FREQUENCY RESPONSE OF HIGHLY NONLINEAR SYSTEMS

By

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1979

Submitted to the Faculty of the Graduate College
of the Oklahoma State University
in partial fulfillment of the requirements
for the Degree of
MASTER OF SCIENCE
May, 1981

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ACKNOWLEDGMENTS

During my graduate studies, I received a great deal of help and encouragement from a number of individuals. I am extremely grateful to all of them for having made my stay at Oklahoma State University a memorable one.

I wish to express my appreciation to my thesis adviser, Dr. J. H. Taylor, for his guidance and assistance throughout this study. I also express my appreciation to the other committee members, Dr. K. N. Reid and Dr. L. D. Zirkle, for their help, advice and criticism.

I thank TOP Services Unlimited for typing the final copy of my thesis.

My parents, my brothers and sisters deserve a special thanks for their advice and encouragement. I dedicate this thesis to my late father, Mr. Ramayya Sankaran, whose advice and encouragement will continue to inspire me throughout my life.

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CHAPTER I

INTRODUCTION

1.1 Purpose

Frequency response techniques are very useful in the analysis of linear systems. Block diagram and transfer function representations are easy to manipulate. Frequency response representations such as Bode plots, Nyquist plots and Nichol's charts are widely used to determine the degree of stability of linear systems. The versatility of the techniques is due to the applicability of the principle of superposition. In the case of nonlinear systems, the principle of superposition is not applicable. Hence, the usefulness of the frequency response techniques is diminished. Nevertheless, it is desirable to extend these techniques to the analysis of nonlinear systems.

The extension is simple in the case of block diagrams. The presence of the nonlinearities imposes some restrictions on the manipulations of the diagram [1]. The extension of the transfer function representation is more difficult. An equation for the nonlinearity in the frequency domain is required. The equation should be compatible with the transfer function of the linear components. This is done by approximating the effect of nonlinear components with a "describing function" (DF) or a quasi-linear gain. A DF is an amplitude dependent gain, so one important characteristic of a nonlinear system is retained in the model.

The frequency response data of a nonlinear system is useful in many ways. The frequency response of a nonlinear system cannot be directly used to obtain the time response to any given signal input, as is the case for a linear system. Yet, it can provide useful spectral information, such as system bandwidth, resonances, etc. Since the output of a nonlinear system is amplitude dependent, the frequency response is a family of curves rather than a single curve. By applying the Nyguist's criterion and Nichol's chart, the frequency response data could be used in the determination of the stability of a nonlinear system [2, 3]. The frequency response curves are of great interest for systems whose inputs are periodic functions of time. Such systems may exhibit "jump resonance." Thus, for several reasons, it is desirable to obtain the frequency response of a nonlinear system.

1.2 Background

The objective of the study is to develop a method for obtaining the frequency response of a nonlinear system when the system equations are of the form:

$$\dot{\underline{x}} = \underline{f}\,(\underline{x}, \underline{u})x \qquad (1)$$

where

$\underline{x}$ = state vector of dimension [n];

$\underline{u}$ = input vector of dimension [m];

$\underline{f}$ = function of $\underline{x}$ and $\underline{u}$ and is of dimension [n]; and

$\dot{\underline{x}}$ = derivative of the state vector.

The nonlinear elements are replaced by their describing functions and the frequency response is obtained. The theoretical basis of this approach is outlined below.

Krylov and Bogoliubov [4] derived the "equivalent linear parameters" method, which is analogous to the describing function methods, from consideration of a second-order differential equation. Johnson [5] developed a general definition for describing functions. Sridhar [6] has developed describing function equations for various forms of nonlinearities.

Johnson [5] showed that the frequency response of a nonlinear system could be used in the determination of limit cycles in systems containing one nonlinearity.

Gelb and Vander Welde [2] suggest an analytical method for finding the frequency response of a nonlinear system using sinusoidal input describing functions (SIDF). The method consists of defining a relationship between the input to the nonlinearity and the input to the system, expressing the gain of the linear and nonlinear elements in polar form and solving graphically for the input to the nonlinearity. It becomes cumbersome and tedious for systems containing more than one nonlinearity. Taylor [7] and Hannebrink et al. [8] extended the SIDF analysis to determine the limit cycle oscillations of systems containing many nonlinearities.

The method discussed in this study consists of expressing the system in state variable form and replacing the nonlinear elements with their SIDF gains. The basic concept was first proposed by Taylor [10] and developed in the course of this study. The algebric equations are expressed in matrix form, and then, a numerical method developed by Brown [9] is used to solve for the amplitudes of the state variables at individual frequencies. The computer program developed is capable of finding the frequency response of systems containing many nonlinearities.

CHAPTER II

DESCRIBING FUNCTION APPROACH

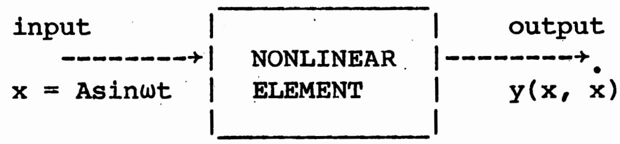
2.1 Introduction

In this chapter, a brief description of the describing function method is presented. The state variable formulation of nonlinear systems is discussed.

2.2 Describing Functions

One of the characteristics of a nonlinear system element is that its "gain ratio" is dependent on both the amplitude and the frequency of the input. If the input to the nonlinearity approximates a sinusoid, then the sinusoidal input describing function (SIDF) adequately describes the gain of the element as a function of the amplitude and the frequency. The SIDF simply replaces the nonlinearity by an approximate "equivalent linear gain," which is dependent on the amplitude as well as the frequency of the input to the nonlinearity. The SIDF technique is usually classified as a frequency domain technique.

Consider a pure sinusoidal signal of constant magnitude and constant frequency applied to a nonlinear element. The output of the nonlinearity would be a periodic nonsinusoidal signal which can be represented by a Fourier series. If the input to the nonlinear element (N) is $x = A \sin\omega t$ (assumed to be a scalar in this discussion, not $\underline{x}$ the state vector) as shown in Figure 1, the output $y(x, \dot{x})$ can be



$$y_1(A, \omega) = A_1 \sin(\omega t + \phi_2)$$

Figure 1. Definition of SIDF

represented by the Fourier series expansion:

$$y(A \sin\omega t, A\omega\cos \omega t) = \sum_{n=1}^{\infty} A_n(A, \omega) \sin [n\omega t + \phi_n (A, \omega)] \tag{2}$$

where:

A = the amplitude of the input;

ω = frequency of the input;

A_n = amplitude of the nth harmonic of the output; and

ϕ_n = phase angle of the nth harmonic of the output.

The sinusoidal input describing function (SIDF) is defined in terms of the fundamental or first harmonic as:

$$N(A, \omega) = \frac{\text{phasor representation of the fundamental component of the output}}{\text{phasor representation of the input}} = \frac{A_1(A, \omega)e^{j\phi_1(A, \omega)}}{A} \tag{3}$$

or

$$N(A, \omega) = j/\pi A \int_0^{2\pi} y(A \sin \Psi, A\, \omega \cos \Psi)\, e^{j\Psi} d\, \Psi \tag{4}$$

where:

$\Psi = \omega t$;

$j = \sqrt{-1}$; and

$e^{j\Psi} = \cos \Psi + j \sin \Psi$.

In cases where the SIDF depends on the input and its derivative, the SIDF is complex valued. There are several ways of writing such a DF:

1. Proportional plus derivative gain

$$N(A, \omega) = n_p(A, \omega) + \frac{n_q(A, \omega)}{\omega} s \tag{5}$$

where:

n_p = the in-phase component of the gain

$$= 1/\pi A \int_1^{2\pi} y(A \sin \Psi, A\omega \cos \Psi) \sin \Psi \, d \Psi; \tag{6}$$

n_q = the quadrature component of the gain

$$= 1/\pi A \int_0^{2\pi} y(A \sin \Psi, A\omega \cos \Psi) \cos \Psi \, d \Psi; \tag{7}$$

and s is the Laplace transform variable.

2. Complex gain:

$$N(A, \omega) = n_p(A, \omega) + j\, n_q(A, \omega)$$

or

$$N(A, \omega) = \rho_N(A, \omega)\, e^{j \theta_N(A, \omega)}$$

where:

$\rho_N = \sqrt{n_p{}^2 + n_q{}^2} = A_1/A$; and

$\theta_N = \tan^{-1}(n_q/n_p)$.

This latter formulation is especially convenient when using the phasor representation of sinusoidal variables, as is often done in frequency domain analysis.

The accuracy of the SIDF analysis is based on the following assumptions:

1. The system must have reached the steady state.
2. The input to the nonlinearity must be approximately sinusoidal.

This can be achieved only if the rest of the system acts as a low-pass filter or if the effect of the nonlinearity is mild.

The characteristics of the SIDF of a nonlinear device are that it is dependent on the amplitude of the input and possibly on the frequency of the input as well. If the nonlinearity is static and single-valued (that is, the output of the nonlinearity is not dependent on the derivative of the input and the nonlinearity is memoryless), the SIDF is real-valued and is independent of the frequency of the input. The derivation of the SIDF for many types of nonlinear devices are discussed by Gelb and Vander Welde [2]. The SIDF is generally a nonlinear algebraic function of amplitude A.

The SIDF allows the advantages of a linear approximation to be retained while continuing to reflect the amplitude-dependence of a nonlinear element acting on an input signal of finite size. The approximation of a nonlinear operation by a linear one is called quasilinearization. The SIDF as defined in Equation (4) provides the static quasi-linear gain which yields the minimum mean squared approximation error. The minimum mean squared error concept provides the general extension of the describing function to other problems such as random inputs and biases [2].

2.3 State Variable Formulation

The definition of the SIDF as a gain permits the output y in Figure 1 to be represented as $y = N(A, \omega)\, x$. The input to the nonlinearity in a control system could be a combination of the state variables $\underline{x}$ and the system inputs $\underline{u}$. If the input to the system is sinusoidal, the state variables $\underline{x}$ could be assumed to be nearly sinusoidal under the conditions stated previously. Then, the state vector and input vector may be represented as:

$\underline{x} = \mathrm{Im}[\underline{r}e^{j\omega t}]$

$\underline{u} = \mathrm{Im}[\underline{c}e^{j\omega t}]$.

Where:

$\underline{r}$ = complex amplitude vector (phasor) of the state variables;

$\underline{c}$ = complex amplitude vector (phasor) of the input vector; and

Im represents the imaginary part.

The complex vector $\underline{r}$ is represented as:

$$\underline{r} = \underline{p} + j\underline{q} \qquad (9)$$

where:

$\underline{p}$ = in-phase component of the amplitude vector $\underline{r}$; and

$\underline{q}$ = quadrature component of the amplitude vector $\underline{r}$

and the input vector $\underline{c}$ is represented as:

$$\underline{c} = \underline{a} + j\underline{b} \qquad (10)$$

where:

$\underline{a}$ = in-phase component of the input vector; and

$\underline{b}$ = quadrature component of the input vector.

Using Fourier transform methods, the operator d/dt can be replaced by $j\omega$. Thus, Equation (1) can be represented as:

$$j\omega I\underline{x} = F_{DF}(\underline{r}, \underline{c})\underline{x} + G_{DF}(\underline{r}, \underline{c})\underline{u} \tag{11}$$

where:

I = identity matrix of dimension (n x n);

$F_{DF} = F_{DFR} + jF_{DFI}$ = complex-valued quasi-linear state dynamics matrix of dimension (n x n); and

$G_{DF} = G_{DFR} + jG_{DFI}$ = complex-valued quasi-linear input allocation matrix of dimension (n x m).

Using Equations (9) and (10), Equation (11) can be written as:

$$j\omega I(\underline{p} + j\underline{q}) = F_{DF}(\underline{p} + j\underline{q}) + G_{DF}(\underline{a} + j\underline{b}) \tag{12}$$

The matrices F_{DF} and G_{DF} are obtained by substituting the SIDF's for the nonlinearities and they are dependent on the complex amplitude vector $\underline{r}$ and the input vector $\underline{c}$. Their imaginary parts exist only if nonlinearities with memory (e.g., hysteresis) are present in the system.

Separating the real and imaginary parts in Equation (12),

$$\begin{aligned} -\omega I\underline{q} - F_{DFR}\,\underline{p} + F_{DFI}\,\underline{q} - G_{DFR}\,\underline{a} + G_{DFI}\,\underline{b} &= 0 \\ \omega I\underline{p} - F_{DFR}\,\underline{q} - F_{DFI}\,\underline{p} - G_{DFR}\,\underline{b} - G_{DFI}\,\underline{a} &= 0 \end{aligned} \tag{13}$$

Equation (13) is a set of 2n real-valued nonlinear algebraic equations in $\underline{p}$ and $\underline{q}$. The elements of $\underline{p}$ and $\underline{q}$ together constitute 2n variables. The solution of Equation (13) will be considered in the next chapter.

Once the values of p and q are obtained the amplitude and phase of the state variables x are obtained from the relationship:

$$pq_i = \sqrt{p_i^2 + q_i^2},\ i = 1, 2, \ldots, n$$

$$\theta_i = \tan^{-1}(q_i/p_i),\ i = 1, 2, \ldots, n$$

where:

pq = magnitude of the amplitude; and

θ = phase angle of the amplitude.

CHAPTER III

SOLUTION OF SYSTEM OF EQUATIONS

3.1 Introduction

In this chapter, the numerical solution method for the system of equations in Equation (13) is presented. The advantages and disadvantages of the numerical method are also discussed.

3.2 Numerical Solution

One powerful method available for the numerical solution of simultaneous, nonlinear algebraic equations is the quadratically convergent Newton-like method based upon Gaussian elimination developed by Brown [9]. It is an iterative method which is a variation of Newton's method using Gaussian elimination. In each row, the variable whose corresponding partial derivative is largest in absolute value is eliminated and an iterative function is formed as discussed in the Appendix. After obtaining the value of the last variable, the values of the other variables are obtained by back substitution in the iteration function. The latest values of the functions are used in each step. The partial derivative is evaluated numerically. A subroutine ZSYSTM developed by ISML [12] based upon this method was used to obtain the solution. Other methods such as Newton's method and Gauss-Seidel method were considered, but this algorithm was chosen because of the following considerations as discussed by Brown [9].

3.3 Advantages and Disadvantages of the Quadratically Convergent Newton-Like Method

The advantages are:

1. It requires only $(N^2/2 + 3N/2)$ function evaluations per iterative step as compared with $(N^2 + N)$ evaluations of the discretized Newton's method.
2. It uses a numerical method to evaluate the partial derivatives, thus eliminating the need for defining the corresponding equations, which may become quite cumbersome for higher order systems. In addition, it saves storage space.
3. Although rigorous convergence results are yet to be obtained, experimental evidence shows a quadratic type of convergence behavior.
4. The computation and inversion of a Jacobian is avoided.
5. It is rapidly convergent compared to the Gauss-Seidel method.

The disadvantages of the method are:

1. Rigorous convergence for the method is yet to be proved.
2. Since the method is only locally convergent, the initial solution guess has to be reasonably close to the final solution in order to obtain convergence.
3. As the number of nonlinearities increases, the number of iterations required for convergence increases substantially.

The method is very useful for the solution of algebraic simultaneous equations which contain mildly nonlinear relations. Many practical nonlinear systems contain mild nonlinear equations; in addition, the

SIDF's for most nonlinearities are milder than the original nonlinearity, so using quasi-linear gains should enhance the effectiveness of this method.

CHAPTER IV

EXAMPLES

4.1 Introduction

In this chapter, two examples illustrating the various aspects of the method are presented. The results obtained were verified by other methods and the results of the verification are also provided.

4.2 Example 1

As a simple case, the Duffing's equation,

$$\ddot{x} + 2\alpha\dot{x} + \omega_0{}^2x + hx^3 = u \tag{14}$$

where:

ω_0 = natural frequency of the system; and

α and h are constants

is considered. If the input u is a sinusoid B sinωt, then the states will also be nearly sinusoidal. If x is nearly sinusoidal, its amplitude A can be determined using SIDF analysis. The SIDF of hx^3 is 3/4(h) $|A|^2$. Substituting this in Equation (14) and choosing $x = x_1$ and $\dot{x} = x_2$ as the state variables yields

$$F_{DF} = \begin{bmatrix} 0 & 1 \\ -(\omega_0{}^2 + \frac{3}{4} h\ |A|^2) & -2\alpha \end{bmatrix}, \quad G_{DF} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Selecting $h = 1$, $\alpha = 0.1$ and $\omega_0 = 5$, the frequency response was obtained. When the system is driven by a sinusoidal input, the solution for A can be shown to be the roots of

$$[(\omega_0^2 - \omega_1^2)\ |A| + \frac{3}{4}|A|^3 h]^2 + [2\alpha\omega_1 |A|]^2 = B^2 \qquad (15)$$

The system exhibits jump resonance [11]. This can be seen in Figure 2 which is the frequency response plot for Equation (14). The amplitudes obtained by using the Newton-like method are shown in Table I. Those amplitudes satisfy Equation (15) as shown in Table I. This example illustrates the effectiveness of the solution method of this study even where multiple solutions exist, as occurs for values of ω between 8 and 21 rad/sec. At frequencies where there is more than one solution, different initial solution guesses have to be used for obtaining the different amplitudes. Whenever a discontinuity is encountered in a frequency response plot (as at $\omega = 21$ rad/sec), the analyst ought to seek other roots.

4.3 Example 2

In this example, the system shown in Figure 3 is considered. The system is part of a gimbal drive system. The system is of fourth order and contains two nonlinearities, a coulomb friction type nonlinearity (N_1) and a limiter type nonlinearity (N_2), shown in Figures 4 and 5. The frequency response of the system is desired, where the input to the system is pressure and the output is angular displacement θ. The state variables ($\underline{x}$) chosen are indicated in Figure 3.

First, the effects of the nonlinearities were suppressed and the frequency response of the linear system was obtained in order to

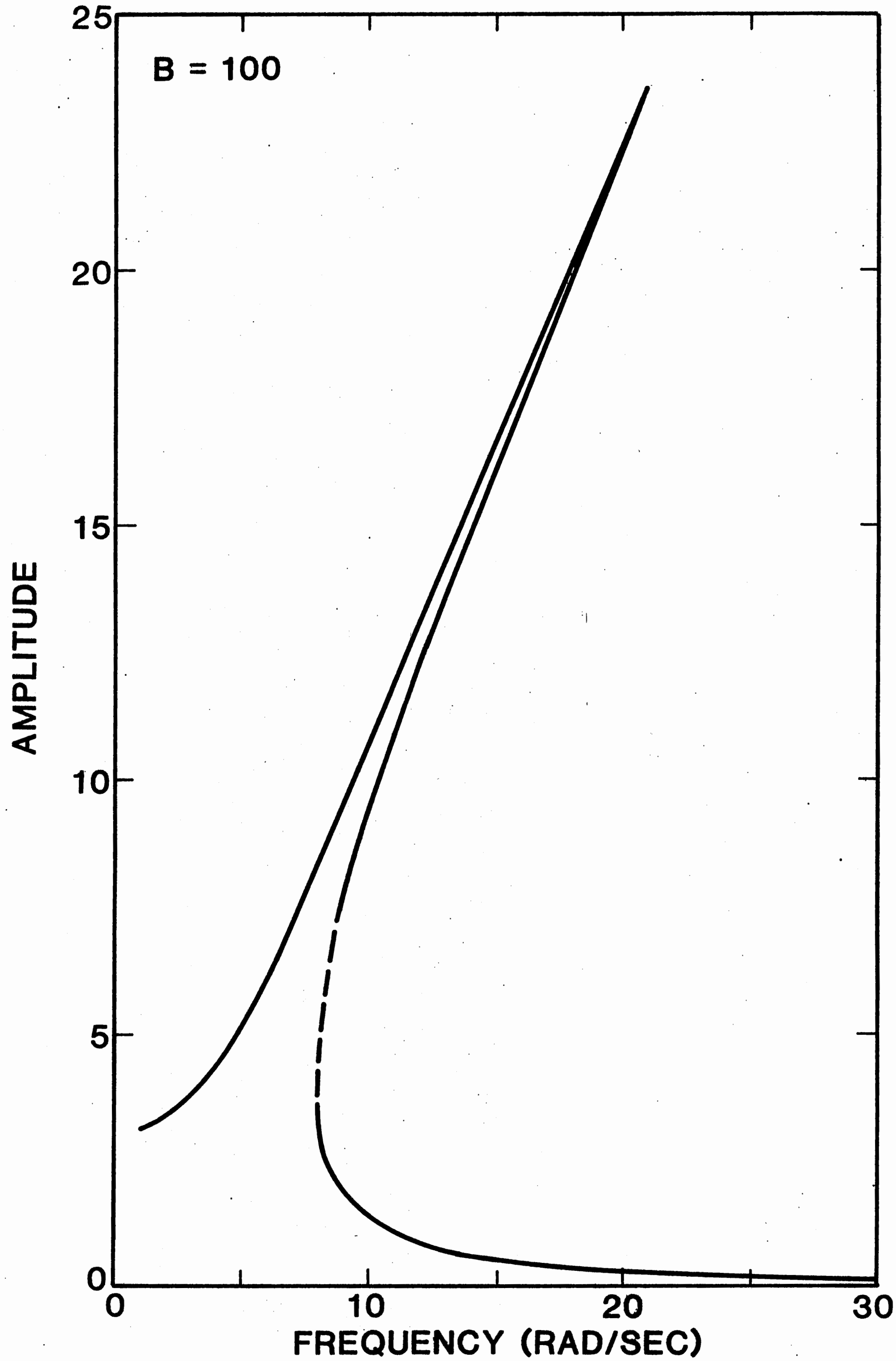


Figure 2. Frequency Response for Duffing's Equation

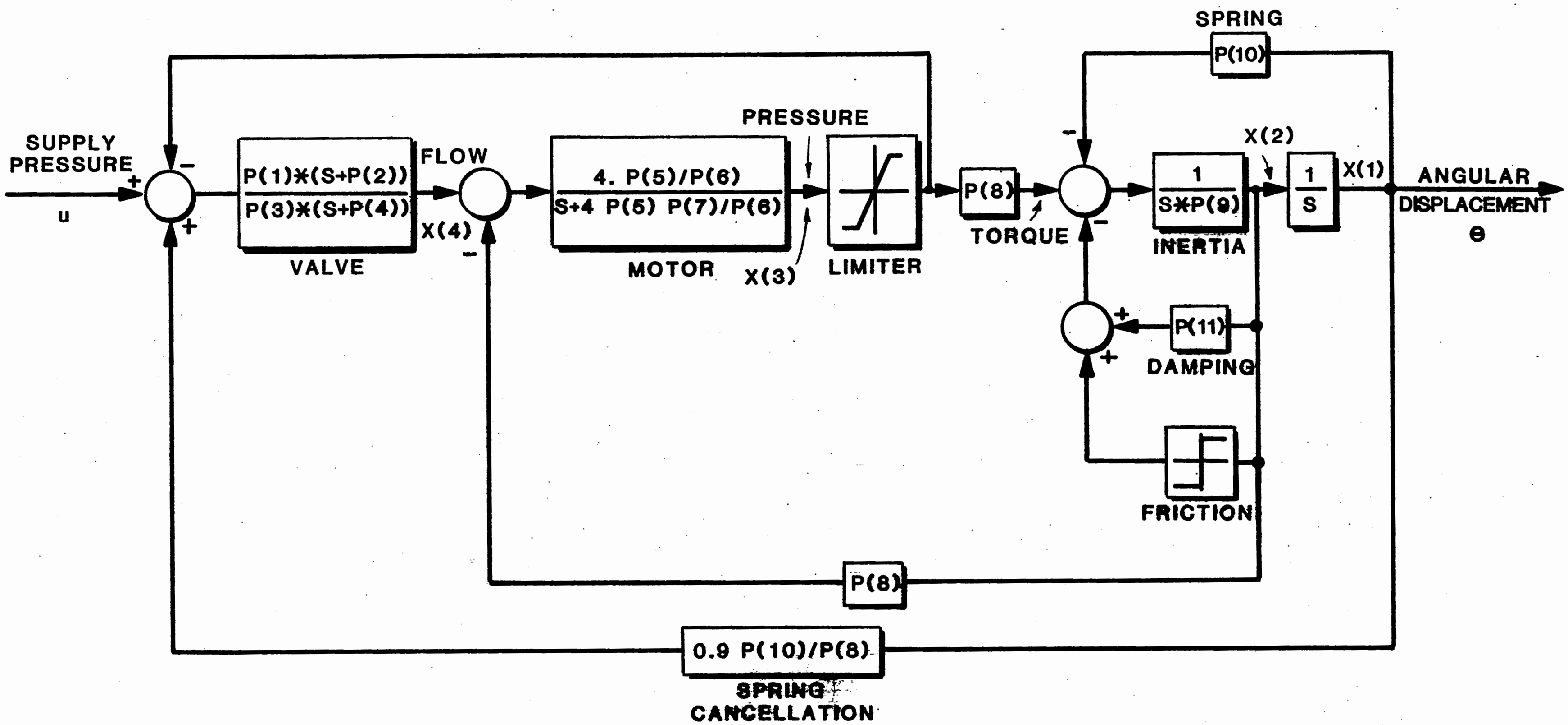
TABLE I

AMPLITUDES FOR B = 100 IN DUFFING'S EQUATION

Frequency (RAD/SEC)	Amplitude of x (A)	Input Amplitude B, Computed From A, Eq. (15)
1	3.17057	100.
2	3.38107	100.
3	3.7588	100.
4	4.33162	100.
5	5.10651	100.
6	6.05283	100.
7	7.11638	99.9998
8	8.24771	100.
	3.17808	100.
9	9.41409	100.0006
	7.56317	99.9998
	1.87265	100.2732
10	10.597	100.0005
	9.26606	99.9997
	1.35788	100.0001
12	12.97803	99.999
	12.1734	100.0004
	0.84395	99.9997
15	16.5428	100.003
	16.10649	99.9994
	0.50041	99.9993
18	20.08119	99.9995
	19.84861	99.9987
	0.33452	100.0007
20	22.419445	100.
	22.419442	99.9989
	0.26669	100.0002
21	23.56841	100.0005
	0.2404	100.0011

TABLE I (Continued)

Frequency (RAD/SEC)	Amplitude of x (A)	Input Amplitude B, Computed From A, Eq. (15)
22	0.21787	99.9992
23	0.19842	100.002
24	0.18149	100.0003
25	0.16667	100.002
30	0.11428	99.9962



P(1) = 3.3798

P(2) = 1847

P(3) = 12485

P(4) = 25

P(5) = 40000

P(6) = 0.05

P(7) = 6.0×10^{-4}

P(8) = 2.24

P(9) = 268.5

P(10) = 9.21×10^{5}

P(11) = 18000

Figure 3. Block Diagram for Example 2

determine the region of maximum importance. It was found to be in the frequency range of 1.0 to 100 Hz. Then, the frequency response of the system with the nonlinearities was obtained.

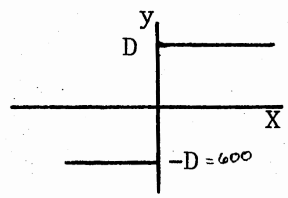


Figure 4. Friction Type Nonlinearity

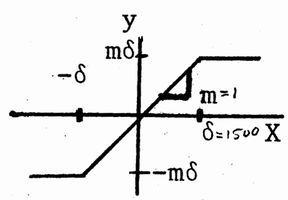


Figure 5. Limiter

The SIDF of a friction type nonlinearity is:

$$N_1 = \frac{4D}{\pi A_1} \tag{16}$$

where:

D = value of friction level

and

A_1 = amplitude of the input to the nonlinearity.

The SIDF of a limiter is:

$$N_2 = mf(\delta/A_2) \tag{17}$$

where:

A_2 = amplitude of the input to the limiter;

δ = saturation limit;

m = gain of the element before saturation;

and

$$f(\delta/A_2) = 2/\pi \left(\sin^{-1}(\delta/A_2) + (\delta/A_2)\sqrt{1-(\delta/A_2)^2}\right) \quad \text{for } |\delta/A_2| < 1$$

$$= 1 \qquad\qquad \text{for } |\delta/A_2| > 1$$

Using the above SIDF's for the two nonlinearities, it can be shown that:

$$F_{DF} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 3430 & -(67 + N_1) & 0.0083xN_2 & 0 \\ 0 & -7.168x10^6 & 2786 & 2.3x10^6 \\ 1.82x10^5 & 0 & 0.4932xN_2 & -25 \end{bmatrix} \tag{18}$$

and

$$G_{DF} = \begin{bmatrix} 0 \\ 0 \\ 2.71x10^{-3} \\ 0.4932 \end{bmatrix} \tag{19}$$

where:

$N_1 = 4D/\pi A_2$;

$N_2 = f(\delta/A_3)$;

A_2 = amplitude of x_2, the input to the friction nonlinearity; and

A_3 = amplitude of x_3, the input to the limiter.

Note that F_{DF} and G_{DF} are both real.

The gain of the system was obtained for the following cases:

1. The nonlinearities absent.
2. For input amplitudes of 250, 500, and 1000 psi when both the nonlinearities are present.
3. For input amplitudes of 500 and 1000 psi when only N_1 is present.
4. For input amplitudes of 500 and 100 psi when only N_2 is present.

The results for cases (1) and (2) are shown in Figure 6. At low frequencies, the amplitude of the input has a significant effect on the gain of the system. This is because the limiter saturates. At high frequencies, the gain does not change with input amplitude as the limiter does not saturate.

The effect of the nonlinearities is considerable at a frequency of 1 Hz and an input amplitude of 1000 psi. So, a more detailed study was undertaken for those conditions. The frequency response gains predicted by the SIDF method were compared with the fundamental component of the output obtained using Fourier analysis of the time doman simulation. The results and the percentage error between the two methods are shown in Table II. The percentage error is defined as (amplitude gain using SIDF-amplitude gain using Fourier analysis)/(amplitude gain using Fourier analysis).

The two methods compare quite well for only one nonlinearity being present. The error when both nonlinearities are present is 11.2

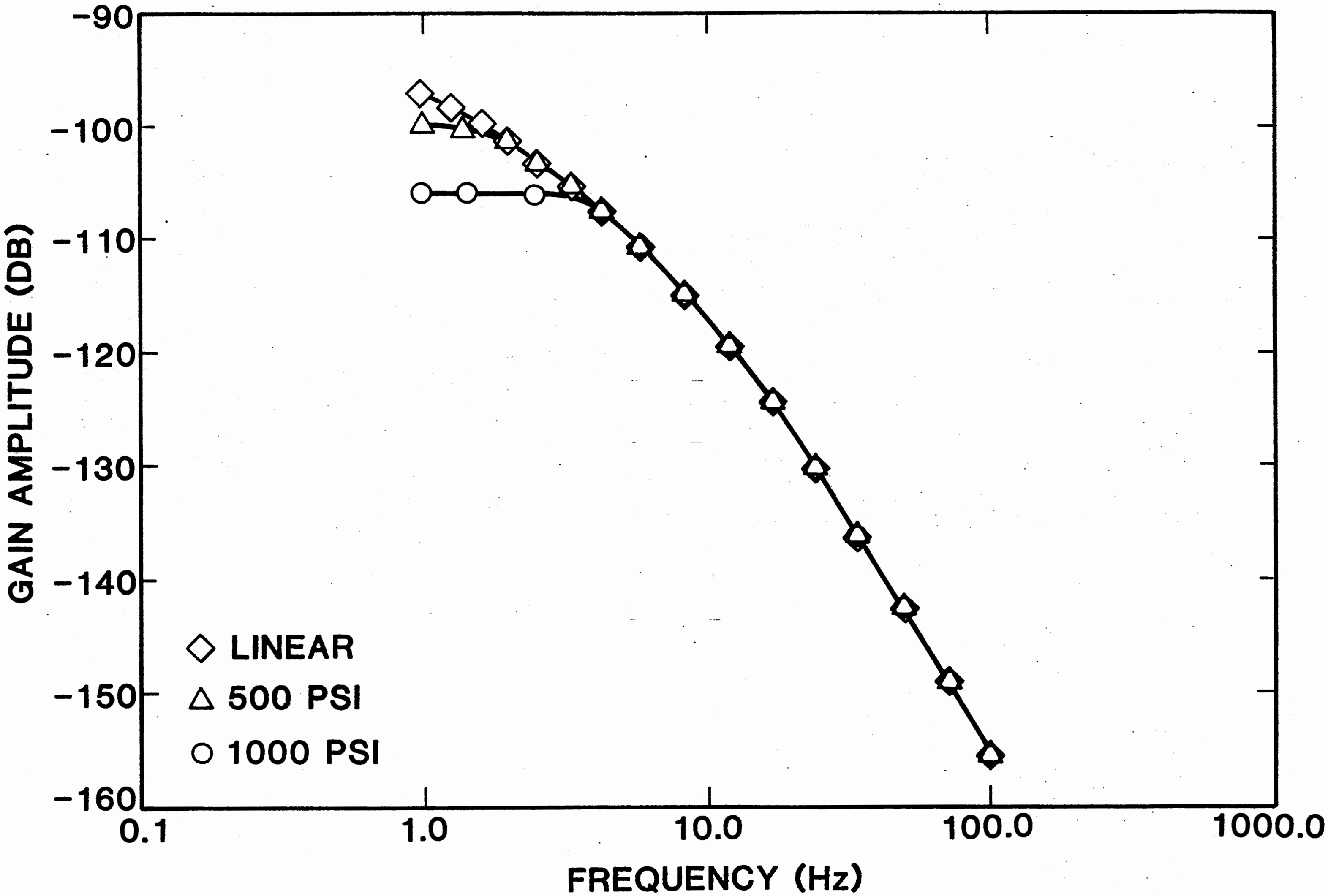


Figure 6. Results of Example 2 When Both Nonlinearities are Present

percent, which is fairly typical of SIDF analysis results. This error is due to the inputs to the nonlinearities not being nearly sinusoidals, as shown in their time histories in Figures 7 and 8. Considering the large higher harmonic contact the SIDF method did quite well.

TABLE II

COMPARISON OF FOURIER ANALYSIS AND SIDF PREDICTIONS

Nonlinearity Present	Amplitude Gain Using SIDF Method	Amplitude Gain Using Fourier Analysis	% Error
No Nonlinearity	0.1358E-4	0.1357E-4	0.00
Friction (N_1) Only	0.962E-5	0.974E-5	1.23
Limiter (N_2) Only	0.4973E-5	0.4824E-5	-3.08
Both Nonlinearities	0.47951E-5	0.4314E-5	-11.15

The frequency response curves for cases (3) and (4) are shown in Figures 9 and 10. The friction level used (D = 600) is not a realistic operating condition; it was chosen because the signals are quite non-sinosoidal and the nonlinearity effect is substantial at this level. The nonlinearities have considerable effect on the gain at low frequencies and high input amplitudes.

The time domain simulation of the system also indicated that there were no small limit cycles or self-starting large limit cycles. The

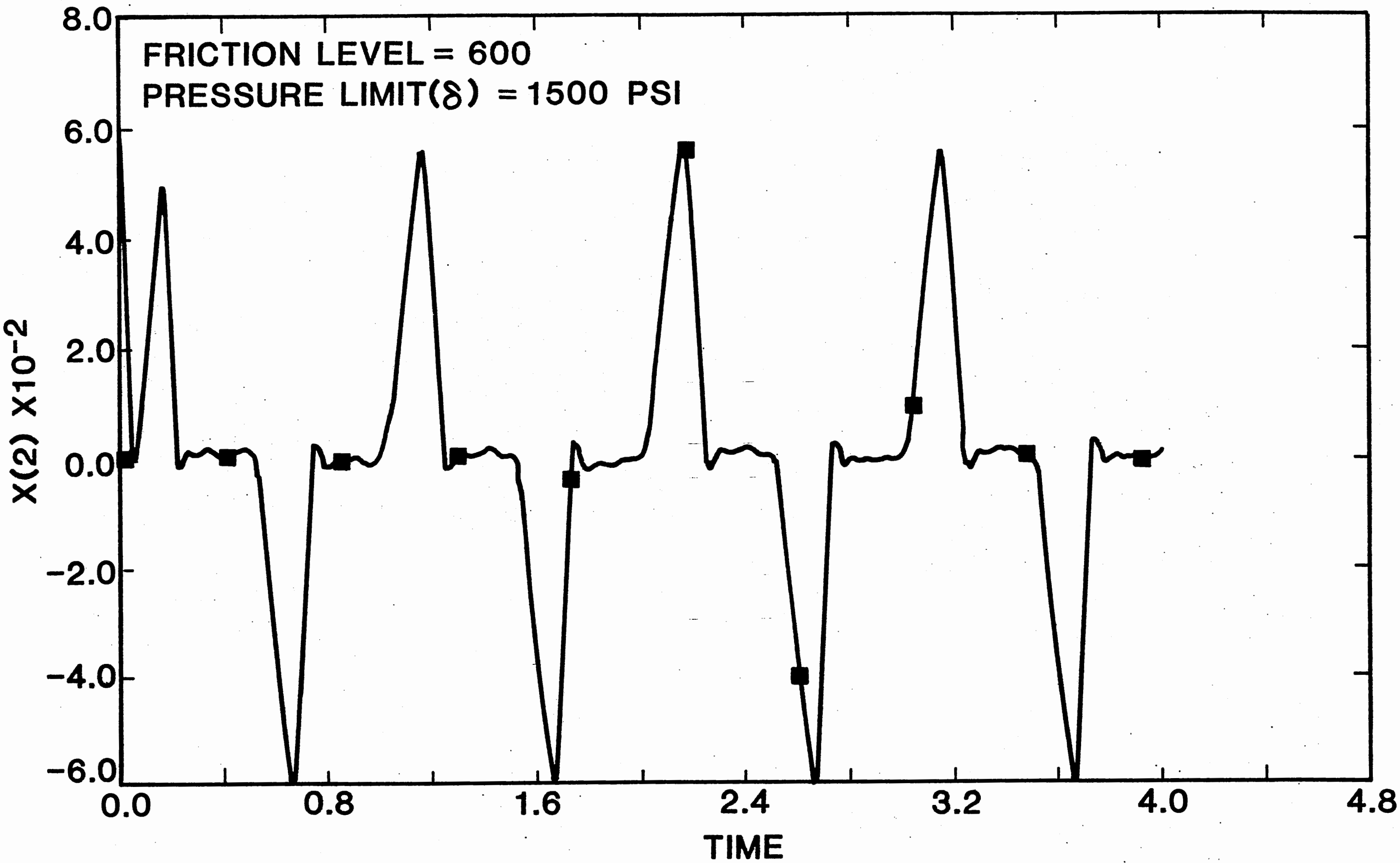


Figure 7. Time History of Input to N_1

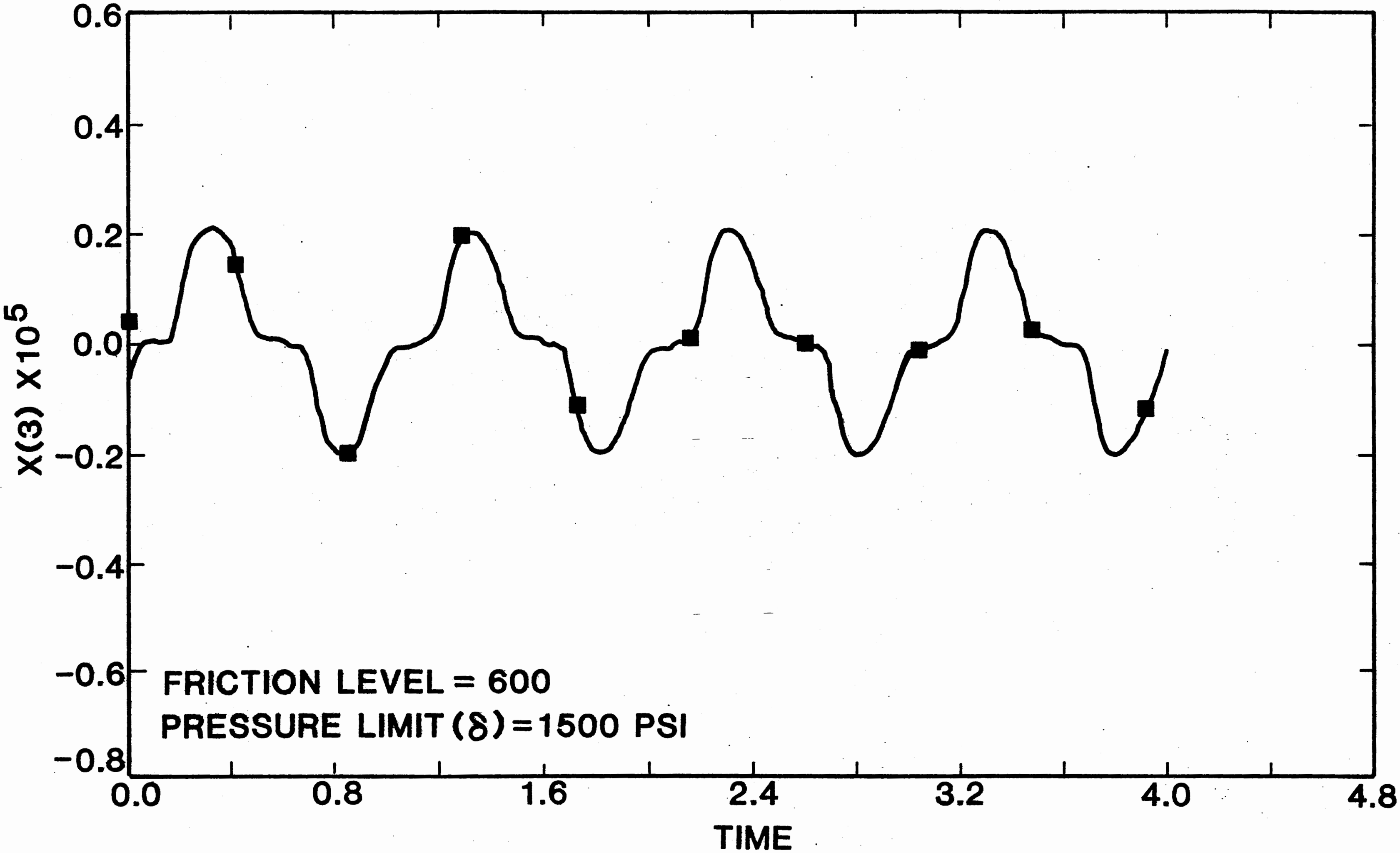


Figure 8. Time History of Input to N_2

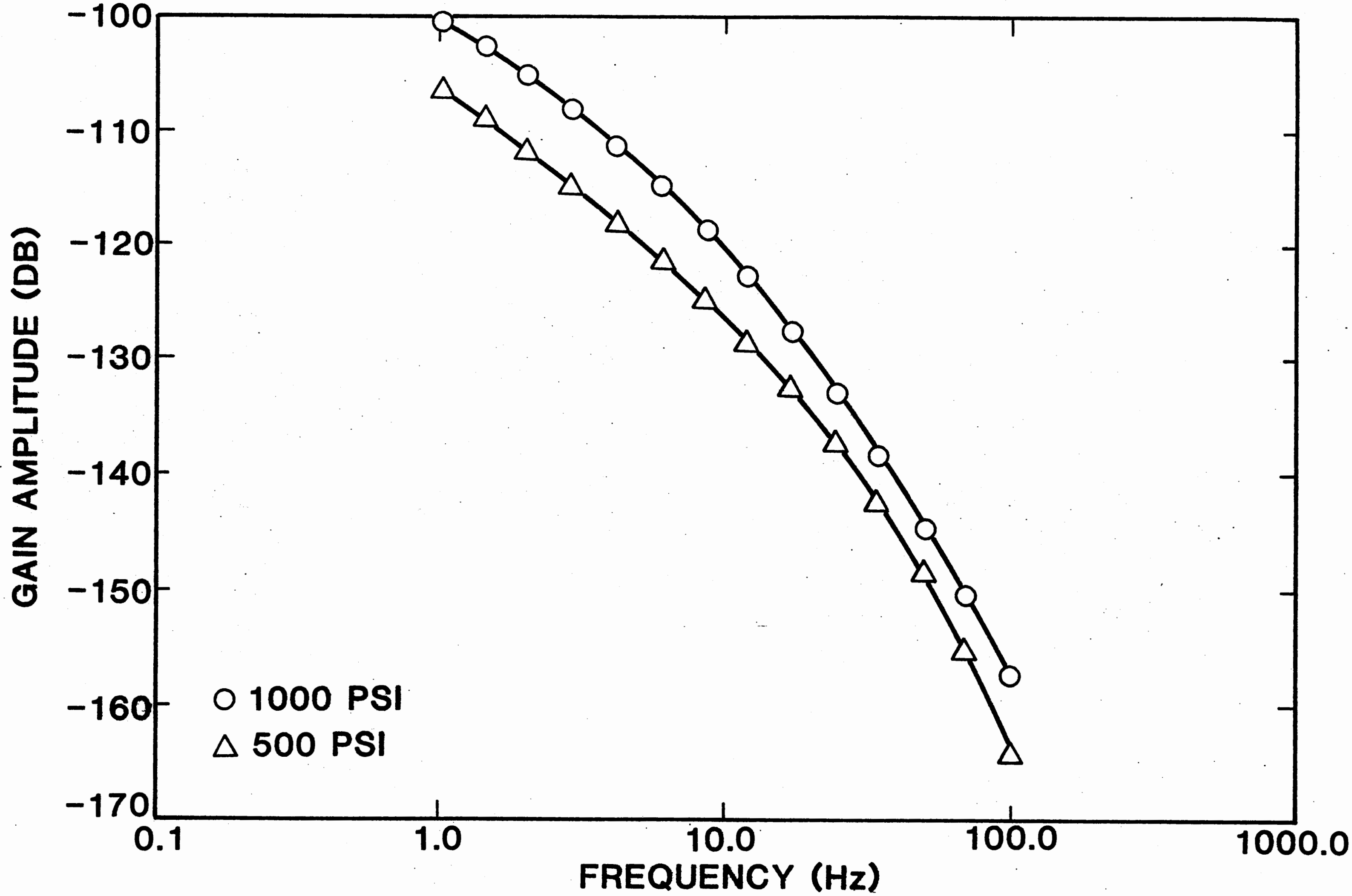


Figure 9. Frequency Response for Example 2 With N_1 Only Present

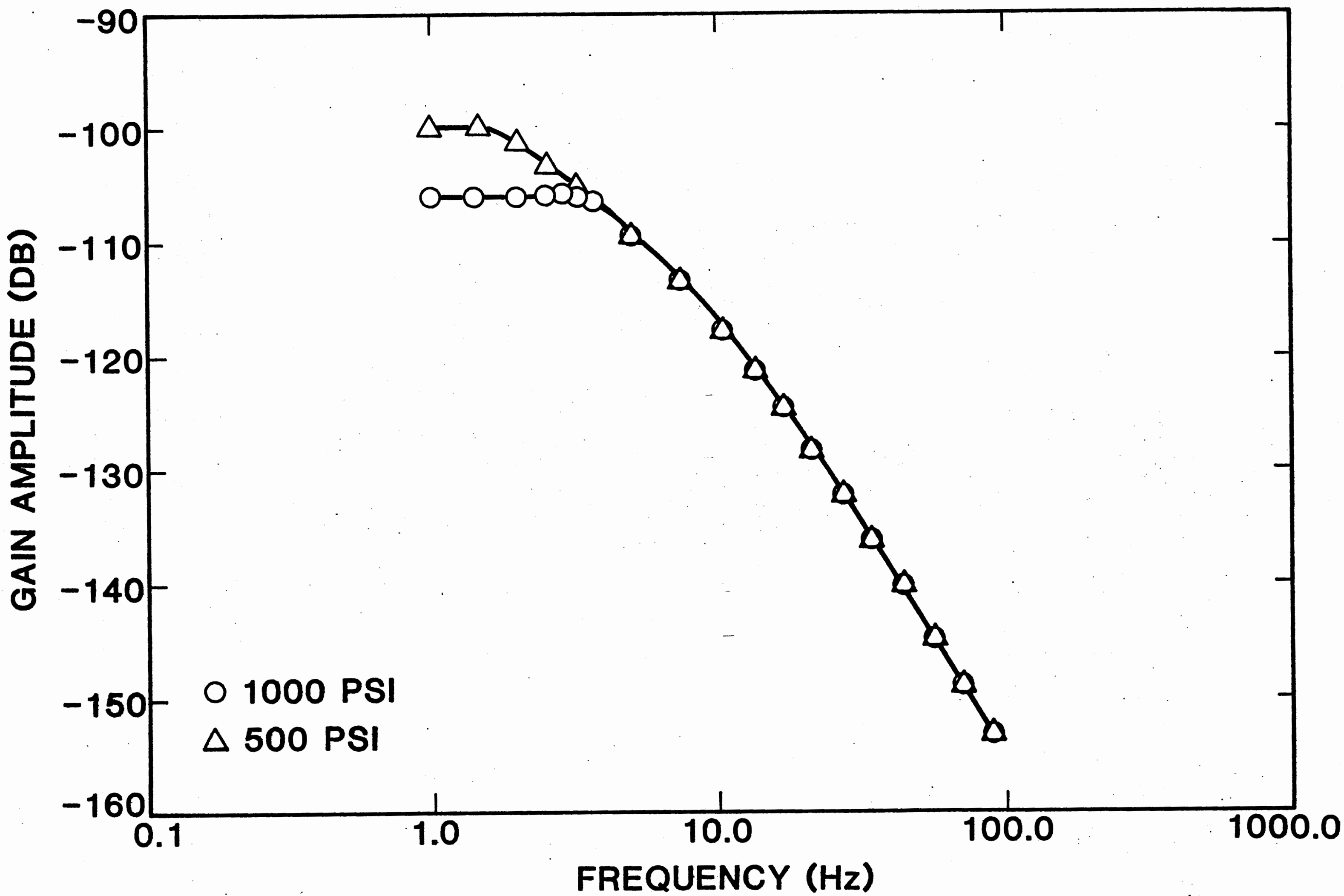


Figure 10. Frequency Response for Example 2 With N_2 Only Present

determination of the possibility of large limit cycles involves the use of two sinusoidal input describing functions and is beyond the scope of this study.

This example illustrates the amplitude dependency of the frequency response of a nonlinear system. It also illustrates the error involved when the input to the nonlinearity is not exactly sinusoidal.

CHAPTER V

CONCLUSIONS AND RECOMMENDATIONS

A technique for finding the frequency response of a nonlinear system using the digital computer has been developed. The method involves expressing the system equations in state variable form, replacing the nonlinearities in the system with their respective sinusoidal input describing functions and solving the resultant set of nonlinear algebraic equations numerically. The numerical method that was employed is effective in solving systems containing mildly nonlinear equations. It requires fewer evaluations per iteration than the Newton's Method and less iterations than the Gauss-Seidel method [9]. This technique can be used only when the system considered does not exhibit limit cycles.

Taking rigorous care to eliminate the possibility of large limit cycles existing in addition to the response to the sinusoidal input is a difficult task, in general. Extensive simulation (using a variety of initial conditions) is required, or two-sinusoidal input DF methods may be employed. Smaller limit cycles can be ruled out with considerable confidence if the frequency response plots do not exhibit sharp resonant péaks (so that F_{DF} does not have eigenvalues near the $j\omega$-axis), or if the response is simulated without limit cycles showing up. Both of these latter conditions have been met in the studies described here.

A restriction imposed on this technique is that the rest of the system must be capable of filtering the higher order harmonics generated by the nonlinearities so that the input to the nonlinearities are nearly sinusoidal. This can be checked by performing a few time domain simulations, as was done in this study.

The extension of the technique to systems containing static elements with memory needs further investigation. There is no apparent reason why the method used here will not be effective, but examples should be studied. Although the quadratically convergent Newton-like method is quite effective in solving mildly nonlinear equations, no proof of convergence has been obtained. The usefulness of the technique would be increased if a more powerful method than the quadratically convergent Newton-like method is used.

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APPENDIX

QUADRATICALLY CONVERGENT NEWTON-LIKE METHOD BASED UPON GAUSSIAN ELIMINATION

A powerful method available for the solution of nonlinear simultaneous algebraic equations is the quadratically convergent Newton-like method based upon Gaussian elimination developed by Brown [9].

Consider a real-valued twice continuously differentiable system of n nonlinear equations in n unknowns. Let it be expressed as:

$$
\begin{aligned}
f_1(\underline{x}) &= f_1(x_1, x_2, x_3, \ldots, x_n) = 0 \\
f_2(\underline{x}) &= f_2(x_1, x_2, x_3, \ldots, x_n) = 0 \\
&\vdots \\
f_n(\underline{x}) &= f_n(x_1, x_2, x_3, \ldots, x_n) = 0
\end{aligned}
\tag{20}
$$

or in vector form as:

$$\underline{f}(\underline{x}) = 0 \tag{21}$$

It is assumed that Equation (20) satisfies the following conditions.

1. In a closed region R whose interior contains a root $\underline{x} = \underline{r}$ of (21), $\underline{f}(\underline{x})$ is twice continously differentiable.
2. The Jacobian $J = d\underline{f}/d\underline{x}$ is nonsingular at $\underline{x} = \underline{r}$.
3. The initial guess $\underline{x}^0$ is chosen in R sufficiently close to $\underline{r}$.

If $\underline{x}^i$ denoted an ith approximation to the root $\underline{x} = \underline{r}$ of (20) and if the above conditions hold, the following steps can be applied.

Step 1. $f_1(\underline{x})$ is expanded in a Taylor series about the point $\underline{x}^i$, retaining only the first order terms, to give the linear approximation

$$
\begin{aligned}
f_1(\underline{x}) \simeq f_1(\underline{x}^i) + f_{1x_1}(\underline{x}^i)(x_1 - x_1^i) + f_{1x_2}(\underline{x}^i)(x_2 - x_2^i) + \ldots + \\
f_{1x_n}(x^i)(x_n - x_n^i)
\end{aligned}
\tag{22}
$$

where f_{kx_j} is the partial derivative of the kth function with respect to x_j.

Equating the right side of (25) to zero and solving for that variable, say x_p, whose corresponding partial derivative is largest in absolute value, one can obtain

$$x_p = x_p^i - \sum_{j=1}^{n-1} (f_{1x_j}^i / f_{1x_p}^i)(x_j - x_j^i) - f_1^i / f_{1x_p}. \qquad (23)$$

Assumption (2) above guarantees such a solution. The constants $f_{1x_j}/f_{1x_p}(\underline{x}^i)$, $j = 1, 2, p-1, p+1, n-1$, and f_1^i/f_{1x_p} are saved for future use. The left-hand side of Equation (23) is renamed b_p:

$$b_p(x_1, x_2, \ldots, x_{p-1}, x_{p+1}, \ldots, x_n) = \text{right hand side of (23)}. \qquad (24)$$

<u>Step 2</u>. A function g_2 of just (n - 1) variables, $x_1, x_2, \ldots, x_{p-1}, x_{p+1}, \ldots, x_n$ is defined as:

$$g_2 = f_2(x_1, x_2, x_{p-1}, b_n, x_{p+1}, x_n) \qquad (25)$$

Expanding g_2 in Taylor series about the point $(x_1, x_2, x_{p-1}, x_{p+1}, \ldots, x_n)^i$, linearizing and solving for that variable whose corresponding partial derivative is largest in absolute value, one repeats the process until new values of all state variables have been calculated. The point $\underline{b}$ thus obtained is used as the improved approximation x^{i+1} to the root $\underline{r}$. Brown [9] has shown that for the conditions stated above, this process is well defined.

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