FAIRCHILD, Jack Elmer, 1928–
A UNIFIED METHOD OF AEROSPACE VEHICLE DYNAMIC ANALYSIS BASED ON TENSOR CONCEPTS.

The University of Oklahoma, Ph.D., 1964
Engineering, aeronautical

University Microfilms, Inc., Ann Arbor, Michigan

THE UNIVERSITY OF OKLAHOMA

GRADUATE COLLEGE

A UNIFIED METHOD OF AEROSPACE VEHICLE DYNAMIC ANALYSIS BASED ON TENSOR CONCEPTS

A DISSERTATION

SUBMITTED TO THE GRADUATE FACULTY

in partial fulfillment of the requirements for the

degree of

DOCTOR OF PHILOSOPHY

BY

JACK ELMER FAIRCHILD

Norman, Oklahoma

1964

A UNIFIED METHOD OF AEROSPACE VEHICLE DYNAMIC ANALYSIS

BASED ON TENSOR CONCEPTS

APPROVED BY

[illegible]

L. A. Comp

George M. Ewing

C. W. Bert

J. D. Palmer

DISSERTATION COMMITTEE

ACKNOWLEDGMENT

The writer would like to express his sincere appreciation to his principal advisor, Dr. H. W. Bergmann, for the cooperation, encouragement, and assistance given during the preparation of this work.

The thoughtful help of Dr. C. E. Springer with the tensor concepts employed is gratiously acknowledged.

Special praise is due to Mrs. Irma R. Tomberlin, Information Librarian, for her efficient assistance in obtaining microfilms of the many references not available in our Library, and to Mrs. Louise M. Sizemore, Engineering Librarian, for her ceaseless help in locating the references used in the preparation of this work.

The writer would also like to thankfully acknowledge the generous financial support provided by two National Science Foundation Science-Faculty Fellowships and a Ford Foundation Grant, without which this work would have been impossible.

Finally, thanks are due to Mrs. Gail Blancett for her expert and efficient typing of the manuscript.

Jack Elmer Fairchild

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

A UNIFIED METHOD OF AEROSPACE VEHICLE DYNAMIC ANALYSIS
BASED ON TENSOR CONCEPTS

CHAPTER I

SURVEY OF DEVELOPMENTS IN FLIGHT DYNAMICS AND RELATED DISCIPLINES

Developments in Flight Dynamics

Flight dynamics shall be construed to include the areas variously identified by the titles "flight mechanics," "space or orbital mechanics," and "stability, guidance, and control." Although the field of flight dynamics was really born after the first successful flight by the Wright brothers in 1903, it would not be fair to neglect the important contributions in mechanics which form the theoretical basis of this modern applied field. For this reason, important developments in the field of flight dynamics may conveniently be divided into two periods--before and after the Wright brothers' first successful flight.

The earlier period furnished virtually all of the theoretical bases of early as well as current methods. Although the classical origin of dynamics as we know it today began with Isaac Newton in the mid-seventeenth century, justice demands that the ideas of Euclid in the third century (B.C.), Copernicus and Kepler in the sixteenth century, and of Galileo and Descartes in the early seventeenth century be mentioned as a necessary prelude to Newton's work. Next in the development chain came the great mathematicians Leibniz, the Bernoullis, Euler, Laplace and others to systematize and formalize the ideas of Newton by mathematics. The generalization of the laws of motion was accomplished by Lagrange in 1778 and later by Hamilton in the first half of the nineteenth century. Routh (1), working in the latter half of the nineteenth century, developed his important criterion for stability. In the last quarter of the nineteenth century and the first few years of the twentieth century, books by Rayleigh (2), Thomson (Lord Kelvin) and Tait (3), Whittaker (5), and Webster (6) can be found to contain all the essentials necessary for the study of the dynamics of an airplane. In addition, during this period, Heaviside developed his operational calculus which would find heavy application later in flight dynamics. Ricci (4) developed the tensor calculus which forms the basis for the mathematics of this dissertation. Liapounov and Poincaré made their important contributions to nonlinear system stability theory in this period also.

The period following the Wright brothers' first flight is characterized by a gradual refinement and sophistication of treatment of the subject as required by the advancement of flight vehicles into new regimes of flight. Bryan and Williams (7) analyzed the longitudinal stability of aerial gliders in 1904, showing that the motion of airplanes could be analyzed purely by mathematics. Lanchester (8) published the first important analysis of airplane motion in 1908 based more on physical considerations, in which he described the "phugoid" motion. Bryan (9) published the first comprehensive book on airplane stability in 1911. Bryan was the first to apply Routh's stability criterion to airplanes. Bairstow's (10) book along with Williams' (11) book were important general treatises in the 1920's and 1930's until Jones (21) published his comprehensive work in 1934 based on the work of Lanchester and Bryan. Jones justified and utilized non-dimensional aerodynamic coefficients and equations throughout and included detailed solutions of typical problems.

All of the aforementioned references used scalar differential equations for the analysis of the motion. Matrix methods were used extensively by Frazier, Duncan, and Collar (23) in their book published first in 1938. Although this book was largely concerned with the dynamics of airplane structures, the rigid body equations of motion of an airplane were developed with matrix procedures. Use was made of Lagrangean equations in the formulation of the equations of motion of the complex systems considered. Unfortunately, acceptance and use of this important work in America occurred only recently. Perkins and Hage (33) published a book containing an excellent treatment of static and dynamic stability in 1949 which soon became the best single reference book for stability and control work in schools and industry. The treatment was classical, using scalar differential equations to describe the motion and Routh's criterion for stability determinations. Extensive use was made of NACA data to provide information useful for preliminary design estimates. However, this book failed to show, or to properly reference, a derivation of the Eulerian equations of motion and to point out the assumptions made in deriving these equations.

Goldstein (34) published his book on classical mechanics in 1950 in which vectors, matrices, dyadics, and to a limited extent, tensors, were used. The value of this book was not recognized in the airplane dynamics field until about 1956, when Abzug (59) applied the methods for handling the coordinate transformations in the equations of airplane motion.

Duncan (38) published his book on stability and control in 1952, in which transform techniques were developed extensively. The concepts of "admittances" and "impulsive admittances" were introduced to describe airplane responses.

In the same year, Ashkenas, Bates, and McRuer (45) completed a classified Navy Bureau of Aeronautics Report containing a thorough derivation of the airplane equations of motion, a discussion of all the important stability derivatives, and an extensive development of the transfer function concept, transform methods of solution, and phase-plane analysis methods. Although its security classification restricted its use to agencies involved in military contracts, it rapidly became a widely accepted reference for stability and control analyses. The adoption of transform methods by aeronautical engineers was slow because of their unfamiliarity with transform techniques. However, the necessity for electronic stabilization devices to artificially stabilize and control an airplane in the early 1950's forced aeronautical engineers to become familiar with these powerful methods. The approach in the BuAer Report was classical although some use was made of vectors in the derivation of the equations of motion. The "black box" method of stability analysis presented in this book is still the basic method employed today in airplane systems analyses. The book was recently declassified and published privately and is now more widely available.

In the early 1950's, a sudden increase in publications on various new methods of analysis revealed that the complexities of modern flight vehicles exceeded the capabilities of the standard methods. An early interest in "time vector methods" of dynamic stability analysis was short-lived because of the clear superiority of electronic analog computers which emerged at about the same time. Doetsch (63), Breuhaus (46), and Sternfield (50), published papers in this area.

Chang (44) developed the equations of motion of a rigid missile by vector methods in 1952. Bolz (42) and Charters (53) used vector notation in developing the equations of motion for their studies in 1952 and 1955, respectively. Young (68) utilized vector analysis to obtain the kinematical relations necessary for his generalized missile dynamics analysis in 1958.

The application of vector methods for stability and control analyses was firmly established by 1959, as evidenced by the publication of Etkin's (73) book. This comprehensive book utilizing modern mathematical methods quickly replaced Perkins and Hage's work as the standard textbook and reference book on stability and control. Etkin improved upon the BuAer Report in that he used vector analysis completely in deriving the equations of motion, and the Lagrangean equations to derive the control mode relations. This book also utilized transform techniques for the solution of the equations of motion, as well as phase-plane plots to describe the nature of the motion.

Vectors and limited matrix notation was used by

Roberson (61, 62, 66, 67, 74) in 1957-1959 in his description of the perturbed attitude motions of satellites. A group of Boeing engineers (81) made extensive use of vector analysis and Lagrangean methods in their publication in 1960. The work by Vedrov, Romanov, and Surina (78), Suddath (79), Klein (75), and Hoelker (87) between 1959 and 1961 indicates, however, that the use of vector analysis or matrix methods was not universal. Their papers all utilized classical scalar differential equations with geometrical resolution methods.

Miele's (90) work, although characterized by extensive use of modern mathematical methods, including vectors, does not use matrices or tensors.

Donegan and Pearson (39) in 1952, after developing the required equations by conventional methods, recast them into a matrix form to facilitate their solution by a computer method reported in NACA Report 1000. A similar approach was made by Donegan (49) in 1954. Apparently the employment of matrix notation to facilitate the computer solution of linear equations preceded their use in the formulation of these equations. Shultz's (51) paper in 1954 made extensive use of matrix methods. In 1955, Byrum and Grady (58) used matrix, vector, and limited tensor notation in their derivation of the general equations of motion of a rigid body of constant mass, moving over a flat earth. Lagrange's equations were utilized to generate these equations. Abzug (59) pointed out in his paper the advantages of matrix notation for the kinematics of airplane motion in 1956. In 1957, Doolin ([illegible]) showed how matrix methods can be applied to various coordinate transformations occurring in airplane motion studies. An example of the reluctance to apply matrix methods in stability analysis can be seen in Abramson's (65) book published in 1958, where no use was made of matrix methods at all in the chapter on airplane stability, although matrices are extensively used throughout the remainder of the book which dealt with structural dynamics.

In 1961, the recognition of the value of matrix notation culminated in the publication of Kolk's (83) book, which made extensive use of matrix methods in the analysis of airplane stability. It is curious, however, that Kolk still relied on an unwieldy trigonometric resolution method to evolve the Eulerian rotation angle relations. Rothman and Pittell (88) presented a general matrix equation which related the various coordinate axis systems. In 1961 also, Buglia, Young, Timmons, and Brinkworth (86) utilized the matrix methods presented by Abzug in their study of a spinning vehicle with variable mass and inertia properties. Cannon (91) utilized vector and matrix methods in developing basic response relations for reaction wheel attitude control in his paper published in 1962. Knox (92) suggested corrections to Rothman and Pittell's work. Matrix methods of analysis are employed exclusively by Swaim (94) in his general theory of dynamic stability of a flexible-bodied missile presented in 1962. Also in 1962, Margulus and Goodman (93) used tensor and matrix methods in deriving the dynamical equations for the attitude matrix of an orbiting satellite.

Developments in the Tensor Calculus

Tensor calculus, or the absolute differential calculus, as it was called initially, began in 1888 with Ricci's first publication on the subject. The theory was extended and applied by Ricci and Levi-Civita (4) in 1901. The tensor concept was and still is rather slow in gaining acceptance. Einstein, however, recognized its value immediately for his theory of relativity and this remained the principal area of application until the 1930's. Other important contributions were made in the 1920's by Eisenhart (13, 15), Appell (14), and Veblen (16, 18). In 1931, McConnell (17) made many applications of tensor analysis in the dynamics of particles and rigid bodies, electricity and magnetism, the mechanics of continuous media, as well as the special theory of relativity. In a book published in 1943, Craig (28) showed similar applications in classical dynamics, and special and general relativity; he also introduced extensors, which Kondo (52) found useful in describing certain aerodynamic coefficients in his work.

The first book on tensor applications specifically for the aeronautical field appeared in 1947 by Michal (30), in which applications were made in elasticity, hydrodynamics, classical mechanics, and boundary-layer theory. Unfortunately, the treatment of classical mechanics was quite brief, consisting only of a tensor presentation of Lagrange's equations. Synge and Schild (32) presented applications to classical dynamics, hydrodynamics, elasticity, and electromagnetic radiation in a book published in 1949. Goldstein's (34) book on classical mechanics used tensors in the development of the rigid body equations of motion and in the discussion of special relativity. Sokolnikoff's (36) book in 1951 presented applications in analytical mechanics, relativistic mechanics, and the mechanics of continuous media. His discussion of analytical mechanics was quite extensive, including the Lagrangean equations in generalized coordinates, Hamilton's canonical equations, Newton's law of gravitation, the principle of least action, Gauss's theorem, and the two body problem.

The Japanese became interested in unifying methods in the engineering sciences in the mid-50's. The results of government sponsored studies of the systematic use of differential geometry, algebraic topology, matrix and tensor theory in all domains of engineering science were published in 1955 which included Professor Kondo's (52) work.

Stigant (72) published a book in 1959 in which he describes the use of tensors in electrical engineering problems, utilizing Kron's techniques.

The most recent books on tensors are the ones by Thomas (85) and Springer (89), published in 1961 and 1962, respectively. Both books are introductions to tensor analysis and differential geometry. However, Springer's book is somewhat more extensive since it includes not only vector analysis but also a chapter on geodesics and union curves.

Developments of Kron's Method

First of all, it must be recognized that Gabriel Kron is a pioneer and innovator who has been the subject of abuse, as well as praise, in a manner somewhat reminiscent of Oliver Heaviside and his development of operational calculus. Kron (20, 25, 27, 57) originated a novel method of analyzing complicated electrical systems based on tensor concepts which, as he has demonstrated, can solve difficult electrical problems. Yet, as an electrical engineer, he has admitted his inability to prove the validity of his method with a rigor satisfactory to mathematicians.

Kron is an enthusiastic and prolific writer of articles on his method, having published over seventy papers from 1932 to the present time, covering the use of his method in many different applications in many diverse fields. A complete bibliography of his published papers can be found on page xiii of Reference 27 and need not be reproduced here. Stigant (31, 60) has reviewed all of Kron's work in a series of papers beginning in 1948. Excluding purely electrical applications, Kron has applied his method to the following areas:

1. Elastic and plastic structural dynamic analysis

2. Solution of ordinary and partial differential equations and time varying equations

3. Problems involving Schrodinger's equation and Poisson's equation

4. Compressible and incompressible fluid flow

5. Control systems

6. Problems involving the electromagnetic field equations

7. Numerical analysis of large systems by "tearing" and "interconnecting "

It is probable that the popular use of electrical analogies for the solution of problems in different fields has hastened the application of his method to so many diverse fields. Kron promoted and defended his method virtually alone until 1944 when Banesh Hoffman (29) of Queens College (Mathematics Department) published a paper in the Quarterly of Applied Mathematics in which he called Kron's method a "method of subspaces" and defended its validity. He showed that if Kron's method were invalid, so also are Lagrange's equations, since the proof of Kron's method is a direct corollary of the proof of Lagrange's equations and the theory of subspaces.

In 1950, Le Corbeiller (35) of Harvard University published a book describing Kron's method of analysis of stationary electric networks. In the preface, Le Corbeiller opined that Kron's work "is the most significant advance in electrical engineering analysis since the introduction of impedances by Kennelly and Steinmetz and the two-reaction method by A. Blondel."

J. L. Synge (37, 48) stimulated by his review of Le Corbeiller's book, felt that the significance of Kron's method could be clarified by appealing to the topological nature of the work. Synge presented an "intuitive proof" of the method which he readily admitted is not acceptable to a mathematician, but will "carry conviction" to engineers and physicists.

In a paper published in 1952, Le Corbeiller and Yeung (40), while pointing out the existence and role of duality in the analysis of mechanical as well as electrical systems, suggested that the research done by Kron on electrical networks could successfully be carried over into a broad class of mechanical systems by recognizing their duality properties.

Langefors (43, 47) published papers in 1952 and 1953 describing his analysis of elastic structures by matrix transformation by using a modification of Kron's method for electrical networks.

In 1954, Professor K. Kondo (52), University of Tokyo, delivered a lecture at the Memorial Meeting for the Pioneers of Aeronautical Science and Engineering in Japan in which he showed that if the dynamics of the disturbed motion of an airplane are derived by the tensor theory of modern differential geometry, the resulting system of equations are similar to Kron's equations of hunting of rotating electrical machinery. He also pointed out that the path which the airplane follows can be defined in terms of non-Riemannian geometry. He suggests that his work is the aerodynamic counterpart of Kron's non-Riemannian dynamics of rotating machinery. Kondo's paper utilized formal tensor methods throughout and ended with a comparison of the resulting tensor equations with the conventional equations developed in Jones' (21) work.

Kron's method was apparently placed on a rigorous mathematical base in 1955 when J. P. Roth (54, 55) of the University of California at Berkley published in a pair of papers proofs of the existence of a solution of the network problem and the validity of Kron's method of tearing based on algebraic topology. About this same time, Kron's (57) interest shifted from applying his method to non-electrical systems to extending it to the tearing of topological models.

Trent (41, 56) recognized that the formulation of two sets of system equations for any lumped system by formal matrix methods as first presented by Kron could also be developed from the properties of linear graphs. His presentation relied heavily on algebraic topology theory, and utilized matrices for operational developments.

In 1958, Professor Higgins (69) devoted a sub-part of his survey article to Kron's method of tearing and its use in multiplying grid-network efficacy. His papers contain a comprehensive bibliography on Kron's method. J. H. Argyris (71) suggested in a survey article in 1958 on the analysis of complex elastic structures that Kron's method of tearing is directly related to his method of "statically redundant basic systems." He opined, concerning Langefors' work, that it would appeal to applied mathematicians who are less interested in a geometrico-physical approach.

In 1959, J. P. Roth (76), now at the Institute for Advanced Study, Princeton, demonstrated in a paper in the Quarterly of Applied Mathematics the advantages of Kron's method of tearing in numerical analysis problems. Solutions of several network problems were obtained by various methods and compared with respect to the number of multiplications required for solution. These examples indicated the clear superiority of Kron's method over K-partitioning, standard partitioning, and standard inversion techniques. In 1958, Kron became interested in applying his method to the study of multidimensional space filters.

In 1960, Weinzweig (80) presented a precise mathematical formulation of Kron's procedure utilizing topological methods, simplified and extended its validity, and showed a dual method which he called the "method of identification."

Koenig and Blackwell (84) published a book in 1961 on electromechanical system analysis which extended and clarified Trent's work. Although the book is applicable to all types of physical systems, the author's electrical background is clearly in evidence. No reference to Kron's work is made in this book, although his influence is obvious, though perhaps indirect (by way of Trent).

The most recent publication of Kron's methods is a book by Bewley (82) of Lehigh University, published in 1961, which presents the essential principles, with applications, of the matrix-tensor methods of analysis of electric circuits and machines introduced by Kron. A recent paper on matrix analysis of substructures by Przemienieki (95) in January, 1963, utilizes a method very similar in concept, at least, to Kron's, although no reference to Kron's work or use of tensors is made.

Conclusions

The foregoing resumés of developments in flight dynamics, tensor analysis, and Kron's work imply to this writer the following:

(1) There is a need for a unified approach to the analysis of vehicle dynamic stability capable of handling within a single framework the motion of an elastic flight vehicle, controlled by electronic stabilization systems, and subjected to forces due to gravity, aerodynamics, radiation pressure, magnetic fields, solar winds, and propulsive-system thrusts. Such a unified approach should possess a "growth potential" for including new effects as they are found to be important.

(2) Tensor concepts seem to be ideally suited for the formulation of the problem mentioned above. In addition, since the presently used vector and matrix methods are but restricted cases of a tensor approach, it is probable that the generalized nature of tensor concepts will allow the easy incorporation of new phenomena into the analysis where they become significant in a particular flight regime (e.g. relativistic effects).

(3) Kron's method, or suitable modifications thereof, seem to offer promise as a technique of exploiting tensor concepts in conjunction with modern computing machines. The method will allow the analysis of a complex system in terms of simpler sub-systems, and leave to computing machines the tedious numerical work of their interrelation. This analysis by simple sub-systems will better

enable the analyst to introduce his physical understanding into a problem, an ability that is virtually lost when a complex problem is formulated all at once.

(4) These tensor methods make it possible to relate the essential features of diverse fields of engineering. This is particularly valuable today where aerospace systems demand the blending of many diverse fields.

CHAPTER II

EXPOSITION OF THE METHOD OF ANALYSIS

Introduction

First of all, it should be understood that the purpose of this dissertation is to reorganize the methods of analyzing aerospace vehicle dynamics utilizing newer mathematical concepts. No new solutions to formerly unsolved problems are claimed. The new analysis procedure is hoped to result in a clearer recognition of the fundamental nature of the problem and thus to aid in its understanding. Max Planck (12) said in 1921 that the highest aim of the physicist "will always be the correlating of various physical observations into a unified system, and, where possible, into a single formula." It is hoped that this dissertation will exhibit this spirit in the engineering field by presenting a general analysis method which is equally applicable to all types of physical systems.

As was indicated in Chapter I, the present technique used for the analysis of vehicle motion is essentially the same as that presented by Routh and Whittaker by the turn of the 19th Century. The only significant changes have been the introduction of vector and matrix methods to facilitate the solution of a complex problem. Although none of the current textbooks or standard reference books utilize tensors, some application of tensors is beginning to appear in the technical literature. The probable reason is that in many instances the communication between mathematicians and engineers leaves much to be desired. Thus, a tendency to continue the use of familiar methods exists in both groups.

From the perspective of the history of science and a consideration of the world as it actually exists, this unfortunate situation must not continue if we are to stay abreast of the scientific and technological revolution now taking place. The profoundness of new ideas is gauged largely by the impact they make upon the world, which can only occur when those ideas serve a utilitarian purpose. Discovery for its own sake is edifying to the individual, but inconsequential to mankind. Thus, a serious effort must be made by both mathematicians and engineers to communicate more effectively.

It is the writer's conclusion after a comprehensive literature search that aspects of the following subject areas could be combined to produce a new approach for the study of aerospace vehicle dynamics:

(1) Tensor analysis

(2) Affine geometry of n dimensions

(3) Topology

(4) Classical mechanics

(5) Aerodynamics

(6) Non-stationary electrical network theory

Although it is true that an analysis could be carried forward successfully by remaining entirely within a particular discipline, it seems to this writer that each discipline has something to offer which the others do not. A better understanding will certainly result if more view-points are considered. This dissertation is therefore an attempt to assemble appropriate elements from these diverse fields to present a more comprehensive and powerful method of analysis which is capable of extension as future needs arise.

The tensor analysis furnishes the basic notation and mathematical analysis technique. The nature of the system can be described geometrically in terms of an affine geometry of n dimensions. Topology provides a clearer picture of the formal relations which exist between the various different analytical approaches. Of course, classical mechanics and aerodynamics furnish the fundamental physical concepts necessary for the analysis. Finally, since at least a partial analogy exists between vehicle dynamic analysis and non-stationary electrical-network analysis, the fruit of investigations in this area can be utilized to advantage. Particularly applicable are the tensor concepts of rotating electrical machinery by Kron and the topological outlook of Firestone, Trent, and of Koenig and Blackwell in systems analysis.

Definition of the General Problem

The generalized vehicle to be analyzed is a three-dimensional unsymmetrical rigid body whose mass center position can be defined in space by three coordinates, and whose orientation in space can be defined by three coordinates. Three additional coordinates may be used to represent the control mode, the changes of which result, in general, in forces (or torques) in all nine coordinate directions. The "state" of the system is defined in relation to a parameter, time, or non-dimensional time parameter. The vehicle is subject to a gravitational force field, aerodynamically induced potential and dissipative forces, and to either holonomic or non-holonomic constraints, depending on the particular case considered. It is anticipated that the method of analysis can be applied with equal success to space vehicles and atmospheric flight vehicles. The method could be extended to include the influence of magnetic, electrostatic, or radiative fields on space vehicle motion, although this is not considered here.

Geometrical Considerations

From the axiomatic point of view (see Veblen [18]), affine geometry is obtained by associating a space of n dimensions with the affine transformation group, which includes all linear transformations. A space is defined as that set of objects which are in a one-to-one reciprocal correspondence with the totality of ordered sets of n real or n complex numbers (which are called "points" here). Other types of geometry are obtained by associating the space with other transformation groups. Thus, metric geometry is obtained by an association with special linear transformations; projective geometry, with fractional linear transformations; and topology, with all continuous reciprocal one-to-one transformations. In terms of generality, affine geometry therefore belongs between metric and projective geometry. A general affine connection may be broken into two components--a "symmetric" and a "skew-symmetric" component. Cartan referred to these two components as "without torsion" and "with torsion," respectively. The law of transformation of the general affine connection is

$$\Gamma^{n}_{lm} = \Gamma^{k}_{ij} \frac{\partial x^i}{\partial x^l} \frac{\partial x^j}{\partial x^m} \frac{\partial x^n}{\partial x^k} + \frac{\partial^2 x^i}{\partial x^l \partial x^m} \frac{\partial x^n}{\partial x^i} .$$

The presence of the second term implies that the affine connection is not a tensor. The role of the affine connection in affine geometry is similar to the role of the Christoffel symbols in Riemannian geometry.

In fact, if the components of the fundamental affine connection are the Christoffel symbols, affine geometry specializes to Riemannian geometry. This condition occurs when the first covariant derivative of the fundamental covariant tensor of the quadratic form,

$$g_{ij}\, dx^i\, dx^j ,$$

vanishes (i.e., $g_{ij,k} = 0$). The affine connection is transitive since successive transformations of the affine connection have the same form. The affine transformation group includes all linear transformations.

Each dimension of the assumed affine space of n dimensions will represent one of the degrees of freedom of the physical problem. All dimensions are handled in an identical manner although each dimension may represent a different physical entity. Thus if n is the number of physical degrees of freedom, and if time is considered in an abstract sense as another dimension, an affine geometry of (n + 1) dimensions can be used to describe the motion of the system. Alternately, an affine geometry of n dimensions can be used if time is considered as an independent parameter. Kondo (52) uses the former definition in his work. Thus the motion of a vehicle can be described as a path in (n + 1) dimensional affine space, or alternately as a path in n dimensional affine space, the path being a function of the parameter time. A "path" in affine space plays a role similar to a "geodesic" in Riemannian space (i.e., a line of shortest distance between points).

It is possible to view the relation of two points in a space in two ways: Either the second point is considered a "mapping" of the first point to a new position, or, the two points are considered to be at the same position, but referred to different coordinate systems. Springer (89) uses the terms "alibi" and "alias," respectively, to denote these two views. In this dissertation, the latter view is adopted. The transformed coordinates need not be, however, the rectangular cartesian coordinates of Euclidean geometry.

Gabriel Kron used the term "non-Riemannian space" in his analysis of rotating electrical machinery. A non-Riemannian space is a space devoid of a metric relation of the form

$$ds^2 = g_{ij}\, dx^i\, dx^j \ .$$

In ordinary Euclidean space, which is a special case of a Riemannian space of three dimensions, the metric relation follows from the Pythagorean Theorem for distances in a three dimensional space. Of course, the usual formulation of the problem can be obtained as a special case of the general problem presented here.

If the kinetic energy of a dynamic system of n dimensions is

$$T = \tfrac{1}{2} M_{ij} \dot{x}^i \dot{x}^j \text{ , } (i, j = 1, \ldots, n),$$

then

$$2T\, dt^2 = M_{ij}\, dx^i\, dx^j = ds^2 \text{ ,}$$

and the M_{ij} may be considered as the "metric" of a space of n dimensions through which a "path" describing the motion takes place. This concept of a "metric" is used in this dissertation.

Topological Considerations

Topology, or "analysis situs," is the study of the topological properties of a topological space. It is often referred to as "rubber sheet geometry" because topological properties can be distinguished from non-topological properties by visualizing the objects under study as points, lines, etc., on a rubber sheet. Only those properties which do not change under stretching are topological. For example, considering a set of points interconnected by lines, the distances between points and the angles between lines are not topological properties; however, the number of points and the nature of their interconnection are topological properties. The only restrictions on the stretching is that no breaks or tears occur and the sheet is not folded or twisted upon itself, thus assuring the uniqueness of points.

While topology may be considered a type of geometry, the terms describing similar concepts are nevertheless not uniform. In topology, two spaces are said to be "isomorphic" if a one-to-one reciprocal correspondence exists between them. A transformation of one isomorphic space into another is called an "isomorphism." A transformation of one isomorphic space into itself is called an "automorphism." A single-valued, one-to-one, continuous, reciprocal transformation is called a "homeomorphism."

The identity transformation is thus an automorphism. An affine transformation is an isomorphism, since it is both one-to-one and linear.

Method of Analysis

In the conventional analysis of the motion of aerospace vehicles, the equations are derived from a "dynamical" point of view. In this approach, the system is divided into "free bodies" or "mass points," and the equations of motion are derived from Newton's Law. This method is analogous to the "nodal" analysis in electric network theory, in which currents incident to a junction point are summed and equated to zero by Kirchhoff's current law. A "dual" of the method based on Newton's Law, utilizes a "kinematical" concept for obtaining the same equations of motion. This "kinematical" method in mechanics is analogous to the "mesh," "loop," or "circuit" analysis method of electrical network theory, where the voltage differences between junction pairs are summed around closed loops and equated to zero by Kirchhoff's voltage law. If applied to dynamics, this law would assume the following form:

"The sum of the velocity differences around any closed loop is zero."

Although this law was recognized by Trent (56), it is seldom used and no name has become associated with it.

As might be expected, neither method is superior in all cases. Thus, familiarity with both methods is desirable for selecting the best approach for the solution of a particular problem. The interrelationship between the two methods has only recently been fully understood by the use of topological concepts.

Before developing the method of analysis, it is necessary to make a distinction between two types of variables present in a particular physical problem. Firestone's (19) definitions, which are based on how the variable is physically measured, and which are also used by Trent and Koenig and Blackwell, are adopted for this purpose.

A "thru-variable" is a variable which to be measured must go "through" the measuring instrument. Force or torque are mechanical examples, current is an electrical example, and heat flow is a thermal example.

An "across-variable" is a variable whose measurement requires the measuring device to be placed "across" two points of the system. Displacement, velocity, acceleration are mechanical examples, potential difference or voltage is an electrical example, and temperature is a thermal example.

A consideration of the velocity in a dynamic system shows the importance of using the ground, or another inertial reference, as a necessary node of the system. Otherwise, the velocity would have no meaning since, in the physical world, the concept of velocity implies its definition and measurement in relative terms with respect to some reference.

With the above definitions made, two sets of equations can be derived for a particular system. One set of equations, called "vertex" or "node" equations, describes the relations among thru-variables. The other set of equations, called "circuit," "loop," or "mesh" equations, describes the relations among across-variables. The description of a dynamical system is completed by relating the thru-variables to the across-variables. These equations are called "terminal equations" by Koenig and Blackwell (84), "canonical equations" by Bewley (82), and "Ohm's Law" by Kron (25).

It is helpful to utilize linear graphs as an aid in formulating the necessary vertex or circuit equations for a particular system. The linear graph shows how the various elements in a physical system are combined. Trent (56) has rigorously shown that any physical system composed of lumped parameters is isomorphic with respect to an oriented linear graph. Thus the properties of linear graphs can be utilized in the study of physical systems.

A linear graph consists of a collection of oriented line segments, called "elements," which are represented by lines with distinct end points (called "vertices," "nodes," or "junction points") and marked with an arrow to denote their orientation. The term "orientation" as used in the linear graph literature is synonymous with the more familiar term "direction." The conventions adopted to define the orientation of an element are illustrated in Figure 1. The element in Figure 1(a) shows that its orientation corresponds to the displacement of a point resulting from the application of a positive force on the point, whereas the orientation of the element in Figure 1(b) corresponds to the internal force on a point resulting from a positive displacement of the point.

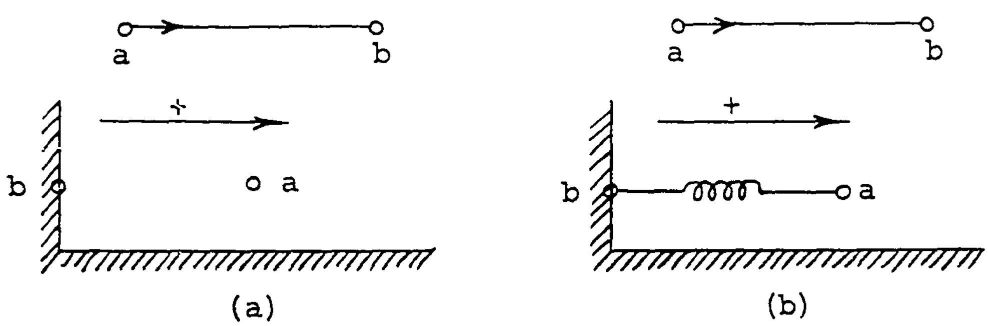

Figure 1.--Linear Graph Representations of Physical Elements

A linear graph may be subdivided into "subgraphs." The elements not included in the subgraph are called the "complement" of the subgraph. A "circuit" ("loop," or "mesh") is a subgraph in which only two elements are incident at each vertex of the graph. A "separate part" is a subgraph which has no vertices in common with its complement. When a linear graph is used to represent the various elements and their interconnection in a particular physical system, it is called a "system graph." Figure 2 shows a general mechanical system consisting of two masses (a,b), three springs (1,5,7), three dampers (2,6,8) and two velocity generators (3,9), and its system graph.

The linear graph representing the mechanical system has ten elements and three vertices. It should be noted that there is one more vertex than degrees of freedom because of the necessity of considering a ground reference "c" in order to correctly define the nature of the system.

In order to arrive at an independent set of vertex or circuit equations, it is necessary to define some additional features. A "tree" is a singly-connected subgraph of the system graph containing all vertices but no circuits. The elements of a tree are called "branches." One of the possible trees of the graph is shown in Figure 2 by heavy lines. Although different trees can be assumed

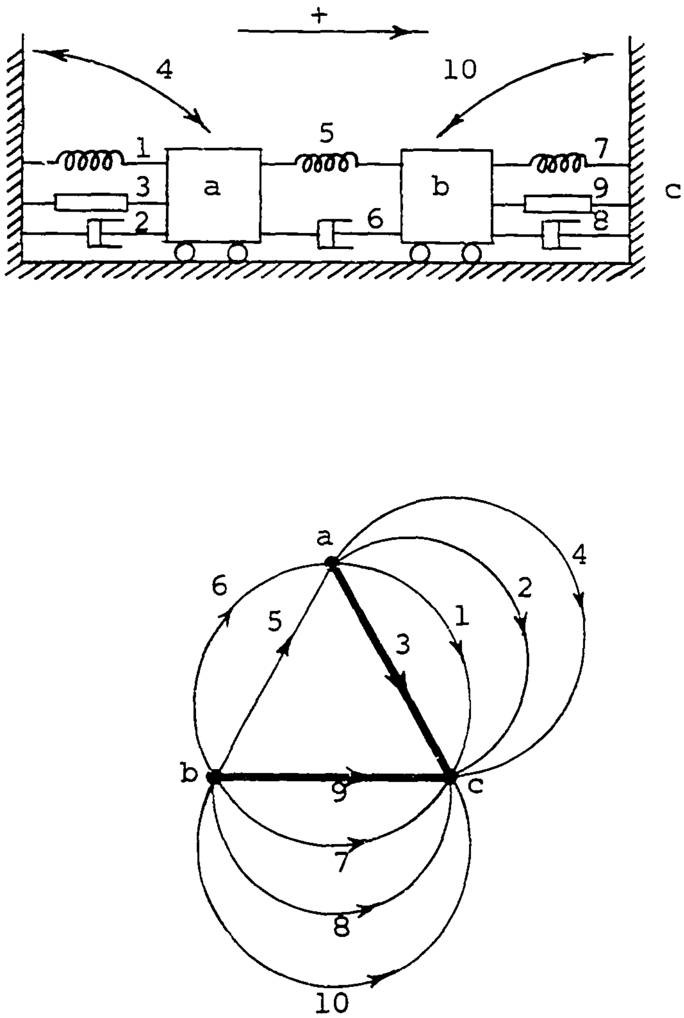

Figure 2.--Mechanical System and Its System Graph.

in a particular graph, it is convenient to use all the known elements as branches of the formulation tree.

A tree in which all branches are incident at the same vertex is called a "Lagrangean tree" and has special significance. "Chords" are the elements of a graph complementary to the tree. A "forest" is a collection of trees resulting from a system graph comprised of two or more separate parts. A "fundamental circuit" is a circuit consisting of branches and no more than one chord. If the vertices of a graph are divided into two groups (for example, by a closed curve not passing through any vertex), the elements may be separated into two sets--a set of elements both of whose vertices are in the same group, and a set of elements with one vertex in each group. This latter set is referred to as a "segregate set." A "cut set" is a segregate set with only one branch.

Four theorems which are useful in the subsequent development are here stated without proof, although their validity is easily demonstrated (see Reference 37).

THEOREM 1: The number of branches in a tree is always one less than the number of vertices.

THEOREM 2: The number of chords (C) in a system graph consisting of N separate parts is always N more than the number of elements (E) less the number of vertices (V).

$$C = E - V + N . \tag{2.5}$$

THEOREM 3: The number of independent circuit equations for a system graph is equal to the number of chords.

THEOREM 4: The number of independent cut set equations for a system graph is equal to the number of branches.

The dual nature of Theorems 3 and 4 should be noted. Either theorem can be derived from the other by interchanging the words "cut set" and "circuit" and the words "chords" and "branches."

For the mechanical system of Figure 2, $V = 3$, $E = 10$, $N = 1$, and $C = 10 - 3 + 1 = 8$. Thus, eight independent circuit equations can be derived for the system. The number of independent cut set equations for this system is $(3 - 1) = 2$.

Referring to Figure 2, it is seen that all possible trees are Lagrangean trees, since two branches are always incident at any vertex.

In order to obtain the cut set equation for the problem in Figure 2, the cut sets at vertices "a" and "b" are utilized. The set of elements cut by a small circle around "c" is a segregate set, not a cut set, since more than one tree branch is present in the set.

The type of general elements considered here are potential and kinetic energy storing elements and energy dissipating elements. These basic types of elements can be defined by relations existing between the thru- and across-variables.

A potential energy storing element is one in which the across-variable is proportional to the time derivative of the thru-variable. An example of this type in a mechanical system is a linear spring which is governed by the equation

$$f = K\,x \ . \tag{2.6}$$

Taking the derivative on both sides,

$$\frac{df}{dt} = K\,\frac{dx}{dt} \ ,$$

and setting $dx/dt = v$,

$$\frac{df}{dt} = K\,v \ ,$$

or

$$v = \frac{1}{K}\,\frac{df}{dt} \ . \tag{2.7}$$

In an electrical system, a coil, or inductance, would be an analogous element.

A kinetic energy storing element is one in which the thru-variable is proportional to the time derivative of the across-variable. The mechanical "mass" or "inertia," or the electrical "capacitor," are examples of this type. The relation for the mechanical system follows directly from Newton's Law (considering a time-independent mass),

$$f = M\,\frac{dv}{dt} \ . \tag{2.8}$$

In an electrical system, the "capacitance" is analogous to the mass.

An energy dissipating element is one in which the thru-variables are proportional to the across-variables. A mechanical viscous damper, where

$$f = C \frac{dx}{dt} = C\ v\ , \tag{2.9}$$

or an electrical resistor are examples of this type. The constant of proportionality is the "damping constant" for the mechanical system and the "conductance" for the electrical system.

These elements are discussed for both mechanical and electrical systems to emphasize their generality and their analogous character, since it is frequently easier to solve problems by using an analogous system. Beyond the use of electrical analogies to solve mechanical problems, the generality of these concepts facilitates the formulation of analogies between other physical systems.

The difference between an "analogy" and a "dual" should be understood. An "analogy" is a system of a different physical type which has the same topological model (system graph) and thus has system equations of the same form. A "dual" is a different system of the same physical type whose topological model is different but related in a definite way. Although a dual has the same number of elements, the number of vertices corresponds to the number of its counterpart's chords, and the number of circuits corresponds to the number of its counterpart's branches. The dual's system graph is its counterpart's "turned inside out." While in many instances duals and analogies are conceptually feasible, their formulation is not always practical, or possible.

Of course, more than one analogy can be found for a particular system. For example, if the forces and currents are selected as analogous thru-variables and velocities and voltages as analogous across-variables for an electromechanical analogy, the relations shown in Table 1 result.

TABLE 1

FIRESTONE ELECTROMECHANICAL ANALOGIES

Type	Mechanical	Electrical
Thru-variable	force, f	current, i
Across-variable	velocity, v	voltage, v
Potential energy storer	$f = K\ v/s$	$i = \Gamma\ v/s = (1/L)\ v/s$
Kinetic energy storer	$f = M\ s\ v$	$i = C\ s\ v$
Energy dissipator	$f = B\ v$	$i = G\ v = (1/R)\ v$

It is seen that the analogous quantities in this set of equations are stiffness (K) and inverse self-inductance (Γ), mass (M) and capacitance (C), and damping (B) and conductance (G). This topological analogy has been called the "Firestone" analogy, and is not the one usually found in standard works.

A dual set of equations may be obtained by formally interchanging thru- by across-variables but maintaining the same definitions for the element relations as before. When this is done, the character of the variables changes, and the terms "thru-" and "across-variable" no longer have the same significance as before. This dual set is shown in Table 2.

TABLE 2

DUAL ELECTROMECHANICAL ANALOGIES

Type	Mechanical	Electrical
Thru-variable	velocity, v	voltage, v
Across-variable	force, f	current, i
Potential energy storer	v = (1/M) f/s	v = S i/s
Kinetic energy storer	v = (1/K) s f	v = L s i
Energy dissipator	v = (1/B) f	v = R i

The unity and completeness of this four-set of equations was first emphasized by Trent in 1955 (56). Gardner and Barnes (26) presented both electrical analogies to the mechanical system in Table 1, but did not present the dual mechanical system shown in Table 2.

Firestone (19), of course, suggested the "non-classical" Firestone analogy in 1933. The classical analogy usually found in treatises on electromechanical analogies is obtained by coupling (or "cross-coupling") the systems shown in both tables. The classical analogy therefore considers force analogous to voltage, velocity to current, mass to inductance, stiffness to inverse capacitance, and damping to resistance. It is also seen that, topologically speaking, the conceptual role of the spring and mass or the inverse capacitance and inductance are interchanged. For these reasons, the classical analogy is not a topologically correct analogy.

Whereas the usual manner of formulating mechanical equations is to sum all forces (including inertia forces) and equate them to zero, the duality property implies that it is equally valid to sum velocities (velocity differences) and equate them to zero. The analogous formulations in electrical network theory are given by Kirchhoff's current and voltage equations. These two laws can be stated for all types of systems as follows:

(1) The sum of all thru-variables incident at a vertex is zero.

(2) The sum of all across-variables around any closed circuit is zero.

Thus, the dual law is obtained by substituting "across" for "thru," and "circuit" for "vertex."

It is clear that a given physical system will have (conceptually, at least) two analogies for each other physical system considered and one dual system of the same physical type. For the mechanical system considered previously, a mechanical dual and two electrical analogies were derived on the basis of its system graph and the relations in Tables 1 and 2. The complete four-set of analogies and duals with the two system graphs are shown in Figure 3.

The system equations for the mechanical system and its dual will next be developed in detail to serve as background for the following general tensor analysis of dynamically similar, though more complex, systems.

Two techniques exist for formulating the system equations from a particular system graph. The better way is usually determined by the nature of the "known" variables. If, as in the system considered here, the "known" variables are across-variables (displacements, x^3, x^9) it is best to write the canonical equations in a form that gives the thru-variables as functions of the across-variables. Introducing "s" as either the Laplacean operator or the differential operator d/dt, these equations are

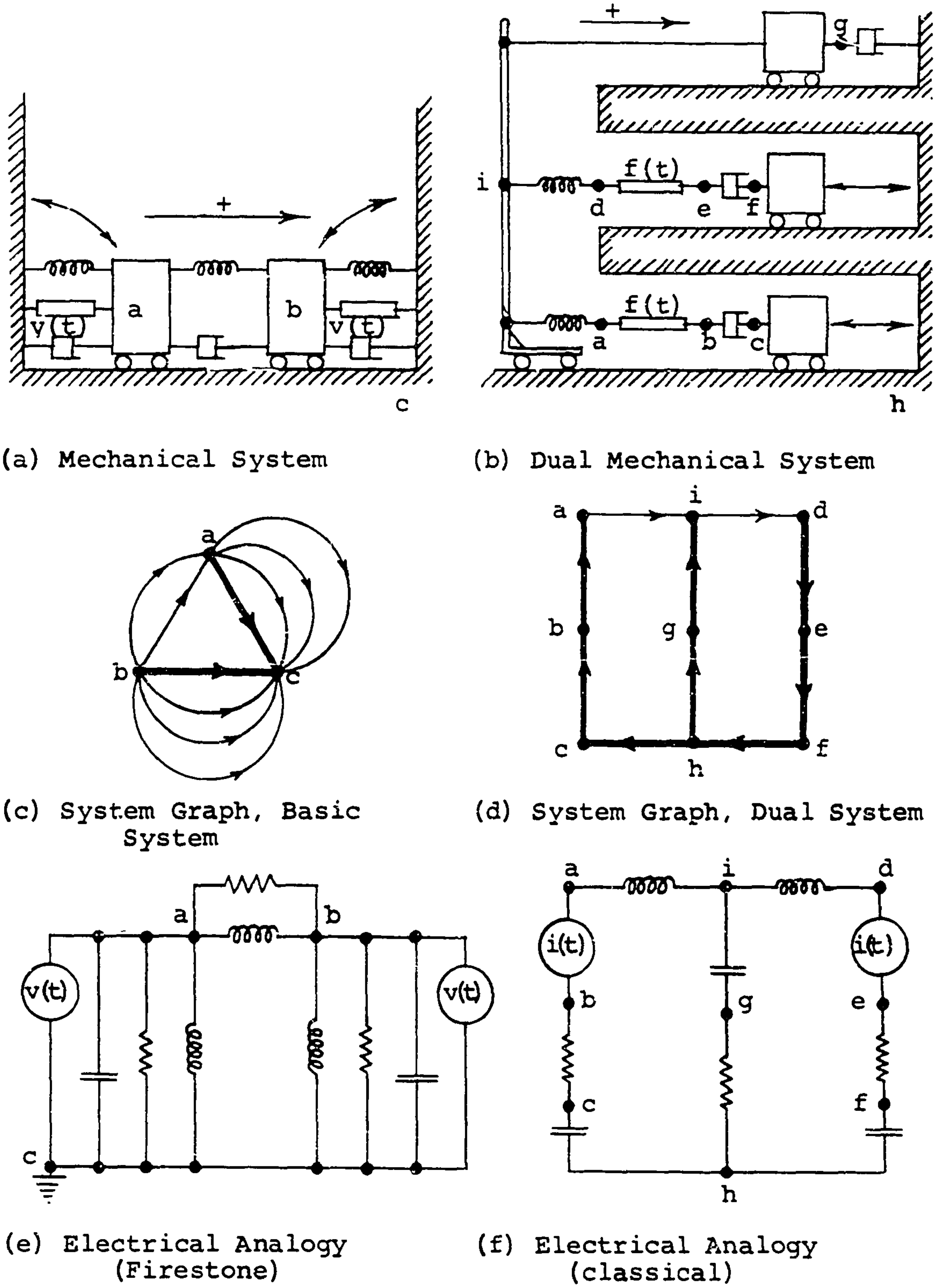

Figure 3.--A Complete Four-set of Analogies, Duals, and System Graphs.

$$\begin{bmatrix} f_1 \\ f_2 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ f_8 \\ f_{10} \end{bmatrix} = \begin{bmatrix} K_{11} & & & & & & & \\ & B_{11}s & & & & & & \\ & & M_{11}s^2 & & & & & \\ & & & K_{12} & & & & \\ & & & & B_{12}s & & & \\ & & & & & K_{22} & & \\ & & & & & & B_{22}s & \\ & & & & & & & M_{22}s^2 \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \\ x^4 \\ x^5 \\ x^6 \\ x^7 \\ x^8 \\ x^{10} \end{bmatrix} . \qquad (2.10)$$

The cut set equations are obtained by reference to the system graph, with the branch variables (f_3, f_9) listed first. Elements whose arrows point in the same direction as the branch element are considered positive and those directed opposite to the branch element are considered negative. Then, referring to Figure 4, where the cut sets are identified by the dotted circles α and β, the cut set equations are

$$\begin{array}{c} \alpha \\ \beta \end{array} \left[\begin{array}{cc|cccccccc} 1 & 0 & 1 & 1 & 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{array}\right] \left[\begin{array}{c} f_3 \\ f_9 \\ \hline f_1 \\ f_2 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ f_8 \\ f_{10} \end{array}\right] = 0 . \qquad (2.11)$$

The fundamental circuit equations are obtained also with reference to the graph by summing the elements in each loop. These elements are defined as positive if they have the same orientation as that of the chord. Referring to Figure 4, where the loops are identified by the dotted paths a,b,...h, the following circuit equations result:

$$\begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \end{array} \left[\begin{array}{rr|rrrrrrrr} -1 & & 1 & & & & & & & \\ -1 & & & 1 & & & & & & \\ -1 & & & & 1 & & & & & \\ & -1 & & & & & & & & 1 \\ & -1 & & & & & & & 1 & \\ & -1 & & & & & & 1 & & \\ 1 & -1 & & & & 1 & & & & \\ 1 & -1 & & & & & 1 & & & \end{array}\right] \left[\begin{array}{c} x^3 \\ x^9 \\ \hline x^1 \\ x^2 \\ x^4 \\ x^5 \\ x^6 \\ x^7 \\ x^8 \\ x^{10} \end{array}\right] = 0 \ . \qquad (2.12)$$

Thus, eighteen independent equations in eighteen unknown variables are obtained, for which a solution is possible. In fact, Roth (54) pointed out that a unique solution always exists in physical systems if the dissipative power is assumed positive definite. This assumption is, of course, well justified and unanimously accepted. If the branch and chord variables of Equation (2.11) are separated, then the canonical equation (2.10) may be substituted therein and a set of ten equations and ten unknowns

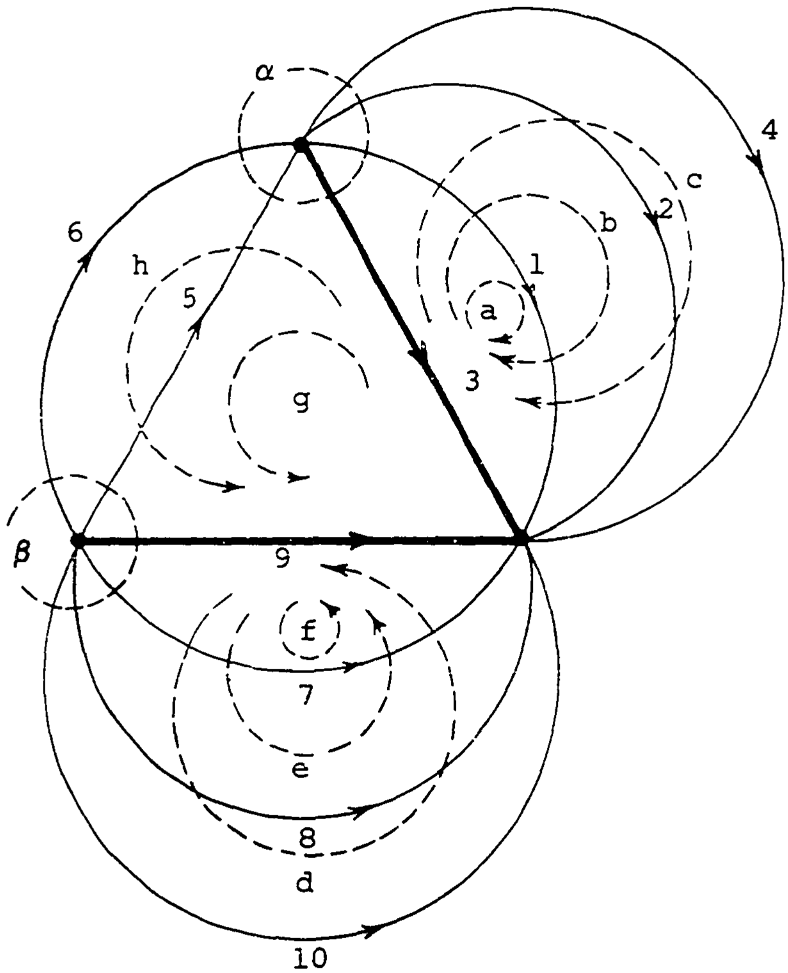

Figure 4.--System Graph for Mechanical System Showing
Cut Sets and Loops Used in Formulation

results:

$$\begin{bmatrix} f_3 \\ f_9 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} K_{11} & & & & & & & \\ & B_{11}s & & & & & & \\ & & M_{11}s^2 & & & & & \\ & & & K_{12} & & & & \\ & & & & B_{12}s & & & \\ & & & & & K_{22} & & \\ & & & & & & B_{22}s & \\ & & & & & & & M_{22}s^2 \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \\ x^4 \\ x^5 \\ x^6 \\ x^7 \\ x^8 \\ x^{10} \end{bmatrix} = 0 . \quad (2.13)$$

Rearranging the rows of Equation (2.12) such that a unit matrix exists to the right of the vertical dotted line separating branch from chord variables, Equation (2.12) may be rewritten such that the chord variables are given as functions of the branch variables,

$$\begin{bmatrix} x^1 \\ x^2 \\ x^4 \\ x^5 \\ x^6 \\ x^7 \\ x^8 \\ x^{10} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ -1 & 1 \\ -1 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x^3 \\ x^9 \end{bmatrix} . \quad (2.14)$$

Replacing the chord variables of Equation (2.13) by the above equation yields

$$\begin{bmatrix} f_3 \\ f_9 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} K_{11} & & & & & & & \\ & B_{11}s & & & & & & \\ & & M_{11}s^2 & & & & & \\ & & & K_{12} & & & & \\ & & & & B_{12}s & & & \\ & & & & & K_{22} & & \\ & & & & & & B_{22}s & \\ & & & & & & & M_{22}s^2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ -1 & 1 \\ -1 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x^3 \\ x^9 \end{bmatrix} = 0 \tag{2.15}$$

Performing the indicated matrix multiplications, the system equations in final form are:

$$\begin{bmatrix} f_3 \\ f_9 \end{bmatrix} + \left[\begin{array}{c|c} K_{11}+B_{11}s+M_{11}s^2+K_{12}+B_{12}s & -K_{12} - B_{12}s \\ \hline - K_{12} - B_{12}s & K_{12}+B_{12}s+K_{22}+B_{22}s+M_{22}s^2 \end{array}\right] \begin{bmatrix} x^3 \\ x^9 \end{bmatrix} = 0 \tag{2.16}$$

The transient solution is obtained by letting $s = d/dt$, substituting the known values of x^3 and x^9, and solving the resulting set of differential equations. The steady state solution (for $t > 0$) is obtained by letting $s = j\omega$ ($j = \sqrt{-1}$ and ω = excitation frequency). The static solution is obtained by setting $s = 0$ and solving the resulting set of algebraic equations.

The formulation of the equations for the dual mechanical system results in equations of identical form if a substitution of appropriate dual elements is made. Since in the dual mechanical system, the across-variables (now forces, f_3 and f_9) are known, the canonical equations express the thru-variables (now displacements) as functions of the across-variables.

$$\begin{bmatrix} x^1 \\ x^2 \\ x^4 \\ x^5 \\ x^6 \\ x^7 \\ x^8 \\ x^{10} \end{bmatrix} = \begin{bmatrix} \frac{1}{K_{11}} &&&&&&& \\ & \frac{1}{B_{11}s} &&&&&& \\ && \frac{1}{M_{11}s^2} &&&&& \\ &&& \frac{1}{K_{12}} &&&& \\ &&&& \frac{1}{B_{12}s} &&& \\ &&&&& \frac{1}{K_{22}} && \\ &&&&&& \frac{1}{B_{22}s} & \\ &&&&&&& \frac{1}{M_{22}s^2} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ f_8 \\ f_{10} \end{bmatrix} . \quad (2.17)$$

The fundamental circuit equations of the dual system are similar to the cut set equations of the original system. Referring to the system graph for the dual system, Figure 5, where the loops are identified by the dotted paths α and β, the circuit equations are

$$\begin{array}{c} \alpha \\ \beta \end{array} \left[\begin{array}{cc|cccccccc} 1 & 0 & 1 & 1 & 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{array}\right] \left[\begin{array}{c} x^3 \\ x^9 \\ \hline x^1 \\ x^2 \\ x^4 \\ x^5 \\ x^6 \\ x^7 \\ x^8 \\ x^{10} \end{array}\right] = 0 . \quad (2.18)$$

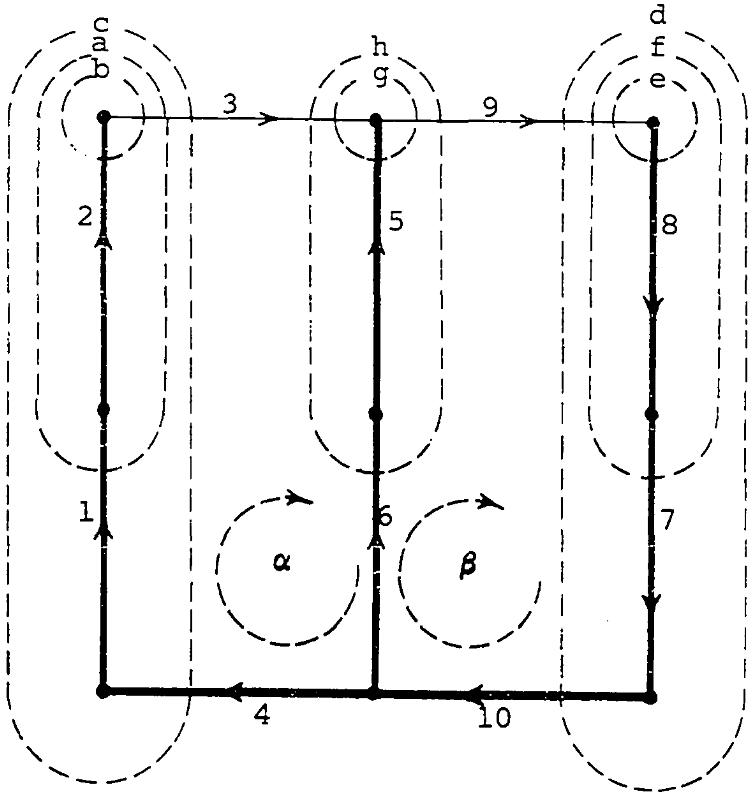

Figure 5.--System Graph for Dual Mechanical System
Showing Cut Sets and Loops Used in Formulation

The cut set equations for the dual system are similar to the fundamental circuit equations of the original system. Referring again to the dual system graph, where the cut sets are identified by dotted curves a, b, ..., h,

$$\begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \end{array}\left[\begin{array}{cc|cccccccc} -1 & & 1 & & & & & & & \\ -1 & & & 1 & & & & & & \\ -1 & & & & 1 & & & & & \\ & -1 & & & & & & & & 1 \\ & -1 & & & & & & & 1 & \\ & -1 & & & & & & 1 & & \\ 1 & -1 & & & & 1 & & & & \\ 1 & -1 & & & & & 1 & & & \end{array}\right]\left[\begin{array}{c} f_3 \\ f_9 \\ \hline f_1 \\ f_2 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ f_8 \\ f_{10} \end{array}\right] = 0 . \qquad (2.19)$$

Now, if the chord variables (x^3, x^9) of Equation (2.18) are separated, and the canonical equation (2.17) substituted therein, the following set of ten equations results:

$$\begin{bmatrix} x^3 \\ x^9 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{K_{11}} & & & & & & & \\ & \frac{1}{B_{11}s} & & & & & & \\ & & \frac{1}{M_{11}s^2} & & & & & \\ & & & \frac{1}{K_{12}} & & & & \\ & & & & \frac{1}{B_{12}s} & & & \\ & & & & & \frac{1}{K_{22}} & & \\ & & & & & & \frac{1}{B_{22}s} & \\ & & & & & & & \frac{1}{M_{22}s^2} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ f_8 \\ f_{10} \end{bmatrix} = 0 . \qquad (2.20)$$

Rewriting Equation (2.19) such that the branch variables are functions of the chord variables,

$$\begin{bmatrix} f_1 \\ f_2 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ f_8 \\ f_{10} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ -1 & 1 \\ -1 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} f_3 \\ f_9 \end{bmatrix} . \tag{2.21}$$

Substituting this equation into Equation (2.20),

$$\begin{bmatrix} x^3 \\ x^9 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{K_{11}} & & & & & & & \\ & \frac{1}{B_{11}s} & & & & & & \\ & & \frac{1}{M_{11}s^2} & & & & & \\ & & & \frac{1}{K_{12}} & & & & \\ & & & & \frac{1}{B_{12}s} & & & \\ & & & & & \frac{1}{K_{22}} & & \\ & & & & & & \frac{1}{B_{22}s} & \\ & & & & & & & \frac{1}{M_{22}s^2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ -1 & 1 \\ -1 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} f_3 \\ f_9 \end{bmatrix} = 0. \tag{2.22}$$

Performing the indicated matrix multiplication, the final form of the system equations is

$$\begin{bmatrix} x^3 \\ x^9 \end{bmatrix} + \left[\begin{array}{c|c} \frac{1}{K_{11}}+\frac{1}{B_{11}s}+\frac{1}{M_{11}s^2}+\frac{1}{K_{12}}+\frac{1}{B_{12}s} & -\frac{1}{K_{12}}-\frac{1}{B_{12}s} \\ \hline -\frac{1}{K_{12}}-\frac{1}{B_{12}s} & \frac{1}{K_{12}}+\frac{1}{B_{12}s}+\frac{1}{K_{22}}+\frac{1}{B_{22}s}+\frac{1}{M_{22}s^2} \end{array}\right] \begin{bmatrix} f_3 \\ f_9 \end{bmatrix} = 0. \tag{2.23}$$

Comparing Equation (2.23) with Equation (2.16), it is seen that both equations are of identical form, except that displacements become forces and dual physical parameters are interchanged.

At this time, the concepts of "impedance" and "admittance" are introduced. In electrical network theory, the impedance (Z) is defined as the coefficient of the current when the voltage across a particular element is given as a function of the current through it. In a general sense, impedance is defined as the coefficient of the thru-variable when the across-variable is written as a function of the thru-variable. Correspondingly, the admittance (Y) is defined as the coefficient of the across-variable when the thru-variable is written as a function of the across-variable. For example, if the current is considered the thru-variable, the electrical admittance would be defined as

$$i = (\Gamma/s + Cs + G)v = Y\ v\ . \tag{2.24}$$

If the Firestone analogy is used, the equation,

$$f = (K/s + Ms + B)v = Y\ v, \qquad (2.25)$$

defines the mechanical admittance. Of course, in both electrical and mechanical systems, the impedance is the inverse of the admittance,

$$Z = Y^{-1}\ . \qquad (2.26)$$

Therefore,

$$v = Z\ i\ , \quad \text{(Electrical System)} \qquad (2.27)$$

$$v = Z\ f\ , \quad \text{(Mechanical System)} \qquad (2.28)$$

The canonical equations for mechanical and electrical systems may then be written in general tensor form using the ordinary summation convention,

<u>Mechanical</u>	<u>Electrical</u>	
$f_i = Y_{ij}\ v^j$	$i_i = Y_{ij}\ v^j$	(2.29, 2.30)
$v^j = Z^{ij}\ f_i$	$v^j = Z^{ij}\ i_i$	(2.31, 2.32)

In terms of a mechanical system, the mechanical admittance of an element is the ratio of the force through the element to the velocity across the element. The mechanical impedance of an element is the ratio of the velocity across the element to the force through the element. These definitions correspond to the analogous definitions in electrical network theory. In mechanical practice, much confusion exists regarding these definitions. What is defined as impedance here is often called "mobility," and what is defined here as admittance is often called "impedance." This topologically inconsistent terminology will be avoided.

Applying these concepts to the mechanical system under consideration, the expressions for the mechanical admittance are given by

$$Y_{11} = M_{11}s^2 + B_{11}s + K_{11} \,, \tag{2.33}$$

$$Y_{12} = B_{12}s + K_{12} \,, \tag{2.34}$$

$$Y_{22} = M_{22}s^2 + B_{22}s + K_{22} \,, \tag{2.35}$$

and the system graph simplifies to that shown in Figure 6.

The development of the system equations parallels that given before for the previous system. The canonical equations are

$$\begin{bmatrix} f_1 \\ f_2 \\ f_4 \end{bmatrix} = \left[\begin{array}{c|c|c} Y_{11} & & \\ & Y_{12} & \\ & & Y_{22} \end{array}\right] \begin{bmatrix} x^1 \\ x^2 \\ x^4 \end{bmatrix} . \tag{2.36}$$

The cut set equations are

$$\begin{array}{c} \alpha \\ \beta \end{array} \left[\begin{array}{cc|ccc} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{array}\right] \left[\begin{array}{c} f_3 \\ f_9 \\ \hline f_1 \\ f_2 \\ f_4 \end{array}\right] = 0 \,. \tag{2.37}$$

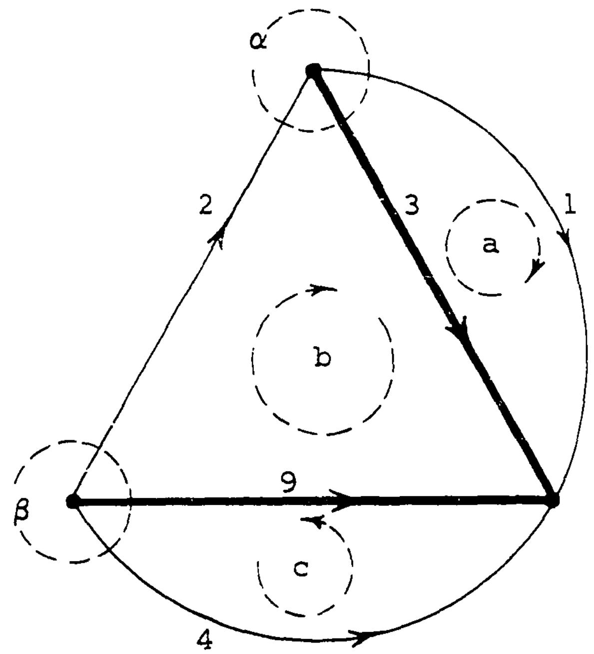

Figure 6.--System Graph of Simplified System Showing Cut Sets and Loops Used in Formulation

The loop equations are

$$\begin{array}{c} a \\ b \\ c \end{array}\left[\begin{array}{cc|ccc} -1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{array}\right]\left[\begin{array}{c} x^3 \\ x^9 \\ \hline x^1 \\ x^2 \\ x^4 \end{array}\right] = 0 \; . \tag{2.38}$$

Separating branch and chord equations from Equation (2.37), and substituting Equation (2.36) therein, we obtain

$$\begin{bmatrix} f_3 \\ f_9 \end{bmatrix} + \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} Y_{11} & 0 & 0 \\ 0 & Y_{12} & 0 \\ 0 & 0 & Y_{22} \end{bmatrix}\begin{bmatrix} x^1 \\ x^2 \\ x^4 \end{bmatrix} = 0 \; . \tag{2.39}$$

Expressing the chord variables (x^1, x^2, x^4) as functions of the branch variables (x^3, x^9)

$$\begin{bmatrix} x^1 \\ x^2 \\ x^4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x^3 \\ x^9 \end{bmatrix} \; . \tag{2.40}$$

Substituting these into Equation (2.39),

$$\begin{bmatrix} f_3 \\ f_9 \end{bmatrix} + \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} Y_{11} & 0 & 0 \\ 0 & Y_{12} & 0 \\ 0 & 0 & Y_{22} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x^3 \\ x^9 \end{bmatrix} \tag{2.41}$$

which, after performing the indicated matrix multiplications, produce the final system equation

$$\begin{bmatrix} f_3 \\ f_9 \end{bmatrix} + \left[\begin{array}{c|c} Y_{11} + Y_{12} & -Y_{12} \\ \hline -Y_{12} & Y_{12} + Y_{22} \end{array}\right] \begin{bmatrix} x^3 \\ x^9 \end{bmatrix} = 0 \ . \qquad (2.42)$$

With the substitution of the proper quantities for the admittances, it is seen that this equation is identical to the previously derived Equation (2.16). A similar simplification can be made for the solution of the dual system.

Particular note should be taken of the fact that the matrix relating the branch thru-variables (forces) to the chord thru-variables in Equation (2.37) is the negative transpose of the matrix relating the branch across-variables (displacements) to the chord across-variables in Equation (2.38). Thus, in formulating the problem, only one of the two sets of equations need be derived. These "transformation tensors" can systematically be obtained by the use of the system graph as shown in this section.

Tensor Analysis of the Mechanical System

In order to parallel the foregoing development, the branch variables are given by the indices i, j, and k, and the chord variables, by the indices l, m, and n. Displacements are considered contravariant tensors of first order (vectors) and are denoted by superscripts (e.g., x^m). The admittance is considered a second order covariant tensor and is denoted as Y_{mn}. Forces are considered covariant tensors and denoted as f_n. With these definitions, the canonical equations (2.36) in tensor notation are

$$f_m = Y_{mn}\, x^n \;. \tag{2.43}$$

The displacements x^n are contravariant and are transformed by the transformation law

$$x^m = C^m_p\, x^p \;. \tag{2.44}$$

Although C^m_p is not a tensor in the usual mathematical sense, the term "transformation tensor," which is used in the literature by Kron and Bewley, will be used here. If the transformation tensor C^m_p is constant, then

$$\dot{x}^m = C^m_p\, \dot{x}^p \;. \tag{2.45}$$

The transformation law for the f_m and the Y_{mn} can be determined by recognizing the invariance of power, first shown to be valid for electrical networks by Kron (27) and later proved to be valid for all physical systems by Roth (54). Thus,

$$P = \dot{x}^p\, f_p = \dot{x}^m\, f_m \;. \tag{2.46}$$

Substituting Equation (2.45) into the right side of the above equation,

$$\dot{x}^p\, f_p = C^m_p\, \dot{x}^p\, f_m \;, \text{ or}$$

$$f_p = C^m_p\, f_m \;, \tag{2.47}$$

which is the law of transformation for forces f_m.

Substituting Equation (2.43) into the right side of (2.47),

$$f_p = C_p^m Y_{mn} x^n .$$

But since

$$x^n = C_q^n x^q ,$$

it follows that

$$f_p = C_p^m Y_{mn} C_q^n x^q .$$

With

$$f_p = Y_{pq} x^q ,$$

the law of transformation for the Y_{mn} is then given by

$$Y_{pq} = C_p^m C_q^n Y_{mn} , \tag{2.48}$$

The cut set equation (2.37) in tensor notation is

$$\delta_j^i f_i + C_j^m f_m = 0 , \tag{2.49}$$

where the δ_j^i is the Kronecker delta, and C_j^m is a transformation tensor describing the manner of connection of the chord thru-variables. The transformation tensor is given formally as

$$C_j^m = \frac{\partial f_m}{\partial f_j} . \tag{2.50}$$

To evaluate this tensor (i.e., determine its components), a set of equations relating the branch thru-variables (f_j) to the chord thru-variables (f_m) of the form

$$f_m = f_m(f_j) \tag{2.51}$$

is necessary. The components of the transformation tensor are then given as the partial derivatives of Equation (2.51) as defined by Equation (2.50). This operation is best accomplished by properly summing the chord thru-variables of each cut set of the system graph.

Substitution of Equation (2.43) into (2.49) yields

$$f_j + C_j^m Y_{mn} x^n = 0 .$$

Utilizing Equation (2.44), it is recognized that

$$f_j + C_j^m Y_{mn} C_k^n x^k = 0 ,$$

or by virtue of (2.48),

$$f_j + Y_{jk} x^k = 0 . \tag{2.52}$$

This tensor equation corresponds to Equation (2.42) derived before.

In the problem statement, the x^k are given and thus the f_j may be determined from Equation (2.52). The x^n are found from the x^k by the transformation law

$$x^n = C_k^n x^k . \tag{2.53}$$

The forces, f_n, can now be established from Equation (2.43).

The existence of a unique solution to this "network problem" has been recently proven rigorously by Roth (54) and Weinzweig (80) by an appeal to algebraic topology theory.

From the "transformation" viewpoint adopted in this dissertation, it is seen that the chord across-variables x^n are transformed into the chord thru-variables f_m by the second order tensor Y_{mn}. The branch across-variables x^k are transformed into the chord across-variables x^n by the mixed tensor of contravariant order one and co-variant order one, C^n_k, and the chord thru-variables f_m are transformed into the branch thru-variables f_j by the mixed tensor of contravariant order one and covariant order one, C^m_j. Since the elements of the system graph represent both thru- and across-variables, the components of the two transformation tensors, C^m_j and C^n_k are related. In fact, a comparison of Equation (2.44) and (2.47) shows that one is simply the transpose of the other (i.e., $C^m_j = C^k_n$).

If forces rather than displacements were given, the problem could most easily be solved by utilizing the dual system solution. It should be noted that in either case, a solution may be obtained without performing any inversions.

It should also be noted that the system equations in admittance form, Equation (2.52), can be expanded into the form of Equation (2.16),

$$f_j + (M_{jk}s^2 + B_{jk}s + K_{jk})\ x^k = 0\ , \qquad (2.54)$$

where the components of the M_{jk}, B_{jk}, and K_{jk} are found from laws of transformation similar to Equation (2.48).

The transformation tensors can be thought of as defining the constraints of the system. Since in the system discussed here, these constraints involve relations between the system variables alone, and no non-integrable relations exist, the system is termed "holonomic."

Tensor Analysis by Lagrange's Equations

The previous developments utilized force-displacement methods based on Newton's Law or current-voltage methods based on Ohm's laws to derive the system equations. These equations may also be derived by the use of energy methods, which employ the generalized method of analysis first introduced by Lagrange. In this method, the various forms of energy existing in the system are defined and then interrelated by Lagrange's equations to produce the correct system equations. Of course, the final set of equations is the same as that derived by force-displacement methods.

Three forms of energy must be recognized and defined: kinetic energy (T), potential energy (V), and dissipative energy (F).

In conventional analyses of mechanical systems, the terms "absolute" and "relative" are frequently used to describe quantities whose value is dependent on the frame of reference. An "absolute" velocity or displacement is the velocity or displacement measured with respect to an inertial reference system. A "relative" velocity or displacement is the relative velocity or displacement of two points in this system. This dichotomy is not necessary if the relevant topological properties of the system are recognized.

The method utilized in this dissertation makes this recognition by considering the ground as a vertex of the system graph. Thus, when the energy relations are defined, all velocities and displacements become "across-variables." The velocities and displacements represented on the system graph between any of the vertices and the "ground" vertex correspond to the conventional "absolute" velocities and displacements. Passive devices between these vertex pairs, such as springs and dampers, would conventionally be called "absolute" springs and dampers. Similarly, velocities, displacements, and passive devices between any two "non-ground" vertices would conventionally be called "relative" quantities or devices. A consideration of the topology of the system, however, shows that the conventional "absolute" velocity is in fact a "relative" velocity with respect to the ground. This distinction is automatically made when the topological properties of the system are recognized.

In analyzing the previously discussed problem by

Lagrangean techniques, of the three across-variables present, only two are necessary to completely define the state of the system. In the following definitions, the general variables (the three-set) are denoted by the indices l, m, and n, and the independent variables (the two-set) are denoted by the indices i, j, and k. The same Lagrangean formulation tree is used as before. The existing energy forms normally are functions of all three general across-variables. Thus, the three forms of mechanical energy are:

$$T = \tfrac{1}{2} M_{mn} \dot{x}^m \dot{x}^n \ . \quad \text{(Kinetic Energy)} \tag{2.55}$$

$$V = \tfrac{1}{2} K_{mn} x^m x^n \ . \quad \text{(Potential Energy)} \tag{2.56}$$

$$F = \tfrac{1}{2} B_{mn} \dot{x}^m \dot{x}^n \ . \quad \text{(Dissipative Energy)} \tag{2.57}$$

Lagrange's equation for a holonomic system is

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}^k}\right] - \frac{\partial T}{\partial x^k} + \frac{\partial V}{\partial x^k} + \frac{\partial F}{\partial \dot{x}^k} + f_i = 0 \ . \tag{2.58}$$

The general variables x^m are related to the independent variables x^j by

$$x^m = C^m_{\cdot j} \, x^j \ , \tag{2.59}$$

where the $C^m_{\cdot j}$ is a transformation tensor identical to that defined in the previous section; it can be evaluated in the same way. Similarly, the M_{mn}, K_{mn}, and B_{mn} are related to the M_{ij}, K_{ij}, and B_{ij} by

$$M_{mn} = C^i_m \, C^j_n \, M_{ij} \ , \tag{2.60}$$

$$K_{mn} = C_m^i \, C_n^j \, K_{ij} \,, \qquad (2.61)$$

$$B_{mn} = C_m^i \, C_n^j \, B_{ij} \,. \qquad (2.62)$$

With these transformations, Equations (2.55) through (2.57) become

$$T = \tfrac{1}{2} \, C_m^i \, C_n^j \, M_{ij} \, C_j^n \, \dot{x}^j \, C_i^m \, \dot{x}^i \,.$$

$$T = \tfrac{1}{2} \, M_{ij} \, \dot{x}^i \, \dot{x}^j \,. \quad (C_m^i \, C_i^m = C_n^j \, C_j^n = 1) \qquad (2.63)$$

Similarly, V and F are

$$V = \tfrac{1}{2} \, K_{ij} \, x^i \, x^j \,, \qquad (2.64)$$

$$F = \tfrac{1}{2} \, B_{ij} \, \dot{x}^i \, \dot{x}^j \,. \qquad (2.65)$$

The various terms of Lagrange's equation then assume the form

$$\frac{\partial T}{\partial \dot{x}^k} = \tfrac{1}{2} \, M_{ij} \left[\dot{x}^i \left(\frac{\partial \dot{x}^j}{\partial \dot{x}^k} \right) + \dot{x}^j \left(\frac{\partial \dot{x}^i}{\partial \dot{x}^k} \right) \right] = \tfrac{1}{2} \, M_{ij} (2 \, \dot{x}^j) = M_{ij} \, \dot{x}^j .$$

$$\frac{d}{dt} \left[\frac{\partial T}{\partial \dot{x}^k} \right] = M_{ij} \, \ddot{x}^j \,. \quad (\text{for constant } M_{ij})$$

$$\frac{\partial T}{\partial x^k} = 0 \,.$$

$$\frac{\partial V}{\partial x^k} = K_{ij} \, x^j \,.$$

$$\frac{\partial F}{\partial \dot{x}^k} = B_{ij} \, \dot{x}^j \,.$$

The system equation follows as

$$M_{ij}\ \ddot{x}^j + 0 + K_{ij}\ x^j + B_{ij}\ \dot{x}^j + f_i = 0\ . \qquad (2.66)$$

Evaluating the M_{ij}, K_{ij}, and the B_{ij},

$$\left[M_{ij}\right] = \left[C_i^m\ C_j^n\ M_{mn}\right] = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \left[\begin{array}{c|c|c} M_{11}s^2 & & \\ \hline & 0 & \\ \hline & & M_{22}s^2 \end{array}\right] \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} = \left[\begin{array}{c|c} M_{11}s^2 & 0 \\ \hline 0 & M_{22}s^2 \end{array}\right] .$$

$$\left[B_{ij}\right] = \left[C_i^m\ C_j^n\ B_{mn}\right] = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \left[\begin{array}{c|c|c} B_{11}s & & \\ \hline & B_{12}s & \\ \hline & & B_{22}s \end{array}\right] \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} =$$

$$\left[B_{ij}\right] = \left[\begin{array}{c|c} B_{11}s + B_{12}s & -B_{12}s \\ \hline -B_{12}s & B_{12}s + B_{22}s \end{array}\right] .$$

$$\left[K_{ij}\right] = \left[\begin{array}{c|c} K_{11} + K_{12} & -K_{12} \\ \hline -K_{12} & K_{12} + K_{22} \end{array}\right] .$$

Upon substitution of these relations into Equation (2.66),

$$\begin{bmatrix} M_{11}s^2 & 0 \\ 0 & M_{22}s^2 \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} + \left[\begin{array}{c|c} B_{11}s + B_{12}s & -B_{12}s \\ \hline -B_{12}s & B_{12}s + B_{22}s \end{array}\right] \begin{bmatrix} x^1 \\ x^2 \end{bmatrix}$$

$$+ \left[\begin{array}{c|c} K_{11} + K_{12} & -K_{12} \\ \hline -K_{12} & K_{12} + K_{22} \end{array}\right] \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} .$$

After collecting terms, the final equation

$$\left[\begin{array}{c|c} M_{11}s^2+B_{11}s+K_{11}+K_{12} & -B_{12}s \;\; -K_{12} \\ \hline -B_{12}s \;\; -K_{12} & M_{22}s^2+B_{12}s+B_{22}s+K_{12}+K_{22} \end{array}\right]\begin{bmatrix} x^1 \\ x^2 \end{bmatrix}+\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

is obtained, which is identical to Equation (2.16) derived before.

Koenig and Blackwell (70) pointed out that the Lagrangean techniques are not self-sufficient for the analysis of complex physical systems since no general procedure for determining the relationship between the variables defining the energy function (called general variables here) and the generalized coordinates (called independent variables here) is given. However, a similar lack of self-sufficiency also exists when Newton's force-displacement methods are employed alone. Such self-sufficiency relations can be obtained if a Lagrangean tree is selected as the formulation tree for the physical system under analysis, since the topological properties derived therefrom provide automatically the remaining relations. Thus, the recognition of topological properties of a given system is important for both force-displacement and Lagrangean formulation methods.

Extension to More General Cases

In the preceding section, the M_{ij} were considered constants. If the M_{ij} are functions of the variables x_i (as in the case of general vehicle motion), Lagrange's equation assumes a slightly different form. The various terms of Lagrange's equation then become

$$\frac{\partial T}{\partial \dot{x}^j} = M_{ij}\ \dot{x}^j\ .$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}^k}\right) = M_{ij}\ \ddot{x}^j + \dot{x}^j\left(\frac{\partial M_{ij}}{\partial x^k}\right)\frac{dx^k}{dt} = M_{ij}\ \ddot{x}^j + \dot{x}^j\dot{x}^k\ \frac{\partial M_{ij}}{\partial x^k}\ .$$

But,

$$\dot{x}^j\dot{x}^k\ \frac{\partial M_{ij}}{\partial x^k} = \tfrac{1}{2}\ \frac{\partial M_{ij}}{\partial x^k}\ \dot{x}^j\dot{x}^k + \tfrac{1}{2}\ \frac{\partial M_{ki}}{\partial x^j}\ \dot{x}^k\dot{x}^j\ .$$

Thus,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}^k}\right) = M_{ij}\ddot{x}^j + \tfrac{1}{2}\left(\frac{\partial M_{ij}}{\partial x^k} + \frac{\partial M_{ki}}{\partial x^j}\right)\dot{x}^k\dot{x}^j.$$

$$\frac{\partial T}{\partial x^k} = \tfrac{1}{2}\ \dot{x}^i\dot{x}^j\ \frac{\partial M_{ij}}{\partial x^k} = \tfrac{1}{2}\ \frac{\partial M_{jk}}{\partial x^i}\ \dot{x}^j\dot{x}^k\ .$$

$$\frac{\partial V}{\partial x^k} = K_{ij}\ x^j\ .$$

$$\frac{\partial F}{\partial \dot{x}^k} = B_{ij}\ \dot{x}^j\ .$$

Collecting terms,

$$M_{ij}\ddot{x}^j + \tfrac{1}{2}\left(\frac{\partial M_{ij}}{\partial x^k} + \frac{\partial M_{ki}}{\partial x^j} - \frac{\partial M_{jk}}{\partial x^i}\right)\dot{x}^j\ \dot{x}^k$$

$$+ K_{ij}x^j + B_{ij}\dot{x}^j + f_i = 0\ .$$

$$M_{ij}\ddot{x}^j + B_{ij}\dot{x}^j + K_{ij}x^j + [jk,i]\dot{x}^j\dot{x}^k + f_i = 0, \quad (2.68)$$

where $[jk, i]$ is the Christoffel symbol of the first kind, i.e.,

$$[jk,\ i] = \tfrac{1}{2}\left(\frac{\partial M_{ij}}{\partial x^k} + \frac{\partial M_{ki}}{\partial x^j} - \frac{\partial M_{jk}}{\partial x^i}\right) . \tag{2.69}$$

If the inertia tensor is independent of the variables x^k, the components of the Christoffel symbols vanish.

Lagrange's equation may be extended to include non-holonomic systems by introducing the transformation tensor C^j_m relating differentials of the holonomic and non-holonomic variables, as a function of the variables x^i. That is,

$$dx^j = C^j_m\, dx^m \tag{2.70}$$

Utilizing these relations, and denoting the non-holonomic variables by the indices l, m, and n, Equation (2.68) transforms into

$$C^l_i M_{lm} C^m_j \frac{d}{dt}\left(C^j_m \dot{x}^m\right) + C^l_i B_{lm} C^m_j \dot{x}^m C^j_m + C^l_i K_{lm} C^m_j C^j_m x^m$$

$$+ [jk,\ i]\, C^j_m \dot{x}^m C^k_n \dot{x}^n + C^l_i f_l = 0 .$$

Multiplying through by C^i_l, simplifying, and rearranging yields

$$f_l + B_{lm}\dot{x}^m + K_{lm}x^m + M_{lm}\frac{d\dot{x}^m}{dt} + M_{lm}C^m_j \dot{x}^m \frac{\partial C^j_m}{\partial x^n}\frac{dx^n}{dt}$$

$$+ [jk,\ i]\, C^i_l C^j_m C^k_n \dot{x}^m \dot{x}^n = 0 \text{ , or}$$

$$f_l + B_{lm}\dot{x}^m + K_{lm}x^m + M_{lm}\frac{d\dot{x}^m}{dt} +$$

$$+ \left(C^m_j M_{lm} \frac{\partial C^j_m}{\partial x^n} + [jk,\ i]\ C^j_m C^i_l C^k_n \right) \dot{x}^m \dot{x}^n = 0\ . \tag{2.71}$$

Recognizing the terms within the parenthesis to be the affine connection $\Gamma_{mn,l}$, the non-holonomic form of Lagrange's equation becomes

$$f_l + B_{lm}\dot{x}^m + K_{lm}x^m + M_{lm}\ddot{x}^m + \Gamma_{mn,l}\dot{x}^m\dot{x}^n = 0\ . \tag{2.72}$$

These equations are referred to in the literature as the Boltzmann-Hamel equations (20). It is seen from the definition of the affine connection $\Gamma_{mn,l}$ that, if the transformation tensors C^j_m are independent of the variables x^n, Equation (2.72) reduces to Equation (2.68). Further, if the M_{ij} are independent of the variables x^j, the equation reduces to Equation (2.66), i.e., the usual form of Lagrange's equation.

Small Perturbation Equations

The small perturbation equations of motion for a non-holonomic system are obtained by differentiating Equation (2.72). This is most easily done by recognizing that

$$M_{lm} \frac{\delta \dot{x}^m}{\delta t} = M_{lm} \left(\frac{d\dot{x}^m}{dt} + \Gamma^m_{pq} \dot{x}^p \frac{dx^q}{dt} \right) = M_{lm}\ddot{x}^m + \Gamma_{pq,l}\dot{x}^p\dot{x}^q\ ,$$

and rewriting Equation (2.72) as

$$f_l + B_{lm}\dot{x}^m + K_{lm}x^m + M_{lm}\frac{\delta\dot{x}^m}{\delta t} = 0\ . \tag{2.73}$$

Now, the differential of Equation (2.73) is

$$df_l + d(B_{lm}\,\dot{x}^m) + d(K_{lm}\,x^m) + d(M_{lm}\,\frac{\delta\dot{x}^m}{\delta t}) = 0. \qquad (2.74)$$

It should be noted that this equation of motion for small perturbations is not a tensor equation since none of the terms, or combination of terms, are tensors. A tensor equation can, however, be derived by replacing the ordinary differentials of Equation (2.73) by "intrinsic" or "absolute" differentials. The pertinent intrinsic differentials are defined as follows:

$$\delta f_l = df_l - \Gamma_{pq,l}\, f_p\, dx_q\,, \qquad (2.75)$$

$$\delta x^m = dx^m + \Gamma^m_{pq}\, x^p\, dx^q\,, \qquad (2.76)$$

$$\delta B_{lm} = dB_{lm} - \Gamma^p_{lq}\, B_{pm}\, dx^q - \Gamma^p_{qm}\, B_{lp}\, dx^q\,. \qquad (2.77)$$

Thus, utilizing these definitions

$$\begin{aligned}
\delta(B_{lm}\,\dot{x}^m) &= \dot{x}^m\,\delta B_{lm} + B_{lm}\,\delta\dot{x}^m \\
&= \dot{x}^m\left(\frac{\partial B_{lm}}{\partial x^q}\,dx^q - \Gamma^p_{lq}\,B_{pm}\,dx^q - \Gamma^p_{qm}\,B_{lp}\,dx^q\right) \\
&\quad + B_{lm}(d\dot{x}^m + \Gamma^m_{pq}\,\dot{x}^p\,dx^q) \\
&= \left(\frac{\partial B_{lm}}{\partial x^q} - \Gamma^p_{lq}\,B_{pm} - \Gamma^p_{qm}\,B_{lp}\right)\dot{x}^m\,dx^q \\
&\quad + B_{lm}\,(d\dot{x}^m - \Gamma^m_{pq}\,\dot{x}^p\,dx^q)\,. \\
&= B_{qml}\,\dot{x}^m\,dx^q + B_{lm}(d\dot{x}^m + \Gamma^m_{pq}\,\dot{x}^p\,dx^q)\,, \qquad (2.78)
\end{aligned}$$

where the B_{qml} is a third order damping tensor defined by

$$B_{qml} = \left(\frac{\partial B_{lm}}{\partial x^q} - \Gamma^p_{lq} B_{pm} - \Gamma^p_{qm} B_{lp} \right) . \qquad (2.79)$$

Similarly,

$$\delta(K_{lm} x^m) = K_{qml} x^m \, dx^q + K_{lm}(dx^m + \Gamma^m_{pq} x^p \, dx^q) , \qquad (2.80)$$

where the K_{lmq} is a third order stiffness tensor defined by

$$K_{qml} = \left(\frac{\partial K_{lm}}{\partial x^q} - \Gamma^p_{lq} K_{pm} - \Gamma^p_{qm} K_{lp} \right) . \qquad (2.81)$$

$$\delta\left(M_{lm} \frac{\delta \dot{x}^m}{\delta t}\right) = \frac{\delta \dot{x}^m}{\delta t} (\delta M_{lm}) + M_{lm} \delta\left(\frac{\delta \dot{x}^m}{\delta t}\right) = M_{lm} \delta\left(\frac{\delta \dot{x}^m}{\delta t}\right) ,$$

since the intrinsic derivative of the tensor M_{lm} is zero. Using Kron's technique (20) for evaluating the above differential,

$$\delta\left(M_{lm} \frac{\delta \dot{x}^m}{\delta t}\right) = M_{lm} \left[\delta \left(\frac{\delta \dot{x}^m}{\delta t} \right) - \frac{\delta}{\delta t}(\delta \dot{x}^m) + \frac{\delta}{\delta t}(\delta \dot{x}^m) \right] \qquad (2.82)$$

The terms inside the brackets are

$$\delta\left(\frac{\delta \dot{x}^m}{\delta t}\right) = d\left(\frac{\delta \dot{x}^m}{\delta t}\right) + \Gamma^m_{pq} \left(\frac{\delta \dot{x}^p}{\delta t}\right) dx^q$$

$$= d\left(\frac{d\dot{x}^m}{dt}\right) + \Gamma^m_{pq} (\dot{x}^p \, d\dot{x}^q + \dot{x}^q \, d\dot{x}^p) + \dot{x}^p \, d\Gamma^m_{pq} \, \dot{x}^q$$

$$+ \Gamma^m_{pq} \frac{d\dot{x}^p}{dt} dx^q + \Gamma^m_{pq} \Gamma^p_{rs} \dot{x}^r \dot{x}^s \, dx^q . \qquad (2.83)$$

$$\frac{\delta}{\delta t}(\delta \dot{x}^m) = \frac{d}{dt}(\delta \dot{x}^m) + \Gamma^m_{pq} \delta \dot{x}^p \frac{dx^q}{dt}$$

$$= \frac{d}{dt}(d\dot{x}^m + \Gamma^m_{pq}\,\dot{x}^p dx^q) + \Gamma^m_{pq}(d\dot{x}^p + \Gamma^p_{rs}\,\dot{x}^r dx^s)\dot{x}^q$$

$$= \frac{d}{dt}(d\dot{x}^m) + \Gamma^m_{pq}\left(\dot{x}^p \frac{d}{dt}(dx^q) + dx^q \frac{d\dot{x}^p}{dt}\right)$$

$$+ \dot{x}^p dx^q \frac{d\Gamma^m_{pq}}{dt} + \Gamma^m_{pq}\, d\dot{x}^p \dot{x}^q + \Gamma^m_{pq}\Gamma^p_{rs}\,\dot{x}^r dx^s \dot{x}^q \ . \tag{2.84}$$

Substracting Equation (2.84) from (2.83) gives

$$\delta\left(\frac{\delta \dot{x}^m}{\delta t}\right) - \frac{\delta}{\delta t}(\delta \dot{x}^m) = d\Gamma^m_{pq}\,\dot{x}^p\dot{x}^q + \Gamma^m_{pq}\Gamma^p_{rs}\,\dot{x}^r\dot{x}^s dx^q$$

$$- \dot{x}^p dx^q \frac{d\Gamma^m_{pq}}{dt} - \Gamma^m_{pq}\Gamma^p_{rs}\,\dot{x}^r dx^s \dot{x}^q$$

$$= \frac{\partial \Gamma^m_{pq}}{\partial x^r} dx^r \dot{x}^p \dot{x}^q + \Gamma^m_{pq}\Gamma^p_{rs}\,\dot{x}^r\dot{x}^s dx^q$$

$$- \frac{\partial \Gamma^m_{pq}}{\partial x^r}\frac{dx^r}{dt}\,\dot{x}^p dx^q - \Gamma^m_{pq}\Gamma^p_{rs}\,\dot{x}^r\dot{x}^q dx^s \ .$$

Upon changing the dummy indices such that a collection of terms is possible

$$\delta\left(\frac{\delta \dot{x}^m}{\delta t}\right) - \frac{\delta}{\delta t}(\delta \dot{x}^m) = \frac{\partial \Gamma^m_{pq}}{\partial x^r}\,\dot{x}^p\dot{x}^q dx^r + \Gamma^m_{sr}\Gamma^s_{pq}\,\dot{x}^p\dot{x}^q dx^r$$

$$- \frac{\partial \Gamma^m_{qr}}{\partial x^p}\,\dot{x}^p\dot{x}^q dx^r - \Gamma^m_{sq}\Gamma^s_{pr}\,\dot{x}^p\dot{x}^q dx^r \ .$$

$$= \left(\frac{\partial \Gamma^m_{pq}}{\partial x^r} - \frac{\partial \Gamma^m_{qr}}{\partial x^p} + \Gamma^m_{sr}\Gamma^s_{pq} - \Gamma^m_{sq}\Gamma^s_{pr}\right)\dot{x}^p\dot{x}^q dx^r$$

$$= K^m_{pqr}\,\dot{x}^p\dot{x}^q dx^r \ , \tag{2.85}$$

where K^m_{pqr} is the Riemann-Christoffel symbol of the second kind.

Considering the definition of the Riemann-Christoffel symbols of the first kind,

$$K_{pqrm} = M_{lm} K^{m}_{pqr} , \tag{2.86}$$

it follows that

$$\begin{aligned} \delta(M_{lm} \frac{\delta x^m}{\delta t}) &= M_{lm} [K^{m}_{pqr} \dot{x}^p \dot{x}^q dx^r - \frac{\delta}{\delta t}(\delta \dot{x}^m)] \\ &= K_{pqrl} \dot{x}^p \dot{x}^q dx^r + M_{lm} \frac{\delta}{\delta t}(\delta \dot{x}^m) . \end{aligned} \tag{2.87}$$

Combining the terms given in Equations (2.75), (2.78), (2.80), (2.84) and (2.87), the tensor form of the small perturbation equation of motion is obtained:

$$\begin{aligned} & df_l - \Gamma_{pq,l} f_p dx_q + B_{qml} \dot{x}^m dx^q + B_{lm}(d\dot{x}^m + \Gamma^m_{pq} \dot{x}^p \dot{x}^q) \\ & + K_{qml} x^m dx^q + K_{lm}(dx^m + \Gamma^m_{pq} x^p dx^q) + K_{pqrl} \dot{x}^p \dot{x}^q dx^r \\ & + M_{lm} \frac{d}{dt}(d\dot{x}^m) + M_{lm} \Gamma^m_{pq} [\dot{x}^p \frac{d}{dt}(dx^q) + dx^q \frac{d\dot{x}^p}{dt} + d\dot{x}^p \dot{x}^q] \\ & + M_{lm} \frac{\partial \Gamma^m_{pq}}{\partial x^r} \dot{x}^r \dot{x}^p dx^q + M_{lm} \Gamma^m_{pq} \Gamma^p_{rs} \dot{x}^r \dot{x}^q dx^s = 0 . \end{aligned} \tag{2.88}$$

($\dot{x}^i = \frac{\delta x^i}{\delta t}$ in this derivation)

In summary, this general tensor equation is valid for describing the small perturbation motion of an n dimensional holonomic or non-holonomic dynamic system whose inertia properties (or metric properties) may be functions of not only the coordinates but also the time (if the time is treated as an additional independent coordinate).

CHAPTER III

APPLICATION TO AN ATMOSPHERIC FLIGHT VEHICLE

Introduction

In Chapter II, a general method of analysis based on tensor concepts which utilizes linear graphs to properly recognize the topological properties of the system was presented. As an aid to understanding, as well as to indicate the application of the method, the equations of motion of a generalized mechanical system were formulated. In this chapter, the method is applied to the more complex problem of determining the small perturbation equations of motion of an aerospace vehicle.

In developing these equations, the vehicle is conceived as a lumped parameter system. Therefore, the methods of analysis presented in Chapter II for mechanical or electrical systems are applicable. However, in changing from a stationary electrical network or ground-anchored mechanical system to an aerospace vehicle free to move in three-dimensional space, some complications occur due to the complexities of the aerodynamic forces and inertial characteristics. Whereas gravitational and inertial forces are defined in terms of inertial frames of reference, the aerodynamic forces are determined with respect to stability axes which are fixed to the mass center of the vehicle. The treatment of the complexities arising in determining the absolute motion of the vehicle is facilitated by the matrix-tensor methods presented in this dissertation.

Two essential aspects recur in studies of this nature. One is the determination of the various stability derivatives by wind tunnel testing where the model motion is restricted by the mechanical constraints of the mounting structure; the other is the study of the motion of a vehicle in free space acted upon by aerodynamic, propulsive, inertial, and gravitational forces. An analytical procedure for both of these aspects is presented in this chapter.

Coordinate Systems

Three coordinate systems are useful in aerospace vehicle motion analyses: inertial coordinates, stability coordinates, and principal axis coordinates. The first system serves well to describe the inertial forces; the second, aerodynamic forces; the third, the inertia properties of the vehicle. In this dissertation, each entity will be defined accordingly. Then, all quantities are transformed into a common frame of reference.

All of the above properties are defined in relation to three right-handed orthogonal axes. The inertial axes are fixed to the earth, which is assumed flat and non-rotating. The stability axes are fixed to the vehicle with the x^1 axis aligned in the initial direction of motion of the vehicle mass center. The air is considered motionless with respect to the earth. The principal axes are fixed to the mass center of the vehicle.

The orientation of the stability axes relative to the inertial axes is defined by three successive Euler rotation angles. It should be noted that the "successive" Euler angles are not the same as the "repetitive" Euler angles used in classical mechanics, where they, by repeating one rotation, result in a "line of nodes." Similarly, the orientation of the principal axes with respect to the stability axes is defined by a second set of three Euler angles.

Because of the complexity of the aerodynamic forces, the coordinates describing the motion of atmospheric flight vehicles are usually based on the stability axes. For space vehicles, where aerodynamic forces are neglected, coordinates based on principal axes may be advantageously employed. In this dissertation, two sets of such coordinates are utilized, each set containing six components.

The first set, which will be called "inertial-Eulerian coordinates," consists of three displacement coordinates along the inertial axes, and three coordinates representing the Euler angles. While the three displacement coordinates are stationary coordinates, those representing the Euler angles are not. In the tensor notation, the inertial-Eulerian coordinates are denoted by the indices l,m,n.

The second set contains the "stability coordinates" consisting of displacements and rotations referred to the three stability axes. All these coordinates are moving coordinates since they are fixed to the moving vehicle. The inertial characteristics are defined with respect to the principal axes and then transformed into the stability coordinate system by the second set of Euler angles. These angles are constants since the inertial properties are considered time-independent. The stability coordinates are denoted in tensor notation by the indices i,j,k.

The relation of the stability coordinates x^i to the inertial-Eulerian coordinates x^l is given by

$$x^l = x^l(x^i) \quad . \tag{3.1}$$

Differentiating with respect to time,

$$\frac{dx^l}{dt} = \frac{\partial x^l}{\partial x^i}\frac{dx^i}{dt} \text{ , or}$$

$$\dot{x}^l = \frac{\partial x^l}{\partial x^i}\dot{x}^i \quad . \tag{3.2}$$

By defining $\partial x^l/\partial x^i = C_i^l$,

$$\dot{x}^l = C_i^l \dot{x}^i \text{ ,} \tag{3.3}$$

where C_i^l is a "transformation tensor," according to Kron.

When $i, l = 1,2,3$, the $\dot{x}^l$ are related to the $\dot{x}^i$ by the successive Euler rotation

$$C^l_i = C^l_b \, C^b_a \, C^a_i \; . \qquad (3.4)$$

In matrix form, these tensors can be written

$$\left[C^a_i\right] = \begin{array}{l} {}_{i}\diagdown^{a} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{bmatrix} \end{array} , \qquad (3.5)$$

$$\left[C^b_a\right] = \begin{array}{l} {}_{a}\diagdown^{b} \\ \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \end{array} , \qquad (3.6)$$

$$\left[C^l_b\right] = \begin{array}{l} {}_{b}\diagdown^{l} \\ \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} . \qquad (3.7)$$

Performing the operations in Equation (3.4) by matrix multiplication, and abbreviating the sine by "s" and the cosine by "c",

$$\left[C^l_i\right] = \begin{array}{l} {}_{i}\diagdown^{l} \\ \begin{bmatrix} c\theta c\psi & c\theta s\psi & -s\theta \\ s\varphi s\theta c\psi - c\varphi s\psi & s\varphi s\theta s\psi - c\varphi c\psi & s\varphi c\theta \\ s\theta c\varphi c\psi + s\varphi s\psi & s\theta c\varphi s\psi - s\varphi c\psi & c\varphi c\theta \end{bmatrix} \end{array} , \; (i, l = 1,2,3) \qquad (3.8)$$

which is an orthogonal transformation since

$$C^l_j \, C^i_l = \delta^i_j \; . \qquad (3.9)$$

Because of the orthogonality of C_i^l (i,l = 1,2,3), the inverse C_l^i is merely the transpose of C_i^l .

When i,l = 4,5,6, the elements, C_i^l, are more difficult to obtain. In this case, the x^l coordinates represent non-orthogonal Eulerian rotations, whereas the x^i are rotations about the orthogonal stability axes. These elements are easily derived by considering a point P (representing the end point of a vector) as fixed with respect to the x^l system. Thus,

$$x^i = C_l^i \, x^l \text{ , and}$$

$$\frac{dx^i}{dt} = C_l^i \frac{dx^l}{dt} + \frac{dC_l^i}{dt} x^l \ .$$

But, since P is fixed with respect to x^l, $dx^l/dt = 0$, and it follows that

$$\dot{x}^i = \frac{dC_l^i}{dt} x^l \ .$$

Recalling that

$$x^l = C_j^l \, x^j \ ,$$

the velocities are

$$\dot{x}^i = \frac{dC_l^i}{dt} C_j^l \, x^j \ . \tag{3.10}$$

Defining

$$E_j^i = \frac{dC_l^i}{dt} C_j^l \ , \tag{3.11}$$

which may be interpreted as an angular rotation tensor,

$$\dot{x}^i = E_j^i \, x^j \ . \tag{3.12}$$

The evaluation of the components of E^i_j is simplified if the skew-symmetry of the E^i_j is recognized by taking the time derivative of both sides of Equation (3.9).

$$\frac{d}{dt}(C^1_j\, C^i_1) = C^1_j \frac{dC^i_1}{dt} + C^i_1 \frac{dC^1_j}{dt} = 0 \ .$$

Rearranging terms,

$$C^1_j \frac{dC^i_1}{dt} = -\, C^i_1 \frac{dC^1_j}{dt} \ . \tag{3.13}$$

Therefore, because of this skew-symmetry, the components of E^i_j on the main diagonal vanish, and only three independent components need to be computed.

To evaluate these independent components of E^i_j, the summations of Equation (3.11) are performed, resulting in

$$E^i_j = C^1_j \frac{dC^i_1}{dt} + C^2_j \frac{dC^i_2}{dt} + C^3_j \frac{dC^i_3}{dt} \ . \tag{3.14}$$

Introducing then the C^1_i from Equation (3.8), the following results are obtained:

$$E^2_1 = C^1_1 \frac{dC^2_1}{dt} + C^2_1 \frac{dC^2_2}{dt} + C^3_1 \frac{dC^2_3}{dt}$$

$$\begin{aligned} E^2_1 = {} & c\theta c\psi\,(-s\varphi s\theta s\psi\dot{\psi} + s\theta c\psi c\varphi\dot{\varphi} + s\varphi c\psi c\theta\dot{\theta} - c\varphi c\psi\dot{\psi} + s\psi s\varphi\dot{\varphi}) \\ & c\theta s\psi\,(s\varphi s\theta c\psi\dot{\psi} + s\theta s\psi c\varphi\dot{\varphi} + s\varphi s\psi c\theta\dot{\theta} - c\varphi s\psi\dot{\psi} - c\psi s\varphi\dot{\varphi}) \\ & -s\theta\,(-s\varphi s\theta\dot{\theta} + c\theta c\varphi\dot{\varphi}) \end{aligned}$$

$$E^3_1 = C^1_1 \frac{dC^3_1}{dt} + C^2_1 \frac{dC^3_2}{dt} + C^3_1 \frac{dC^3_3}{dt}$$

$$E_1^3 = c\theta c\psi(-s\theta c\varphi s\psi\dot{\psi}+c\varphi c\psi c\theta\dot{\theta}-s\theta c\psi s\varphi\dot{\varphi}+s\varphi c\psi\dot{\psi}+s\psi c\varphi\dot{\varphi})$$
$$+c\theta s\psi(s\theta c\varphi c\psi\dot{\psi}+c\varphi s\psi c\theta\dot{\theta}-s\theta s\psi s\varphi\dot{\varphi}+s\varphi s\psi\dot{\psi}-c\psi c\varphi\dot{\varphi})$$
$$-s\theta(-c\varphi s\theta\dot{\theta}-c\theta s\varphi\dot{\varphi})$$

$$E_2^3 = C_1^1 \frac{dC_1^3}{dt} + C_1^2 \frac{dC_2^3}{dt} + C_1^3 \frac{dC_3^3}{dt}$$

$$E_2^3 = (s\varphi s\theta c\psi-c\varphi s\psi)(-s\theta c\varphi s\psi\dot{\psi}+c\varphi c\psi c\theta\dot{\theta}-s\theta c\psi s\varphi\dot{\varphi}+s\varphi c\psi\dot{\psi}+s\psi c\varphi\dot{\varphi})$$
$$+(s\varphi s\theta s\psi+c\varphi c\psi)(s\theta c\varphi c\psi\dot{\psi}+c\varphi s\psi c\theta\dot{\theta}-s\theta s\psi s\varphi\dot{\varphi}+s\varphi s\psi\dot{\psi}-c\psi c\varphi\dot{\varphi})$$
$$+(s\varphi c\theta)(-c\varphi s\theta\dot{\theta}-c\theta s\varphi\dot{\varphi})$$

Or in simplified form,

$$E_1^2 = \dot{\theta}s\varphi - c\varphi c\theta\dot{\psi}$$

$$E_1^3 = \dot{\psi}c\theta s\varphi + \dot{\theta}c\varphi$$

$$E_2^3 = \dot{\psi}s\theta - \dot{\varphi}$$

From the skew-symmetry of E_j^i ,

$$E_1^1 = E_2^2 = E_3^3 = 0$$

$$E_2^1 = -E_1^2 = \dot{\psi}c\varphi c\theta - \dot{\theta}s\varphi$$

$$E_3^1 = -E_1^3 = -\dot{\psi}c\theta s\varphi - \dot{\theta}c\varphi$$

$$E_3^2 = -E_2^3 = \dot{\varphi} - \dot{\psi}s\theta$$

Displaying these results in matrix form,

$$\left[E_j^i\right] = \left[\begin{array}{c|c|c} 0 & \dot{\theta}s\varphi - \dot{\psi}c\varphi c\theta & \dot{\psi}c\theta s\varphi + \dot{\theta}c\varphi \\ \dot{\psi}c\varphi c\theta - \dot{\theta}s\varphi & 0 & \dot{\psi}s\theta - \dot{\varphi} \\ -\dot{\psi}c\theta s\varphi - \dot{\theta}c\varphi & \dot{\varphi} - \dot{\psi}s\theta & 0 \end{array}\right] \qquad (3.15)$$

The corresponding angular velocities are defined with respect to the x^i coordinates by the following array:

$$\left[E^i_j\right] = \begin{bmatrix} 0 & -\dot{x}^6 & \dot{x}^5 \\ \dot{x}^6 & 0 & -\dot{x}^4 \\ -\dot{x}^5 & \dot{x}^4 & 0 \end{bmatrix} . \qquad (3.16)$$

The $\dot{x}^l$ $(l = 4,5,6)$ correspond to the Eulerian angular velocities $\dot{\varphi}$, $\dot{\theta}$, and $\dot{\psi}$. The angular velocities with respect to the x^i coordinates, $\dot{x}^4$, $\dot{x}^5$, $\dot{x}^6$, correspond to p, q, r in conventional aerodynamics notation. By comparing terms of Equations (3.15) and (3.16), the relationships between $\dot{x}^i$ and $\dot{x}^l$ can be expressed as

$$\begin{bmatrix} \\ \dot{x}^i \\ \\ \end{bmatrix} = \begin{bmatrix} 1 & 0 & -s\theta \\ 0 & c\varphi & c\theta s\varphi \\ 0 & -s\varphi & c\varphi c\theta \end{bmatrix} \begin{bmatrix} \\ \dot{x}^l \\ \\ \end{bmatrix} ,$$

or inversely,

$$\begin{bmatrix} \\ \dot{x}^l \\ \\ \end{bmatrix} = \begin{bmatrix} 1 & s\varphi t\theta & c\varphi t\theta \\ 0 & c\varphi & -s\varphi \\ 0 & s\varphi/c\theta & c\varphi/c\theta \end{bmatrix} \begin{bmatrix} \\ \dot{x}^i \\ \\ \end{bmatrix} . \qquad (3.17)$$

Therefore,

$$\left[C^l_i\right] = \begin{bmatrix} 1 & s\varphi t\theta & c\varphi t\theta \\ 0 & c\varphi & -s\varphi \\ 0 & s\varphi/c\theta & c\varphi/c\theta \end{bmatrix} . \; (i,l = 4,5,6) \qquad (3.18)$$

Combining Equations (3.8) and (3.18) the complete transformation tensor C_i^1 is

$$\left[C_i^1\right] = \begin{array}{cc} & 1 \\ i & \left[\begin{array}{c|c|c|c|c|c} c\theta c\varphi & c\theta s\psi & -s\theta & & & \\ s\varphi s\theta c\psi - c\varphi s\psi & s\varphi s\theta s\psi + c\varphi c\psi & s\varphi c\theta & & 0 & \\ s\theta c\varphi c\psi + s\varphi s\psi & s\theta c\varphi s\psi - s\varphi c\psi & c\varphi c\theta & & & \\ \hline & & & 1 & s\varphi t\theta & c\varphi t\theta \\ & 0 & & 0 & c\varphi & -s\varphi \\ & & & 0 & s\varphi/c\theta & c\varphi/c\theta \end{array}\right] \end{array} \qquad (3.19)$$

Most of the elements of C_i^1 are trigonometric functions of the Euler angles and therefore non-linear unless they are linearized. However, since Equation (3.2) may be interpreted as defining relations valid at a point in a "tangent space," the C_i^1 may be thought of as constants at a particular point in space. Of course, the value of these constants may differ at different points; however, in a stability and control study, the interest centers on small perturbation behavior about a particular reference point. The behavior at other points is considered an independent problem.

These relations may be linearized by assuming each Euler angle to be the sum of a reference, or steady-state, angle and a perturbation angle. For example,

$$\theta = \theta_o + d\theta .$$

Thus,

$$\sin\theta = \sin(\theta_o + d\theta) = \sin\theta_o \cos d\theta + \cos\theta_o \sin d\theta .$$

Assuming the perturbation quantities to be sufficiently small to neglect products of the perturbation quantities, then,

$$\sin d\theta = d\theta,\ \cos d\theta = 1\ ;$$

$$\sin \theta = \sin \theta_o + d\theta \cos \theta_o \quad . \tag{3.20}$$

Similarly,

$$\cos \theta = \cos(\theta_o + d\theta) = \cos \theta_o \cos d\theta - \sin \theta_o \sin d\theta$$

$$\cos \theta = \cos \theta_o - d\theta \sin \theta_o \ . \tag{3.21}$$

$$\sin \varphi = \sin \varphi_o + d\varphi \cos \varphi_o \ . \tag{3.22}$$

$$\cos \varphi = \cos \varphi_o - d\varphi \sin \varphi_o \ . \tag{3.23}$$

$$\sin \psi = \sin \psi_o + d\psi \cos \psi_o \ . \tag{3.24}$$

$$\cos \psi = \cos \psi_o - d\psi \sin \psi_o \ . \tag{3.25}$$

If these simplifying relations are introduced into Equation (3.8), a set of linear transformation tensors is obtained.

An alternate method which results in considerable simplification places the inertial axes at the mass center of the vehicle and orients it such that at the instant under consideration the inertial-Eulerian and the stability axes coincide. Although this method is appropriate for stability and control studies where the direction of flight is unimportant, it is impractical for a trajectory or flight path analysis.

In this case, the reference components of the Euler angles vanish ($\varphi_o = \theta_o = \psi_o = 0$) and the following simplifications are possible:

$$\sin\varphi = d\varphi;\ \cos\varphi = 1\ ;$$

$$\sin\theta = d\theta;\ \cos\theta = 1\ ;$$

$$\sin\psi = d\psi;\ \cos\psi = 1\ .$$

If these approximations are substituted into Equation (3.8) and products of perturbation quantities neglected, a linearized transformation tensor involving infinitesimals results:

$$\left[C^{1}_{i}\right] = {}_{i}\diagdown^{1}\left[\begin{array}{ccc|ccc} 1 & d\psi & -d\theta & & & \\ -d\psi & 1 & d\varphi & & 0 & \\ d\theta & -d\varphi & 1 & & & \\ \hline & & & 1 & 0 & d\theta \\ & 0 & & 0 & 1 & -d\varphi \\ & & & 0 & d\varphi & 1 \end{array}\right] \tag{3.26}$$

Similarly, the inverse transformation C^{i}_{1} can be found as

$$\left[C^{i}_{1}\right] = {}_{1}\diagdown^{i}\left[\begin{array}{ccc|ccc} 1 & -d\psi & d\theta & & & \\ d\psi & 1 & -d\varphi & & 0 & \\ -d\theta & d\varphi & 1 & & & \\ \hline & & & 1 & 0 & -d\theta \\ & 0 & & 0 & 1 & d\varphi \\ & & & 0 & -d\varphi & 1 \end{array}\right]\ . \tag{3.27}$$

The Inertia Tensor

The inertial properties of the vehicle are described in terms of mass for the translational displacements, and in terms of mass moments and products of inertia for rotational displacements, which generally are functions of the Euler angles. While the inertial properties are independent of the kinematic variables, in some cases a variation with time is possible (e.g., rocket vehicles with time-varying masses). Here, the inertial properties are considered as time-independent.

The power of the tensor concept can be demonstrated by the following method of obtaining the moments and products of inertia with respect to any arbitrary axes. The essential feature is to recognize the invariance of the inertia tensor under all allowable coordinate transformations. Rectangular Cartesian coordinates will be utilized in the derivation although the results may be transformed into any other coordinate system. Restricted versions of these relations without derivations are often found in the literature when only two axes are coincident with the principal axes (see Ref. 73, p. 104). In this dissertation, a general tensor derivation of the inertia tensor components is given when none of the three reference axes coincide with the principal axes. The results are identical (with appropriate changes in symbols, of course) to those presented by Whittaker (5) by a more tedious summation process.

The relative orientation of the principal axes x^p (p = 1,2,3) with respect to the stability axes x^i (i = 1,2,3) is expressed by a second set of successive Euler angles γ, α, β.

The relations between the displacements x^p and x^i are given by the Euler rotation matrix derived previously (using only the first three coordinates)

$$x^p = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta & -s\alpha \\ s\gamma s\alpha c\beta - c\gamma s\beta & s\gamma s\alpha s\beta + c\gamma c\beta & s\gamma c\alpha \\ s\alpha c\gamma c\beta + s\gamma s\beta & s\alpha c\gamma s\beta - s\gamma c\beta & c\gamma c\alpha \end{bmatrix} \begin{bmatrix} x^i \end{bmatrix} . \tag{3.28}$$

The inertia tensor is a second order covariant tensor and obeys the tensor transformation formula

$$I_{ij} = I_{pq} \frac{\partial x^p}{\partial x^i} \frac{\partial x^q}{\partial x^j} = I_{pq} C_i^p C_j^q . \tag{3.29}$$

Since the products of inertia vanish in principal axis coordinates, the only non-zero components occur when p = q. Therefore,

$$I_{ij} = I_{pp} C_i^p C_j^p , \tag{3.30}$$

or in expanded form

$$I_{ij} = I_{11} C_i^1 C_j^1 + I_{22} C_i^2 C_j^2 + I_{33} C_i^3 C_j^3 . \tag{3.31}$$

The symmetry of the inertia tensor is evident from the form of Equation (3.29). Thus,

$$I_{ij} = I_{ji} . \tag{3.32}$$

When $i \neq j$, Equation (3.31) gives the products of inertia; when $i = j$, the moments of inertia. By substituting appropriate partial derivatives obtained from Equation (3.28), the components of the inertia tensor become (indicating the principal moments of inertia by bars)

$$I_{11} = \bar{I}_{11}(c\alpha c\beta)^2 + \bar{I}_{22}(c\alpha s\beta)^2 + \bar{I}_{33}(-s\alpha)^2 . \tag{3.33}$$

$$I_{22} = \bar{I}_{11}(s\gamma s\alpha c\beta - c\gamma s\beta)^2 + \bar{I}_{22}(s\gamma s\alpha s\beta + c\gamma c\beta)^2 + \bar{I}_{33}(s\gamma c\alpha)^2 . \tag{3.34}$$

$$I_{33} = \bar{I}_{11}(c\gamma s\alpha c\beta + s\gamma s\beta)^2 + \bar{I}_{22}(c\gamma s\alpha s\beta - s\gamma c\beta)^2 + \bar{I}_{33}(c\gamma c\alpha)^2 . \tag{3.35}$$

$$I_{21} = I_{12} = \bar{I}_{11}(c\alpha c\beta)(s\gamma s\alpha c\beta - c\gamma s\beta) + \bar{I}_{22}(c\alpha s\beta)(s\gamma s\alpha s\beta + c\gamma c\beta) + \bar{I}_{33}(-s\alpha)(s\gamma c\alpha) . \tag{3.36}$$

$$I_{31} = I_{13} = \bar{I}_{11}(c\alpha c\beta)(c\gamma s\alpha c\beta + s\gamma s\beta) + \bar{I}_{22}(c\alpha s\beta)(c\gamma s\alpha s\beta - s\gamma c\beta) + \bar{I}_{33}(-s\alpha)(c\gamma c\alpha) . \tag{3.37}$$

$$I_{32} = I_{23} = \bar{I}_{11}(s\gamma s\alpha c\beta)(c\gamma s\beta)(c\gamma s\alpha c\beta + s\gamma s\beta) + \bar{I}_{22}(s\gamma s\alpha s\beta + c\gamma c\beta)(c\gamma s\alpha s\beta - s\gamma c\beta) + \bar{I}_{33}(s\gamma c\alpha)(c\gamma c\alpha) . \tag{3.38}$$

By setting $\gamma = \beta = 0$, and writing the trigonometric functions in the usual manner,

$$I_{11} = \bar{I}_{11} \cos^2\alpha + \bar{I}_{33} \sin^2\alpha . \tag{3.39}$$

$$I_{22} = \bar{I}_{22} \quad . \tag{3.40}$$

$$I_{33} = \bar{I}_{11} \sin^2\alpha + \bar{I}_{33} \cos^2\alpha \quad . \tag{3.41}$$

$$I_{12} = I_{23} = 0 \quad . \tag{3.42}$$

$$I_{13} = I_{31} = \bar{I}_{11} \sin\alpha \cos\alpha + \bar{I}_{33}(-\sin\alpha)\cos\alpha =$$

$$= \tfrac{1}{2} (\bar{I}_{11} - \bar{I}_{33}) \sin 2\alpha \quad . \tag{3.43}$$

These equations are the same as those given by Etkin (73) without derivation.

If Equation (3.29) rather than the simplified Equation (3.30) is used, the technique will relate the components of the inertia tensor in any two arbitrary axis systems. It should be noted that these tensor relations allow the computation of a single component without the need of deriving the complete set of expressions.

In the case where only one of the stability axes is not coincident with the principal axes, the inertia tensor in matrix form is given by

$$M_{ij} = \left[\begin{array}{ccc|ccc} m & & & & & \\ & m & & & 0 & \\ & & m & & & \\ \hline & & & I_{11} & 0 & I_{13} \\ & 0 & & 0 & I_{22} & 0 \\ & & & I_{31} & 0 & I_{33} \end{array}\right] \tag{3.44}$$

where the elements in the lower right quadrant are given by Equations (3.39) through (3.43).

Consequently, Bewley's statement that "there is no such thing as a 'mutual' moment of inertia" (see Ref. 82, p. 172) is incorrect. The products of inertia, which are the off-diagonal components of the inertia tensor, are "mutual" moments of inertia and truly analogous to mutual inductances in electrical network theory.

Aerodynamic Forces

Three types of aerodynamic forces are usually considered in vehicle motion studies: forces proportional to displacements, to velocities, and to accelerations of the vehicle with respect to the fluid mass. The latter forces increase the effective mass of the vehicle; they are usually neglected, but included here.

Since a slight abstraction is used in this dissertation to emphasize the unity of the concepts, no distinction will ordinarily be made between forces and torques. A torque is simply a "rotational force" in one of the n dimensions of the space which corresponds to a rotational coordinate. Similarly, a displacement can be either translational or rotational depending on which of the n dimensions is being considered. Of course, all n dimensions are handled in an identical manner, irrespective of the nature of the particular dimension.

It is customary in aerodynamics to use non-dimensional stability derivatives to describe the forces acting on a particular vehicle. Force coefficients (C_F) and moment coefficients (C_M) are conventionally defined as

$$C_F = F/qS \ , \tag{3.45}$$

$$C_M = M/qSl \ , \tag{3.46}$$

where F is the aerodynamic force in pounds, M the aerodynamic moment in foot-pounds, q is the "dynamic pressure" in pounds per square foot, S is a reference area (usually the wing area for winged vehicles and the cross-sectional area for non-winged vehicles), and l is a reference length (e.g., the mean aerodynamic chord for pitching, the wing semi-span for yawing or rolling, the body diameter or length for non-winged vehicles). The dynamic pressure is given by

$$q = \tfrac{1}{2}\, \rho\, V^2, \ \ (V^2 = \dot{x}^i\, \dot{x}^i \ , \ i = 1,2,3) \tag{3.47}$$

where ρ is the density of the fluid in slugs per cubic foot and V is the true velocity of the fluid in feet per second relative to the vehicle.

In contrast to common practice, no distinction is made in this dissertation between force and torque since a torque is merely a "generalized force" in one of the n dimensions of the space considered. Similarly, translations and rotations are "generalized displacements" in one of the n dimensions of space. Thus, moment and force coefficients are defined by the single relation

$$C_{F_i} = \frac{F_i}{q\,\bar{S}} \ , \tag{3.48}$$

where $\bar{S}$ is a "shape reference" which is interpreted as a reference area for translational coordinates (i = 1,2,3) and as a reference area-moment for rotational coordinates (i = 4,5,6).

The conventional "stability derivatives" are then defined as the partial derivatives of the force and moment coefficients with respect to the coordinate displacements, velocities, and accelerations. The partial derivatives with respect to displacements are conventionally called "static stability derivatives," while those taken with respect to velocities and accelerations are "dynamic stability derivatives." For a particular analysis, the stability derivatives are often thought of as constants and the results considered valid for small perturbations of the variables.

In the literature, the usual symbols for forces, F_i (i = 1,2,...6), are X, Y, Z, L, M, N, and those for velocities, $\dot{x}^i$ (i = 1,2,...6), u, v, w, p, q, r, respectively.

Then the generalized force expression is

$$F_i = C_{F_i} \tfrac{1}{2} \rho \bar{S} (u^2 + v^2 + w^2) , \qquad (3.49)$$

or, if the velocities u, v, and w, are decomposed into steady-state or reference values, u_o, v_o, and w_o, and perturbation quantities, $\bar{u}$, $\bar{v}$, and $\bar{w}$,

$$F_i = C_{F_i} \tfrac{1}{2} \rho \bar{S}[(u_o + \bar{u})^2 + (v_o + \bar{v})^2 + (w_o + \bar{w})^2] .$$

Since with reference to the stability coordinates, $v_o = w_o = 0$,

$$F_i = C_{F_i} \tfrac{1}{2} \rho \bar{S}(u_o{}^2 + u_o\bar{u} + \bar{u}^2 + \bar{v}^2 + \bar{w}^2)$$

$$= C_{F_i} \tfrac{1}{2} \rho \bar{S}u_o{}^2(1 + \frac{\bar{u}}{u_o} + \frac{\bar{u}^2}{u_o^2} + \frac{\bar{v}^2}{u_o^2} + \frac{\bar{w}^2}{u_o^2}) \; .$$

Neglecting all second order terms of the perturbation quantities,

$$F_i = C_{F_i} \tfrac{1}{2} \rho \bar{S} u_o^2 \; (1 + \frac{\bar{u}}{u_o}) \; . \tag{3.50}$$

Thus, even in a linearized version, the F_i depends on $\bar{u}$; C_{F_i} is a constant, or a function of $\bar{u}$ if compressibility effects are significant. In conventional stability analyses, the assumption $\bar{u} = 0$ is made, which renders F_i independent of $\bar{u}$.

In this dissertation, in place of the conventional non-dimensional stability derivatives, three "aerodynamic tensors" will be defined and utilized: an aerodynamic stiffness tensor, an aerodynamic damping tensor, and an aerodynamic inertia tensor. Whereas Kondo (52) uses Craig's (28) extensor concept for defining equivalents of these aerodynamic entities, this writer feels that the use of three conventional tensors is more appropriate although the elegance of the extensor concept is not denied. The components of these aerodynamic tensors are merely the partial derivatives of the forces with respect to the displacements, velocities, and accelerations. For denoting the tensor components, conventional aerodynamic symbols are used, e.g., X_u for the partial derivative of F_1 (i.e., X) with respect to $\dot{x}^1$ (i.e., u). The tensor notation for the same component is B_{11}.

Since aerodynamic force data are usually obtained from wind tunnel tests, the force components are usually measured with respect to either the stability or the body axes. Conventional wind tunnel balance systems employed in low speed tests orient the stability axes according to the direction of the free stream. High speed tests require a "sting mounting" downstream of the model and force data are obtained with respect to a set of body axes by a strain gage technique. Since the aerodynamic forces have tensor properties, the same transformation techniques used for the inertia tensor apply for the transformation of the aerodynamic tensor. For example, the components of the aerodynamic force tensor, which may be thought of as dimensionalized stability derivatives, are transformed from the stability axes to the principal axes by

$$A_{pq} = A_{ij} \, C^i_p \, C^j_q \, , \tag{3.51}$$

where the components of the aerodynamic force tensor are denoted by the symbol "A_{ij}." The C^i_p are inverses of the C^p_i derived in the previous section for the inertia tensor.

The utilization of Lagrangean concepts requires the definition of an aerodynamic kinetic energy function

(T_A), an aerodynamic potential energy function (V_A), and an aerodynamic dissipation function (F_A):

$$T_A = \tfrac{1}{2} A_{ij} \dot{x}^i \dot{x}^j , \tag{3.52}$$

$$V_A = \tfrac{1}{2} K_{ij} x^i x^j , \tag{3.53}$$

$$F_A = \tfrac{1}{2} B_{ij} \dot{x}^i \dot{x}^j , \tag{3.54}$$

where

$$A_{ij} = \partial F_i / \partial \ddot{x}^j , \quad \text{(aerodynamic inertia tensor)}$$

$$K_{ij} = \partial F_i / \partial x^j , \quad \text{(aerodynamic stiffness tensor)}$$

$$B_{ij} = \partial F_i / \partial \dot{x}^j , \quad \text{(aerodynamic damping tensor)}$$

For the general problem considered here, the three aerodynamic tensors will have 36 components each. These three individual tensors may be combined into a single "aerodynamic admittance tensor" defined as

$$Y_{ij} = A_{ij} s + B_{ij} + K_{ij}/s . \tag{3.55}$$

In free flight, the effects due to aerodynamic and inertial accelerations cannot be separated. For this reason, it is convenient to consider the sum of the aerodynamic and vehicle inertia as a single "effective inertia." Denoting this effective inertia by $\bar{M}_{ij}$, a "vehicle admittance" $\bar{Y}_{ij}$ may be expressed by

$$\begin{aligned} \bar{Y}_{ij} &= (A_{ij} + M_{ij}) s + B_{ij} + K_{ij}/s \\ &= \bar{M}_{ij}\, s + B_{ij} + K_{ij}/s \end{aligned} \tag{3.56}$$

and the vehicle forces become

$$F_i = \bar{Y}_{ij} \dot{x}^j . \tag{3.57}$$

The aerodynamic admittances may be found by "dynamic" wind tunnel testing, or "dynamic" flight testing. In wind tunnel testing, forces are measured from known displacements of the model, while in flight testing the displacements of the vehicle resulting from known applied forces are measured. Therefore, to define completely the admittance tensor, the dynamic response to all of the six generalized forces, or all of the six generalized displacements would have to be obtained. This is not currently done, although the state of the art of dynamic testing procedures in other disciplines would indicate that it is entirely feasible.

The admittance data from wind tunnel tests produce $\bar{M}_{ij}$, but not A_{ij} or M_{ij}. Thus, to obtain A_{ij}, the M_{ij} of the model must be subtracted from the $\bar{M}_{ij}$. Then, the $\bar{M}_{ij}$ of the actual vehicle is established by adding its M_{ij} to A_{ij}.

In practice, the admittance tensor is defined by measuring the response of each of the coordinates to successive sinusoidal or impulsive excitations. Under the simplifying assumptions of a conventional aerodynamic analysis (see Ref. 73, p. 124), the tensors K_{ij}, B_{ij}, A_{ij}, and Y_{ij} in matrix notation assume the form

$$\left[K_{ij}\right] = {}^{i\,\diagdown\,j}\left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & 0 \end{array}\right] \tag{3.58}$$

$$\left[B_{ij}\right] = {}^{i\,\diagdown\,j}\left[\begin{array}{ccc|ccc} X_u & 0 & X_w & 0 & 0 & 0 \\ 0 & Y_v & 0 & Y_p & 0 & Y_r \\ Z_u & 0 & Z_w & 0 & Z_q & 0 \\ \hline 0 & L_v & 0 & L_p & 0 & L_r \\ M_u & 0 & M_w & 0 & M_q & 0 \\ 0 & N_v & 0 & N_p & 0 & N_r \end{array}\right] \tag{3.59}$$

$$\left[A_{ij}\right] = {}^{i\,\diagdown\,j}\left[\begin{array}{ccc|c} 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 \\ 0 & 0 & Z_{\dot{w}} & \\ \hline 0 & 0 & 0 & \\ 0 & 0 & M_{\dot{w}} & 0 \\ 0 & 0 & 0 & \end{array}\right] \tag{3.60}$$

$$\left[Y_{ij}\right] = {}^{i\,\diagdown\,j}\left[\begin{array}{ccc|ccc} X_u & 0 & X_w & 0 & 0 & 0 \\ 0 & Y_v & 0 & Y_p & 0 & Y_r \\ Z_u & 0 & (Z_w+Z_{\dot{w}}s) & 0 & Z_q & 0 \\ \hline 0 & L_v & 0 & L_p & 0 & L_r \\ M_u & 0 & (M_w+M_{\dot{w}}s) & 0 & M_q & 0 \\ 0 & N_v & 0 & N_p & 0 & N_r \end{array}\right] \tag{3.61}$$

It may be perplexing that the aerodynamic stiffness tensor has only zero components with reference to the stability coordinates. This is due to the fact that in a conventional aerodynamic analysis, new coordinates are introduced by assuming $\bar{u} = 0$ (thus, $u = u_o$) and by defining

$$\sin \beta = \frac{v}{u_o} \quad , \tag{3.62}$$

$$\sin \alpha = \frac{w}{u_o} \quad . \tag{3.63}$$

When these angles are considered small,

$$\beta = \frac{v}{u_o} \quad , \tag{3.64}$$

$$\alpha = \frac{w}{u_o} \quad . \tag{3.65}$$

Thus, it is apparent that the "displacement" coordinates (α, β) replace the "velocity" coordinates of the former system, and components formerly considered damping components (i.e., proportional to velocities) now become stiffness components (i.e., proportional to displacements).

While each component of the aerodynamic admittance tensor has a functional relation to one or more of the conventionally defined stability derivatives, the establishment of the relations themselves is not necessary for a proper dynamic stability analysis. In fact, the tensor approach used here shows clearly that dynamic testing procedures cannot be used to isolate individual stability derivatives, a fact that has only recently been recognized by investigators in the field. One important advantage of the stipulated testing method is that the dynamic and static components are obtained simultaneously in the same test.

In this dissertation, the order of the matrices is kept small for practical considerations. Additional degrees of freedom may be introduced by merely adding additional sets of coordinates. For example, if control modes were to be considered about each of the three stability axes, three new coordinates representing the control mode displacements would be added, and the Y_{ij} given by a 9 by 9 order matrix. No new relations need be formulated since the tensor relations are valid for n dimensions.

The preceding definition of the admittance tensor, Y_{ij}, for conventional analyses assumes that the fluid density, ρ, is a constant. Generally, however, the density is a function of the inertial displacement coordinates (e.g., a known function of altitude, x^3). Thus, non-zero components of the aerodynamic stiffness tensor appear, since F_i is a function of the displacement coordinates. Of course, since the density function is most easily written in inertial coordinates, it must be transformed into stability coordinates to obtain the correct F_i.

If the flight regime under consideration is such that C_{F_i} is a function of Mach Number as well as the coordinates and their derivatives (e.g., the transonic speed range), provisions for treating these effects can easily be made.

If Equations (3.44) and (3.61) are combined in accordance with Equation (3.56) to obtain $\bar{Y}_{ij}$, and these tensor components are rearranged such that the elements are in the order (1,3,5,2,4,6) instead of (1,2,3,4,5,6), then

$$
\left[\bar{Y}_{ij}\right] = \left[\begin{array}{ccc|ccc}
X_u+ms & X_w & 0 & 0 & 0 & 0 \\
Z_u & Z_w+(Z_{\dot{w}}+m)s & Z_q & 0 & 0 & 0 \\
M_u & M_w+M_{\dot{w}}s & M_q+I_{22}s & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & Y_v+ms & Y_p & Y_r \\
0 & 0 & 0 & L_v & L_p+I_{11}s & L_r+I_{13}s \\
0 & 0 & 0 & N_v & N_p+I_{13}s & N_r+I_{33}s
\end{array}\right] \tag{3.66}
$$

Rearranging the tensor components in this manner shows that no coupling exists between the (1,3,5) coordinates and the (2,4,6) coordinates; therefore each set may be considered independently. This is, of course, the common approach in a conventional stability and control analysis when the longitudinal equations ($i,j = 1,3,5$) are separated from the lateral-directional equations ($i,j = 2,4,6$). It should be noted that this is valid only when coupling components do not exist. Also, only one additional component, X_q, may be added in this system without coupling the coordinates. Additional coupling components, however, would not present any difficulties, excepting the tediousness of the calculations.

Gravitational Forces

The gravitational forces are most easily defined with respect to inertial coordinates x^a whose x^1-x^2 plane is parallel to the earth's surface. Since the gravitational forces, F_{G_a}, act straight down when the earth is considered flat, their components with respect to these coordinates are

$$F_{G_a} = 0, \quad a \neq 3 \;;$$

$$F_{G_a} = mg, \quad a = 3 \;. \tag{3.67}$$

The gravitational force components in the inertial coordinate system whose axes are coincident with the stability axes x^i, are found from

$$F_{G_1} = C_1^a \, F_{G_a} \;, \tag{3.68}$$

where C_1^a is the Euler rotation describing the orientation of the x^1 and x^a coordinates. These angles are considered fixed for a particular problem. Performing the operations in Equation (3.68), the components of the gravitational forces expressed in inertial coordinates x^1 are

$$\left[F_{G_1}\right] = {}^{1\diagdown a}\left[\begin{array}{ccc|ccc}
c\Theta c\Psi & c\Theta s\Psi & -s\Theta & & & \\
s\Phi s\Theta c\Psi - c\Phi s\Psi & s\Phi s\Theta s\Psi + c\Phi c\Psi & s\Phi c\Theta & & 0 & \\
s\Theta c\Phi c\Psi + s\Phi s\Psi & s\Theta c\Phi s\Psi - s\Phi c\Psi & c\Phi c\Theta & & & \\
\hline
& & & 1 & s\Phi t\Theta & c\Phi t\Theta \\
& 0 & & 0 & c\Phi & -s\Phi \\
& & & 0 & s\Phi / c\Theta & c\Phi / c\Theta
\end{array}\right]
\begin{bmatrix} 0 \\ 0 \\ mg \\ 0 \\ 0 \\ 0 \end{bmatrix} .$$

$$\left[F_{G_1}\right] = \begin{bmatrix} -mg & \sin\Theta \\ mg & \sin\Phi\cos\Theta \\ mg & \cos\Phi\cos\Theta \\ & 0 \\ & 0 \\ & 0 \end{bmatrix} . \quad (3.69)$$

The Equations of Motion

In the most general case of a non-holonomic system, the equations of motion in any coordinate system are given by the Boltzman-Hamel equation (Equation [2.72]), which is re-stated here for convenience:

$$f_i = \bar{M}_{ij} s\,\dot{x}^j + B_{ij}\,\dot{x}^j + K_{ij}\,\dot{x}^j/s + \Gamma_{jk,i}\,\dot{x}^k\,\dot{x}^j . \quad (2.72)$$

If the terms are grouped as

$$f_i = (\bar{M}_{ij}\,s + B_{ij} + K_{ij}/s + \Gamma_{jk,i}\,\dot{x}^k)\,\dot{x}^j$$

and $\bar{Y}_{ij}$ is defined by

$$\bar{Y}_{ij} \equiv \bar{M}_{ij}\,s + B_{ij} + K_{ij}/s + \Gamma_{jk,i}\,\dot{x}^k , \quad (3.70)$$

then

$$f_i = \bar{Y}_{ij}\,\dot{x}^j . \quad (3.71)$$

The small perturbation equations of motion follow from Equation (3.71) by differentiating both sides,

$$df_i = \bar{Y}_{ij}\,d\dot{x}^j + d\bar{Y}_{ij}\,\dot{x}^j .$$

Since the $\bar{Y}_{ij}$ are functions only of the $\dot{x}^k$,

$$df_i = \bar{Y}_{ij}\, d\dot{x}^j + \frac{\partial \bar{Y}_{ij}}{\partial \dot{x}^k}\, d\dot{x}^k\, \dot{x}^j \ .$$

Evaluating the partial derivative in the above equation yields

$$df_i = \bar{Y}_{ij}\, d\dot{x}^j + \Gamma_{jk,i}\, d\dot{x}^k\, \dot{x}^j \ ,$$

which, after rearranging the dummy indices, takes the form

$$df_i = \bar{Y}_{ij}\, d\dot{x}^j + \Gamma_{kj,i}\, \dot{x}^k\, d\dot{x}^j \ .$$

Introducing $\bar{Y}_{ij}$ from Equation (3.70),

$$df_i = (\bar{M}_{ij}s + B_{ij} + K_{ij}/s + \Gamma_{jk,i}\, \dot{x}^k + \Gamma_{kj,i}\, \dot{x}^k)\, d\dot{x}^j \ . \tag{3.72}$$

or, denoting the terms inside the parenthesis as $\bar{\bar{Y}}_{ij}$,

$$df_i = \bar{\bar{Y}}_{ij}\, d\dot{x}^j \ . \tag{3.73}$$

Following Kron's terminology in his analysis of electrical machinery (20), $\bar{\bar{Y}}_{ij}$ is called the "transient admittance." The definition of $\bar{\bar{Y}}_{ij}$ implies that if measurements are taken with respect to coordinate systems in which the $\Gamma_{jk,i}$ do not vanish, $\dot{x}^k$ must be held constant. Thus, if measurements are made in inertial-Eulerian coordinates, a "constant speed" evaluation of $\bar{\bar{Y}}_{ij}$ is required. However, it is evident that if $\dot{x}^k = 0$,

$$\bar{\bar{Y}}_{ij} = \bar{Y}_{ij} \ . \tag{3.74}$$

These relations suggest that the components of $\bar{\bar{Y}}_{ij}$ could be evaluated in a wind tunnel by rotating the model at a constant $\dot{x}^k$ (k = 4,5,6) and by measuring its response to an excitation. With the $\bar{\bar{Y}}_{ij}$ thus determined, the stability, transient, and steady-state characteristics of a vehicle may be found by routine techniques.

In electrical machinery analyses, the $\dot{x}^k$ is referred to as the "synchronous speed"; this corresponds to the "equilibrium rolling, yawing, or pitching velocity" in airplane dynamics.

The dynamic stability of the system is determined by evaluating the roots of the "characteristic" or "secular" equation,

$$\left| \bar{\bar{Y}}_{ij} \right| = 0 \quad . \tag{3.75}$$

If the question is not the degree of stability or instability but rather whether or not the vehicle is simply stable, then Routh's stability criterion (1) is applicable. This criterion establishes the number of roots with positive real parts which, of course, corresponds to the number of unstable modes of motion. This procedure is well known and need not be discussed further here.

The transient response is obtained by setting $s = d/dt$ and solving the resulting set of linear differential equations and the steady-state response to a sinusoidal excitation is obtained by setting $s = j\omega$, where ω is the frequency of the excitation, and $j = \sqrt{-1}$.

The dual form of Equation (3.73)

$$d\dot{x}^j = \bar{\bar{Z}}^{ij}\, df_i \tag{3.76}$$

provides the motional response to applied forces (as in flight testing). The $\bar{\bar{Z}}^{ij}$ is properly denoted as the "transient impedance."

Since $\Gamma_{jk,i}$ vanishes in the stability coordinate system,

$$\bar{\bar{Y}}_{ij} = \bar{Y}_{ij}\ , \tag{3.77}$$

and the equations of motion become

$$df_i = \bar{Y}_{ij}\, d\dot{x}^j\ . \tag{3.78}$$

These equations could have been derived from the general tensor form of the small perturbation equations of motion, Equation (2.88), since the Christoffel symbols vanish and $B_{qji} = K_{qji} = K_{pqri} = 0$. Therefore, Equation (3.78) is a tensor equation. $\bar{Y}_{ij}$ is obtained by adding Equations (3.44) and (3.61); thus,

$$\left[\bar{Y}_{ij}\right] = \left[\begin{array}{c|c|c|c|c|c} X_u+ms & 0 & X_w & 0 & 0 & 0 \\ 0 & Y_v+ms & 0 & Y_p & 0 & Y_r \\ Z_u & 0 & Z_w+(Z_{\dot{w}}+m)s & 0 & Z_q & 0 \\ \hline 0 & L_v & 0 & L_p+I_{11}s & 0 & L_r+I_{13}s \\ M_u & 0 & M_w+M_{\dot{w}}p & 0 & M_q+I_{22}s & 0 \\ 0 & N_v & 0 & N_p+I_{13}s & 0 & N_r+I_{33}s \end{array}\right], \tag{3.79}$$

$$\left[dx^j\right] = \begin{bmatrix} u \\ v \\ w \\ p \\ q \\ r \end{bmatrix} . \tag{3.80}$$

To evaluate df_i, the equation

$$f_i = C_i^l \, f_l$$

is differentiated and yields

$$df_i = C_i^l \, df_l + dC_i^l f_l \ .$$

Since the f_l are constants, $df_l = 0$, and

$$df_i = \frac{\partial C_i^l}{\partial x^m} \, dx^m \, f_l \ . \tag{3.81}$$

Evaluating $(\partial C_i^l/\partial x^m)\,dx^m$,

$$\left[\frac{\partial C_i^l}{\partial x^m}\, dx^m\right] = \left[\begin{array}{ccc|ccc} 0 & d\psi & -d\theta & & & \\ -d\psi & 0 & d\varphi & & 0 & \\ d\theta & -d\varphi & 0 & & & \\ \hline & & & 0 & 0 & d\theta \\ & 0 & & 0 & 0 & -d\varphi \\ & & & 0 & d\varphi & 0 \end{array}\right] , \tag{3.82}$$

and upon multiplication by f_l ,

$$\left[df_i\right] = \left[\begin{array}{ccc|ccc} 0 & d\psi & -d\theta & & & \\ -d\psi & 0 & d\varphi & & 0 & \\ d\theta & -d\varphi & 0 & & & \\ \hline & & & 0 & 0 & d\theta \\ & 0 & & 0 & 0 & -d\varphi \\ & & & 0 & d\varphi & 0 \end{array}\right] \left[\begin{array}{c} -mg\ \ s\Theta \\ mg\ s\Phi c\Theta \\ mg\ c\Phi c\Theta \\ 0 \\ 0 \\ 0 \end{array}\right] , \quad (3.83)$$

or more explicitly,

$$\left[df_i\right] = \left[\begin{array}{c} mg(d\psi\ s\Phi c\Theta - d\theta\ c\Phi c\Theta) \\ mg(d\varphi\ c\Phi c\Theta - d\psi\ s\Theta\ ) \\ -mg(d\theta\ s\Theta + d\varphi\ s\Phi c\Theta) \\ 0 \\ 0 \\ 0 \end{array}\right] . \quad (3.84)$$

Therefore, the linearized small perturbation equations of motion are

$$mg(d\psi\ s\Phi c\Theta - d\theta c\Phi c\Theta) = X_u du + ms\ du + X_w dw \quad , (3.85)$$

$$mg(d\varphi\ c\Phi c\Theta - d\psi s\Theta\ ) = Y_v dv + ms\ dv + Y_p dp + Y_r dr \quad , (3.86)$$

$$-mg(d\theta\ s\Theta + d\varphi s\Phi c\Theta) = Z_u du + Z_w dw + (Z_{\dot{w}} + m)s\ dw + Z_q dq , (3.87)$$

$$0 = L_v dv + L_p dp + I_{11} s\ dp + L_r dr + I_{13} s\ dr \quad , (3.88)$$

$$0 = M_u du + M_w dw + M_{\dot{w}} s\ dw + M_q dq + I_{22} s\ dq \quad , (3.89)$$

$$0 = N_v dv + N_p dp + I_{13} s\ dp + N_r dr + I_{33} s\ dr \quad . (3.90)$$

Since no allowance is made for a free control mode, these equations are termed "stick-fixed" or "controls-fixed" equations. The "stick-free" or "controls-free" equations can be developed by merely adding coordinates to represent the desired free control motion. This is illustrated for a longitudinal control in the next section. In the general case, a total of nine coordinates are needed--the six already employed and three new coordinates representing the longitudinal, lateral, and directional controls.

It should be noted that linear graphs were not employed in the developments in this section. In Chapter II, the use of linear graphs was shown to be an expeditious way for determining the transformation tensors in electrical and mechanical problems. The aerospace vehicle problem, however, differs in two important ways: the aerodynamic tensor is neither diagonal nor symmetrical, and the variables are coupled by the Euler rotation relations. Consequently, the transformation tensor is not composed of zeros, plus ones, and minus ones, as in the systems considered in Chapter II, but rather of complicated trigonometric functions of the variables (Equation [3.19]) or linear functions of the variables (Equation [3.26]). Therefore, the linear graph theory, while useful in many cases, is less powerful than the general tensor methods presented here. Trent (55) called this type of coupling by Euler rotations "inverse coupling" and indicated that, since no satisfactory electrical "inverse coupler" had been invented (as of 1955), an exact electrical analogy was impossible. This is one important reason why electronic differential analyzers have pre-empted direct electrical analogies for studies of this type.

Incorporation of a Stabilization System

Equations of motion which include the effects of an arbitrary stabilization system are obtained by a coordinate transformation from the basic system to a new system, denoted here by the indices p, q, and r. To demonstrate this procedure, the appropriate equations of motion of a conventional vehicle with a generalized longitudinal stabilization system are derived. As indicated by the discussion following Equation (3.66), the longitudinal motion of a conventional vehicle may be described with respect to three stability coordinates, $\dot{x}^1$, $\dot{x}^3$, and $\dot{x}^5$ (u, w, and q, resp.). The introduction of a control mode coordinate, $\dot{x}^7$ ($\dot{\delta}$ in conventional aerodynamic analyses), increases the number of necessary coordinates to four. The Y_{ij} therefore becomes, using the standard control parameters,

$$\left[Y_{ij}\right] = \begin{array}{c} {}_{i}\diagdown^{j} \\ \left[\begin{array}{cccc} X_u & X_w & 0 & X_\delta/s \\ Z_u & Z_w+Z_{\dot{w}}s & Z_q & Z_\delta/s \\ M_u & M_w+M_{\dot{w}}s & M_q & M_\delta/s+M_{\dot{\delta}} \\ H_u & H_w+H_{\dot{w}}s & H_q & H_\delta/s+H_{\dot{\delta}} \end{array}\right] \end{array} \qquad (3.91)$$

where δ is the longitudinal control deflection and H is the control hinge moment.

A longitudinal stabilization system which affects the vehicle motion in all the x^i coordinates by displacements of the control surface is defined as "generalized." The new coordinates x^p are functions of the coordinates x^i, i.e.,

$$x^i = x^i(x^p) . \tag{3.92}$$

Explicitly,

$$u = \dot{x}^1 \tag{3.93}$$

$$w = \dot{x}^2 \tag{3.94}$$

$$q = \dot{x}^3 \tag{3.95}$$

$$\dot{\delta} = K_u \ddot{x}^1 + K_w \dot{x}^2 + K_q \dot{x}^3 , \tag{3.96}$$

where K_u, K_w, and K_q represent stabilization system transfer functions for flight path velocity stabilization, angle of attack stabilization (since $\alpha \approx w/u_o$), and pitch stabilization, respectively. The nature of the transfer function determines the nature of the stabilization system. For example, the pitch stabilization system transfer function has the form

$$K_q = K_1/s + K_2 + K_3 s . \tag{3.97}$$

In the case of a pure pitch damper,

$$K_q = K_2 , \quad (K_1 = K_3 = 0) \tag{3.98}$$

A pure pitch stiffener would lead to

$$K_q = K_1/s , \quad (K_2 = K_3 = 0) . \tag{3.99}$$

In the present development, pure pitch damping is assumed; thus,

$$K_u = K_w = 0 \ , \tag{3.100}$$

$$K_q = K_2 \ . \tag{3.101}$$

By differentiating Equation (3.92),

$$\dot{x}^p = C^p_i \, \mathring{x}^i \ .$$

Consequently,

$$\left[C^i_p\right] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & K_q \end{bmatrix} . \tag{3.102}$$

$\bar{Y}_{pq}$ now follows from the transformation equation

$$\bar{Y}_{pq} = C^i_p \, \bar{Y}_{ij} \, C^j_q \tag{3.103}$$

which corresponds to the matrix equation,

$$\left[\bar{Y}_{pq}\right] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & K_q \end{bmatrix} \begin{bmatrix} X_u & X_w & 0 & X_\delta/s \\ Z_u & Z_w+Z_{\dot{w}}s & Z_q & Z_\delta/s \\ M_u & M_w+M_{\dot{w}}s & M_q & M_\delta/s+M_{\dot{\delta}} \\ H_u & H_w+H_{\dot{w}}s & H_q & H_\delta/s+H_{\dot{\delta}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & K_q \end{bmatrix} \tag{3.104}$$

Performing this operation,

$$\left[\bar{Y}_{pq}\right] = \left[\begin{array}{c|c|c} X_u & X_w & K_q X_\delta/s \\ \hline Z_u & Z_w+Z_{\dot{w}}s & Z_q + K_q Z_\delta/s \\ \hline M_u+K_q H_u & M_w+M_{\dot{w}}s + & M_q+K_q H_q + K_q(M_\delta/s+M_{\dot{\delta}}) \\ & K_q(H_w+H_{\dot{w}}s) & +(K_q)^2(H_\delta/s+H_{\dot{\delta}}) \end{array}\right] \tag{3.105}$$

df_p is obtained from

$$\left[df_p\right] = \left[C_p^i \, df_i\right] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & K_q \end{bmatrix} \begin{bmatrix} mg(d\psi s\Phi c\Theta - d\theta c\Phi c\Theta) \\ -mg(d\theta s\Theta \;\; + d\varphi s\Phi c\Theta) \\ 0 \\ 0 \end{bmatrix} \quad (3.106)$$

$$\left[df_p\right] = \begin{bmatrix} mg(d\psi s\Phi c\Theta - d\theta c\Phi c\Theta) \\ -mg(d\theta s\Theta \;\; + d\varphi s\Phi c\Theta) \\ 0 \end{bmatrix} \quad (3.107)$$

The equations of motion of a vehicle with a pure pitch damper are therefore

$$df_p = \bar{Y}_{pq} \, d\dot{x}^q , \quad (3.108)$$

where the components of the df_p and the $\bar{Y}_{pq}$ are given by Equations (3.107) and (3.105), respectively. The equations of motion with a more sophisticated stabilization system are similar although more involved.

Transformation to Inertial-Eulerian Coordinates

The small perturbation equations of motion in inertial-Eulerian coordinates are given by Equation (2.88). However, rather than deriving them directly from this equation, they will be developed by transforming the previously established equations from the stability coordinate system to the inertial-Eulerian coordinate system. In this process, $\bar{Y}_{lm}$ is obtained from the $\bar{Y}_{ij}$ by the transformation

$$\bar{Y}_{lm} = C^i_l \, \bar{Y}_{ij} \, C^j_m \, . \tag{3.109}$$

A similar transformation produces $\bar{M}_{lm}$, while $\Gamma_{mn,l}$ is obtained from the transformation equation for affine connections

$$\Gamma_{mn,l} = C^m_j \, \bar{M}_{lm} \frac{\partial C^j_m}{\partial x^n} + [jk,i] \, C^j_m \, C^i_l \, C^k_n \, . \tag{3.110}$$

Since the $[jk,i]$ vanish in the stability coordinate system where $\bar{M}_{ij}$ is constant,

$$\Gamma_{mn,l} = C^m_j \, \bar{M}_{lm} \frac{\partial C^j_m}{\partial x^n} = \bar{M}_{lj} \frac{\partial C^j_m}{\partial x^n} \tag{3.111}$$

The computation of the $\bar{Y}_{lm}$ is best accomplished by recognizing that C^i_l is the sum of a unit matrix $[I]$ and a matrix $[S]$, i.e.,

$$[C] = [I] + [S] \, . \tag{3.112}$$

Similarly,

$$[C_t] = [I] + [S_t] \, . \tag{3.113}$$

Since the matrix equivalent of Equation (3.109) is

$$[\bar{Y}_{lm}] = [C][\bar{Y}_{ij}][C_t] \, , \tag{3.114}$$

$$[\bar{Y}_{lm}] = ([I] + [S])[\bar{Y}_{ij}]([I] + [S_t]) \, .$$

Expanding this equation,

$$[\bar{Y}_{lm}] = [\bar{Y}_{ij}] + [S][\bar{Y}_{ij}] + [\bar{Y}_{ij}][S_t] + [S][\bar{Y}_{ij}][S_t] \, .$$

Since all non-zero terms of $[S_t]$ and $[S][\bar{Y}_{ij}]$ are homogeneous functions of infinitesimal Euler angles,

$$[S][\bar{Y}_{ij}][S_t] = 0 \;, \tag{3.115}$$

if products of angles are neglected. Therefore,

$$[\bar{Y}_{lm}] = [\bar{Y}_{ij}] + [S][\bar{Y}_{ij}] + [\bar{Y}_{ij}][S_t] \;. \tag{3.116}$$

$[\bar{Y}_{ij}]$ is given by Equation (3.79) and $[S]$ and $[S_t]$ by Equation (3.27). Performing the operations indicated by Equation (3.116), $[\bar{Y}_{lm}]$ takes the form shown in Figure 7.

Considering that

$$\bar{M}_{lj} = C_j^m \, \bar{M}_{lm} \;,$$

or equivalently

$$\bar{M}_{lj} = C_l^i \, \bar{M}_{ij} \;, \tag{3.118}$$

it follows that

$$\left[\bar{M}_{lj}\right] = \begin{array}{c} {}_{l}\diagdown{}^{j} \\ \left[\begin{array}{ccc|ccc} m & -md\phi & d\theta(m+Z_{\dot{w}}) & & & \\ md\phi & m & -d\varphi(m+Z_{\dot{w}}) & & 0 & \\ -md\theta & md\varphi & (m+Z_{\dot{w}}) & & & \\ \hline 0 & 0 & 0 & I_{11} & 0 & I_{13} \\ 0 & 0 & M_{\dot{w}} & I_{13} & I_{22} & d\varphi I_{33} \\ 0 & 0 & -d\varphi M_{\dot{w}} & d\theta I_{11}+I_{13} & -d\varphi I_{22} & d\theta I_{13}+I_{33} \end{array}\right] \end{array} \;. \tag{3.119}$$

Since C_l^i is independent of the x^l when $l = 1,2,3$ (i.e., φ, θ, ϕ, respectively),

$$\frac{\partial C_m^j}{\partial x^1} = \frac{\partial C_m^j}{\partial x^2} = \frac{\partial C_m^j}{\partial x^3} = 0 \;, \tag{3.120}$$

$$
\begin{array}{c|c}
l \backslash m & \\
\hline
 & \left[\begin{array}{c|c|c|c|c|c}
X_u+ms+d\theta(Z_u-X_w) & d\psi(X_u-Y_v)-d\varphi X_w & X_w+d\theta Z_w+d\theta(Z_{\dot{w}}s-X_u) & -d\psi Y_p & d\theta Z_q & -d\psi Y_r \\
\hline
d\psi(X_u-Y_v)-d\varphi Z_u & Y_v+ms & d\psi X_w-d\varphi Z_w+d\varphi(Y_v-Z_{\dot{w}}s) & -d\psi Y_p & d\theta Z_q & -d\psi Y_r \\
\hline
Z_u+d\theta Z_w+d\theta(Z_{\dot{w}}s-X_u) & d\psi Z_u-d\varphi Z_w+d\varphi(Y_v-Z_{\dot{w}}s) & Z_w+(Z_{\dot{w}}+m)s-d\theta(X_w+Z_u) & d\varphi Y_p & Z_q & d\varphi(Y_r-Z_q) \\
\hline
-d\psi L_v & L_v & d\varphi L_v & L_p+I_{11}s & d\varphi(L_r+I_{13}s) & L_r+I_{13}s+d\theta(L_p+I_{11})s \\
\hline
M_u+d\theta(M_w+M_{\dot{w}}s) & d\psi M_u+d\varphi N_v-d\varphi(M_w-M_{\dot{w}}s) & M_w+M_{\dot{w}}s-d\theta M_u & d\varphi(N_p+I_{13}s) & M_q+I_{22}s & d\varphi(N_r-M_q)-d\varphi(I_{22}-I_{33})s \\
\hline
-d\varphi M_u-d\psi N_v & N_v+d\theta L_v & -d\varphi(M_w+M_{\dot{w}}s)+d\varphi N_v & N_p+I_{13}s+d\theta(L_p+I_{11}s) & d\varphi(N_r-M_q)-d\varphi(I_{22}-I_{33})s & dN_r+I_{33}s+d\theta(L_r+N_p+2I_{13})s
\end{array}\right]
\end{array}
\tag{3.117}
$$

Figure 7.--The Vehicle Admittance Tensor, $\bar{Y}_{lm}$, in Inertial-Eulerian Coordinates.

and

$$\left[\frac{\partial c_m^j}{\partial x^4}\right] = \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & & & \\ 0 & 0 & 1 & & 0 & \\ 0 & -1 & 0 & & & \\ \hline & & & 0 & 0 & 0 \\ & 0 & & 0 & 0 & -1 \\ & & & 0 & 1 & 0 \end{array}\right] , \tag{3.121}$$

$$\left[\frac{\partial c_m^j}{\partial x^5}\right] = \left[\begin{array}{ccc|ccc} 0 & 0 & -1 & & & \\ 0 & 0 & 0 & & 0 & \\ 1 & 0 & 0 & & & \\ \hline & & & 0 & 0 & 1 \\ & 0 & & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \end{array}\right] , \tag{3.122}$$

$$\left[\frac{\partial c_m^j}{\partial x^6}\right] = \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & & & \\ -1 & 0 & 0 & & 0 & \\ 0 & 0 & 0 & & & \\ \hline & & & & & \\ & 0 & & & 0 & \\ & & & & & \end{array}\right] . \tag{3.123}$$

Because of the equalities in Equations (3.120),

$$\Gamma_{m1,1} = \Gamma_{m2,1} = \Gamma_{m3,1} = 0 . \tag{3.124}$$

$\Gamma_{m4,1}$, $\Gamma_{m5,1}$, and $\Gamma_{m6,1}$ are determined by combining Equation (3.119) with Equations (3.121), (3.122), and

(3.123), respectively, as indicated by Equation (3.111). Performing these operations,

$$\left[\Gamma_{m4,1}\right] = \left[\begin{array}{ccc|ccc} 0 & -d\theta(m+Z_{\dot{w}}) & -md\psi & & & \\ 0 & d\varphi(m+Z_{\dot{w}}) & m & & 0 & \\ 0 & -(m+Z_{\dot{w}}) & md\varphi & & & \\ \hline 0 & 0 & 0 & 0 & I_{13} & 0 \\ 0 & -M_{\dot{w}} & 0 & 0 & d\varphi I_{33} & -I_{22} \\ 0 & d\varphi M_{\dot{w}} & 0 & 0 & d\theta I_{13}+I_{33} & d\varphi I_{22} \end{array}\right], \quad (3.125)$$

$$\left[\Gamma_{m5,1}\right] = \left[\begin{array}{ccc|ccc} d\theta(m+Z_{\dot{w}}) & 0 & -m & & & \\ -d\varphi(m+Z_{\dot{w}}) & 0 & -md\psi & & 0 & \\ (m+Z_{\dot{w}}) & 0 & md\theta & & & \\ \hline 0 & 0 & 0 & 0 & 0 & I_{11} \\ M_{\dot{w}} & 0 & 0 & 0 & 0 & I_{13} \\ -d\varphi M_{\dot{w}} & 0 & 0 & 0 & 0 & d\theta I_{11}+I_{13} \end{array}\right], \quad (3.126)$$

$$\left[\Gamma_{m6,1}\right] = \left[\begin{array}{ccc|c} md\psi & m & 0 & \\ -m & md\psi & 0 & 0 \\ -md\varphi & -md\theta & 0 & \\ \hline & & & \\ & 0 & & 0 \end{array}\right]. \quad (3.127)$$

Since, in terms of inertial-Eulerian coordinates, the F_{G_i} are constants,

$$dF_{G_1} = 0 \ . \quad (3.128)$$

The small perturbation equations of motion may thus be written in inertial-Eulerian coordinates as

$$df_l = (\bar{Y}_{lm} + \Gamma_{mn,l}\ \dot{x}^m)\ d\dot{x}^n\ , \qquad (3.129)$$

where $\bar{Y}_{lm}$ and $\Gamma_{mn,l}$ are given by Equation (3.117), and Equations (3.125), (3.126), and (3.127), respectively. Obviously these equations are much more complex than those expressed in stability coordinates. This, of course, is the reason why stability coordinates are usually used in deriving the equations of motion. Equation (3.129) is useful, however, for particular cases where ground oriented measurements are involved (e.g., photographic tracking methods). Although only one such transformation is shown here, the equations of motion in any desired coordinate system may be obtained by a transformation similar to the one employed here.

CHAPTER IV

CONCLUSIONS

A unified method of analysis for aerospace vehicles utilizing tensor concepts has been presented. Throughout the development, generalized concepts were employed so that the method is applicable to any type of physical system, or a mixed system of two or more types.

In general, all systems analysis methods are ultimately based either on force-displacement or Lagrangean techniques. Koenig and Blackwell's contribution, for example, which is an elaboration of the pioneering work of Firestone and Trent, used a matrix formulation of the force-displacement technique, whereas Kron introduced the use of tensor concepts based on the Lagrangean technique. The equivalence of these two approaches was demonstrated by developing the equations of motion for a generalized mechanical system by both methods, and the importance of a system's topological properties stressed. The concepts of "duals" and "analogies" served to emphasize the generality of both approaches.

Because of the nature of the moving axis systems, the Lagrangean technique, formulated in tensor notation, was found to be more appropriate for the analysis of aerospace vehicles. Paralleling Kron's tensor analysis of rotating electrical machinery, the following relations were obtained for a general dynamic system:

(1) general tensor equations of motion for non-holonomic systems

(2) small perturbation equations of motion for non-holonomic systems

With these equations and by defining various "aerodynamic tensors," a general tensor method of analysis for aerospace vehicles was established. In the development of this method, only six degrees of freedom were considered; however, a technique for adding degrees of freedom (e.g., to represent control modes) was included. The equations of motion were formulated in stability coordinates, and the equivalent equations in inertial-Eulerian coordinates derived by a transformation technique. A transformation method was used to accommodate the effects of an arbitrary stabilization system and illustrated by analyzing the longitudinal motion of an airplane with a pure pitch damper system. Tensor relations for the evaluation of the individual components of the inertia tensor and the aerodynamic tensor (conventionally referred to as "dimensional stability derivatives") were given.

The writer concludes that the use of tensor concepts leads to a unified and powerful method of analysis of aerospace vehicle dynamics, capable of incorporating various effects. While some of the general tensor equations (e.g., Equation [2.88]) are more elaborate than those encountered in the highly simplified analyses usually made, their advantage is evident in more general studies. In fact, these tensor equations may well add to a better understanding of the nature of the simplifying assumptions. The tensor analysis, in general, proved to be a very convenient method of mathematical notation, excepting, perhaps, in the derivation of the angular rotation matrix, which can be more easily obtained by the vector method used by Etkin (73).

Three areas of further study are suggested by this work. First, the method could advantageously be extended to provide for aero-elastic and thermo-elastic phenomena of aircraft structures. Second, a detailed development of the suggested dynamic wind-tunnel testing procedure should be pursued. Problems of model design, instrumentation, wind-tunnel modification, data aquisition and handling, etc., in this new approach pose many challenging problems. Third, it would seem promising to apply this method to the analysis of space vehicle motion. Much confusion exists in the literature because of the many different coordinate systems used for space vehicle motion studies. The tensor transformation techniques developed in this dissertation would easily show the equivalence of various different approaches.

Forces encountered in the space environment due to magnetic and electrostatic fields, radiation pressures, solar winds, etc., could also be incorporated in these tensor methods.

LIST OF REFERENCES

1. Routh, E. J., Dynamics of a System of Rigid Bodies. Part II. Reprinting of the 6th ed. of work first published by the Macmillan Co., London, in 1860. (Dover Publications, Inc., New York, 1955).

2. Rayleigh, J. W. S., The Theory of Sound. 2 vols. Reprinting of 2nd ed. of work first published by the Macmillan Co., London, in 1877. (Dover Publications, Inc., New York, 1945).

3. Thomson, W., and Tait, P. G., Principles of Mechanics and Dynamics. 2 parts. Reprinting of last ed. of work first published by the Cambridge University Press, Cambridge, in 1879. (Dover Publications, Inc., New York, 1962).

4. Ricci, G. and Levi-Civita, T., "Methodes de calcul differentiel absolu et leurs applications," Math. Ann. 54, 125-201 (1901).

5. Whittaker, E. T., Analytical Dynamics. Reprinting of 4th ed. of work first published in 1904. (Cambridge University Press, Cambridge, 1961).

6. Webster, A. G., The Dynamics of Particles. Reprinting of 2nd ed. of work first published by B. G. Teubner, Leipzig, in 1904. (Dover Publications, Inc., New York, 1959).

7. Bryan, G. H., and Williams, W. E., "The longitudinal stability of aerial gliders," Proc. Roy. Soc. London (A) 73, 100-116 (1904).

8. Lanchester, F. W., Aerodonetics (A. Constable and Co., Inc., London, 1908).

9. Bryan, G. H., Stability in Aviation (Macmillan Co., London, 1911).

10. Bairstow, L., Applied Aerodynamics (Longmans, Green, and Co., London, 1920).

11. Williams, K. P., The Dynamics of the Airplane (John Wiley and Sons, Inc., New York, 1921).

12. Planck, Max, A Survey of Physical Theory. Reprinting of work first published by Constable and Co., London, in 1921. (Dover Publications, Inc., New York, 1960).

13. Eisenhart, L. P., Riemannian Geometry (Princeton University Press, Princeton, 1926).

14. Appell, P., Cours de mecanique rationelle. Vol. 5. (Gauthier-Villars, Paris, 1926).

15. Eisenhart, L. P., Non-Riemannian Geometry (American Mathematical Society, New York, 1927).

16. Veblen, O., Invariants of Quadratic Differential Forms (Cambridge University Press, Cambridge, 1927).

17. McConnell, A. J., Applications of Tensor Analysis. Reprinting of work first published by Blackie & Sons, Toronto, in 1931. (Dover Publications, Inc., New York, 1957).

18. Veblen, O., and Whitehead, J. H. C., The Foundations of Differential Geometry (Cambridge University Press, London and New York, 1932).

19. Firestone, F. A., "A new analogy between mechanical and electrical systems," J. Acoust. Soc. Amer. 4, 249-267 (1933).

20. Kron, G., "Non-Riemannian dynamics of rotating electrical machinery," J. Math. and Phys. 13, 103-194 (1934).

21. Jones, B. M., Aerodynamic Theory. Vol. 5. (Springer, Berlin, 1935), Div. N.

22. Concordia, C., "The use of tensors in mechanical engineering problems," G. E. Rev. 39, 335-341 (1936).

23. Frazer, R. A., Duncan, W. J., and Collar, A. R., Elementary Matrices. Reprinting of 1st ed. first published in 1938. (Cambridge University Press, Cambridge, 1963).

24. Firestone, F. A., "The mobility method of computing the vibration of linear mechanical and acoustical systems: mechanical-electrical analogies," J. Appl. Phys. 9, 373-387 (1938).

25. Kron, G., Tensor Analysis of Networks (John Wiley and Sons, Inc., New York, 1939).

26. Gardner, M. F., and Barnes, J. L., Transients in Linear Systems (John Wiley and Sons, Inc., New York, 1942).

27. Kron, G., Tensors for Circuits. Reprinting of work published by the General Electric Company in 1942. (Dover Publications, Inc., New York, 1959).

28. Craig, H. V., Vector and Tensor Analysis (McGraw-Hill Book Co., Inc., New York, 1943).

29. Hoffman, B., "Kron's method of subspaces," Quart. Appl. Math. 2, 218-231 (1944).

30. Michal, A. D., Matrix and Tensor Calculus (John Wiley and Sons, Inc., New York, 1947).

31. Stigant, S. A., "Gabriel Kron on tensor analysis I: a bibliographical record," BEAMA J. 55, 276-285 (1948).

32. Synge, J. L., and Schild, A., Tensor Calculus (University of Toronto Press, Toronto, 1949).

33. Perkins, C. D., and Hage, R. E., Airplane Performance, Stability, and Control (John Wiley and Sons, Inc., New York, 1949).

34. Goldstein, Herbert, Classical Mechanics (Addison-Wesley Publishing Co., Inc., Reading, Mass., 1950).

35. Le Corbeiller, P., Matrix Analysis of Electrical Networks (Harvard University Press, Cambridge, Mass., 1950).

36. Sokolnikoff, I. S., Tensor Analysis (John Wiley and Sons, Inc., New York, 1951).

37. Synge, J. L., "The fundamental theorem of electrical networks," Quart. Appl. Math. 9, 113-127 (1951).

38. Duncan, W. J., The Principles of the Control and Stability of Aircraft (Cambridge University Press, New York, 1952).

39. Donegan, J. J., and Pearson, H. A., "Matrix method of determining the longitudinal stability coefficients and frequency response of an aircraft from transient flight data," NACA Rep. 1070 (1952).

40. Le Corbeiller, P., and Yeung, Y. -W., "Duality in mechanics," J. Acoust. Soc. Amer. 24, 643-648 (1952).

41. Trent, H. M., "An alternative formulation of the laws of mechanics," J. Appl. Mech. 19, 147-150 (1952).

42. Bolz, R. E., "Dynamic stability of a missile in rolling flight," J. Aero. Sci. 19, 395-403 (1952).

43. Langefors, Borje, "Analysis of elastic structures by matrix transformation with special regard to semi-monocoque structures," J. Aero. Sci. 19, 451-458 (1952).

44. Chang, T., "General equations of motion of a rigid missile," Cornell Aero. Lab. Rep. 43 (1952).

45. Ashkenas, I. L., Bates, C. L., and McRuer, D. T., "Dynamics of the airframe," U. S. N. BuAer Rep. AE-61-4-II (1952).

46. Breuhaus, W. O., "Resume of the time vector method as a means for analysing aircraft stability problems," WADC Tech. Rep. 52-299 (1952).

47. Langefors, Borje, "An approximate solution of simultaneous equations by means of transformation of variables--applications to aeronautical problems," SAAB Aircraft Co. Linkoping TR-7 (1953).

48. Synge, J. L., "The fundamental theorem of electrical networks," Quart. Appl. Math. 12, 215 (1953-54).

49. Donegan, J. J., "Matrix methods for determining the longitudinal stability derivatives of an airplane from transient flight data," NACA Rep. 1169 (1954).

50. Sternfield, L., "A vector method approach to the analysis of the dynamic lateral stability of aircraft," J. Aero. Sci. 21, 251-256 (1954).

51. Shultz, W., "Motion and disturbance equations in a coordinate system fixed to the airplane," Z. Flugwiss. 2, 157-168 (1954).

52. Kondo, K., "On the dynamics of the aeroplane and non-Riemannian geometry," 2 parts, J. Japan Soc. Aero. Eng. 2; 161-166, 193-196: (1954).

53. Charters, A. C., "The linearized equations of motion underlying the dynamic stability of aircraft, spinning projectiles, and symmetric missiles," NACA TN 3350 (1955).

54. Roth, J. P., "An application of algebraic topology to numerical analysis: On the existence of a solution to the network problem," Proc. Nat. Acad. Sci. 41, 518-521 (1955).

55. Roth, J. P., "The validity of Kron's method of tearing," Proc. Nat. Acad. Sci. 41, 599-600 (1955).

56. Trent, H. M., "Isomorphisms between oriented linear graphs and lumped physical systems," J. Acoust. Soc. Amer. 27, 500-527 (1955).

57. Kron, G., "Tearing and interconnecting as a form of transformation," Quart. Appl. Math. 13, 147-159 (1955).

58. Byrum, B. L., and Grady, E. R., "General airframe dynamics of guided missiles," J. Aero. Sci. 22, 534-540 (1955).

59. Abzug, M. J., "Applications of matrix operators to the kinematics of airplane motion," J. Aero. Sci. 23, 679-684 (1956).

60. Stigant, S. A., "Gabriel Kron on tensor analysis II: a bibliographical record," Matrix and Tensor Quart. 7; 16-24 (1956); 56-58, 72-76 (1957).

61. Roberson, R. E., "Torques on a satellite vehicle from internal moving parts," ASME Paper 57-A-39 (1957).

62. Roberson, R. E., "Attitude control of a satellite vehicle--an outline of problems," Proc. 8th Int. Astro. Congr. Barcelona, 317-339 (1957).

63. Doetsch, K. H., "The time vector method for stability investigations," 2 parts, Aero. Res. Counc. London Rep. Mem. 2945 (1957).

64. Doolin, B. F., "The application of matrix methods to coordinate transformations occurring in systems studies involving large motions of aircraft," NACA TN 3968 (1957).

65. Abramson, H. N., An Introduction to the Dynamics of Airplanes (The Ronald Press, New York, 1958).

66. Roberson, R. E., "Principles of inertial control of satellite attitude," Proc. 9th Int. Astro. Congr. Amsterdam, p. 33 (1958).

67. Roberson, R. E., "Gravitational torques on a satellite vehicle," J. Franklin Institute 265, 13-22 (1958).

68. Young, Dana, "Generalized missile dynamics analysis II--equations of motion," Space Tech. Lab. Rep. EM 8-8 (1958).

69. Higgins, T. J., "Electroanalytic methods VI. Multiplication of grid-network efficacy by use of 'tearing'," Appl. Mech. Rev. 11, 203-206 (1958).

70. Koenig, H. E., and Blackwell, W. A., "On the codification of Lagrangian formulation," Proc. IRE, 1428-1429 (1958).

71. Argyris, J. H., "On the analysis of complex elastic structures," Appl. Mech. Rev. 11, 331-338 (1958).

72. Stigant, S. A., _The Elements of Determinants, Matrices, and Tensors for Engineers_ (Macdonald and Co., Ltd., London, 1959).

73. Etkin, B., _Dynamics of Flight_ (John Wiley and Sons, Inc., New York, 1959).

74. Roberson, R. E., "A unified analytical description of satellite attitude motions," Astronautica Acta 5, 347-355 (1959).

75. Klein, H., "Some notes on dynamics of trajectories," Douglas Aircraft Co. Rep. SM-23288 (1959).

76. Roth, J. P., "An application of algebraic topology: Kron's method of tearing," Quart. Appl. Math. 17, 1-24 (1959).

77. Miele, A., "Lagrange multipliers and quasi-steady flight mechanics," J. Aerospace Sci. 26, 592-598 (1959).

78. Vedrov, V. S., Romanov, G. L., and Surina, V. N., "The airplane as an object of control (Block diagrams of the equations of perturbed airplane motion)," NASA Tech. Transl. F-5 (1959).

79. Suddath, J. H., "A theoretical study of the angular motions of spinning bodies in space," NASA TN R-83 (1960).

80. Weinzweig, A. I., "The Kron method of tearing and the dual method of identification," Quart. Appl. Math. 18, 183-190 (1960).

81. Bohne, Q. R., Clingan, B. E., deCeault, C. W., and Deutschle, P. C., "The dynamic response of advanced vehicles," WADC Rep. 60-518 (1960).

82. Bewley, L. V., Tensor Analysis of Electric Circuits and Machines (The Ronald Press, New York, 1961).

83. Kolk, W. R., Modern Flight Dynamics (Prentice-Hall, Inc., Englewood Clifts, New Jersey, 1961).

84. Koenig, H. E., and Blackwell, W. A., Electromechical System Theory (McGraw-Hill Book Co., Inc., New York, 1961).

85. Thomas, T. Y., Tensor Analysis and Differential Geometry (Academic Press, New York, 1961).

86. Buglia, J. J., Young, G. R., Timmons, J. D., and Brinkworth, H. S., "Analytical method of approximating the motion of a spinning vehicle with variable mass and inertia properties acted upon by several disturbing parameters," NASA TR R-110 (1961).

87. Hoelker, R. F., "Theory of artificial stabilization of missiles and space vehicles with exposition of four control principles," NASA TN D-555 (1961).

88. Rothman, H. L. and Pittel, M., "Transformation of coordinate axis systems of aircraft," J. Aerospace Sci. 28, 583-584 (1961).

89. Springer, C. E., Tensor and Vector Analysis (The Ronald Press, New York, 1962).

90. Miele, A., Flight Mechanics. Vol. 1. (Addison-Wesley Publishing Co., Inc., Reading, Mass., 1962).

91. Cannon, R. H., Jr., "Some basic response relations for reaction-wheel attitude control," ARS J. 32, 264-265 (1962).

92. Knox, E. C., "Comment on transformation of coordinate axis systems of an aircraft," J. Aerospace Sci. 29, 604-605 (1962).

93. Margulus, G., and Goodman, G. S., "Dynamical equations for the attitude matrix of an orbiting satellite," ARS J. 32, p. 1414 (1962).

94. Swaim, R. L., "A general theory and analysis of the dynamic stability of flexible-bodied missiles," AF Flight Dynamics Lab. Tech. Doc. Rep. ASD-TDR-62-627 (1962).

95. Przemienieki, J. S., "Matrix structural analysis of sub-structures," AIAA J. 1, 138-147 (1963).

NOMENCLATURE

A = aerodynamic inertia

B = damping, number of branches

C = electrical capacitance, number of circuits

C^i_j = transformation tensor

E = number of elements

E^i_j = angular rotation tensor

f = force (variable)

F = dissipative energy function

g = acceleration of gravity

g_{ij} = fundamental metric tensor

G = electrical conductance

H = control hinge moment

i = electrical current (variable)

I = rotational inertia

K = stiffness, stabilization system constant

L = electrical inductance

m = vehicle mass

M = general mass

M_{ij} = inertia tensor

$\overline{M}_{ij}$ = effective inertia tensor

N = number of separate parts of graph

q = dynamic pressure

R = electrical resistance

s = differential operator, Laplace transform operator

S = electrical inverse capacitance, reference area

$\overline{S}$ = reference shape dimension

t = time (variable)

T = kinetic energy function

v = velocity, electrical voltage (variables)

V = total velocity, potential energy function, number of vertices

x = displacement (variable)

Y = electrical or aerodynamic admittance

$\overline{Y}$ = vehicle admittance

$\overline{\overline{Y}}$ = vehicle transient admittance

Z = electrical or aerodynamic impedance

$\overline{Z}$ = vehicle impedance

$\overline{\overline{Z}}$ = vehicle transient impedance

X, Y, Z, L, M, N = conventional aerodynamic nomenclature for forces and moments in x^i coordinate directions ($i = 1, \ldots, 6$, resp.)

u, v, w, p, q, r = conventional aerodynamic nomenclature for velocities in x^i coordinate directions ($i = 1, \ldots, 6$, resp.)

α, β, γ = successive Euler angles between x^p and x^i coordinates

φ, θ, ψ = successive Euler angles between x^l and x^i coordinates

Φ, Θ, Ψ = successive Euler angles between x^a and x^l coordinates

Γ = electrical inverse inductance

δ = control displacement

δ^i_j = Kronecker delta

ρ = fluid density

Special Symbols

$[ik,j]$ = Christoffel symbol of the first kind

$\{{}^{\,j}_{ik}\}$ = Christoffel symbol of the second kind

$\Gamma_{jk,i}$ = covariant form of the affine connection

Γ^i_{jk} = contravariant form of the affine connection

$\frac{\delta}{\delta t}$ = intrinsic or absolute derivative

$\frac{d}{dt}$ = ordinary derivative

K_{pqrm} = Riemann-Christoffel symbol of the first kind

K^l_{pqr} = Riemann-Christoffel symbol of the second kind

Subscripts and Superscripts

i,j,k = stability coordinates

l,m,n = inertial-Eulerian coordinates which are coincident with stability axes at $t = 0$.

p,q,r = principal axis coordinates

a,b,c = inertial-Eulerian coordinates with x^1-x^2 plane parallel to the surface of the earth