QUANTUM THEORY OF FREE ELECTRON LASERS

By

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PREFACE

In this study I have tried to present the Quantum Theory of Free Electron Lasers in connection with the classical theories. A model for wiggler wavelength-tapering is proposed as a gain enhancement mechanism and Quantum Mechanical calculations of frequencies and gain have been carried out. In Chapter I, I have tried to give an overall account of Free Electron Lasers especially the experimental aspects before I discussed the theory in later chapters. Chapter II summarizes the Classical theories and can be skipped if one is interested in Quantum Theory only. Chapter III discusses the proposed model for tapering. Quantum Theory is presented in Chapters IV and V.

I would like to express my deepest appreciation and respect to my major adviser, Dr. N. V. V. J. Swamy, for his patient guidance which was needed in the course of my struggle to complete this work. I also thank Dr. Richard Powell and Dr. John Chandler for serving on my committee and Mrs. Janet Sallee for her excellent typing.

I would like to acknowledge the positive influence of my previous physics teachers, Mr. Ömer Incesu who introduced me to physics and Mr. Haydar Çağlayan who enjoyed teaching physics so much that I was inspired to do the same.

I owe special thanks and gratitude to my parents Hasan and Yüksel Saritepe who constantly encouraged me to seek knowledge and pursue an advanced degree. I also owe my thanks to my parents-in-law, Nakip and Nuran Anlar for their support and appreciation of my work.

Most of all I thank my wife, Nilüfer, for her love and understanding which made everything possible. I dedicate this work to her.

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CHAPTER I

FREE ELECTRON LASERS

1. Description of Free Electron Lasers

Free Electron Lasers (FEL) are generators of coherent radiation. FEL does not refer to a physical device or experiment but to various physical mechanisms in which coherent radiation is produced from the energy of relativistic electrons. The essential part of a FEL is a magnet structure into which the relativistic, high energy electrons are injected. There are basically two types of magnet structures, undulators and wigglers. Undulators originally proposed by Motz [1] long time ago are simply arrays of magnets with alternating polarity. Figure 1 shows the undulator. The magnetic field of the undulator near the axis is approximately sinusoidal.

The wiggler structure originally proposed by Madey [2] is really ingenious. Magnets with alternating polarity are set in a helical arrangement. Figure 2 makes this structure more vivid. The best way to describe a wiggler magnetic field analytically is by means of its vector potential which can be written as

$$\vec{A} = (a \cos k_w z)\hat{x} + (a \sin k_w z)\hat{y} \tag{1.1}$$

Here k_w is the wavenumber $(1/\lambda_w)$ of the wiggler.

High energy (relativistic) electrons injected into this magnetic field radiate due to their helical accelerated motion in this field. According to classical electrodynamics the process involved here is synchrotron

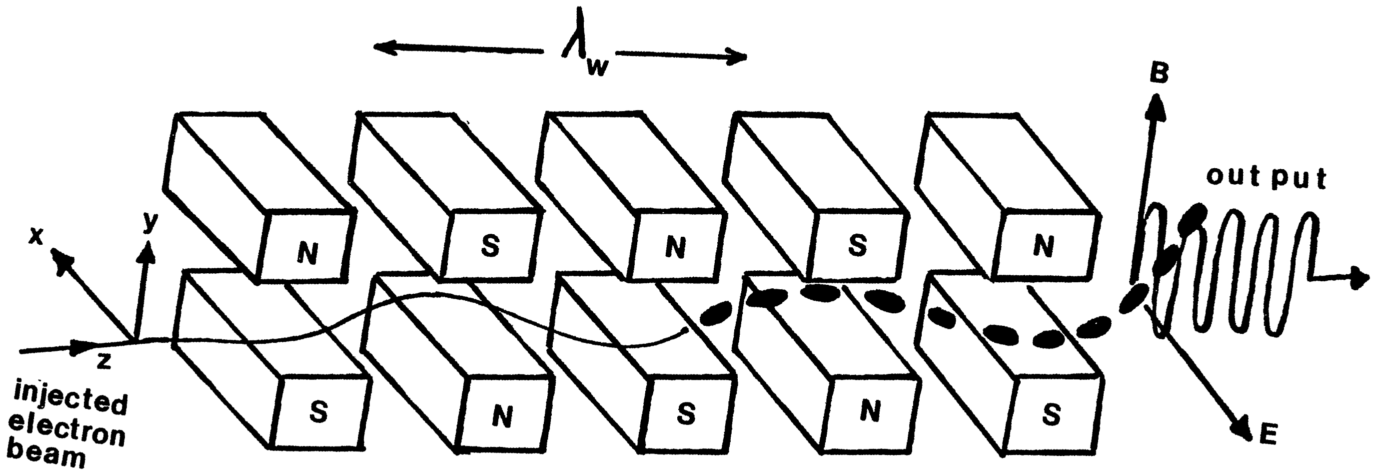

Figure 1 . Undulator

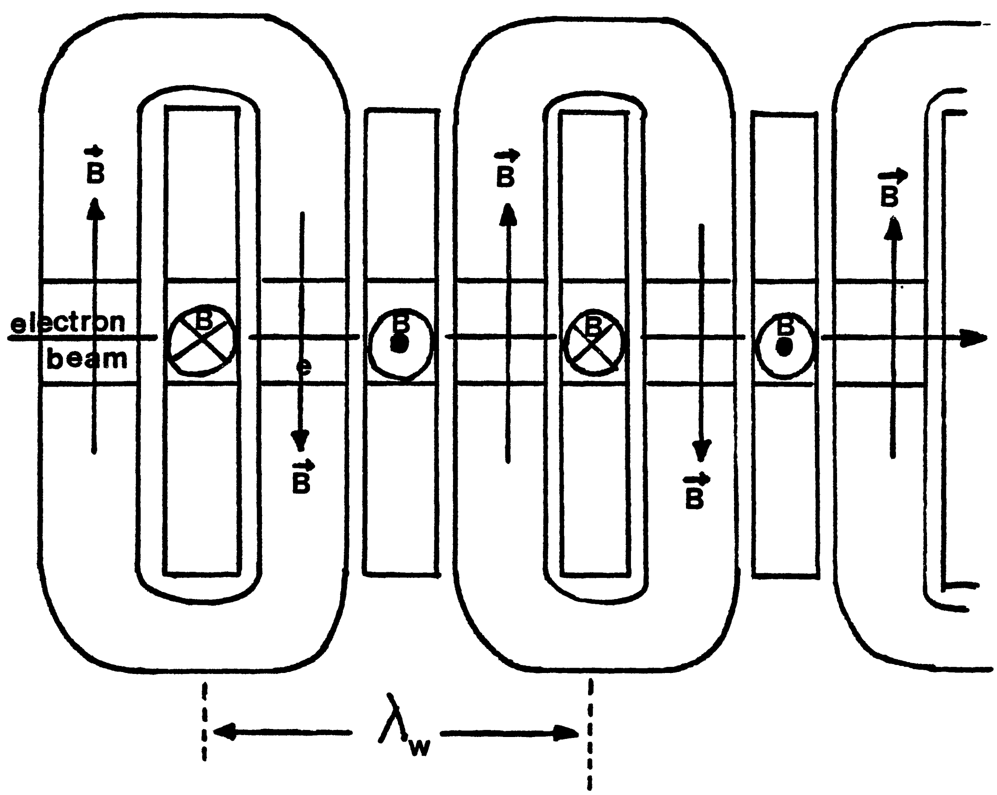

Figure 2 .Wiggler magnets

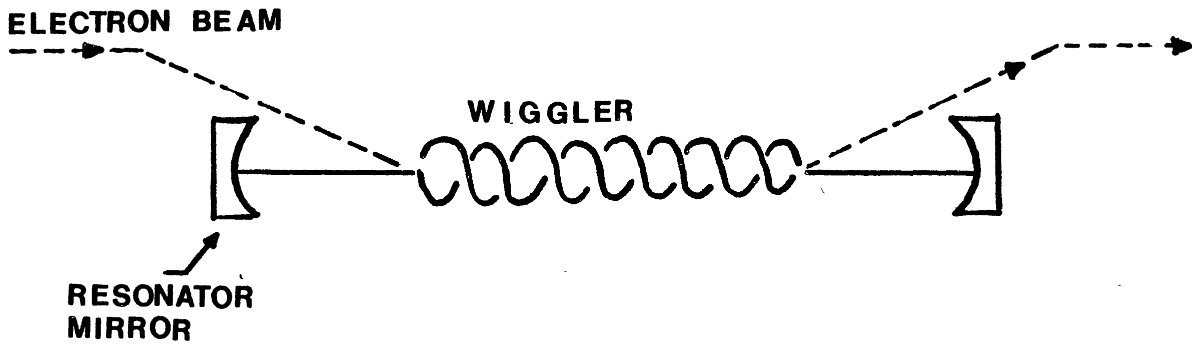

Figure 3 - .Schematic diagram of a FEL oscillator using wiggler

radiation, but it can equivalently be described as Magnetic Bremsstrahlung.

In classical picture, the radiation due to helical motion of electrons self consistently interacts with the electrons and forces them to oscillate at the same frequency. A resonance follows. The result of this resonance between the electron frequency and the radiation frequency is that an observer riding on the electron sees the wiggler wavelength and the optical wavelength as the same. Due to this resonance the repeatedly reflected spontaneous radiation will intensify and become coherent as it stimulates the electrons to radiate at the same resonant wavelength. Laser action results.

Summaries of classical theories of FEL will be given in Chapter II but to facilitate the understanding of this lasing process, let us go into a little more detail, again in the framework of classical electrodynamics. Quantum mechanical discussion will be presented in Chapters IV and V. At the injection point electrons oscillate (wiggle) with random phases. Consequently radiation is incoherent. The $\vec{v} \times \vec{B}$ force on the electrons combined with the radiation field causes a longitudinal density wave which bunches the electrons by decelerating some and accelerating the others. In the literature this longitudinal density wave is called "ponderomotive" or "trapping" wave and acts like the slow traveling electromagnetic wave traveling in a waveguide.

If the axial velocity of electrons is slightly greater than the velocity of the trapping wave, then the average energy of electrons decreases and this means energy is extracted from slowing electrons in the form of radiation. Bunching and extraction of energy from electrons are intimately related. If an electron enters the magnetic field, just at the moment when it is accelerated by the trapping wave it will still be accelerated at the end of the wiggler and also most of the time in between. Similarly if an electron is decelerated at the moment it enters the wiggler, it will still be decelerated when it leaves the wiggler. We can say that whether an electron is accelerated or decelerated by the trapping wave depends on its entrance phase. In order to have a "gain" in radiation the number of decelerated electrons should be greater than the number of accelerated electrons. If there is no spread in the velocity of electrons, then the number of decelerated electrons equals the number of accelerated electrons and hence no gain. That is why a slight spread in the velocity of electrons is necessary. But an excessive spread reduces bunching, obviously, because electrons too fast or too slow can not be trapped by the trapping wave. Long bunches result in incoherence. Untrapped electron means losing another electron to extract radiation. In short, we can say that gain increases as we increase the number of trapped electrons and coherence gets better as we shorten the electron bunches.

To summarize, in the classical picture of FEL, the basic idea is to create electron bunches in the wiggler so that they oscillate in phase to produce coherent light at a certain fundamental frequency. A slight spread in the velocity of electron beam will trigger a gain in radiation. In this way very powerful and intense coherent radiation would be obtained. Furthermore, it could be designed to produce any continuous wavelength. Above all, there would not be any breakdown of the lasing medium.

2. Use and Applications of FEL

Since this work relates to a particular model of a FEL it may not be out of place to review the applications of a FEL to show its importance.

FEL are distinguished from conventional lasers by two important characteristics, wide range continuous tunability and high power. Wide range tunability is desparately needed in spectroscopy. FEL operating in extreme-ultra-violet (xuv) and far-infra-red (fir) bands will especially benefit spectroscopic investigations. The fir range is of particular interest to solid-state experimenters, because almost all the collective excitations of solids, phonons, magnons, occur at these wavelengths. Recently successful FEL experiments have been demonstrated in this range [3]. With kilowatts of tunable fir power it is anticipated that one will be able to excite collective modes in a non-linear fashion. At low power, solid state experimenters have been restricted to exciting phonons and magnons with wavenumbers near zero. Now they will be able to explore the entire Brillouin zone. Also, with high power laser sources operating in fir one can study lattice instabilities which are thought to be important in ferro-electric transitions and excited DNA modes that may play a role in replication.

Conventional sources of coherent radiation have limited optical power output and efficiency at short wavelengths. That is why FEL at xuv are very much sought after. Currently FEL oscillators at this frequency range are plagued by the mirror reflectivity problems but FEL amplifier at xuv is a certain possibility. Explanation of FEL oscillators and amplifiers will be given in Section 4 of this chapter.

The high power lasers promised by FEL sources can be used in a variety of places. One of the most important applications is to use them as heating sources in inertial fusion reactors [4]. Another promising application is in accelerator technology. Accelerator physicists are particularly interested in using high power FEL output in the 1 cm wavelength regime to drive miniature versions of conventional RF-lineacs [5]. Currently large RF-lineacs for High Energy Physics are driven by 10 cm radio-frequency input from powerful klystron tubes. By reducing the driving wavelength by an order of magnitude we could have a miniature SLAC type lineac with an accelerating gradient of several hundred Mev per meter. A two mile lineac of such design could then accelerate electrons to perhaps Tev.

Another much sought after application is that a high peak power FEL operating at submicron wavelengths with an efficiency exceeding 10% would be of considerable interest for military applications. Microwave communications, especially radars require high power sources. FEL are expected to be heavily used in communications.

3. Short History of Experimental and Theoretical Efforts

Madey [1] resurrected the basic idea of Motz [2] after twenty years. As we stated before, Madey proposed wiggler structure and carried out a QED calculation to prove that gain is indeed possible. He treated the problem as the scattering of wiggler-magnetic-photons (Weisacker-Williams method) from the incoming electrons. In 1976 the first FEL amplifier was demonstrated by Elias, Madey et al. [6] at Stanford University. A year later Deacon and Madey et al. [7] reported the first operation of FEL oscillator. In the beginning everyone thought that gain mechanism was due to purely quantum mechanical causes. It was not long before quite a few theoretical papers appeared describing the physics of FEL according to Classical Electrodynamics. Colson [8] derived the now famous gain expression classically. Later Louisell, Colson et al. [9] gave single-particle classical theory of FEL. They showed that equations of motion are considerably simplified if motion with energy in the neighborhood of resonance is discussed. Meanwhile FEL amplifiers and oscillators have been demonstrated in several laboratories. An article in Physics Today [10] summarizes all the experimental set-ups up to that time.

It was soon realized that the central problems in FEL is efficiency enhancement. This led to the formulation of Classical Theory of Variable FELs. Kroll, Morton and Rosenbluth [11] showed in an elegant exposition that by decreasing resonant energy adiabatically more electrons could be trapped and hence one could have increased gain. There are many suggestions on how to increase gain and overall efficiency. Those who are concerned with overall efficiency are concentrating on storage-ring accelerators. The most successful group in this context is the one in France [12] who first demonstrated FEL oscillation in the visible range of the optical spectrum.

One should not neglect the efforts of two Italian physicists Bambini and Renieri [13] in the development of classical theories of FEL. They analyzed the motion of electrons in a moving frame so that the basically relativistic problem could be reduced to a nonrelativistic one.

The current theoretical and experimental research is developing in two directions, efficiency enhancement and higher frequency generation through FEL mechanisms.

After Madey's initial QED effort Quantum Mechanical theories of FEL have been neglected for awhile. It was later developed to some extent by Becker et al. in a series of papers [15]. Quantum Theory of variable wiggler FELs is still missing. It is our intention to fill this gap partially in this thesis.

4. Brief Review of Some Experimental Aspects of FEL

Relativistic electron beams used in FEL are produced by different types of accelerators. Original Stanford experiment used RF-Lineac. Later, storage-ring accelerators have been used. Most recently there is an effort to utilize Van de Graaff accelerators for microwave generation. With the development of Pulsed-RF-Lineacs now there is a new trend towards using Pulsed RF-Lineacs. Depending on the type of accelerator used and beam characteristics there are two modes of FEL mechanisms, FEL amplifiers and FEL oscillators. In FEL amplifiers, output of an external laser is beamed into the wiggler alongside the electron beam from the accelerator. There are no mirrors. Radiation emitted by the wiggling electrons intensify the laser beam, that in turn stimulates the electrons to radiate more intensely. Amplification of the laser beam follows. FEL oscillator does not need an external laser. Radiation emitted by the electrons will be reflected off the mirrors and intensifies inside the wiggler cavity.

There are two regimes of FEL operation, Compton and Raman regimes. Compton regime can be described as high energy but low current electron beam operation. Compton regime FEL use RF-lineacs, microtrons, storage-ring accelerators. These accelerators are characterized by their high energy, low current output. Low current makes the single particle approach plausible. Since they produce very high energy electrons these accelerators are generally used to produce optical and ultra-violet wavelengths. The disadvantage of Compton regime FEL is their low gain and efficiency. That is why only FEL oscillators are designed to operate in Compton regime.

In Raman regime space-charge effects cannot be ignored and single particle approach is not valid. Raman regime FEL use induction lineacs or pulsed transmission-line accelerators which are characterized by their high current but low energy electron beam output. High current dramatically increases the gain, but the low energy electrons can only produce microwave frequencies. In Raman regime amplifier mode is more practical.

Some suggest a third operating regime, High-gain Compton regime. Wiggler field is very strong so that the ponderomotive force on electrons is dominant over the space-charge effects. Strong wiggler makes it possible to use high current accelerators and yet the theoretical calculations can be carried out using the single particle approach which makes the analysis easier.

Wiggler magnet structure is used in most of the FEL experiments mainly because transverse momentum of the electrons is minimal compared to that of undulators. Elias and Madey gave a technical description of the superconducting wiggler they used in the original FEL experiments in Reviews of Scientific Instruments article [16]. Typical values of experimental parameters used in the various FEL experiments are shown in Table I.

If a pulsed accelerator is used, electron pulse and external laser pulses are synchronized so that they interact with each other. Long optical pulse duration is very much desired in applications because the fourier transform of a much longer optical pulse produced by long electron pulse, has a far narrower frequency spectrum, permitting spectroscopic studies with very high energy resolution. Furthermore, long optical pulse implies greater time-averaged power delivered to the experimental sample. Long pulses of electron beam are not only desirable but are necessary as well for FEL oscillators and amplifiers to operate. A reasonable gain is

TABLE I

TYPICAL EXPERIMENTAL PARAMETERS OF FEL USED IN VARIOUS EXPERIMENTS

Experiment	Parameter	Typical Value
Stanford Univ. (Amplifier uses rf lineac	wavelength beam energy peak current	10.6 microns 24 MeV 0.1 A
Stanford Univ. (oscillator) uses rf lineac	wavelength beam energy peak current	3.3 microns 43 MeV 1.3 A
Bell Labs (amplifier) uses microtron	wavelength beam energy peak current	100-400 microns 10-20 MeV 5 A
MIT (pulse-line-generated beams)	peak power wavelength beam energy beam current	1.5 MW 3 mm 1 MeV 5 kA
NRL (uses induction lineac)	wavelength beam energy peak current	8 mm 0.7 MeV 200 A
ORSAY (uses storage-ring)	wavelength beam energy beam current	0.5 microns 240 MeV 2 A

possible only when the radiation bounces back and forth many times before the electron beam disappears. Induction lineac pulses last only on the order of 10 nsec, too short for FEL oscillator requirement. That is why high power FEL are amplifiers and not oscillators.

Most of the FELs are operating in the amplifier mode. Microwave FEL amplifiers have special significance. Optical output of FEL operating at microwave frequencies has a broad bandwidth (Fourier components of an optical pulse) and this makes it possible to amplify, without turning a knob, milimeter wave signals containing a wide spectrum of frequencies (broad-band high-power amplifier). This would be particularly useful for microwave communications. One could shape a desired signal waveform on a low power device and then amplify it to very high power with a FEL.

Lastly there is the possibility of producing optical wavelengths without using high energy beams. (Optical wavelength is proportional to the wiggler wavelength and since the wiggler wavelengths cannot practically be made smaller than a few centimeters, high energy, relativistic electrons are needed to produce optical wavelengths.) One possibility is to use intense laser beam or the output by another FEL as a pump field to stimulate the electrons of not so high energy. This idea is based on the physical principle that electrons behave exactly the same in the field of an electromagnetic plane wave as they do in a static wiggler field (Weizacker-Williams). In the field of an external laser beam, slow electrons will wiggle (oscillate) at higher frequencies and amplification at higher optical wavelengths will follow. The requirement here for the external laser beam is to be intense. Since we now know that only FEL mechanisms can generate great intensities, this set-up is sometimes called "two-stage FEL". The output of another FEL is used as the external laser.

The resonance frequency in the second stage will be much higher than the frequency in first stage because in the frame of moving electro-magnetic plane waves an incoming electron appears to be much faster even though it is slow in the lab frame. As can be seen, a two-stage FEL mechanism is an ingenious frequency multiplication scheme. The price of this frequency multiplication however is the low efficiency. A two-stage FEL is far less efficient than a one-stage FEL.

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CHAPTER II

SUMMARY OF THE CLASSICAL THEORIES OF FEL

1. Synchrotron Radiation

In Chpater I we described the undulator and the wiggler fields. We also discussed the physics of electrons in a wiggler, qualitatively. In this chapter we shall give a more detailed description of the FEL physics in the framework of Classical Electrodynamics.

In order to understand the radiation due to electrons moving in helical (sinusoidal in undulators) orbits we need to look at the energy spectrum. The average energy emitted into frequency range between ω and $\omega + d\omega$ is

$$P_{k\omega}(\vec{r})\ d\omega = \frac{k^2}{2\pi r^2 c} \left| \int J_{\perp_{k\omega}}(\vec{r}\,')e^{-i\vec{k}\cdot\vec{r}\,'}\ d\Omega' \right|^2 d\omega \qquad (2.1)$$

$J_{\perp_{k\omega}}(\vec{r}\,')$ is the Fourier amplitude of the component of the current density vector perpendicular to $\vec{k}$, and $\vec{k}$ is a vector of absolute value $|\vec{k}| = \frac{\omega}{c}$ which points from the current element at $\vec{r}\,'$ to the observer at $\vec{r}$. The expression for $J_{\perp_{k\omega}}$ is derived from the equations of motion. In his pioneering work in which the suggestion of a FEL was first made, Motz [1] demonstrated the radiation characteristics of an electron in an undulating magnetic field.

The relevant equations of motion are

$$\vec{F} = \frac{e}{c} \vec{v} \times \vec{B} = \frac{d}{dt} (m\vec{v}) \tag{2.2}$$

where the undulator magnetic field $\vec{B}$ is

$$B_y = a \sin k_w Z, \quad B_x = 0, \quad B_z = 0 \tag{2.3}$$

and m is the moving mass, then the equations of motion become

$$\begin{aligned} \frac{mc}{e} \ddot{x} &= \dot{z} a \sin k_w Z \\ \frac{mc}{e} \ddot{z} &= -\dot{x} a \sin k_w Z \end{aligned} \tag{2.4}$$

We also know that $\dot{x} << 1$ (motion in x-direction is negligible). Then we can write the equations of motion as

$$\begin{aligned} \frac{d}{dt} (mv_x) &= ea\beta \sin k_w z \\ \frac{d}{dt} (mv_z) &= 0 \\ v_x = \frac{dx}{dt}, \quad v_z &= \frac{dz}{dt} \\ m &= \frac{m_o}{\sqrt{1 - \frac{1}{c^2} (v_x^2 + v_z^2)}} \end{aligned} \tag{2.5}$$

The current densities turn out to be

$$J_x = ev_x \, \delta(y) \, \delta(z-c\beta t) \, \delta(x + \frac{ea\sqrt{1-\beta^2}}{\beta cm_o \omega_o^2} \sin \omega_o t)$$

$$J_z = e\beta c\ \delta(y)\ \delta(z-c\beta t)\ \delta\left(x + \frac{ea\sqrt{1-\beta^2}}{\beta c m_o \omega_o^2} \sin \omega_o t\right) \tag{2.6}$$

β is the longitudinal velocity

$\omega_o = 2\pi\beta c k_w$

$k_w = \frac{1}{\lambda_w}$, λ_w = wavelength of the undulator

The Fourier components will be

$$J_{x\omega} = \frac{ev_x}{\beta c}\ \delta(y)\ \delta\left(x + \frac{ea\sqrt{1-\beta^2}}{\beta c m_o \omega^2} \sin \omega_o z\right) e^{\frac{i\omega z}{\beta c}}$$

$$J_{z\omega} = e\ \delta(y)\ \delta\left(x + \frac{ea\sqrt{1-\beta^2}}{\beta c m_o \omega_o^2} \sin \omega_o z\right) e^{\frac{i\omega z}{\beta c}} \tag{2.7}$$

Let the direction of observation $\vec{k}$ have polar angles ϕ,θ with respect to z-axis, then

$$J_{k\omega} = J_{x\omega}(1 - \cos^2\phi\ \sin^2\theta)^{1/2}\ \hat{U}_1 + J_{z\omega} \sin\theta\ \hat{U}_2$$

where $\hat{U}_1$ and $\hat{U}_2$ are unit vectors in the directions of the components perpendicular to $\vec{k}$. When we substitute (2.7) into (2.1) and evaluate the integral, it is found [1] that the energy radiated per unit area into nth harmonic is

$$\frac{1}{(1-\beta\ \cos\theta)^2} \left(\frac{\omega}{\omega_1}\right)^2 \frac{e^2\beta^2 G_n^2}{4\pi^2 r^2 c} \{\sin^2\theta\ J_n^2(Z) + \frac{1}{4} \sin^2 X[J_{n+1}(2) + J_{n-1}(Z)]^2 \times \frac{e^2 a^2 \ell^2 (1-\beta^2)}{4\pi^2 \beta^2 E_o^2}\} \tag{2.8}$$

where

$$Z = \frac{\omega}{\omega_1} \frac{ea(1-\beta^2)^{1/2} \ \ell \sin\theta \cos\phi}{2\pi\beta E_o(1-\beta\cos\theta)}$$

$$\omega_1 = \frac{2\pi\beta \, ck_w}{1-\beta\cos\theta} \text{ (fundamental frequency)}$$

$$E_o = m_o c^2$$

$$G_n \equiv \int_{-\frac{\ell\pi}{\lambda_w}}^{\frac{\ell\pi}{\lambda_w}} \exp[i(\frac{\omega}{\omega_1} - n)\xi]d\xi$$

$$= \frac{2\sin(\frac{\omega}{\omega_1} - n)\frac{\ell\pi}{\lambda_w}}{(\frac{\omega}{\omega_1} - n)}$$

$$\sin X = \sqrt{1 - \cos^2\phi \sin^2\theta}$$

Note that G_n^2 has a large maximum in a narrow frequency range

$$|\delta_\omega| < (\frac{\lambda_w}{2\ell})\omega_1$$

Also if the observer is on the z-axis, fundamental frequency becomes

$$\omega_1 = \frac{2\pi\beta \, ck_w}{1-\beta} \quad (2.9)$$

This expression is found to be valid for nonrelativistic and relativistic electrons alike. But in a practical FEL, electrons are highly relativistic, that is

$$\beta \approx 1, \; 1-\beta \approx \frac{1}{2\gamma^2}$$

In the relativistic limit the fundamental frequency will be

$$\nu_1 \approx 2\gamma^2 \, ck_w \tag{2.10}$$

In Chapter IV we shall see that (2.10) needs to be corrected for quantum mechanical effects like radiation reaction. Equation (2.10) is derived on the basic assumption that radiation field does not change the path of the electrons in the undulator. In general this assumption is not valid in a FEL cavity. We will discuss the effect of radiation on the electrons in the next section. The relativistic expression for fundamental frequency deserves some attention. ω_1 is proportional to γ^2 not to γ, because of the Relativistic Doppler shift.

An observer riding on the relativistic electron takes a time $t' = \frac{\ell'}{v}$ to cover the distance ℓ'. He sees $n = \frac{\ell}{\lambda_w} = \frac{\ell'}{\lambda'_w}$ magnet poles (waves) as he passes them. Therefore, the frequency he sees is

$$\nu' = \frac{\ell' v}{\lambda'_w \ell'} = \frac{v}{\lambda'_w}$$

but $\lambda'_w = \sqrt{1-\beta^2}\;\lambda_w$, hence he sees a frequency

$$\nu' = \frac{v}{\lambda_w\sqrt{1-\beta^2}} = \frac{v\gamma}{\lambda_w} \tag{2.11}$$

The observer in the laboratory frame sees the same frequency (2.11) multiplied by a factor γ. That is how the γ^2 dependence can be understood [2].

So far we discussed the synchrotron radiation of electrons moving in an undulator. Same approach can be adopted for the radiation of electrons in a wiggler [3]. Kincaid has calculated the power spectrum using the notations and formulas of Jackson which is equivalent to the Fourier expansion formalism.

The brightness per electron is given by [4]

$$\frac{dI(\omega)}{d\Omega} = \frac{e^2\omega^2}{4\pi^2 c} \left| \int_{-\infty}^{\infty} \hat{n} \times (\hat{n} \times \hat{\beta})\, e^{[i\omega(t-\hat{n}\,.\,\frac{r(t)}{c})]}\, dt \right|^2 \tag{2.12}$$

This equation is equivalent to (2.1). Hence $\hat{n}$ is a unit vector pointing from the origin to the observer. $\vec{r}(t)$ is the vector describing the path of the electron, $\beta(t)$ is $\frac{\dot{\vec{r}}(t)}{c}$.

The orbit of a relativistic electron in a wiggler (helical) magnetic field is also a helix, with the same wavelength λ_w. The pitch angle of the helix is given by

$$\theta_{pitch} \approx \frac{2\pi r}{\lambda_w} = \frac{\lambda_w}{2\pi\rho} \tag{2.13}$$

where $\rho = \frac{\gamma\beta mc^2}{e\beta}$ is the cyclotron radius. We can also express the pitch angle as

$$\theta_{pitch} = \frac{\lambda_w e\beta}{2\pi\gamma mc^2} \equiv \frac{k}{\gamma} \tag{2.14}$$

If the vector to the observer is $\hat{n} = \hat{z}\cos\theta + \hat{y}\sin\theta$ and if the radius of the helical orbit is denoted by a, the equations of motion will be

$$\vec{r}(t) = \beta^* ct\,\hat{z} + a \sin \omega_o t\,\hat{y} + a \cos \omega_o t\,\hat{x}$$

$$\beta(t) = \beta^* \hat{z} + \frac{a\omega_o}{c} \cos \omega_o t\,\hat{y} - \frac{a\omega_o}{c} \sin \omega_o t\,\hat{x} \tag{2.15}$$

here $\beta^* = \beta[1 - (\frac{K}{\gamma})^2]^{1/2}$ which is the longitudinal velocity of the electron in wiggler magnetic field. $\frac{K}{\gamma}$ is small for all practical FEL since the strength of the magnetic field K is always negligible compared to the energy (numerically) of the relativistic electrons.

If we look at the motion of an electron from an inertial frame moving in the $+\hat{z}$ direction at the average speed of an electron, $v = \beta^* c$. In this frame an electron travels in a nonrelativistic circular orbit if the magnetic field is weak ($k<1$). Radiation will be a dipole pattern as shown in Figure 4. In the lab frame the radiation pattern will be as shown in Figure 4 due to Doppler effect. If the field is strong ($k>1$) the circular motion in moving frame is also relativistic, hence a cyclotron radiation pattern will be seen in moving frame and the radiation in lab frame will contain harmonics. Figure 5 shows the details of these two cases.

Figure 3 is the result of the evaluation of (2.12). Using (2.15) it is seen that

$$\begin{aligned} \hat{n}x(\hat{n}x\vec{\beta}) = {} & [\frac{K}{\gamma} \sin^2\theta \sin \omega_o t + \frac{K}{\gamma} \cos^2\theta \sin \omega_o t]\,\hat{x} \\ & + [\beta^* \sin\theta \cos\theta - \frac{K}{\gamma} \cos^2\theta \cos \omega_o t]\,\hat{y} \\ & + [-\beta^* \sin^2\theta + \frac{K}{\gamma} \cos\theta \sin\theta \cos \omega_o t]\,\hat{z} \end{aligned} \tag{2.16}$$

and

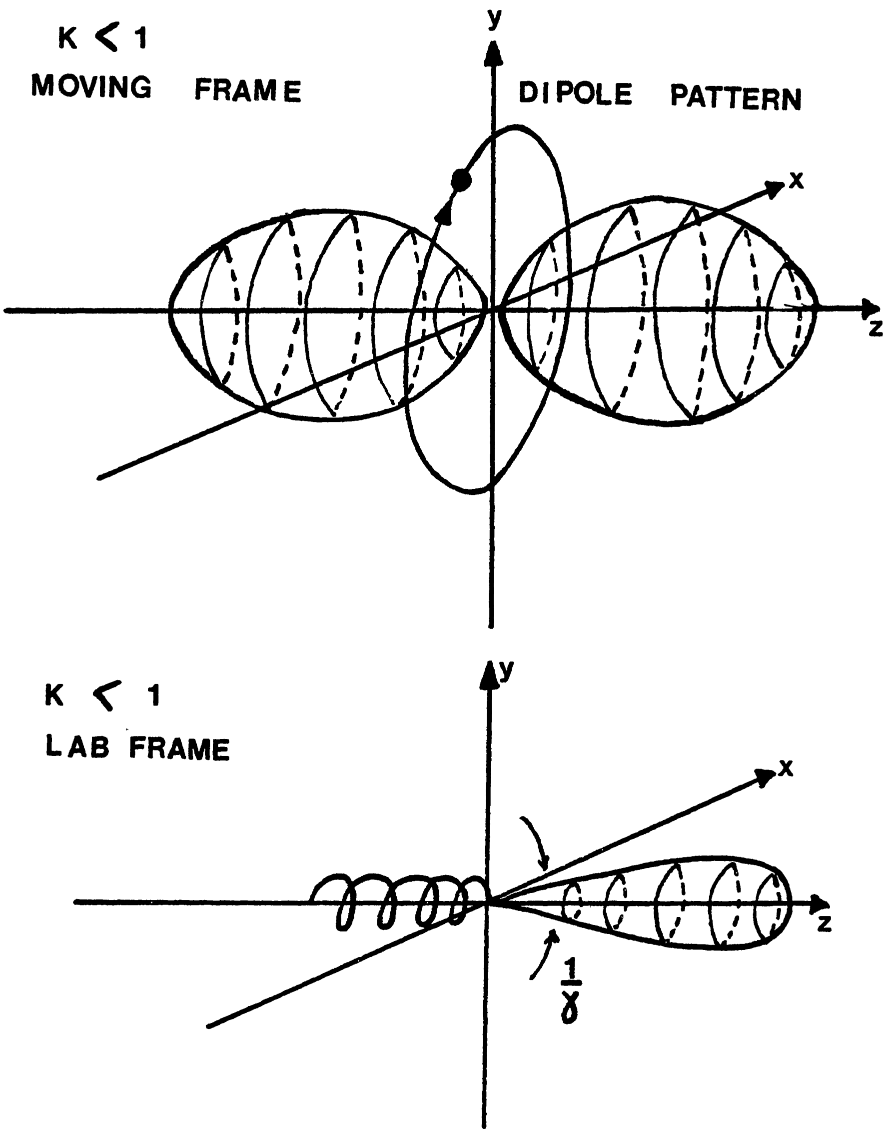

Figure 4 . Radiation pattern produced in a weak wiggler field

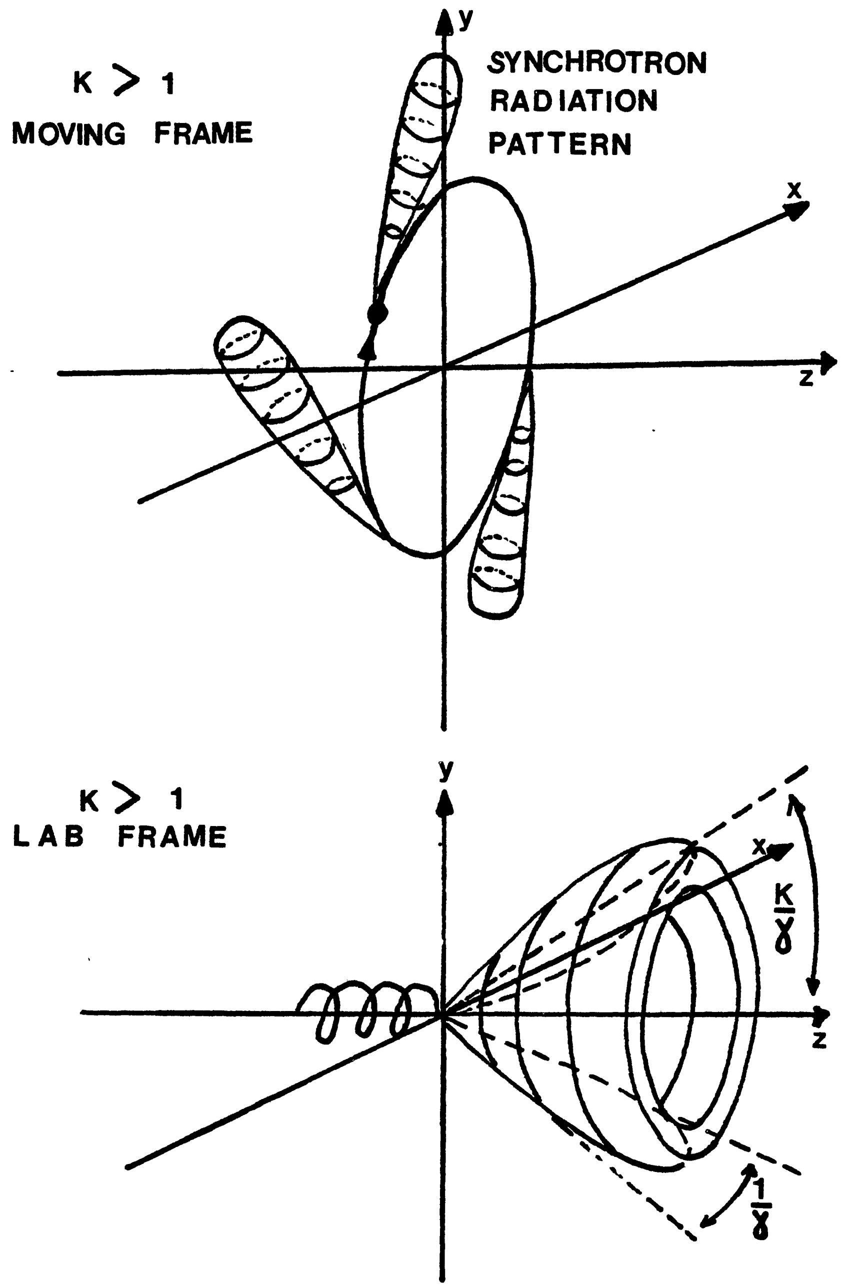

Figure 25 .Radiation pattern produced in strong wiggler

$$\hat{n} \cdot \frac{\vec{r}(t)}{c} = \beta^* t \cos\theta + \frac{a}{c} \sin \omega_o t \tag{2.17}$$

(2.16) and (2.17) are to be substituted in (2.12) and then the somewhat length integral needs to be evaluated. This has been done by Kincaid [3] in the Appendix of his paper. The result is,

$$\frac{dI(\omega)}{d\Omega} = \frac{e^2 \omega^2 k^2}{\pi^2 c \omega_o^2 \gamma^2} \sum_{n=1}^{\infty} [J_n'^2(x) + (\frac{\gamma\theta}{k} - \frac{n}{x})^2] J_n^2(x) \times \frac{\sin^2}{(\frac{w}{w_1} - n)^2} [N\Pi(\frac{\omega}{\omega_1} - n)] \tag{2.18}$$

Here $x = \frac{k_\omega \theta}{\gamma \omega_o}$

$$\omega_1 = \frac{\omega_o}{(1-\beta^* \cos\theta)} = \frac{2\pi\beta^* c k_w}{(1-\beta^* \cos\theta)} \quad \text{fundamental frequency} \tag{2.19}$$

Note that the expression for fundamental frequency is identical in form to the expression we obtained for the undulator.

As far as the fundamental frequency is concerned, wigglers and undulators are not different. Here β of the undulator is replaced by the β^* of the wiggler. When we were discussing the undulator we set $\beta \approx 1$ assuming the magnetic field to be weak. When the magnetic field is strong $\beta \to \beta^* = \sqrt{1 - (\frac{K}{\gamma})^2}$. This expression for β^* is identical in form for undulators and wigglers. The only difference is in K the strength of the magnetic field. K might be different for wigglers and undulators.

In the ultra-relativistic limit (2.19) becomes

$$W_1 \approx \frac{2\gamma^2 \omega_o}{(1 + K^2 + \gamma^2\theta^2)}$$

For $\theta = 0$ (on axis) all the higher harmonics vanish and only fundamental frequency remains

$$\omega = \omega_1 = \frac{2\gamma^2 \omega_o}{(1 + K^2)} \tag{2.20}$$

In terms of wavelength

$$\lambda = \frac{\lambda_w}{2\gamma^2} (1 + K^2)$$

The brightness on the axis will be

$$\left.\frac{dI(\omega)}{d\Omega}\right|_{\substack{\omega=\omega_1 \\ \theta=0}} = \frac{2N^2 e\gamma^2 K^2}{c(1+K^2)} \tag{2.21}$$

N is the number of electrons.

2. Single Particle Electron Dynamics in a FEL

A free electron in vacuum cannot radiate energy since such a process would not conserve both energy and momentum, which is required in all the physical processes. In the case of FEL radiation, necessary momentum is provided by the periodic static magnetic field. As we mentioned in Chapter I, radiation process can be seen either as synchrotron radiation or as Magnetic Bremsstrahlung. In either case one has to consider the effect of the radiation field on the motion of electrons.

The electron motion is obtained from the single particle Hamiltonian

$$H = mc^2\gamma$$

$$\gamma^2 = 1 + \left(\frac{\vec{P}_T - e\vec{A}_T}{m_c}\right)^2 + \left(\frac{P_z}{mc}\right)^2 \tag{2.22}$$

where the transverse vector potential is

$$\vec{A}_T = \vec{A}_S + \vec{A}_W \tag{2.23}$$

with

$$\vec{A}_S = a_s\, e^{-i(kz-wt)}\,\hat{\varepsilon}_- \tag{2.24}$$

being the circularly polarized radiation field. Here $\hat{\varepsilon}_{\mp} = \hat{x} \mp i\hat{y}$, $k = \frac{w}{c}\,\hat{z}$ and a_s is a slowly varying function of z. $\frac{\partial a_s}{\partial z} << ka_s$, $\frac{\partial a_s}{\partial t} << \omega a_s$.

The other term in (2.23) is the wiggler vector potential

$$\vec{A}_w(\vec{r}) = a\, e^{ik_w z}\,\hat{\varepsilon}_- + a\, e^{-ik_w z}\,\hat{\varepsilon}_+ \tag{2.25}$$

Note that (2.25) is identical to (1.1). This way of writing the wiggler vector potential makes it possible to consider the periodic magnetic field as a traveling wave in the moving reference frame which we discussed in Section 1 of this chapter.

The Hamilton's equations

$$\frac{\partial H}{\partial p_i} = \dot{q}_i, \quad \frac{\partial H}{\partial q_i} = -\dot{p}_i \tag{2.26}$$

will lead us to

$$\dot{Z} = c\beta = c\left(1 - \frac{1+\phi}{\gamma^2}\right)^{1/2} \tag{2.27}$$

$$m\gamma\dot{p} = \vec{P}_T - e\,A_T$$

$$\dot{P}_Z = -\frac{mc^2}{2\gamma}\frac{\partial\phi}{\partial Z}$$

where

$$\phi = \left(\frac{P_T - eA_T}{m_c}\right)^2$$

$$\vec{r} = (\vec{\rho}, z), \quad \hat{z}\cdot a(\vec{r}) \equiv 0$$

Generally, $a >> a_s$, that is, the magnitude of the wiggler field is always much greater than the magnitude of the radiation field. Then (2.27) can be written as

$$\beta = \sqrt{1 - \frac{(mc)^2 + (\vec{P}_T - e\vec{A}_w)^2}{(mc\gamma)^2}} \tag{2.28}$$

$$m\gamma\dot{\rho} = \vec{P}_T - e\vec{A}_w$$

Now substituting (2.25) in (2.28) we obtain

$$\dot{\gamma} = -i \frac{e^2 w_s a}{2m^2 c^2 \gamma} [a_s e^{i(k_w z + k_z - wt)} - a_s e^{-i(k_w z + k_z - wt)}] \tag{2.29}$$

Let us define $U \equiv -\omega a_s$ and $\phi \equiv k_w z + kz - wt$. Using the above definitions and $\frac{d}{dt} = v_z \frac{d}{dz}$ we obtain the equations of motion for an electron inside the wiggler cavity

$$\frac{d\gamma}{dz}(\vec{r}) = -\frac{e^2 a}{m^2 c^3 \gamma\beta} \mathrm{Im}[U(\vec{r},t)e^{i\phi}] \tag{2.30}$$

$$\frac{d\phi}{dz}(\vec{r}) = k + k_w - \frac{k}{\beta} \tag{2.31}$$

As we discussed in Chapter I, the phase ϕ of the electron is constant as long as the electron is trapped. In reality, however, as the electron loses energy to the radiation its phase slips. But this means low efficiency. In practice ϕ is always kept constant by either changing k_w with respect to z or the wiggler magnitude a. We shall discuss this concept in more detail in the next section and also in Chapter III. For the time being let us assume that phase ϕ is kept constant somehow. Then (2.31) can be integrated

$$\gamma^2(Z) = \gamma^2(0) - 2\frac{e^2 A_w}{m^2 c^3}\left|U(0)\right| \sin(\phi + \phi_p) \tag{2.32}$$

β is approximated by unity since the electrons are highly relativistic. And $U(\vec{r},t) = |U(0)|\, e^{i\phi_p}$ is the ponderomotive (trapping) potential.

The trapping potential $U(\vec{r},t)$ is not easy to determine [4]. As a matter of fact numerical methods need to be applied to calculate $|U(0)|$ and ϕ_p.

3. Free Electron Lasers With Variable Wigglers

After the first successful operations of FEL amplifiers and oscillators, it was realized that uniform wigglers with constant wavelength and constant magnitude were not capable of trapping most of the electrons. The physical reason is simple. Energy is extracted from the electron in the form of radiation and this means electron is slowing down. If the wavelength of the wiggler is constant, then the observer riding on the electron will see the wavelength of the wiggler increasing as he rides through it. The way out of this difficulty is to either increase the wiggler magnitude slowly in the z-direction so that the electron gains the lost energy from the wiggler or to decrease the wiggler wavelength so that the observer on the electron sees a constant wavelength. Same physical processes can be explained in terms of the equations of motion given in last section. In general $\frac{d\phi}{dz}(\vec{r}) \neq 0$. If the coupled equations of motion (2.30), (2.31) could be solved we could have seen that ϕ was not constant. In other words same physical processes can be described as the slipping of the phase of an electron with respect to the trapping potential. Electron phase slips and goes out of the trap as it radiates.

Kroll, Morton and Rosenbluth proposed a theory taking the above ideas into account [5]. Keeping the electron in resonance, or keeping the electron phase constant is called, "The adiabatic decrease of the resonant energy" in their theory. They discuss the motion about this synchronous energy. They also introduced the concept of "Bucket" which corresponds to the trapping potential well. Figure 6 shows clearly what a bucket is. It is a closed curve and its interior in phase-space. An electron inside a closed curve of phase-space is called "trapped electron".

The motion of electrons about the synchronous energy can be discussed as follows:

$$\gamma = \gamma_{resonance} + \delta\gamma \tag{2.33}$$

where synchronous energy and phase are defined as

$$\gamma_r^2(z) = \frac{k\mu^2}{2k_w(z)} \tag{2.34}$$

$$\frac{d}{dz}[\gamma_r(z)] = -\frac{kaa_s}{\gamma_r}\sin\phi_r \tag{2.35}$$

Here $\mu^2 = 1 + a^2 + a_s^2$.

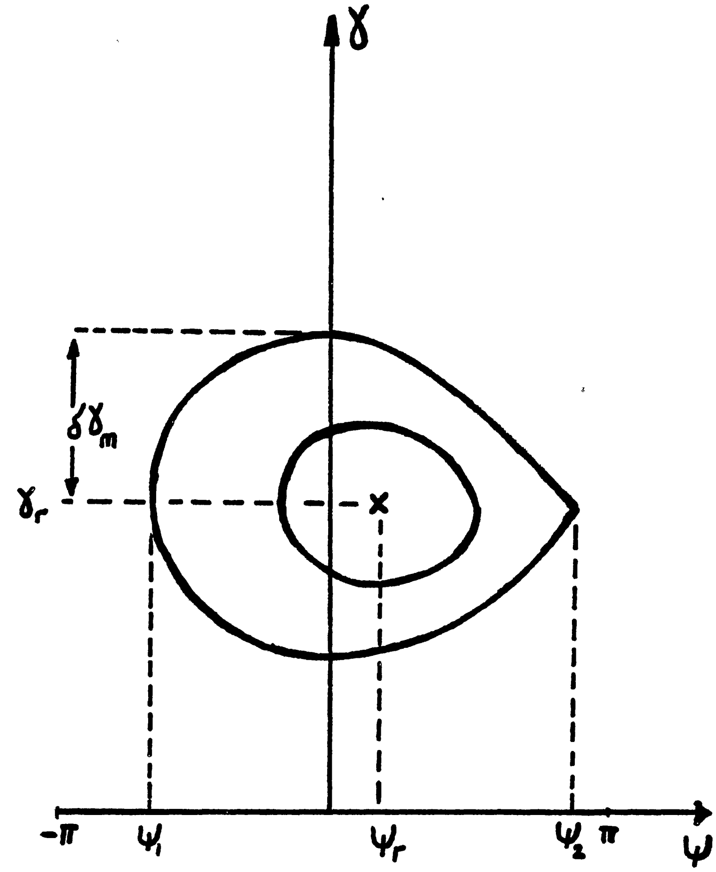

Figure 6 . Stable phase plane trajectories

Note that it is possible to look at Equations (2.34) and (2.35) as either definitions of γ_r and ϕ_r assuming that k_w, a, a_s are known functions of z, or it is possible to consider these equations as design equations where the wiggler functions k_w and a are to be determined to achieve the desired functions γ_r, ϕ_r and a_s.

To continue the discussion of the motion about synchronous energy, if $|\delta\gamma| << \gamma_r$ we can neglect $\frac{a_s}{a}$ terms, and then we have

$$\phi' = 2 \frac{k_w}{\gamma_r} (\delta\gamma) \tag{2.36}$$

$$(\delta\gamma)' = - \frac{kaa_s}{\gamma_r} (\sin\phi - \sin\phi_r) \tag{2.37}$$

Here prime denotes the derivative with respect to longitudinal position z, along the wiggler.

These equations of motion (2.36), (2.37) could be obtained from a Hamiltonian

$$H = \frac{k_w}{\gamma_r} (\delta\gamma)^2 + F(\phi) \tag{2.38}$$

where

$$F(\phi) = - \frac{k_s a a_s}{\gamma_r} (\cos\phi + \phi\sin\phi_r)$$

The closed orbits in phase-space correspond to electrons trapped in buckets which perform stable oscillations about the synchronous value. If the parameters change adiabatically, then the maximum value of $\delta\gamma$ for which a particle may be trapped in a bucket is calculated from the requirement that the area of the closed phase curve remains constant, i.e.,

$$J = \oint \phi(\delta\gamma)\, d\phi = \text{constant}$$

Kroll, Morton and Rosenbluth have calculated this. The results are

$$\delta\gamma_{\text{maximum}} = \frac{2\gamma_r \sqrt{aa_s}}{\mu}\, \Gamma(\phi_r) \tag{2.39}$$

with

$$\Gamma(\phi_r) = \sqrt{\cos\phi_r - (\frac{\pi}{2}\, \mathrm{sgn}\phi_r - \phi_r)\sin\phi_r}$$

The area of the bucket is

$$J = \frac{16\ \gamma_r\ \sqrt{aa_s}}{\mu}\ \alpha(\phi_r) \tag{2.40}$$

$$\alpha(\phi_r) = \frac{\sqrt{2}}{8}\int_{\phi_1}^{\phi_2} [\cos\phi_r + \cos\phi - (\pi-\phi_r-\phi)\sin\phi_r]^{1/2} d\phi$$

The electron will oscillate in the clockwise direction about the synchronous point at $\delta\gamma = 0$, $\phi = \phi_r$ with frequency Ω

$$\Omega = \frac{2k_w}{\mu}\sqrt{a_s a \cos\phi_r} = \frac{\sqrt{\cos\phi_r}}{\Gamma(\phi_r)}\,(\frac{\delta\gamma_m}{\gamma})k_w \tag{2.41}$$

4. Phase Area Displacement

Phase area displacement method refers to an operational mode wherein the phase area occupied by the electrons is degraded downward in energy. This method can be best explained by graphics. Figure 7 shows graphically what happens to a bucket, full of electrons, as they traverse a wiggler of variable parameters.

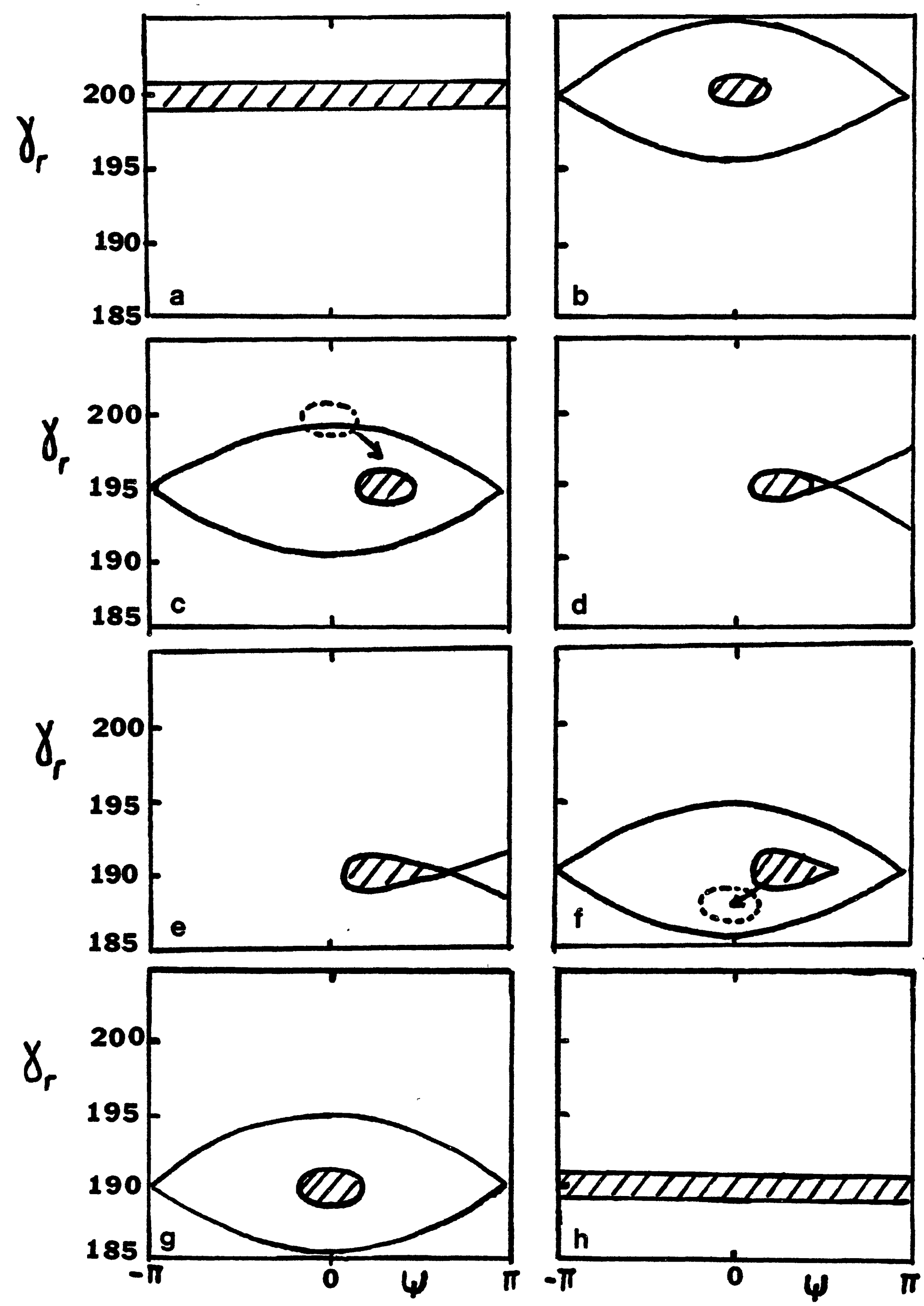

Figure 7 . (a)Initial distribution(b)Capture(c)Increase of average ψ (d),(e)Deceleration(f)Decrease of average ψ (g)Decapture (h)After decapture.

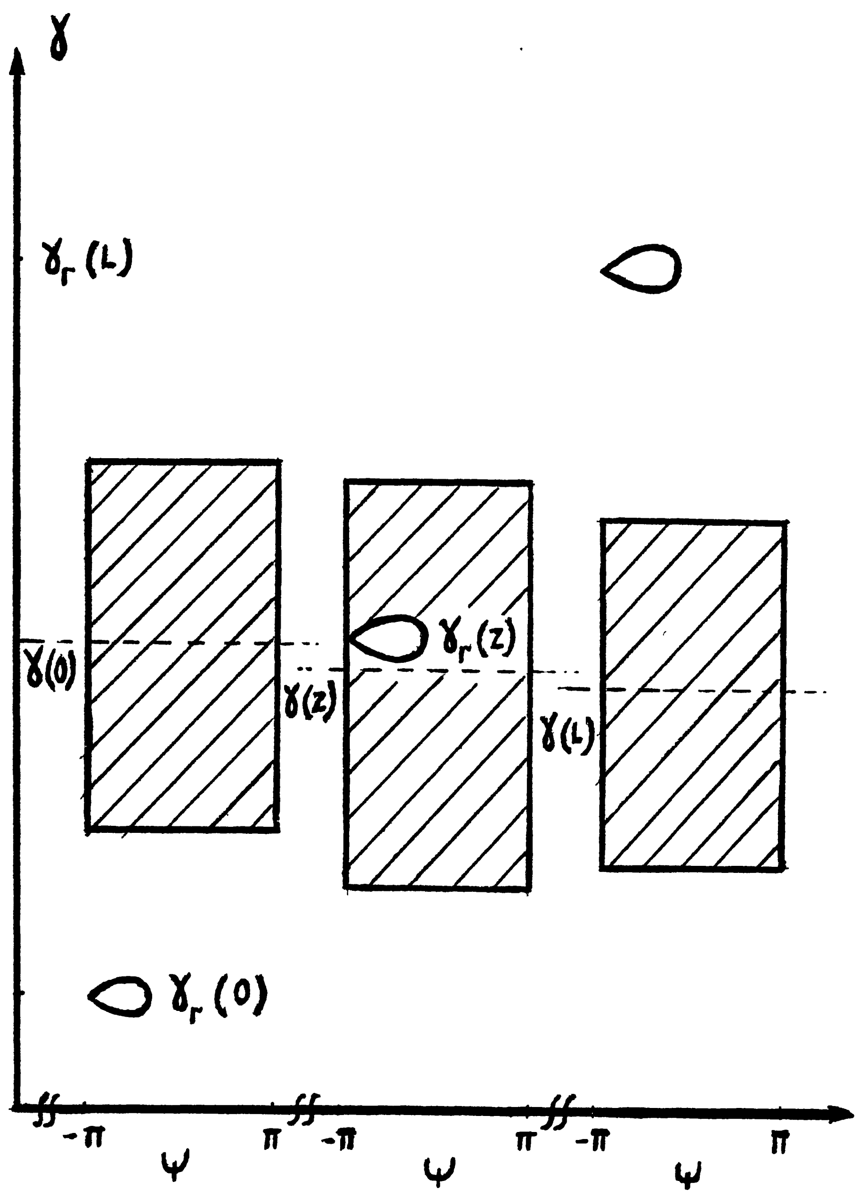

Figure 8 . Position of empty bucket and phase area of electrons at various positions in the FEL

The method illustrated in Figure 7 is called "Deceleration of bucket", end result being the decrease of resonant energy. In this method a certain amount of energy-spread is allowed whereas in the method of "Acceleration of empty bucket" which is illustrated by Figure 8 there is no restriction on the energy-spread of the electron beam. Figure 8 shows how the resonant energy decreases as the empty bucket is accelerated.

The graphical method of FEL physics proved to be very useful in understanding the physics and also in the design of FEL. Almost all the simulations of FEL work with the bucket concept.

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CHAPTER III

DISCUSSION OF THE TAPERING

Currently free electron lasers operate at very low efficiencies. Typical values are between 2% - 5% for the overall efficiency. Overall efficiency is calculated by multiplying the efficiencies of each component of the FEL (accelerator, wiggler, etc.).

Experimenters first attempted to increase the overall efficiency because at that time single-pass gain enhancement mechanisms were not conceived. Main idea was to circulate the electrons. While a storage-ring accelerator would serve the purpose, the problem in this is that the electron beam is very sensitive to any velocity spread (emittance) in the beam. The ponderomotive potential inside the wiggler would certainly increase the emittance dangerously since it accelerates some of the electrons and decelerates the others. It was also found impossible to insert straight wigglers longer than 1.5 m into a storage-ring assembly. The LURE Laboratory in France developed a FEL based on storage-ring accelerator anyway. Their emphasis was on producing optical wavelengths of radiation not on efficiency enhancement. They found very surprisingly that the spread introduced by wiggler did not cause any instabilities.

The recent efforts of physicists at Santa Barbara have been to improve overall efficiency by designing a FEL based on an electro-static accelerator and a circulation mechanism. Although it is successful the electron beam produced by Van de Graaff accelerators is of modest energy and can be run only for short pulse durations. We discussed in Section IV of Chapter

I that FEL oscillators or amplifiers require long electron pulse durations to achieve any reasonable gain.

Meanwhile other ideas came forth on how to increase the single-pass gain and efficiency. There are two factors that could be improved, the first one being the number of electrons trapped in ponderomotive potential. Gain is increased simply because there are more electrons trapped and trapping ensures energy extraction.

The second factor is the slipping of electrons out of resonance as they lose energy to the radiation field. Kroll, Morton and Rosenbluth [1] came up with an idea that could increase the number of trapped electrons and keep them trapped (in resonance) all the way through the wiggler. This idea is to vary parameters like wiggler wavelength and amplitude adiabatically. We discussed this theory in Chapter II. In this chapter we shall concentrate on the experimental features of this theory. Most of the experimenters taper (decrease adiabatically) the wiggler wavelength and others decrease the amplitude of the wiggler and a few taper both [2]. Some experiments use linear taperings, that is, the wiggler wavelength or the amplitude is decreased in a linear fashion. Typical value for the gradient is 9-10%. Almost all the laboratories reported more than 90% efficiency enhancement even with small gradients.

Many physicists so far have attempted to study the physics of FEL by simulation methods. Most notable of those are Mani [3] and Coffey, Lax and Eliot [4] Mani proposed a wiggler-wavelength-tapering which has a functional form

$$\lambda_w = \lambda_o e^{-\left(\frac{z}{z_o}\right)^n} \tag{3.1}$$

For n = 2 and n = 4 simulation showed that almost 100% of the electrons would be trapped. Mani physically argued that initially wiggler wavelength should change slowly and towards the end of the wiggler it should decrease faster because electrons start feeling the effect of radiation field only towards the end.

Coffey et al., reported the results of their simulation which is based on Los Alamos Laser using linear tapering. Results are comparable to Mani's results. Coffey et al., also showed classical mechanically what the functional form of the wavenumber $k_w(z)$ should be to keep the electrons always in resonance. Their conclusion is

$$k_w(Z) = k_{signal} \left(\frac{1}{\beta_z} - 1\right) \tag{3.2}$$

where

$$\beta_z = \sqrt{1 - \frac{1+M^2}{\gamma^2}} \ , \quad \gamma^2 = \gamma^2(0) - mZ \tag{3.3}$$

m is a constant

$M = \frac{ea}{M_o C}$, a: amplitude of the wiggler field.

These classical mechanical calculations motivated us to consider a wiggler-wavelength-tapering of the form

$$k_w(Z) = k_w - \frac{b}{Z} \quad \text{where} \quad Z_{min} >> b$$

We are not allowed to start the z-dependence from zero or other small z value because that would be too steep a change in the wiggler wavelength which is against the basic requirement and assumption that wavelength (wavenumber) of the wiggler changes adiabatically. We show the different taperings graphically in Figure 9. In the figure this curve is concave

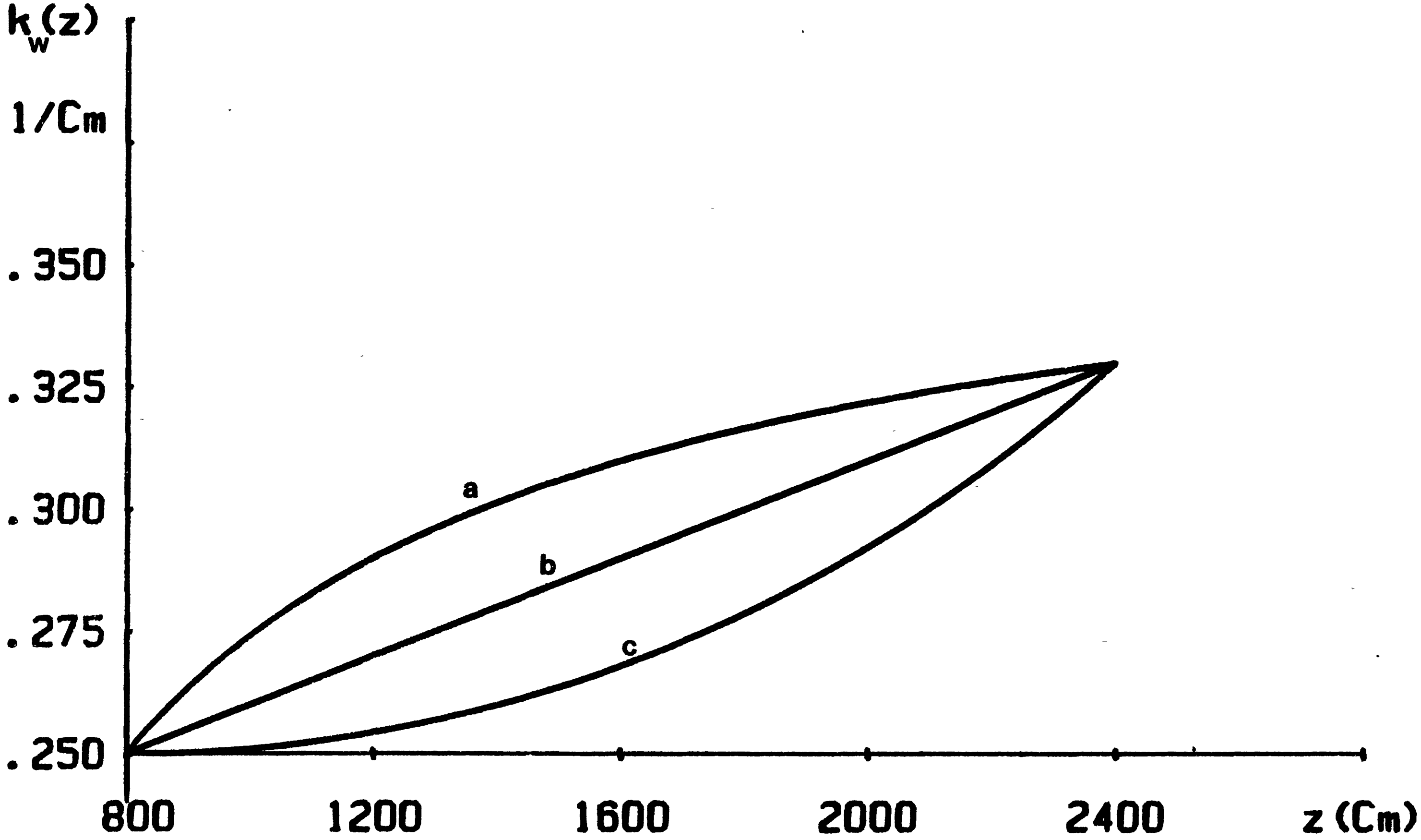

Figure 9 . (a)Proposed tapering(b)Linear tapering(c)Exponential tapering

to the z-axis as against the exponential tapering of Mani but the rise is similar.

We argue that our tapering increases the gain more than the other proposed taperings do. The final gain expression is proportional to the average wavenumber of the wiggler and also to the third power of effective interaction length. It is very easy to see that effective interaction length of a wiggler with a wavenumber increasing concavely to the z-axis, is greater than the one with same geometrical length and a wavenumber increasing linearly or convexly to the z-axis. To illustrate this we calculate the number of magnet elements in a wiggler of 16 m long. We recall that the effective interaction length is really determined by the number of magnet elements and not by the geometrical length.

The number of magnet elements in a wiggler tapered according to $k_w(z) = k_w - \frac{b}{z}$, $z_{min} = 800$, $z_{max} = 2400$ would be

$$\int_{800}^{2400} (k_w - \frac{b}{2})\, dz = 486.5$$

$$k_w = 0.37\ \text{cm}^{-1} \tag{3.4}$$

$$b = 96, \quad \lambda_{initial} = 4\ \text{cm}, \quad \lambda_{final} = 3\ \text{cm}$$

whereas the number of magnet elements for a linearly tapered wiggler and a uniform wiggler would be respectively

$$\int_{0}^{1600} (k_w + mz)\, dz = 464$$

$$m = 0.00005$$

$$k_w = 0.25\ \text{cm}^{-1}, \ \lambda_{initial} = 4\ \text{cm}, \ \lambda_{final} = 3\ \text{cm}$$

and 400. The effective interaction length with our suggested tapering would thus be larger by a factor of 1.216 over uniform wiggler and by 1.048 over a linearly tapered wiggler. It turns out that this would result in a gain-increase by a factor of $(1.216)^4$ over the gain obtained by uniform wiggler.

If we did not use the same geometrical length but the same number of magnet elements, the gain would then increase by a factor of 1.216 since the average wavelength of the tapered wiggler ($k_w(z) = k_w - \frac{b}{z}$) is 1.216 times greater than the wavelength of uniform wiggler.

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CHAPTER IV

QUANTUM THEORY OF AN ELECTRON IN A TAPERED WIGGLER

1. Introduction

Becker and Mitter [1] gave solutions of Dirac equation for the one-dimensional motion of a relativistic electron in a uniform wiggler field. Those solutions were not derived in detail in any of their papers. Our objective is to derive the solutions for motion in a tapered wiggler field for the most general case and then to show that these solutions reduce to Becker-Mitter solutions for a uniform wiggler. We also derive solutions of the Dirac equation for the motion of an electron in 3-dimensions. We would like to point out that our solutions are more explicit and easy to work with.

Classical mechanical calculations have already proved that tapering the wiggler wavelength or amplitude increases gain. We shall calculate this gain enhancement Quantum Mechanically. We propose a model for the tapering. The merit of this particular functional model is that the relativistic motion of the electron in this tapered wiggler and associated harmonics, gain etc., can be studied elegantly from a quantum mechanical viewpoint.

Question of whether Quantum Mechanics is necessary at all to analyze FEL can be answered simply by the fact that it is made imperative by quantizing the electromagnetic field as a photon field. Another important point is that classical theories of FEL do not take into account the spin of the electron. Especially for those FEL using strong wigglers spin-wiggler field interaction is significant. High gain requires strong wiggler field. In future FEL experiments, effects of electron spin will play a much more significant role.

We use Dirac equation instead of the Klein-Gordon equation since only Dirac equation incorporates spin. Using Klein-Gordon equation one obtains the erroneous result that emitted radiation is unpolarized. But as far as calculation of gain is concerned Klein-Gordon equation can be used effectively and this has indeed been done in the past.

As it is discussed in an earlier chapter we propose a tapering of the wiggler wavelength of the form

$$k_w^* = k_w - \frac{b}{z} \tag{4.1}$$

The numerical values given below pertain to Los Alamos FEL.

$$k_w = 0.37 \text{ cm}^{-1}, \; b = 96$$

$$Z_{min} = 800 \text{ cm}, \quad Z_{max} = 2400 \text{ cm}$$

so that

$$\lambda_w \text{ initial} = \frac{1}{k_w^*} = 4 \text{ cm}, \quad \lambda_w \text{ final} = \frac{1}{k_w^*} = 3 \text{ cm}.$$

The choice of limits does not affect the geometrical length but it determines the effective interaction length ℓ' and therefore the region of integration over z of the relevant matrix element which happens to be a function of $k_w^* Z$.

2. Derivation of Solutions in a Tapered Wiggler

The vector potential of the tapered wiggler is

$$\vec{A} = a \cos k_w^* Z\, \hat{x} + a \sin k_w^* Z\, \hat{y} + 0\hat{Z} \tag{4.2}$$

k_w^* is the modulated wavenumber which characterizes the tapering $k_w^* = k_w - \frac{b}{z}$. Here a measures the strength of the field and b (a dimensionless number) the degree of tapering.

In the direct product notation the time independent Dirac equation will be

$$(c\vec{\alpha}\cdot\vec{\underset{\sim}{P}} + \beta\, m_o c^2 - E)\ \psi = (\underset{\sim}{H} - E)\ \psi = 0 \tag{4.3}$$

$$\vec{\alpha} = \rho_1 \otimes \vec{\sigma}, \quad \beta = \mathbf{1} \otimes \rho_3$$

where the kinetic momentum

$$\vec{\underset{\sim}{P}} = (\underset{\sim}{p}_x - \frac{ea}{c} \cos k_w^* Z)\hat{x} + (\underset{\sim}{p}_y - \frac{ea}{c} \sin k_w^* z)\hat{y} + \underset{\sim}{p}_z \hat{Z} \tag{4.4}$$

or

$$\vec{\underset{\sim}{P}} = \vec{\underset{\sim}{p}} - \frac{e}{c}\vec{A} \tag{4.5}$$

Here $\vec{\alpha}$ is a 4x4 and ρ is a 2x2 Dirac matrix. It is sufficient to solve the time independent equation because the vector potential is static. Our technique of solving the equation (4.3) is first to multiply it on the left with $\underset{\sim}{H}$+E and obtain

$$\psi = (c\rho_1\ \vec{\sigma}\cdot\vec{P} + \rho_3 m_o c^2 + E)\phi \tag{4.6}$$

where ϕ satisfies

$$(c^2(\vec{\sigma}\cdot\vec{P})^2 + m_o^2c^4 - E^2)\phi = 0 \tag{4.7}$$

To arrive at this equation we used the well known anti-commutation relations of Dirac operators

$$[\rho_1,\rho_3]_+ = 0 \tag{4.8}$$

We have the identity

$$(\vec{\sigma}\cdot\vec{P})(\vec{\sigma}\cdot\vec{P}) = \vec{P}\cdot\vec{P} + i\vec{\sigma}\cdot(\vec{P}x\vec{P})$$

Defining the modulated wavenumber $k_w^* = k_w - \frac{b}{z}$

$$\begin{aligned}
\vec{P}x\vec{P} = {} & \{\underset{\sim}{p}_y\underset{\sim}{p}_z - \frac{ea}{c}(\sin k_w^*z)\underset{\sim}{p}_z - \underset{\sim}{p}_z\underset{\sim}{p}_y + \frac{ea}{c}\,\underset{\sim}{p}_z \sin k_w^*z\}\hat{x} \\
& + \{\underset{\sim}{p}_z\underset{\sim}{p}_x - \frac{ea}{c}\,\underset{\sim}{p}_z \cos k_w^*z - \underset{\sim}{p}_x\underset{\sim}{p}_z + \frac{ea}{c}\,(\cos k_w^*z)\underset{\sim}{p}_z\}\hat{y} \\
& + \{\underset{\sim}{p}x\underset{\sim}{p}_y - \frac{ea}{c}\,\underset{\sim}{p}_x \sin k_w^*z - \frac{ea}{c}\,(\cos k_w^*z)\underset{\sim}{p}_y + (\frac{ea}{c})^2 \sin k_w^*z \cos k_w^*z \\
& - \underset{\sim}{p}_y\underset{\sim}{p}_x + \frac{ea}{c}\,\underset{\sim}{p}_y \cos k_w^*z + \frac{ea}{c}\,(\sin k_w^*z)\underset{\sim}{p}_z \\
& - (\frac{ea}{c})^2 \sin k_w^*z \cos k_w^*z\}\hat{z}
\end{aligned}$$

Using the commutation relations $[p_i,p_j] = 0$ and

$$[p_z,f(z)] = -i\hbar\frac{\partial}{\partial z}f(z)$$

we obtain

$$\underset{\sim}{\vec{P}}\times\underset{\sim}{\vec{P}} = - i\hbar\, k_w \frac{e}{c} \vec{A} \tag{4-9}$$

$$\vec{P}\cdot\vec{P} = (\underset{\sim}{p}_x - \frac{ea}{c} \cos k_w^* z)(\underset{\sim}{p}_x - \frac{ea}{c} \cos k_w^* z)$$

$$+ \quad (\underset{\sim}{p}_y - \frac{ea}{c} \sin k_w^* z)(\underset{\sim}{p}_y - \frac{ea}{c} \sin k_w^* z) + \underset{\sim}{p}_z^2$$

$$\vec{P}\cdot\vec{P} = (\underset{\sim}{\vec{p}}^2 - \frac{2ea}{c} \underset{\sim}{p}_x \cos k_w^* z - \frac{2ea}{c} \underset{\sim}{p}_y \sin k_w^* z + (\frac{ea}{c})^2$$

$$\vec{P}\cdot\vec{P} = \underset{\sim}{\vec{p}}^2 - \frac{2e}{c} \vec{A}\cdot\vec{p} + (\frac{ea}{c})^2 \tag{4.10}$$

Finally we have

$$(\vec{\sigma}\cdot\vec{P})(\vec{\sigma}\cdot\vec{P}) = \underset{\sim}{\vec{p}}^2 - \frac{2e}{c} \vec{A}\cdot\vec{P} + (\frac{ea}{c})^2 + \hbar k_w \frac{e}{c} \vec{\sigma}\cdot\vec{A} \tag{4.11}$$

The experiments have in the past been such that the electron propogates in the z-direction with negligible transverse momentum. Later we will comment on what is involved in removing this restriction. We therefore assume that $p_x = o = p_y$ and then obtain

$$c^2(\vec{\sigma}\cdot\vec{P})(\vec{\sigma}\cdot\vec{P}) = c^2\underset{\sim}{p}_z^2 + (\hbar k_w\ ec)\ \vec{\sigma}\cdot\vec{A} + e^2a^2 \tag{4.12}$$

The iterated Dirac equation in 2x2 form will be

$$\begin{pmatrix} c^2\underset{\sim}{p}_z^2+\eta^2 & ea\hbar k_w c\, e^{-ik_w^* z} \\ ea\hbar k_w c\, e^{ik_w^* z} & c^2\underset{\sim}{p}_z^2+\eta^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \tag{4.13}$$

where we set for brevity

$e^2a^2 + m_o^2c^4 - E^2 \equiv \eta^2$ which means $(ea)^2 - \eta^2$ is equal to (z-component of momentum)$^2c^2$. Also we used $\phi = \binom{x_1}{x_2}$. Now we have the coupled equations

$$
\begin{aligned}
&(c^2 \underset{\sim}{p}_z^2 + \eta^2) x_1 + ea\hbar k_w c\, e^{-ik_w^* z} x_2 = 0 \\
&ea\hbar k_w c\, e^{ik_w^* z} x_1 + (c^2 \underset{\sim}{p}_z^2 + \eta^2) x_2 = 0
\end{aligned}
\tag{4.14}
$$

We can now express x_2 in terms of x_1

$$
x_2 = - \frac{e^{ik_w^* z} (c^2 \underset{\sim}{p}_z^2 + \eta^2) x_1}{ea\hbar k_w c}
\tag{4.15}
$$

or

$$
x_1 = \frac{e^{-ik_w^* z} (c^2 \underset{\sim}{p}_z^2 + \eta^2)}{ea\hbar k_w c} x_2
\tag{4.16}
$$

Eliminating x_2 leads us to

$$
(ea\hbar k_w c)^2 e^{ik_w^* z} x_1 - c^2 \underset{\sim}{p}_z^2 e^{ik_w^* z} (c^2 \underset{\sim}{p}_z^2 + \eta^2) x_1 - \eta^2 e^{ik_w^* z} (c^2 p_z^2 + \eta^2) x_1 = 0
\tag{4.17}
$$

We have the commutation relation

$$
\underset{\sim}{p}_z^2 e^{ik_w^* z} = e^{ik_w^* z} (\underset{\sim}{p}_z^2 + 2\hbar k_w \underset{\sim}{p}_z + \hbar^2 k_w^2)
\tag{4.18}
$$

Therefore, we shall have an operator equation for x_1

$$e^{ik_w^* z}\ \{(ea\hbar k_w c)^2 - c^2 \underset{\sim}{p}_z^2(c^2 \underset{\sim}{p}_z^2+\eta^2) - 2\hbar k_w c^2(c^2 p_z^2+\eta^2)\underset{\sim}{p}_z$$

$$- (\hbar k_w c)^2(c^2 \underset{\sim}{p}_z^2+\eta^2) - \eta^2 c^2 \underset{\sim}{p}_z^2 - \eta^4\}\ x_1 = 0 \qquad (4.19)$$

Completing this to a perfect square we have

$$e^{ik_w^* z}[\{(ea\hbar k_w c)^2 - (\hbar k_w c\eta)^2\} - \{c^2 \underset{\sim}{p}_z^2 + \hbar k_w c^2 \underset{\sim}{p}_z + \eta^2\}^2]x_1=0 \qquad (4.20)$$

Since the operator operating on x_1 is a function of $\underset{\sim}{p}_z$ only x_1 should be the eigenfunction of $\underset{\sim}{p}_z$ and we can take

$$x_1 = e^{\frac{i}{\hbar} p_1 z}$$

where p_1 is a continuous number. x_2 can now be determined from Equation (4.14), as a function of another continuous number p_2, by eliminating x_1 and using the commutation relation

$$\underset{\sim}{p}_z^2\ e^{-ik_w^* z} = e^{-ik_w^* z}\ (\underset{\sim}{p}_z^2 - 2\hbar k_w\ \underset{\sim}{p}_z + \hbar^2 k_w^2) \qquad (4.21)$$

The operator equation for x_2 is

$$e^{ik_w^* z}\ [\{c^2 \underset{\sim}{p}_z^2 - \hbar k_w c^2 \underset{\sim}{p}_z + \eta^2\}^2 - \{(ea\hbar k_w c)^2 - (\hbar k_w c\eta)^2\}]x_2=0 \qquad (4.22)$$

It is easy to determine p_1 by insisting that $x_1 = e^{\frac{i}{\hbar} p_1 z}$ satisfies the Equation (4.20). With the substitution $\eta^2 = e^2 a^2 - p_z^2 c^2$, we obtain

$$(p_1^2 + \frac{e^2a^2}{c^2} - p_z^2 + \hbar k_w p_1)^2 = (\hbar k_w)^2[e^2a^2 - \eta^2] = (\hbar k_w p_z)^2 \qquad (4.23)$$

$$p_1 = -\frac{\hbar k_w}{2} + \sqrt{(p_z + \frac{\hbar k_w}{2})^2 - (\frac{ea}{c})^2} \qquad (4.24)$$

Using Equation (4.22) we obtain

$$p_2 = \frac{\hbar k_w}{2} + \sqrt{(p_z - \frac{hk_w}{2})^2 - (\frac{ea}{c})^2} \qquad (4.25)$$

p_z, k_w, a are design parameters, so p_1 and p_2 are determined in terms of the design parameters. More meaningfully we could write

$$E^2 = E_o^2 + [- \frac{\hbar c k_w}{2} \mp \sqrt{c^2(p_1 + \frac{\hbar k_w}{2})^2 + (ea)^2}\ ]^2 \qquad (4.26)$$

$$E^2 = E_o^2 + [+ \frac{\hbar c k_w}{2} \mp \sqrt{c^2(p_2 - \frac{hk_w}{2})^2 + (ea)^2}\ ]^2 \qquad (4.27)$$

where $E_o \equiv m_o c^2$. Note that these expressions for energy are equivalent. To complete the solution of the Dirac equation we have (for positive square-root of Equation (4.7))

$$\phi_2 = \frac{1}{E+E_o} \vec{\sigma}\cdot\vec{P}\ \phi_1$$

where

ϕ_1 is a spinor.

Equation (4.20) justifies the following choice of ϕ_1

$$\phi_1 = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} d\, e^{\frac{i}{\hbar} p_1 z} \\ g\, e^{\frac{i}{\hbar}(p_1 + \hbar k_w) z} \end{pmatrix} \tag{4.28}$$

Substituting this into Equation (4.7) we obtain

$$(c^2 p_1^2 + \eta^2) d\, e^{\frac{i}{\hbar} p_1 z} + eac\hbar k_w\, e^{ib}\, e^{\frac{i}{\hbar} p_1 z}\, g = 0$$

$$eac\hbar k_w e^{-ib} d\, e^{\frac{i}{\hbar}(p_1 + \hbar k_w) z} + [c^2 (p_1 + \hbar k_w)^2 + \eta^2] g\, e^{\frac{i}{\hbar}(p_1 + \hbar k_w) z} = 0 \tag{4.29}$$

in order to have a unique solution, determinant must be equal to zero

$$(c^2 p_1^2 + \eta^2)[c^2 (p_1 + \hbar k_w)^2 + \eta^2] - (eac\hbar k_w)^2 = 0 \tag{4.30}$$

also from each equation we have

$$g = - \frac{(c^2 p_1^2 + \eta^2) e^{-ib}}{eac\hbar k_w} d = - \frac{eac\hbar k_w}{c^2 (p_1 + \hbar k_w)^2 + \eta^2} d \tag{4.31}$$

where d is arbitrary. d and g are constants or functions independent of z. p_1 is related to the effective momentum of the electron in the z-direction p_z. Note that p_1 is just a (physical) number. We are not interested in the negative energy solutions with negative square root because they refer to positrons. The complete solution in the bi-spinor will be

$$\psi = \begin{pmatrix} \phi_1 \\ \phi_2 = \dfrac{c\, \vec{\sigma} \cdot \vec{p}}{E + m_o c^2} \phi_1 \end{pmatrix} \tag{4.32}$$

Let us work it out

$$\frac{c\,\vec{\sigma}\cdot\vec{p}}{E+E_o} = \frac{1}{E+E_o}\begin{pmatrix} c\underset{\sim}{p}_z & -ea\,e^{-i(k_w z-b)} \\ -ea\,e^{i(k_w z-b)} & -c\underset{\sim}{p}_z \end{pmatrix}\begin{pmatrix} d\,e^{\frac{i}{\hbar}p_1 z} \\ g\,e^{\frac{i}{\hbar}(p_1+\hbar k_w)z} \end{pmatrix}$$

$$= \frac{1}{E+E_o}\begin{pmatrix} [cp_1 d - eag\,e^{ib}]\,e^{\frac{i}{\hbar}p_1 z} \\ [-eade^{-ib} - c(p_1+\hbar k_w)g]e^{\frac{i}{\hbar}(p_1+\hbar k_w)z} \end{pmatrix} \tag{4.33}$$

where

$$E_o \equiv m_o c^2$$

The complete unnormalized solution will be

$$\psi_p^{\uparrow} = d\begin{pmatrix} 1 \\ -\dfrac{eac\hbar k_w}{c^2(p_1+\hbar k_w)^2+\eta^2}\,e^{ik_w^* z} \\ \dfrac{1}{E+E_o}\left[cp_1 + \dfrac{e^2a^2c\hbar k_w}{c^2(p_1+\hbar k_w)^2+\eta^2}\right] \\ \dfrac{1}{E+E_o}\left[-ea + \dfrac{c(p_1+\hbar k_w)eac\hbar k_w}{c^2(p_1+\hbar k_w)^2+\eta^2}\right]e^{ik_w^* z} \end{pmatrix} e^{\frac{i}{\hbar}(p_1 z-Et)} \tag{4.34}$$

Note that this solution reduces to the solutions of Dirac equation for motion of electrons in free space when the wiggler field is switched off, that is when

$$a \to 0 \qquad p_1 \to p_z$$
$$k_w \to 0 \qquad b \to 0 \tag{4.35}$$

d is arbitrary, we let $d = \sqrt{\frac{E+E_o}{2\,E_o}}$ to normalize $u\,\bar{u}u = 1$.

The second positive energy solution is derived by taking

$$\phi_1 = \begin{pmatrix} d\, e^{\frac{i}{\hbar}(p_2-\hbar k_w)z} \\ \\ g\, e^{\frac{i}{\hbar}p_2 z} \end{pmatrix} \tag{4.36}$$

This choice of ϕ_1 is justified by the operator Equation (4.22). Again, d,g are independent of z. p_2 is interpreted as the effective momentum in the z-direction when the electron enters the wiggler field with spin-down (spin is anti-parallel to z-direction).

Substituting ϕ_1 in the iterated Dirac equation we obtain

$$[c^2(p_2-\hbar k_w)^2 + \eta^2]d\, e^{\frac{i}{\hbar}(p_2-\hbar k_w)z} + eac\hbar k_w\, e^{ib}\, g\, e^{\frac{i}{\hbar}(p_2-\hbar k_w)z} = 0$$
$$eac\hbar k_w e^{-ib}\, d\, e^{\frac{i}{\hbar}p_2 z} + g(c^2p_2^2 + \eta^2)e^{\frac{i}{\hbar}p_2 z} = 0 \tag{4.37}$$

det = 0 leads to

$$[c^2(p_2-\hbar k_w)^2 + \eta^2]\,[c^2p_2^2+\eta^2] - (eac\hbar k_w)^2 = 0 \tag{4.38}$$

We also have

$$d = -\frac{c^2p_2^2+\eta^2)e^{ib}}{eac\hbar k_w}\,g = -\frac{eac\hbar k_w\ e^{ib}}{c^2(p_2-\hbar k_w)^2+\eta^2}\,g \qquad (4.39)$$

where g is arbitrary.

The complete solution in bi-spinor form will be

$$\psi = \begin{pmatrix} \phi_1 \\ \\ \dfrac{c\ \vec{\sigma}\cdot\vec{p}}{E+E_o}\,\phi_1 \end{pmatrix} \qquad (4.40)$$

$$\frac{c\ \vec{\sigma}\cdot\vec{p}}{E+E_o}\,\phi_1 = \frac{1}{E+E_o}\begin{pmatrix} c\underset{\sim}{p}_z & -ea\ e^{-i(k_wz-b)} \\ -ea\ e^{i(k_wz-b)} & -c\underset{\sim}{p}_z \end{pmatrix}\begin{pmatrix} d\ e^{\frac{i}{\hbar}(p_2-\hbar k_w)z} \\ g\ e^{\frac{i}{\hbar}p_2z} \end{pmatrix}$$

$$= \frac{1}{E+E_o}\begin{pmatrix} [c(p_2-\hbar k_w)d - eag\ e^{ib}]\ e^{\frac{i}{\hbar}(p_2-\hbar k_w)z} \\ \\ [-ea\ e^{-ib}\ d - cp_2g]\ e^{\frac{i}{\hbar}p_2z} \end{pmatrix} \qquad (4.41)$$

The complete unnormalized second solution will be

$$\psi_p^{\downarrow} = g \begin{pmatrix} \dfrac{-eac\hbar k_w\, e^{-i(k_w z-b)}}{c^2(p_2-\hbar k_w)^2+\eta^2} \\ \\ 1 \\ \\ \dfrac{-1}{E+E_o}\left[\dfrac{c(p_2-\hbar k_w)eac\hbar k_w}{c^2(p_2-\hbar k_w)^2+\eta^2} + ea\right] e^{-i(k_w z-b)} \\ \\ \dfrac{1}{E+E_o}\left[\dfrac{e^2a^2c\hbar k_w}{c^2(p_2-\hbar k_w)^2+\eta^2} - cp_2\right] \end{pmatrix} e^{\frac{i}{\hbar}(p_2 z-Et)} \tag{4.42}$$

As is well known [3] when the Dirac particle is moving in a field the up or down spin of the particle can be known with precision only when the particle is momentarily at rest. We associate solution (4.42) with down-spin since it reduces to $U_+^{\downarrow}$ when the wiggler field is switched off and arbitrary g is taken to be $g = \sqrt{\dfrac{E+E_o}{2E_o}}$.

There is one delicate calculation of limit that is not so obvious

$$\lim_{\substack{a\to o \\ k_w\to o}} \frac{eac\hbar}{c^2(p_1+\hbar k_w)^2+\eta^2} \neq 0 \tag{4.43}$$

as quoted in the litarature on untapered wiggler [1]. But

$$\lim_{\substack{a\to o \\ k_w\to o}} \frac{eac\hbar k_w}{c^2(p_1+\hbar k_w)^2+\eta^2} = 0 \tag{4.44}$$

similarly,

$$\lim_{\substack{a\to o\\ k_w\to o}} \frac{eac\hbar k_w}{c^2(p_2-\hbar k_w)^2+\eta^2} = 0 \tag{4.45}$$

These limits are not obvious because p_1 and η^2 are rather complicated expressions which should be written explicitly before the limits are evaluated.

3. Comparison of the Solutions to Those of a Uniform Wiggler

Becker [1] gives the solutions of Dirac equation with uniform Wiggler field ($p_x = 0 = p_y$).

$$\psi_p = [N_-(\gamma_z \frac{\varepsilon a}{\sqrt{2}} e^{-ik_w z} - \gamma_- \frac{(\varepsilon a)^2}{2k_w\lambda_-})\gamma_+ e^{-ik_w\lambda_- z}$$

$$+ N_+(\gamma_z \frac{\varepsilon a}{\sqrt{2}} e^{ik_w z} - \gamma_+ \frac{(\varepsilon a)^2}{2k_w\lambda_+})\gamma_- e^{-ik_w\lambda_+ z}]U(p)e^{\frac{i}{\hbar}(p_z z-Et)} \tag{4.46}$$

where

$$\varepsilon = \frac{e}{\hbar c}, \quad \gamma_\pm = \frac{1}{\sqrt{2}}(\gamma^1 \pm i\gamma^2), \quad \vec{\gamma} = \beta\vec{\alpha} \tag{4.47}$$

also,

$$(E\gamma^o - cp_z\gamma_z - E_o)U(p) = 0, \quad E^2 - c^2p_z^2 = E_o^2 \tag{4.48}$$

$$\hbar k_w\lambda_\mp = p_z \mp \frac{\hbar k_w}{2} - [(p_z \mp \frac{\hbar k_w}{2})^2 - (\frac{ea}{c})^2]^{1/2} \tag{4.49}$$

N_-, N_+ are normalization constants.

Let us write the Becker solutions explicitly

$$\psi_p^{\uparrow} = N_+ \begin{pmatrix} \dfrac{(\varepsilon a)^2}{k_w\lambda_+} \\ \varepsilon a \; e^{ik_w z} \\ \dfrac{(\varepsilon a)^2}{k_w\lambda_+} \; \dfrac{cp_2}{E+E_o} \\ \varepsilon a \; e^{ik_w z} \dfrac{cp_z}{E+E_o} \end{pmatrix} e^{-i(k_w\lambda_+ - \frac{p_z}{\hbar})z} \tag{4.50}$$

$$\psi_p^{\downarrow} = N_- \begin{pmatrix} -\varepsilon a \; e^{-ik_w z} \\ \dfrac{(\varepsilon a)^2}{k_w\lambda_-} \\ \varepsilon a \; e^{-ik_w z} \dfrac{cp_z}{E+E_o} \\ -\dfrac{(\varepsilon a)^2}{k_w\lambda_-} \; \dfrac{cp_z}{E+E_o} \end{pmatrix} e^{-i(k_w\lambda_- - \frac{p_z}{\hbar})z} \tag{4.51}$$

In our solutions we chose

$$g = -\frac{eac\hbar k_w}{c^2(p_1+\hbar k_w)^2+\eta^2}$$

Becker and Mitter chose

$$d = -\frac{eac\hbar k_w}{c^2p_1^2+\eta^2}\, g$$

which are equivalent.

Equivalence of Becker's and our solutions is established when we identify the effective momentum terms

$$p_1 = p_z - \hbar k_w \lambda_+ \tag{4.52}$$

$$p_2 = p_z - \hbar k_w \lambda_- \tag{4.53}$$

then d, g relation leads us to

$$d = -\frac{eac\hbar k_w}{c^2p_1^2+\eta^2}\, g, \quad \text{g is arbitrary let } g = 1$$

$$d = -\frac{eac\hbar k_w}{c^2(p_z-\hbar k_w\lambda_+)^2+\eta^2} = -\frac{eac\hbar k_w}{c^2(p_z^2 - 2p_z\hbar k_w\lambda_+ + \hbar^2k_w^2\lambda_+^2) + e^2a^2 - c^2p_z^2}$$

$$d = -\frac{eac\hbar k_w}{e^2a^2 + c^2\hbar^2k_w^2\lambda_+^2 - 2c^2\hbar p_z k_w\lambda_+}$$

since $\hbar k_w\lambda_+ = p_z + \frac{\hbar k_w}{2} - \sqrt{(p_z + \frac{hk_w}{2})^2 - (\frac{ea}{c})^2}$

above express for d reduces to

$$d = +\frac{eak_w}{c\hbar k_w^2\lambda_+} = +\frac{\varepsilon a}{k_w\lambda_+} \tag{4.54}$$

then $\psi_p^\uparrow$ follows. Note that Becker and Mitter multiply all the rows in the bi-spinor by εa in other words they let $g = \varepsilon a$ since g is arbitrary.

To examine whether Becher and Mitter solutions reduce to free-particle solutions we need to know

$$\lim_{a\to o} \frac{ea}{k_w \lambda_+} \quad 0 \tag{4.55}$$

but

$$\lim_{a\to o} \frac{e^2 a^2}{k_w \lambda_+} = 1 \neq 0 \tag{4.56}$$

We notice that defining effective momentum in the wiggler as $p_z - \hbar k_w \lambda_+$ and $p_z - \hbar k_w \lambda_-$ is an unnecessary complication. λ_+ and λ_- unnecessarily complicate the solutions. Another point we would like to make is that Becker solutions are two particular solutions whereas our solutions are more general . We also derived the determining equations for p_1 and p_2 which correspond to the determining equation for $\hbar k_w \lambda_{\mp}$. Becker and Mitter introduced Equation (4.49) as a defining equation for $\hbar k_w \lambda_{\mp}$ somewhat abruptly. It is to be noticed however when the tapering is switched off, i.e., in the limit $b \to o$, Our solutions go over into Becker and Mitter solutions.

5. 3-Dimensional Solutions of Dirac Equation in a Uniform Wiggler

Three-dimensional solutions of Dirac equation need to be considered when $p_x \neq 0$, $p_y \neq 0$. Following the same techniques used in Section (3) of this chapter we obtain

$$c^2 (\vec{\sigma}\cdot\vec{P})(\vec{\sigma}\cdot\vec{p}) = c^2 \underset{\sim}{\vec{p}}^2 + e^2 a^2 - 2e \ \vec{a}\cdot\underset{\sim}{\vec{p}} + \hbar k_w ec \ \vec{\sigma}\cdot\vec{a}$$

Iterated Dirac equation in 2x2 form is

$$\begin{pmatrix} c^2\vec{p}^2 - 2eac(p_x\cos k_w z + p_y\sin k_w z)+\eta^2 & ea\hbar ck_w\, e^{-ik_w z} \\ ea\hbar ck_w\, e^{ik_w z} & c^2\vec{p}^2-2eac(p_x\cos k_w z+p_y\sin k_w z)+\eta^2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \tag{4.57}$$

we substitute

$$\phi_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} d\, e^{\frac{i}{\hbar}\vec{p}_1\cdot\vec{r}} \\ g\, e^{\frac{i}{\hbar}(\vec{p}_1\cdot\vec{r} + \hbar k_w z)} \end{pmatrix} \tag{4.58}$$

After some straightforward algebra we obtain

$$g = -\frac{c^2\vec{p}_1^2 - 2eac(p_{1x}\cos k_w z + p_{1y}\sin k_w z) + \eta^2}{eac\hbar k_w}\, d \tag{4.59}$$

d is arbitrary.

Complete solution will be

$$\psi = \begin{pmatrix} \phi_1 \\ \dfrac{c\ \vec{\sigma}\cdot\vec{p}}{E + E_o}\,\phi_1 \end{pmatrix} \tag{4.60}$$

$$\frac{c\vec{\sigma}\cdot\vec{p}}{E+E_o}\phi_1 = \begin{pmatrix} cp_{\tilde{1}z} & cp_{-}-ea\ e^{-ik_w z} \\ cp_{+}-ea\ e^{ik_w z} & -cp_{\tilde{1}z} \end{pmatrix} \begin{pmatrix} d\ e^{\frac{i}{h}\vec{p}_1\cdot\vec{r}} \\ g\ e^{\frac{i}{h}(\vec{p}_1\cdot\vec{r}+\hbar k_w z)} \end{pmatrix} \tag{4.61}$$

where

$$\begin{aligned} p_{+} &= p_{\tilde{1}x} + ip_{\tilde{i}y} \\ p_{-} &= p_{\tilde{1}x} - ip_{\tilde{1}y} \end{aligned} \tag{4.62}$$

$$\frac{c\vec{\sigma}\cdot\vec{p}}{E+E_o}\phi_1 = \frac{1}{E+E_o}\begin{pmatrix} [cdp_{1z}-eag] + [cgp_{-} + ci\hbar(\frac{\partial g}{\partial x} - i\frac{\partial g}{\partial y})]e^{ik_w z} \\ [cdp_{+}] - [ead + c(p_{1z} + \hbar k_w)g - ci\hbar\frac{\partial g}{\partial z}]e^{ik_w z} \end{pmatrix} e^{\frac{i}{\hbar}(\vec{p}_1\cdot\vec{r} - Et)}$$

The first unnormalized positive energy solution is

$$\psi^{\uparrow}_{\vec{p}1} = \begin{pmatrix} d \\ g\ e^{ik_w z} \\ \frac{1}{E+E_o}\{[cdp_{1z} - eag] + [cgp_{-} + ci\hbar(\frac{\partial g}{\partial x} - i\frac{\partial g}{\partial y})]\ e^{ik_w z}\} \\ \frac{1}{E+E_o}\{[cdp_{+}] - [ead + cg(p_{1z} + hk_w) + ci\hbar\frac{\partial g}{\partial z}]\ e^{ik_w z}\} \end{pmatrix} e^{\frac{i}{\hbar}(\vec{p}_1\cdot\vec{r} - Et)} \tag{4.63}$$

The second positive energy solution is obtained by substituting

$$\phi_1 = \begin{pmatrix} d\ e^{\frac{i}{\hbar}(\vec{p}_2\cdot\vec{r} - \hbar k_w z)} \\ \\ g\ e^{\frac{i}{\hbar}\vec{p}_2\cdot\vec{r}} \end{pmatrix} \tag{4.64}$$

in Equation (4.57).

After some simple algebra we obtain

$$d = -\frac{c^2\vec{p}_2^{\,2} - 2eac(p_{2x}\cos k_w z + p_{2y}\sin k_w z) + \eta^2}{eac\hbar k_w}\, g \tag{4.65}$$

where g is arbitrary.

$$\frac{c\vec{\sigma}\cdot\vec{p}}{E+E_o}\,\phi_1 = \frac{1}{E+E_o}\begin{pmatrix} [cd(p_{2z}-\hbar k_w) + ci\hbar\,\frac{\partial d}{\partial z} - eag]e^{-ik_w z} + [cgp_-] \\ \\ [cdp_+ + ci\hbar(\frac{\partial d}{\partial x} + i\,\frac{\partial d}{\partial y})]e^{-ik_w z} - [ead + cgp_{2z}] \end{pmatrix} e^{\frac{i}{\hbar}(\vec{p}_2\cdot\vec{r}-Et)}$$

The second unnormalized positive energy solution is

$$\psi^{\downarrow}_{\vec{p}_2} = \begin{pmatrix} d\ e^{-ik_w z} \\ \\ g \\ \\ \frac{1}{E+E_o}\ \{[cd(p_{2z}-\hbar k_w z) + ci\hbar\,\frac{\partial d}{\partial z} - eag]e^{-ik_w z} + [cgp_-]\} \\ \\ \frac{1}{E+E_o}\ \{[cdp_+ + ci\hbar(\frac{\partial d}{\partial x} + i\,\frac{\partial d}{\partial y})]e^{-ik_w z} - [ead + cgp_{2z}]\} \end{pmatrix} e^{\frac{i}{\hbar}(\vec{p}_2\cdot\vec{r}-Et)} \tag{4.66}$$

where

$$p'_{+} = p_{2x} + ip_{2y}$$
$$p'_{-} = p_{2x} - ip_{2y} \qquad (4.67)$$

This goes to show that the Dirac equation has analytical solutions for the type of tapering proposed even if there happens to be a transverse component to the motion of the electron.

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CHAPTER V

RADIATION AND GAIN

1. Radiation

Our objective is to calculate the frequencies and gain quantum mechanically. Since the radiation field inside the wiggler is weak compared to the energy of relativistic electrons, perturbation theory can be used to calculate the probability of emission of a single photon from the perturbed electron. We can also calculate the probability of absorption of a photon. Difference between the probabilities of emission and absorption of a photon will lead us to the calculation of gain. Frequencies will be derived from the conservation of energy and momentum principles.

Let us consider the interaction of a charged particle and an electromagnetic field. The nonrelativistic Hamiltonian for the quantum mechanical system of charged particle and electromagnetic field is

$$H = \frac{1}{2m} |\vec{p} - \frac{e}{c} \vec{A}|^2 + V + \frac{1}{8\pi} \int d^3x(\vec{E}^2+\vec{B}^2) \tag{5.1}$$

Same Hamiltonian can be written as

$$H = H_o + H_{rad} + H_I \tag{5.2}$$

where

$$H_o = \frac{1}{2} \vec{p}^2 + V, \quad H_{rad} = \frac{1}{8\pi} \int d^3x(E^2+B^2)$$

$$H_I = - \frac{e}{mc} \vec{p}\cdot\vec{A} + \frac{e^2}{2mc^2} |\vec{A}|^2 \tag{5.3}$$

H_I is called the interaction Hamiltonian and this will be treated as a perturbation. The passage to Quantum Mechanics is done by replacing the measurable quantities in the classical Hamiltonian by the corresponding operators (First quantization). The quantum mechanical Hamiltonian for the electron will be

$$H = \frac{1}{2} m \underset{\sim}{p}^2 + V \left(\frac{e}{mc} \vec{A}\cdot\underset{\sim}{\vec{p}} + \frac{e}{2mc} \underset{\sim}{\vec{p}}\cdot\vec{A} + \frac{e^2}{2mc^2} |\vec{A}|^2\right) \tag{5.4}$$

Our system consists of a charged particle and electromagnetic plane waves (Radiation). One way to obtain the plane waves from the Maxwell equations is to set $\vec{\nabla}\cdot\vec{A} = 0$ (Coulomb Gauge).

Then the Interaction Hamiltonian will become

$$H_I = \frac{i\hbar e}{mc} A_i \partial_i + \frac{e^2}{2mc^2} |\vec{A}|^2 \tag{5.5}$$

To 1st order perturbation theory we can neglect the term $\frac{e^2}{2mc^2} |\vec{A}|^2$. Thus the interaction Hamiltonian in nonrelativistic Quantum Mechanics is

$$H_I = \frac{i\hbar e}{mc} A_i \partial_i \tag{5.6}$$

So far we discussed the interaction Hamiltonian semiclassically inasmuch as the radiation field is not quantized. A complete Quantum Mechanical discussion requires the quantization of electromagnetic fields also. This is done by "second quantization". Classical electromagnetic waves are quantized to yield photons, and the wavefunctions themselves are now operators.

The wavefunction of a photon with definite momentum $\vec{k}$ and a definite polarization $\hat{e}$, normalized to 1 photon per unit volume can be written as

$$A(r,t) = \sqrt{\frac{2\pi}{\omega}}\ \hat{e}\ e^{i\vec{k}\cdot\vec{r}-\omega t} \tag{5.7}$$

After second quantization procedure this wavefunction becomes the radiation field operator.

$$\underset{\sim}{A}(r,t) = \sum_n \sqrt{\frac{2\pi}{\omega_n}}\ \hat{e}_n[\underset{\sim}{a}_n\ e^{i\vec{k}\cdot\vec{r}-\omega_n t} + \underset{\sim}{a}_n^+\ e^{-i(\vec{k}\cdot\vec{r}-\omega_n t)}] \tag{5.8}$$

where a_n^+ and a_n are creation and annihilation operators respectively.

In order to treat the subject most generally we must consider the relativistic interaction Hamiltonian as well as the quantized electromagnetic field. Dirac equation in the presence of electromagnetic field is

$$\{c\vec{\alpha}\cdot(\vec{p} - \frac{e\vec{A}}{c}) + \beta m_o c^2 - (E-c\phi)\}\psi=0 \tag{5.9}$$

terms can be arranged as

$$\{(c\vec{\alpha}\cdot\vec{p} + \beta m_o c^2-E) - e\vec{\alpha}\cdot\vec{A} + e\phi\}\psi=0 \tag{5.10}$$

Here the interaction Hamiltonian will be

$$H_I = e\ \vec{\alpha}\cdot\vec{A} + e\phi \equiv e\alpha_\mu A^\mu \tag{5.11}$$

We take H_I as the perturbation and apply 1st order perturbation theory to calculate the probability of emission of a photon. That is

$$\left(\frac{\text{transition probability}}{\text{time}}\right)_{\text{emission}} = \frac{2\pi}{h} |\langle f|H_I|i\rangle|^2\ \rho(E) \tag{5.12}$$

Let us write the initial and final states more explicitly to illustrate that they are composite states of electron and the photons

$$<f| \quad = \quad <\psi_{f\ \text{electron}}|<n+1| \tag{5.13}$$

$$|i> \quad = \quad |\psi_{i\ \text{electron}}>|n>$$

Then the matrix elements will be

$$<f|H_I|i> \quad = \quad \int \psi_f <n+1| \ e\alpha_\mu A^\mu \ \psi_i |n> \ d^4x \tag{5.14}$$

$$= \quad e \int \psi_f \ \alpha_\mu \psi_\iota \ <n+1|A^\mu|n> \ d^4x$$

In general we substitute (5.8) in (5.14) for A^μ. But since we are interested in emission only it is sufficient to write

$$A^\mu \quad = \quad \sqrt{\frac{2\pi}{\omega_n}} \ \hat{e}_n \ a_n^+ \ e^{-i(\vec{k}\cdot\vec{r}-\omega_n t)} \ , \quad \hbar\omega_n \quad = \quad E_f - E_i \tag{5.15}$$

Now we have

$$<f|H_I|i> \quad = \quad e\sqrt{\frac{2\pi}{\omega}} \int \psi_f \alpha_\mu \psi_i \ <n+1|a_n^+|n> \ \hat{e}_n e^{-i(\vec{k}\cdot\vec{r}-\omega t)} \tag{5.16}$$

since

$$<n+1|a_n^+|n> \quad = \quad \sqrt{n+1} \ <n+1|n+1> \quad = \quad \sqrt{n+1}$$

and

$$\psi_f \ \alpha\psi_i \quad \equiv \quad \psi_f \gamma^o \gamma \ \psi_i \quad \equiv \quad \bar{\psi}_f \gamma \psi_i$$

we shall have

$$<f|H_I|i> \; = \; e\sqrt{\frac{2\pi}{\omega}}\;\sqrt{n+1}\int d^4x(\bar{\psi}_f\gamma\psi_i)\hat{e}_n\; e^{-i(\vec{k}\cdot\vec{r}-\omega t)} \tag{5.17}$$

In our calcuations we shall assume circularly polarized photons. Circular polarization can be represented as

$$\hat{e}_{n_+} \; = \; \frac{1}{\sqrt{2}}\,(\hat{x} + i\hat{y}) \qquad \text{right-circularly polarization}$$

$$\hat{e}_{n_-} \; = \; \frac{1}{\sqrt{2}}\,(\hat{x} - i\hat{y}) \qquad \text{left-circularly polarization}$$

Now let us define $\gamma_{\mp} \; = \; \frac{1}{\sqrt{2}}\,(\gamma^1 \mp i\gamma^2)$, then $\gamma\hat{e}_{n_{\pm}} \; = \; \gamma_{\mp}$

Since we assumed $p_x = p_y = 0$ in Chapter IV, our cavity is basically one-dimensional. Then the matrix elements will be

$$<f|H_I|i> \;\equiv\; M^{\mp} \;=\; e\sqrt{\frac{2\pi}{\omega}}\;\sqrt{n+1}\int_{-\infty}^{\infty} dt \int_{-\ell/2}^{\ell/2} dz\;\psi_f\;\gamma_{\mp}\psi_i\;e^{i(\omega t-kz)} \tag{5.18}$$

where ℓ is the interaction length, and interaction time is assumed infinite to conserve energy. In the next section we will drop the numerical constants for brevity. In Section (4) will reintroduce them when we calculate the actual transition probabilities.

2. Matrix Elements

Before we proceed to calculate the matrix elements it is necessary to write down the solutions of Dirac Equation in a tapered wiggler in a concise form.

$$\psi_p^{\uparrow} = N_1 \sqrt{\frac{E+E_o}{2E_o}} \begin{pmatrix} 1 \\ \lambda_1 e^{ik_w z} e^{-ib} \\ \frac{1}{E+E_o}[cp_1 - ea\lambda_1] \\ \frac{-1}{E+E_o}[ea + c(p_1 + \hbar k_w)\lambda_1] e^{ik_w z} e^{-ib} \end{pmatrix} e^{\frac{i}{\hbar}(p_1 z - Et)}$$

$$\psi_p^{\downarrow} = N_2 \sqrt{\frac{E+E_o}{2E_o}} \begin{pmatrix} \lambda_2 e^{-ik_w z} e^{ib} \\ 1 \\ \frac{1}{E+E_o}[c(p_2 - \hbar k_w)\lambda_2 - ea] e^{-ik_w z} e^{ib} \\ \frac{-1}{E+E_o}[ea\lambda_2 + cp_2] \end{pmatrix} e^{\frac{i}{\hbar}(p_2 z - Et)} \qquad (5.19)$$

$$\lambda_1 \equiv \frac{-eac\hbar k_w}{c^2(p_1 + hk_w)^2 + \eta^2}, \qquad \lambda_2 \equiv \frac{-eac\hbar k_w}{c^2(p_2 - \hbar k_w)^2 + \eta^2} \qquad (5.20)$$

$E_o \equiv m_o c^2$, $N_1\sqrt{\frac{E+E_o}{2E_o}}$, $N_2\sqrt{\frac{E+E_o}{2E_o}}$ are normalization constants. Matrix elements are

$$M_i^{\mp} = L^2 \int_{-\infty}^{\infty} dt \int_{-\ell 1/2}^{\ell 1/2} dz\ \bar{\psi}_{p'}\ \gamma_{\mp}\ \psi_p\ e^{i(\omega t - kz)} \tag{5.21}$$

ℓ': effective interaction length

L^2: cross-sectional area of the electron beam

ω: frequency of the emitted or absorbed radiation

+: right-circularly polarized radiation

-: left-circularly polarized radiation

First, we need to evaluate $\psi_{p'}\ \gamma_{\mp}\ \psi_p$ terms. ($\bar{\psi}$) means Parli adjoint.

$$\bar{\psi}_p\ \gamma_{\mp}\ \psi_p \ \equiv\ \psi^+\ \gamma^o \gamma_{\mp}\ \psi_p$$

$$\bar{\psi}_{p'}^{\uparrow}\ \gamma_-\ \psi_p^{\uparrow} = -\frac{\sqrt{2}}{2E_o} N_1' N_1\ F_1^-\ e^{-i(k_w z - b)}\ e^{-\frac{i}{\hbar}(p_1' - p_1)z}\ e^{\frac{i}{\hbar}(E' - E)t} \tag{5.22}$$

where $\qquad F_1^- = A\lambda_1'[cp_1 - ea\lambda_1] - B[ea + \lambda_1' c(p_1' + \hbar k_w)]$

$$A \equiv \sqrt{\frac{E' + E_o}{E + E_o}}, \qquad B \equiv \sqrt{\frac{E + E_o}{E' + E_o}}, \qquad E_o \equiv m_o c^2$$

$$\bar{\psi}_{p'}^{\downarrow} \gamma_- \gamma_p^{\downarrow} = -\frac{\sqrt{2}}{2E_o} N_2' N_2 F_2^-\ e^{-i(k_w z - b)}\ e^{-\frac{i}{\hbar}(p_2' - p_2)}\ e^{\frac{i}{\hbar}(E' - E)t} \tag{5.23}$$

where $\qquad F_2^- = A[c(p_2 - \hbar k_w)\lambda_2 - ea] - B\lambda_2[ea\lambda_2' + cp_2']$

$$\bar{\psi}^{\downarrow}_{p'}\gamma_-\gamma^{\uparrow}_p = -\frac{\sqrt{2}}{2E_o} N_2'N_1 \; F_3^- \; e^{-\frac{i}{\hbar}(p_2'-p_1)} \; e^{\frac{i}{\hbar}(E'-E)t} \tag{5.24}$$

where $F_3^- = A[cp_1 - \lambda_1 ea] - B[\lambda_2' \, ea + cp_2']$

$$\bar{\psi}^{\uparrow}_{p'}\gamma_-\gamma^{\downarrow}_p = -\frac{\sqrt{2}}{2E_o} N_1'N_2 \; F_4^- \; e^{-2i(k_w z-b)} \; e^{-\frac{i}{\hbar}(p_1'-p_2)} \; e^{\frac{i}{\hbar}(E'-E)t} \tag{5.25}$$

where $F_4^- = A\lambda_1'[c(p_2-\hbar k_w)\lambda_2 - ea] - B[ea + \lambda_1 c(p_1'+\hbar k_w)]$

$$\bar{\psi}^{\uparrow}_{p'} \; \gamma_+ \; \psi^{\uparrow}_p = \frac{\sqrt{2}}{2E_o} N_1'N_1 \; F_1^+ \; e^{i(k_w z-b)} \; e^{-\frac{i}{\hbar}(p_1'-p_1)} \; e^{\frac{i}{\hbar}(E'-E)t} \tag{5.26}$$

where $F_1^+ = A[\lambda_1 c(p_1+\hbar k_w) + ea] - B\lambda_1[cp_1' - ea\lambda_1']$

$$\bar{\psi}^{\downarrow}_{p'} \; \gamma_+ \; \psi^{\downarrow}_p = \frac{\sqrt{2}}{2E_o} N_2'N_2 \; F_2^+ \; e^{i(k_w z-b)} \; e^{-\frac{i}{\hbar}(p_2'-p_2)} \; e^{\frac{i}{\hbar}(E'-E)t} \tag{5.27}$$

where $F_2^+ = A[\lambda_2 ea + cp_2]\lambda_2' - B[c(p_2' - \hbar k_w)\lambda_2' - ea]$

$$\bar{\psi}^{\downarrow}_{p'} \; \gamma_+ \; \psi^{\uparrow}_p = \frac{\sqrt{2}}{2E_o} N_2'N_1 \; F_3^+ \; e^{2i(k_w z-b)} \; e^{-\frac{i}{\hbar}(p_2'-p_1)} \; e^{\frac{i}{\hbar}(E'-E)t} \tag{5.28}$$

where $F_3^+ = A\lambda_2'[ea + \lambda_1 c(p_1+\hbar k_w)] - B\lambda_1[c(p_2'-\hbar k_w)\lambda_2'-ea]$

$$\bar{\psi}^{\uparrow}_{p'} \; \gamma_+ \; \psi^{\downarrow}_p = \frac{\sqrt{2}}{2E_o} N_1'N_2 \; F_4^+ \; e^{-\frac{i}{\hbar}(p_1'-p_2)} \; e^{\frac{i}{\hbar}(E'-E)t} \tag{5.29}$$

where $$F_4^{+} = A[\lambda_2 ea + cp_2] - B[cp_1' - ea\lambda_1']$$

We are now ready to calculate the matrix elements.

$$M_i^{\mp} = \mp L^2 \int_{-\infty}^{\infty} dt\, e^{i(\frac{E'}{\hbar} - \frac{E}{\hbar} + \omega)t} \int_{-\ell'/2}^{\ell'/2} dz\, \frac{\sqrt{2}}{2E_o} N_m' N_k\, F_i^{\mp}\, e^{-\frac{i}{\hbar}(p_k' - p_m + \hbar k)}$$

$$\times\ e^{in(k_w z - b)}$$

$$n = 0, \mp 1, \mp 2$$

$$k = 1,2 \qquad m = 1,2$$

$$M_i^{\mp} = \mp L^2 \delta(\frac{E'}{\hbar} - \frac{E}{\hbar} + \omega)\, \frac{\sqrt{2}}{2E_o} N_k' N_m\, F_i^{\mp}\, G(w_i^{\mp}) \tag{5.30}$$

where

$$G(w_i^{+}) = \frac{1}{W_i^{+}} \sin(\frac{\ell' W_i^{+}}{2})\, e^{-inb}$$

(5.31)

$$G(W_i^{-}) = \frac{1}{W_i^{-}} \sin(\frac{\ell' w_i^{-}}{2})\, e^{inb}$$

$$W_1^{+} = \frac{p_1'}{\hbar} - \frac{p_1}{\hbar} + k - k_w \qquad W_1^{-} = \frac{p_1'}{\hbar} - \frac{p_1}{\hbar} + k + k_w$$

$$W_2^{+} = \frac{p_2'}{\hbar} - \frac{p_2}{\hbar} + k - k_w \qquad W_2^{-} = \frac{p_2'}{\hbar} - \frac{p_2}{\hbar} + k + k_w \tag{5.32}$$

$$W_3^{+} = \frac{p_2'}{\hbar} - \frac{p_1}{\hbar} + k - 2k_w \qquad W_3^{-} = \frac{p_2'}{\hbar} - \frac{p_1}{\hbar} + k$$

$$W_4^{+} = \frac{p_1'}{\hbar} - \frac{p_2}{\hbar} + k \qquad W_4^{-} = \frac{p_1'}{\hbar} - \frac{p_2}{\hbar} + k + 2k_w$$

$F_i^{\mp}$ have already been listed.

$$\ell' = \left[\ell k_w - b\ell n \frac{Z_{max}}{Z_{min}}\right] \frac{1}{k_{wu}} \tag{5.33}$$

To calculate transition rates we shall need $|M_i^{\mp}|^2$

$$|M_i^{\mp}|^2 = [\frac{2L^4}{E_o^2}][N'N\ F_i^{\mp}]^2 |G(W_i^{\mp})|^2 \tag{5.34}$$

We shall evaluate $(N'N)^2$ and $(F_i^{\mp})^2$ terms in the ultrarelativistic limit.

In the ultrarelativistic limit

$$A \equiv \sqrt{\frac{E'+E_o}{E+E_o}} \rightarrow 1, \quad B \equiv \sqrt{\frac{E+E_o}{E'+E_o}} \rightarrow 1$$

λ_1^2, λ_2^2, $\lambda_1\lambda_2$, $\lambda_1'\lambda_2$, $\lambda_1\lambda_2'$ terms can be neglected because these terms are of the order of $(\frac{ea}{cp_z})^2$ which is very small in the ultrarelativistic limit.

$$(F_1^-)^2 \cong e^2a^2 + 2\hbar k_w\ eac\ \lambda_1'$$

$$(F_2^-)^2 \cong e^2a^2 + 2\hbar k_w\ eac\ \lambda_2 \tag{5.35}$$

$$(F_3^-)^2 \cong c^2(p_1-p_2')^2 - 2eac(p_1 - p_2')(\lambda_1 + \lambda_2')$$

$$(F_4^-)^2 \cong 0$$

$$(F_1^+)^2 \cong e^2a^2 - 2\hbar k_w\ eac\lambda_1$$

$$(F_2^+)^2 \cong e^2a^2 - 2\hbar k_w\ eac\lambda_2' \qquad (5.36)$$

$$(F_3^+) \cong 0$$

$$(F_4^+)^2 \cong c^2(p_2-p_1')^2 + 2eac(p_2-p_1)(\lambda_1'+\lambda_2)$$

we also have

$$\lambda_1 \cong -\frac{ea}{3cp_z}, \quad \lambda_2 \cong +\frac{ea}{3cp_z}, \quad \lambda_1' + \lambda_2 \cong 0, \quad \lambda_1 + \lambda_2' = 0$$

In nonrelativistic Quantum Mechanics a plane wave is either delta function normalized or normalized to give unit probability of finding the particle in a finite volume, that is, $\psi^*\psi = 1$. Analogous normalization for the relativistic plane wave is

$$\bar{U}U \equiv \tilde{U}\,\gamma^o U = 1 \qquad (5.37)$$

The factor $\sqrt{\frac{E+E_o}{2E_o}}$ was introduced in the unnormalized solutions so that $\bar{U}U = 1$. When we normalize we would like to keep this factor and state the normalization as

$$\psi_p^+\,\psi_p = 1 \qquad (5.38)$$

In our case

$$N_1^2\,\frac{E+E_o}{2E_o}\,[1 + \lambda_1^2 + \frac{1}{(E+E_o)^2}\,([cp_1 - ea\lambda_1]^2 + [ea + c(p_1 + \hbar k_w)\lambda_1]^2) = 1$$

$$N_2^2 \frac{E+E_o}{2E_o}[\lambda_2^2 + 1 + \frac{1}{(E+E_o)^2}([c(p_2 - \hbar k_w)\lambda_2 - ea]^2 + [ea\lambda_2 + cp_2]^2) = 1$$

Evaluating these expressions in the ultrarelativistic limit we obtain

$$N_1^2 \cong \frac{2E_o}{[2cp_z - c^2\hbar k_w]}, \qquad N_2^2 \cong \frac{2E_o}{[2cp_z + c^2\hbar k_w]} \tag{5.39}$$

then,

$$(N_1'N_1)^2 \cong \frac{E_o^2}{c^2 p_z' p_z}[1 - \frac{1}{2}\hbar k_w \frac{(p_z'+p_z)}{p_z'p_z} + \frac{1}{4}(\hbar k_w)^2 \frac{(p_z'+p_z)^2}{p_z'^2 p_z^2} + \cdots\cdot] \tag{5.40}$$

$$(N_2'N_2) \cong \frac{E_o^2}{c^2 p_z' p_z}[1 + \frac{1}{2}\hbar k_w \frac{(p_2'+p_z)}{p_z'p_z} + \ldots] \tag{5.41}$$

$$(N_2'N_1)^2 \approx (N_1'N_2)^2 \approx \frac{E_o^2}{c^2 p_z' p_z} \tag{5.42}$$

3. Frequencies

The best method to calculate frequencies is to use the conservation of momentum and energy. Let us write the four-vectors of effective momentum

$$P^{\mu}_{eff_1} = (\frac{E}{c},0,0,p_1), \qquad P^{\mu}_{eff_2} = (\frac{E}{c},0,0,p_2) \tag{5.43}$$

Squares of these four-vectors will be

$$(P^{\mu}_{eff_1})^2 = \frac{E^2}{c^2} - p_1^2, \qquad (P^{\mu}_{eff_2})^2 = \frac{E^2}{c^2} - p_2^2 \tag{5.44}$$

After the emission or absorption of a photon, we have

$$(p'^{\mu}_{eff_1})^2 = \frac{E'^2}{\hbar} - p'^2, \quad (p'^{\mu}_{eff_2})^2 = \frac{E'^2}{\hbar^2} - p_2'^2 \tag{5.45}$$

The recoil due to a photon is not enough to conserve the momentum. The additional momentum is provided by the wiggler magnetic field. The observer on the electron sees the wiggler field as an incoming electromagnetic wave even though from the laboratory frame it is just a static magnetic field with ulternating polarity. This motivates us to write the following equation for conservation of momentum

$$p'^{\mu}_{eff} = p^{\mu}_{eff} - \hbar q^{\mu} + n\hbar k^{\mu}_{w} \qquad n = 0, \pm 1, \pm 2 \tag{5.46}$$

where

$$k^{\mu}_{w} = (0,0,0,n\bar{k}_w)$$

$$q^{\mu} = (\tfrac{\omega}{c},0,0,k)$$

Here $n\hbar\bar{k}_w$ is the magnetic quantum of the wiggler field $\hbar k$ is the quantum of the radiation field. In the case of a uniform wiggler $\bar{k}_w = k_w$. If the wiggler is tapered, we take $\bar{k}_w$ as the average wavenumber. In the tapering we proposed, this average wavenumber is

$$\bar{k}_w = k_w - \frac{b}{\ell} \ln \frac{Z_{max}}{Z_{min}} \tag{5.47}$$

We can also show that Equation (5.46) is indeed the correct conservation of momentum equation by looking at the line shape of emission or absorption. We derived the line-shapes when we calculated the matrix elements. We now modify Equations (5.31) and (5.32) by replacing $k_w(z) \rightarrow \bar{k}_w$.

$$G(\bar{W}_i) = \frac{1}{\bar{W}_i}\sin\left(\frac{\ell'\bar{W}_i}{2}\right) \qquad (5.48)$$

$$\bar{W}_i^{\mp} = \frac{p'}{\hbar} - \frac{p}{\hbar} + k \pm n\bar{k}_w, \quad n = 0,1,2 \qquad (5.49)$$

Since the line shape is very sharp and peaks at $\bar{W}_i = 0$ we can obtain the conservation of momentum equation by letting $\bar{W}_i = 0$. We then transformed (5.49) into four-vector notation. This shows that all the information relating to frequencies and conservation laws are contained in the matrix elements.

We now combine Equations (5.44), (5.45) and (5.46) and obtain

$$(E - \hbar\omega_1)^2 - (p_1 - \hbar k + n\hbar\bar{k}_w)^2 = E'^2 - p_1'^2 \qquad (5.50)$$

this leads to

$$\hbar\omega_1 = -\frac{c^2p_1^2 - c^2p_1'^2 + 2np_1\hbar c^2\bar{k}_w + E'^2 - E^2 + (n\hbar c\bar{k}_w)^2}{2(E - cp_1 - \hbar cn\bar{k}_w)} \qquad (5.51)$$

where $n = \mp 1$ -: emission
+: absorption

We used the dispersion relation $\omega = ck$ which is valid for the emission in forward direction. Above expression for $\hbar\omega_1$ is exact. Similarly other frequencies can be calculated

$$\hbar\omega_2 = -\frac{c^2p_2^2 - c^2p_2'^2 + 2np_2\hbar c^2\bar{k}_w + E'^2 - E^2 + (n\hbar c\bar{k}_w)^2}{2(E - cp_2 - n\hbar c\bar{k}_w)}, \ n=\mp 1 \qquad (5.52)$$

$$h\omega_3 = -\frac{c^2p_2^2 - c^2p_1'^2 + 2np_1\hbar c^2\bar{k}_w + E'^2 - E^2 + (n\hbar c\bar{k}_w)^2}{2(E - cp_2 - n\hbar c\bar{k}_w)}, \ n=\mp 2 \qquad (5.53)$$

$$\hbar\omega_4 = -\frac{c^2p_1^2 - c^2p_2'^2 + E'^2 - E^2}{2(E - cp_1)} \quad (5.54)$$

A typical FEL, especially the ones which operate in visible and far infrared, uses highly relativistic electrons (for Los Alamos Laser $\gamma = 200$) [2]. This makes it possible to evaluate the exact expressions for frequencies in the ultra-relativistic limit.

$$E = E_o\gamma \approx cp_z \approx cp_1 \approx cp_2$$

we shall ignore the terms of the order of $\left(\frac{ea}{cp_z}\right)^2$. Also,

$$\frac{\bar{k}_w c\hbar}{E_o\gamma} << 1, \quad \frac{\bar{k}_w c\hbar}{cp_z} << 1, \quad \frac{1}{\gamma^2} \equiv \left(\frac{E_o}{cp_z}\right)^2 << 1$$

In the ultrarelativistic limit determining equations for p_1 and p_2 become

$$P_1 = \sqrt{\left(p_z + \frac{\hbar\bar{k}_w}{2}\right)^2 - \left(\frac{ea}{c}\right)^2} - \frac{\hbar k_w}{2} \approx p_z - \left(\frac{ea}{c}\right)^2 \frac{1}{(2p_z + \hbar\bar{k}_w)} \quad (5.55)$$

$$P_2 = \sqrt{\left(p_z - \frac{\hbar\bar{k}_w}{2}\right)^2 - \left(\frac{ea}{c}\right)^2} + \frac{\hbar\bar{k}_w}{2} \approx p_z - \left(\frac{ea}{c}\right)^2 \frac{1}{(2p_z - \hbar\bar{k}_w)} \quad (5.56)$$

Applying these approximations and using

$$E = \sqrt{c^2p_z^2 + E_o^2} \cong cp_z + \frac{1}{2}\frac{E_o^2}{c^2p_z^2} + \cdots$$

the frequencies can be written as

$$\uparrow\uparrow \quad \hbar\omega_1 = \frac{-2n\gamma^2\hbar\bar{k}_w c}{\left[1 + \left(\frac{ea}{E_o}\right)^2 - 2n\gamma\,\frac{\hbar\bar{k}_w c}{E_o}\right]} \qquad \begin{array}{l} n = \mp 1 \text{ (fundamental)} \\ -\text{: emission} \\ +\text{: absorption} \end{array} \tag{5.57}$$

$$\downarrow\downarrow \quad \hbar\omega_2 = \frac{-2n\gamma^2\hbar\bar{k}_w c}{\left[1 + \left(\frac{ea}{E_o}\right)^2 - 2n\gamma\,\frac{\hbar\bar{k}_w c}{E_o}\right]} \qquad n = \mp 1 \text{ (fundamental)} \tag{5.58}$$

$$\downarrow\uparrow \quad \hbar\omega_3 = \frac{-2n\gamma^2\hbar\bar{k}_w c}{\left[1 + \left(\frac{ea}{E_o}\right)^2 - 2n\gamma\,\frac{\hbar\bar{k}_w c}{E_o}\right]} \qquad n = \mp 2 \text{ (1st harmonic)} \tag{5.59}$$

$$\uparrow\downarrow \quad \hbar\omega_4 = \frac{2\gamma^2\hbar\bar{k}_w c}{\left[1 + \left(\frac{ea}{E_o}\right)^2\right]} \qquad \text{emission only} \quad \text{(0'th harmonic)} \tag{5.60}$$

The fundamental optical wavelength $(\lambda = \frac{2\pi}{\omega})$ will be

$$\gamma = -\frac{n\lambda_w}{2\gamma^2}\left[1 + \left(\frac{ea}{E_o}\right)^2 - 2n\gamma\,\frac{\hbar\bar{k}_w c}{E_o}\right] \qquad \begin{array}{l} n = \mp 1 \\ \lambda_w = \frac{1}{\bar{k}_w} \end{array} \tag{5.61}$$

The reason n = -1, -2 gives emission is because negative frequency means absorption. In other words positive energy of quanta necessarily implies emission.

The terms in the expression for fundamental optical wavelength can be explained physically.

1st term: This term is the same as the expression we obtain from relativistic doppler shift argument in Chapter II. In that argument it was assumed that wiggler magnetic field does not affect the path of the ultrarelativistic electron.

2nd term: Magnetic field actually modifies the motion of the electron, second term shows the effect of this interaction.

3rd term: Lowest order correction due to recoil. When the electron emits photon in the forward direction it recoils. This is purely Quantum Mechanical, that is, photon is a Quantum Mechanical concept. That is why classical mechanical calculations cannot account for this term.

Most important of all is the fact that this term is the basis of gain mechanism. Expressions for emission and absorption are identical except this term. In other words the difference between the emitted and absorbed radiation frequency is due to this term and this makes the transition rates for absorption adn emission different, hence the gain in radiation follows. Details of gain mechanism will be elaborated in the next section.

Finally in this section we would like to examine the correspondence between frequencies and matrix elements. We know that peak of the lineshape occurs when $\bar{W} = 0$ which determines the frequency.

If

$$
\begin{aligned}
\bar{W}_1^- &= 0 \quad \text{then} \quad p_1' = p_1 - \hbar k - \hbar\bar{k}_w && (5.62)\\
\bar{W}_2^- &= 0 \quad \text{then} \quad p_2' = p_2 - \hbar k - \hbar\bar{k}_w\\
\bar{W}_4^- &= 0 \quad \text{then} \quad p_1' = p_2 - \hbar k - 2\hbar\bar{k}_w\\
\bar{W}_3^- &= 0 \quad \text{then} \quad p_2' = p_1 - \hbar k\\
\bar{W}_1^+ &= 0 \quad \text{then} \quad p_1' = p_1 - \hbar k + \hbar\bar{k}_w\\
W_2^+ &= 0 \quad \text{then} \quad p_2' = p_2 - \hbar k + \hbar\bar{k}_w\\
W_3^+ &= 0 \quad \text{then} \quad p_2' = p_1 - \hbar k + 2\hbar\bar{k}_w\\
\bar{W}_4^+ &= 0 \quad \text{then} \quad p_1' = p_2 - \hbar k
\end{aligned}
$$

When we compare these expressions to

$$P'^{\mu}_{eff} = P^{\mu}_{eff} - \hbar q^{\mu} + n\hbar k^{\mu}_{w}$$

we shall find out that $\bar{W}_1^- = 0$, $\bar{W}_2^- = 0$ corresponds to $n = -1$ which indicates emission. By similar arguments we conclude that matrix elements associated with emission are M_1^-, M_2^-, M_3^-, M_4^-, and M_4^+. But $|M_4^-|^2 \approx 0$ so we choose M_1^-, M_2^-, M_3^- and M_4^+ as the 4 matrix elements associated with 4 different emission frequencies.

4. Transition Rates, Gain

In general transition probability per second is given by

$$d\Gamma = \frac{2\pi}{\hbar} |M|^2 \rho(E) \qquad \text{(Fermi's golden rule)} \tag{5.63}$$

where

$$|M|^2 = \frac{1}{2} \sum_{i=1}^{4} |M_i|^2 \tag{5.64}$$

and $\rho(E)$: density of final states.

The reason we need to sum the mod-squares of matrix elements over final states and average this sum over initial spin states is dictated by the conditions of the experiment. In general the incoming beam of electrons will be unpolarized and the spins of the outgoing electrons are not observed.

We previously calculated the matrix elements $M_i^{\mp}$ without the proper numerical factors. We now introduce them. Each coupling of an electron to a real photon introduces a factor $\frac{e\sqrt{4\pi}}{\sqrt{2\omega}}$ ($\sqrt{2\omega}$ is in the denominator since the photon wavefunction is normalized to 1 photon per unit volume), and each coupling of an electron to a virtual photon introduces $-i\,\frac{4\pi e^2}{2\omega}$. In our case we neglected space charge effects mainly because we look at the Compton regime where single particle approach is adopted. We replace

$$|M|^2 \rightarrow \frac{4\pi}{2\omega} e^2 \frac{1}{2} \Sigma |M_i|^2$$

Density of final states for the emitted photons is $\frac{d^3k}{r^3(2\pi)^3}$ in 3-dimensions. But in most practical experiments of FEL $p_x = p_y = 0$ which is what we assumed in the beginning. Then our problem is basically 1-dimensional. In 1-dim density of final states for photons is $dk/2\pi V$. This factor is already incorporated in the numerical factor $\frac{4\pi}{2\omega}$ since the photon wavefunctions are normalized to 1 photon per unit volume.

Density of final states for electron making free-free transition is taken care of by integrating the transition rate with respect to p_z', since p_z' is continuously distributed around a fixed p_z value.

$$\int d\Gamma = \Gamma_{\text{per second}} = \frac{2\pi}{\hbar} \int \frac{4\pi e^2}{2\omega} (\frac{1}{2} \Sigma |M_i|^2 \, dp_z' \, \frac{1}{V^3}$$

$$\Gamma_{\text{single pass}} = \frac{\ell'}{c} \frac{2\pi\ell^2}{\omega\hbar} \int \frac{dp_z'}{V^3} \frac{1}{2} \Sigma |M_i|^2 \tag{5.65}$$

We shall evaluate the transition rates for emission only. Transition rate of absorption is given by $\Gamma(\omega+\delta)$ where δ is the difference between absorption and emission frequencies (fundamental frequency).

$$\delta = \omega(\text{absorption}) - \omega(\text{emission}) \cong c\bar{k}_w (2\gamma)^3 (\frac{k_w \hbar c}{E_o}) [1 + (\frac{ea}{E_o})^2]^{-2} \tag{5.66}$$

$\langle\omega\rangle$ = average frequency = resonance frequency

$$= \frac{2c\bar{k}_w \gamma^2}{[1 + (\frac{ea}{E_o})^2]}$$, then δ can be expressed in terms of $<\omega>$.

$$\delta = 4 \frac{\bar{k}_w \hbar c}{E_o} \frac{\gamma<\omega>}{[1 + (\frac{ea}{E_o})^2]} \tag{5.67}$$

Suppose there are N photons in the wiggler cavity, then

$$\Gamma_{\text{stimulate emission}} = N\,\Gamma_{\text{spontaneous emission}} \tag{5.68}$$

$$\Gamma_{\text{stimulated absorption}} = \Gamma_{\text{stim. emis.}}(\omega+\delta) \tag{5.69}$$

The average energy loss of an electron passing the wiggler field in ℓ'/c secinds is

$$\Delta E = \hbar\omega \frac{\ell'}{c} (\Gamma_{\text{stim. emiss.}}(\omega) - \Gamma_{\text{stim. emiss.}}(\omega+\delta))$$

$$= \hbar\omega \frac{\ell}{c} N\delta \frac{\partial\Gamma_{\text{spont}}(\omega)}{\partial\omega} \tag{5.70}$$

The gain is defined as the ratio of the energy loss/cm^3 and the energy density of the stimulating field.

$$g = \frac{\Delta E\ \rho eV}{\hbar c\omega N} \qquad \rho e\text{: electron density}$$

therefore,

$$g = -\frac{\ell}{c} V\delta\rho_e \frac{\partial\Gamma_{\text{spont.}}}{\partial\omega} \tag{5.80}$$

Let us evaluate $\frac{1}{2}\Sigma|M_i|^2$ in the ultrarelativistic limit

$$\frac{1}{2}\Sigma|M_i|^2 = \frac{1}{2}\{\frac{2L^4}{E_o^2}([N_1'N_1F_1^-G(\bar{W}_1^-)]^2 + [N_2'N_2F_2G(\bar{W}_2^-)]^2$$

$$+ [N_2'N_1F_3^-G(\bar{W}_3^-)]^2 + [N_1'N_2F_4^+G(\bar{W}_4^+)]^2)\} \quad (5.81)$$

We kept only the emission terms.

$$\frac{1}{2}\Sigma|M_i|^2 = \frac{L^4}{E_o^2}\{\frac{E_o^2}{c^2p_z'p_z}[1 - \frac{1}{2}\hbar\bar{k}_w\frac{p_z'+p_z}{p_z'p_z} + \cdots](e^2a^2 + 2\hbar\bar{k}_w eac\lambda_1')G^2(\bar{W}_1^-)$$

$$+ \frac{E_o^2}{c^2p_z'p_z}[1 + \frac{1}{2}\hbar\bar{k}_w\frac{p_2'+p_2}{p_z'p_z} + \cdots](e^2a^2+2\hbar\bar{k}_w eac\lambda_2)G^2(\bar{W}_2^-)$$

$$+ \frac{E_o^2}{c^2p_z'p_z}[c^2(p_1-p_2')^2 - 2eac(p_1-p_2')(\lambda_1+\lambda_2')]G^2(\bar{W}_3^-)$$

$$+ \frac{E_o^2}{c^2p_z'p_z}[c^2(p_z-p_1')^2 + 2eac(p_2-p_1')(\lambda_1'+\lambda_2)]G^2(\bar{W}_4^+)\} \quad (5.82)$$

in the ultra-relativistic limit

$$\lambda_1 \cong -\frac{ea}{3cp_z'}, \quad \lambda_2 \cong \frac{ea}{3cp_z} \quad (5.83)$$

then,

$$\lambda_1' + \lambda_2 \cong 0, \quad \lambda_1 + \lambda_2' \approx 0, \quad p_2-p_1' \approx 0$$

$$p_1-p_2' \approx 0$$

Also $\bar{W}_1^- \cong \bar{W}_2^-$ since $p_1 \approx p_2$, then we can denote $W \equiv \bar{W}_1^- = \bar{W}_2^-$. Therefore,

$$\frac{1}{2}\Sigma|M_i|^2 = \frac{2L^4e^2a^2}{c^2p_z'p_z}[1 - \frac{1}{2}\hbar\bar{k}_w \frac{p_z'+p_z'}{p_z'p_z} + \frac{1}{4}\hbar^2\bar{k}_w^2 \frac{(p_z'+p_z)^2}{p_z'^2p_z^2} + \cdots]G^2(W)$$

where

$$G^2(W) = \frac{1}{W}\sin(\frac{\ell'W}{2}), \quad W = \frac{p_1'}{\hbar} - \frac{p_1}{\hbar} + k + \bar{k}_w$$

For the transition rate per single pass we have

$$\Gamma = \frac{e^2}{\hbar}\frac{\ell'}{c^3}\frac{4\pi L^4}{\omega p_z}\frac{e^2a^2}{V^3}\int\frac{1}{p_z'}[1 - \frac{1}{2}\hbar\bar{k}_w\frac{p_z'+p_z}{p_z'p_z} + \cdots]G^2(W)\ dp_z' \qquad (5.85)$$

The first term will give us the rate for fundamental frequency. Therefore,

$$\Gamma^{(f)}_{\text{spontaneous}} = \frac{e^2}{\hbar}\frac{\ell'}{c^3}\frac{4\pi L^4}{\omega p_z}\frac{e^2a^2}{V^3}\int dp_z'\frac{G^2(w)}{p_z'} \qquad (5.86)$$

Fortunately we do not have to evaluate the integral explicitly because of the sharp maximum of function $G^2(w)$. $G^2(w)$ behaves much like a Dirac delta function. So we can replace all p_z' outside $G(w)$ by the corresponding values of the maximum. That is,

$$\Gamma^{(f)}_{\text{spont. emiss.}} = \frac{e^2}{\hbar}\frac{\ell'}{c^3}\frac{4\pi L^4}{p_z p\omega}\frac{e^2a^2}{V^3}G^2(w), \quad cp \equiv \sqrt{(E-\hbar\omega_1)^2 - E_o^2}$$

Before we take the derivative of this expression with respect to ω, we need to look at the derivative of W with respect to ω.

In terms of the longitudinal velocity β_a of the electron in the wiggler field

$$\gamma_a^2 = \frac{\gamma^2}{[1 + (\frac{ea}{E_o})^2]} \qquad \beta_a^2 = \beta^2 - (\frac{ea}{E_o\gamma})^2 \tag{5.87}$$

we have

$$W = \bar{k}_w + \frac{\omega}{c} - \frac{\omega}{c\beta_a} \qquad (\text{since } \frac{p_1'}{\hbar} - \frac{p_1}{\hbar} \approx \frac{k}{\beta_a})$$

$$\frac{\partial w}{\partial \omega} = \frac{[1 + (\frac{ea}{E_o})^2]}{2\gamma^2} \tag{*5.88}$$

we used the approximations $\gamma >> 1$, $1-\beta \simeq \frac{1}{2\gamma^2}$, also

$$\frac{\partial G^2(w)}{\partial w} = \frac{1}{W^3}[1 - \cos \ell' w - \frac{1}{2}\ell' w \sin \ell w] \tag{5.89}$$

finally gain expression can be written as

$$g = 16\pi^2(\frac{e^2}{E_o})\bar{k}_w(\frac{ea}{E_o})^2 \frac{\rho e}{(\gamma w)^3}[1 - \cos \ell' w - \frac{1}{2}\ell w \sin \ell w] \tag{5.90}$$

where

$$\bar{k}_w = k_w - \frac{b}{\ell} \ln \frac{Z_{max}}{Z_{min}}$$

$$\ell' = \ell[\frac{k_w}{k_{wu}} - \frac{b}{\ell k_{wu}} \ln \frac{Z_{max}}{Z_{min}}]$$

k_{wu}: wavenumber of the untapered wiggler.

It is easily seen that when $b \to 0$ gain expression reduces to the standard gain expression for uniform wiggler.

With the numerical values pertaining to the Los Alamos FEL it can be shown that the gain increases by a factor of 2.19 if the same geometrical length is used and by a factor of 1.216 if the same number of magnet elements are used.A better way of stating the amount of increase would be as follows.

If the same geometrical length is used

$$\text{Increase factor of single-pass gain} = (k_w / k_{wu})^4$$

If the same number of magnet elements are used

$$\text{Increase factor of single-pass gain} = k_w / k_{wu}$$

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CHAPTER VI

SUMMARY AND CONCLUSIONS

Free electron laser physics has attracted the attention of quite a few theorists and experimenters in recent years, plasma physicists, laser QED physicists, computer programmers and electrical engineers. We can attribute this phenomenon to the fact that physicists in different fields had found something interesting in FEL theory and experiment.

The different approaches to FEL are equally valid and practical as far as the end results are concerned and hence there is no distinct advantage in using classical theory over quantum mechanical theory. The most crucial word in FEL physics is "Gain". Although classical and quantum theories arrive at approximately the same gain expression which is given in Equation (5.90), the conceptual physical mechanisms explaining the "gain" are somewhat different in the two theories. Gain in quantum theory of FEL is related to the concepts of "photon" and "recoil". Gain is possible because the frequency of the emitted photon is less than the frequency of the absorbed photon due to recoil. This frequency difference causes a difference between the rates of emission and absorption and since the rate for emission is greater than the rate for absorption, gain follows. In the classical theory gain is explained in terms of the number of decelerated electrons being greater than the number of accelerated electrons. The average energy loss of the electron beam is the gain in radiation field.

Energy extraction from the electrons in the form of radiation is a consequence of energy and momentum conservation requirements. If an electron is to emit or absorb electromagnetic radiation, then the following conditions must be satisfied.

a) Energy must be conserved between the initial and final states of the electron and radiation.

b) Momentum must be conserved between the initial and final states of the electron and radiation.

c) The dispersion relationships for the interacting particles must be satisfied.

The emission or absorption of electromagnetic radiation by charged particles in vacuum is an impossibility because there is nothing to conserve momentum. There are numerous mechanisms that satisfy the above conditions. We would like to name a few of these mechanisms which have been used in various FEL experiments.

a) Cerenkov effect

b) Compton scattering

c) Magnetic Bremsstrahlung

d) Static fields

e) Limited interaction length

The first four of the mechanisms are familiar and are to be found in standard texts. The last one is interesting and simple enough to clarify here. If a free electron were to emit a photon in the forward direction, imposing energy conservation leads to a momentum gap

$$\delta_p \cong \hbar\omega \left[\frac{1}{v} - \frac{1}{c}\right] \approx \frac{\hbar\omega}{2c\gamma^2}$$

where it has been assumed that photon energy is much less than the electron energy. If the interaction length is limited, then the uncertainity principle requires.

$$(\delta_z)(\delta_p) \leq \frac{\hbar}{2}$$

This means that an uncertainity exists in the momentum that could allow single photon emission from a free electron. If γ is sufficiently large, then the interaction length can be many wavelengths long. This is the mechanism on which quantum theories of FEL have been based. In Chapter V when we calculated the matrix elements we used an infinite interaction time but a finite interaction length. The finite interaction length makes the line shape G(W) sharp enough to cause difference in the rates of emission and absorption.

In this thesis we derived the solutions of Dirac Equation for the motion of an electron in a tapered wiggler field. We showed that these solutions reduce to those of uniform wiggler when the tapering is removed ($b \rightarrow o$). Our proposed tapering is such that the wavenumber of the wiggler increases in the z-direction as

$$k_w(z) = k_w - \frac{b}{z} \qquad Z_{min} = 800 \text{ cm} \qquad b = 96$$

$$Z_{max} = 2400 \text{ cm}$$

for a 1600 cm wiggler, these parameters having a bearing on experiment. Using the solutions for motion in a tapered wiggler we calculated the matrix elements, radiation frequencies and the gain. Final expression for gain shows that the tapering we proposed enhances the gain over the one with linear tapering. With the numerical values pertaining to Los Alamos FEL it can be shown that gain increases by a factor of 2.19 if the same geometrical length is used and bv a factor of 1.216 if the same number of magnet elements are used.

We conclude that the simplicity of quantum mechanical methods can easilv compete with the definiteness of the classical mechanical calculations.Furthermore the quantum theory of FEL has the advantage of including the effects of quantized electromagnetic field and the electron spin.In other words the quantum theorv of FEL provides a deeper and more detailed understanding of the FEL physics.

The natural extension of this study would be the investigation of the xuv and x-ray FEL.In these frequencv regions it is necessary to use the quantum theorv because as the photons become more energetic the probability of the pair creations and annihilations increase.

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