GEOMETRIC CONSTRUCTIONS

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GEOMETRIC CONSTRUCTIONS

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PREFACE

The solutions of geometric construction problems have always intrigued me. The simplicity of the problems can frustrate the begining geometer. Much to the suprise of most mathematicians, algebra plays a fundamental role in the understanding of the solutions.

This paper analyzes the properties of a construction tool called the Mira. Also included is a discussion of the constructible polygons a Mira can construct. Later, compass and straightedge constructions in the hyperbolic plane are discussed.

At this time, I wish to express my gratitude and deepest appreciation to my major adviser, Dr. Joel K. Haack, who has been an invaluable mentor and friend. Also, I would like to thank the other members of my committee, Dr. Paul F. Duvall, and Dr. David C. Ullrich, for their assistance.

I would like to thank Professor Phil Briggs from East Central University for his help in locating an importamt resource. A special thanks are due: Bobby Winters, for his encouragement along the way; Tom Potter, for his help in drawing several figures; and Steve Roy, for helping me to relax by pulling me away to play golf.

Of course, I thank my family for their constant encouragement and support. My wife, Angela, deserves much

credit for her constant love, devotion, and support for me while I finished the paper. This paper is dedicated to my son, Marc who has been a source of inspiration.

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CHAPTER I

INTRODUCTION

The idea of drawing geometrical figures is not new. Certainly, when people first started to draw geometric figures, they must have wondered what figures they could draw and which they could not. It was the Greeks, however, who first formalized and popularized such problems. Today these problems are often presented to high-school geometry classes.

The tools most commonly used in the drawing of the geometrical figures are the compass and straightedge. Much has been proved concerning the efficacy of these tools, hence we will not dwell on this. Rather, we introduce and analyze a new tool, called the Mira, produced by Creative Publications, Inc. The Mira consists of a rectangular piece of red plastic that is supported vertically by two end pieces; it resembles an I-beam. The reason the Mira is made of red plastic is so that one can see through it but, at the same time, one can see the reflection of objects on the other side. In this manner, the reflection of a line in the plane can be constructed.

Accompanying the Mira is an exercise booklet. In the booklet, the student is led through several construction techniques, culminating in the trisection of an arbitrary angle. It was this capability of the Mira, along with the ease of its use, that make it an attractive target of analysis. This work analyzes the constructions possible with the Mira, along with a consideration of constructions in the hyperbolic plane.

Chapter 2 considers constructions with a Mira and results in a characterization of the numbers a Mira can construct: construct second-degree and third-degree extensions of the rationals.

Chapter 3 examines the class of regular polygons the Mira can construct. The Mira can construct regular polygons with a prime number of sides when the prime is of the form $2^k \; 3^j + 1$.

Chapter 4 then proceeds to the hyperbolic plane. Here a theorem is proved that states the relationship between constructions in the hyperbolic plane and constructions using Euclidean tools in the upper-half-plane model.

The paper concludes with some suggestions for further research.

CHAPTER II

CONSTRUCTIONS IN THE EUCLIDEAN PLANE WITH A MIRA

The compass and straightedge are familiar tools to the beginning geometer. Nevertheless, a new approach is needed to solve some of the construction problems presented by the ancient Greeks. By a construction with a compass and straightedge, we mean a finite list of operations used to obtain a locus of points. The operations allowed are

1) Locate the intersection of two lines;

2) Locate the intersection of two circles;

3) Locate the intersection of a line and a circle.

A number N is constructible if, given a unit segment, a construction can be given that produces a segment of length N.

Although most mathematicians are familiar with the characterization of numbers that are constructible via compass and straightedge, we will state it here for reference. A number is constructible if and only if it is an element of a field obtained in a finite tower of quadratic extensions of the rationals [7, p. 21]. An early formulation of this theorem was given by Descartes in 1637 [11, p.106]. The remainder of this section is devoted to developing a similar characterization for the numbers constructible with a Mira. A number N is said to be <u>Mira constructible</u> if, given a unit segment, a construction with a Mira can be given that produces a segment of length N. The operations allowed in a construction with a Mira are

1) Connect two points with a straightedge
2) Locate the intersection of two nonparallel lines
3) Reflect a point through a line
4) Reflect a line through a line

A fifth operation will be discussed in connection with the trisection of an angle. This operation is not needed for the first two results.

First, we compare the field E of numbers constructible with compass and straightedge with the set F of numbers that are constructible with the Mira.

<u>Lemma 1</u>

The set F of Mira constructible numbers is a field. If r is a nonnegative element of F, then the square root of r is also an element of F.

<u>Proof</u>

We will demonstrate three of the operations: addition, multiplication, and extraction of square roots. The others are similar. For the addition of two segments AC and DE, the Mira is positioned to reflect one segment AC to A'C' so that one endpoint A'=D is common to both segments. Then the Mira is used to reflect the segment A'C' to A''C'' so that both segments A'C''and DE lie on a line; see Figure 1.

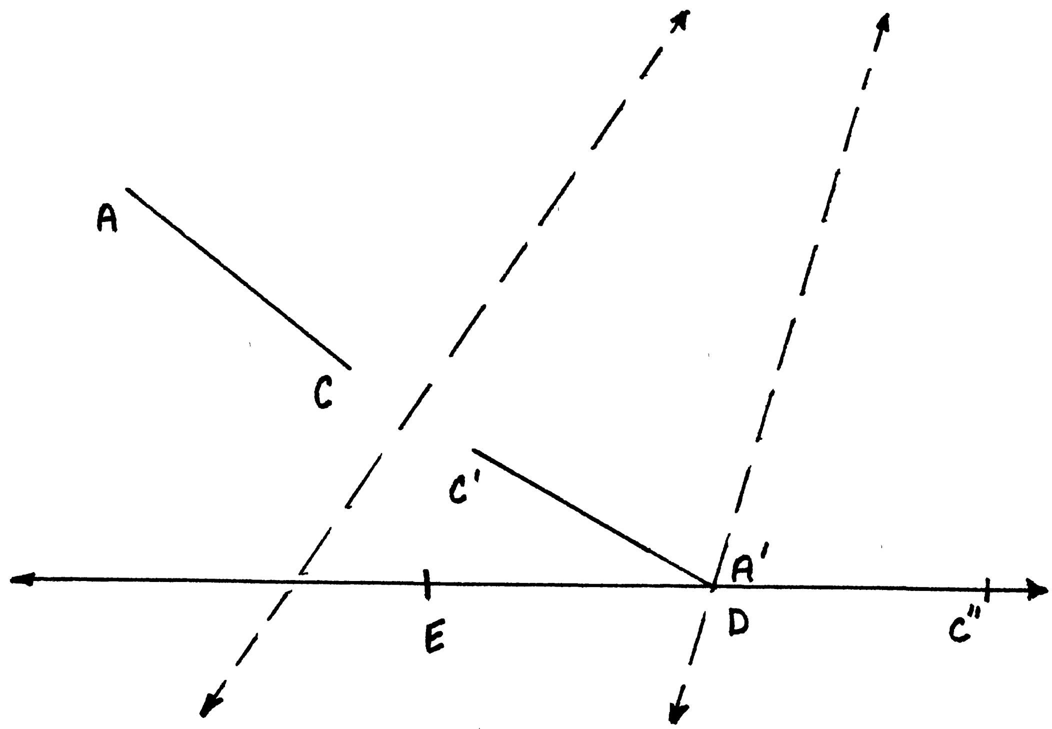

Figure 1. Addition of segments with a Mira

The multiplication of two segments is almost as simple. Lay off one of the given segments OA on a ray OX. On the same ray, lay off the unit segment OI, such that both segments have the vertex of the ray as a common endpoint. Erect a perpendicular OY at the vertex O and lay off the other given segment OB. Now erect perpendiculars at A, I, and B. Construct the line through the vertex O and the intersection D of the perpendicular at I and at B. Then drop a perpendicular from the point E of intersection of the line and the perpendicular at A. Call the base of the perpendicular thus constructed C. Then the segment from the vertex to C is the required segment; see Figure 2.

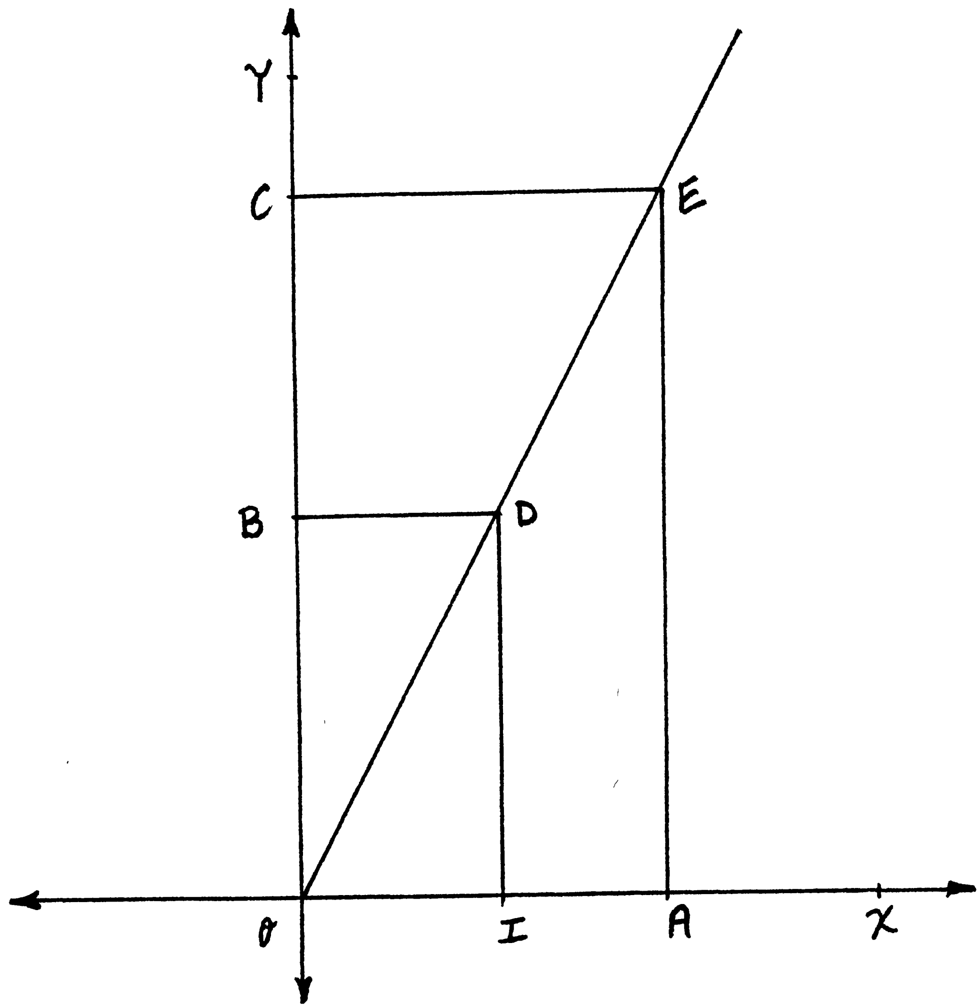

Figure 2. Multiplication of Segments

To perform the extraction of square roots, first add the unit segment AB to the given segment BC. Erect a perpendicular at the common endpoint B. Using a line of reflection through the midpoint M of the segment AC, use the Mira to reflect A onto the perpendicular, call the point X. Then the segment BX is the square root of segment BC; see Figure 3 [3, p.29].

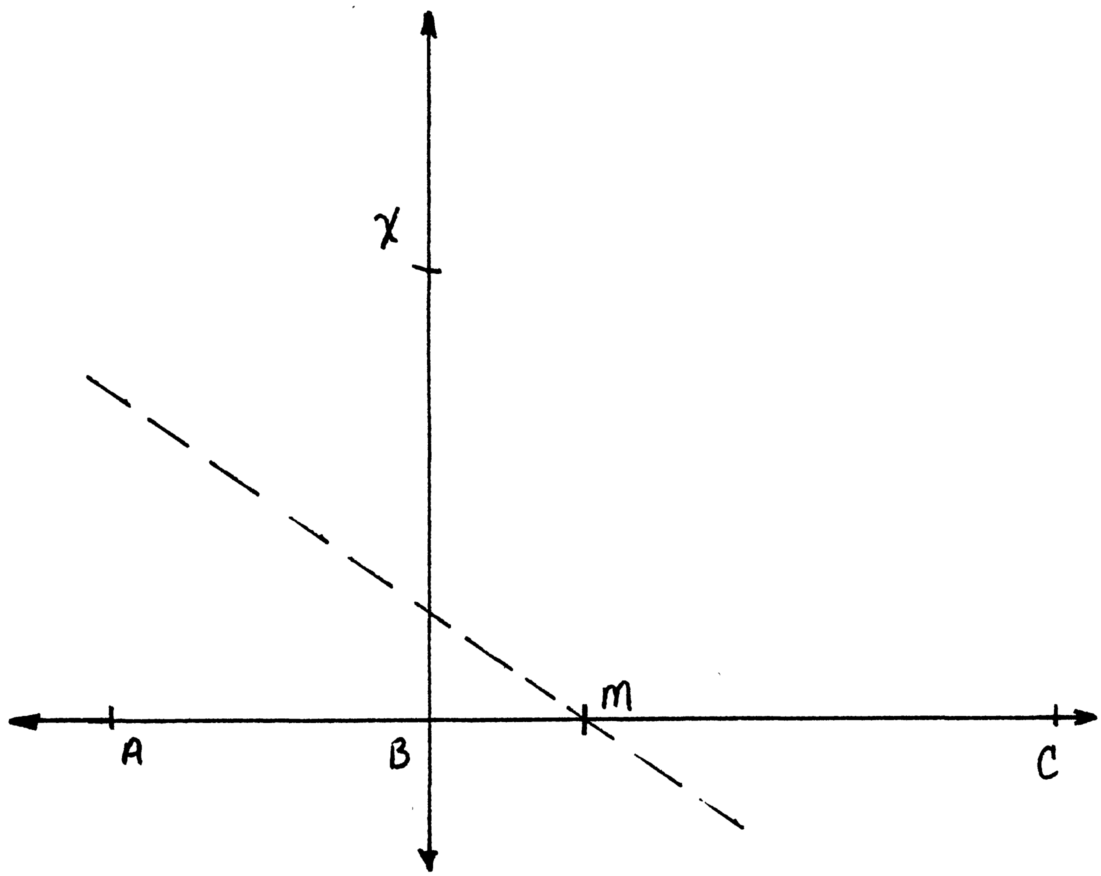

Figure 3. Extraction of Square Roots

Using this lemma and the characterization of compass and straightedge numbers, one can show that any compass and straightedge construction can be performed with the Mira.

Theorem 2

The field of Mira constructible numbers, F, is an extension of the field of compass and straightedge numbers, E.

In fact, the inclusion in Theorem 2 is proper since F properly contains E. We are now able to perform constructions with the Mira that the compass and straightedge cannot. For example, the trisection of an arbitrary angle cannot be done with a compass and straightedge [7, p.26], but can in fact be accomplished with the Mira by following a construction due to Pappus (300 AD); see Figure 4. In this construction, the fifth operation appears.

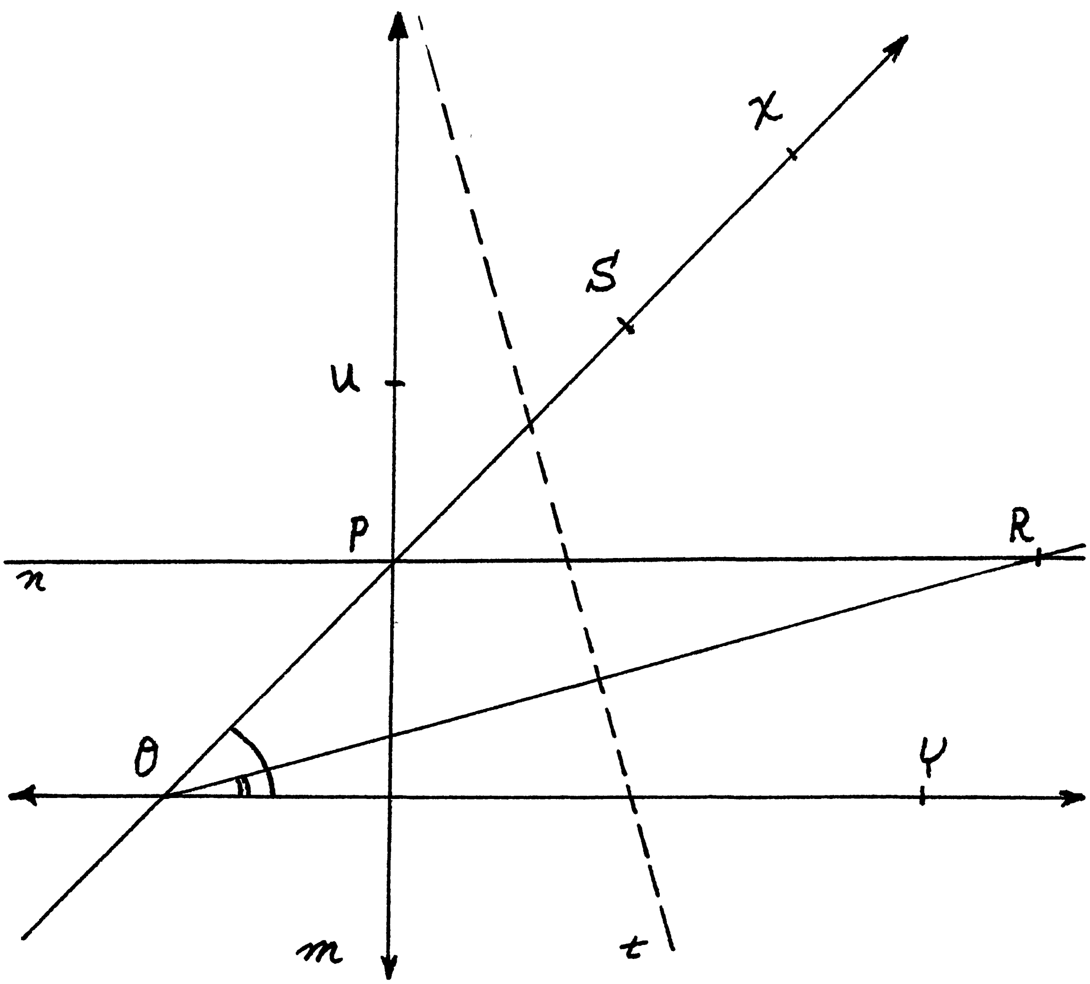

Figure 4. Angle Trisection

Given $\measuredangle$ XOY, choose P on ray OX and locate S on ray OX such that OP = PS. Through P construct line n parallel to OY and line m perpendicular to line OY. Find the line t that reflects S onto m and the line n through O. Locate R, the image of O under reflection in t. Then $\measuredangle$ ROY is one-third of $\measuredangle$ XOY [3 p.54].

The operation that was performed above which could not be accomplished by the compass and straightedge was that of locating a line that would reflect a point onto a line and a line through a point simultaneously. We will call this operation <u>simultaneous reflection</u> and now list all of the operations we will allow the Mira to perform:

1) Connect two points with a straight line
2) Locate the intersection of two non-parallel lines
3) Reflect a point through a line
4) Reflect a line through a line
5) Perform a simultaneous reflection

In other words, these operations, and only these, are those used to perform any Mira construction.

Now we must analyze the operation of simultaneous reflection. Then, without loss of generality, we may assume that we are given two lines, one the y-axis, the other with equation $y = ax + r$, intersecting the first at the point $(0,r)$, with r an element of F. Also, we may assume the points are $O = (c,0)$, and $S = (s,t)$, where c, s, and t are all elements of F. This is possible since all the various rotations and translations are accomplished by a compass and straightedge, and Theorem 2 assures us that these are possible with the Mira also. Then Q, the reflection of the point O onto the line $y = ax + r$, and H the image of S reflected onto the y-axis, are the new points. Let the coordinates of H be $(0,u)$ and of Q be (v,w); see Figure 5.

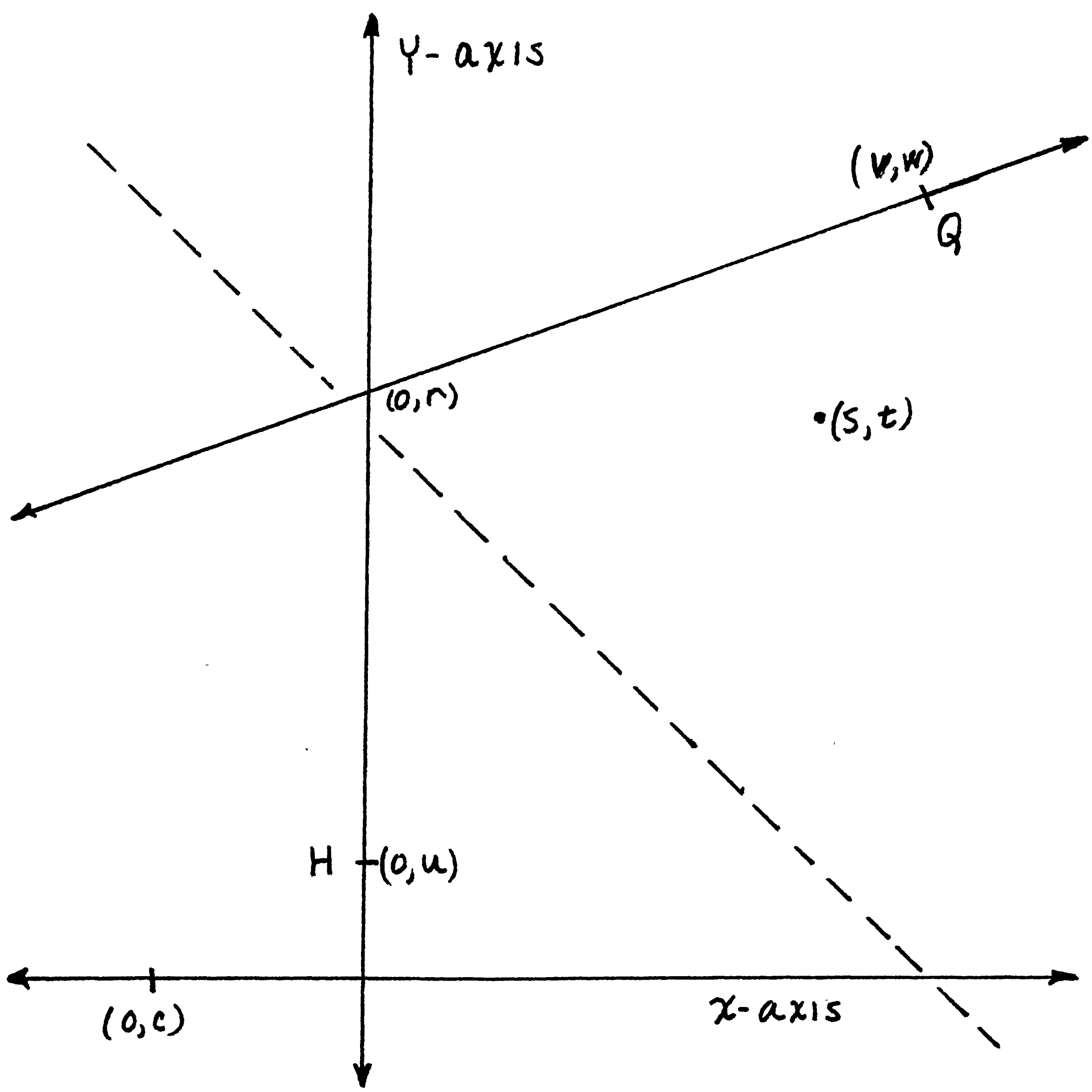

Figure 5. Simultaneous Reflection

Consider the different expressions for the slope of the Mira line so located by simultaneous reflection. One formula for the slope m of the Mira line is given by

$$m = (u + t - w)/(s - c - v), \tag{1}$$

since the Mira line passes through the midpoints of OQ and HS. Also, since the Mira line is perpendicular to line HS,

$$m = s/(u - t). \tag{2}$$

Finally, the Mira line is also perpendicular to line OQ, so

$$m = (c - v)/w. \tag{3}$$

Our known parameters are a,c,r,s, and t. We solve for v. An equation for v in the general case of simultaneous reflection is

$$\begin{aligned} 0 = {} & (a^2+1)v^3 + (a^2s-2at+2ar+a^2c-s-c)v^2 + \\ & (2ars+2act-2rt-2acr+r^2+c^2)v + (r^2s+2crt-cr^2-sc^2+c^3) \end{aligned} \tag{4}$$

Note the above equation is cubic, not quadratic, so if the polynomial is irreducible over the extension $E(a,c,r,s,t)$, of E, then a compass and straightedge can not construct v.

Consider now the case in which the given lines are parallel; assume their equations are $y = 0$ and $y = r$. Also, given are two distinct points, call them $B = (b,c)$ and $S = (s,t)$. Then reflect B onto $y = r$, and S onto $y = 0$. If we let the point $R = (v,r)$ be the reflection of B under the reflection in the Mira, then using the same type of analysis as above, we obtain the following equation for v:

$$0 = (c-r-t)v^2 + 2(rs-cs+tb)v - (rs+cs). \tag{5}$$

This polynomial is of lesser degree than a cubic, hence, nothing new is gained.

It is not obvious that the paramenters a,c,r,s,t can be manipulated so that equation 4 can represent an arbitrary cubic polynomial with coefficients in F. Thus, further investigation is necessary to determine that we can in fact, construct the roots of any cubic polynomial with coefficients in F using the Mira.

We shall start by solving an equation of the form

$$v^3 - w = 0, \tag{6}$$

where w is a nonnegative element of F; that is, we wish to show that we can find a real cube root of an arbitrary element of F. Now, equation 4 becomes equation 6 if the following four conditions are satisfied:

$$1 = 1 + a^2, \tag{7}$$

$$0 = a^2s-2at-2ar-a^2c-s-c, \tag{8}$$

$$0 = 2ars+2act-2rt-2acr+r^2+c^2, \tag{9}$$

$$-w = r^2s+2crt-cr^2-sc^2+c^3. \tag{10}$$

Whence equation 7 forces a = 0; then equation 8 results in s = −c and equations 9,10 become

$$0 = -2rt+r^2+c \tag{11}$$

$$-w = -2cr^2+2crt \tag{12}$$

This results in two equations in three unknowns. Let c=1. Solving the resultant system we have

$$r^2 = 1+w, \tag{13}$$

which can then be used to find the values of the other parameters. Thus, equation 4 can be manipulated to obtain a cube root of any element of F.

Now that the Mira can find real cube roots of arbitrary constructible real numbers, the question of the ability of the Mira to solve arbitrary cubic equations whose coefficients are in F arises. Examining the cubic formula below, we notice that the formula contains cube roots of square roots.

CUBIC EQUATIONS

A cubic equation, $y^3 + py^2 + qy + r = 0$ may be reduced to the form,—

$$x^3 + ax + b = 0$$

by substituting for y the value, $x - \frac{p}{3}$. Here

$$a = \tfrac{1}{3}(3q - p^2) \text{ and } b = \tfrac{1}{27}(2p^3 - 9pq + 27r).$$

For solution let,—

$$A = \sqrt[3]{-\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}, \quad B = \sqrt[3]{-\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}},$$

then the values of x will be given by,

$$x = A + B, \quad -\frac{A+B}{2} + \frac{A-B}{2}\sqrt{-3}, \quad -\frac{A+B}{2} - \frac{A-B}{2}\sqrt{-3}.$$

Figure 6. The Cubic Formula [10, p.103]

Thus, there may be occasions in which we need to take the cube root of a complex number of the form a + bi, where a,b are in F. We denote the set of such complex numbers F(i). To be able to solve arbitrary cubics with coefficients in F by means of the cubic formula, we must be able to take the cube root of these complex numbers.

Consider the polar representation of a complex number, that is, ze^{it}, where z is an element of F, and t is an angle between 0 and 2π. Since we can construct cube roots, we can find the cube root of such a number. Simply take the cube root of z and trisect t; both of the operations can be done with a Mira. Hence, the Mira can construct the cube root of any complex number in F(i). Therefore, we may use the Mira to solve <u>any</u> cubic equation with coefficients in F.

The quartic formula, as in the case of the cubic formula, involves only cube and square roots. Hence we may use the Mira to solve any quartic equation whose coefficients are in F. The above arguments have just proved Theorem 3.

<u>Theorem 3</u>

The Mira can be used to construct the real and imaginary parts of the roots of any quartic equation whose coefficients are in F.

Now that the algebraic significance of the operation called simultaneous reflection is known, let us formally state the characterization of the field of numbers the Mira operations will produce.

<u>Theorem 4</u>

A number p is Mira constructible if and only if there is a finite tower of extensions of the rationals, with each intermediate extension of degree three or less, such that p is in the last field of the tower.

<u>Proof</u>

Suppose p is constructible. Then p can be located by the finite application of operations the Mira can perform. If operations 1-4 are used, then the extension is at most quadratic. If simultaneous reflection is used, we have proved that we have formed at most a cubic extension of a quadratic extension of our field. Hence, application of the various operations leads to a finite tower of extensions of degree two or three.

Conversely, let N be the order of a tower of quadratic and cubic extensions of the rationals. The proof is by induction on the order of the tower. If N=0, then the point is rational and is constructible. Assume the theorem is true for N=M. Let a be in the M+1 st field in the tower; then a is of the form $d+e\sqrt{c}$, or of the form $g+j\sqrt{h}+k\sqrt[3]{h}$, where d,c,e, or g,h,j,k are in the M th field. By the inductive hypothesis, a is constructible.

Now we are ready to apply these results to the Greek construction problems introduced earlier. As we have shown, the trisection problem can be solved by the Mira. A similar result holds for the duplication of the cube,since theorem 3 tells us that we may now solve the equation $x^3 - 2 = 0$. However, the squaring of the circle is still not solvable by the Mira. We know that pi is transcendental, not the root of any polynominal with rational coefficients [7, p.48]. Hence, the Mira cannot square the circle.

CHAPTER III

CONSTRUCTION OF REGULAR POLYGONS

The problem of the construction of regular polygons is well known. With a compass and straightedge, we know that a regular polygon with n sides is constructible if and only if $n = (2^k)pq\ldots rst$, where $p,q,\ldots,r,s,t$ are distinct Fermat primes, that is, primes of the form $2^m + 1$, [1, p.186].

However, with the Mira, the answer takes a new form. We know that a nonagon can not be constructed with a compass and straightedge, but an angle of 120 degrees can be constructed and trisected with the Mira, hence constructing an angle of 40 degrees and from that a nonagon. We would like to characterize the polygons the Mira can construct.

The problem may be viewed as constructing the p-th roots of unity, for p prime, because the vertices of the regular p-gon inscribed in the unit circle in the complex plane are the roots of the equation

$$x^p - 1 = 0. \tag{14}$$

If the left side of equation 14 is factored, the irreducible factor other than $x - 1$ is called the pth cyclotomic polynomial. The degree of this cyclotomic polynomial is $p - 1$. The roots of this polynomial generate an extension of the rationals; we would like to characterize this extension to compare it to the field of Mira constructible numbers. Following the characterization of the constructibility of a regular polygon with a prime number of sides with a compass and straightedge, we are led to the following theorem.

THEOREM 5

For a prime p, a regular p-gon is constructible by a Mira if and only if p is of the form $2^n 3^k + 1$, with n and k nonnegative integers.

Proof

Let p be prime and suppose a regular p-gon is constructible by the Mira. Then the degree of the cyclotomic polynomial is p-1. The Galois group of the polynomial will have order p-1 and be cyclic [8, p.299], hence solvable. Then the composition factors in the normal series for the group correspond to intermediate extensions of prime degree in a tower of fields over the rationals. Since the Mira can only construct extensions of order 2 or 3, the composition factors must have order 2 or 3. Hence, p-1 is of the form $2^n 3^k$.

Conversely, let p be of the form indicated. Then the Galois group of the splitting field of the cyclotomic polynomial over the rationals will have order $2^n 3^k$. Furthermore, the group will be cyclic, hence solvable. Any normal series for the group will have composition factors that are of prime order 2 or 3. These factors correspond to extensions of degree 2 or 3. By our characterization of Mira constructible numbers, the Mira can construct extensions of degree 2 or 3. Therefore, the regular p-gon is constructible.

We will call the primes of Theorem 5 Mira primes. Now that the problem is solved for these primes, it is easy to determine for which n a regular n-gon is Mira constructible.

THEOREM 6

For general n, a regular n-gon is constructible with a Mira if and only if n can be factored as

$$n = 2^k 3^l \prod p_i ,$$

where the p_i are distinct Mira primes.

Proof

The primative n-th roots of unity are constructible if and only if the primative m-th roots of unity are constructible, for $m = 2^k$, 3^l, p_i.

Consider the following example. Certainly, a heptagon is not constructible with compass and straightedge, but seven is a Mira prime. So consider the 7th cyclotomic polynomial, [5, section 4.4],

$$0 = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1. \qquad (15)$$

Now substitute $y = x + 1/x$ in equation 13 to produce a cubic equation in y:

$$0 = y^3 + y^2 + 2y - 1. \qquad (16)$$

By the proof of Theorem 3, we can actually exhibit a construction of the solutions of equation 16, hence exhibit solutions of equation 15; see Figure 7.

The Mira cannot solve all construction problems, as we have seen. For further example, consider the construction

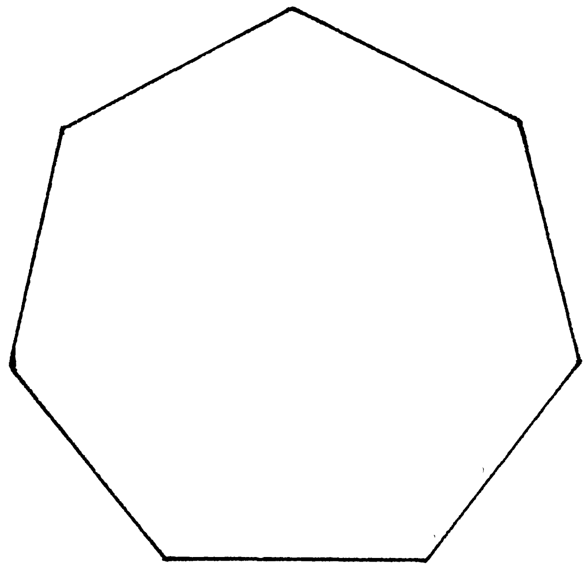

Figure 7. Heptagon

of a regular 11-gon. Using an analysis similar to that above, we find our substitution reduces the degree of the polynomial to five. In fact, the resultant irreducible fifth degree polynomial cannot be solved by the Mira, since the Mira cannot produce elements of a fifth degree extension.

Therefore, the Mira again can produce more geometric figures than the compass and straightedge. Also, we see that algebra again is used to produce the necessary proofs.

CHAPTER IV

CONSTRUCTIONS IN THE HYPERBOLIC PLANE

In order to answer questions about constructions in the hyperbolic plane [6, Ch.6], we will use a model of hyperbolic geometry in the Euclidean plane. In this way, we hope to convert questions about hyperbolic construction with hyperbolic tools into those constructions that involve only the more familiar Euclidean tools. The model we choose to use is the upper-half-plane model. In this model, the hyperbolic plane is the set of all the points in the real plane with positive y-coordinates. Then we interpet "line" as "an open vertical ray with its vertex on the x-axis or an open semi-circle with its center on the x-axis". The usual interpetations of incidence and betweenness are still used [6, p.191]. In this model, one can easily see how it is possible for the negation of the parallel postulate to hold.

However, more needs to be said about the interpretation of congruence. Two angles are congruent in the model if they are congruent in the Euclidean sense. A model of this type is said to be <u>conformal</u>. Two segments are congruent if their hyperbolic lengths are equal. To the casual observer, the distance seems to be distorted in the vertical. The Euclidean length of a hyperbolic segment of length one that is closer to the x-axis, called the line at infinity, is less than the Euclidean length of a hyperbolic segment of length one that is farther from the x-axis. For an exact formula for the distance between two points in terms of their Euclidean coordinates, see the appendix.

The relationship between Euclidean circles and hyperbolic circles is easy to state. The locus of points of the circle is the same, but the hyperbolic center is closer to the line at infinity than is the Euclidean center, because of the distortion of distance in the model.

There are curves in the hyperbolic plane that have no Euclidean counterparts; these are called hypercircles and horocircles. If we take all the points that are equidistant to a given hyperbolic line (on one side of the line), the resultant locus of points is not a line, but a curve called a <u>hypercircle</u>. To obtain a horocircle, take a point off a given line and draw the circle with the point as its center and the distance from the point to the line as its radius. As the point recedes from the line a larger circle is drawn. In the Euclidean plane, the limit of these circles is the line itself. In the hyperbolic plane, however, the limiting locus of points is not the line, but rather a curve called the <u>horocircle</u>.

Now, we say that a point is <u>constructible</u> in the hyperbolic plane if we can locate it as the intersection of any of the following hyperbolic figures either with a similar figure or with another of the list: a line through two points previously constructed, a circle with a center and radius given, a hypercircle with given radius, or a horocircle with given radius. The tools employed in the hyperbolic plane include the hypercompass, to draw the hypercircle through a point off a line, and the horocompass, which is used to draw the horocircle with any given ray as radius. Now our list of hyperbolic tools is complete: a straightedge, a compass, a hypercompass, and a horocompass.

Unfortunately, we cannot lay our hands on these hyperbolic tools. Therefore, we wish to show that we can use our Euclidean tools in a hyperbolic model to simulate any hyperbolic construction. We will use the notation H-line for a hyperbolic line, and H-circle for a hyperbolic circle. Similarly, we use E-line and E-circle to denote a Euclidean line and circle, respectively.

Theorem 7

A construction is possible in the hyperbolic plane if and only if it is possible in the upper-half-plane model with Euclidean tools.

Proof

We have two cases to consider in the forward implication: an H-line through two points, and a hyperbolic circle with a given radius.

Given two points, it is simple to construct the H-line through them with a Euclidean compass and straightedge. If the points have the same x-coordinate, then draw a vertical E-ray through them; this will be the H-line through them also. If the points are otherwise positioned, we must construct the Euclidean circle centered on the x-axis that passes through them. First, draw the E-segment that connects the two points. Next, draw the E-perpendicular bisector of the segment. The point in which the perpendicular bisector intersects the x-axis is the center of the semicircle that is the desired H-line. The perpendicular bisector must intersect the x-axis since the points are not vertical.

Next, if we are given a radius (i.e. a hyperbolic segment) and a center, we must be able to construct the hyperbolic circle at that point. Unfortunately, the hyperbolic and Euclidean centers of a circle in this model never coincide. If we can construct a vertical diameter of the hyperbolic circle, we could bisect it to find the Euclidean center; hence we can then draw the circle.

Without loss of generality, we may assume that we are given an H-segment CA where C is the center of the hyperbolic circle. First, we assume that CA is a vertical segment. Consider the H-line perpendicular to segment CA at the point C. It will be a Euclidean semicircle. If we reflect CA about this H-line, we will obtain a vertical diameter of the circle. Reflections through H-lines in the upper-half-plane model are given by Euclidean inversions in the circles that represent H-lines or by Euclidean reflections through vertical rays that represent H-lines. (For this and other results concerning inversions, see

[6, p.194ff and p.287ff].) Hence, all we need do is invert the segment; see Figure 8.

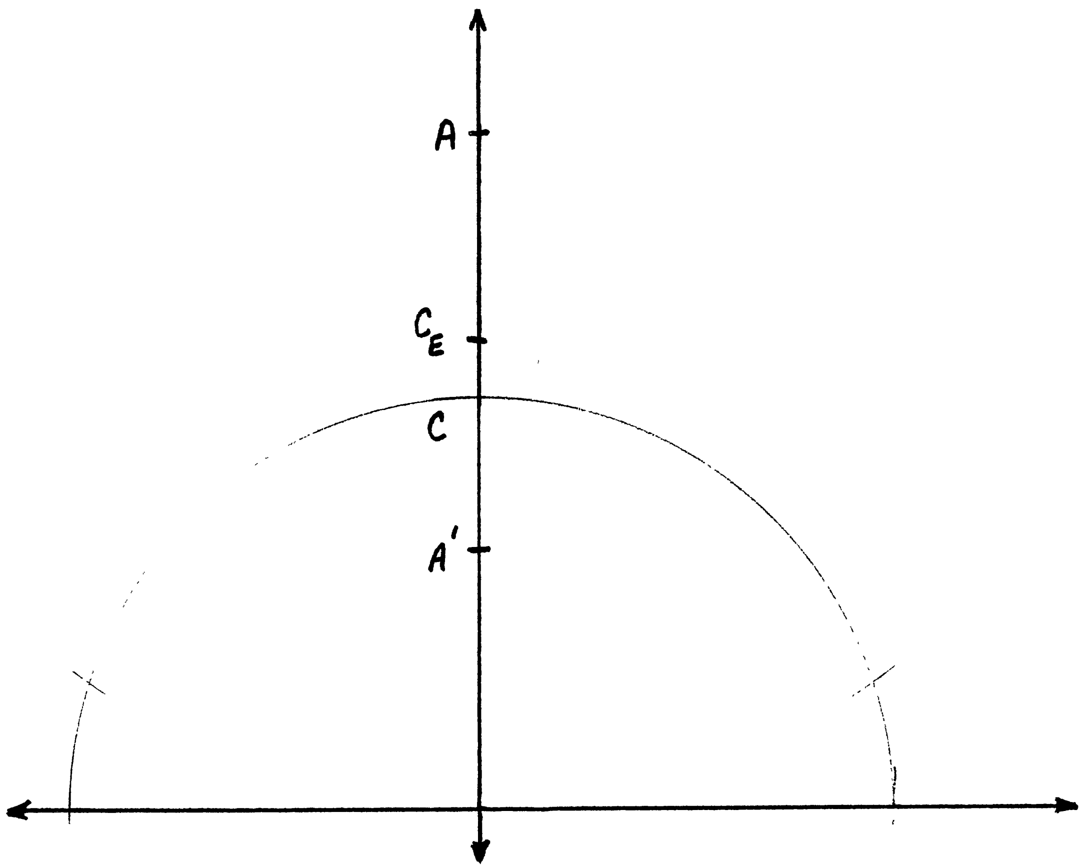

Figure 8. Vertical Radius Case

Next, assume that CA is not a vertical segment. We will now reflect the segment CA so that it is a vertical radius of the circle, using Euclidean tools.

Let O' and O'' be the ideal points where the H-line CA strikes the x-axis. Consider the semi-circle with radius O''C. If we invert CA through this semi-circle, then CA will become a vertical segment with C as an endpoint, [6, Prop 7.8]. Now we are done by the case above; see Figure 9.

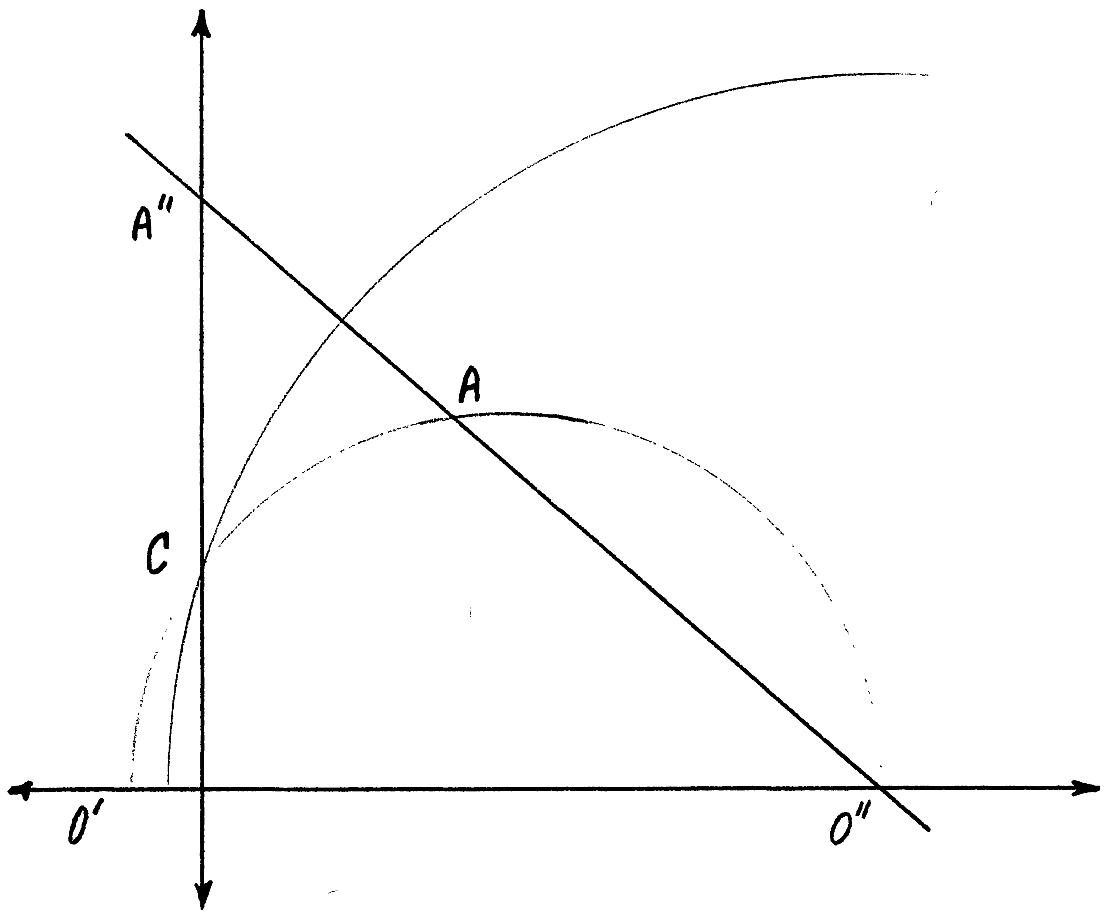

Figure 9. Nonvertical Radius Case

Conversely, we show that any point constructible by Euclidean tools in the upper-half-plane model can be constructed by hyperbolic compass and straightedge. By the Mohr-Mascheroni theorem, all Euclidean constructions can be carried out using a compass alone, [4, p.172]. This result is proved by means of inversion through a selected circle. For example, consider a segment of an E-line in the model. Let us take an arbitrary circle in the model and invert the segment through the circle. We know that the inversion takes the line to a circle, that is inside the given circle but passing through the center of the given circle [6, Prop 7.7]. Note that in the proof of the Mohr-Mascheroni Theorem, we may invert in any arbitrary circle in the model, ensuring that all circles required in a construction will be entirely contained in the upper-half-plane.

Thus, it suffices to show that, given two points in the upper half-plane, we are able to construct the Euclidean circle having one of the points as its center (with hyperbolic tools), if the circle lies entirely in the upper half-plane. For this construction we will employ more than just compass and straightedge. As discussed before, the other tools commonly used in the hyperbolic plane are the horocompass and hypercompass. In 1944, Nestorowitsch proved that the three kinds of compasses, together with a straightedge, are of efficacy equal to that of standard H-compass and straightedge, [2, p.290]. With this result, we may use all of the compasses in our construction.

Assume we are given point C as the Euclidean center of a circle through a point. Note that the locus of points on the circle is the same for the hyperbolic and Euclidean circles, the only difference being the location of the center. We assume that we are given a "vertical" H-line l through the point C. For case 1, assume that the point A is not on this line. Then construct a horocircle through A with the vertical ray from A away from the axis as its diameter. This produces a horizontal E-line m that intersects the vertical line through C at a point D. Using the compass, lay off the E-segment DA on the horocircle m on the opposite side of l to produce the point B. Note that the Euclidean length of AD is equal to that of BD since the E-line m is horizontal. Then B is also on the circle. Next, drop a perpendicular n to the E-line m at point A; this will be another "vertical" H-line. Using the hypercompass, draw the hypercircle through the points C and B. Let the intersection of the hypercircle and line n be called E.

Then E is on the Euclidean circle, hence on the hyperbolic circle. Consider the Euclidean triangles BDC and CFE; certainly $\measuredangle$ DBC = $\measuredangle$ FCE and $\measuredangle$ BCD = $\measuredangle$ CEF. Also, the Euclidean segments BD = DA = CF, hence the triangles are congruent. Then the Euclidean distances BD and CF are congruent. Therefore, E is on the circle. To find the hyperbolic center, use the hyperbolic compass to construct the H-perpendicular bisector of the H-segment AE. The intersection of this line with the H-line DC is the hyperbolic center of the circle; see Figure 10.

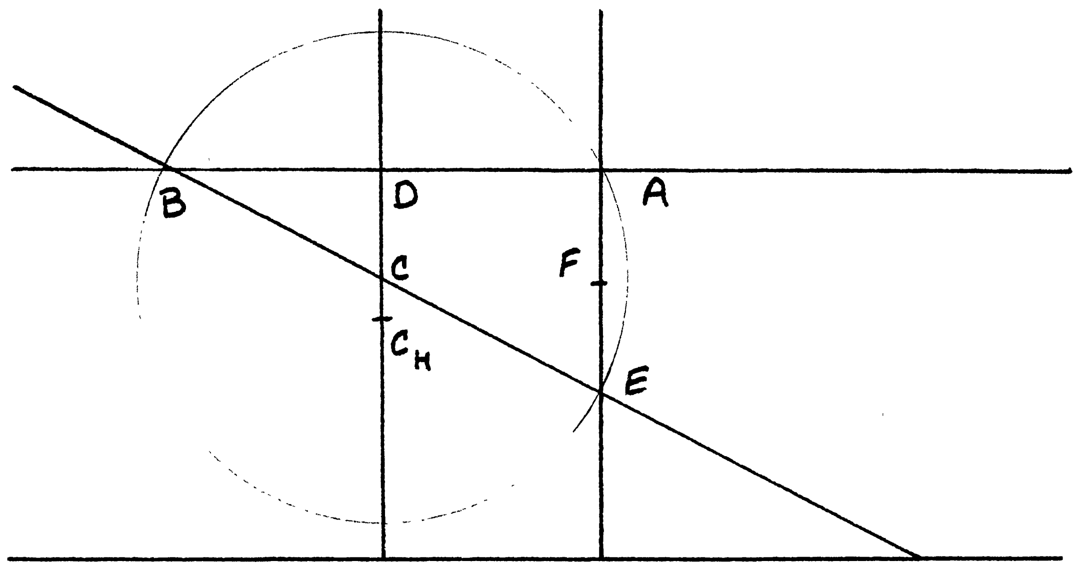

Figure 10. Hyperbolic Center of a Circle - Case 1

For case 2, assume A is on the vertical H-line through C. Draw the H-lines perpendicular to the line AC at points A and C. Draw the horocircle through C and the point at infinity; this intersects line AC at point C. Then, the horocircle and the perpendicular at A intersect at the point B. Next, draw the hypercircle (equidistant curve) to AC with radius AB. This hypercircle intersects the perpendicular at C in a point D. The E-segment DB is not vertical. By case 1, we can draw a circle with D as its center and the Euclidean segment DB as its radius. The hypercircle intersects this circle in a point E. If we draw a perpendicular to AC through E, the foot of the perpendicular F produces the vertical diameter of the circle centered at C; see Figure 11.

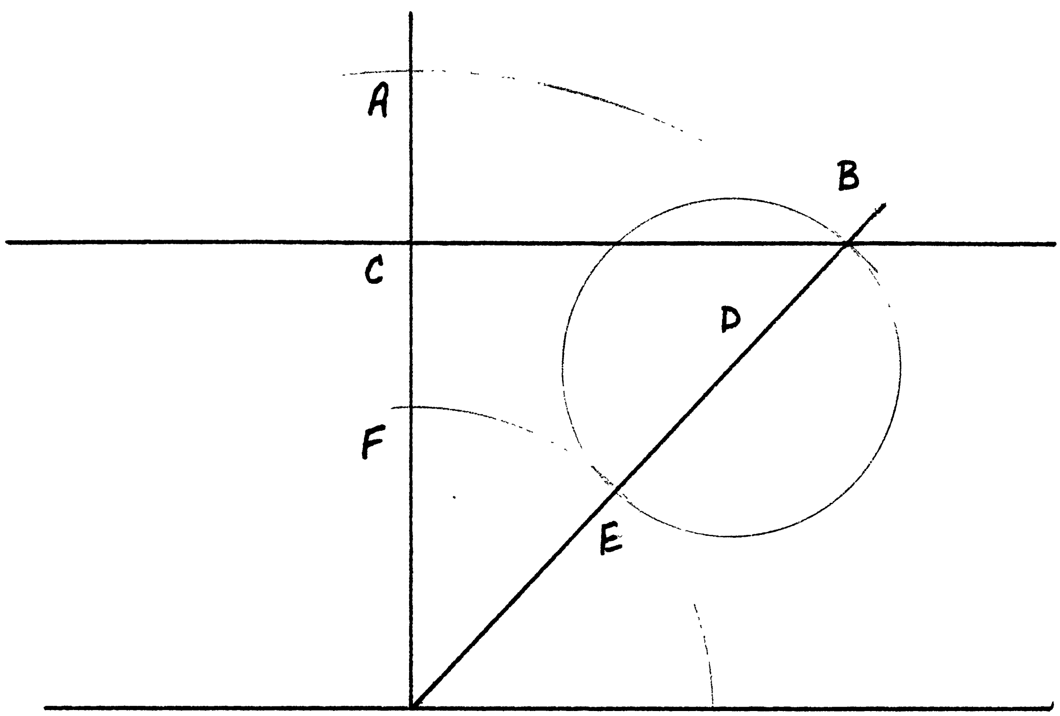

Figure 11. Hyperbolic Center of a Circle - Case 2

One last case must be considered, that which occurs when A and C lie on the horocircle through C and the point at infinity. As we have seen, the horocircle is a E-horizontal line perpendicular to the vertical H-line at C. Let O be the ideal point, i.e. the vertex of the vertical H-line through C. Let O' be the ideal point produced by the vertical through A. Let O'' be the ideal point produced by the perpendicular through C. Now draw the vertical through the point O''. Draw perpendiculars to the vertical line through O'' from the points O and O'and; call these points D and B respectively. The E-segment DB is congruent to the E-segment CA. Hence, the segment DB is vertical. By case 2, we may draw the circle centered at B with radius DB. Next, draw the horocircle through D and the point at infinity. The horocircle intersects the vertical through C at the point E. Then the E-segment CE is congruent to the E-segment CA, and by case 2, we may draw the circle centered at C; see Figure 12.

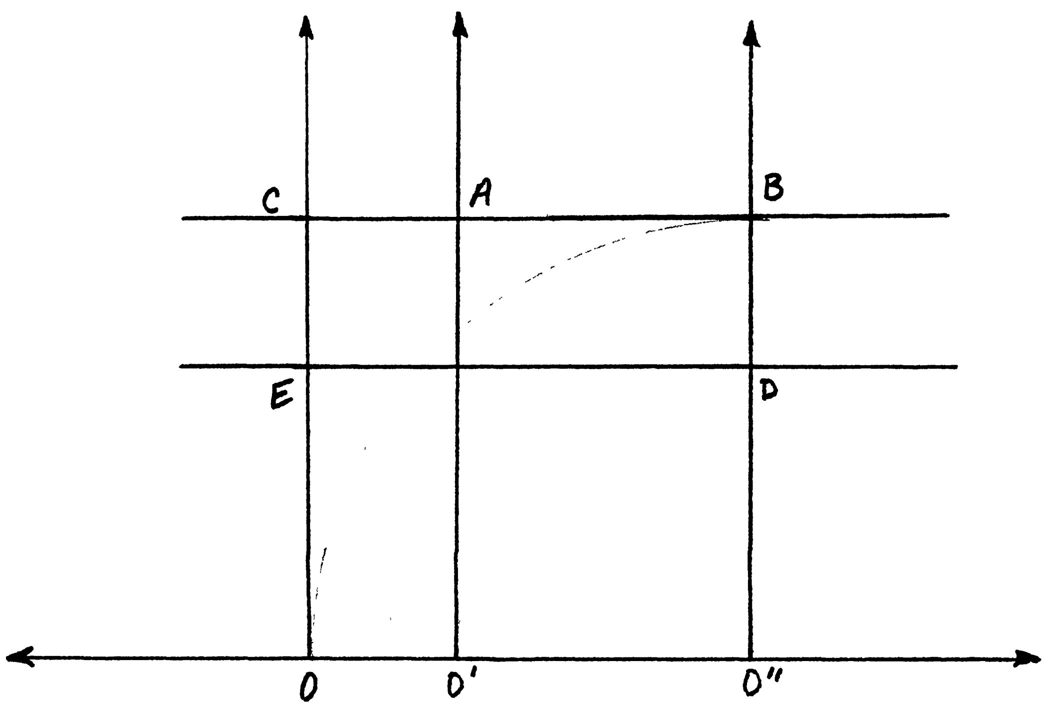

Figure 12. Hyperbolic Center of a Circle - Case 3

With Theorem 7, hyperbolic constructions may be explored with ease. For example, consider the simple construction of dividing a segment into three congruent parts. Assume that the coordinates of the endpoints of the segment are (0,1) and (0,2). Using the formula for the hyperbolic distance between these points (see the appendix), we find the distance to be ln 2. But, the coordinates of the point that is one-third of this distance can be found by letting q be the y-coordinate of the point, and letting d be the distance between that point and (0,1). We then set $d=(1/3)\ln 2$, $k=1$, and $h=p=0$ in the formula for the hyperbolic distance between the two points (see the appendix):

$$e^{2(1/3)\ln 2} = \frac{1 + q^2 + \sqrt{(1 - q^2)^2}}{1 + q^2 - \sqrt{(1 - q^2)^2}} \tag{17}$$

This reduces to:

$$\sqrt[3]{2} = q \tag{18}$$

Since we cannot construct the cube root of two with a compass and straightedge in the Euclidean plane, the segment cannot be trisected in the hyperbolic plane.

CHAPTER V

CONCLUDING REMARKS

Theorem 7 opens the door for many more questions about the hyperbolic plane, the foremost question being, is there an analogous version of Theorem 7 for the "hyperbolic mira"? Also, can the numbers the Mira can construct in the hyperbolic plane be characterized?

Theorem 5 states which regular polygons with a prime number of sides which the Mira can construct in the Euclidean plane; note that not all primes are included. We conjecture that there is a limitation on which polygons can be constructed with any tool. Regular polygons also occur in the hyperbolic plane; can we charactertize the ones we can construct?

As with most areas of mathematics, geometry increasingly uses other areas of mathematics to solve more difficult problems. We have seen a good example of this in the construction problems. In effect, mathematics has become all one field, with no one sub-field completely self-contained.

Geometry, however, affords one the satisfaction of a visual completion of the problem. Along with this visual satisfaction, there are problems stemming from physically carring out the constructions, mainly errors. Although we speak of lines with no width, physical lines have width. This causes error in the construction. The Mira is a major offender in producing drawing errors. We found it hard to draw an accurate construction. The best results were obtained in a well-lighted room, with a very bright light behind the Mira, so that the reflection could be easily seen. In this way the construction of the heptagon was carried out. The error caused the first construction to be wildly inaccurate. The second construction was carried out with the help of a ruler for checking the length of the segments. This improved the accuracy of the construction greatly, and produced the correct figure.

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APPENDIX

One of the various models of the hyperbolic plane is called the Cayley-Klein unit disk model. The distance formula is known for this model [9, p.282]. If we let $P=(x,y)$ and $Q=(w,z)$, then the distance d between P and Q is give by:

$$d = (1/2)\ln\left(\frac{1-xw-yx+[(w-x)^2+(z-y)^2-(xz-wy)^2]^{1/2}}{1-xw-yx+[(w-x)^2+(z-y)^2-(xz-wy)^2]^{1/2}}\right) \tag{19}$$

Also given in [9, p.285] is the isomorphism between the Cayley-Klein model and our upper-half-plane model. The map is defined by sending (x,y) to (x',y') where

$$x' = x/(1 - y), \tag{20}$$

$$y' = t/(1 - y), \tag{21}$$

and t is defined to be $\sqrt{1-x^2-y^2}$. If we invert this isomorphism and substitute into equation 19, we have the distance formula for the upper-half-plane model. The distance d between two distinct points (h,k) and (p,q) in the upper-half-plane model is given by

$$d = (1/2)\ln\left(\frac{(h-p)^2+(k^2+q^2) + \sqrt{T}}{(h-p)^2+(k^2+q^2) - \sqrt{T}}\right), \tag{22}$$

where

$$T = (k-q)^2(k+q)^2+(h-p)^4+2(h-p)^2(k^2+q^2). \tag{23}$$

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