearly if $a = (0,\gamma_n^0)$ or

$d_i = (0,\gamma_n^0)$ for some i then $b = (0,\gamma_n^0)$. Suppose $a > (0,\gamma_n^0)$ and $d_i >$ $(0,\gamma_n^0)$ for every i. Then $a = (x,\gamma_n^0)$ where $x_\lambda > 0$ for λ the maximum index where $x_i \neq 0$ and λ belongs to Z, and $d_i = (t^i,\gamma_n^0)$ where $t^i_{\lambda_i} > 0$ for λ_i the maximum index where $t^i_j \neq 0$ and λ_i does not belong to Z (see 3.15). We have $\lambda_i \leq t_i$ for every i since $d_1 << d_2 << \ldots << d_n$ and $d_i > (0,\gamma_n^0)$ for every i. If $\lambda_i = t_i$, then $b = (0,\gamma_n^0)$ otherwise if $\lambda_i < k_i$ then $d_1 =$ $(0,\gamma_n^0)$ and $b = (0,\gamma_n^0)$ since $b << d_1$.

Again, the condition in Lemma 3.26 can be broken into an equivalent set of implications.

Let a,b,c, d_1, $d_2,\ldots,d_n$ be positive elements of M_n^- and consider the implications:

(L1) $b \leq d_1 \leq d_2 \leq \ldots \leq d_n \leq a \leq c$, $d_i = d_i^c$ for every i and $a \leq a^c$ implies $b = (0,\gamma_n^0)$.

(L2) $b^2 \leq d_1 \leq d_1^2 \leq d_2 \leq \ldots \leq d_n \leq d_n^2 \leq a \leq a^2 \leq c$, $b = b^c$, $d_i = d_i^c$ for every i and $a^2 \leq a^c$ implies $b = (0,\gamma_n^0)$.

⋮

(Li) $b^i \leq d_1 \leq d_1^i \leq d_2 \leq \ldots \leq d_n \leq d_n^i \leq a \leq a^i \leq c$, $b = b^c$, $d_i = d_i^c$ for every i and $a^i \leq a^c$ implies $b = (0,\gamma_n^0)$.

⋮

(LP) $b^p \leq d_1 \leq d_1^p \leq d_2 \leq \ldots \leq d_n \leq d_n^p \leq a \leq a^p \leq c$, $d_i = d_i^c$ for every i and $a << a^c$ then $b = (0,\gamma_n^0)$.

⋮

Proposition 3.27: An ℓ-group G satisfies the condition "if b, d_1, d_2, ..., d_n, a,c are positive elements of G with $b << d_1 << d_2 << \ldots << d_n << a << c$, $d_i = d_i^c$ for every i and $a << a^c$ implies $b = 1_G$" iff it satisfies the implications (L1), (L2),...,(Ln),... .

Next, we will prove that the P_n^-'s are pairwise different and form an infinite chain between A and $R \cap A^2$.

<u>Lemma 3.28</u>: For every $n \geq 1$, M_n^- strictly contains W^- and M_{n-1}^-. Hence, P_n^- strictly contains P_o and P_{n-1}^-.

<u>Proof</u>: Initially, let us prove that W^- is strictly contained in M_n^-. Let $\phi : W^- \to M_n^-$ defined by $\phi(x,m) = (y,\gamma_o^m)$ where $y_i = x_i$ if i belongs to Z and $y_i = 0$ otherwise. Clearly, ϕ is a well defined injective map. It suffices to prove that ϕ is a homomorphism (i.e., we need to prove that $\phi((x,m)(h,p)) = \phi((x,m))\ \phi((h,p)))$.

First, note that $(x,m)(h,p) = (t, m+p)$ where $t_i = x_i + h_{i+m}$. So $\phi((x,m)(h,p)) = (t', \gamma_o^{m+p})$ where $t'_i = t_i$ if i belongs to Z and $t'_i = 0$ otherwise. Secondly, $\phi((x,m)) = (y, \gamma_o^m)$ and $\phi((h,p)) = (\ell,\gamma_o^p)$ where $y_i = x_i$ and $\ell_i = h_i$ if i belongs to Z and $y_i = \ell_i = 0$ otherwise. Thus, $\phi((x,m)) \cdot \phi((h,p)) = (y,\gamma_o^m)(\ell,\gamma_o^p) = (s,\gamma_o^{m+p})$ where $s_i = y_i + \ell_{\gamma_o^m(i)}$. Note that if $i \in N_n$ belongs to Z, then $s_i = y_i + \ell_{\gamma_o^m(i)} = y_i + \ell_{i+m} =$

$x_i + h_{i+m} = t_i$. Otherwise $s_i = t_i = 0$. Hence, $\phi((x,m)(h,p)) = \phi((x,m))\ \phi((h,p))$. Thus, $W^- \leq M_n$ and $Q_o^- \leq P_n^-$. Next, we need to prove the $M_{n-1}^- \lneq M_n^-$. Let $\bar{\gamma}_{n-1}$ and $\bar{\gamma}_n$ be the o-permutation of N_{n-1} and N_n respectively as defined above. Let γ_o be the map from N_{n-1} into N_n

$$\begin{tikzcd}
N_{n-1} \arrow[r, "\gamma_o"] \arrow[d, "\bar{\gamma}_{n-1}"'] & N_n \arrow[d, "\bar{\gamma}_n"] \\
N_{n-1} \arrow[r, "\gamma_o"'] & N_n
\end{tikzcd}$$

defined by:

$$\gamma_o(i) = i \quad \text{if} \quad i \text{ belongs to } Z$$

$$\gamma_o(i) = i+1 \quad \text{otherwise}$$

It can be easily verified that $\gamma_n o \gamma_o = \gamma_o o \gamma_{n-1}$. Therefore $M^-_{n-1} \leq M^-_n$ and $P^-_{n-1} \leq P^-_n$. In order to prove the strict inclusion between P^-_{n-1} and P^-_n, let $a,b,c, d_1,d_2,\ldots,d_{n-1}$ be positive elements of M^-_n such that for $k > 0$, $c = (0,\gamma^k_n)$, $b = (y,\gamma^0_n)$, $a = (x,\gamma^0_n)$, $d_i = (t^i,\gamma^0_n)$ where $y_i = 0$ if $i \neq t_1$ and $y_i = 1$ otherwise, $x_i > 0$ if i belongs to Z and $x_i = 0$ otherwise, $t^i_j = 0$ for every $j \neq t_{i+1}$ and $t^i_j = 1$ otherwise. Note that $b \ll d_1 \ll d_2 \ll \ldots \ll d_{n-1} \ll a \ll c$, $d_i = d^c_i$, and $a \ll a^c$ but $b = b^c$ and $b \neq (0,\gamma^0_n)$. Hence, $M^-_{n-1} \lneq M^-_n$ and $P^-_{n-1} \lneq P^-_n$. From Example 3.25 we get that $M^-_1 \gneq W^-$ so now we have $W^- \lneq M^-_1 \lneq M^-_2 \lneq \ldots \lneq M^-_n \lneq \ldots$ and $Q^-_o \lneq P^-_1 \lneq P^-_2 \lneq \ldots \lneq P^-_n \lneq \ldots$.

It is interesting to compare the two families (P^-_n) and (Q^-_n).

<u>Theorem 3.28</u>: For every $n \geq o$ we have $P^-_n \lneq Q^-_n$ and $P^-_n \neq Q^-_i$ for all $o < i \leq n-1$.

<u>Proof</u>: We start by proving that $M^-_n \lneq W^-_n$. Let Λ_n, N_n, $\bar{\alpha}_n$, $\bar{\gamma}_n$ be as defined before. Let γ_o be the map from N_n into Λ_n defined by

$$\gamma_o(i) = i \quad \text{if} \quad i = t_j$$

$$\gamma_o(i) = i \quad \text{if} \quad i \text{ in } Z$$

where $\gamma_o(i)$ belongs to $Z_2 \leq \Lambda_n$. It can be easily verified that $\bar{\alpha}_n o \gamma_o = \gamma_o o \bar{\gamma}_n$. Hence $M^-_n \leq W^-_n$. In order to prove the strict inclusion, let $a,b,c, d_1,d_2,\ldots,d_n$ be such that for $k > 0$, $c = (0,k)$, $a = (x,\alpha^0_n)$, $b = (y,\alpha^0_n)$, and $d_i = (t^i,\alpha^0_n)$ where $x_i > 0$ if i belongs to Z_2 and $x_i = 0$ otherwise, $y_i > 0$ if i belongs to Z_1 and $y_i = 0$ otherwise, $t^i_j = 0$ for

$j \neq t_i$ and $t^i_j = 1$ if $j = t_i$. Thus $b \ll d_1 \ll d_2 \ll \ldots \ll d_n \ll a \ll c$, $d_i = d_i^c$ for every $i = 1,\ldots,n$ and $a \ll a^c$ but $b \neq (0,\alpha^0_n)$ and $b \ll b^c$. Hence, $P^-_n \lneq Q^-_n$.

$$\begin{tikzcd} N_n \arrow[r, "\gamma_o"] \arrow[d, "\bar{\gamma}_n"'] & \Lambda_n \arrow[d, "\bar{\alpha}_n"] \\ N_n \arrow[r, "\gamma_o"'] & \Lambda_n \end{tikzcd}$$

It remains to prove that P^-_n is different than Q^-_{n-1}. Let a,b,c, $d_1,d_2,\ldots,d_{n-1}$, be positive elements in M^-_n chosen as follows: Fix $k > 0$, and $c = (0,\gamma^k_n)$, $b = (y,\gamma^0_n)$, $a = (x,\gamma^0_n)$, $d_i = (t_i,\gamma^0_n)$ where $y_i = 0$ if $i \neq t_1$, and $y_i = 1$ otherwise, $x_i > 0$ if $i \in Z$ and $x_i = 0$ otherwise, $t^i_j=0$ for every $j \neq t_{i+1}$ and $t^i_j = 1$ otherwise. Then we have $b \ll d_1 \ll \ldots \ll d_{n-1} \ll a \ll c$, $d_i = d_i^c$ for every $i = 1,\ldots,n-1$ and $a \ll a^c$, but $b = b^c$. Then b cannot be infinitely smaller than b^c. Therefore M^-_n does not satisfy the implications that W^-_{n-1} satisfies so $P^-_n \lneq Q^-_{n-1}$. Now P^-_n cannot be equal to Q_i for $i < n-1$ since $Q^-_i \lneq Q^-_{n-1}$.

<u>Remark</u>: For every n, P^-_n belongs to $\mathcal{R}\cap A^2$ since P^-_n is contained in Q^-_n and Q^-_n belongs to $\mathcal{R}\cap A^2$. Then $(P^-_n)_{n\in Z\pm\{0\}}$, is another countable infinite chain of different quasi-varieties containing A and contained in $\mathcal{R}\cap A^2$.

Using similar techniques we can construct an infinite chain of quasi-varieties between A and $\mathcal{R} \cap A^2$. Let Q^+_o be the quasi-variety generated by M^+. We know Q^+_o covers A. First, we consider some properties of W^+. The most important one is the following: If a,b,c are positive elements of W^+ with $b \ll a \ll c$ and $a^c \ll a$, then $b^c \ll b$. Again, this condition can be stated in an equivalent way by using an infinite set of implications. They are:

(T1) $b \le a \le c$ and $a^c \le a$ implies $b^c \le b$.

(T2) $b^2 \le a \le a^2 \le c$ and $(a^c)^2 \le a$ implies $(b^c)^2 \le b$

$\vdots$

(Tλ) $b^\lambda \le a \le a^\lambda \le c$ and $(a^c)^\lambda \le a$ implies $(b^c)^\lambda \le b$

$\vdots$

These implications give a characterization of W^+ and provide a way of determining whether an ℓ-group is different than W^+.

There are ℓ-groups that fail the condition above. Now, consider $G(\gamma_1)$ (see Example 3.25) with the following order: $(0,\gamma_1^0) \le (g,\gamma_1^n)$ iff $n > 0$ or $n = 0$ and $g_i \ge 0$ where $i \in Z$ is the minimum index with respect to $g_i \ne 0$, or $n = 0$, $g_i = 0$ for every $i \in Z$ and $g_{t_1} \ge 0$. Let M_1^+ be $G(\gamma_1)$ with this order, and P_1^+ the quasi-variety generated by M_1^+.

Now, M_1^+ fails the implications. In fact, consider a,b,c to be three special positive elements of M_1^+. Fix $n > 0$ and let $c = (0,n)$, $a = (x,0)$, and $b = (y,0)$ where $x_i = 0$ if $i = t_1$ and $x_i > 0$ otherwise, $y_i = 0$ if $i \ne t_1$ and $y_i > 0$ otherwise. Notice that $b \ll a \ll c$ and $a^c \ll a$ but $b = b^c$. Hence, b^c is not infinitely smaller than b.

The ideas of this example can be generalized as in 3.25 to find other ℓ-groups which fail the condition above. Hence, as for the P_n^-'s we have a family $(P_n^+ | n \in Z^+ - \{0\})$ of countably infinite distinct quasi-varieties between A and $R \cap A^2$. Therefore, the theorem follows.

<u>Theorem 3.29</u>: There is an infinite chain of distinct quasi-varieties between A and $R \cap A^2$.

To complete this chapter we look at a general way of constructing ℓ-groups using the techniques that have been developed. This is done similar to the construction of W_n^- but with a different index set.

Let $\Lambda_{n_1,n_2,\ldots,n_k} = Z_1 \cup \{t_1^1,\ldots,t_{n_1}^1\} \cup Z_2 \cup \{t_1^2,\ldots,t_{n_2}^2\} \cup \ldots \cup$ $Z_k \cup \{t_1^k,\ldots,t_{n_2}^k\} \cup Z_{k+1}$ where $Z_i = Z$ for every $i = 1,\ldots,k+1$. Let $\bar{\alpha}_{n_1,n_2,\ldots,n_k}$ be the o-permutation of $\Lambda_{n_1,\ldots,n_k}$ defined by $\bar{\alpha}_{n_1,n_2,\ldots,n_k}(i) = i+1$ if i belongs to $Z_1 \cup Z_2 \cup \ldots \cup Z_{k+1}$ and $\bar{\alpha}_{n_1,n_2,\ldots,n_k}(i) = i$ otherwise. Note that the index set $\Lambda_{n_1,\ldots,n_k}$ is totally ordered as follows: The Z_i's carry the natural order of Z, and if i_j is an element of Z_j then $i_j \le t_1^j \le t_2^j \le \ldots \le t_{n_j}^j \le i_{j+1}$. Let $G = \bigoplus_{\Lambda_{n_1,\ldots,n_k}} Z$ and $\alpha_{n_1,\ldots,n_k}$ be the o-automorphism of G such that $\phi(g) = g'$ where $g_i' = g_{\bar{\alpha}_{n_1,\ldots n_k}(i)}$. Consider $W_{n_1,\ldots,n_k}^-$ to be the cyclic extension of G by $\alpha_{n_1,\ldots,n_k}$, and $Q_{n_1,\ldots,n_k}^-$ the quasi-variety generated by $W_{n_1,\ldots,n_k}^-$.

We first see how to compare $W_{n_1,n_2\ldots,n_k}^-$ with the Q_n^-'s constructed before.

<u>Theorem 3.30</u>: If $n_1+n_2+\ldots+n_k = n$, then $W_{n_1\ldots,n_k}^-$ belongs to Q_n^- and in fact $Q_{n_1,\ldots,n_k}^- \ge Q_n^-$.

Let ϕ': $W_n^- \to W_{n_1,\ldots,n_k}^-$ be defined by

$\phi'((g,\alpha^m_n)) = (g',\alpha^m_{n_1,\ldots,n_k})$ where $g'_i = 0$ if $i \in Z_2 \cup \ldots \cup Z_k$,

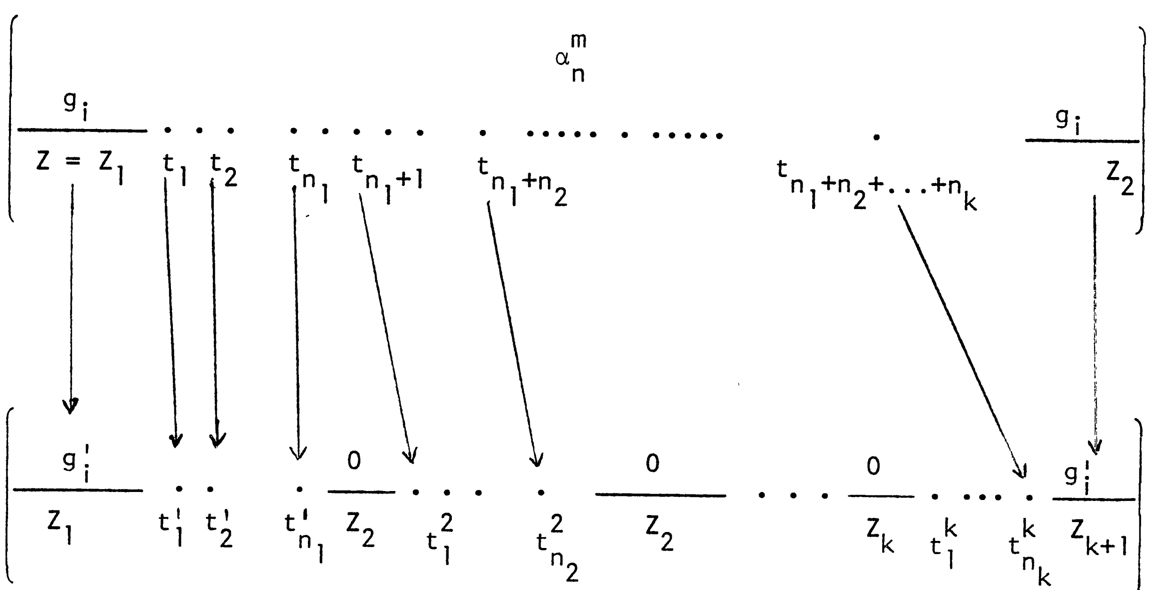
$g'_i = g_j$ where $j \in Z_1 \leq \Lambda_n$, $i = j$ and $i \in Z_1 \leq \Lambda_{n_1,\ldots,n_k}$.

$g'_i = g_j$ where $j \in Z_2 \leq \Lambda_n$, $i = j$ and $i \in Z_{k+1} \leq \Lambda_{n_1,\ldots,n_k}$,

and $g'_{t^1_1} = g_{t_1}$, $g'_{t^1_{n_1}} = g_{t_{n_1}}$, $g'_{t^2_1} = g_{t_{n_1+1}}, \ldots, \ldots, g'_{t^2_{n_2}} = g_{t_{n_1+n_2}}$.

Thus, $g'_{t^i_j} = g_{t_{n_1+\ldots+n_{i-1}+j}}$ for $1 \leq i \leq k$ and $1 \leq j \leq n_i$.

It appears easier if we draw the following picture:



The map ϕ' is clearly well-defined. We need to prove that ϕ' is a homomorphism; i.e., that $\phi'((g,\alpha_n^m)(h,\alpha_n^p)) = \phi'((g,\alpha_n^m))\cdot\phi'((h,\alpha_n^p))$. First, $\phi'((g,\alpha_n^m)) = (g',\alpha^m_{n_1,\ldots,n_k})$ and $\phi'((h,\alpha_n^p)) = (h',\alpha^p_{n_1,\ldots,n_k})$. Now,

$\phi'((g,\alpha_n^m)(h,\alpha_n^p)) = \phi'((r,\alpha_n^{m+p}))$ where $r_i = g_i + h_{\alpha_n^m(i)}$. Then

$\phi'((r,\alpha_n^{m+p})) = (r',\alpha^{m+p}_{n_1,\ldots,n_k})$. But $\phi'((g,\alpha_n^m))\cdot\phi'((h,\alpha_n^p)) =$

$(g',\alpha^m_{n_1,\ldots,n_k})(h',\alpha^p_{n_1,\ldots,n_k}) = (s,\alpha^{m+p}_{n_1,\ldots,n_k})$ where $s_i = g_i' +$

$h'_{\alpha^m_{n_1,\ldots,n_k}(i)}$. It suffices to prove that $s_i = r_i'$ for every $i \varepsilon \Lambda_{n_1,..,n_k}$.

First, if $i \varepsilon Z_2 \cup \ldots \cup Z_k$, $r_i' = s_i = 0$. Secondly, if $i \varepsilon Z_1 \leq$ $\Lambda_{n_1,\ldots,n_k}$ then $r_i' = r_j$ where $i = j$ and $j \varepsilon Z_1 \leq \Lambda_n$. Thus, $r_i' = r_i = g_i +$ $h_{\alpha_n^m(i)} = g_i' + h'_{\alpha_n^m(i)} = g_i' + h'_{\alpha^m_{n_1,\ldots,n_k}(i)} = s_i$. Similarly, if $i \varepsilon Z_{k+1}$, $r_i' = s_i$. Finally, if $\lambda = t^i_j$ with $1 \leq i \leq k$ and $1 \leq j \leq n_i$ then

$r_\lambda' = r_{t_{n_1+n_2+\ldots+n_{i-1}+j}} = g_{t_{n_1+n_2+\ldots+n_{i-1}+j}} + h_{\alpha_n^m(t_{n_1+n_2+..+..+n_{i-1}+j})} =$

$g_{t_{n_1+n_2+\ldots+n_{i-1}+j}} + h_{t_{n_1+n_2+\ldots+n_{i-1}+j}} = g_\lambda' + h_\lambda' = g_\lambda' + h'_{\alpha^m_{n_1,..,n_k}(\lambda)} =$

s_λ. Thus, ϕ' is a homomorphism and clearly a monomorphism.

Last, we need to prove that ϕ' is an ℓ-homomorphism. It suffices to show that $\phi'(x \vee y) = \phi'(x) \vee \phi'(y)$ for every $x,y \varepsilon W_n^-$. Let $x = (g,\alpha_n^m)$ and

$y = (h, \alpha_n^p)$. We know that W_n^- is totally ordered. Hence, we can assume that $x \leq y$. So $x v y = y$ and $\phi'(xvy) = \phi'(y)$. Two possibilities arise. First, if $m < p$, then $\phi'(x) < \phi'(y)$ and $\phi'(x) v \phi'(y) = \phi'(y) = \phi'(xvy)$. Secondly, if $m = p$, then let η_1, η_2 be the indexes of Λ_n maximal with respect to $g_{\eta_1} \neq 0$ and $h_{\eta_2} \neq 0$, respectively. Now $x \leq y$ implies that one of the following is true.

(i) $\eta_1 \leq \eta_2$ and $\eta_1, \eta_2 \in Z_2 \leq \Lambda_n$

(ii) $\eta_1 < \eta_2$ with $\eta_1 \in \{t_1, \ldots, t_n\} \leq \Lambda_n$ and $\eta_2 \in Z_2 \leq \Lambda_n$

(iii) $\eta_1 \leq \eta_2$ with $\eta_1, \eta_2 \in \{t_1, \ldots, t_n\} \leq \Lambda_n$

(iv) $\eta_1 < \eta_2$ with $\eta_1 \in Z_1 \leq \Lambda_n$ and $\eta_2 \in \{t_1, \ldots, t_n\} \leq \Lambda_n$

(v) $\eta_1 < \eta_2$ with $\eta_1, \eta_2 \in Z_1 \leq \Lambda_n$

Easily, we can verify that in all cases we have $\phi'(x) \leq \phi'(y)$, and hence $\phi'(x) v \phi'(y) = \phi'(y)$. Thus, ϕ' is an ℓ-monomorphism. Hence, $W_n^- \leq W_{n_1, \ldots, n_k}^-$ and $Q_n^- \leq Q_{n_1, \ldots, n_k}^-$.

Using the same proof techniques, it can be shown that if $n_1 + n_2 + \cdots + n_k > n$, then $Q_n^- \leq Q_{n_1 + n_2 + \cdots + n_k}^-$.

CHAPTER IV

NON-REPRESENTABLE ℓ-GROUPS

The goal of this chapter is to construct an infinite chain of different quasi-varieties each of which contains A and is contained in the variety $L_n \cap A^2$, where L_n is defined by $[x^n, y^n] = 1$ for any fixed integer $n \geq 2$.

Recall some of the better studied ℓ-group varieties are E = the trivial variety, A = the variety of abelian ℓ-groups, R = the variety of representable ℓ-groups, N = the variety of normal-valued and ℓ-groups, and L = the variety of all ℓ-groups. It has been shown that all non-trivial varieties contain A (E. C. Weinberg [23] and that all proper varieties are contained in N (W. C. Holland [8]).

Since N plays an important role in this chapter, we first define normal-valued ℓ-groups and give some characterizations of such ℓ-groups.

Definition 4.1: Let G be an ℓ-group and C a subgroup of G. C is solid iff C is a convex ℓ-subgroup of G (i.e., if $x, y \in C$ are such that $|y| \leq |x|$ then $y \in C$).

Let $C(G)$ be the set of all solid subgroups of G ordered by inclusion.

Definition 4.2: Let M be a subgroup of G. M is regular iff $M \in C(G)$ and if $M = {}_{i \in I} C_i$ for $C_i \in C(G)$ then there exists $i \in I$ such that $M = C_i$.

Let $R(G)$ be the set of all regular subgroups of an ℓ-group G, and for every $R \in R(G)$, let R^* be the element of $C(G)$ that covers R.

Definition 4.3: Let G be an ℓ-group, R an element of $R(G)$. R is a normal value if R is normal in R^*. If for every $R \in R(G)$, R is a normal value then G is a normal-valued ℓ-group.

Any representable ℓ-group is normal-valued but the converse is not true. In fact, the ℓ-group in Example 3.5 (ii) is normal valued but not representable.

There are numerous characterizations of normal valued ℓ-groups. The most useful ones are the following: An ℓ-group G is normal-valued iff G satisfies one of the relations:

(i) For every $a,b \varepsilon G$, $a^{-1}b^{-1}ab << |a|v|b|$.

(ii) If $a,b \varepsilon G^+$ then $ab \leq b^2a^2$.

It is known that the varieties $A^2 = AA,\ldots,A^n = A^{n-1}A$ form a proper chain from A to N with $N = V\{A^n;n\varepsilon N\}$ (see A. M. W. Glass [5]). Since A plays such an important role, it is natural to attempt to generalize it. Therefore, for $n \geq 2$ let L_n be the variety defined by the equation $[x^n,y^n] = 1$. These L_n's form a family with several properties of interest:

(1) $L_m < L_n$ iff m is a proper divisor of n (J. Martinez [11]),

(2) $L_m \wedge L_n = A$ iff $(m,n) = 1$ (E. B. Scrimger [21]), and

(3) $L_n \wedge R = A$ for all $n \geq 2$ (J. Martinez [11]).

Also, within each L_n lies a subdirectly irreducible ℓ-group of particular interest: $G_n = \overset{n-1}{\underset{0}{\pi}} Z \overleftarrow{\times} Z$ where for

$$(a_0,\ldots,a_{n-1};b),\ (c_0,\ldots,c_{n-1};d) \text{ in } G_n,$$

$$(a_0,\ldots,a_{n-1};b) \leq (c_0,\ldots,c_{n-1};d) \text{ iff } b<d \text{ or } b=d, \text{ and}$$

$$a_i \leq c_i \text{ for } 0 \leq i \leq n-1, \text{ and}$$

$$(a_0,\ldots,a_{n-1};b)(c_0,\ldots,c_{n-1};d) = (a_0 + c_{0+b},\ldots,a_{n-1} + c_{n-1+b};b+d)$$

with all subscripts read modulo n. This ℓ-group may also be viewed as an ℓ-subgroup of ZWrZ and is called the Scrimger ℓ-group (see Smith [22] for more details on G_n).

Let S_n be the variety of ℓ-groups generated by G_n (i.e. the smallest variety of ℓ-groups containing G_n). We know that $S_n \leq \mathcal{N}\text{-}\mathcal{R}$ and that for every prime number n S_n covers $\mathcal{A}$ (E. B. Scrimger [21]). It is also readily seen by considering the generating ℓ-groups that for p, q distinct primes,

$$S_p, S_q \lneq S_p \vee S_q \lneq S_{pq}$$

Thus, $\{S_p | p \text{ is prime}\}$ generates a sublattice of L isomorphic to the lattice of finite sets of primes. The work done thus far with S_n varieties has relied primarily on knowledge of their generating ℓ-groups. Some equations that are satisfied in G_n for $n \geq 2$ have been brought to light. They include:

(i) $[a^n,b^n] = 1.$

(ii) $[[a,b],[c,d]] = 1.$

(iii) $[a,b,c^n] = 1.$

(iv) $[a,b,c]^{1+c+c^2+\cdots+c^{n-1}} = 1.$

(v) $[a^{1+b+\cdots+b^{n-1}}, c^{1+d+\cdots+d^{n-1}}] = 1.$

(vi) $[a,kb]^{1+b+\cdots+b^{n-1}} = 1$ for any $k \geq 2$.

(vii) $[a^{1+b_1+b_2+\cdots+b_{n-1}}, c^{1+d_1+\cdots+d_{n-1}}] = 1.$

Little has been said of the quasi varieties generated by the G_n's.

We know that quasi-varieties are weaker algebraic structures than varieties. Although, if n is a prime number, then there are not any varieties between A and S_n, and $A \lneq S_n \leq L_n \cap A^2$. We will construct an infinite number of different quasi-varieties between A and $L_n \cap A^2$.

First, we establish the analog of Scrimger's theorem for quasi-varieties. But, before doing so we need the following definitions, results, and notations. First, let H_o be the quasi-variety generated by G_n.

<u>Definition 4.4</u>: Let G be an ℓ-group and N a solid subgroup of G. N is prime iff for any arbitrary elements $a,b \varepsilon N$ with $a \vee b = e$, where e is the identity of G, we have either $a \varepsilon N$ or $b \varepsilon N$. Also, N is a minimal prime subgroup iff there exists no nontrivial ℓ-subgroup of G strictly contained in N, and is called in Scrimger [21] a representing subgroup.

<u>Definition 4.5</u>: Let G be an ℓ-group and F an nonempty subset of G such that

(i) The meet of finitely many elements of F belongs to F.

(ii) If $x \varepsilon F$, and $t \geq x$, then $t \varepsilon F$.

(iii) The smallest element e of G^+ does not belong to F.

Then F is called a <u>filter</u> over G. If F is maximal, then F is called an <u>ultrafilter</u>.

<u>Definition 4.6</u>: Let G be an ℓ-group, e the identity of G and x an arbitrary element of G, and $A \leq G$. Then,

$$x^{\perp} = \{g | g \varepsilon G \text{ and } |g| \wedge |x| = e\}, \text{ and}$$

$$A^{\perp} = \{g | g \varepsilon G \text{ and } |g| \wedge |a| = e \text{ for every } a \varepsilon A\}.$$

$x^{\perp}$ and $A^{\perp}$ are called the <u>polars</u> of x and A, respectively.

The following theorems (see Bigard [2]) demonstrate the significance of ultrafilters and polars in generating minimal prime subgroups.

<u>Theorem 4.7</u>: Let G be an ℓ-group, M an arbitrary minimal prime

subgroup of G. The map ϕ defined by $\phi(M) = G^{+}-M$ is a bijection between the set of all minimal prime subgroups of G and the set of all ultrafilters over G^{+}.

Theorem 4.8: Let G be an ℓ-group, C a solid subgroup of G, and U an ultrafilter over C^{+}. The set $M = \cup\{x^{\perp} | x \in U\}$ is a minimal prime subgroup of G, and any minimal prime subgroup M of G that does not contain C is of this form (i.e. $M = \cup\{x^{\perp} | x \in U\}$ for some ultrafilter U over G^{+})

Proof: Let V be the set of all upper bounds in G of elements of U (i.e. $V = \{x \in G$ such that there exists $y \in U$ with $y \leq x\}$).

Clearly $U \leq V$. To see that V is a filter over G^{+} we show that V satisfies the conditions in 4.6. First, if x,y are elements of V, then there exists $x', y' \in U$ such that $x \geq y'$ and $y \geq y'$. So, $x \wedge y \geq x' \wedge y' \in U$, and then $x \wedge y \in V$. Now, if $x \in V$ and $t \geq x$, then there exists $g \in U$ such that $x \geq g$. Thus, $t \geq x \geq g \in U$ so $t \in V$.

Now suppose $e \in V$. Then there exists $u \in U$ such that $u \leq e$. Thus, $e \in U$ contrary to the definition of a filter since e is also the smallest element of C^{+}. Hence, $e \notin V$. Note that $G^{+} - V \leq G^{+} - U$. Next, we will prove that V is an ultrafilter over G^{+}. It suffices to prove that for every $g \in G^{+} - V$ there exists $v \in V$ such that $g \wedge v = e$ Considering $g \in G^{+}-V$ and $x \in U$.

If $g \wedge x \in U$, then $g \in U$ since $g \geq g \wedge x$. But $G^{+} - V \leq G^{+} - U$, so $g \in G^{+} - U$, and we have a contradiction. Hence, $g \wedge x \notin U$. Since U is an ultrafilter there exists $y \in U$ such that $(g \wedge x) \wedge y = e$. But $x \wedge y \in U$ since U is closed under taking the meet of a finite number of elements of U. Let $v = x \wedge y$. Then $g \in v^{\perp}$ and $v \in V$. Now, every filter T over G^{+} strictly containing V contains the smallest element e. Suppose T is such a filter and $t \in T-V$. Then there exists $s \in V$ such that $t \wedge s = e$. But $s \in T$ since $V \leq T$. Hence, $s \wedge t = e \in T$. Therefore, V is an ultrafilter over G^{+}

and $\cup\{x^{\perp}|x\varepsilon V\} = \cup\{x^{\perp}|x\varepsilon U\} = G^{+}-V$ is a minimal prime subgroup of G.

Conversely, let M be a minimal prime subgroup of G that does not contain C, $V = G^{+}-M$, and $U = V \cap C$. Clearly, U is an ultrafilter over C^{+}. Now, the set of all upper bounds in G of elements of U is an ultrafilter so it must be equal to V.

A similar reasoning leads to the following proposition.

<u>Proposition 4.9</u>: Let G be an ℓ-group, g an element of G, X the set of all elements of G^{+} smaller than or equal to $|g|$, and U an ultrafilter over X. The set $\cup\{x^{\perp}|x\varepsilon U\}$ is a minimal prime subgroup of G, and every minimal prime subgroup M of G that does not contain g is of this form (i.e. $M = \cup\{x^{\perp}|x\varepsilon U\}$ where U is an ultrafilter over X).

By Scrimger [21], if n is a prime number S_n covers A. Hence, it is very important to determine when a given ℓ-group G contains an ℓ-subgroup isomorphic to G_n. Therefore, the following lemma is useful.

<u>Lemma 4.10</u>: If G is an ℓ-group of L_n, and C a representing subgroup of G, which has n distinct conjuguates of the form $x^{-i}\,C\,x^{i}$ for some $x\varepsilon G$ and $x > e$ where e is the identity of G, then G contains an ℓ-subgroup isomorphic to G_n.

<u>Proof</u>: See Scrimger [21].

Using this Lemma we can prove the analog of Scrimger's theorem for quasi-varieties.

<u>Theorem 4.11</u>: The quasi-variety H_o covers A in the lattice of quasi-varieties of ℓ-groups.

<u>Proof</u>: Suppose there exists a quasi-variety Q such that $A \lneqq Q \lneqq H_o$. Then there exists an ℓ-group $G \varepsilon Q$ such that G is nonabelian. Hence, there are two positive elements $x,y \varepsilon G$ such that $xy \neq yx$ and $x \wedge y \neq e$.

Note that G contains always a nontrivial minimal prime subgroup C.

In fact, let X be the set of all elements in G^+ smaller than or equal to x, $X \neq \{e\}$ since x and $x \wedge y \in X$, and U an ultrafilter over X that contains x and $x \wedge y$. Then, by 4.9 $C = \cup\{a^{\perp} | a \in U\}$ is a minimal prime subgroup of G. C is nontrivial since x^{-1}, y^{-1} and $x^{-1} \wedge y^{-1}$ are in C.

Now, let C be a nontrivial minimal prime subgroup of G. It suffices to prove that C has n distinct conjuguates of the form $x^{-i}\ C\ x^i$ for some $x > e$, since then by Lemma 4.10 G contains an ℓ-subgroup isomorphic to G_n and $Q = H_o$. First, since $G \in S_n \leq L_n$ we have $x^{-n}\ C\ x^n = C$ for all $x \in G$. In fact, suppose $e < c \in C$. Then $e < c < c^2 < \ldots < c^n = x^{-n} c^n x^n \in x^{-n}\ C\ x^n$, so $c \in x^{-n}\ C\ x^n$, and hence $C \leq x^{-n}\ C\ x^n$. Similarly, $C \leq x^n\ C\ x^{-n}$, so $C = x^{-n}\ C\ x^n$. Next, the number t of distinct conjuguates of C of the form $x^{-i}\ C\ x^i$ is a divisor of n. In fact, let i be the smallest positive integer such that $x^{-i}\ C\ x^i = C$. If i is not a divisor of n, then there are integers r and s such that $rn+si = k$, where $1 \leq k < i$, and k is the greatest common divisor of n and i. Then, $x^{-k}\ C\ x^k = x^{-rn-si}\ C\ x^{rn+si}$ $= C$, contradicting the minimality of i. Hence, if n is prime, then t is 1 or n. Now, we will show that there exist at least two distinct conjuguates of C of the form $x^{-i}\ C\ x^i$. There exists $c < x \in G$ such that $x^{-1}Cx \neq C$. Suppose $x^{-1}Cx = C$ for all $c < x \in G$. Then, C is an ℓ-ideal of G, so since it is also a representing subgroup, $C = \{e\}$, and G is totally ordered and therefore representable. But $\mathcal{R} \cap L_n = \mathcal{A}$, so G must be abelian since $G \in \mathcal{R} \cap L_n$. This is a contradiction. Therefore there is a positive x such that $x^{-1}\ Cx \neq C$. Thus, $x^{-1}\ Cx$ and $x^{-0}Cx^0 = C$ are two distinct conjuguates of C. Hence, $t = n$ and the theorem is proved. Since by 4.10 G contains an ℓ-subgroup isomorphic to G_n and $Q \leq H_o$. Therefore, $Q = H_o$ and H_o covers $\mathcal{A}$ in the lattice of quasi-varieties.

It is not always easy to find the implications that define a given quasi-variety. But it is possible to find a set of implications that are satisfied by all elements of the quasi-variety. We consider first some properties of G_n.

Proposition 4.12: If a,b,c are positive elements of G_n with $b \ll a \ll c$, then $b = (0,\dots,0;0)$.

Proof: Clearly the property is true if $a = b = c = (0,\dots,0;0)$. Let $a = (a_0,\dots,a_{n-1};q)$, $b = (b_0,\dots,b_{n-1};p)$ and $c = (c_0,\dots,c_{n-1},k)$. First, if $k=0$ then $p = q = 0$ since $b \ll a \ll c$. Now, $a \ll c$ implies $a_i = 0$ for every $i = 0,\dots,n-1$.

Suppose there exists i, $0 \le i \le n-1$, such that $a_i > 0$. We have either $c_i = 0$ or $c_i > 0$. But, if $c_i = 0$ then $a_i > c_i$ and this is contradiction with $a \le c$. Otherwise, there exists $m \in Z^+-\{0\}$ such that $ma_i > c_i$ contrary to $a \ll c$. Hence, $a_i = o$ for every $i = 0,\dots,n-1$, and $b_i = 0$ for every $i = 0,\dots,n-1$ since $b \ll a$. Hence, $b = (0,\dots,0;0)$. Next, if $k > 0$ then $q = p = 0$. Otherwise, if $q > 0$, then there exists $\lambda \in Z^+-\{0\}$ such that $\lambda q > k$ and $a^\lambda > c$ contrary to $a \ll c$. So, $p = q = 0$, and $b_i = 0$ for every $i = 0,\dots,n-1$ since $b \ll a$. Hence, $b = (0,\dots,0;0)$.

The condition in this proposition can be stated in an equivalent way by using an infinite set of implications.

Proposition 4.13: An ℓ-group G satisfies the property "If a,b,c are positive elemnts of G with $b \ll a \ll c$ then $b = 1_G$", iff G satisfies the implication.

(I) $b^n \le a \le a^n \le c$ for all $n \in Z$, $n \ge 1$ implies $b = 1_G$.

Proof: Clearly if (I) is true for every $n \ge 1$ then the condition holds.

Conversely assume the property is satisfied. If $b^n \le a \le a^n \le c$ for all positive integers n, then $b \ll a \ll c$ so $b = 1_G$.

Note that, for every $\lambda \geq 1$ the implication $b^{\lambda} \leq a \leq a^{\lambda} \leq c$ implies b is the identity of G_n is not always true in G_n. For this we will give the following example:

Example 4.14: Fix $\lambda \geq 1$, and let $a,b,c \in G_n$ such that $c = (0,0,\ldots;k)$ where $k > 0$, $a = (n\lambda,\ldots,n\lambda;0)$ where λ, $n \in Z^{+}-\{0\}$, and $b = (1,\ldots,1;0)$. Now, $b^{\lambda} = (\lambda,\ldots,\lambda;0) \leq (n\lambda,\ldots,n;0) = a \leq (n\lambda^2,\ldots,n\lambda^2;0) = a^{\lambda} \leq (0,\ldots,0;k) = c$. But $b \neq (0,\ldots,0;0)$ and (I) fails in G_n for every $\lambda \geq 1$.

Propositions 4.12 and 4.13 give a characterization of G_n and provide a way of determining whether an ℓ-group is different than G_n.

Generally, to show that an ℓ-group is not in H_o it suffices to find elements in the ℓ-group that do not satisfy (I) as we will see in the following example:

Example 4.15: Let $H_n^1 = \prod_0^n Z \overleftarrow{\times}_{-} Z$ where for $(a_0, a_1,\ldots,a_n;b)$; and $(c_0, c_1,\ldots,c_n;d)$ in H_n^1, $(a_0,\ldots,a_{n-1},a_n;b) \leq (c_0,\ldots,c_{n-1},c_n;d)$ iff $b<d$ or $b=d$ and $a_n < c_n$ or $b = d$, $a_n = c_n$ and $a_i \leq c_i$ for $i = 0,\ldots,n-1$, and $(a_0, a_1,\ldots,a_n;b)\,(c_0,\ldots,c_n;d) = (a_0 + c_{a+b},\ldots,a_{n-1} + c_{n-1-b}, a_n+c_n;$ $b+d)$ with all subscripts smaller than or equal to $n-1$ read modulo n. Let H_1 be the quasi-variety generated by H_n^1. H_n^1 carries a natural group structure. Note that $(a_0,\ldots,a_{n-1},a_n;b)^{-1} = (-a_{0-b},\ldots,-a_{n-1-b},-a_n;-b)$ where all subscripts smaller than or equal to $n-1$ are read modulo n, and the identity in H_n^1 is $(0,\ldots0,0;0)$.

Also, the order $\leq$ defines a lattice order on H_n^1. In fact,

$$(a_0,\ldots,a_n;b)\wedge(b_0,\ldots,b_n;d) = \begin{cases}(a_0,\ldots,a_n;b) & \text{if } b<d\\ (b_0,\ldots,b_n;d) & \text{if } d<b\\ (a_0,\ldots,a_n;b) & \text{if } b=d \text{ and } a_n<b_n\\ (b_0,\ldots,b_n;d) & \text{if } b=d \text{ and } b_n<a_n\\ (a_0\wedge b_0,\ldots,a_{n-1}\wedge b_{n-1},a_n;b) & \text{if } b=d \text{ and } b_n=a_n\end{cases}$$

where $a_i \wedge b_i$ = minimum (a_i,b_i), and

$$(a_0,\ldots,a_n;b)\vee(b_0,\ldots,b_n;d) = \begin{cases}(a_0,\ldots,a_n;b) & \text{if } b>d\\ (b_0,\ldots,b_n;d) & \text{if } b<d\\ (a_0,\ldots,a_n;b) & \text{if } b=d \text{ and } a_n>b_n\\ (b_0,\ldots,b_n;d) & \text{if } b=d \text{ and } a_n<b_n\\ (a_0\vee b_0,\ldots,a_{n-1}\vee b_{n-1},a_n;b) & \text{if } b=d \text{ and } a_n=b_n\end{cases}$$

where $a_i \vee b_i$ = maximum (a_i,b_i).

To show H_n^1 is an ℓ-group it only remains to verify that if $a,b,c \in H_n^1$ where $a \leq b$ then $ac \leq bc$ and $ca \leq cb$. Since a,b,c are in H_n^1 write $a = (a_0,\ldots,a_n;k)$, $b = (b_0,\ldots,b_n;p)$, and $c = (c_0,\ldots,c_n;d)$. Now $a \leq b$ implies that $k < p$ or $k = p$ and $a_n < b_n$, or $k = p$, $a_n = b_n$ and $a_i < b_i$ for $i = 0,\ldots,n-1$. If $k < p$ then $k+d < p+d$ and $ac = (a_0,\ldots,a_n;k)(c_0,\ldots,c_n;d) = (a_0 + c_{0+k},\ldots,a_n+c_n;k+d) \leq (b_0 + c_{0+k},\ldots,b_n+c_n;p+d) = (b_0,\ldots,b_n;p)(c_0,\ldots,c_n;d) = bc$. Otherwise, if $k=p$ and $a_n < b_n$ then

$a_n+c_n < b_n+c_n$ and $ac \leq bc$. At last, if $k=p$, $a_n=b_n$ and $a_i \leq b_i$ for $i = 0,\ldots n-1$ then $a_i + c_{i+k} \leq b_i+c_{i+k}$ and $ac \leq bc$. We can verify in a similar way that $ca \leq cb$. Note that H_n^1 is an ℓ-group that fails (I). To see this let a,b,c be 3 elements of G such that $a = (0,0,\ldots,0,1;0)$, $b = (1,0,0,\ldots,0,0;0)$, and $c = (0,0,\ldots 0,0;m)$ with $m > 0$. Clearly $b << a << c$. But $b \neq (0,\ldots 0,0;0)$. Hence, (I) fails to be true in H_n^1.

By introducing the above example, we know of the existence of ℓ-groups different than G_n and therefore the existence of quasi-varieties different than H_o. It is important to first construct the ℓ-groups and then take the quasi-varieties generated by them. In order to establish a nested sequence of ℓ-groups, first we need the following definitions and notations:

Let k be a positive integer, and $H_n^k = \overset{n+k-1}{\underset{0}{\pi}} Z \overleftarrow{x}_\Theta Z$ where for $(a_0,a_1,\ldots,a_n,\ldots,a_{n+k-1};b)$, $(c_0,\ldots,c_{n+k-1};d)$ in H_n^k, $(a_0,\ldots,a_{n+k-1};b) \leq (c_0,\ldots,c_{n+k-1};d)$ iff $b < d$ or $b = d$ and $a_{n+k-1} < c_{n+k-1}$ or $b=d$, $a_{n+k-1} = c_{n+k-1}$ and $a_{n+k-2} < c_{n+k-2}$ or .... or $b=d$, $a_{n+i} = c_{n+i}$ for every $i = 0,\ldots,k-1$ and $a_i \leq c_i$ for $i = 0,\ldots,n-1$, and $(a_0,a_1,\ldots,a_{n-1}, a_n,\ldots,a_{n+k-1};b)$. $(c_0,\ldots,c_{n-1}, c_n,\ldots,c_{n+k-1};d) = (a_0 + c_{0+b},\ldots, a_{n-1} + c_{n-1+b}, a_n + c_n,\ldots,a_{n+k-1} + c_{n+k-1};b+d)$ with all subscripts less than or equal to n-1 read modulo n. Let H_k be the quasi-variety generated by H_n^k.

For every $k \in Z^+ - \{0\}$, H_n^k is an ℓ-group. To prove this we will use induction over k. If k=1 we know that H_n^1 is an ℓ-group so the claim is true for k=1. Suppose that H_n^k is an ℓ-group and let us prove that H_n^{k+1} is also an ℓ-group. Clearly H_n^{k+1} is a group. Now, the order $\leq$ defined on H_n^{k+1} is a lattice order. For this let a,b,c be elements of H_n^{k+1}. Write $a = (a_0,..,a_{n+k};p)$, $b = (b_0,...,b_{n+k};q)$, and $c = (c_0,...,c_{n+k};d)$. Consider the elements a',b',c' of H_n^k such that $a' = (a_0,...,a_{n+k-1};p)$, $b' = (b_0,...,b_{n+k-1};q)$ and $c' = (c_0,...,c_{n+k-1};d)$. Since H_n^k is an ℓ-group then $a'\wedge b'$ and $a'\vee b'$ exist. Let $x = (x_0,...,x_{n+k-1};t) = a'\vee b'$ and $y = (y_0,...,y_{n+k-1};s) = a'\wedge b'$. Now, the meet and the join of a and b exist in H_n^k. In fact

$$(a_0,a_1,...,a_{n+k};p)\wedge(b_0,...,b_{n+k};q) = \begin{cases} (a_0,...,a_{n+k};p) & \text{if } p<q \\ (b_0,...,b_{n+k},q) & \text{if } p>q \\ (a_0,...,a_{n+k};p) & \text{if } p=q \text{ and } a_{n+k} < b_{n+k} \\ (b_0,...,b_{n+k};q) & \text{if } p=q \text{ and } a_{n+k} > b_{n+k} \\ (y_0,...,y_{n+k-1},a_{n+k};p) & \text{if } p=q \text{ and } a_{n+k} = b_{n+k}. \end{cases}$$

and

$$(a_0,...,a_{n+k};p)\vee(b_0,...,b_{n+k};q) = \begin{cases} (a_0,...,a_{n+k};p) & \text{if } p>q \\ (b_0,...,b_{n+k};q) & \text{if } p<q \\ (a_0,...,a_{n+k};p) & \text{if } p=q \text{ and } a_{n+k} > b_{n+k} \\ (b_0,...,b_{n+k};q) & \text{if } p=q \text{ and } a_{n+k} < b_{n+k} \\ (x_0,...,x_{n+k-1},a_{n+k};p) & \text{if } p=q \text{ and } a_{n+k} = b_{n+k}. \end{cases}$$

Next, we need to verify only that if $a \le b$ then $ac \le bc$. Now, $a \le b$ implies that $p<q$ or $p=q$ and $a_{n+k} < b_{n+k}$ or $p=q$, $a_{n+k} = b_{n+k}$ and $a_{n+k-1} < b_{n+k-1}$ or,...,$p=q$, $q_{n+i} = b_{n+i}$ for $i=0,\ldots,k$ and $a_i \le b_i$ for $i=0,\ldots,n-1$. If $p<q$ then $p+d < q+d$ and $ac = (a_0,\ldots,a_{n+k};p)$. $(c_0,\ldots,c_{n+k};d) = (a_0 + c_{0+d},\ldots,a_{n-1} + c_{n-1+d}, a_n+c_n,\ldots,a_{n+k} + c_{n+k}; p+d) \le (b_0 + c_{+d}, \ldots,b_{n-1} + c_{n-1+d}, b_n + c_n,\ldots,b_{n+k} + c_{n+k}; q+d) = (b_0,\ldots,b_{n+k};q)$ $(c_0,\ldots,c_{n+k};d) = bc$. Otherwise, if $p=q$ and $a_{n+k} < b_{n+k}$ then $a_{n+k} + c_{n+k} < b_{n+k} + c_{n+k}$ and $ac \le bc$. At last, if $p=q$ and $a_{n+k} = b_{n+k}$, then $ac \le bc$ since $a' \le b'$ implies $a'c' \le b'c'$ in H_n^k. We can verify in a similar way that $ca \le cb$.

The family of ℓ-groups $(H_n^k | k \in Z^+-\{0\})$ forms a nested sequence of ℓ-groups. And if they are pairwise different then we have an infinite chain of quasi-varieties each of which contains H_o and is contained in $L_n \cap A^2$. Hence, it is necessary to prove that the H_n's are pairwise different. Therefore, we need to find one or more implications satisfied by each H_n and make sure that the implications will fail in H_k if $k > n$.

Lemma 4.16: If $a,b,c, d_1,\ldots,d_k$ are positive elements of H_n^k with $b << d_1 << d_2 << \ldots << d_k << a << c$ then $b = (0,\ldots,0,0;0)$.

Proof: Clearly, the lemma is true if $a = b = c = d_1 = \ldots = d_n = (0,\ldots,0;0)$.

Let $a,b,c, d_1,\ldots,d_k$ be positive elements of H_n^k. Write $a = (a_0,\ldots,a_{n+k-1};p)$, $b = (b_0,\ldots,b_{n+k-1};q)$, $c = (c_0,\ldots,c_{n+k-1};m)$ and $d_i = (x_0^i,\ldots,x_{n+k-1}^i;n_i)$. Let $s,r,t, \lambda_1,\ldots,\lambda_k$ be the maximum indices for which $a_s \ne 0$, $b_r = 0$, $c_t \ne 0$, $x_{\lambda_i}^i \ne 0$, respectively. First, if $m\ge 0$ then $p = q = n_i = 0$ for $i = 0,\ldots,k$ and $r < \lambda_1 < \lambda_2 < \lambda_3 <\ldots<\lambda_k < s < t$

since $b << d_1 << d_2 << \cdots << d_k, << a << c$. Now, if $b_r = 0$ then $b = (0,0,\cdots 0;0)$ since b is positive and r is maximal with respect to $b_r \neq 0$. Two cases arise. First if m = other we have $t \leq n+k-1$, $s \leq n+k-2$ and $\lambda_i \leq n+i-3$. Hence, $\lambda_1 \leq n-2$, and therefore $d_1 = (x_0^1,\ldots, x_{n-2}^1,0,0,\ldots,0;0)$. Since $b << d_1$ then $b_i = 0$ for every $i = 0,\ldots,n-1$ (see Proof of 4.12) and $b_r = 0$. So $b = (0,0,\ldots,0;0)$. Now, if $m > 0$ then we have $s \leq n+k-1$, and $\lambda_i \leq n+i-2$. So $\lambda_1 \leq n-1$ and $d_1 = (x_0',\ldots, x_{n-1}',0,000,0;0)$. Now, $b << d_1$ implies $b = (0,\ldots 0;0)$ (see Proof of 4.12), and the lemma is proved.

The condition in the above lemma determines a set of implications that are simultaneously satisfied by H_k. Let a,b,c, d_1, $d_2,\ldots,d_k$ be positive elements of H_k and consider the implication.

> (I^k) If $b^n < d_1 \leq d_1^n \leq \ldots . \leq d_k \leq d_k^n \leq a \leq a^n \leq c$ for all positive integers n then $b = (0,0,\ldots,0;0)$.

<u>Proposition 4.17</u>: The condition "if a,b,c, $d_1,\ldots,d_k$ are positive elements of an ℓ-group G such that $b << d_1 << \cdots << d_k << a << c$ then b is the identity of G" is satisfied iff the implication (I^k) is satisfied.

<u>Proof</u>: Clearly, (I^k) implies that the condition holds. Conversely, assume the condition is satisfied. If $b^\lambda \leq d_1 \leq d_1^\lambda \leq \cdots \leq d_k \leq d_k^\lambda \leq a \leq a^\lambda \leq c$ for all positive integers λ, then $b << d_1 << \cdots << d_k <<$

$a << c$ so b is the identity of G. Hence, the implication (I^k) is satisfied.

It is important that every H_k contains H_o and that any pair H_n^k and $H_n^{k'}$ will generate two different quasi-varieties.

Lemma 4.19: For every $K \in Z^+ - \{0\}$. H_{k+1} strictly contains H_k.

Proof: Let ϕ be the map from H_n^k into H_n^{k+1} defined by: $\phi((a_o,\ldots,a_{n+k-1};p)) = (a_0,\ldots,a_{n+k-1},0;p)$. Clearly, ϕ is a well-defined injective map. We need to show that ϕ is a homomorphism. Let a,b be elements of H_n^k. Then $a = (a_0,\ldots,a_{n+k-1};p)$ and $b = (b_0,\ldots,b_{n+k-1};q)$. We have $\phi(a) = (a_0,\ldots,a_{n+k-1},0;p)$, $\phi(b) = (b_0,\ldots,b_{n+k-1},0;q)$, and $ab = (a_0 + b_{+p},\ldots,a_{n-1} + b_{n-1+p}, a_n+b_n,\ldots,a_{n+k-1} + b_{n+k-1};p+q)$. Thus,

$$\phi(ab) = (a_0 + b_{+p},\ldots,a_{n-1} + b_{n-1+p}, a_n + b_n,\ldots,a_{n+k-1} + b_{n+k-1},0;p+q).$$

$$= (a_0,\ldots,a_{n+k-1},0;p)\ (b_0,\ldots,b_{n+k-1},0;q) = \phi(a).\ \phi(b).$$

Hence, ϕ is a homomorphism. ϕ clearly preserves lattice operations so we get $H_n^k \le H_n^{k+1}$ and $H_k \le H_{k+1}$.

In order to prove the strict inclusion it suffices to find positive elements $b, d_1,\ldots,d_k$, a and c of H_n^{k+1} such that the hypothesis in Lemma 4.16 is true for k but the conclusion of the lemma fails. Let $b, d_1,\ldots, d_k$, a,c be such that $b = (b_0,b_1,\ldots,b_{n+k};0)$, $d_i = (x_0^i,\ldots,x_{n+k}^i;0)$ for $i = 1,\ldots,k$, $a = (x_0,\ldots,x_{n+k};0)$, $c = (0,0,0,\ldots,0;m)$ where the following are true:

$b_i = 0$ for $i \neq 0$ and $b_i = 1$ otherwise,

$x_j^i = 0$ for $j \neq n-1+i$ and $x_j^i = 1$ otherwise,

$x_i = 0$ for $i \neq n+k$ and $x_i = 1$ otherwise, and $m > 0$.

Note that $b \ll d_1 \ll d_2 \ll \dots \ll d_k \ll a \ll c$ but $b \neq (0,\dots,0;0)$. Therefore, we have the following:

$$G_n = H_n^0 \lneq H_n^{\prime} \lneq \dots \lneq H_n^{k-1} \lneq \dots \lneq H_n^k \lneq H_n^{k+1} \leq \cdots, \text{ and}$$

$$H_o \lneq \dots \lneq H_{k-1} \lneq H_k \lneq H_{k+1} \leq \dots \quad .$$

Lemma 4.20: For every k, $H_k \leq L_n \cap A^2$.

Proof: For every $x,y \in H_n^k$ we have $x^n y^n = y^n x^n$.

Let $x = (x_0,\dots,x_{n-1},\dots,x_{n+k-1};p)$ and $y = (y_0,\dots,y_{n-1},\dots,y_{n+k-1};q)$. We have $x^n = (x_0 + x_{0+p} + \dots + x_{0+(n-1)p}, x_1 + x_{1+p} + \dots + x_{1+(n-1)p},\dots, x_{n-1} + x_{n-1+p} + \dots + x_{n-1+(n-1)p}, nx_n, \dots, nx_{n+k-1};np)$ and $y^n = (y_0 + y_{0+p} + \dots + y_{0+(n-1)p}, y_1 + y_{1+p} + \dots + y_{1+(n-1)p}, \dots, y_{n-1} + y_{n-1+p} + \dots + y_{n+1+(n-1)p}, ny_n,\dots,ny_{n+k-1};nq)$, where indices less than or equal to $n-1$ are read modulo n.

Now, $x^n y^n = (x_0 + x_{0+p} + \dots + x_{0+(n-1)p} + y_0 + y_{0+p} + \dots + y_{0+(n-1)p},\dots,x_{n-1} + x_{n-1+p} + \dots + x_{n-1+(n-1)p} + y_{n-1} + y_{n-1+p} + \dots + y_{n-1+(n-1)p}, nx_n + ny_n,\dots,nx_{n+k-1} + ny_{n+k-1}; np+nq)$, and $y^n x^n = (y_0 + y_{0+p} + \dots + y_{0+(n-1)p} + x_0 + x_{0+p} + \dots + x_{0+(n-1)p}, \dots, y_{n-1} + y_{n-1+p} + \dots + y_{n-1+(n-1)p} + x_{n-1} + x_{n-1+p} + \dots + x_{n-1+(n-1)p}, ny_n + nx_n, \dots, nx_{n+k-1} + ny_{n+k-1}; nq + np)$. Thus, $x^n y^n = y^n x^n$ and $H_n^k \in L_n$. Hence, $H_k \leq L_n$.

It remains to show that $H_n^k \in A^2$. For this, let $H = \{(a_0,\dots,a_{n+k-1};0) \mid a_i \in Z \text{ for } i = 0,\dots,n+k-1\}$. Clearly, H is an abelian ℓ-ideal of H_n^k, and $H_n^k/H \simeq Z$. Thus, $H, H_n^k/H \in A$. Therefore, $H_n^k \in A^2$ and $H_k \leq A^2$. Hence, $H_k \in L_n \cap A^2$.

Combining Theorem 4.11 Lemmas 4.18 and 4.19 we have the following important result:

<u>Theorem 4.20</u>: Let n be a prime number. Then there is an infinite chain of distinct quasi-varieties between A and $L_n \cap A^2$.

<u>Proof</u>: Clearly $(H_k | k \in Z^+)$ forms a countably infinite chain of different quasi-varieties of non-representable ℓ-groups between A and $L_n \cap A^2$.

Note that if for every $k \in Z^+, H_n^k \in S_n$, then $(H_k | k \in Z^+)$ forms a countably infinite chain of distinct quasi-varieties between A and S_n.

To complete this chapter we look at a special type of cyclic extensions that can be embedded in S_n for some $n \geq 2$.

<u>Example 4.22</u>: Let $\Lambda = R$ (the set of real numbers), $G = \pi_{\Lambda} Z$ and $\alpha : R \to R$ defined as follows: If k is a fixed irrational number inside the interval $(0,1), \alpha(n+x) = n + (x+k) \bmod 1$. Consider $G(\alpha)$ the cyclic extension of G by α. Then any finitely generated subgroup of $G(\alpha)$ can be embedded in S_n for some n. In fact, if H is a finitely generated subgroup of $G(\alpha)$ then it is generated by elements of the form: $(\ldots 0, x_{i_1}, 0, \ldots ; \alpha^{k_1})$ where $x_{i_1} = 1$, $\ldots$ $(\ldots, 0, \ldots, 0, x_{i_s}, 0, \ldots, \alpha^{k_s})$ where $x_{i_s} = 1$, $i_1 < i_2 < \ldots < i_s$, and $k_i \geq 0$ for $i = 1, \ldots, s$. Note that s is the number of generators.

Now, two possibilities arise. First, if $i_1, \ldots, i_s \in [n, n+1]$ we may assume that n=0 and $i_0, \ldots, i_s \in [0,1]$. Next, let $\phi : G(\alpha) \to G_{sn}$ where G_{sn} is the Scrimger ℓ-group of order sn, defined by $\phi(\ldots, 0, \ldots, 0, 1_{i_\lambda}, 0, \ldots ; \alpha^{k_\lambda}) = (0, \ldots, 1_\lambda, 0, \ldots, 0; k_\lambda s)$ for $i_\lambda \leq i_s$. Note that if

$\phi(\ldots,0,\ldots,0,\ 1_{(i_\lambda+tk)\bmod 1},\ 0,\ldots;\alpha^{k_\lambda}) = (0,\ldots,0,\ 1_{\lambda-ts},0,\ldots,0,\ldots,0;$ $k_\lambda s)$, where subscripts are read modulo sn, then ϕ is an ℓ-homomorphism. It suffices to prove that $\phi(h_1h_2) = \phi(h_1)\phi(h_2)$, where $h_1 = (\ldots,0,\ldots,0,$ $1_{i_r},0\ldots;\alpha^{k_r})$ and $h_2 = (\ldots,0,\ldots,0,\ 1_{i_t},0\ldots;\alpha^{k_t})$ with $i_1 \le i_r < i_t \le$ i_s. We have $\phi(h_1) = \phi(\ldots,0,\ldots,0,\ 1_{i_r},0\ldots;\alpha^{k_r}) = (0,\ldots,0,\ 1_r,0\ldots,0;$ $k_rs)$, and $\phi(h_2) = (0,\ldots,0,1_t,0,\ldots,0;k_ts)$. Hence, $\phi(h_1)\cdot\phi(h_2) =$ $(0,\ldots,0,\ 1_r,\ldots,\ 1_{t-k_rs},0,\ldots,0;k_rs + k_ts)$. Now, $t - k_rs \ne r + \lambda sn$. Suppose $t - k_rs = r + \lambda sn$ then $t-r = \lambda sn + k_rs = (\lambda n + k_r)s$. But $0 < \lambda n + k_r < 1$ since $0 < t - r < s$. Hence, $-\lambda n < k_r < -\lambda n+1$ and this is a contradiction since k_r is an integer. Hence, $t - k_rs \ne r \bmod(sn)$. Next, $h_1h_2 = (\ldots,0,1_{i_r},\ldots,0,\ 1_{\alpha^{k_r}(i_t)},0,\ldots;\alpha^{k_r+k_t}) = (\ldots0,1_{i_r},\ldots,$ $1(i_t+k_rk)\bmod 1,0,\ldots;\alpha^{k_r+k_t})$, and $\phi(h_1h_2) = (0,\ldots,1_r,\ldots,0,1_{t-k_rs},0\ldots,$ $0;(k_r+k_t)s)$. Thus, $\phi(h_1h_2) = \phi(h_1)\phi(h_2)$. Hence, ϕ is a homomorphism and clearly ϕ is an ℓ-monomorphism.

Secondly, if $i_1,\ldots,i_s$ belong to different intervals such that $i_\lambda \in$ $[n_\lambda,\ n_\lambda+1]$. Let c_λ be the number of generators inside the interval $[n_\lambda,n_\lambda+1]$, i and let $i_{\lambda_1},\ i_{\lambda_{c_\lambda}},\ldots i_{\lambda_{c_\lambda}} \in [n_\lambda,n_\lambda+1]$ with $i_{\lambda_1} < i_{\lambda_2} < \ldots <$ $i_{\lambda_{c_\lambda}}$ be all the generators inside $[n_\lambda,n_\lambda + 1]$. Suppose $c_1 + c_2 + \ldots +$ $c_p = s$. Define $\phi{:}G(\alpha) \to G_{c_1n} \times G_{c_2n} \times \ldots \times G_{c_pn}$ by:

$\phi(\ldots.0,\ldots,0,\ 1_{i_\lambda},\ 0,\ldots;\alpha^{k_\lambda}) = ((0;k_\lambda c_1),\ldots(\ldots0\ldots,0\ 1_k,0\ldots;k_\lambda c_\lambda),$ $\ldots,(0;k_\lambda c_p))$ where $i_\lambda = i_{\lambda_k} \varepsilon [n_\lambda,n_\lambda+1]$. Again, $\phi(h_1h_2) = \phi(h_1)\phi(h_2)$. First, if i_r, $i_t \varepsilon [n_r, n_r+1]$ then $\phi(h_1h_2) = \phi(h_1)\phi(h_2)$. Now, suppose that $i_r \varepsilon [n_r, n_r+1]$ and $i_t \varepsilon [n_t, n_t+1]$ with $i_r < i_t$. We have $\phi(h_1) = ((0;k_rc_1),\ldots,(\ldots,0,1_i,0,\ldots,0;k_rc_r),\ldots,(0;k_rc_t),\ldots(0;k_rc_p)$ where $1_{i_r} = 1_{r_i} \varepsilon [n_r, n_r+1]$, and $\phi(h_2) = ((0;k_tc_1)\ldots(0;k_tc_r)\ldots,$ $(\ldots,0,1_j,\cdot,\ldots,0;k_tc_t)\ldots(0;k_tc_p))$ where $1_{i_t} = 1_{t_j} \varepsilon [n_t,n_t+1]$. Thus, $\phi(h_1)\cdot\phi(h_2) = ((0;k_rc_1+k_tc_1),\ldots,(\ldots,0,1_i,0,\ldots,0;k_rc_r + k_tc_r),\ldots$ $(0,\ldots,1_{j-k_rc_t},0,\ldots,0;k_rc_t + k_tc_t),\ldots,(0;k_rc_p + k_tc_p))$. Now, $\phi(h_1h_2) = \phi((\ldots,0,1_{i_r},\ldots,0,1_{(i_t+k_rk)\mathrm{mod}1},0,\ldots;\alpha^{k_r+k_t}) = ((0;(k_r+k_t)c_1),$ $\ldots,(0,\ldots,0,1_i,0,\ldots,\ ;(k_r+k_t)c_r),\ldots,\ldots,\ldots,(0,\ldots,1_{j-k_rc_t},0,\ldots,0;$ $(k_r+k_t)c_t),\ldots,(0;(k_r+k_t)c_p))$ since $(i_t+k_rk)\mathrm{mod}1$ belongs to $[n_t,n_t+1]$. Hence, $\phi(h_1h_2) = \phi(h_1)\ \phi(h_2)$ and ϕ is a homomorphism. Clearly ϕ is an ℓ-monomorphism.

Finally, let m be the least common multiple of $c_1,\ldots,c_p$ then H is embedded in $G_{mn} \times G_{mn} \times \ldots \times G_{mn}$ and $H \varepsilon S_{mn}$.

CHAPTER V

THE AMALGAMATION PROPERTY

In this chapter we consider the problem of determining whether or not certain classes of ℓ-groups satisfy the amalgamation property. It is known that the class of groups and the class of lattices satisfy the amalgamation property, so it is natural to ask whether various classes of ℓ-groups have this property. For the class A of abelian ℓ-groups there are several proofs showing that the amalgamation property holds. We will discuss the relationship between this property and other important constructions of ℓ-groups and then show it fails for all the quasi-varieties of ℓ-groups constructed in the preceeding two chapters.

Definition 5.1: Let U be a class of ℓ-groups and $(G_i | i \varepsilon I)$ a family of ℓ-groups in U, then G is called the U-free product of $(G_i | i \varepsilon I)$ if the following three conditions are satisfied:

1) G belongs to U

2) $\bigcup_{i \varepsilon I} G_i$ generates G

3) Whenever α_i is a given arbitrary ℓ-homomorphism from G_i into H where H is an ℓ-group of U, then the entire collection of ℓ-homomorphisms α_i can be extended to a suitable ℓ-homomorphism from G into H: i.e., there exists a homomorphism γ from G into H such that $\gamma o j_i = \alpha_i$ for every

i, where j_i is the natural embedding of G_i into G. The U-free product G of $(G_i \mid i\varepsilon I)$ is denoted by $\overset{U}{\underset{i\varepsilon I}{\bigsqcup}} G_i$.

<u>Definition 5.2</u>: Let U be a variety of ℓ-groups and X a nonempty set. The ℓ-group F is called the <u>U-free ℓ-group</u> on X if the following are true.

(1) There exists an injection $\alpha: X \to F$ such that $\alpha(X)$ generates F.

(2) If $G \varepsilon U$ and $\beta: X \to G$ is any map, then there is an ℓ-homomorphism $\gamma: F \to G$ such that $\gamma o \alpha = \beta$. A diagram of the situation is given below:

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & F \arrow[d, dashed, "\gamma"] \\ & G \end{tikzcd}$$

Since $\alpha(X)$ generates F, we can show that γ is unique. The U-free ℓ-group always exists inside a variety of ℓ-groups.

<u>Note</u>: If x has only one element, the free abelian group on X is Z, but Z cannot be the free-abelian ℓ-group on X since Z is totally ordered.

<u>Theorem 5.3</u>: If U is a variety of ℓ-groups then the U-free ℓ-group on $\{x\}$ is $Z \boxplus Z$.

<u>Proof</u>: Let $X = \{x\}$ and define $\alpha: X \to Z \boxplus Z$ by $\alpha(x) = (1,-1)$. We first note show that $\alpha(x)$ generates $Z + Z$. If (n,m) is an element of $Z + Z$, we may write it as $(n,m) = n((1,-1)\vee(0,0)) - m((1,-1)\wedge(0,0))$. Next, let G be an element of U and $\beta: X \to G$ be a map. Define $\gamma: Z \boxplus Z \to G$ by $\gamma(n,m) = n(\beta(x)\vee 0) - m(\beta(x)\wedge 0)$. Note that $\gamma o \alpha(x) = \gamma(\alpha(x) = \gamma((1,-1))$ $= (\beta(x)\vee 0) - (-1)(\beta(x)\wedge 0) = \beta(x)^+ - (-\beta(x)\vee 0) = \beta(x)^+ - \beta(x)^- = \beta(x)$. Thus, $\gamma o \alpha = \beta$. It remains to prove that γ is an ℓ-homomorphism. Let (n_1,m_1), (n_2,m_2) be elements of $Z \boxplus Z$. Then

$$\gamma((n_1,m_1)\ \ (n_2,m_2)) = \gamma((n_1+n_2,m_1+m_2))$$

$$= (n_1+n_2)(\beta(x)v0) - (m_1+m_2)(\beta(x)\wedge 0)$$

$$= n_1(\beta(x)v0) + n_2(\beta(x)v0) - m_1(\beta(x)\wedge 0) - m_2(\beta(x)\wedge 0)$$

$$= n_1(\beta(x)v0) - m_1(\beta(x)\wedge 0) + n_2(\beta(x)v0) - m_2(\beta(x)\wedge 0)$$

$$= \gamma(n_1,m_1) + \gamma(n_2,m_2),$$

hence, γ is a homomorphism.

Now, let (n_1,m_1) and (n_2,m_2) be elements of $Z \boxplus Z$. We have

$$\gamma((n_1,m_1)v(n_2,m_2)) = \gamma((n_1vn_2,m_1vm_2))$$

$$= (n_1vn_2)(\beta(x)v0) - (m_1vm_2)(\beta(x)\wedge 0)$$

$$= (n_1(\beta(x)v0)\ v\ n_2(\beta(x)v0)) - (m_1(\beta(x)\wedge 0)\ v\ m_2(\beta(x)\wedge 0))$$

$$= [n_1(\beta(x)v0) - (m_1(\beta(x)\wedge 0)\ v\ m_2(\beta(x)\wedge 0))]\ v\ [n_2(\beta(x)v0)$$

$$- (m_1(\beta(x)\wedge 0)\ v\ m_2(\beta(x)\wedge 0))]$$

$$= [n_1(\beta(x)v0) - m_1(\beta(x)\wedge 0)]\ v\ [n_1(\beta(x)v0) - m_2(\beta(x)\wedge 0)]$$

$$v\ [n_2(\beta(x)v0) - m_1(\beta(x)\wedge 0)]\ v\ [n_2(\beta(x)v0)$$

$$- m_2(\beta(x)\wedge 0)] = \gamma(n_1,m_1)\ v\ \gamma(n_1,m_2)\ v\ \gamma(n_2,m_1)$$

$v\ \gamma(n_2,m_2) = \gamma(n_1,m_1)\ v\ \gamma(n_2,m_2)$ since $\gamma(n_1,m_2)$,

$\gamma(n_2,m_1) \le \gamma(n_1,m_1)\ v\ \gamma(n_2,m_2)$.

Similarly, $\gamma((n_1,m_1)\wedge(n_2,m_2)) = \gamma((n_1,m_1))\wedge\ \gamma((n_2,m_2))$. Thus, γ is an ℓ-homomorphism. Hence $Z\boxed{+}Z$ is the u-free ℓ-group on $\{x\}$.

<u>Definition 5.4</u>: Let P be a partially ordered group and U a variety of ℓ-groups. If F is an element of U such that

i) there exists an o-homomorphism α from P into F such that $\alpha(P)$ generates F and,

ii) for each G in U and $\beta:P \to G$ such that β is an o-homomorphism. there exists an ℓ-homomorphism γ from F into G such that $\gamma\circ\alpha = \beta$,

$$\begin{tikzcd} P \arrow[r] \arrow[dr, "\beta"'] & G \arrow[d, "\gamma"] \\ & G \end{tikzcd}$$

then F is called the <u>u-free extension</u> of P and is denoted by $F_U(P)$.

Note that free extensions do not always exist, but free abelian ℓ-groups always exist. The following establishes a relation between free extensions and free abelian ℓ-groups.

<u>Theorem 5.5</u>: Let X be a nonempty set and $P = \bigoplus_{|X|} Z$ with the trivial order. Then the free abelian ℓ-group on X is isomorphic to the abelian free extension over P.

<u>Definition 5.6</u>: A group G is <u>divisible</u> if given any $n \in Z$ and $x \in G$ there exists a $y \in G$ such that $x = y^n$. If the group is written additively, then there exists $y \in G$ such that $x = ny$.

Note that every abelian group G can be embedded in a divisible abelian group. In fact, G is isomorphic to F/K for some free abelian group F with subgroup K. But F is isomorphic to $\bigoplus Z$, so F can be embedded in $H \cong \bigoplus Q$. We can consider K as a subgroup of H so that G can be embedded in H/K. Since homomorphic images of divisible groups are divisible, factor groups of divisible groups are divisible. Therefore, H/k is divisible.

Next, we consider three properties for ℓ-group varieties because of their importance.

<u>The subalgebra property for free products</u>: If $(G_i \mid i \varepsilon I)$ and $(H_i \mid i \varepsilon I)$ are families in U with each H_i an ℓ-subgroup of G_i, then $\overset{U}{\bigsqcup}_{i \varepsilon I} H_i$ is the ℓ-subgroup of $\overset{U}{\bigsqcup}_{i \varepsilon I} G_i$ generated by $\bigcup_{i \varepsilon I} H_i$.

<u>The amalgamation property</u>: Let U be a class of ℓ-groups, then U satisfies the amalgamation property if whenever A, B_1, $B_2 \ \varepsilon\ U$ and $\alpha_1 : A \to B_1$, $\alpha_2 : A \to B_2$ are ℓ-monomorphisms, then there exists $C \ \varepsilon\ U$ and ℓ-monomorphisms $\beta_1 : B_1 \to C$ and $\beta_2 : B_2 \to C$ such that $\beta_1 o \alpha_1 = \beta_2 o \alpha_2$. The diagram below illustrates this situation.

$$\begin{tikzcd} & B_1 \arrow[dr, dashed, "\beta_2"] & \\ A \arrow[ur, "\alpha_1"] \arrow[dr, "\alpha_1"'] & & C \\ & B_2 \arrow[ur, dashed, "\beta_2"'] & \end{tikzcd}$$

<u>The divisible embedding property</u>: Every ℓ-group in U can be embedded in a divisible ℓ-group of U.

The subalgebra property is of importance for free products of ℓ-groups because of the rich embedding theory available (see for example Conrad et al. [3] and Bernau [1]). The amalgamation property is a very powerful device in establishing specific embeddings. In fact the amalgamation property implies the divisible embedding property (K. R. Pierce [15]). The amalgamation and the subalgebra properties are closely related.

It is known that the amalgamation property implies the subalgebra property (Jonsson [9, Theorem 1.3]). Further, in the presence of the following condition, it can be shown that the two are equivalent (Grätzer and Lakser [6, Theorem 4]).

<u>The congruence extension property</u>: If $G \varepsilon U$, where U is a class of ℓ-groups and H is a subalgebra of G, then every congruence on H can be extended to a congruence on G. The congruences on ℓ-groups correspond to normal, solid subgroups. Since normal subgroups of normal subgroups need not be normal the above property does not hold in general for classes of ℓ-groups. However, in the abelian case, congruence extension is easily established.

Now using the representation theorem 3.3 in W. B. Powell and C. Tsinakis [18], it can be shown directly that the subalgebra property holds for A.

Thus, in view of the preceeding discussion we get the next result which has also been proved outside the context of free-products in K. R. Pierce [15, Theorem 2.3] and [17, Theorem 1] (see also K. R. Pierce [16]).

<u>Theorem 5.7</u>: The variety A satisfies the amalgamation property and the subalgebra property.

Consider next the varieties M^+, M^-, and L_n = the variety of all ℓ-groups satisfying the law $[x^n,y^n] = 1$ where $n \varepsilon Z^+$. It has been proved that M^+, M^-, and L_n for every $n > 1$ fail the amalgamation property. We will give an argument proving that L_n fails the amalgamation property that can be found in W. B. Powell and C. Tsinakis [19 , Section 3 and 4.2].

<u>Proposition 5.8</u>: The varieties L_n do not satisfy the subalgebra property or the amalgamation property.

<u>Proof</u>: For each $n > 1$ we have $A \lneq L_n$ and so $F_{L_n}(\{x_1,x_2\}) = (Z \boxed{+} Z)^{L_n} \sqcup (Z \boxed{+} Z)$ is not abelian. Let H_1 and H_2 be the ℓ-subgroups of $F_{L_n}(\{x_1,x_2\})$ generated by $\{x_1\}$ and $\{x_2\}$, respectively. Then $H_1 \cong H_2 \cong$

$Z \boxed{+} Z$ so $\coprod\limits_{i=1,2}^{L_n} H_i$ is isomorphic to $F_{L_n}(\{x_1,x_2\})$. But the ℓ-subgroup of $F_{L_n}(\{x_1,x_2\})$ generated by $H_1 \cup H_2$ is abelian and hence not isomorphic to $\coprod\limits_{i=1,2}^{L_n} H_i$. Hence L_n fails the subalgebra property and so also the amalgamation property.

A proof of the following theorem can be found in K. L. Pierce [15, Theorem 3.1].

Theorem 5.9: The variety L fails the amalgamation property.

Up to this time A was the only known ℓ-group variety satisfying the amalgamation property. The subsequent considerations will lead to the main result of this chapter which asserts that all of the quasi-varieties $(Q_n^- | n \geq o)$, $(P_n^- | n \geq o)$, and $(H_n | n \geq o)$ constructed in the third and fourth chapters fail the amalgamation property.

We start by introducing some additional terminology. Let U be a class of ℓ-groups. By the amalgamation base, AMAL(U), we mean those ℓ-groups A of U such that, for all B_1, B_2 of U and all embeddings $\theta_1 : A \to B_1$, $\theta_2 : A \to B_2$, there exists C in U and embeddings $\mu_1 : B_1 \to C$, $\mu_2 : B_2 \to C$ such that $\mu_1 \theta_1 = \mu_2 \theta_2$. It is clear that U has the amalgamation property if and only if AMAL(U) = U. The next result is implicit in K. R. Pierce [15], (see Theorem 5.1).

Proposition 5.10: If U is an ℓ-group quasi-variety and $Z \in$ AMAL(U), then U has the divisible embedding property. In particular, if U satisfies the amalgamation property, then it also satisfies the divisible embedding property.

Let us remark that proposition 5.10 yields an alternative proof of the fact that the varieties L_n, $n>1$, do not satisfy the amalgamation property. Indeed, $A \leq L_n$ if $n>1$ (Weinberg [23]), and clearly a divisible

ℓ-group in L_n is abelian. It follows that L_n does not satisfy the divisible embedding property and hence the amalgamation property.

Let us describe briefly Reilly's construction of quasi-representable varieties. Consider a countable infinite set $X' = X \cup \{z\}$, $z \notin X$, and let F be the free group on X. For each $w \in F$, $\ell(w)$ will denote the ℓ-group law $z^+ \wedge (w^{-1}z^- w)$. For any non-empty subclass U of L define $F(U) = \{w \in F \mid G \models \ell(w)$ for every $G \in U\}$. Also for any non-empty subset W of F define $\mathcal{Q}(w) = \{G \in L \mid G \models \ell(w)$ for every $w \in W\}$. It is immediate that if W is a non-empty subset of F, then $\mathcal{Q}(W)$ is an ℓ-group variety, and less trivially that if U is a non-empty subclass of L, then $F(U)$ is a fully invariant subgroup of F (Reilly [20, Theorem 2.1]). Following, Reilly we shall call these varieties quasi-representable.

In regard to the position of these varieties in the lattice of ℓ-group varieties, we note that $\mathcal{Q}(F) = \mathcal{R}$ is the least quasi-representable variety, and hence each of these varieties contains $\mathcal{R}$.

For each positive integer n, let B_n denote the fully invariant subgroup of F defining the Burnside variety of exponent n. The quasi-representable varieties $\mathcal{Q}(B_n)$ will play a key role in our considerations. Note that G belongs to $\mathcal{Q}(B_n)$ if and only if for all elements a,b of G, $a^+ (b^{-n}a^- b^n) = 0$.

<u>Theorem 5.11</u>: Let U be an ℓ-group quasi-variety such that $U \nleq \mathcal{R}$, and $U \leq \mathcal{Q}(B_n)$ for some positive integer n. Then U does not have the divisible embedding property and hence in particular, the amalgamation property.

<u>Proof</u>: Suppose, G is a divisible ℓ-group in $U \leq \mathcal{Q}(B_n)$, and let a,b be elements of G. There exists $c \in G$ such that $b = c^n$. Hence, $a^+ \wedge (b^{-1}a^- b) = a^+ \wedge (c^{-n}a^- c^n) = 0$. But this means that G belongs to $\mathcal{R}$

(see for example Conrad [3 , Theorem 1.8]). Thus since $U \nleq \mathcal{R}$, U cannot have the divisible embedding property. That U fails the amalgamation property follows from proposition 5.10.

Note that each H_k contains H_o and $H_o \nleq \mathcal{R}$, but $H_o \leq L_n \leq \mathcal{Q}(B_n)$. Hence, according to the above H_o fails the divisible embedding property, and so also the amalgamation. Thus, every H_k fails the amalgamation property, since $H_o \leq H_k \leq L_n$ for every k.

Therefore the following theorem holds.

Theorem 5.12: For every k, the quasi-variety H_k constructed in Chapter IV fails the amalgamation property.

Proof: Clearly, $H_k \nleq \mathcal{R}$ and $H_k \leq L_n \leq \mathcal{Q}(B_n)$. So by 5.11 H_k fails the amalgamation property.

Next, we need to show that the two chains of distinct quasi-varieties constructed in the third chapter fail the amalgamation property. But, first we need the following lemmas:

Lemma 5.13: There exists a totally ordered set I with distinct o-permutations α,β,γ such that $\alpha = \beta^n = \gamma^n$.

Proof: Let G be any ℓ-group which is not an R-group (see 3.6). Then there exist x, y, z ε G and $n > 1$ such that $x = y^n = z^n$ and $x \neq y \neq z$. By Holland's theorem G can be embedded in the ℓ-group A(I) of o-permutations of some totally ordered set I. Let α,β,γ in A(I) be the elements corresponding to x,y,z. Then $\alpha = \beta^n = \gamma^n$ and α,β,γ are distinct.

Lemma 5.14: There exists a totally ordered set I with α,β,γ as in 5.13 where $\alpha(i) > i$ and $\beta(i) > i$ for each $i \varepsilon I$.

Suppose $\alpha(i) \leq i$ for all $i \varepsilon I$ then replace α with α^{-1} so that $\alpha(i) > i$ for every $i \varepsilon I$. We know that $\alpha(i) \neq i$ for all i, for if not α would be the identity and thus so would β and γ. Now let $I' = \{i \varepsilon I | \alpha(i) > i\}$. So $I' \neq \phi$. If $\beta(i) \leq i$, for some $i \varepsilon I'$, then $\beta^2(i) \leq$

$\beta(i) < i$. Thus, $\beta^n(i) \le i$. But $\alpha(i) = \beta^n(i)$, so $\beta^n(i) > i$. Thus $\beta(i) > i$ for all $i \in I'$.

To complete this chapter a final important result will prove the existence of algebraic structures closer to A than the Medvedev varieties M^+, M^- which fail the amalgamation property.

Theorem 5.15: For every $n \ge 0$, Q_n^- and P_n^- constructed in the third chapter fail the amalgamation property.

Proof: The techniques used prove that Q_o^- (see Chapter III) fails the amalgamation property will also establish the same result for Q_n^- since $Q_o^- \le Q_n^- \le R$. The proof is similar for P_n^-.

Let Λ be a totally ordered set such that it admits three distinct o-automorphisms $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ such that $\bar{\alpha} = \bar{\beta}^n = \bar{\gamma}^n$ for some $n > 0$, such that $\bar{\alpha}(i) > i$ for every $i \in \Lambda$. By lemmas 5.12 and 5.13 we know of the existence of such Λ, $\bar{\alpha}$, $\bar{\beta}$, and $\bar{\gamma}$.

Now, let $G = \bigoplus_{\Lambda} Z$ ordered by: $a \ge 0$ iff $a_i \ge 0$ where i is the maximum index with respect to $a_i \ne 0$. Also, let α, β, γ be o-automorphisms of G defined by: $\alpha(a) = b$ where $b_i = a_{\bar{\alpha}(i)}$, $\beta(a) = c$ where $c_i = a_{\bar{\beta}(i)}$ and, $\gamma(a) = d$ where $d_i = a_{\bar{\gamma}(i)}$. Note that $\alpha = \beta^n = \gamma^n$ since $\bar{\alpha} = \bar{\beta}^n = \bar{\gamma}^n$. Consider $G(\alpha)$, $G(\beta)$ and $G(\gamma)$ that are generated respectively by:

$$m = (\bar{0},\alpha) \text{ and } n_i = (\bar{a},\alpha^0) \text{ where } \bar{a}_j = \begin{cases} 1 & \text{if } j=i \\ 0 & \text{if } j\ne i \end{cases},$$

$$r = (\bar{0},\beta) \text{ and } s_i = (\bar{b},\beta^0) \text{ where } \bar{b}_j = \begin{cases} 1 & \text{if } j=i \\ 0 & \text{if } j\ne i \end{cases},$$

and

$$t = (\bar{0},\gamma) \text{ and } u_i = (\bar{c},\gamma^0) \text{ where } \bar{c}_j = \begin{cases} 1 & \text{if } j=i \\ 0 & \text{if } j\neq i \end{cases} .$$

Now, let G_1, G_2, G_3 be the ℓ-subgroups of $G(\alpha)$, $G(\beta)$ and $G(\gamma)$, generated respectively by: m and n_i, s and t_i, t and u_i for a fixed $i \in \Lambda$. Note that $G_1, G_2, G_3 \in Q_o^-$ since each of these is generated by two elements x and y, such that $y << y^x << x$ and the set $\{y^{x^n} \mid n \text{ in } Z\}$ is independent. Next, we will prove that if $J_1:G_1 \to G_2$ and $J_2:G_1 \to G_3$ are the natural embeddings, then there is no $H \in Q_o^-$, such that there exists $\sigma_1:G_2 \to H$ and $\sigma_2:G_2 \to H$ with $\sigma_1 J_1 = \sigma_2 J_2$. Note that $Q_o^- \leq \mathcal{R}$ and every ℓ-group in Q_o^- is an R-group (i.e. if $x^n = y^n$ then $x = y$). Assume there exists such an H and that the amalgamation property holds. Then, there exists $\sigma_1:G_2 \to H$ and $\sigma_2:G_3 \to H$ such that $\sigma_1 J_1 = \sigma_2 J_2$. So $(\sigma_1(0,\beta))^n = \sigma_1((0,\beta)^n =$ $\sigma_1((0,\beta^n)) = \sigma_1 J_1(0,\alpha) = \sigma_2 J_2(0,\alpha) = \sigma_2((0,\gamma^n)) = (\sigma_2(0,\gamma))^n$. Thus, $\sigma_1(0,\beta) = \sigma_2(0,\gamma)$ since H is an R-group. Note that for every $g \in G$ we have $(0,\beta)(g,\beta^0)(0,\beta^{-1}) = (\beta(g),\beta)(0,\beta^{-1}) = (\beta(g),\beta^0)$. Similarly, $(0,\gamma)(g,\gamma^0)(0,\gamma^{-1}) = (\gamma(g),\gamma 0)$. Thus $\alpha_1 J_1(\beta(g),\alpha 0) = \alpha_1(\beta(g),\beta^0) =$ $\sigma_1((0,\beta)(g,\beta^0)(0,\beta^{-1})) = \sigma_2((0,\gamma)(g,\gamma^0)(0,\gamma^{-1})) = \sigma_2(\gamma(g),\gamma^0) =$ $\sigma_2 J_2(\gamma(g),\gamma^0) = \sigma_1 J_1(\beta(g),\alpha^0)$. So $\sigma_1 J_1(\gamma(g),\gamma^0) = \sigma_1 J_1(\beta(g),\beta^0)$. Then $\beta(g) = \gamma(g) =$ since $\sigma_1 J_1$ is a monomorphism. So $\beta = \gamma$ since g was chosen arbitrarily and we have a contradiction to β and γ being distinct. So the amalgamation property fails in Q_o^-. Hence, it fails in P_n^- and Q_n^- since all ℓ-groups in P_n^- or Q_n^- are R-groups.

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2

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