THE HYPOLATTICE OF JOIN ENDOMORPHISMS AND THE HYPOLATTICE FORMED BY THE APPLICATION OF THE COMPLEMENTARY ORDER

By

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THE HYPOLATTICE OF JOIN ENDOMORPHISMS AND THE HYPOLATTICE FORMED BY THE APPLICATION OF THE COMPLEMENTARY ORDER

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PREFACE

This study of the structure of hypolattices, join endomorphisms and the complementary order which may be applied to a hypolattice was a vehicle used to extend my own knowledge of a specific area of mathematics. However, it was also a means of gaining some knowledge of some of the fundamental approaches to research in mathematics. Therefore, the knowledge gained was twofold, one part being concerned with the actual mathematics and the other my growth as a student.

I would like to extend my sincere thanks to my major adviser, Dr. Wayne Powell, for his guidance throughout this project.

I would also like to thank the Math Department at Oklahoma State University for the research assistantship which made it possible for me to have the time to complete the research for this project.

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NONMENCLATURE

$\vee$	supremum of two elements, or the join
$\wedge$	infimum of two elements, or the meet
$\bigvee$	supremum of a set of elements, or the join
$\bigwedge$	infimum of a set of elements, or the meet
$L_{\vee}$	the set of join endomorphisms for a hypolattice, L
L'	the hypolattice L considered under the complementary order
$\vee'$	the join of two elements under the complementary order
$\wedge'$	the meet of two elements under the complementary order

CHAPTER I

INTRODUCTION

A hypolattice, while like a lattice many ways, is a much more general structure. Therefore it is possible to obtain information about this more general structure through the extension of material concerning lattices. Facts obtained in this manner may then be combined with present information in order to gain more knowledge of the structure. This paper deals with the definition of and general facts concerning hypolattices and explores material presented by Grätzer and Schmidt [2] on lattices and join endormorphisms for the more general structure of the hypolattice. Finally, this paper considers results obtained by combining the material derived from Grätzer and Schmidt [2] with a suborder presented by Powell [4] which is defined and explained as applicable.

In order to form a basis for discussion some concepts concerning hypolattices are defined and clarified in this chapter. These include a formal definition of a hypolattice, several classifications of hypolattices based on certain properties they possess, and definitions of meet and join endormorphisms in terms of their action on a hypolattice.

The starting point is the definition of a hypolattice. A <u>hypolattice</u> L is a poset such that every closed interval $[k, \ell] \subseteq L$ is a lattice where if $x, y \in [k_1, \ell_1]$ and $x, y \in [k_2, \ell_2]$ then $x \vee y$ exists and is the same in each interval and dually. This last condition will periodically be referred to as the property of intervals. Figure 1(a) is an example of a poset which is a hypolattice while Figure 1(b) is a poset which is not since in this case x, y $\in$ [0, a] and x, y $\in$ [0, b] but $x \vee y = a$ in [0, a] and $x \vee y = b$ in [0, b].

Figure 1. Posets

A structure like this in which each closed interval is a lattice is said to be a <u>weak hypolattice</u>. One should note that a poset which satisfies the definition of a lattice will automatically satisfy the definition of a hypolattice while a hypolattice will satisfy the definition of a weak hypolattice. The definition of a hypolattice indicates that a distinction exists between the types of meets and joins which can occur. Specifically, there are some meets and joins which must exist because they are contained in a closed interval which is by definition a lattice. These types of meets and joins are said to be <u>de jure</u>. However, it is also possible for a meet or join to exist when x and y are not contained in a common closed interval; i.e., this meet or join does not exist as a result of the definition of the hypolattice. Meets and joins which occur in this manner are said to be <u>de facto</u>. The implications of these meets and joins will be discussed more fully later, but the distinction allows for the definition of a subhypolattice. A hypolattice L is said to be a <u>subhypolattice</u> of another hypolattice K provided any meet or join which exists de jure in L exists in K and is the same in L as it is in K. So, for example Figure 2(a) is a subhypolattice of Figure 2(b), but not a subhypolattice of Figure 2(c) since $a \vee b = 1$ in Figure 2(a) and $a \vee b = c$ in Figure 2(c).

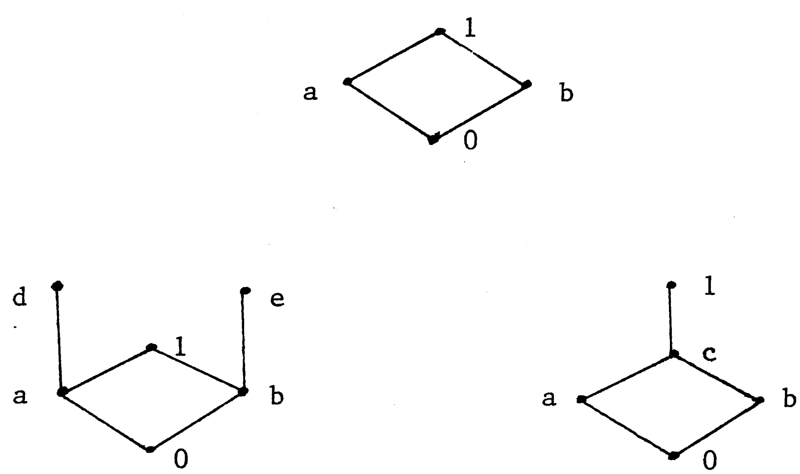


Figure 2. Hypolattices and Subhypolattices

As with many structures, it is possible to classify a hypolattice based on the properties it displays. One such classification is that of a star. A <u>star</u> is a hypolattice L with the property that for every x and y in L such that $x, y > 0$, $x \nleq y$. Another classification is based on the concept of completeness of a lattice from which it is possible to define the concept of relative completeness in a hypolattice. A hypolattice is said to be <u>relatively</u>

complete provided every closed interval is a complete lattice. Similarly, a hypolattice is said to be relatively complemented or relatively distributive provided that every closed interval is a complemented or distributive lattice, respectively. From this a Boolean hypolattice is one which is relatively distributive and relatively complemented.

In the discussion of hypolattices, the definitions of join and meet endomorphisms and join and meet irreducible elements differ only slightly from those for lattices. In each case the difference arises because of the distinction between de jure and de facto joins and meets. A join (meet) endomorphism is defined exactly as for lattices with the exception that it need only preserve de jure joins (meets); i.e., it need not preserve de facto joins (meets). This will be discussed in Chapter II. Similarly, an element is join (meet) irreducible provided it is so in the usual sense with respect to de jure joins (meets). For example in Figure 1(a), the element z is meet irreducible even though $x \wedge y = z$ since $x \wedge y$ exists de facto.

CHAPTER II

GENERAL PROPERTIES OF HYPOLATTICES AND JOIN ENDOMORPHISMS

Later discussions in this paper are dependent upon fundamental ideas concerning the existence of meets and joins in a hypolattice and the action of join endomorphisms on these operations. These ideas are presented in this chapter in order to form a basis for those discussions. Facts concerning the structure of a hypolattice are presented in the format of statement followed by justification or in some cases examples are used to illustrate a particular property.

As a starting point, one should note that if two elements in a hypolattice have a common upper bound and a common lower bound then their meet and join must exist because the elements are contained in the closed interval from that common lower bound to common upper bound. If two elements have a common lower bound and the hypolattice is finite then this property is a little stronger.

PROPOSITION 2.1. If L is a finite hypolattice and x and y have common lower bound then $x \wedge y$ will exist.

Proof. The justification of this is a little more involved although the concept is relatively simple. Let x, $y \in L$ and suppose $a \leq x, y$. If one considers the closed intervals [a, x] and [a, y] then $K = [a, x] \cap [a, y]$ contains all lower bounds for x and y which are greater than or equal to a. Now K is nonempty so $K = \{x_1, x_2, \ldots x_n\}$ for some $x_i \in L$, $1 \leq i \leq n$. Then $x_1 \vee x_2 \vee \ldots \vee x_n \in K$ since $x_i \leq x, y$ for every i based on earlier comments. Also $x_i \leq x_1 \vee x_2 \vee \dots \vee x_n$ for every i and therefore $x \wedge y$ exists and is in fact $x_1 \vee x_2 \vee \dots \vee x_n$. Note that this is dually true for $x \vee y$.

Based on these facts it is possible to construct examples of the four ways in which de jure and de facto meets and joins can occur in a hypolattice.

(i) If both $x \wedge y$ and $x \vee y$ exist then they are de jure by definition. For example

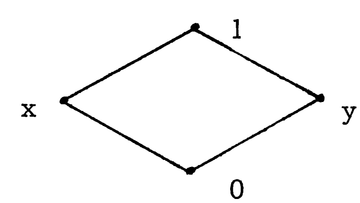


Figure 3. De Jure Meets and Joins

(ii) If $x \wedge y$ exists but x and y have no common upper bound then $x \wedge y$ is de facto. For example

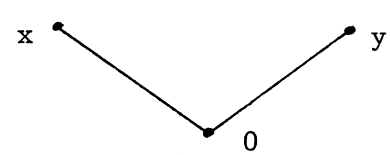


Figure 4. De Facto Meet

(iii) If $x \vee y$ exists but x and y have no common upper bound then $x \vee y$ is de facto. For example

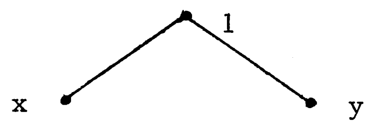


Figure 5. De Facto Join

(iv) If $x \wedge y$ and $x \vee y$ do not exist then x and y are not contained in a common interval. For example

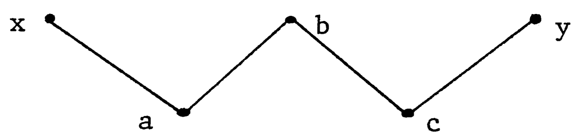


Figure 6. The Meet and Join of x and y Do Not Exist

If L is an infinite hypolattice, it is possible for x and y to have either a common upper or lower bound, but not both, and for $x \wedge y$ and $x \vee y$ not to exist. For example let $L = \{a \in \mathbb{R} \mid 0 \leq a < 1\} \cup \{x, y\}$ where $a \leq x, y$ for every $a \in [0, 1)$. This situation is depicted in Figure 7. Note that 0 is a common lower bound but that $x \wedge y$ does not exist. However, if L is finite this cannot occur based on the results above.

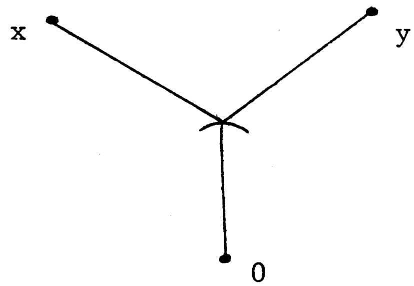


Figure 7. The Meet of x and y Does Not Exist but They Have a Common Lower Bound

Another interesting result concerns the application of join endomorphisms to hypolattices. The definition of a join endomorphism requires only that de jure joins be preserved, therefore the question arises as to whether or not de facto joins will also be preserved. This is not the case in general as can be seen in the following example. Consider the hypolattice

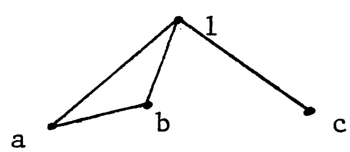


Figure 8. A Join Endomorphism and De Facto Join

with the join endomorphism $\Phi(x) = \begin{cases} x & x \neq c \\ b & x = c \end{cases}$. Now $a \vee c = 1$ is a de facto join but $\Phi(a \vee c) = 1 \neq b = \Phi(a) \vee \Phi(c)$. So as stated a join endomorphism need not preserve de facto joins.

CHAPTER III

THE HYPOLATTICE OF JOIN ENDOMORPHISMS

Through study of material presented by Grätzer and Schmidt [2] concerning the lattice formed by the join endomorphisms of a lattice one finds that similar results hold for hypolattices under like conditions. Essentially, Grätzer and Schmidt [2] showed that for lattices there are specific conditions under which the join endomorphisms will form a lattice and, based on this, that a lattice is distributive if and only if the lattice of join endomorphisms is distributive. This chapter will show that these results do indeed hold for hypolattices under similar conditions.

First, in order to discuss the hypolattice of join endomorphisms, the ordering of the join endomorphisms must be clearly understood and notation agreed upon for the purpose of simplification. The order, $\leq$, will be defined for join endormorphisms Φ and θ as follows, $\Phi \leq \theta$ provided for every $x \in L$, a hypolattice, $\Phi(x) \leq \theta(x)$ in the order on L. For notational purposes L will denote a hypolattice and $L_{\vee}$ will denote the set of join endomorphisms of L.

According to Grätzer and Schmidt [2], the join endomorphisms of an arbitrary lattice do not form a lattice, and this is also true for hypolattices. The counter example given by Grätzer and Schmidt [2] is found by considering the lattice $[0,1] \times [0,1]$ with the exception of the point (0,1). Let this lattice be L and note that L is a sublattice of the lattice $[0, 1]\times[0, 1]$. Now consider the following:

$$\theta(x, y) = \begin{cases}(1, 1) & (x, y) = (1, y);\ y \neq 0 \\ (0, 0) & x \neq 1\end{cases}, \text{ and}$$

$$\Phi(x, y) = \begin{cases}(0, 0) & (x, y) = (x, 0);\ x \neq 1 \\ (1, y) & y \neq 0\end{cases}$$

which are both join endomorphisms. If $\theta \wedge \Phi$ does not exist then $L_\vee$ is not a lattice while if θ and Φ are contained in some closed interval then $L_\vee$ will not be a hypolattice either. If one considers every $\Psi \in L_\vee$ such that $\Psi \leq \theta, \Phi$ then each Ψ will be of the form

$$\Psi(x, y) = \begin{cases}(0, 0) & x \neq 1 \\ (a, 0) & y \neq 0\end{cases}$$

Note that each Ψ is a join endomorphism and also that of these there is no greatest one because $a \neq 1$ as the point $(0, 1) \notin L$. Therefore, $\theta \wedge \Phi$ does not exist and $L_\vee$ is not a lattice. If 0 and I are defined by $0(x, y) = (0, 0)$ and $I(x, y) = (1, 1)$ then $0 \leq \theta, \Phi \leq I$; i.e., $\theta, \Phi \in [0, I]$. But from above $\theta \wedge \Phi$ does not exist. Therefore, this closed interval is not a lattice which implies $L_\vee$ is not a hypolattice either. Recalling that any lattice is a hypolattice, this shows that for an arbitrary hypolattice L, $L_\vee$ need not be a hypolattice.

According to Grätzer and Schmidt [2] a sufficient condition for $L_\vee$ to be a lattice is that L be a complete lattice in which case $L_\vee$ is also complete. In reality this condition is both necessary and sufficient the proof of which is similar to the following generalization to relatively complete hypolattices.

THEOREM 3.1 The join endomorphisms of a hypolattice L form a relatively complete hypolattice if and only if L is relatively complete.

<u>Proof</u>. Let L be a relatively complete hypolattice. Further, let $L_\vee$ be the set of join endomorphisms of L. Now $L_\vee$ will be a relatively

complete hypolattice if the interval $[\Phi, \theta]$ is a complete lattice and the property of intervals holds for all $\Phi, \theta \in L_V$ such that $\Phi \leq \theta$. To show this, suppose $\Psi_1, \Psi_2 \in [\Phi \ \ \theta]$ which implies $\Phi(x) \leq \Psi_1'(x)$, $\Psi_2(x) \leq \theta(x)$ for every $x \in L$. For any $x \in L$ it is clear that $\Psi_1(x) \vee \Psi_2(x)$ and $\Psi_1(x) \wedge \Psi_2(x)$ exist in $[\Phi(x), \ \theta(x)]$ since $\Psi_1, \Psi_2 \in L_V$ and L is a relatively complete hypolattice. So let $\Psi(x) = \Psi_1(x) \vee \Psi_2(x)$ and $\Gamma(x) = \Psi_1(x) \wedge \Psi_2(x)$ for each $x \in L$. A primary concern at this point is that Ψ and Γ be join endomorphisms. In the case of Ψ this is relatively clear since

$$
\begin{aligned}
\Psi(x \vee y) &= \Psi_1(x \vee y) \vee \Psi_2(x \vee y) \\
&= \Psi_1(x) \vee \Psi_1(y) \vee \Psi_2(x) \vee \Psi_2(y) \\
&= \Psi_1(x) \vee \Psi_1(x) \vee \Psi_2(y) \vee \Psi_2(y) \\
&= \Psi(x) \vee \Psi(y)
\end{aligned}
$$

For Γ, however, the process is a little more complicated. Begin by considering $H = \{\Gamma' \mid \Phi \leq \Gamma' \leq \Psi_1, \Psi_2\}$. Then by the relative completeness of L the supremum of H must exist and is in fact Γ by definition; i.e., $\Gamma = \bigvee_{\Gamma' \in H} \Gamma'$. Based on this relationship consider

$$
\begin{aligned}
\Gamma(x \wedge y) &= \bigvee_{\Gamma' \in H} \Gamma'(x \wedge y) \\
&= \bigvee_{\Gamma' \in H} (\Gamma'(x) \wedge \Gamma'(y)) \\
&= \bigvee_{\Gamma' \in H} \Gamma'(x) \wedge \bigvee_{\Gamma' \in H} \Gamma'(y) \\
&= \Gamma(x) \wedge \Gamma(y)
\end{aligned}
$$

Therefore, Γ is indeed an element of L_V which shows that $[\Phi, \ \theta]$ is a lattice. To show that $[\Phi, \theta]$ is complete one must consider an arbitrary set $H \subseteq [\Phi, \theta]$. Now the supremum of H exists and is a join endomorphism in the same manner as above. So the interval $[\Phi, \ \theta]$ is complete. Finally, if the property of intervals holds then L_V will indeed be a relatively complete hypolattice as desired. To demonstrate this, let $\Psi_1, \Psi_2 \in [\Phi_1, \theta_1] \cap [\Phi_2, \theta_2]$. Now, for any $x \in L$ $\Psi_1(x), \Psi_2(x) \in [\Phi_1(x), \theta_2(x)] \cap [\Phi_1(x), \theta_2(x)]$ and $\Psi_1(x) \vee \Psi_2(x)$ and $\Psi_1(x) \wedge \Psi_2(x)$ exist and are the same in each interval by the fact that L is a hypolattice and satisfies the property of intervals. If $\Psi(x) = \Psi_1(x) \vee \Psi_2(x)$ and $\Gamma(x) = \Psi_1(x) \wedge \Psi_2(x)$ for all $x \in L$, then $\Psi, \Gamma \in L_V$ from above and satisfy the property of intervals by definition. So L_V satisfies the property of intervals and as desired is a relatively complete hypolattice.

To prove the converse suppose that L is not a relatively complete hypolattice and note that this can occur in two ways. In one case L is not a hypolattice while in the other L is a hypolattice but is not relatively complete. Suppose first that L is not a hypolattice. Then some interval $[k, \ell] \subseteq L$ is not a lattice or the property of intervals would not hold. Assume that $[k, \ell]$ is not a lattice. This implies that there exist $a, b \in [k, \ell]$ such that $a \vee b$ or $a \wedge b$ do not exist. If $a \vee b$ does not exist consider the join endomorphisms $\Phi(x) = k$, $\theta(x) = \ell$, $\Psi(x) = a$ and $\Gamma(x) = b$ for all $x \in L$ and note that $\Psi, \Gamma \in [\Phi, \theta]$ but $\Psi \vee \Gamma$ does not exist which implies that L_V is not a hypolattice. A similar argument holds if $a \wedge b$ does not exist. If the property of intervals does not hold in L then for some $a, b \in [k_1, \ell_1]$ and $[k_2, \ell_2]$ $a \vee b$ or $a \wedge b$ are not the same in $[k_1, \ell_1]$ and $[k_2, \ell_2]$. Once again consider the join endomorphisms $\Phi_1(x) = k_1$, $\Phi_2(x) = k_2$, $\theta_1(x) = \ell_1$, $\theta_2(x) = \ell_2$, $\Psi(x) = a$ and $\Gamma(x) = b$ for all $x \in L$. Then corresponding to the values of $a \vee b$ and $a \wedge b$ in the respective intervals $\Psi \vee \Gamma$ or $\Psi \wedge \Gamma$ are not the same in $[\Phi_1, \theta_1]$ and $[\Phi_2, \theta_2]$. In each case L_V is not a hypolattice. In the second case suppose that L is a hypolattice which is not relatively complete. This implies that there is some interval $[k, \ell] \subset L$ which is not a complete lattice; i.e., there is a subset H of $[k, \ell]$ such that $\bigvee H$ or $\bigwedge H$ does not exist. Suppose that $\bigvee H$ does not exist and define Φ and θ as in the first case. Further let $\Gamma_\alpha(x) = h_\alpha$ where $h_\alpha \in H$ and $\alpha \in A$ and note that $\Gamma_\alpha \in [\Phi, \theta]$ for all $\alpha \in A$ but that $\bigvee_{\alpha \in A} \Gamma_\alpha$ does not exist. This indicates that $L_\vee$ is not relatively complete either. Therefore, if L is a relatively complete hypolattice then $L_\vee$ is a relatively complete hypolattice as desired.

According to Grätzer and Schmidt [2] the lattice of join endomorphisms will necessarily have a smallest element 0. However, this is not true for a hypolattice simply because a relatively complete hypolattice need not have a least element (consider Figure 6, page 7).

As a corollary to this, if L is a finite hypolattice $L_\vee$ will also be a hypolattice since a finite hypolattice is relatively complete. Because relatively complete hypolattices are the only hypolattices for which $L_\vee$ is also a hypolattice discussion will be limited to relatively complete hypolattices for the rest of this chapter.

One goal of this chapter, as in Grätzer and Schmidt [2], is to describe when $L_\vee$ will be a relatively distributive hypolattice. This will be accomplished by a series of lemmas establishing criteria for embedding $L_\vee$ in L based on the number of join irreducible elements contained in L. This leads to the statement that $L_\vee$ will be a relatively distributive hypolattice if and only if L is a relatively distributive hypolattice.

Recall that in a hypolattice an element is considered to be join irreducible if it is join irreducible with respect to de jure joins in the usual sense. Also, note that for a finite hypolattice a join endomorphism is determined by where it takes the join irreducible elements since every element is the join of join irreducible elements. Based on this Lemma 3.2 follows.

LEMMA 3.2. If $a_1, a_2, \ldots, a_k$ are the join irreducible elements in a finite, relatively distributive hypolattice L with smallest element 0 and $b_1, b_2, \ldots, b_k$ are arbitrary in L, then a necessary and sufficient condition for the existence of a join endomorphism θ with $\theta(a_i) = b_i$ is that $a_i \le a_j$ implies that $b_i \le b_j$.

<u>Proof</u>. Let L, $a_1, a_2, \ldots, a_k$ and $b_1, b_2, \ldots, b_k$ be as described. Further let r(a) be the set of all join irreducible elements less than or equal to a, that is, $r(a) = \{a_i \mid a_i \le a\}$. This implies that $a = \bigvee_{a_i \in r(a)} a_i$. Suppose that $\theta(a_i) = b_i$ as specified and based on this define $\theta(a) = \bigvee_{a_i \in r(a)} \theta(a_i)$. It is necessary to show that θ is a join endomorphism; i.e., that $\theta(a \vee b) = \theta(a) \vee \theta(b)$ for $a, b \in L$ where $a \vee b \in L$. This holds if $r(a) \cup r(b) = r(a \vee b)$. Clearly $r(a) \cup r(b) \subseteq r(a \vee b)$ therefore all that remains is to show that $r(a \vee b) \subseteq r(a) \cup r(b)$. Suppose $x \in r(a \vee b)$ but $x \notin r(a) \cup r(b)$. Note that $x \wedge a < x$ and $x \wedge b < x$ since $x \not\le a, b$ by definition. Note $x \wedge a$ and $x \wedge b$ exist because $x, a, b \in [0, a \vee b]$. Further, by the distributive law $x = x \wedge (a \vee b) = (x \wedge a) \vee (x \wedge b)$ but this is not possible since x is join irreducible by definition. Therefore, $x \in r(a) \cup r(b)$. This implies $r(a \vee b) = r(a) \cup r(b)$. Therefore θ is a join endomorphism.

Using this result it is possible to show that $L_\vee$ may be embedded in the direct product of copies of L based on the number of join irreducible elements in L.

LEMMA 3.3. Let k be the number of join irreducible elements in L, a finite, relatively distributive hypolattice with smallest element 0. Then the hypolattice L_V can be embedded in the direct product of k copies of L.

<u>Proof</u>. Let $f: L_V \to \prod_k L$ be defined by $f(\theta) = (b_1, b_2, \ldots b_k)$ where $\theta(a_1, a_2, \ldots, a_k) = (b_1, b_2, \ldots, b_k)$ and $a_1, a_2, \ldots, a_k$ are the join irreducible elements of L. If $\theta, \Phi \in L_V$ such that

$$\theta(a_1, a_2, \ldots, a_k) = (b_1, b_2, \ldots, b_k), \text{ and}$$

$$\Phi(a_1, a_2, \ldots, a_k) = (c_1, c_2, \ldots, c_k), \text{ then}$$

$$\theta \vee \Phi = (b_1 \vee c_1, b_1 \vee c_2, \ldots, b_k \vee c_k), \text{ and}$$

$$\theta \wedge \Phi = (b_1 \wedge c_1, b_1 \wedge c_2, \ldots, b_k \wedge c_k)$$

whenever these meets and joins exist. That is, b and c are contained in some common interval. Assuming this and that $\theta \wedge \Phi$ exists clearly implies that f is a monomorphism, and therefore f embeds L_V in k copies of L as desired.

Now consider a relatively distributive hypolattice L. The direct product of L k times is also relatively distributive and by Lemma 3.2 L_V is a subhypolattice of this product. Therefore the fact that L is relatively distributive implies the same for L_V. Conversely, if L_V is relatively distributive then L must be also since L can be embedded in L_V. This is done by defining $f: L \to L_V$ by $f(a) = \theta$ where $\theta \in L_V$ is such that $\theta(x) = a$ for all $x \in L$. This proves the following result which is the focus of this chapter.

THEOREM 3.4. The join endomorphisms of a finite hypolattice L form a relatively distributive hypolattice if and only if L is relatively distributive.

CHAPTER IV

THE COMPLEMENTARY ORDER

The partial order which defines a hypolattice describes a relationship between the elements of a set. If there are two partial orders $\leq_1$ and $\leq_2$ on the same set, then $\leq_1$ will be called a <u>suborder</u> of $\leq_2$ if $a \leq_1 b$ implies that $a \leq_2 b$ for any a and b in the set. A suborder can be defined in any manner as long as this condition is satisfied. This section deals with a particular suborder, defined by Powell [4], which is determined by the original order on a relatively distributive hypolattice $(L, \leq)$ with smallest element 0. Under the order $\leq'$, $a \leq' b$ is equivalent to the existence of some $c \in L$ such that $a \vee c = b$ and $a \wedge c = 0$. This is called the <u>complementary order</u>. For notational purposes $(L, \leq)$ will be denoted by L and $(L, \leq')$ will be denoted by L' The supremum and infimum of a, $b \in L$ will be denoted in the usual manner, $a \vee b$ and $a \wedge b$, if they exist while the supremum and infimum of a, $b \in L'$ will be denoted $a \vee' b$ and $a \wedge' b$ respectively if they exist. As noted by Powell [4], L' is a partially ordered set and $\leq'$ is a suborder of $\leq$. A significant result upon which this section is dependent is Theorem 1 [4].

> Let $(L, \leq)$ be a relatively distributive hypolattice with 0. Then L with the complementary order $\leq'$ is a Boolean hypolattice. If a, $b \in L$ then $a \wedge' b$ exists and whenever $a \vee' b$ exists $a \vee' b = a \vee b$.

An example of the complementary order is given below.

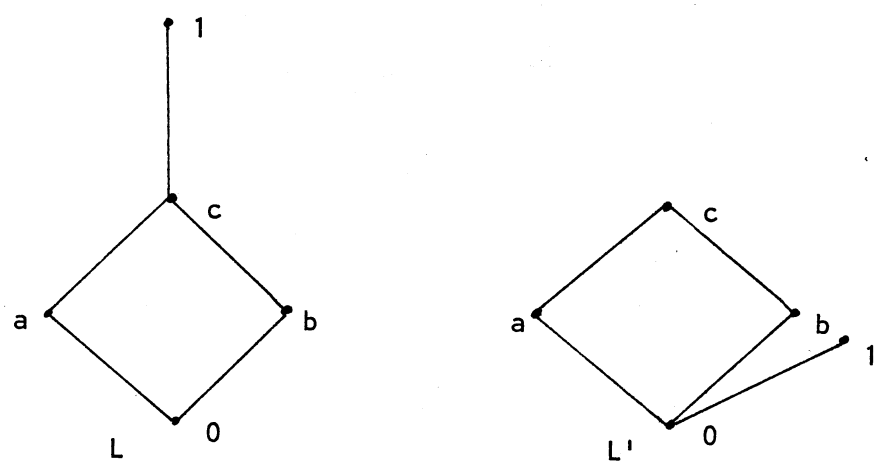


Figure 9. A Relatively Distributive Hypolattice and the Complementary Order

Further, although the Theorem is not applicable to hypolattices which are not relatively distributive it is worth noting that for finite hypolattices it is possible to construct the hypolattice which results from the application of the complementary order simply by considering the relationships of the elements. For example,

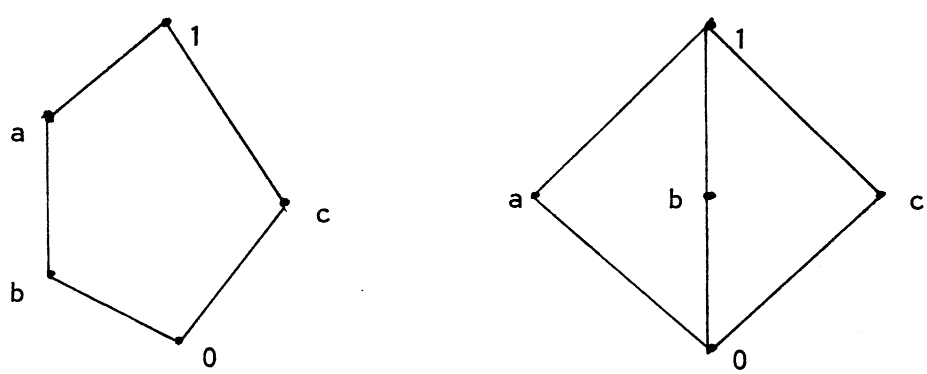


Figure 10. A Relatively Non-distributive Hypolattice and the Complementary Order

While it is possible to apply the complementary order to any finite hypolattice from this point on only relatively distributive hypolattices will be considered.

The desired result of this chapter is the classification of the hypolattices which arise from the application of the complementary order based on properties of the original hypolattice. There are three basic types of configurations which can occur when the hypolattice is considered under the complementary order. The complementary order may form a star, preserve the original hypolattice, or form a configuration which is a combination of these two.

In the next theorem recall that a star is a hypolattice in which for every $x, y \in L$ such that $x, y > 0$, $x \not\leq y$.

THEOREM 4.1. For L, a relatively distributive hypolattice with smallest element 0 the following are equivalent:

(i) L' is a star.

(ii) 0 is meet irreducible in L'.

(iii) 0 is meet irreducible in L.

Proof. The form of this proof is to show that (i) implies (ii) which implies (iii) which in turn implies (i).

In the first case assume that L' is a star and suppose $x \wedge' y = 0$ and that $x \vee' y$ exists. Note $x \wedge' y$ is de jure. Now $x, y \geq x \wedge' y$ by definition, but since L' is a star and $x \neq 0$, $x = x \vee' y$ which implies that $x \wedge' y = y$ or $y = 0$. Therefore 0, is meet irreducible in L'.

In the second case assume 0 is meet irreducible in L' and consider $x \wedge y = 0$ where $x \vee y = z$ for $x, y, z \in L$. This implies that $x, y \leq' z$ and that $x \vee' y = z$ and $x \wedge' y = 0$ by Theorem 1 from Powell [4].

However, since 0 is meet irreducible in L', x or y must be 0, and therefore 0 is meet irreducible in L.

Finally, assume 0 is meet irreducible in L and suppose $x \leq' z$ for $x \neq 0$ and $z \in L$. Then there exists some $y \in L$ such that $x \wedge y = 0$ and also $x \vee y = z$. But 0 is meet irreducible in L and $x \neq 0$ which implies that $y = 0$ so $x = z$. Therefore, by definition L' is a star.

This theorem indicates the importance of the positioning of the zero element in relation to the other elements in the original order. Essentially, any time 0 is meet irreducible in the original hypolattice the complementary order reduces L to a star.

There are also hypolattices which are completely preserved by the application of the complementary order.

LEMMA 4.2. Let L be a relatively distributive hypolattice with smallest element 0. Then every interval $[0, y] \subseteq L$ is complemented if and only if $L = L'$.

<u>Proof</u>. Assume that L is a relatively distributive hypolattice with smallest element 0 such that every interval $[0, y] \subseteq L$ is complemented. Suppose $x \leq y$; i.e., $x \in [0, y]$. Then because the interval is complemented there exists $z \in [0, y]$ such that $x \vee z = y$ and $x \wedge z = 0$. This implies that $x \leq' y$ and so L is the same as L'.

Suppose that L and L' are the same. Then if $x \in [0, y] \subset L$, $x \leq y$ which implies that $x \leq' y$. Therefore, there exists $z \in L$ with $x \vee z = y$ and $x \wedge z = 0$, and the interval is complemented by definition.

While these are interesting results they are very limited in the types of hypolattices to which they apply, as there are many hypolattices which are actually a combination of subhypolattices which are of these types. Therefore, before discussing the manner in which the complementary order effects these hypolattices one must consider the role that subhypolattices play in the formation of a hypolattice under the complementary order. First, for any hypolattice L which is relatively distributive with smallest element 0, L' will be a subhypolattice of L simply by definition. While this is helpful in the understanding of subhypolattices and the complementary order the following is more applicable.

LEMMA 4.3. If L is a subhypolattice of K, a relatively distributive hypolattice with smallest element 0, such that $0 \in L$, then L' is a subhypolattice of K'.

<u>Proof</u>. Let L and K be as described and let a, $b \in L$ such that $a \vee' b$ and $a \wedge' b$ exist. By Theorem 1 from Powell [4] $a \vee' b = a \vee b$ and $a \wedge' b = a \wedge b$, but these are de jure in L and are therefore the same in K. Clearly $a \vee' b$ and $a \wedge' b$ must exist in K' since they exist in L' and L is contained in K. Therefore, the values are equal and L' is a subhypolattice of K'.

Based on this information about subhypolattices it is possible to describe the hypolattice which results from the application of the complementary order to an arbitrary relatively distributive hypolattice with smallest element 0.

THEOREM 4.4. If L is a relatively distributive hypolattice with smallest element 0, then L' will be the union of a star and those intervals [0, y] which are complemented and therefore preserved under the application of the complementary order.

<u>Proof</u>. Let L be as specified and note that any complemented interval [0, y] is a subhypolattice of which L will be preserved under the complentary order. This follows from Lemma 4.2.

What remains to be shown is that $K = \{ x \in L \mid x \notin [0, y)$ for any y such that [0, y] is complemented$\} \cup \{0\}$ is a subhypolattice of L and that 0 is meet irreducible in K. To proceed let a, $b \in K$ such that a, $b \in [c, d] \subseteq K$. Now $a \vee b \in [c, d] \subseteq L$ because L is a hypolattice. Suppose $a \vee b \notin K$. This would imply that $a \vee b \in [0, y]$, a complemented interval for some $y \in L$. But then a, $b \in [a \wedge b, a \vee b] \subset [0, y]$ which is a contradiction of the choice of a and b. Therefore $a \vee b \in K$. In the case of $a \wedge b$, if $a \wedge b \notin K$, then $a \wedge b$ is in a complemented interval [0, y] for some $y \in L$. Hence, $c \leq a \wedge b \in [0, y]$ or $c \in [0, y]$ in which case $c = a \wedge b = y$. Therefore, $a \wedge b \in K$ and K is a subhypolattice of L. To show that 0 is meet irreducible in K let a, b, $c \in K$ with $a \vee b = c$, $a \wedge b = 0$ and a, $b < c$. Since a, b, $c \in K$ the interval [0, c] is not complemented so there exists some $d \in [0, c]$ such that there does not exist $e \in [0, c]$ with $d \vee e = c$ and $d \wedge e = 0$. Now, either d is not related to either a or b, d is related to exactly one of a and b, or d is related to both a and b. In the first and second case the interval [0, c] is not distributive which contradicts the hypothesis. In the third case if $a \leq d \leq b$ then [0, c] is once again not distributive and if $d \leq a$, b or a, $b \leq d$ then $a \wedge b \leq d$ or $d \leq a \wedge b$ which is also a contradiction. Therefore, c = a or c = b. If c = a, then 0 = b which implies that 0 is meet irreducible in K.

By Lemma 4.1 K' will be a star and by Lemma 4.3 both the complemented intervals and K' are subhypolattices of L'. In fact L' is their union.

A particular result of the preceding theorem is that L' is relatively distributive, and also that if L" is found by applying the complementary order to L', then L" must always be ismomorphic to L'.

CHAPTER V

JOIN ENDOMORPHISMS AND THE COMPLEMENTARY ORDER

Combining the information about join endomorphisms and the structure of the hypolattice L' leads to several questions concerning how these topics effect one another. These are basically extensions of ideas which were presented earlier but can now be explored in terms of the other information discussed. This section will deal with the possibility of embedding the hypolattice L'_V in the hypolattice L, the correspondence between the join endomorphisms of L and those of L', and finally the implication of $L_V \cong K_V$ for hypolattices L and K.

Lemma 3.3 in Chapter III states that the hypolattice of join endomorphisms can be embedded in k copies of the original hypolattice where k is the number of join irreducible elements in a finite hypolattice. This leads to the question of whether or not the hypolatice resulting from the application of the complementary order or its hypolattice of join endomorphisms may also be embedded in the original hypolattice. One should note that since the hypolattice obtained by application of the complementary order will retain the properties of the original hypolattice it is possible to embed L'_V in L'. This is the key for embedding L'_V in L.

THEOREM 5.1. Let L be a finite, relatively distributive hypolattice with smallest element 0 then the hypolattice L'_V can be embedded in k copies of L where k is the number of join irreducible elements in L'.

Proof. This follows as a direct result of Lemma 3.3.

This theorem reveals that the hypolattice of join endomorphisms of the complementary order is a structure which is contained in the direct product of copies of the original hypolattice.

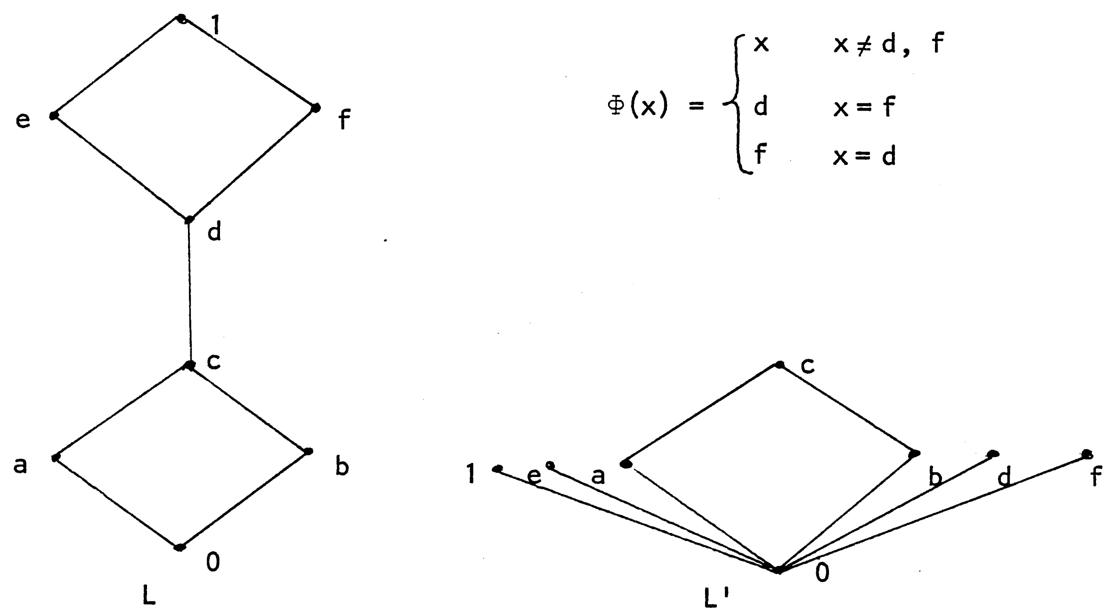
The second section of this chapter deals with the relationship between the join endomorphisms of the hypolattices L and L'. There is in fact some correspondence between the two sets, and the purpose of this section is to show that $L_{\vee} \subsetneq L'_{\vee}$ for hypolattices where $L \neq L'$. The fact that $L_{\vee} \subseteq L'_{\vee}$ will be established by formal proof and an example will be given where $L_{\vee} \subsetneq L'_{\vee}$.

The proof of the containment of $L_{\vee}$ in L' is actually fairly straightforward. The results of Theorem 4.4 establish that L' is the union of a star and the complemented intervals of the form [0, y] contained in L which are preserved under the complementary order. This in turn implies that L' is a subhypolattice of L, and this is the basis for proving that $L_{\vee} \subseteq L'_{\vee}$.

LEMMA 5.2. The join endomorphisms of a relatively distributive, finite hypolattice L with smallest element 0 are contained in the join endomorphisms of the corresponding hypolattice L'.

Proof. Let L be a relatively distributive hypolattice with smallest element 0 and let $\theta \in L_{\vee}$. By definition, in order for θ to be a join endomorphism of L' it need only preserve de jure joins in L'. Let a, b $\in$ L' such that $a \vee' b$ is de jure in L' and $\theta(a \vee' b)$ and $\theta(a) \vee' \theta(b)$ exist. Since $a \vee' b$ is de jure $a \wedge' b$ exists, and furthermore $a \vee' b = a \vee b$ and $a \wedge' b = a \wedge b$ which implies that $a \vee b$ is de jure in L also. Therefore, $a \vee b$ is preserved by θ ; that is, $\theta(a \vee' b) = \theta(a \vee b) = \theta(a) \vee \theta(b) = \theta(a) \vee' \theta(b)$ by [4]. So $\theta \in L'_{\vee}$ and as desired $L_{\vee} \subseteq L'_{\vee}$.

The next thing to show is that $L_\vee \subsetneq L'_\vee$ for a relatively distributive, finite hypolattice L where $L \not\cong L'$. This is done simply by considering an example. Let L be the hypolattice given and Φ the join endomorphism for the corresponding hypolattice L'.



$$\Phi(x) = \begin{cases} x & x \neq d,\ f \\ d & x = f \\ f & x = d \end{cases}$$

Figure 11. A Join Endomorphism of L' but not of L

Considering the action of Φ on the hypolattice L yields that $\Phi \notin L_\vee$ so $L_\vee \subsetneq L'_\vee$ since $\Phi(d \vee f) = (d) \neq (f) = \Phi(d) \vee \Phi(f)$.

This section demonstrates that with respect to join endomorphisms L' is in effect a simplification of L, that is if L and L' are not the same.

The last point to consider is the implication of $L_\vee \cong K_\vee$ for finite hypolattices L and K which are relatively distributive. From this L and K cannot be completely determined, however it is possible to determine some of the characteristics they have in common which are direct consequences of theorems in Chapter III. The characteristics are that L and K are relatively complete and relatively distributive by Theorem 3.1 and Theorem 3.4, respectively. Recall this arises from the fact that L and K can be embedded in copies of themselves.

CHAPTER VI

SUMMARY

The goal of this paper has been to extend one's knowledge of the structure of a hypolattice. To accomplish this goal early chapters dealt with basic facts concerning the properties of hypolattices and their structure. In Chapter III the emphasis shifted to extend these basic facts by patterning the discussion of hypolattices after that of lattices presented by Grätzer and Schmidt [2]. Chapter IV focused on describing the hypolattice which arises when the elements are considered under the complementary order defined by Powell [4]. Finally in Chapter V the ideas from Chapters III and IV were combined to consider questions that arose as a result of the study in the previous chapters.

Chapter III followed the pattern of Grätzer and Schmidt [2]. They had shown through a series of lemmas and examples that the set of join endomorphisms of a finite, distributive lattice forms a complete, distributive lattice. With this guide to follow, while some steps may have varied slightly, it was possible to show that these same results hold for the more general structure of the hypolattice under similar conditions. This generalization from the more specific to the more general is important in mathematics and in this case yielded interesting results.

The information obtained in Chapter IV focused on a basic problem in mathematics which is to classify structures in terms of the properties they possess or the manner in which they originate. In this case the goal was to describe the hypolattice L' which originates from the application of the complementary order described by Powell [4] to a relatively distributive hypolattice L with smallest element 0. The result from this indicates that L' is in effect a simplification of L in that the only intervals which remain intact in L' are those of the form [0, y] which are complemented while the rest of L becomes a star which is a very simple hypolattice.

The consideration in Chapter V of questions that arose as a result of the previous chapters showed that there are some implications which are direct results of the theorems proved throughout the paper. For example being able to embed L' into copies of the hypolattice L is based mainly on the fact that L' is a subhypolattice of L and that $L'_{\vee}$ can be embedded in copies of L' directly from Chapter III. The other question concerning the implications for hypolattices L and K when $L_{\vee} \cong K_{\vee}$ is basically a matter of considering the properties of L and K in light of Chapter III, in which most of the results were both necessary and sufficient. Therefore, L and K will have some of the same properties that L and K have which are both necessary and sufficient.

This effort to more fully understand hypolattices, the hypolattice of join endomorphisms and the action of the complementary order has yielded several interesting results. However, it has also been a means for introducing a few methods of approaching research in mathematics. The generalization of the material from Grätzer and Schmidt [2] to the more general structure of a hypolattice is one such method. And exposure to these methods leads to a better understanding of the processes involved in research in mathematics.

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