A GENERAL METHOD FOR THE CONSTRUCTION OF FREE LATTICE-ORDERED ALGEBRAS

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PREFACE

A method for the construction of free lattice-ordered algebras was developed in the setting of a variety of abstract algebras in which subdirectly irreducible algebras are totally ordered. The method developed simplified the problem of construction of free ℓ-algebras in that the free unordered algebra, which is in general easier to find, is used.

Various examples of varieties of algebras for which this method of construction is valid were presented and noted as the motivating factors for this paper. One particular construction, that of a Boolean algebra, led me to be able to characterize the cardinality of finite free Boolean algebras. This characterization in turn led to a result that described the order of a free ℓ-algebra in a somewhat more general setting.

I wish to express my appreciation and thanks to the members of my graduate committee and especially to my advisor, Dr. Wayne B. Powell, for their help to me and perserverance with me.

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LIST OF SYMBOLS

Symbol:

π	-	direct or subdirect product
ϕ	-	null set
$\cap$	-	set intersection
$\cup$	-	set union
$\subseteq$	-	set inclusion
$\leq$	-	less than or equal to
$\cong$	-	isomorphism
$\circ$	-	function composition
X	-	cardinality of set X
$\wedge$	-	\`meet´, greatest lower bound
$\vee$	-	\`join´, least upper bound
∞	-	infinity
$\phi: S \to T$	-	ϕ a function from set S to set T
$\mathbf{Z}; \mathbf{Z}^+$	-	the set of integers; the positive integers
$\mathbf{R}; \mathbf{R}^+$	-	the real numbers; the positive reals
P(X)	-	the power set of set X.

CHAPTER I

INTRODUCTION

Free algebraic structures have long been studied in the specific cases of free groups, free rings and free modules and, in the recent past, in a more generalized setting. Birkhoff initiated a push toward mathematical abstraction in the 1930's and 1940's trying to universalize mathematical concepts from areas that at the time were considered diverse. The concept of abstract algebras was spawned by this shift in mathematical thought. The study of lattice-ordered abstract algebras is a logical progression of this shift in mathematics, due to the importance of lattice theory in mathematics, and it is in this setting that our study of free algebras takes place.

Working primarily with Birkhoff's Abstract Algebra [3] to define the general mathematical setting in which this paper deals and with Bernau [1] and Powell [7] for concrete examples, this paper develops a general method for giving explicit constructions of free lattice-ordered algebras.

Chapter II presents the preliminaries needed to develop the study. Generalized definitions of algebras, morphisms and free algebras are given.

Chapter III develops a method for constructing free algebras in the general setting described in Chapter II.

Chapter IV presents the reader how to use the strategy developed in Chapter III to create actual examples of free ℓ-algebras. Free abelian ℓ-groups, free f-modules and free Boolean algebras are discussed.

Chapter V discusses the cardinality of finite free ℓ-algebras. The order of finite free Boolean algebras is characterized.

Chapter VI concludes with questions for further research.

CHAPTER II

PRELIMINARIES

An (abstract) algebra is a pair $[S,F]$ with S a non-empty set and F a collection of functions, each $f_\alpha \varepsilon \{f\alpha\}_{\alpha\varepsilon A} = F$ mapping $\underset{N(\alpha)}{\Pi} S$ to S, where each $N(\alpha)$ is a non-negative integer. Familiar algebraic structures are encompassed by this definition. For instance, consider an additive group $(G,+)$. We may represent this group in terms of the above definition as $[G,\{+, -, 0\}]$ where $+$ maps $G\times G$ to G and is called a binary operation, $-$ maps G to G and is refered to as unary, and 0 is the empty function mapping ϕ to G and is called nullary. Similarly, we may represent the field $(S, +, .)$ by $[S,\{+, -, 0, ., ^{-1}, 1\}]$.

If $[S,F]$ is an algebra, $T\subseteq S$ and $F' \subseteq F$, then $[T,F']$ is and F'-subalgebra of $[S,F]$ provided T is closed under the operations of F', i.e. T is F'-closed. If $T\subseteq S$ and $F'\subseteq F$, the F'-subalgebra of $[S,F]$ generated by T is the intersection of all F'-subalgebras of $[S,F]$ which contain T. This intersection is easily shown to be an F'-subalgebra of $[S,F]$ and will be called the F'-closure of T in S.

Let $\{[S,F]\}$ be a collection of algebras with variable S and constant F, i.e. the operations for each

$[S_1,F] \in \{[S,F]\}$ correspond to those of each $[S_2,F] \in \{[S,F]\}$ and so we use the same symbols for both. Such a collection is said to be a collection of similar algebras. Note that the collection of F'-subalgebras of an algebra $[S,F]$ forms a collection of similar algebras. The collection of all rings forms a collection of similar algebras.

Consider $[S,F']$ and $[T,F]$ to be algebras with $F' \subseteq F$ and ϕ a function mapping S to T. If ϕ has the property

$$\phi(f_\alpha(s_1, \ldots, s_{N(\alpha)})) = f_\alpha(\phi(s_1), \ldots, \phi(s_{N(\alpha)})) \qquad (1)$$

for all $f_\alpha \in F'$ and $s_1, \ldots, s_{N(\alpha)} \in S$, the ϕ is said to be an F'-homomorphism. If ϕ is injective it is said to be an F'-monomorphism, if surjective an F'-epimorphism and, if both, and F'-isomorphism. If there exists an F-isomorphism from $[S,F]$ to $[T,F]$, $[S,F]$ and $[T,F]$ are said to be isomorphic, denoted $[S,F] \cong [T,F]$. The homomorphic image of $[S,F']$ under the F'-homomorphism ϕ is defined as $[\phi[S], F']$ where $\phi(s) = \{\phi(S) \mid s \in S\}$. $[\phi(S),F']$ is easily shown to be an F'-subalgebra of $[T,F]$.

Define a relation, $\equiv$, on a set S to be a subset R of SxS; if $s_1, s_2 \in S$ then s_1 is related to s_1, $s_1 \equiv s_2$, if and only if $(s_1, s_2) \in R$. An equivalence relation is a relation on a set S which is reflexive, symmetric and transitive. That is, for all $s_1, s_2, s_3 \in S$

i) $s_1 \equiv s_1$ (reflexive)

ii) $s_1 \equiv s_2$ implies $s_2 \equiv s_1$ (symmetric)

iii) $s_1 \equiv s_2$ and $s_2 \equiv s_3$ implys $s_1 \equiv s_3$ (transitive).

An equivalence relation forms a partition of the set. Denote the disjoint elements of this partition as $\{[s_i] \mid s_i \varepsilon S\}$ and call these elements equivalence classes. It is generally known that $s_1 \equiv s_2$ if and only if s_1, $s_2 \varepsilon [s_i]$ for some i. Symmetry of the equivalence relation and the above result provide that the representation of equivalence classes we use is well-defined.

An equivalence relation, $\equiv$, on the set S of the algebra [S,F] is a congruence relation provided, for each $f_\alpha \varepsilon F$ and $x_1,\ldots,x_{N(\alpha)}$, $y_1,\ldots,y_{N(\alpha)}$ S such that $x_i \equiv y_i$ for $1 \leq i \leq N(\alpha)$, we have

$$f\alpha(x_1,\ldots,x_{N(\alpha)}) \equiv f\alpha(y_1,\ldots,y_{N(\alpha)}) . \tag{2}$$

If ϕ is a homomorphism of [S,F] to [T,F] we may construct a congruence on [S,F] using ϕ and the following theorem.

<u>Theorem II. 1</u>: Let [S,F] and [T,F] be algebras and ϕ an F-homomorphism from [S,F] to [T,F]. The relation on S defined by $s_1 \equiv s_2$ if and only if $\phi(s_1) = \phi(s_2)$, $s_1,s_2 \varepsilon S$, is a congruence relation on S.

Proof: Let s_1, s_2, $s_3 \varepsilon S$. Since $\phi(s_1) = \phi(s_1)$, $s_1 \equiv s_1$ hence $\equiv$ is reflexive. Since $\phi(s_1) = \phi(s_2)$ implies $\phi(s_2) = \phi(s_1)$, if $s_1 \equiv s_2$ then $s_2 \equiv s_1$ hence $\equiv$ is symmetric. Since $\phi(s_1) = \phi(s_2)$ and $\phi(s_2) = \phi(s_3)$ imply $\phi(s_1) = \phi(s_2)$, if $s_1 \equiv s_3$ and $s_2 \equiv s_3$ then $s_1 \equiv s_3$ hence $\equiv$ is transitive. Thus $\equiv$ is an equivalence relation on S. Let $f\alpha \varepsilon F$ and $x_1,\dots,x_{N(\alpha)},y_1,\dots y_{N(\alpha)}$ S such that $x_i \equiv y_i$ for $1 \leq i \leq N(\alpha)$.

Then

$$\phi(f\alpha(x_1,\dots,x_{N(\alpha)})) = f\alpha\ (\phi(x_1),\dots, \phi(x_{N(\alpha)}))$$
$$= f\alpha(\phi(y_1),\dots, \phi(y_{N(\alpha)})) = \phi(f\alpha(y_1,\dots,y_{N(\alpha)})) \tag{3}$$

hence $f\alpha(x_1,\dots,x_{N(\alpha)}) \equiv f\alpha(y_1,\dots,y_{N(\alpha)})$. (4)

Given an algebra and a homomorphism from it we may define a similar algebra on the congruence classes of the relation defined in Theorem II 1. The next theorem is proved in [2,p.136].

Theorem II. 2: Let [S,F] be an algebra and ϕ an F-homomorphism from [S,F]. Let S/ϕ denote the set of congruence classes of S determined by ϕ as in Theorem II. 1. Then $[S/\phi,F]$ forms an algebra with operations on S/ϕ defined by the formula

$$f\alpha([s_1],\dots,[s_{N(\alpha)}]) = [f(s_1,\dots, s_{N(\alpha)})] \tag{5}$$

for each $f\alpha \varepsilon F$ and $[s_1],\dots,[s_{N(\alpha)}] \varepsilon S/\phi$.

The following corollary to the above two theorems is a generalization of the Fundamental Theorem of Group Homomorphisms [6,p.233] and the Fundamental Theorem of Ring Homomorphisms [6,p.243].

Corollary II. 3: If $[S,F]$ is an algebra there exists a one-to-one corresondence between congruence relations on $[S,F]$ and F-epimorphic images of $[S,F]$. Moreover, if $[T,F]$ is an epimorphic image of $[S,F]$ under ϕ then $[T,F] \cong [S/\phi,F]$ under $\bar{\phi}$ defined by

$$\phi([S_1] = \phi(S_1) \tag{6}$$

for all $[s_1] \varepsilon S/\phi$.

Throughout this paper homomorphisms and their associated congruence relations will be denoted by the same greek letter.

Let $\{[S_\beta,F]\}_{\beta \varepsilon B}$ be a collection of similar algebras and define the direct product of this collection of algebras to be the set $\prod_{\beta \varepsilon B} S_\beta$ with $f\alpha \varepsilon F$ defined on this set by

$$f\alpha(b_1,\ldots, b_{N(\alpha)}) = b \tag{7}$$

where $b_1,\ldots, b_{N(\alpha)}$, $b\varepsilon \prod_{\beta \varepsilon B} S_\beta$ with

$$b(\beta) = f\alpha(b_1(\beta),\ldots, b_{N(\alpha)}(\beta)) \text{ for } \beta\varepsilon B. \tag{8}$$

Note that the formation of a product algebra produces an algebra similar to each of its component algebras. It can easily be seen that if $[T_\beta,F']$ is an F'-subalgebra of $[S_\beta,F]$, $\prod_{\beta \varepsilon B} [T_\beta, F']$ is an F'subalgebra of $\prod_{\beta \varepsilon B} [S_\beta,F]$.

Such a subalgebra is said to be a subdirect product of the algebras $\{[S_\beta,F]\}_{\beta\varepsilon B}$ provided for each $s_\beta \varepsilon S_\beta$ there exists $b\varepsilon \prod_{\beta\varepsilon B} [T_\beta,F']$ such that $b(\beta) = s_\beta$. That is, if for each $\beta\varepsilon B$ we define a project map

π_β: $\prod_{\beta\varepsilon B} [T_\beta,F'] \to S_\beta$ by $\pi_\beta(b) = b(\beta)$, π_β is surjective for each $\beta\varepsilon B$. The next theorem is easily verified.

<u>Theorem II. 4</u>: If π_β is a projection map from the direct product of algebras

$\prod_{\beta\varepsilon B} [T_\beta,F'] \subseteq \prod_{\beta\varepsilon B}[S_\beta,F]$ to S_β, $T_\beta \subseteq S_\beta$, then π_β is an F'-homomorphism. If this product is a subdirect product then π_β is an F'-epimorphism.

Monomorphisms into products of algebras can be characterized as follows:

<u>Theorem II. 5</u>: If ϕ: $[S,F] \to \prod_{\beta\varepsilon B} [S_\beta,F]$ is an F-homomorphism and $\phi(s)$ is a subdirect product of $\prod_{\beta\varepsilon B} S_\beta$ then ϕ is a F-monomorphism if and only if for each collection $\{s_\beta |\ s_\beta \varepsilon s_\beta\ ,\ \beta\varepsilon B\}$ we have that the cardinality of $\bigcap_{\beta\varepsilon B} \phi_\beta^{-1}(s_\beta)$ is 0 or 1, where

$$\phi_\beta^{-1}(s_\beta) = \{\ s\varepsilon S \mid \phi(s) = s_\beta,\ \phi = \prod_{\beta\varepsilon B} \phi_\beta\} \ . \qquad (9)$$

Proof: Suppose ϕ is injective and there exists a set $\{s_\beta \mid s_\beta \varepsilon S_\beta, \beta \varepsilon B\}$ such that the cardinality of $\bigcap_{\beta \varepsilon B} \phi_\beta^{-1}(s_\beta)$ is greater than 1. Let x and y be elements of this intersection such that $x \neq y$. Then $\phi(x) = \phi(y)$ yet $x \neq y$ hence ϕ is not injective.

Conversely suppose $x, y \varepsilon S$ such that $\phi(x) = \phi(y)$. Then $\phi_\beta^{-1}(\phi_\beta(x)) = \phi_\beta^{-1}(\phi_\beta(y))$ for all $\beta \varepsilon B$ implying $\phi_\beta^{-1}(\phi_\beta(x)) = \phi_\beta^{-1}(\phi_\beta(y))$ for all $\beta \varepsilon B$. Since $x \varepsilon \bigcap_{\beta \varepsilon B} \phi_\beta^{-1}(\phi_\beta(x))$, $y \varepsilon \bigcap_{\beta \varepsilon B} \phi_\beta^{-1}(\phi_\beta(y))$, $\bigcap_{\beta \varepsilon B} \phi_\beta^{-1}(\phi_\beta(x)) = \bigcap_{\beta \varepsilon B} \phi_\beta^{-1}(y))$ and each of these intersections has cardinality less than or equal to 1, $x = y$. Hence ϕ is injective hence a monomorphism.

An algebra is said to be subdirectly irreducible provided for each isomorphism from the algebra to a subdirect product of algebras at least one of the associated projections out of the product back to the algebra is an isomorphism itself. Simply, if an algebra is subdirectly irreducible, no information can be gained by representing the algebra as a subdirect product of algebras since the algebra is isomorphic to one of the components of the product. Recalling the Fundamental Theorem of Finitely Generated Abelian Groups [8,p.131] we may readily see that finite abelian groups of prime order are subdirectly irreducible.

Define a variety to be a collection of similar algebras which is closed with respect to the formation homomorphic images, subalgebras and direct products of algebras. For instance, if {G} is the collection of all abelian groups then {G} is a variety since homomorphic images of abelian groups are abelian groups, subgroups of abelian groups are abelian groups and direct products of abelian groups are abelian groups. The following classic theorem of Birkhoff [2,p.92] is crucial to the development of this paper.

Theorem II. 6: Let η be a variety of algebras and $[S,F]\varepsilon\eta$. Then $[S,F]$ has an isomorphic representation as a subdirect product of subdirectly irreducible algebras in η.

This theorem in essence says that in a variety, each algebra can be broken down as a subdirect product of algebras each of which can be reduced no further, much the same way that an algebraic number can be factored into a product of irreducibles in a given field.

Consider a collection of similar algebras $\eta=\{[S,F]\}$, $[S,F]\varepsilon\eta$ and a set X. $[S,F]$ is said to be the free algebra generated by X in η provided that if there exists and injection $a:X \rightarrow [S,F]$ such that the F-closure of $a(X)$ is $[S,F]$, for any $[T,F]\varepsilon\eta$ and function b; $X \rightarrow [T,F]$ there exists a unique F-homomorphism ϕ: $[S,F] \rightarrow [T,F]$ such that $b = \phi o a$. That is, the following diagram commutes.

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[dr, "b"'] & {[S,F]} \arrow[d, "\exists!\phi"] \\ & {[T,F]} \end{tikzcd}$$

Figure 1: Commutative Diagram of a Free Algebra

Perhaps the most common example of a free algebraic structure is a vector space generated by its basis in the class of all vector spaces over a given field. In this instance the basis serves as the set X with unique homomorphisms between vector spaces being determined merely be defining functions, b, between basis sets.

A generalization of the above definition is that of the free extension of an algebra. Let $\eta' = \{[T,F']\}$ and $\eta = \{[S,F]\}$ be collections of similar algebras with F' F. If $[T,F']\varepsilon\eta'$, $[S,F]\varepsilon\eta$ and ϕ is an F-monomorphism from $[T,F']$ to $[S,F]$ such that the F-closure of $\phi(T)$ in S is S, then $[S,F]$ is called the free extension of the algebra $[T,F']$ in η provided for each $[R,F]\varepsilon\eta$ and F'-homomorphism θ: $[T,F'] \rightarrow [R,F]$ there exists a unique F-homomorphism $\psi : [S,F] \rightarrow [R,F]$ such that $\theta = \psi \circ \phi$. That is, the following diagram commutes.

$$\begin{tikzcd}
{[T,F']} \arrow[r, "\phi"] \arrow[dr, "\Theta"'] & {[S,F]} \arrow[d, "\exists!\Psi"] \\
& {[R,F]}
\end{tikzcd}$$

Figure 2: Commutative Diagram of a Free Algebraic Extension

This is a generalization of the previous definition in which X = T and F = $\emptyset$. The next two theorems on the existence and uniqueness of free algebras can be found in Birkhoff [2,p.143].

<u>Theorem II. 7</u>: If η is a variety of algebras and X is a set then the free algebra generated by X in η exists.

<u>Theorem II. 8</u>: If η is a collection of similar algebras and X is a set and [S,F] is the free algebra generated by X in η, then [S,F] is the unique such algebra in η.

Lastly, recall that a relation on a set S is a subset of the product S x S and define a partial order relation, $\leq$, on S to be a relation that is reflexive, antisymmetric and transitive. That is, for all $s_1, s_2, s_3 \varepsilon S$

i) $s_1 \leq s_1$ (refexive)

ii) $s_1 \leq s_2$ and $s_2 \leq s_1$ imply $s_1 = s_2$ (antisymmetric)

iii) $s_1 \leq s_2$ and $s_2 \leq s_3$ imply $s_1 \leq s_3$ (transitive).

A partial order, $\leq$, on a set S is called a total order provided it has the comparability property, either $s_1 \leq s_2$ or $s_2 \leq s_1$ for all s_1, $s_2 \varepsilon S$. Sets with total orders are often referred to as chains.

If $\leq$ is a partial order on set S define for each $x,y \varepsilon S$ the least upper bound of x and y, $x \vee y$ or `x join y´, to be the element z of S which satisfies the two following criteria if such an element exists:

i) $x \leq z$ and $y \leq z$, and

ii) if $w \varepsilon S$ such that $y \leq w$ and $y \leq w$, then $z \leq w$.

Similarly, z is the greatest lower bound of x and y, $x \wedge y$ or `x meet y´, provided:

i) $z \leq x$ and $z \leq y$, and

ii) if $w \varepsilon S$ such that $w \leq x$ and $w \leq y$, then $w \leq z$.

Note that $x \wedge y$ and $x \vee y$ may not exist for all $x,y \varepsilon S$ if S is partially ordered. If, however, $x \wedge y$ and $x \vee y$ exists for all x and y in a partially ordered set S this set is said to form a lattice. Note that a totally ordered set is necessarily a lattice.

Consider an algebra [S,F]. [S,F] is a lattice-ordered algebra, ℓ-algebra, provided

i) there exists a partial order $\leq$ on S,

ii) $\{\Lambda, V\} \varepsilon F$ where Λ and V are considered binary functions on S, and

iii) if $f\alpha \varepsilon F$ then for any $\{S_{ij}\}_{i=1}^{N(\alpha)}\ {}_{j=1}^{K_{N(\alpha)}}$ we have

$$f\alpha\ (S_{1_1} * \ldots * S_{1_{K_1}}, \ldots, S_{N(\alpha)_1} * \ldots * S_{N(\alpha)_{K_{N(\alpha)}}})$$

$$= f\alpha\ (t_1, \ldots, t_{N(\alpha)_1}) * \ldots * f\alpha\ (t_{1_\ell}, \ldots, t_{N(\alpha)_\ell}) \qquad (10)$$

where each occurrence of * can be either Λ or V and

$$\{t_{11}, \ldots, t_{N(\alpha)_1}, \ldots, t_{1_\ell}, \ldots, t_{N(\alpha)_\ell}\} \quad \{S_{ij}\}_{i=1}^{N(\alpha)}\ {}_{j=1}^{K_{N(\alpha)}} .$$

Condition iii is required in order to tie the order relation of the set S into the algebraic structure of [S,F]. For instance in a lattice ordered group G we will see that for all a, b, $c\varepsilon G$ the following hold:

i) $a(b \wedge c) = ab \wedge ac$

ii) $a(b \vee c) = ab \vee ac$

iii) $(b \vee c)^{-1} = b^{-1} \wedge c^{-1}$

iv) $(b \wedge c)^{-1} = b^{-1} \vee c^{-1}$.

CHAPTER III

CONSTRUCTION OF FREE LATTICE-ORDERED ALGEBRA

In this chapter we will develop a method for the construction of free ℓ-algebras using the definitions discussed in Chapter II. The construction will follow directly from the following four lemmas.

<u>Lemma III. 1</u>: Let η be a variety of ℓ-algebras in which subdirectly irreducibles are totally ordered, $\eta = \{[S,F]\}$, and define $\eta' = \{[S',F'] \mid F' = F \setminus \{\wedge, \vee\}$, $S' \subseteq S$ for some $[S,F] \in \eta$ and S' is F'-closed $\}$. Let X be a set and α an injection from X to $[S,F] \in \eta$ such that the F-closure of $\alpha(X)$ is S. Then the F'-closure of S' in S is S and is given by $\{\bigvee_i \bigwedge_j s'_{ij} \mid s'_{ij} \in S'\}$.

<u>Proof</u>: It will suffice to show $[\{\bigvee_i \bigwedge_j s'_{ij}\},F]$ is an F-subalgebra of $[S,F]$ since $\alpha(X) \subseteq \{\bigvee_i \bigwedge_j s'_{ij}\}$ and S is the F-closure of $\alpha(X)$.

Clearly $\{\bigvee_i \bigwedge_j s'_{ij}\}$ is closed under $\vee$ and $\wedge$. Consider $f \in F'$, f n-ary, and $\{s'_{11},\ldots, s'_{1K_1},\ldots, s'_{N_1},\ldots, s'_{NK_N}\} \subsetneq S'$. Then, if $\{s_{i_r j_r}\} = \{s_{r_1},\ldots,s_{r_{k_r}}\}$, $1 \leq r \leq N$, we have

$$f(\bigvee_i \bigwedge_j s'_{i,j,},\ldots, \bigvee_i \bigwedge_j s'_{i_N j_N})$$

$$= f(y_{11}, \ldots , y_{N_1}) * \ldots . * f(y_{1_\ell}, \ldots , y_{N_\ell}) \qquad (11)$$

where $\{y_{11},\ldots, y_{N_\ell}\}$ $\{s'_{11},\ldots, s_{NK_n}\}$ and $*$ is either $\wedge$ or . Since S' is F'-closed, each $f(y_{1_r},\ldots, y_{N_r})$, $1 \leq r \leq \ell$, is in S', hence $\{\bigvee_i \bigwedge_j s'_{ij}\}$ is F'-closed. Thus $S = \{\bigvee_i \bigwedge_j s'_{ij} \varepsilon S'\}$.

Lemma III. 1 gives us a method to generate and represent the elements of an ρ-algebra using the elements of a simpler underlying unordered algebraic system.

<u>Lemma III. 2</u>: If η, η', $[S'F']$, $[S,F]$, X and α are as in Lemma III. 1 there exists a collection of nontrivial congruence relations $\Gamma' = \{\phi'_\gamma\}$ on $[S',F']$ such that

i) $[S',F']/\phi'_\gamma$ can be totally ordered as an algebra in η for each $\phi'_\gamma \varepsilon \Gamma'$, and

ii) for any collection $\{[s'_\gamma] \mid [s'_\gamma] \varepsilon [S',F']/\phi'_\gamma$ for each $\phi'_\gamma \varepsilon \Gamma'\}$ we have $|\bigcap_{\Gamma'} [S'_\gamma]| \leq 1$.

<u>Proof</u>: By Theorem II. 6 there exists an isomorphism $\phi: [S,F] \to \prod_\Gamma [S_\gamma,F]$, $\phi = \prod_\Gamma \phi_\gamma$, with each $[S_\gamma,F]$ subdirectly irreducible hence totally ordered. Define a partition of S' by $\phi'_\gamma = \{S' \quad [s_\gamma] \mid [s_\gamma] \varepsilon [S,F]/\phi_\gamma\}$ for each $\phi_\gamma \varepsilon \Gamma$. Note that each ϕ'_γ is a congruence on $[S',F']$ since

$F' \subseteq F$. Let $\Gamma' = \{\phi'_\gamma \mid \phi_\gamma \varepsilon \Gamma\}$. Since $[S,F]/\phi_\gamma \cong [S_\gamma,F]$, $[S,F]/\phi_\gamma$ is totally ordered. The order defined on $[S',F']/\phi_\gamma$ given by $S' \cap [s_1] \leq S' \cap [s_2]$ if and only if $[s_1] \leq [s_2]$ in $[S,F]/\phi_\gamma$ gives a total order on $[S',F']/\phi'_\gamma$. Since ϕ is an isomorphism it is an injection, we have $|\bigcap_\Gamma [s_\gamma]| \leq 1$ for any collection $\{[s_\gamma] \mid [s_\gamma] \varepsilon [S,F]/\phi_\gamma\}$. Since $S' \cap [s_\gamma] \subsetneq [s_\gamma]$ for all γ, we have $|\bigcap_\Gamma (S' \cap [s_\gamma])| = 1$ for any collection $\{S' \cap [s_\gamma] \mid S' \cap [s_\gamma] \varepsilon [S',F']/\phi'_\gamma\}$. Hence Γ' satisfies conditions i) and ii) above.

<u>Lemma III. 3</u>: Let η, η', $[S,F]$, $[S',F']$, X and α be as in Lemma III. 1. Let $[T,F]$ be an element of η, β a function from X to $[T,F]$ and T' the F'-closure of $\beta(X)$ in T. If θ' is an F'-homomorphism from $[S',F']$ to $[T',F']$ such that $\beta = \theta' \circ \alpha$, then θ' can be uniquely extended to an F-homomorphism θ from $[S,F]$ to $[T,F]$ such that $\beta = \theta \circ \alpha$.

<u>Proof</u>: Let Γ' be a collection of congruence relations on $[S',F']$ maximal with respect to conditions i) and ii) of Lemma III. 2. Denote $\Gamma' = \{\phi'_\gamma\}$. Let ϕ' denote the canonical F'-monomorphism from $[S',F']$ to $\Pi_{\Gamma'} [S',F']/\phi'_\gamma$, i.e. $\phi'(s) = \Pi_{\Gamma'} [S_\gamma]$ where $[s_\gamma] = [s] \varepsilon [S',F']/\phi'_\gamma$. Since each $[S',F']/\phi'_\gamma$ can be totally ordered as an algebra we can define $\wedge$ and $\vee$ on $\Pi_{\Gamma'} [S',F']/\phi'_\gamma$ and hence consider it as an ℓ-algebra in η. Recall that $S = \{\bigvee_i \bigwedge_j s'_{ij} \mid s'_{ij} \varepsilon S'\}$ by Lemma II. 1 hence we can embed $[S,F]$ in $\Pi_{\Gamma'} [S',F']/\phi'_\gamma$ by

$\bigvee_i \bigwedge_j s'_{ij} \to \bigvee_i \bigwedge_j \phi'(s'_{ij})$. Represent $[T,F]$ by $\prod_\tau [T,F]/\psi_t$ with each $[T,F]/\psi_t$ subdirectly irreducible hence totally ordered. It will suffice to show that if θ' is a homomorphism from $\phi'(S')$ to $\prod_\tau [T',F']/\phi'_t$ then we may extend θ' to θ from $\{\bigvee_i \bigwedge_j \phi(s'_{ij}) \mid s'_{ij} \varepsilon S'\}$ to $\prod_\tau [T,F]/\psi_t$.

Note that since Γ' contains all congruences on $[S',F']$ such that $[S',F']/\phi'_\gamma$ can be totally ordered and since $[S',F']/\psi_t \circ \theta'$ is a totally ordered ℓ-algebra, $\psi_t \circ \theta'$ is an element of Γ' for each t. That is, for each ψ_t, there exist $\phi'_{\gamma(t)} \varepsilon \Gamma'$ such that $\phi'_{\gamma(t)} = \psi_t \circ \theta'$.

Define θ: $\{\bigvee_i \bigwedge_j \phi'(s'_{ij})\} \to \prod_\tau (T',F')/\psi_t$ by

$$\theta\left(\bigvee_i \bigwedge_j \phi'(s'_{ij})\right) = \bigvee_i \bigwedge_j \theta'(\phi'(s'_{ij})). \tag{12}$$

Let us show θ is well defined. Suppose

$\bigvee_i \bigwedge_j \phi'(s'_{ij}) = \bigvee_r \bigwedge_s \phi'(s'_{rs})$, $s'_{rs} \varepsilon S'$. Let $\psi_t \varepsilon \tau$. Then

$$\psi_t \circ \theta\left(\bigvee_i \bigwedge_j \phi'(s'_{ij})\right) = \psi_t\left(\bigvee_i \bigwedge_j \theta'(s'_{ij})\right)$$

$$= \bigvee_i \bigwedge_j (\psi_t \circ \theta'(s'_{ij})) = \bigvee_i \bigwedge_j (\phi'_{\gamma(t)}(s'_{ij}))$$

$$= \bigvee_r \bigwedge_s (\phi'_{\gamma(t)}(s'_{rs})) = \bigvee_r \bigwedge_s (\psi_t \circ \theta(\bigvee_r \bigwedge_s \theta'(s'_{rs}))$$

$$= \psi_t\left(\bigvee_r \bigwedge_s \theta'(s'_{rs})\right) = \psi_t \circ \theta\left(\bigvee_r \bigwedge_s \sigma'(s'_{ij})\right). \tag{13}$$

Hence $\psi_t \circ \theta\,(\bigvee_i \bigwedge_j \phi'(s'_{ij})) = \psi_t \circ \theta\,(\bigvee_r \bigwedge_s \phi'(s'_{rs}))$ for all ψ_t implying $\theta(\bigvee_i \bigwedge_j \phi'(s'_{ij})) = \theta\,(\bigvee_r \bigwedge_s \phi'(s'_{rs}))$. Thus θ is well defined.

Let us show θ is a homomorphism. Suppose $f \varepsilon F$, f N-ary, and $\bigvee_{i_1} \bigwedge_{j_1} \phi'(s'_{i_1 j_1}), \ldots, \bigvee_{i_N} \bigwedge_{j_N} \phi'(s'_{i_N j_N}) \varepsilon \{\bigvee_i \bigwedge_j \phi'(s'_{ij})\}$. From Lemma III. 1 we have

$$f(\bigvee_{i_1} \bigwedge_{j_1} \phi'(s'_{i_1 j_1}), \ldots, \bigvee_{i_N} \bigwedge_{j_N} \phi'(s'_{i_N j_N})) = \bigvee_r \bigwedge_s \phi'(s'_{rs})$$

for some $\{s'_{rs}\} \varepsilon S'$. Then

$$\psi_t \circ \theta\,(f\,(\bigvee_{i_1} \bigwedge_{j_1} \phi'(s'_{i_1 j_1}), \ldots, \bigvee_{i_N} \bigwedge_{j_N} \phi'(s'_{i_N j_N})))$$

$$= \psi_t \circ \theta\,(\bigvee_r \bigwedge_s \phi'(s'_{rs})) = \psi_t\,(\bigvee_r \bigwedge_s \theta'(s'_{rs}))$$

$$= \bigvee_r \bigwedge_s (\psi_t \circ \theta'(s'_{rs})) = \bigvee_r \bigwedge_s (\phi'_{\gamma(t)}(s'_{rs}))$$

$$= f\,(\bigvee_{i_1} \bigwedge_{j_1} \phi'_{\gamma(t)}(s'_{i_1 j_1}), \ldots, \bigvee_{i_N} \bigwedge_{j_N} \phi_{\gamma(t)}(s'_{i_N j_N}))$$

$$= f(\bigvee_{i_1} \bigwedge_{j_1} \psi_t \circ \theta'(s'_{i_1 j_1})), \ldots, \bigvee_{i_N} \bigwedge_{j_N} \psi_t \circ \theta'(s'_{i_N j_N}))$$

$$= f(\psi_t\,(\bigvee_{i_1} \bigwedge_{j_1} \theta'(s'_{i_1 j_1})), \ldots, \psi_t\,(\bigvee_{i_N} \bigwedge_{j_N} \theta'(s'_{i_N j_N})))$$

$$= f(\,\psi_t \circ \theta\,(\bigvee_{i_1} \bigwedge_{j_1} \phi'(s'_{i_1 j_1})), \ldots, \psi_t \circ \theta\,(\bigvee_{i_N} \bigwedge_{j_N} \phi'(s_{i_N j_n}))) \quad (14)$$

Hence $\psi_t \circ \theta\,(f(\bigvee_{i_1} \bigwedge_{j_1} \phi'(s'_{i_1 j_1}),\ldots, \bigvee_{i_N} \bigwedge_{j_N} \phi'(s'_{i_N j_N})))$

$$= f\,(\,\theta\,(\bigvee_{i_1} \bigwedge_{j_1} \phi'(s'_{i_1 j_1})),\ldots, \theta(\bigvee_{i_N} \bigwedge_{j_N} \phi'(s'_{i_N j_N}))) \qquad (15)$$

for all ψ_t implying

$$\theta\,(f\,(\bigvee_{i_1} \bigwedge_{j_1} \phi'(s'_{i_1 j_1}),\ldots, \bigvee_{i_N} \bigwedge_{j_N} \phi'(s'_{i_N j_N})))$$

$$= f(\,\theta\,(\bigvee_{i_1} \bigwedge_{j_1} \phi'(s'_{i_1 j_1})),\ldots, \theta(\bigvee_{i_N} \bigwedge_{j_N} \phi'(s'_{i_N j_N}))).\qquad (16)$$

Thus θ is a homomorphism.

Clearly $\beta = \theta \circ \alpha$ since $\beta = \theta' \circ \alpha$. Let us show θ is unique. Suppose w is another homomorphism from $\prod_{\Gamma} [S',F']/\phi'_{\gamma}$ to $\prod_{\tau} (T,F)/\psi_t$ such that w agrees with θ' on $\phi'(S')$ and $\beta = w \circ \alpha$. Consider $\bigvee_i \bigwedge_j w \circ \phi'(s'_{ij})$. Then

$$w(\bigvee_i \bigwedge_j \phi'(s'_{ij})) = \bigvee_i \bigwedge_j w \circ \phi'(s'_{ij})$$
$$= \bigvee_i \bigwedge_j \theta \circ \phi'(s'_{ij}) = \theta\,(\bigvee_i \bigwedge_j \phi'(s'_{ij})) \qquad (17)$$

and hence θ is unique.

<u>Lemma III. 4</u>: Let η be a variety of ℓ-algebras in which subdirectly irreducibles are totally ordered. Define η' as in Lemma III. 1 and let X be a set. Then the free algebra generated by X in η' exists.

Proof: Let $[M,F]$ be the free ℓ-algebra over X in η, its existence following from Theorem II. 7. Let α be an injection from X to M such that the F-closure of $\alpha(X)$ is M. Let M' be the F'-closure of $\alpha(X)$ in M where $F' = F \setminus \{ \vee, \wedge \}$. Then $[M',F']$ is the free algebra over X in η:

Let $[T',F'] \varepsilon \eta'$ and $[T,F] \varepsilon \eta$ such that $[T',F']$ is an F'subalgebra of $[T,F]$. Let β be a function X to T' whence from X to T. Recall that $\alpha(X)$ generates M' under F'. Since $[M,F]$ is free over X in η there exists a unique F-homomorphism θ from M to T such that $\beta = \theta \circ \alpha$. Let θ' be the restriction of θ to M' and note θ' is an F'-homomorphism from M' to T'; since $F' \leq F$. The uniqueness of θ' follows from Lemma III. 3. Hence $[M',F']$ is the free algebra over X in η'.

$$
\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & {[M',F']} \arrow[d, "\exists ! \theta'"] & \subseteq & {[M,F]} \arrow[d, "\exists ! \theta"] \\
 & {[T',F']} & \subseteq & {[T,F]}
\end{tikzcd}
$$

Figure 3: Commutative Diagram of the Free Ordered and Unordered Algebras over a Set X

<u>Theorem III. 5</u>: Let η be a variety of ℓ-algebras in which subdirectly irreducible ℓ-algebras are totally ordered. Define $\eta' = \{[S',F'] \mid S' \varepsilon S$ for some $[S,F] \varepsilon \eta$ and S' is F'-closed where $F' = F \setminus \{ \vee, \wedge\}\}$. Let X be a set and $[M',F']$ the free algebra over X in η'. Let Γ' be a collection of congruence relations on $[M',F']$ such that Γ' is maximal with respect to the following two conditions:

i) $[M',F']/\phi'_\gamma$ can be totally ordered as an algebra in η for all $\phi'_\gamma \varepsilon \Gamma'$, and

ii) For any collection $\{[S_\gamma] \mid [S_\gamma] \varepsilon [M',F']/\phi'_\gamma,\ \phi'_\gamma \varepsilon \Gamma'\}$ we have $|\bigcap_{\Gamma'} [S_\gamma]| \leq 1$.

Define $F_{M'} = \{ \bigvee_i \bigwedge_j \phi' (m'_{ij}) \mid \phi' = \Pi_{\Gamma'} \phi'_\gamma,\ m'_{ij} \varepsilon M'\}$. Then $[F_{M'},F]$ is the free ℓ-algebra over X in η.

<u>Proof</u>: (Consider Figure 4.) Let $[T,F] \varepsilon \eta$ and β a function from X to T. Let T' be the F'-closure of $\beta(X)$ in T. Let α be an injection from X to $[M',F']$ such that $\alpha(X)$ generates M' under F'. Since $[M',F']$ is free over X in η' there exists and F'-homomorphism θ' such that $\beta = \theta' \circ \alpha$.

Consider now the function $\theta' \circ \alpha$ from X to F_M'. $\phi' \circ \alpha(X)$ generated F_M' under F by the definition of F_M' and since $\alpha(X)$ generates M' under F'. We may extend θ' to a homomorphism θ from $[F_m',F]$ to $[T,F]$ by Lemma III. 3 such that

$\beta = \theta \circ \phi' \circ \alpha$. The uniqueness of θ follows from the uniqueness of $\theta'$and the uniqueness of the extension of θ'. Hence $[F_{M'},F]$ is the free ℓ-algebra over X in η.

$$
\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & {[M',F']} \arrow[r, "\phi'"] \arrow[d, "\Theta'"] & F_{M'} \subseteq \prod_{\Gamma'}{}^{[M',F']}/\phi\gamma \arrow[d, "\Theta"] \\
 & {[T',F']} \subseteq & {[T,F]}
\end{tikzcd}
$$

Figure 4. Commutative Diagram of the Construction of a Free ℓ-Algebra

CHAPTER IV

SPECIFIC EXAMPLES OF FREE ℓ-ALGEBRAS

In this chapter we apply the results of Chapter III to give constructions for specific examples of ℓ-algebras. If η is a variety of ℓ-algebras then η' as defined earlier is a difficult object to work with, difficult in the sense that we are not assured of η' being a variety and that we do not have a firm idea of what is contained in η'. It is much easier to work instead with a variety of F'-algebras since varieties are closed under the formation of subalgebras, direct products and homomorphic images; η' may have none of these properties.

The strategy then for constructing actual examles of ℓ-algebras will be as follows: starting with a set X and variety of ℓ-algebras $\eta=\{[S,F]\}$ in which subdirectly irreducibles are totally ordered we will let η' be the variety of all F'-algebras where $F'=F\setminus\{\vee, \wedge\}$. Then, letting $[M',F']$ be the free algebra over X in η', if there exists a collection Γ' of congruences on $[M',F']$ that is maximal with respect to the conditions of Lemma III. 2. then F_M' as defined in Theorem II. 5 is the free ℓ-algebra over X in η. This method leaves two things to construct - $[M',F']$ and Γ'.

Consider the case of lattice ordered abelian groups (abelian ℓ-groups). An abelian ℓ-group is an abelian group $(G,\leq)$ such that for all a, b, $c \in G$, $a(b \wedge c) = (ab) \wedge (ac)$ and $a(b \vee c) = (ab) \vee (ac)$. (Earlier in Chapter II conditions were given for an abelian group to be an abelian ℓ-group. These are an equivalence to the above definition.) Familiar examples include:

i) $(\mathbb{R}, +, \leq)$ with $\leq$ the natural order of the reals, and

ii) $(p(X), \Delta, \leq)$ with X a nonempty set, Δ the symmetric difference of sets and $\leq$ being set inclusion.

If η is the collection of all abelian ℓ-groups then η is a variety of ℓ-algebras in which subdirectly irreducibles are totally ordered. Since η satisfies the hypothesis of Theorem III. 5 the method described above can be used to construct free abelian ℓ-groups. Theorems similar to those in Chapter III but specific to abelian ℓ-groups can be found in Bernau [0,p.48] and were used as motivation for this paper. Let us explicitly construct the free ℓ-group over $X = \{a\}$ in η, the variety of all ℓ-groups.

Let η' be the variety of all abelian groups and clearly $M' = Z$, the free group on one generator in η'. If G is an abelian ℓ-group it is easily shown that for o, $a \in G$ with $o < a$, then $a < a + a$. Hence if G is nontrivial and totally ordered, G is not finite. The only congruence ϕ then on Z for which Z/ϕ can be totally ordered as a group and is infinite is the trivial congruence, i.e. $\phi = \{\{x\} | x \varepsilon Z\}$, with $Z/\phi \cong Z$. Since Z can be totally ordered as a group in only two ways, the usual order and the inverse order given by $a \leq b$ if and only if $b \leq a$ in the usual order, there exist only two congruences in Γ', each trivial. Then $F_M' \subseteq Z \times Z$. It is easy to see that $F_m' = Z \times Z$ since $(1,-1), (0,0) \varepsilon F_M'$. Hence the free ℓ-group on one generator is $Z \times Z$.

Consider now the case of f-modules. Define the positive cone of an abelian ℓ-group $(G, \cdot, \leq)$ to be all $g \varepsilon G$ such that $g \geq 1$. A lattice ordered ring, ℓ-ring, $(R, +, \cdot \leq)$ is a ring $(R, +, \cdot)$ with a lattice order $\leq$ such that $(R, +, \leq)$ is an abelian ℓ-group and, if a, b, $c \varepsilon R$ with $a \leq b$ and $c \varepsilon R+$ then $ac \leq bc$ and $ca \leq cb$. An F-ring is an ℓ-ring such that if a, $b \varepsilon R$ with $a \wedge b = 0$ and $r \varepsilon R+$ then $(ra) \wedge b = 0$. Define a lattice ordered module, ℓ-module, $(M, R, +, \cdot, \leq)$ to be a module with $(M, +, \leq)$ an abelian ℓ-group, $(R, +, \cdot, \leq)$ an ℓ-ring and if a, $b \varepsilon M$ such that $a \leq b$ and $c \varepsilon R+$ then $ca \leq cb$ and $ac \leq bc$. Define an f-module to be an ℓ-module such that if $a, b \varepsilon M$ such that $a \wedge b = 0$ and $r \varepsilon R+$, then $(ra) \wedge b = 0$. Examples of f-modules are:

i) abelian ℓ-groups as modules over the integers Z, and

ii) any vector space which satisfies the conditions of being an ℓ-module is also an f-module.

The collection of all f-modules over a given f-ring forms a variety in which subdirectly irreducibles are totally ordered, hence the method for construction of free ℓ-algebras we have developed can be used for free f-modules. Theorems similar to those found in Chapter III for the specific case of f-modules may be found in Powell [7,p29]. These also served as motivation for this paper. Let us explicitly construct the free f-module over one generator, $X = \{a\}$, in the variety η of all f-modules over the totally ordered f-ring of reals, R.

Let η' be the variety of all modules over R and clearly $M' = R$, the free module over X in η'. Since R has no proper nontrivial submodules and can only be ordered in two ways, the usual order and the inverse order, Γ' has only two elements, both trivial. Hence $F_M' \subseteq R \times R$ and, again, it is easily seen that $F_M' = R \times R$.

Consider, as a last example, Boolean algebras. A lattice L is said to be complemented provided there exists elements 0 and 1 in L such that, for all $\ell \varepsilon L$, $0 \leq \ell$ and $\ell \leq 1$ and there exists $\ell \varepsilon L$ such that $\ell \wedge \ell' = 0$ and $\ell \vee \ell' = 1$. L is said to be distributive provided $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ and $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ for all $a,b,c \varepsilon L$. If L is both complemented and distributive then L is said to be a Boolean algebra. Examples are:

i) the power set of a nonempty set X, P(X), forms a Boolean algebra with $A \leq B$ if and only if A B for $A, B \varepsilon P(X)$, and

ii) Any totally ordered set with two elements is a Boolean algebra. In fact a Boolean algebra is subdirectly irreclucible if and only if B is a two element chain [2,p.162].

If η is the collection of all Boolean algebras then η is a variety in which subdirectly irreducibles are totally ordered. Hence the construction of Chapter III is valid for the construction of free Boolean algebras. Let us explicitly construct the free Boolean algebra over two generators.

Let $X = \{a,b\}$ and define $M' = \{a, a', b, b', 0, 1\}$ ordered as follows: $0 \leq a, a', b, b' \leq 1$. Define V and Λ on M' as follows: $a \vee a' = b \vee b' = 1$ and $a \wedge a' = b \wedge b' = 0$. M' may be realized in conventional lattice representation as Figure 5.

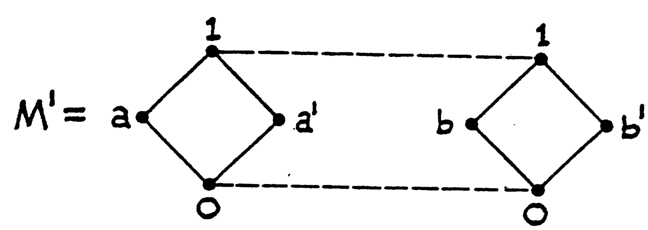

Figure 5: The Free Complemented Set with 0 and 1 over Two Generators

We may denote the set of operations in a Boolean algebra as $F = \{ \vee, \wedge, ', 0, 1\}$. If $F' = F \setminus \{ \vee, \wedge\}$ then $[M', F']$ is the free algebra over two generatiors ($X = \{a,b\}$) in the collection of all F'-algebras (see Lemma V. 2.). Note $\wedge$ and $\vee$ are still defined for some pairs of elements in M' yet are not elements of F' since they are not defined for all pairs. Let us find all congruences ϕ'_γ on $[M',F']$ such that $(m',F']/\phi'_\gamma$ can be totally ordered as a Boolean algebra. Since totally ordered Boolean algebras have order 2 we must only find congruences on $[M',F']$ such that $|[M',F']/\phi'_\gamma| = 2$. Visually inspecting $[M',F']$ we see there are four:

$$\phi'_1 = \{ \{ 1, a, b \}, \{0, a', b' \}\}, \tag{18}$$

$$\phi'_2 = \{ \{ 1, a', b' \}, \{ 0, a, b\}\}, \tag{19}$$

$$\phi'_3 = \{ \{ 1, a', b\}, \{0, a, b'\}\}, \text{ and} \tag{20}$$

$$\phi'_4 = \{ \{ 1, a, b'\}, \{0, a', b\}\}. \tag{21}$$

Order these using the order induced by that of M', i.e.

$$\{1, a, b\} \geq \{0, a', b'\} \text{ in } [M',F']/\phi'_1,$$
$$\{1, a', b'\} \geq \{0, a, b\} \text{ in } [M',F']/\phi'_2,$$
$$\{1, a', b\} \geq \{0, a, b'\} \text{ in } [M',F']/\phi'_3, \text{ and}$$
$$\{1, a, b'\} \geq \{0, a', b'\} \text{ in } [M',F]/\phi'_4.$$

$c = \phi'(a) \vee \phi'(b') =$

$d = \phi'(a) \vee \phi'(b) =$

$e = \phi'(a') \vee \phi'(b') =$

$f = \phi'(a') \vee \phi'(b) =$

$g = \phi'(a) \wedge \phi'(b') =$

$h = \phi'(a') \wedge \phi'(b') =$

$i = \phi'(a) \wedge \phi'(b) =$

$j = \phi'(a') \wedge \phi'(b) =$

$k = c \wedge f =$

$l = g \vee j =$

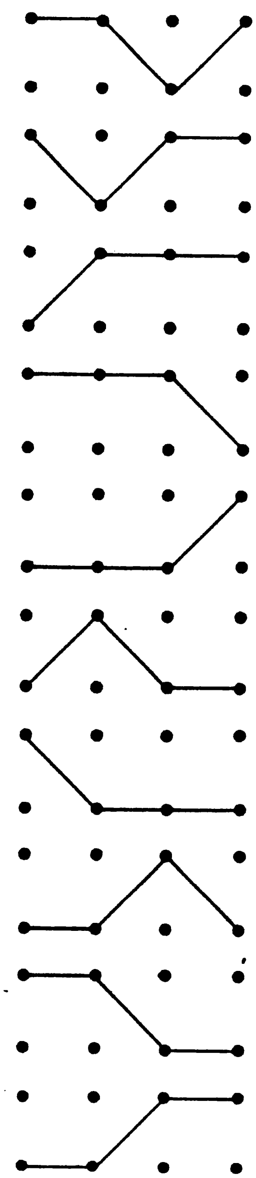

Figure 8. Elements of $F_{M'}$ Generated by $\phi'(M')$ in $\prod_{i=1}^{4} (M', F']/\phi'_i$

If $\Gamma' = \{ \phi'_i \}_{i=1}^{4}$ then Γ' satisfies the hypothesis of Theorem III. 5. Denote for simplicity the larger element of ϕ'_i as 1_i and the smaller element as 0_i. We may then represent $\prod_{i=1}^{4} [M',F']/\phi'_i$ as in Figure 6 and, if ϕ is the cononical embedding of Γ' into $\prod_{i=1}^{4}[M',F']/\phi'_i$ we have the elements depicted in Figure 7.

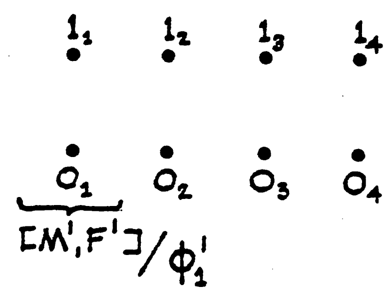

Figure 6. The Direct Product of Quotients of $[M',F']$ over Γ'

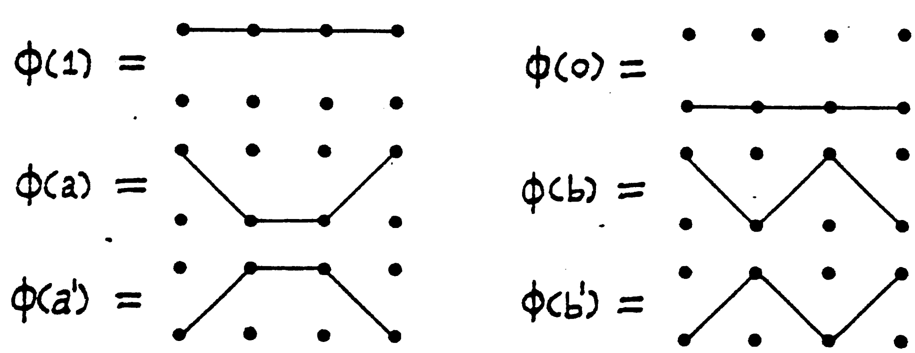

Figure 7: The Embedding of $[M',F']$ into $\prod_{i=1}^{4}[M',F']/\phi'_i$

Letting $F_M' = \{\bigvee_i \bigwedge_j \phi'(M_{ij}) \mid M_{ij} \;\; M'\}$ as in Theorem III. 5 we generated the rest of the elements of $\prod_{i=1}^{4} (M',F']/\phi'_i$, Figure 8. By Theorem III. 5 we have $F_M' = \prod_{i=1}^{4} [M',F']/\phi'_i$ is the free Boolean algebra over two generators. Figure 9 gives a conventional lattice representation of F_M'.

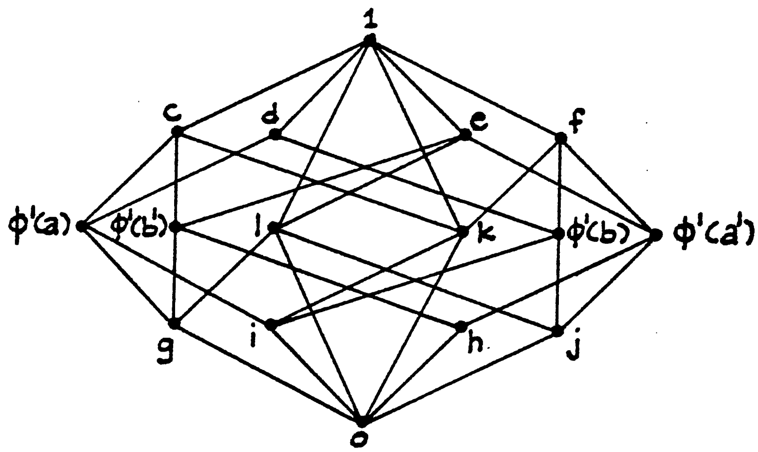

Figure 9 - Conventional Lattice Representation of the Free Boolean Algebra over Two Generators

CHAPTER V

THE CARDINALITY OF FINITE FREE ℓ-ALGEBRAS:

A Special Case

Consider the following special case:

<u>Theorem V. 1</u>: Let $\eta = \{[S,F]\}$ be a variety of ℓ-algebras in which subdirectly irreducibles are totally ordered and have cardinality $N < \infty$. Then if $[F_M',F]$ is a finite free algebra in η over a set X, then $|F_M'| = N^m$ for some positive integer $m < \infty$.

<u>Proof</u>: Construct F_M' over X as in Theorem III. 5, $F_M' \subseteq \prod_{\Gamma'} [M',F']/\phi'_\gamma$, with each $[M',F']/\phi'_\gamma$ having cardinality N. Let us argue that the cardinality of Γ' is finite. Since F_M' is finite and $[M',F']$ can be embedded in $[F_M',F]$, $[M',F']$ is finite. Since $[M',F']$ is finite it has only a finite number of congruences hence Γ' must be finite. Let $|\Gamma'| = m$, then $F_M' \prod_{i=1}^{m} [M',F']/\phi'_i$. It will suffice to show that $\phi'(M)$ generates all of $\prod_{i=1}^{m} [M',F']/\phi'_i$ under $\vee$ and $\wedge$.

For simplicity denote $[M',F']/\phi'_i$ by the set $\{0_i, 1_i, \ldots, m_i\}$ ordered as the integers; do not forget, however, that each r_S for $1 \leq r \leq N$ and $1 \leq S \leq m$ is a subset (congruence class) of M'. Using this notation it will suffice to show that for each $1 \leq r \leq N$ and $1 \leq s \leq M$, the element b_{r_s} of $\prod_{i=1}^{m} [M',F']/\phi'_i$ given by

$$b_{r_s}(x) = \begin{cases} r_s \ , & x = s \\ 0_x \ , & 1 \leq x \leq m \ , \ x = s \end{cases} \tag{22}$$

is in $F_{M'}$, since these elements clearly generate $\prod_{i=1}^{m} [M',F']/\phi'_i$ under V and Λ.

Choose such an r and s and let $m_s \varepsilon r_s$. Define $b_s = \phi'(m_s)$ and $M_s = \bigcup_{i=r}^{m} i_s$. Now, for $1 \leq x \leq m$ with $x \neq s$, define b_x as follows: if $0_x \cap M_s \neq \emptyset$ then $b_x = \phi'(m_s) \wedge \phi'(m_x)$ $m_x \varepsilon 0_x \cap M_s$; if $0_x \cap M_s = \emptyset$, let $b_x = \phi'(m_S) \vee \phi'(m_x)$ such that $m_x \varepsilon 0_x$. Then $b_{r_s} = \bigwedge_{x=1}^{m} b_x$ and we are done.

From the last example of Chapter IV we recall that in the variety of all Boolean algebras, totally ordered elements have cardinality two. From the above theorem we see that finite free Boolean algebras have order 2^m for some positive integer m. In this case m is determined by the cardinality of X as follows.

<u>Lemma V. 2</u>: Let X be a set, $X = \{x_i\}_{i \varepsilon I}$, and define a partially ordered set $(M', \leq)$ by $M' = X \cup X' \cup \{0,1\}$, $X' = \{x'_i\}_{i \varepsilon I}$, with $0 \leq x_i, x'_i \leq 1$ and $x_i \wedge x'_i = 0$ and $x_i \vee x'_i = 1$ for all $i \varepsilon I$. If $F' = \{', 0, 1\}$ then $[M',F']$ is the free algebra over X in the collection of all F'-algebras, η'.

<u>Proof</u>: Let $[T',F']\varepsilon\eta'$ and α an injection from X to M' given by $\alpha(X_i) = X_i$ for $i\varepsilon I$. Note $\alpha(X)$ generates M' under F'. Let β be a function from X to $[T',F']$. Define θ': $M' \to T'$ by

$$\theta' (x) = \begin{cases} 1, & x = 1 \\ 0, & x = 0 \\ \beta(x), & x \in X \\ \beta(x)', & x \in X' \end{cases} \qquad (23)$$

Clearly θ' preserves $'$, 1 and 0 hence is an F'-homomorphism. Also, it is obvious that $\beta = \theta' \circ \alpha$ and that θ' is unique. Hence $[M',F']$ is free over X in η'.

<u>Theorem V. 3</u>: Let X and $[M',F']$ be as in Lemma V. 2 with X finite. Then $[M',F;]$ has exactly 2^X distinct congruence relations ϕ' such that $[M',F']/\phi'$ is a totally ordered Boolean algebra.

<u>Proof</u>: Since a Boolean algebra is totally ordered if and only if it has cardinality 2, it will suffice to show that $[M',F']$ has exactly 2^X congruences ϕ' such that $|[M',F']/\phi'| = 2$. By induction:

if $X = 1$, $X = \{X_i\}$, $M' = \{x, x', 1, 0\}$ has only two such congruences $\phi'_1 = \{\{1,x\}, \{x', 0\}\}$ and $\phi'_2 = \{\{1, x'\}, \{x,0\}\}$. These are evident by inspection of Figure 10.

Assume now for $m < N$ that if $|X| = m$, then M' has 2^m such congruence relations. Suppose that $|X| = N$, $X = \{x_1, \ldots, x_N\}$. Let $X^* = \{X_1, \ldots, X_{N-1}\}$. Since $|X^*| < N$ $M^* = X \cup X^* \cup \{0,1\}$ has 2^{N-1} such congruences. Denote these by $\phi'_i = \{1_i, 0_i\}$, $1 \leq i \leq 2^{N-1}$. Let $M' = X \cup X' \cup \{0,1\}$. Then for each $1 \leq i \leq 2^{N-1}$, the paritions of M' given by $\phi'_{i_1} = \{1_i \cup \{x_N\}, 0_i \cup \{x_N'\}\}$ and $\phi'_{i_2} = \{1_i \cup \{x'_N\}, 0_i \cup \{x_N\}\}$ are congruences on M' such that $|[M',F']/\phi_{i_1}| = 2$ and $|[M',F']/\phi_{i_2}| = 2$. Hence $[M',F']$ has at least $2^{N-1} \cdot 2 = 2^N$ such congruences. Let us show these are all the congruences ϕ' on $[M',F]$ such that $|[M',F']/\phi'| = 2$.

Let $\phi'_i = \{1_{i_1} 0_i\}$ be a congruence on $[M',F']$ such that $|[M',F']/\phi'| = 2$. Note that if $x_N \varepsilon 1_i$ then $x'_N \not\varepsilon 0_i$. (Otherwise, suppose x_N, $x_N' \varepsilon 1_i$ implying $0 = x_N \wedge x_N' \varepsilon 1_i$ and $1 = x_N \vee x_N' \varepsilon 1_i$. Since ϕ'_i has two elements there exists $x \varepsilon M'$ such that $x \varepsilon 1_i$. Then $x \equiv x$ and $1 \equiv 0$ yet $x \vee 0 = x \neq 1 = x \vee 1$, hence a contradiction. Similarly for x_N, $x_N' \varepsilon 0_i$.) Let $\phi_i^* = \{1_i \setminus \{x_N\}, 0_i \setminus \{x'_N\}\}$ if $x_N \varepsilon 1_i$ or, if $x_N \varepsilon 0_i$, $\phi_i^* = \{1_i \setminus \{x'_N\}, 0_i \setminus \{x_N\}\}$. Then ϕ_i^* is one of the 2^{N-1} congruences on M^*, hence ϕ_i is one of the 2^N congruences on M' and we are done.

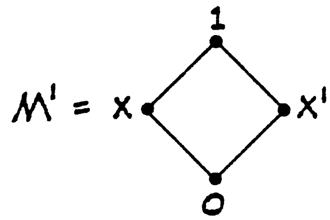

Figure 10 - The Free Complemented Poset with 0 and 1 over One Generator

<u>Corollary V. 4</u>: If X is a set of cardinality N, then the free Boolean algebra over X has order 2^{2^N}.

CHAPTER VI

CONCLUSION

The idea that a general method for the construction of free ℓ-algebras arose when it was realized that a similar strategy had been used in at least two separate instances the construction of free abelian ℓ-groups by Bernau [0] and the construction of free f-modules by Powell [7]. I adapted this construction to that of Boolean algebras and then to a general case of abstract algebras in Chapter III. The strategy developed simplifies the problem of construction in that the free unordered algebra is used to build the lattice-ordered algebra and useful methods of construction and characterizations of free unordered algebras have been developed in the past.

Several questions have arisen since the initial work in this paper was completed. In the theorem and lemmas of Chapter III, how strongly is the fact used that the operations excluded from F' are $\wedge$ and $\vee$? Can this method of construction be generalized even further to exclude operations other than $\wedge$ and $\vee$ from F'?

When considering the cardinality of certain cases of free ℓ-algebras in Chapter V, Theorem V. 1 is far from satisfactory. Can m in Theorem V. 1 be determined by η and X in general?

One of the greatest pleasures in doing mathematics is to abstract situations from specific cases, as in this paper, in order to develop theory to encompass as many of these specific cases as possible. The most difficult aspect of doing this, as I found, is trying to find the common threads which tie these cases together in order that a generalization can be made.

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