

PRONY'S METHOD: DETERMINING THE NUMBER
OF EXPONENTIAL MODES AND THE
OPTIMAL SAMPLE PERIOD

By

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CHAPTER I

THE RESEARCH PROBLEM

The problem of fitting sampled data to an exponential curve occurs in many applications. The particular application which motivated this research was the need to identify the system transfer function for a specific class of ecological systems (Mulholland, 1981). There are likewise many methods for fitting data to an exponential curve. One such technique, Prony's method, is examined in this research.

The particular application in view is clarified by reference to Figure 1. The system transfer function $H(s)$ is desired. The system is assumed to be linear, and is known to be in steady state with a constant unit input $u(t)$. This input is removed at time $t=0$. If the system has N poles, then the output $x(t)$ will have N exponential modes. The output is corrupted by system noise $w(t)$ and by measurement noise v_k . Output measurements z_k , $k = 0, 1, \dots$ are periodically spaced with period τ .

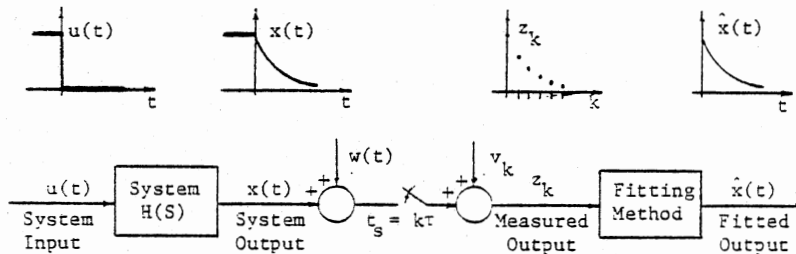


Figure 1. Determining a System Transfer Function

If the output measurements can be fit to a continuous exponential function $\hat{x}(t)$, then the transfer function can be approximated by

$$h(t) = -\dot{\hat{x}}(t) \approx -\dot{\hat{x}}(t) \quad (1.1)$$

$$H(s) = \mathcal{L}[h(t)] \quad (1.2)$$

The negative sign is necessary because of the inverted step input. $\mathcal{L}[\cdot]$ represents the Laplace transform, $h(t)$ is the impulse response, and $\dot{x}(t)$ is the time derivative of $x(t)$. Thus, to find the system transfer function, it is first necessary to fit the periodic measurements z_k to a continuous exponential curve.

The research presented in this paper has been limited to two specific questions. First, given the measured output of the system z_k , can the number of exponential modes N in the data be determined? That is, how many poles does the system have? Second, what is the optimal sample period τ for a given system?

To answer the first question, a number of methods are examined, and the singular value factorization is found to give the most reliable estimate of the number of exponential modes. The singular value factorization, when used with Prony's method, is also found to provide a better fitting function for the data than do other factorizations. To answer the second question, a number of data functions are simulated, and the sample period which provides the most accurate fitting function is observed. This sample period is found to be within the bounds predicted by a sampling theory.

CHAPTER II

REVIEW OF PREVIOUS RESEARCH

Prony's (1795) method is a technique for fitting exponential functions to periodically spaced data. Hildebrand (1956) and Hamming (1962) provide excellent developments of the technique along with illustrative examples. Householder (1949) discusses an iterative refinement to Prony's method which makes it a true least-squares curve fit. Other books on numerical methods tend to overlook Prony's method or dismiss it as being unpractical and prone to large errors. It is suspected that a lack of an adequate sampling theory is responsible for this systematic neglect of what could otherwise be a powerful tool for model identification.

Prony's method is very useful for analyzing experimental data when the full state vector is not measured. The regression analysis technique, by contrast, requires data for all components of the state. Lin and Yu (1977) have used Prony's method to solve for the companion matrix. Mulholland (1981) has generalized this technique.

The problem of determining the number of exponential modes was addressed by Householder (1949). Van Blaricum and Mitra (1977) compared the method suggested by Householder with a method of their own. Householder's method is referred to later in this paper as the FG factorization. Van Blaricum's method is the eigenvector analysis discussed later, and is related to the singular value factorization. The problem

of determining the number of modes becomes that of determining the rank of a matrix A . This rank determination problem is addressed in a wide body of literature. Of particular use have been Dongarra et al. (1979), Stewart (1973), and Lawson and Hanson (1974).

The problem of determining the optimal sample period for Prony's method is addressed by Mulholland (1981). Chapter V is an extension of this work. Chapter III is also based upon the work of Mulholland (1981).

The computer programs used for the numerical aspect of this research were programmed in FORTRAN IV. The programs were run in double precision on an IBM 370. These programs made extensive use of the LINPACK subroutines available for nominal cost from IMSL, Inc., Sixth Floor, NBC building, 7500 Bellaire Blvd., Houston, Texas 77036. In particular, the QR factorization and singular value factorization were done by the subroutines DQRDC and DSVDC. The LINPACK subroutines include a subset of the Basic Linear Algebra Subroutines (BLAS), which were also much used. See Dongarra et al. (1979) for complete details.

CHAPTER III

PRONY'S METHOD FOR FITTING

EXPONENTIAL FUNCTIONS

Given a scalar measurement of periodically spaced data, defined by $x(t)$, Prony's method seeks an exponential decomposition of the form

$$\hat{x}(t) = \hat{p}_1 e^{\hat{\lambda}_1 t} + \hat{p}_2 e^{\hat{\lambda}_2 t} + \dots + \hat{p}_N e^{\hat{\lambda}_N t} \quad (3.1)$$

where N is a fixed integer. The $\hat{}$ notation is used to distinguish the fitting function $\hat{x}(t)$ from the data function being fitted, $x(t)$. The $\hat{}$ notation is dropped for p and λ excepting where confusion may result.

If $x(t)$ is to be correctly fitted by $\hat{x}(t)$, some information about the number N of exponential modes in $x(t)$ must be known. The derivation of Prony's method is simplified if it is originally assumed that the fitting function is exact, that is $\hat{x}(t) = x(t)$. Prony's method requires periodic measurements of $x(t)$, so (3.1) can be rewritten as

$$x_k = x(k\tau) = p_1 \mu_1^{k\tau} + \dots + p_N \mu_N^{k\tau} \quad (3.2)$$

where τ is the sample period, k is any integer, and

$$\mu_i = e^{\lambda_i \tau}, \quad i = 1, 2, \dots, N. \quad (3.3)$$

In (3.2), the integer k indicates the k -th periodic sample of $x(t)$, and by letting k range from 0 to $(M-1)$ the data set is given by

$$X = (x_0, x_1, x_2, \dots, x_{M-1}). \quad (3.4)$$

Writing (3.2) for $k = 0, 1, \dots, (M-1)$ gives

$$\begin{aligned}
x_0 &= p_1 + \dots + p_N \\
x_1 &= p_1 \mu_1 + \dots + p_N \mu_N \\
x_2 &= p_1 \mu_1^2 + \dots + p_N \mu_N^2 \\
&\vdots \\
&\vdots \\
x_{M-1} &= p_1 \mu_1^{M-1} + \dots + p_N \mu_N^{M-1}
\end{aligned} \tag{3.5}$$

where the data set of (3.4) containing M knowns has been set equal to the $2N$ unknown constants: p_1, \dots, p_N and μ_1, \dots, μ_N . Thus, no unique solution of (3.5) is possible for $M < 2N$. For $M \geq 2N$ a solution is possible, with an over determined system resulting when the strict inequality holds.

If the fitting function is not exact then $x(t) \neq \hat{x}(t)$ and equations (3.5) become approximations. In this event, Prony's solution to (3.5) is approximately a least squares error solution. That is, the residual defined by

$$\zeta = \left[\sum_{k=0}^{M-1} [x_k - \hat{x}(k\tau)]^2 \right]^{1/2} \tag{3.6}$$

is approximately minimized by Prony's solution (Householder, 1949). In equation (3.6) the x_k are the values being fitted, and the $\hat{x}(k\tau)$ are values from the fitting function. If the x_k in (3.6) represent measured values from a system output, then they would correspond to the z_k in Figure 1.

Prony's method does not attempt to solve (3.5) directly, but instead develops a solution algorithm which requires the computation of the roots of an N -th order polynomial and the solution of two N -th order linear algebraic equation systems. Let μ_1, \dots, μ_N be the roots of

$$c_0 \mu^N + c_1 \mu^{N-1} + \dots + c_{N-1} \mu + c_N = 0 \tag{3.7}$$

In factored form this is given by:

$$(\mu - \mu_1)(\mu - \mu_2) \dots (\mu - \mu_N) = 0. \quad (3.8)$$

Equation (3.7) assumes $c_0 = 1$. In order to determine the unknown coefficients (c_1, \dots, c_N) in (3.7), the first equation of (3.5) is multiplied by c_N , the second by c_{N-1} , and so forth, until the N -th equation is multiplied by c_1 and the $(N+1)$ -th by c_0 . The results of these multiplications are then added to give

$$\begin{aligned} & c_N(p_1 + \dots + p_N) + c_{N-1}(p_1\mu_1 + \dots + p_N\mu_N) + \dots \\ & + c_1(p_1\mu_1^{N-1} + \dots + p_N\mu_N^{N-1}) + c_0(p_1\mu_1^N + \dots + p_N\mu_N^N) \quad (3.9) \\ & = c_N x_0 + \dots + c_1 x_{N-1} + c_0 x_N. \end{aligned}$$

This assumes $M > N$. In (3.9) it should be noted that each μ_i , for $i = 1$ to N , satisfies the polynomial (3.7), so the left side of (3.9) sums to zero and

$$c_0 x_N + c_1 x_{N-1} + \dots + c_N x_0 = 0. \quad (3.10)$$

The same multiply and sum procedure can be applied to any of the equations of (3.5). This gives the generalized equation:

$$c_0 x_{i+N} + c_1 x_{i+N-1} + \dots + c_N x_i = 0, \quad i = 0, 1, \dots, M-N-1. \quad (3.11)$$

In particular, the following set of $(M-N)$ equations in N unknown polynomial coefficients may be formed. (Since a scale factor is arbitrary, c_0 is assumed equal to 1.)

$$\begin{aligned} -x_N &= c_1 x_{N-1} + \dots + c_N x_0 \\ -x_{N+1} &= c_1 x_N + \dots + c_N x_1 \\ &\vdots \\ -x_{M-1} &= c_1 x_{M-2} + \dots + c_N x_{M-N-1} \end{aligned} \quad (3.12)$$

A unique solution of (3.12) can be obtained only for $M > 2N$. If $M = 2N$, (3.12) has an exact solution. A least squares estimate can be obtained when $M > 2N$.

A solution of (3.12) for the coefficients c_1, c_2, \dots, c_N defines the N -th order polynomial (3.7). The N roots of this polynomial, in turn, define the exponential (3.1), and according to (3.3)

$$\lambda_k = (1/\tau) \ln(\mu_k), \quad k = 1, 2, \dots, N. \quad (3.13)$$

Also, the N roots computed from (3.7) define the coefficient powers in (3.5), from which values for p_1, p_2, \dots, p_N can be computed.

The procedure outlined above assumes that the λ_k do not include multiple poles. Van Blaricum and Mitra (1977, p. 174-175) discussed the multiple pole case, and showed that Prony's method generally produces good estimates for the λ_k and poor estimates for the p_k when multiple poles are present.

Prony's method for single poles is summarized in the following algorithm:

Algorithm 3.1: Prony's Method.

Given M data points: x_0, x_1, \dots, x_{M-1} representing periodically spaced data of period τ , fit the data to an exponential function of the form:

$$\hat{x}(t) = p_1 e^{\lambda_1 t} + p_2 e^{\lambda_2 t} + \dots + p_N e^{\lambda_N t}. \quad (3.14)$$

1. Define the matrix and the vectors:

$$W = \begin{bmatrix} x_0 & x_1 & & x_{N-1} \\ x_1 & x_2 & & x_N \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ x_{M-N-1} & x_{M-N-2} & \dots & x_{M-2} \end{bmatrix} \quad (3.15)$$

$$\underline{c}'_N = (c_N, c_{N-1}, \dots, c_1)^T \quad (3.16)$$

$$\underline{w}_{N+1} = (x_N, x_{N+1}, \dots, x_{M-1})^T. \quad (3.17)$$

2. Solve for c_i , $i = 1, 2, \dots, N$ from the linear system:

$$W \underline{c}'_N = -\underline{w}_{N+1}. \quad (3.18)$$

3. Solve for the roots μ_i , $i = 1, 2, \dots, N$ of the polynomial:

$$\mu^N + c_1 \mu^{N-1} + \dots + c_{N-1} \mu + c_N = 0. \quad (3.19)$$

4. Solve for λ_i , $i = 1, 2, \dots, N$ from:

$$\lambda_i = (1/\tau) \ln(\mu_i). \quad (3.20)$$

5. Define the matrix and vectors:

$$V_M = \begin{bmatrix} 1 & 1 & \dots & 1 \\ \mu_1 & \mu_2 & \dots & \mu_N \\ \mu_1^2 & \mu_2^2 & \dots & \mu_N^2 \\ \vdots & \vdots & \dots & \vdots \\ \mu_1^{M-1} & \mu_2^{M-1} & \dots & \mu_N^{M-1} \end{bmatrix} \quad (3.21)$$

$$\underline{x} = (x_0, x_1, \dots, x_{M-1})^T \quad (3.22)$$

$$\underline{p} = (p_1, p_2, \dots, p_N)^T. \quad (3.23)$$

6. Solve for p_i , $i = 1, 2, \dots, N$ from the linear system:

$$V_M \underline{p} = \underline{x}. \quad (3.24)$$

Prony's method reduces the nonlinear exponential fit of (3.1) to two standard problems of numerical analysis. Two linear systems, (3.18) and (3.24), must be solved, and one N-th order polynomial (3.19) must be solved.

CHAPTER IV

DETERMINING THE NUMBER OF EXPONENTIAL MODES

A. Introduction

The purpose of this chapter is to find a way of estimating the number of exponential modes N in the measured data. Section B shows that this problem reduces to that of determining the rank of a particular matrix. Sections C through E examine various means of determining the rank of a matrix. Section F examines the effectiveness of these methods as they are applied to Prony's method. Of the methods examined, it is found that when the singular value factorization is incorporated into Prony's method, it produces the best estimate for N , and also produces the most accurate estimate for the exponential fitting function (3.1).

B. Prony's Method and the Rank of a Matrix

The difference equation (3.11) must be satisfied for vectors $\underline{w}_1, \underline{w}_2, \dots, \underline{w}_N$ defined as follows:

$$\underline{w}_1 = \begin{bmatrix} x_0 \\ x_1 \\ \cdot \\ \cdot \\ x_{\gamma-1} \end{bmatrix}, \quad \underline{w}_2 = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_{\gamma} \end{bmatrix}, \quad \dots, \quad \underline{w}_N = \begin{bmatrix} x_{N-1} \\ x_N \\ \cdot \\ \cdot \\ x_{N+\gamma-1} \end{bmatrix} \quad (4.1)$$

If $\gamma \geq N$, then these N vectors will be independent. Any $N+1$ of these vectors, however will form a dependent set (Householder, 1949, p. 10). Thus, the problem of determining the number of exponential modes reduces to that of determining the rank of the matrix given by

$$W_{\gamma_i} = \begin{bmatrix} x_0 & x_1 & \cdots & x_{i-1} \\ x_1 & x_2 & & x_i \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ x_{\gamma-1} & x_{\gamma} & & x_{\gamma+i-1} \end{bmatrix}, \quad \gamma \geq i. \quad (4.2)$$

If $N \geq i$ then W_{γ_i} will be of full rank. If $N < i$, then W_{γ_i} will be of rank N .

Methods for detecting dependency in W_{γ_i} involve factoring W_{γ_i} into a triangular matrix T and some other matrix or matrices. Then W_{γ_i} is of rank K only if the first K diagonal elements of T are non-zero. In practice, the data points x_0, x_1, \dots will be noisy, and the test for a zero diagonal element will be replaced with a test for a negligible element. Section D will define this test.

Such matrix factorizations also are useful in finding the least squares error solution to a linear system. The linear system with m equations in $n \leq m$ variables may be represented by

$$\underline{Ax} = \underline{b} \quad (4.3)$$

where A is a $m \times n$, \underline{x} is a n -vector of the unknown variables, and \underline{b} is a m -vector. When $m < n$, then (4.3) does not have a unique solution. When $m = n$ (4.3) has an exact solution if A is not singular. If $m > n$ (4.3) represents an over determined system. In this case, the residual \underline{r} is given by

$$\underline{r} = \underline{Ax} - \underline{b}. \quad (4.4)$$

The least squares error solution to (4.3) when $m \geq n$ is the best in the

sense that it will minimize the Euclidean length of the residual \underline{r} . The least squares problem shall be denoted as

$$\underline{Ax} \doteq \underline{b}. \quad (4.5)$$

Prony's solution may use such a least square error solution to equations (3.11) when $M \geq 2N$. The survey of matrix factorizations which follows will include indication of how the factorization is used to solve (4.5).

C. Survey of Matrix Factorizations

1. Normal Equations with Cholesky Factorization

The least squares problem (4.5) may be shown to be equivalent to solving the system (Rice, p. 148):

$$A^T A \underline{x} = A^T \underline{b}. \quad (4.6)$$

These are the "normal equations," and must be solved for \underline{x} .

One may attempt to solve for \underline{x} in (4.6) as:

$$\underline{x} = (A^T A)^{-1} A^T \underline{b}. \quad (4.7)$$

However, matrix inversion is comparatively inaccurate, very expensive, and rarely necessary. It is much better to solve (4.6) by Gaussian elimination. Gaussian elimination factors the matrix A into two matrices: $A = LU$, where L is a triangular matrix. Thus it is also called the "LU factorization." Gaussian elimination is extensively described in almost any book dealing with computer matrix manipulations (Rice, p. 33-34; Dongarra et al., p. 1.10-1.11; Forsythe et al., p. 32-41).

It is even better to note that $A^T A$ is positive definite if A is non-singular. The normal equations (4.6) may thus be solved by Cholesky factorization. Cholesky factorization is about half as expensive as Gaussian elimination (See Table I).

The Cholesky factorization of a positive definite matrix H is given by:

$$H = R^T R \quad (4.8)$$

where R is upper triangular with positive diagonal elements. In our case,

$$H = A^T A = R^T R. \quad (4.9)$$

To solve system (4.6) one may first solve

$$R^T \underline{w} = A^T \underline{b} \quad (4.10)$$

for \underline{w} , and then

$$R \underline{x} = \underline{w} \quad (4.11)$$

for \underline{x} . These two systems are triangular, and may be solved by back substitution rather than by matrix inversion.

A Cholesky factorization of the augmented matrix

$$\tilde{H} = (A, \underline{b})^T (A, \underline{b}) \quad (4.12)$$

can give both the solution \underline{x} and the norm of the residual \underline{r} with no further computation. See Section B.2.

The Cholesky factorization is widely described in the literature (Dongarra, et. al., p. 3.1, 3.9-3.10, 8.1-8.3; Rice, p. 46-48)

2. The QR Factorization

A square matrix H is orthogonal if $H^T H = I$, where I is the identity matrix. For an orthogonal matrix H , $H^T = H^{-1}$ and $HH^T = I$.

The QR decomposition of a matrix A is given by:

$$A = Q' \begin{bmatrix} R \\ 0 \end{bmatrix}. \quad (4.13)$$

Matrix A is m by n ; Q' is orthogonal and m by m , and R is upper triangular and n by n . It is assumed $m \geq n$.

If Q' is partitioned as $Q' = (Q, Q'')$, then the QR factorization of matrix A is given by:

$$A = QR, \quad (4.14)$$

where Q is m by n . This factorization is often all that is needed, so that the full decomposition need not be computed.

Then to solve $A\underline{x} \doteq \underline{b}$, multiply by Q^T on both sides to obtain:

$$\text{or} \quad Q^T A \underline{x} = Q^T \underline{b} \quad (4.15)$$

$$\underline{R} \underline{x} = Q^T \underline{b}. \quad (4.16)$$

This is a triangular system which may be solved by back substitution.

The residual \underline{r} may be found from the full QR decomposition (4.13). See Dongarra et. al., (1979, p. 9.2) for details.

If the diagonal elements of R are chosen to be positive, then for a non-singular matrix A , (4.14) represents a unique factorization, and the R matrix in (4.14) is identical to the R matrix obtained by Cholesky factorization. This is proved from

$$A^T A = (QR)^T QR = R^T Q^T QR = R^T R. \quad (4.17)$$

If Q and A are partitioned as $A = (A_1, A_2)$ and $Q = (Q_1, Q_2)$ where A_1 and Q_1 are k by n , and if R_{11} is the k by k leading principal submatrix of R , then the QR factorization of the truncated matrix A_1 is given by

$$A_1 = Q_1 R_{11}. \quad (4.18)$$

The least squares problem $A\underline{x} \doteq \underline{b}$ may be solved in a somewhat different manner as follows:

Form the augmented matrix

$$\tilde{A} = (A, \underline{b}) \quad (4.19)$$

The QR factorization of this matrix is given by

$$\tilde{A} = \tilde{Q}\tilde{R} \quad (4.20)$$

$$\tilde{Q} = (Q, q) \quad (4.21)$$

$$\tilde{R} = \begin{bmatrix} R & z \\ 0 & \rho \end{bmatrix} . \quad (4.22)$$

Then the solution \underline{x} is obtained from

$$R\underline{x} = \underline{z} . \quad (4.23)$$

The residual is

$$\underline{r} = A\underline{x} - \underline{b} = \rho q . \quad (4.24)$$

The norm of the residual is

$$\|\underline{r}\| = \rho . \quad (4.25)$$

There are several methods for obtaining the factorization $A = QR$. The most commonly known and most easily understood is Gram-Schmidt orthogonalization. This method, however, has numerical difficulties. Modified Gram-Schmidt orthogonalization overcomes the numerical difficulties of the Gram-Schmidt method, uses less storage, and lends itself to a pivoting scheme. It produces a Q matrix which may be numerically far from orthogonal; but the least-squares solution is nevertheless accurate (Rice, 1981, p. 152; Stewart, 1973, p. 217). Another method of obtaining the QR decomposition uses elementary plane rotations (Givens transformations). Perhaps the most widely used method at present uses elementary reflections (Householder transformations). This method is best with regard to speed and storage requirements. It produces an orthogonal Q matrix. It provides essentially the same accuracy as modified Gram orthogonalization, although numerical research indicates that the modified Gram-Schmidt algorithm may produce slightly more accurate results to the least squares problem as the residual size increases (Jordan, 1968; Wampler, 1969). These methods are widely described in the literature (Rice, 1981, p. 149-155;

Lawson and Hanson, 1974, p. 9-17, 53-62; Dongarra et al., 1979, p. 9.1-9.3, 9.13-9.17). Each of the methods above theoretically produces the same Q and R matrices, to within the signs of the rows and columns. The difference is in the numerical accuracy of the methods, and their speed and storage requirements.

3. The FG Factorization

The FG factorization of a matrix A is given by:

$$A = FG. \quad (4.26)$$

A is m by n, F is m by n with orthogonal columns, and G is n by n and upper triangular with ones on its diagonal.

Since F has orthogonal columns,

$$F^T F = D \quad (4.27)$$

where D is a diagonal matrix.

If we assume the QR factorization produces a R matrix with positive diagonal elements, then the QR and FG factorizations are related as follows:

$$\begin{aligned} FG &= QR \\ (FG)^T FG &= (QR)^T QR \\ G^T DG &= R^T R \\ \sqrt{D}G &= G\sqrt{D} = R \\ G &= R\sqrt{D}^{-1} \end{aligned}$$

Furthermore,

$$\begin{aligned} FG &= QR \\ FG &= Q\sqrt{D}G \\ F &= Q\sqrt{D}. \end{aligned}$$

Thus we have the following two relationships:

$$G = R\sqrt{D}^{-1} \quad (4.28)$$

$$F = Q\sqrt{D}. \quad (4.29)$$

Furthermore, since G has ones on its diagonal

$$\sqrt{d_{ii}} = r_{ii}. \quad (4.30)$$

Thus the FG factorization may be regarded simply as a variant of the QR factorization. The solution to the least-square problem $Ax \doteq b$ is given by:

$$G\underline{x} = D^{-1}F^T\underline{b}. \quad (4.31)$$

Alternately, if we let

$$\tilde{A} = (A, \underline{b}) \quad (4.32)$$

$$\tilde{F} = (F, \underline{f}) \quad (4.33)$$

$$\tilde{G} = \begin{bmatrix} G & \underline{w} \\ 0 & 1 \end{bmatrix} \quad (4.34)$$

then the solution to $A\underline{x} \doteq b$, where A is m by n , is given by:

$$G\underline{x} = \underline{w}. \quad (4.35)$$

The residual is

$$\underline{r} = A\underline{x} - \underline{b} = \underline{f} \quad (4.36)$$

The norm of the residual is

$$\|\underline{r}\| = \|\underline{f}\| = \sqrt{d_{n+1, n+1}}. \quad (4.37)$$

where $d_{n+1, n+1}$ is the last diagonal element of $\tilde{D} = \tilde{F}^T\tilde{F}$.

The FG factorization is of special interest due to treatment of it in literature regarding Prony's method (Householder, 1949; Van Blaricum and Mitra, 1977). A. S. Householder suggested an algorithm for computing the FG factorization. It is suspected that the algorithm he suggested was for instructive purposes only, and was never intended for numerical

use. The algorithm he suggested is very similar to the unmodified Gram-Schmidt algorithm, and would thus have the same numerical difficulties. For comparative purposes, the algorithm suggested by Householder for the FG factorization, and the unmodified Gram-Schmidt algorithms are given below. It must be emphasized that these algorithms are inferior and should not be used. A modified FG factorization which has the advantages of the modified Gram-Schmidt algorithm can easily be developed by comparing the two algorithms below with the modified Gram-Schmidt algorithm presented in Stewart (1973, p. 217).

The algorithms assume A is m by n and non-singular. The notation $\underline{a}_k \leftarrow \underline{f}_k$ indicates that \underline{f}_k should replace \underline{a}_k .

Algorithm 4.1: FG Factorization: $A = FG$ (Householder, 1949, p. 11-12).

For $k = 1, 2, \dots, n$ do:

$$g_{ik} = (1/d_{ii}) \underline{f}_i^T \underline{a}_k \quad (i = 1, 2, \dots, k-1)$$

$$\underline{f}_k = \underline{a}_k - \sum_{i=1}^{k-1} g_{ik} \underline{f}_i$$

$$g_{kk} = 1$$

$$d_{kk} = \underline{f}_k^T \underline{f}_k .$$

Algorithm 4.2: Gram-Schmidt Orthogonalization: $A = QR$ (Stewart, 1974, p. 216).

For $k = 1, 2, \dots, n$ do:

$$r_{ik} = \underline{q}_i^T \underline{a}_k \quad (i = 1, 2, \dots, k-1)$$

$$\underline{q}_k = \underline{a}_k - \sum_{i=1}^{k-1} r_{ik} \underline{q}_i$$

$$r_{kk} = \|\underline{q}_k\|$$

$$q_k \leftarrow \frac{q_k}{r_{kk}}$$

4. Singular Value Factorization

The singular value decomposition of a matrix A is given by

$$(U')^T A V = \begin{bmatrix} S \\ 0 \end{bmatrix} \quad (4.38)$$

for $m \geq n$. A is m by n , U is an orthogonal m by m matrix, V is an orthogonal n by n matrix, and S is a diagonal n by n matrix with non-negative diagonal elements. The diagonal elements s_{ii} are the singular values of A and are uniquely determined by A .

If U' is partitioned as $U' = (U, U'')$, then the singular value factorization is given by

$$A = USV^T \quad (4.39)$$

where U is m by n . This factorization is often all that is needed, so that the full decomposition need not be computed.

The singular values of A are the square roots of the eigenvalues of $A^T A$, and the columns of V are the eigenvectors of $A^T A$. The eigenvectors of AA^T are the columns of U . That is, the eigenvector decompositions of $A^T A$ and AA^T are given by

$$A^T A = V(S^T S)V^T \quad (4.40)$$

and

$$AA^T = U(SS^T)U^T \quad (4.41)$$

The spectral norm of A is given by

$$\|A\| = \max \{ \|\underline{Av}\| : \|v\| = 1 \} = \max \{ s_{ii} \} \quad (4.42)$$

and the Frobenius norm is given by

$$\|A\|_F = \left[\sum_{i=1}^m \sum_{j=1}^n a_{ij}^2 \right]^{1/2} = \left[\sum_{i=1}^n s_{ii}^2 \right]^{1/2} \quad (4.43)$$

For a non-singular A, a useful condition number is given by

$$\kappa(A) = \max \{ s_{ii} \} / \min \{ s_{ii} \}. \quad (4.44)$$

This condition number is a measure of the sensitivity of the solution \underline{x} to the matrix A:

$$\min \{ s_{ii} \} \leq \frac{\|A\underline{x}\|}{\|\underline{x}\|} \leq \max \{ s_{ii} \}. \quad (4.45)$$

The solution to the least squares problem $A\underline{x} \doteq \underline{b}$ is obtained by solving for \underline{x} in the diagonal system:

$$S\underline{V}^T \underline{x} = \underline{U}^T \underline{b} \quad (4.46)$$

This is equivalent to solving for \underline{z} in:

$$S\underline{z} = \underline{U}^T \underline{b} \quad (4.47)$$

and then for \underline{x} in:

$$\underline{x} = \underline{V}\underline{z}. \quad (4.48)$$

The residual vector $\underline{r} = A\underline{x} - \underline{b}$ can be found from the full singular value decomposition (4.38). See Dongarra et. al., (1979, p. 11.3) for details.

The singular value factorization of $\tilde{A} = (A, \underline{b})$ is of interest. If A is m by n, then \tilde{A} is m by (n+1). Suppose further that \tilde{A} has rank n; i.e., \tilde{A} is singular. Then the n + 1 singular value is zero. Furthermore, since the columns of \tilde{V} are the eigenvectors of $\tilde{A}^T \tilde{A}$, and the singular values are the square roots of the eigenvalues of $\tilde{A}^T \tilde{A}$ (equation 4.40.) then by the definition of an eigenvector,

$$\tilde{A}^T \tilde{A} \underline{v}_{n+1} = 0 \quad (4.49)$$

where \underline{v}_{n+1} is the last column of \tilde{V} .

Now consider equations (3.12) in the form given in (3.18):

$$\underline{W} \underline{c}'_{-N} = -c_0 \underline{w}_{-N+1}. \quad (4.50)$$

The least squares solution by the normal equations is given by

$$\underline{W}^T \underline{W} \underline{c}'_{-N} = \underline{W}^T (-c_0) \underline{w}_{-N+1}. \quad (4.51)$$

Define:

$$\tilde{W} = (W, \underline{w}_{N+1}) \quad (4.52)$$

and

$$\underline{c}_N = \begin{bmatrix} c'_N \\ c_0 \end{bmatrix}. \quad (4.53)$$

Then (4.48) can be expressed equivalently as

$$\tilde{W}^T \tilde{W} \underline{c}_N = 0. \quad (4.54)$$

Comparison of (4.49) with (4.54) indicates that the coefficients for the polynomial (3.7) are given by setting $n = N$ and identifying $\underline{v}_{N+1} = \underline{c}_N$, that is:

$$c_0 = v_{N+1, N+1} \quad (4.55)$$

$$c_1 = v_{N, N+1}$$

.

.

$$c_N = v_{1, N+1}$$

(Van Blaricum and Mitra, p. 179).

In general, a least squares solution to $A\underline{x} \doteq \underline{b}$, where A is dimensioned m by n , is given from the singular value decomposition of $\tilde{A} = (A, \underline{b})$ as:

$$\begin{aligned} x_1 &= v_{1, n+1} / v_{n+1, n+1} \\ x_2 &= v_{2, n+1} / v_{n+1, n+1} \\ &\vdots \\ x_n &= v_{n, n+1} / v_{n+1, n+1}. \end{aligned} \quad (4.56)$$

However, (4.56) is a solution to the least squares problem $A\underline{x} \doteq \underline{b}$ only if (A, \underline{b}) is of rank n , where A is dimensioned m by n . If (A, \underline{b}) is of rank $n+1$, but is very nearly of rank n , then the singular value decomposition of (A, \underline{b}) provides an approximation to the least squares solution. The approximation becomes more accurate as (A, \underline{b}) comes "closer" to being

of rank n ; that is, as the $n+1$ singular value becomes closer to zero.

The singular value factorization is extensively described in the literature (Dongarra et al., p. 11.1-11.23; Lawson and Hanson, p. 18-27, 107-120, 196-198; Forsythe et al., p. 201-235; Stewart, 1973, p. 317-326).

D. The Measurement Rank of a Matrix

1. Definition

If a matrix A represents measured data, then it is known only to a finite "measurement precision." If b represents an actual data value, and a represents the measured value, then the measurement precision ϵ will satisfy

$$\epsilon \leq |a - b| . \quad (4.57)$$

To within the precision of the measurements then, the matrix A cannot be distinguished from the matrix B , where:

$$A = B + E. \quad (4.58)$$

A , B , and E are m by n matrices, and E is a completely arbitrary matrix, with

$$\max \{|e_{ij}|\} \leq \epsilon . \quad (4.59)$$

If B represents the actual matrix known to infinite precision, then A represents the measured matrix, and E represents the error in these measurements. A is the matrix B perturbed by E .

Note that in this definition it is assumed that each element of A is known to the same absolute precision. This is consistent with many physical measurement processes. For example, a voltmeter which is accurate to three significant digits, and truncates after the third digit would have (after normalization), $\epsilon = 10^{-3}$. The matrix E has elements uniformly distributed between 0 and 10^{-3} . If in addition we can subtract

the truncation bias (or the voltmeter rounds to the nearest digit rather than truncates) then $\epsilon = 10^{-3}/2$, and E has elements uniformly distributed between $-10^{-3}/2$ and $+10^{-3}/2$.

The rank K of the matrix A is the number of columns of A which are linearly independent. But A is known only to finite precision. Thus, we must define a "measurement rank" of matrix A as the rank of matrix $B = A - E$, over all choices of the matrix E. That is, the measurement rank of A is the true rank of any possible perturbation of A, where "possible" perturbations must be less than E.

The true rank of a matrix A known to infinite precision must be greater than or equal to the measurement rank of the same matrix A known only to finite precision.

2. An Upper Bound for the K+1 Singular Value

In order to determine the measurement rank of matrix A, we must determine when a diagonal element is "negligible." For the Cholesky factorization or the QR factorization, the negligible element will be some r_{ii} . For the FG factorization, the negligible element will be $r_{ii} = \sqrt{d_{ii}}$. For the singular value factorization the negligible element will be some s_{ii} . An element is negligible if some perturbation of A could make that element equal to zero.

If matrix A is of measurement rank K, then $s_{K+1,K+1}$ should be negligible. In this section an upper bound is obtained for $s_{K+1,K+1}$. In the following section an expected value for $s_{K+1,K+1}$ is obtained. In section 4 this will be related to $r_{K+1,K+1}$.

Let $A = B + E$. If their respective singular values are b_i , a_i , and e_i , and each set is labelled in non-increasing order, then

$$|b_i - a_i| \leq \|A - B\| = \|E\| = e_1 \quad (4.60)$$

(Lawson and Hanson, 1974, p. 25; Stewart 1973, p. 321).

It follows that the smallest matrix E such that B + E is singular satisfies $a_n = e_1$, where A is m by n, and it is assumed $m \geq n$. It can be shown that

$$\alpha \leq \|A\| \leq \sqrt{m} \alpha \quad (4.61)$$

where $\alpha = \max \{\|a_{ij}\|\}$ (Stewart, 1979, p. 183). For the singular value factorization of A, a_i is identified as s_{ii} , and s_{ii} is negligible when

$$s_{ii} \leq \sqrt{m} \epsilon \quad (4.62)$$

where ϵ is the measurement precision. Matrix A has measurement rank K if for only the first K elements, $s_{ii} > \sqrt{m} \epsilon$.

3. An Expected Value for the K+1 Singular Value

Consider once again the matrix $A = B + E$. E represents a noise matrix. Assume that the noise is uncorrelated with standard deviation σ . Assume that B is of rank K and dimensioned m by K+1. Reasoning similar to that used for equation (4.49) will produce

$$(B^T B) \underline{v}'_{K+1} = 0 \quad (4.63)$$

where \underline{v}'_{K+1} is the K+1 column of matrix V' of the singular value factorization of B, and is to be distinguished from \underline{v}_{K+1} which will be defined later.

The matrix $H = A^T A$ will have the i, j element given by

$$\begin{aligned} h_{ij} &= \underline{a}_i^T \underline{a}_j \\ h_{ij} &= (\underline{b}_i + \underline{e}_i)^T (\underline{b}_j + \underline{e}_j) \\ h_{ij} &= \underline{b}_i^T \underline{b}_j + \underline{e}_i^T \underline{b}_j + \underline{b}_i^T \underline{e}_j + \underline{e}_i^T \underline{e}_j \end{aligned} \quad (4.64)$$

The expected value of the first term in (4.64) is deterministic. The expected value of the second and third terms is zero because the noise is uncorrelated. For the same reason, the expected value of the fourth term is given by

$$\begin{aligned} E \{ e_i^T e_j \} &= 0 & ; i \neq j \\ &= m \sigma^2 & ; i = j. \end{aligned} \quad (4.65)$$

Thus, the expected value of H is:

$$E \{ H \} = B^T B + m \sigma^2 I, \quad (4.66)$$

where I is the identity matrix.

If η_{K+1} is the K+1 eigenvalue of H, and \underline{v}_{K+1} is the K+1 eigenvector of H, then

$$H \underline{v}_{K+1} = \eta_{K+1} \underline{v}_{K+1}. \quad (4.67)$$

The expected value of the left hand side is

$$E \{ H \underline{v}_{K+1} \} = E \{ (A^T A) \underline{v}_{K+1} \} \quad (4.68)$$

$$\approx E \{ (A^T A) \} E \{ \underline{v}_{K+1} \} \quad (4.69)$$

$$\approx (B^T B + m \sigma^2 I) \underline{v}'_{K+1} \quad (4.70)$$

$$= m \sigma^2 \underline{v}'_{K+1}. \quad (4.71)$$

Equation (4.69) follows from (4.68) only if the noise in $(A^T A)$ is uncorrelated with the noise in \underline{v}_{K+1} . This is not strictly the case, but it is a reasonable approximation. Equation (4.70) follows from (4.69) if $E \{ \underline{v}_{K+1} \} \approx \underline{v}'_{K+1}$. This is again a reasonable approximation.

Comparison of (4.71) with (4.67) would indicate that the expected value of η_{K+1} is approximately $m \sigma^2$. Since singular values of A are the square roots of eigenvalues of $A^T A$, the expected value for the K+1 singular value of A is approximately

$$E \{ s_{K+1,K+1} \} = \sqrt{m} \sigma. \quad (4.72)$$

Thus, (4.72) would suggest that a reasonable test for a negligible singular value would be

$$s_{ii} \leq \xi \sqrt{m} \sigma. \quad (4.73)$$

where $\xi \geq 1$ is a safety factor to allow for singular values above the mean. A good value for ξ would need to be determined empirically.

Equation (4.73) represents a slight improvement over (4.62). For example, if E represents uniform noise distributed between $-w/2$ and $+w/2$, then $\xi = w/2$, and $\sigma = w/\sqrt{12}$. Therefore, for $\xi < 2/\sqrt{12} = 1/\sqrt{3}$ (4.73) gives a closer bound than (4.62). For E representing Gaussian noise the improvement is more dramatic, since an upper bound for ξ will be placed at perhaps 2σ or 3σ .

The derivation of this section for the expected value has closely followed the derivation in Van Blaricum and Mitra (1979, p. 179-180).

4. An Approximation for the K+1 Singular Value

Rigorous bounds for $r_{K+1,K+1}$ of the QR factorization can also be determined (Stewart, 1977, p. 509-517). These bounds are not nearly as strict as those for $s_{K+1,K+1}$. However, generally r_{ii} is approximately the same size as s_{ii} .

In particular, r_{11}/r_{ii} is a good approximation to s_{11}/s_{ii} . The ratio r_{11}/r_{ii} is always an underestimate of s_{11}/s_{ii} , usually by a factor of less than three, and very rarely by a factor of more than ten. (Dongarra et al., p. 9.5, 9.25). Numerical examples in section F will illustrate this.

E. Comparison of the Matrix Factorizations

1. Speed and Accuracy

For arithmetic of a given precision, those methods which involve the formation of $H = A^T A$ (Gaussian elimination; Cholesky factorization; eigenvalue analysis) cannot provide as great an accuracy as the other methods. Lawson and Hanson (1974, p. 126-129) provide an example demonstrating this, and Golub and Wilkinson (1966, p. 143-144) provide a precise analysis on the bounds of the error. In the worst case, these methods require a computer word length twice that required for the other methods in order to provide the same accuracy.

In many cases, however, error due to machine round-off will be negligible compared to perturbation error of the matrix A . If machine round-off is negligible, those methods which form $A^T A$ are faster and should be used. Otherwise, those methods which do not use the normal equations are preferable.

Table I provides a summary of the number of arithmetic operations required for each of the methods. Gaussian elimination and Cholesky factorization provide equivalent accuracy, but since the Cholesky factorization is faster and provides the R matrix it is preferable. Also recall that for the factorization $A = QR$, the Householder transformation is fastest but the modified Gram-Schmidt algorithm may provide very slightly more accurate results.

It should be noted that for use with Prony's method, the matrix to be factored is the matrix defined in (4.2). This matrix has a very specialized structure, so it is possible that any of the algorithms could be modified to take advantage of that structure for improved speed or

TABLE I
OPERATIONS COUNTS FOR VARIOUS FACTORIZATIONS

Factor	Method	Operations Counts	Comments
$H=A^T A=LU$	Normal equations plus Gaussian elimination	$mn^2/2 + n^3/3$	Requires twice word length for same accuracy.
$H=A^T A=R^T R$	Normal equations plus Cholesky factorization	$mn^2/2 + n^3/6$	Requires twice word length for same accuracy.
$A=QR$	Gram-Schmidt orthogonalization (or modified Gram-Schmidt)	mn^2	Unmodified form should not be used.
$A=QR$	Orthogonalization by Givens transformations	$2n^3/3$	
$A=QR$	Householder transformations	$mn^2 - n^3/3$	
$A=FG$	Algorithm 4.1 (or modified Algorithm)	mn^2	Unmodified form should not be used.
$A=USV^T$	Singular value factorization	$2mn^2 + t$	
$A=QR$ $R=USV^T$	Householder transformations followed by Singular value factorization of R.	$mn^2 + 5n^3/3$ $+ t$	
$H=A^T A=VS^2V^T$	Eigenvalue analysis of normal equations	$4n^3/3 + t$	Requires twice word length for same accuracy.

The Operations Counts are taken directly from Lawson and Hanson (1974, p. 122). They are also derived elsewhere (Rice, 1981). Terms of lower order are neglected. t corresponds to the iterative phase of the QR algorithm.

memory requirements.

2. Ability to Determine Rank

Section D derived an upper bound and an expected value for the $K+1$ singular value. The Cholesky factorization, the QR factorization and the FG factorization all provide a value r_{ii} which is approximately the same size as s_{ii} . (In the FG factorization $r_{ii} = \sqrt{d_{ii}} = \|f_i\|$). These r_{ii} values cannot provide as much information as the s_{ii} . The numerical results in section F will demonstrate this. However, often the r_{ii} do provide adequate information to estimate the rank K .

F. Prony's Method and the Matrix Factorizations

1. Using the Factorizations

To clarify the preceding material of the chapter, two algorithms are presented. Algorithm 4.3 incorporates the singular value decomposition into Algorithm 3.1. Algorithm 4.4 incorporates the QR decomposition into Algorithm 3.1. It should be apparent from these two algorithms how to incorporate any of the other factorizations into Algorithm 3.1. It should be noted that Algorithm 4.3 produces only an approximation for the least squares solution for (3.12), which provides the coefficients for polynomial (3.7). (See equations 4.49 to 4.55). To obtain the true least squares solution would require a second factorization. By contrast, Algorithm 4.4 provides the true least squares solution. (See equations 4.19 to 4.25). This is because the QR factorization of A is equal to the truncated QR factorization of $\tilde{A} = (A, \underline{b})$. This is not true of the singular value factorization. However, in Section 2 it will be found

that the singular value factorization actually produces better coefficients for polynomial (3.7) than does the QR factorization.

Algorithm 4.3: Prony's Method Using the Singular Value Factorization.

1. Define the matrix:

$$W_{\gamma k} = \begin{bmatrix} x_0 & x_1 & \cdots & x_{k-1} \\ x_1 & x_2 & \cdots & x_k \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ x_{\gamma-1} & x_{\gamma} & \cdots & x_{k+\gamma-1} \end{bmatrix} \quad (4.74)$$

2. Factor W as:

$$W_{\gamma k} = USV^T \quad (4.75)$$

3. Search for a negligible s_{ii} .

If a negligible s_{ii} is not found, increase k and begin from step 1.

If s_{ii} is negligible and $i > k$, let $k=i$, and begin from step 1.

If s_{kk} is the first negligible value, then $N = k-1$, and continue.

4. Let

$$\begin{aligned} c_0 &= v_{N+1, N+1} \\ c_1 &= v_{N, N+1} \\ &\cdot \\ &\cdot \\ c_N &= v_{1, N+1} \end{aligned} \quad (4.76)$$

(This is an approximation).

The vector \underline{c}'_N is given by

$$\underline{c}'_N = (c_N, c_{N-1}, \dots, c_1)^T. \quad (4.77)$$

5. Solve for the roots μ_i , $i=1, 2, \dots, N$ of the polynomial:

$$c_0 \mu^N + c_1 \mu^{N-1} + \dots + c_{N-1} \mu + c_N = 0. \quad (4.78)$$

6. Solve for the λ_i , $i=1, 2, \dots, N$ from

$$\lambda_i = (1/\tau) \ln(\mu_i) \quad (4.79)$$

7. Define the matrix and vectors:

$$V_M = \begin{bmatrix} 1 & 1 & \dots & 1 \\ \mu_1 & \mu_2 & & \mu_N \\ \mu_1^2 & \mu_2^2 & & \mu_N^2 \\ \vdots & \vdots & & \vdots \\ \mu_1^{M-1} & \mu_2^{M-1} & & \mu_N^{M-1} \end{bmatrix} \quad (4.80)$$

$$\underline{x} = (x_0, x_1, \dots, x_{M-1})^T \quad (4.81)$$

$$\underline{p} = (p_1, p_2, \dots, p_N)^T \quad (4.82)$$

where $M = N + \gamma$

8. Solve for p_i , $i=1, 2, \dots, N$ from the linear system

$$V_M \underline{p} = \underline{x}. \quad (4.83)$$

Algorithm 4.4: Prony's Method Using the QR Factorization.

1. Define the matrix and vectors:

$$W_{\gamma k} = \begin{bmatrix} x_0 & x_1 & \dots & x_{k-1} \\ x_1 & x_2 & \dots & x_k \\ \vdots & \vdots & & \vdots \\ x_{\gamma-1} & x_{\gamma} & \dots & x_{k+\gamma-1} \end{bmatrix} \quad (4.84)$$

2. Factor W as:

$$W_{\gamma k} = QR \quad (4.85)$$

3. Search for a negligible r_{ii} .

If a negligible r_{ii} is not found, increase k and begin from step 1.

If r_{ii} is the first negligible value, let $N = i - 1$, and continue.

4. Let

$$\underline{c}'_N = (c_N, c_{N-1}, \dots, c_1)^T \quad (4.86)$$

$$c_0 = 1.0.$$

Let R_{N+1} be the leading principle submatrix of R , and express

R_{N+1} as:

$$R_{N+1} = \begin{bmatrix} R' & z \\ 0 & \rho \end{bmatrix} \quad (4.87)$$

Then solve for \underline{c}'_N from the triangular system

$$R' \underline{c}'_N = \underline{z}. \quad (4.88)$$

5. Continue with step 5 of Algorithm 4.3

2. The Effectiveness of the Factorizations

The major question of this chapter is whether the various factorizations can produce a good estimate of the number of exponential modes, N , in the original data. The preceding material in the chapter has been preparatory to answering this question. The remainder of the chapter examines four specific data functions in order to give a qualitative answer to this question. The first example primarily is intended to show the inferior numerical properties of Algorithm 4.1. This algorithm is the FG variant of unmodified Gram-Schmidt orthogonalization. The remaining examples compare the QR factorization (via Householder transformations) with the singular value factorization.

Each data function is of the form

$$z_k = p_1 e^{\lambda_1 k \tau} + p_2 e^{\lambda_2 k \tau} + \dots + p_N e^{\lambda_N k \tau} + v_k, \quad k = 0, 1, \dots, M-1 \quad (4.89)$$

Prony's method is used to fit each of these data functions to a function

of the form

$$\hat{z}(t) = \hat{p}_1 e^{\hat{\lambda}_1 t} + \hat{p}_2 e^{\hat{\lambda}_2 t} + \dots + \hat{p}_N e^{\hat{\lambda}_N t}. \quad (4.90)$$

If reference is made to Figure 1, then $\hat{z}(t)$ may be identified with $\hat{x}(t)$, and $x(t)$ is identified as

$$x(t) = p_1 e^{\lambda_1 t} + p_2 e^{\lambda_2 t} + \dots + p_n e^{\lambda_n t}. \quad (4.91)$$

Thus, the function in (4.91) is corrupted in (4.89) by noise v_k . It is assumed that the noise $w(t)$ in Figure 1 may be modelled as being incorporated in the v_k . Excepting where there may be confusion, the $\hat{}$ notation for p_i and λ_i will be dropped. The Prony solution (4.90) is computed in a fashion similar to Algorithm 4.3 or 4.4. Matrix (4.74) or (4.84) is formed dimensioned γ by ψ , and the values for γ and ψ are noted in each table. For a correct Prony solution, $\psi = N+1$, and $\gamma = M-N$. Step 3 of each algorithm was omitted. The sampling period is τ .

The v_k are produced by a pseudo-random number generator, and are either uniformly distributed with zero mean and width w (equivalent to standard deviation σ_v), or normally distributed with zero mean and standard deviation σ_v . The "strength" of the noise refers to the size of σ_v . Uniform noise could represent round-off error (truncation error) of a measuring device. In this case, the comments of section C.1 are relevant. If the measuring device has d digits of accuracy, and if any truncation bias can be subtracted from the measurements, then

$$\epsilon = w/2 = 10^{-d}/2. \quad (4.92)$$

The relationship between the width w of a uniform distribution and its standard deviation σ_v is given by

$$\sigma_v = (1/\sqrt{12})w. \quad (4.93)$$

The d_i are the diagonal terms given by

$$\begin{aligned} d_{i(\text{FG})} &= \sqrt{d_{ii}} \\ d_{i(\text{QR})} &= r_{ii} \\ d_{i(\text{SV})} &= s_{ii} \end{aligned} \quad (4.94)$$

for the FG, QR, and singular value factorizations respectively.

The residual ρ is the residual norm from computing the c_i in step 4 of either algorithm, that is,

$$\rho = \left\| W_{\gamma N} \underline{c}'_N - \underline{w}_{N+1} \right\| \quad (4.95)$$

where \underline{w}_{N+1} is the $N+1$ column of $W_{\gamma, N+1}$, and the remaining terms are identified in Algorithm 4.3 or 4.4. This residual was explicitly computed, but it is seen to be about equal to d_{N+1} for the FG and QR factorizations. (Equations 4.25 and 4.37). The residual ζ is the norm of the vector difference of the measured data and the fitting function, that is

$$\zeta = \left\| \hat{\underline{z}} - \underline{z} \right\| \quad (4.96)$$

where

$$\hat{\underline{z}} = (\hat{z}(0), \hat{z}(\tau), \dots, \hat{z}[(M-1)\tau])^T \quad (4.97)$$

$$\underline{z} = (z_0, z_1, \dots, z_{M-1})^T. \quad (4.98)$$

These residuals give an indication of the accuracy of the solution.

All computations were performed in double precision on an IBM 370. Double precision represents 14 hexadecimal digits (about 16.8 decimal digits). Values in the table are shown to whatever precision is necessary for the relevant discussion. The programming language used was FORTRAN IV. Algorithm 4.1 was used for the FG factorization. The QR factorization and singular value factorization were performed with DQRDC and DSVDC of the LINPACK subroutines. DQRDC performs the QR decomposition

by means of Householder transformations. DSVDC performs the singular value decomposition by means of the "QR algorithm" which also uses Householder transformations. See Dongarra et al. (1979, p. 9.13-9.21, 11.6-11.17) for details.

It is useful in the discussion which follows to refer to the mean and standard deviation of the d_i values. Let these be denoted as

$$\begin{aligned}\mu_{d_i} &= \text{Mean of } d_i \\ \sigma_{d_i} &= \text{Standard Deviation of } d_i.\end{aligned}\quad (4.99)$$

In all of the research which was done, the following relations were found to be true:

$$\mu_{d_{N+1}(SV)} < \sqrt{\gamma} \varepsilon \quad (4.100)$$

$$\mu_{d_{N+1}(SV)} \approx \sqrt{\gamma} \sigma_v \quad (4.101)$$

$$\sigma_{d_{N+1}(SV)} \approx \sigma_v. \quad (4.102)$$

Relation (4.100) follows from (4.62) and (4.101) follows from (4.72).

Example 1: It was stated earlier that unmodified Gram-Schmidt orthogonalization (or the FG variant of it) shows inferior numerical properties. Table II demonstrates this. The data function examined is

$$z_k = 1.0e^{-3.0k\tau} + 1.0e^{-3.5k\tau} + 1.0e^{-4.0k\tau}, \quad k = 0, 1, \dots, M-1. \quad (4.103)$$

No noise is added. The QR and singular value factorizations are seen to extract the λ_i and p_i with about the same accuracy. Unmodified Gram-Schmidt orthogonalization (represented by the FG factorization) is considerably less accurate. The values for ρ and ξ also indicate its poor accuracy.

The d_i values for the FG and QR algorithms are seen to be about equal except in the critical $N+1$ value. This is the threshold value for

TABLE II

PRONY SOLUTION VALUES TO DEMONSTRATE INFERIORITY OF
UNMODIFIED GRAM-SCHMIDT ORTHOGONALIZATION

Data Function: $Z_k = e^{-3.0k\tau} + e^{-3.5k\tau} + e^{-4.0k\tau}$,
 $k = 0, 1, \dots, M-1$ (4.103)
 $M = 53, \gamma = 50, \psi = 4, \tau = 0.1$
 No Noise

Item	True Value	Unmodified Gram-Schmidt (FG factorization)	Householder Transforms (QR factorization)	Singular Value Factorization
λ_1	-3.0	-2.996	-2.99999 99996 39	-2.99999 99996 38
λ_2	-3.5	-3.481	-3.49999 99980 78	-3.49999 99980 74
λ_3	-4.0	-3.994	-3.99999 99993 53	-3.99999 99993 51
p_1	1.0	0.975	0.99999 99976 34	0.99999 99976 29
p_2	1.0	0.994	0.99999 99988 74	0.99999 99988 69
p_3	1.0	1.029	1.00000 00034 91	1.00000 00035 00
d_1	---	4.239	4.239	5.803
d_2	---	7.022 E-03	7.022 E-03	1.261 E-02
d_3	---	5.734 E-06	5.734 E-06	6.523 E-06
d_4	0.0	1.093 E-08	1.354 E-15	4.872 E-16
ρ	0.0	1.093 E-08	5.854 E-15	5.987 E-15
σ	0.0	2.421 E-07	3.383 E-14	3.423 E-14

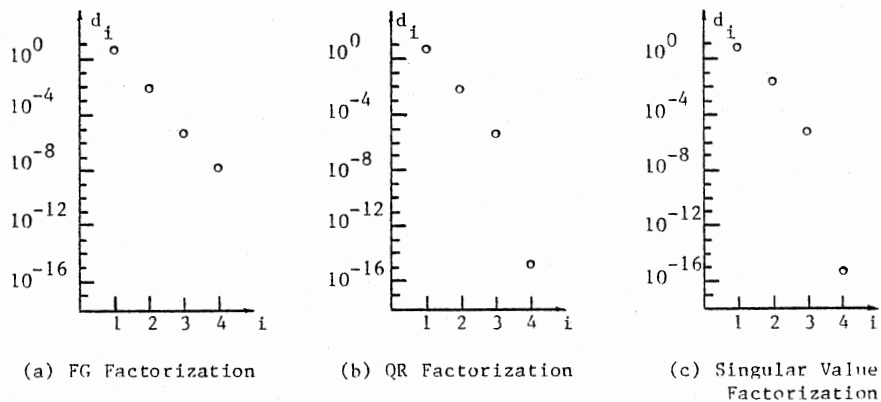


Figure 2. Diagonal Values from Table II

determining N . Figure 2 plots the values of d_i . The decrease in their magnitudes shows increasing dependency in the columns of matrix (4.2). Nevertheless, for the QR and singular value factorizations the $N+1$ value drops significantly more than the other values.

Since all computations were performed in double precision, it is questionable whether single precision computation for unmodified Gram-Schmidt orthogonalization would have produced meaningful results.

Example 2: Even with very noisy data a good estimate of the number of modes N can be obtained. Either the QR or the singular value factorization may be used, but greater confidence in the rank estimate will be obtained with the singular value factorization. Table III demonstrates Prony's solution for the data function

$$z_k = 1.0e^{-0.062k\tau} + 1.0e^{-0.402k\tau} + v_k, \quad k = 0, 1, \dots, M-1. \quad (4.104)$$

Four noise distributions are used with various σ_v . In every case, relations (4.100) to (4.102) hold true. A good test for N would be:

$$d_{N+1}(\text{SV}) \leq 1.5\sqrt{\gamma} \sigma_v, \quad (4.105)$$

according to equation (4.73). For the examples in Table III, this test would be accurate even for singular values which deviate by as much as $3\sigma_{d_i}$ from their mean μ_{d_i} .

The QR factorization also gives a good estimate of N . The suggested test is

$$d_{N+1}(\text{QR}) \leq 2.0 \sqrt{\gamma} \sigma_v. \quad (4.106)$$

However, for part (d) of Table III with uniform noise of width 0.1, this test is seen to be inferior to (4.105).

Part (d) of the table is also of interest because it compares the ability of Prony's method to extract the λ_i and p_i , with the ability of

TABLE III
 PRONY SOLUTION VALUES FOR AN EXAMPLE
 WITH N=2

Data Function: $z_k = e^{-0.062k\tau} + e^{-0.402k\tau} + v_k$, $k = 0, 1, \dots, M-1$ (4.104)
 $M = 27$; $r = 25$; $\psi = 3$; $\tau = 30$; 400 Monte Carlo trials

(a) $v_k = \text{Gaussian Noise}$, $\sigma_v = 0.000289$ (b) $v_k = \text{Uniform Noise}$, $w = 0.001$
 $(\sigma_v = 0.000289)$

Item	True Value	QR Factorization		Singular Value Factorization		QR Factorization		Singular Value Factorization		Item
		Mean	S. Dev.	Mean	S. Dev.	Mean	S. Dev.	Mean	S. Dev.	
λ_1	-0.062	-0.062007	0.00009	-0.062002	0.00009	-0.062011	0.00010	-0.062006	0.00010	λ_1
λ_2	-0.402	-0.402055	0.00094	-0.402029	0.00094	-0.402112	0.00099	-0.402086	0.00099	λ_2
p_1	1.00	1.00014	0.0017	1.00007	0.0017	1.00020	0.0018	1.00013	0.0018	p_1
p_2	1.00	0.99989	0.0017	0.99997	0.0017	0.99979	0.0018	0.99987	0.0018	p_2
d_1	---	2.64	0.30 E-3	3.40	0.42 E-3	2.64	0.28 E-3	3.40	0.41 E-3	d_1
d_2	---	2.67 E-1	0.25 E-3	3.17 E-1	0.26 E-3	2.67 E-1	0.27 E-3	3.17 E-1	0.28 E-3	d_2
d_3	---	2.13 E-3	0.38 E-3	1.39 E-3	0.25 E-3	2.14 E-3	0.33 E-3	1.40 E-3	0.21 E-3	d_3
$\sqrt{\gamma\sigma_v}$	---	---	---	1.44 E-3	---	---	---	1.44 E-3	---	$\sqrt{\gamma\sigma_v}$
ρ	---	2.13 E-3	0.38 E-3	2.13 E-3	0.38 E-3	2.14 E-3	0.33 E-3	2.14 E-3	0.32 E-3	ρ
ζ	---	1.49 E-3	0.27 E-3	1.49 E-3	0.27 E-3	1.52 E-3	0.22 E-3	1.52 E-3	0.21 E-3	ζ

(c) $v_k = \text{Uniform Noise}$, $w = 0.01$
 $(\sigma_v = 0.00289)$ (d) $v_k = \text{Uniform Noise}$, $w = 0.1$
 $(\sigma_v = 0.0289)$

λ_1	-0.062	-0.0624	0.0010	-0.0619	0.0011	-0.087	0.007	-0.0605	0.0108	λ_1
λ_2	-0.402	-0.4045	0.0110	-0.4018	0.0109	-0.777	0.276	-0.4296	0.1403	λ_2
p_1	1.00	1.0065	0.0189	0.9987	0.0192	1.380	0.101	0.976	0.190	p_1
p_2	1.00	0.9933	0.0186	1.0014	0.0188	0.618	0.106	1.027	0.191	p_2
d_1	---	2.64	0.29 E-2	3.40	0.42 E-2	2.65	0.28 E-1	3.40	0.42 E-1	d_1
d_2	---	2.68 E-1	0.27 E-2	3.18 E-1	0.25 E-2	3.14 E-1	0.23 E-1	3.50 E-1	0.25 E-1	d_2
d_3	---	2.16 E-2	0.32 E-2	1.42 E-2	0.21 E-2	1.95 E-1	0.26 E-1	1.39 E-1	0.21 E-1	d_3
$\sqrt{\gamma\sigma_v}$	---	---	---	1.44 E-2	---	---	---	1.44 E-1	---	$\sqrt{\gamma\sigma_v}$
ρ	---	2.16 E-2	0.32 E-2	2.16 E-2	0.33 E-2	1.95 E-1	0.26 E-1	2.14 E-1	0.33 E-1	ρ
ζ	---	1.56 E-2	0.25 E-2	1.52 E-2	0.23 E-2	2.26 E-1	0.39 E-1	1.55 E-1	0.29 E-1	ζ

the singular value factorization to estimate N . The standard deviation of λ_2 using the singular value factorization is about 30 percent of the value for λ_2 . But the estimate for N , as stated earlier, is accurate for d_i which vary from their mean μ_{d_i} even by as much as $3\sigma_{d_i}$. Thus, the Prony solution becomes meaningless (not even one significant digit of accuracy remaining) before the estimate for N becomes inaccurate. This conclusion is supported by other examples in this research.

The data function of part (a) of the table differs from that of part (b) only in that the noise distribution is Gaussian rather than uniform. An equivalent σ_v is used. The results are not seen to be significantly different for the mean and standard deviation of the Prony solution values.

In every case the residuals ρ for the QR factorization and the SV factorization are seen to be approximately equal, but the ρ for QR is very slightly smaller than the ρ for SV. This is apparent in part (d) of the table; in parts (a), (b), and (c) it is not apparent because adequate digits are not displayed. The inequality

$$\rho_{QR} < \rho_{SV} \quad (4.107)$$

is true not only for this example, but also for every example which occurred in this research. This result is expected since the singular value factorization of an augmented matrix produces only an estimate for the least squares problem (Section C.4).

It would thus seem that the singular value factorization would provide a less accurate estimate for the λ_i and p_i than does the QR factorization. But the opposite is true, as can be seen from part (c) of the table. In particular, the residual ζ is seen to be significantly lower

for the singular value factorization, and the estimates for λ_i and p_i are seen to be much better for the singular value factorization. Since Prony's method is not a true least squares technique (Chapter III), and the singular value factorization (as incorporated in Algorithm 4.3) is likewise not a true least squares technique, apparently the "errors" made in the singular value factorization have the effect of causing Prony's method to come closer to being a true least squares technique. The reason for this is unknown. It can only be attributed to interaction of a positive nature between the singular value factorization and Prony's method. It is possible that Prony's method by the singular value factorization is better than by the QR factorization only for some specific types of functions. These issues are areas for further research.

The singular value factorization can produce an exact least squares solution by equations (4.46) to (4.48). This entails factorization of matrix (4.74) dimensioned N by γ rather than $N+1$ by γ . Such a factorization should produce exactly the same least squares solution as the QR factorization. Thus, to obtain the improved estimates for λ_i and p_i , it is not sufficient simply to use the singular value factorization. Rather, the singular value factorization must be used as specified in Algorithm 4.3.

Example 3: When the λ_i are more closely spaced, the columns of the matrix (4.2) come closer to dependency. Thus, it is more difficult to obtain a good estimate for N . It is also more difficult to extract the λ_i and p_i . Table IV illustrates these difficulties. The data function being examined is

$$z_k = 1.0e^{-0.062k\tau} + 1.0e^{-0.200k\tau} + 1.0e^{-0.402k\tau} + v_k, \quad k=0, 1, \dots, M-1 \quad (4.108)$$

TABLE IV

PRONY SOLUTION VALUES FOR AN EXAMPLE WITH N=3

Item	True Value	QR Factorization		Singular Value Factorization	
		Mean	S. Dev.	Mean	S. Dev.
λ_1	-0.062	-0.0637	0.0007	-0.0618	0.0010
λ_2	-0.200	-0.226	0.016	-0.198	0.018
λ_3	-0.402	-0.434	0.030	-0.404	0.020
p_1	1.00	1.058	0.025	0.993	0.038
p_2	1.00	1.167	0.145	1.008	0.119
p_3	1.00	0.775	0.167	0.999	0.156
d_1	---	3.80	0.28 E-3	4.89	0.43 E-3
d_2	---	2.85 E-1	0.24 E-3	4.36 E-1	0.30 E-3
d_3	---	8.70 E-3	0.31 E-3	9.23 E-3	0.22 E-3
d_4	---	2.96 E-3	0.51 E-3	1.41 E-3	0.25 E-3
$\sqrt{\gamma\sigma_v}$	1.44 E-3				
ρ	---	2.96 E-3	0.51 E-3	3.03 E-3	0.55 E-3
ζ	---	4.63 E-3	0.15 E-3	2.15 E-3	0.82 E-3

This data function differs from that of (4.104) only in the addition of the second term of (4.108). The noise strength used is the same as that in Table III (b). Yet, the values for λ_i and p_i are not nearly as accurate in Table IV.

If (4.105) is again used as the test for N, an accurate N will be obtained. However, the $d_{N(SV)}$ and $d_{N+1(SV)}$ values are separated by less than an order of magnitude in Table IV, whereas in Table III (a) they were separated by over 2 orders of magnitude. The QR factorization could also give an accurate N if the test were changed to

$$d_{N+1(QR)} \leq 3.0 \sqrt{\gamma} \sigma_v. \quad (4.109)$$

But this illustrates the difficulty with the QR factorization. Exactly where should the threshold be? The singular value factorization has a clearly defined threshold. The QR factorization does not. Where there is little noise, or where the λ_i are widely separated, the QR factorization could provide an accurate estimate for N by the test

$$d_{N+1(QR)} \leq 5.0 \sqrt{\gamma} \sigma_v. \quad (4.110)$$

or even

$$d_{N+1(QR)} \leq 10.0 \sqrt{\gamma} \sigma_v. \quad (4.111)$$

However, where there is greater noise or there are closely spaced λ_i , the singular value factorization should be used.

The value ξ in equation (4.73) was chosen empirically to be 1.5 in equation (4.105). In the lack of any a priori knowledge, how would a proper value for ξ be chosen? If it is desired to set the threshold to detect $d_{N+1(SV)}$ which differ from the mean by $3\mu_{d_{N+1(SV)}}$, then the value for ξ would be

$$\xi = \frac{\mu_d + 3\sigma_d}{\mu_d}, \quad (4.112)$$

where the $N+1(SV)$ notation has been suppressed. But from relations (4.101) and (4.102), equation (4.112) becomes

$$\xi \approx \frac{\sqrt{\gamma} + 3}{\sqrt{\gamma}} . \quad (4.113)$$

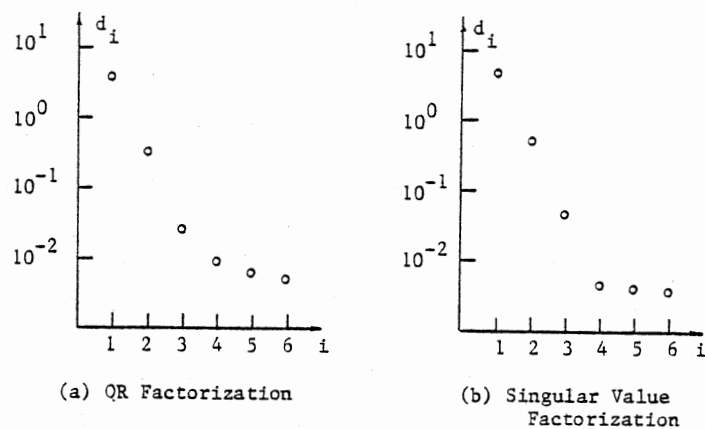
This would have given a value of 1.6 rather than 1.5 for equation (4.105). For small values of γ , and especially if the noise may be modelled as uniform noise, a better test may be given by:

$$d_{N+1(SV)} \geq \sqrt{\gamma} \varepsilon \quad (4.114)$$

according to equation (4.62). The only problem with these schemes is whether the d_N value will also fall below $\xi \sqrt{\gamma} \sigma_v$ or $\sqrt{\gamma} \varepsilon$.

Since the test for the d_{N+1} value involves either ε or σ_v , it is implied that these quantities are known. Often, however, the statistics of the noise v_k are unknown. In this case how can an estimate for N be made? Figure 3 shows the diagonal values from the factorizations of matrix (4.74) or (4.84) for data function (4.108). Values of $\gamma = 23$ and $\psi = 6$ were used. It is apparent that d_4 is almost an order of magnitude below d_3 , but then the d_i values decrease at a much slower rate. This would indicate that the proper choice for N is 3; and this would be a correct estimate. The value for $d_{4(SV)}$ is about 0.00153, and $\gamma = 23$. This would result in an estimate from (4.101) of $\sigma_v = 0.00032$. If the noise is assumed to be uniform, then from (4.93), $w = 0.0011$. These values for σ_v and w are very close to the actual values. Thus, the singular value factorization can produce an estimate for the standard deviation of the noise σ_v , as well as an estimate for the number of exponential modes N .

Example 4: The final example uses a function suggested by Van Blaricum and Mitra (1977). The Prony solution is given in Table V.



$$\text{Data Function: } z_k = e^{-0.062k\tau} + e^{-0.200k\tau} + e^{-0.402k\tau} + v_k,$$

$$k = 0, 1, \dots, M-1 \quad (4.108)$$

$M = 28, \gamma = 23, \psi = 6, 400$ Monte Carlo trials

$v_k = \text{Uniform Noise}, w = 0.001 \quad (\sigma_v = 0.000289)$

Figure 3. Mean Diagonal Values for an Example with $\psi = N + 3$

TABLE V

PRONY SOLUTION VALUES FOR AN EXAMPLE WITH COMPLEX POLES

Item		True Value	QR Factorization		Singular Value Factorization	
			Mean	S. Dev.	Mean	S. Dev.
Data Function: $z_k = e^{\lambda_1 k\tau} + e^{\lambda_2 k\tau} + \dots + e^{\lambda_{12} k\tau} + v_k,$						
$k = 0, 1, \dots, M-1 \quad (4.115)$						
$M = 212, \gamma = 200, \psi = 13, \tau = 1.0, 20$ Monte Carlo trials						
$\lambda_2 = \lambda_1^*, \lambda_4 = \lambda_3^*, \dots, \lambda_{12} = \lambda_{11}^*$						
$v_k = \text{Gaussian Noise}, \sigma_v = 0.005$						
Real Part	λ_1	-0.082	-0.0815	0.0007	-0.0821	0.0007
	λ_3	-0.147	-0.1387	0.0036	-0.1467	0.0036
	λ_5	-0.188	-0.1772	0.0063	-0.1870	0.0065
	λ_7	-0.220	-0.2159	0.0059	-0.2190	0.0061
	λ_9	-0.247	-0.2577	0.0071	-0.2483	0.0067
	λ_{11}	-0.270	-0.3092	0.0166	-0.2710	0.0144
Imaginary Part	λ_1	0.926	0.9268	0.0009	0.9250	0.0009
	λ_3	2.874	2.8770	0.0021	2.8734	0.0023
	λ_5	4.835	4.8334	0.0042	4.8352	0.0042
	λ_7	5.800	5.7969	0.0038	5.3003	0.0038
	λ_9	8.767	8.7340	0.0113	8.7561	0.0100
	λ_{11}	10.733	10.7351	0.0141	10.7305	0.0130
	d_{11}	---	0.477	0.008	0.374	0.008
	d_{12}	---	0.327	0.007	0.293	0.007
	d_{13}	---	0.117	0.010	0.068	0.006
	$\sqrt{\sigma_v}$	0.0707				
	σ	---	0.117	0.010	0.119	0.010

The data function is given by

$$z_k = 1.0e^{\lambda_1 k\tau} + 1.0e^{\lambda_2 k\tau} + \dots + 1.0e^{\lambda_{12} k\tau} + v_k, \quad k = 0, 1, \dots, M-1 \quad (4.115)$$

The λ_i are complex and occur in complex conjugate pairs. They are listed in the table. The p_i are not computed. The data function has 12 exponentials, but they are fairly widely separated because they are complex. If the test for N using the singular value factorization is once again:

$$d_{N+1} \leq 1.5 \sqrt{\gamma} \sigma_v,$$

the correct value for N would be obtained even for d_i which differ from the mean by as much as 5 standard deviations. For the QR factorization, the test could once again be

$$d_{N+1} \leq 3.0 \sqrt{\gamma} \sigma_v.$$

The λ_i estimates returned by the singular value factorization are significantly better than the estimates from the QR factorization, even though the residual ρ is slightly smaller for the QR factorization. This would again support the conclusion from Example 2 that the singular value factorization works better for Prony's method than the QR factorization.

To the precision shown in the table, the values obtained by the FG factorization (unmodified Gram-Schmidt) were the same as those indicated in the table for the QR factorization. Van Blaricum and Mitra (1977) however did not obtain meaningful results from the FG factorization for this identical data function. Their calculations were probably done in single precision. If so, this would account for the discrepancy because of the inferior numerical properties of unmodified Gram-Schmidt factorization.

G. Summary

The problem of determining the number of exponential modes N in a

data function has been reduced to the problem of determining the rank of a matrix. Various matrix factorizations can provide an estimate of the rank of a matrix, but the singular value factorization has been shown to provide the best estimate. When incorporated into Prony's method, the singular value factorization has also been shown to provide a better fitting function for the data than any of the other factorizations. The ability of the singular value factorization to provide an estimate for N surpasses the ability of Prony's method to provide an accurate fitting function.

CHAPTER V

DETERMINING THE OPTIMAL SAMPLE INTERVAL

A. Introduction

The purpose of this chapter is to determine an optimal sampling period for Prony's method. First, a sampling theory for Prony's method is presented. This sampling theory indicates bounds for the sample period, but does not indicate the optimal sample period. Second, some specific functions are simulated on the computer, and an optimal interval for their sample period is observed.

B. A Sampling Theory for Prony's Method

In Prony's Method the data function is assumed to be of the form

$$x_k = p_1 e^{\lambda_1 k \tau} + p_2 e^{\lambda_2 k \tau} + \dots + p_N e^{\lambda_N k \tau}, \quad k = 0, 1, \dots, M-1 \quad (5.1)$$

where τ is the (constant) sampling period. The data must be viewed over a total time interval of at least

$$T = (2N - 1) \tau.$$

During this time interval, the sample period τ must be chosen such that the slow exponential mode defined by some λ_i in (5.1) changes from one sample to another, while the fast exponential mode cannot be equal to zero at $t = T$ within the measurement precision ϵ defined in (4.57).

Assume first that the real parts of the λ_i are negative. That is, they represent decaying exponential functions. This assumption is consistent for any stable system. Assume further that the λ_i are real.

This assumption is not strictly necessary, but it simplifies the derivation to follow. For complex λ_i , the bounds stated later in (5.10) may be regarded as being for the real parts of the λ_i . Finally, assume that the λ_i are known, and ordered as follows:

$$\lambda_N < \lambda_{N-1} < \dots < \lambda_1 < 0 \quad (5.2)$$

where λ_N defines the fast exponential mode and λ_1 defines the slow exponential mode (λ_N is more negative than λ_1). The sample period τ must be chosen so that

$$p_1 - p_1 e^{\lambda_1 \tau} \geq \epsilon \quad (5.3)$$

where ϵ is the measurement precision defined in (4.57). Solving for τ yields

$$\tau \geq (1/\lambda_1) \ln(1 - \epsilon/p_1) \quad (5.4)$$

For the slow exponential mode this expression implies that $\lambda_1 \tau$ must be such that it is not rounded to zero to within the measurement precision defined by ϵ . The alternative is

$$0 \leq 1 - e^{\lambda_1 \tau} \leq \epsilon/p_1 \quad (5.6)$$

which implies

$$e^{\lambda_1 \tau} \doteq 1 \quad (5.7)$$

where the dot indicates "equal to" in the finite precision arithmetic. The use of these data, defined by τ such that (5.7) holds, implies that $\mu \doteq 1$ in equation (3.5). This could lead to numerical problems with Prony's method since matrix (3.13) could become singular.

The fast exponential also affects the choice of a proper sample period. In order to compute the exponential mode defined by λ_N , it must not be rounded to zero at the end of the data set defined by (3.4). That is, for $M = 2N$ in (3.5), the exponential mode defined by μ_N must appear.

Thus,

$$p_N e^{\lambda_N T} = p_N e^{(2N-1)\lambda_N \tau} \geq \epsilon \quad (5.8)$$

Solving for τ gives

$$\tau \leq 1/(2N-1)\lambda_N \ln(\epsilon/p_N) \quad (5.9)$$

Thus, the value of τ must be chosen according (5.9) so that the data set contains the fast mode.

The maximum value for the sample period is given by (5.9), while the minimum value is defined by (5.4). Combining these two produces the following bounds for τ :

$$(1/\lambda_1) \ln(1-\epsilon/p_1) \leq \tau \leq [1/(2N-1)\lambda_N] \ln(\epsilon/p_N) \quad (5.10)$$

The choice of a proper sample period is only possible when

$$(1/\lambda_1) \ln(1-\epsilon/p_1) \leq 1/(2N-1)\lambda_N \ln(\epsilon/p_N) \quad (5.11)$$

Inequality (5.11) can be written as

$$(2N-1)\lambda_N/\lambda_1 \leq \ln(\epsilon/p_N)/\ln(1-\epsilon/p_1) \quad (5.12)$$

where N is the number of modes in the data function, ϵ is the measurement precision, and λ_N and λ_1 define the fast and slow exponential modes respectively, and p_N and p_1 are the coefficients associated with λ_N and λ_1 . If the inequality defined by (5.12) cannot be satisfied, then there is no sample period for which Prony's method will work.

Equation (5.11) requires knowledge of the λ_i and p_i , which are not available when the identification problem is under study. However, often bounds on the λ_i will be known, and p_i may be assumed approximately equal to 1.0. In particular, many systems, including ecological systems, may be described by compartment models. Compartment models have definite bounds on the upper and lower eigenvalues, corresponding to λ_N and λ_1 .

Mulholland and Gowdy (1977, pp. 321-344) derived a sampling theory for the regression analysis method for such compartment models. Mulholland (1981, p. 51) has shown that the sampling bounds for Prony's method are more strict than for the regression analysis method. For M large, the maximum sample period for the regression analysis method may be nearly twice that of Prony's method. This may cause problems with the practical application of Prony's method which may explain why it is neglected in the literature. On the other hand, the regression analysis method requires all components of the state vector, whereas Prony's method requires only scalar measurements.

Under some conditions the upper bound for τ may be too strict. For example, consider a case where $N=2$, and

$$e^{(2N-2)\lambda_N\tau} \geq \varepsilon/p_N \geq e^{(2N-1)\lambda_N\tau} \quad (5.13)$$

so that τ does not satisfy (5.8). Then the equations (3.5) become

$$\begin{aligned} x_0 &= p_1 + p_2 \\ x_1 &= p_1\mu_1 + p_2\mu_2 \\ x_2 &= p_1\mu_1^2 + p_2\mu_2^2 \\ x_3 &= p_1\mu_1^3 + 0 \end{aligned} \quad (5.14)$$

where the last term $p_2\mu_2^3$ is missing. Then if λ_1 and λ_2 are sufficiently separated ($\mu_1^3 \gg \mu_2^3$) a Prony solution of (5.14) may still give a good estimate for λ_1 and λ_N . For example, consider the data functions given in Table VI. Functions (a) and (c) are as (5.14). Function (b) is as (5.14), but also missing the term $p_2\mu_2^2$. The Prony solution for (a) is relatively accurate, even though τ does not satisfy (5.8). The Prony solution for (b) is relatively inaccurate, for apparently the $p_2\mu_2^2$ term is crucial. The Prony solution for (c) is also poor apparently because the λ_i are too

closely spaced.

It would thus seem that the term $(2N - 1)$ in the inequalities (5.9) through (5.12) could be replaced with $(2N - \delta)$ where δ is determined by the value of N and the separation of the λ_i . In particular, (5.10) becomes

$$(1/\lambda_1) \ln(1 - \epsilon/p_1) \leq \tau \leq [1/(2N - \delta)\lambda_N] \ln(\epsilon/p_N) \quad (5.15)$$

The minimum value for δ is 1, and the maximum value for δ is $(N-1)$. It is to be expected that in general, δ will be much closer to 1 than to $(N-1)$.

TABLE VI
PRONY SOLUTION VALUES FOR τ TOO LARGE

Data Function	Prony Solution	
	λ_1	λ_2
(a) $x_k = \begin{cases} e^{-0.062k\tau} - e^{-0.402k\tau}, & k = 0, 1, 2 \\ e^{-0.062k\tau}, & k = 3 \end{cases}$	-0.060	-0.412
(b) $x_k = \begin{cases} e^{-0.062k\tau} - e^{-0.402k\tau}, & k = 0, 1 \\ e^{-0.062k\tau}, & k = 1, 2 \end{cases}$	-0.076	-0.306
(c) $x_k = \begin{cases} e^{-0.200k\tau} - e^{-0.402k\tau}, & k = 0, 1, 2 \\ e^{-0.200k\tau}, & k = 3 \end{cases}$	-0.180	-0.468

C. Some Numerical Investigation

The data functions examined in this section were simulated on a computer as

$$z_k = z(k\tau) = p_1 e^{\lambda_1 k\tau} + p_2 e^{\lambda_2 k\tau} + \dots + p_N e^{\lambda_N k\tau} + v_k$$

$$k = 0, 1, \dots, M-1. \quad (5.16)$$

The λ_i are assumed to be real. For notational definiteness equations will always be expressed such that $\lambda_1 > \lambda_2 > \dots > \lambda_N$ (λ_N is the most negative value).

Prony's method is used to fit each data function of the form (5.16) to an exponential function of the form

$$\hat{z}(t) = \hat{p}_1 e^{\hat{\lambda}_1 t} + \hat{p}_2 e^{\hat{\lambda}_2 t} + \dots + \hat{p}_N e^{\hat{\lambda}_N t}. \quad (5.17)$$

Equations (5.16) and (5.17) are identical to (4.89) and (4.90). The discussion regarding those equations in Section IV.F.2 is therefore relevant. In particular, equations (4.89) through (4.93) remain true. The method of generating v_k remains the same. The $\hat{}$ notation for p_i and λ_i in equation (5.17) is dropped excepting where it may cause confusion.

Numerous simulations of (5.16) were performed in order to obtain the information presented in the following discussion. Most of these simulations involved repeated Monte Carlo trials in order to obtain meaningful statistics. The number of Monte Carlo trials used is indicated in the relevant figures. Prony's solution for the data points simulated by (5.16) was performed using the singular value factorization as indicated in Algorithm 4.3. Step 3 of the algorithm was omitted. Matrix (4.74) was formed dimensioned γ by ψ , and the values for γ and ψ are given in each table or figure. For a correct Prony solution, $\psi = N+1$, and $\gamma = M-N$.

Where Monte Carlo trials are used, it is useful to refer to the mean and standard deviation of various estimates. If the item of interest is y , and its estimate is \hat{y} , then the mean and standard deviation of y will be denoted as

$$\begin{aligned}\sigma_{\hat{y}} &= \text{mean of } y \\ \mu_{\hat{y}} &= \text{standard deviation of } \hat{y}.\end{aligned}\quad (5.18)$$

The normalized mean error e_y and normalized standard deviation s_y are often of greater interest than $\mu_{\hat{y}}$ and $\sigma_{\hat{y}}$. These unitless quantities are defined as

$$\begin{aligned}e_y &= (\mu_{\hat{y}} - y) / y \\ s_y &= \sigma_{\hat{y}} / y.\end{aligned}\quad (5.19)$$

Note that the $\hat{}$ notation on the left side of equations (5.19) is dropped. For application to Prony's method, the variable y will typically be replaced with λ_i or p_i .

The quantities of (5.19) provide a standard by which to judge how well \hat{y} estimates y . The smaller that the quantities of (5.19) are, the better is the estimate. If e_y is small in comparison to s_y , then s_y alone provides the standard. Typically, when s_y is plotted against τ , the curve is concave upward, as illustrated in Figure 4. The optimal sample period then occurs at the single point where s_y is a minimum. An interval for a near-optimal sample period is defined by

$$\tau_0^- \leq \tau \leq \tau_0^+$$

Where τ_0^- and τ_0^+ are defined in Figure 4. This interval will be referred to as the optimal sample interval. The factor of 2 in Figure 4 is arbitrary, but provides a reasonable standard for comparisons similar to the "3-decibel" point of filter functions. Often, the optimal sample interval

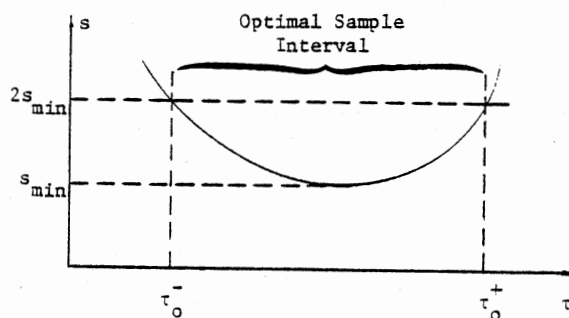


Figure 4. Definition of Optimal Sample Interval

for estimating λ_1 will be different from that for estimating λ_2 . In this case the overall optimal sample interval is given by the strictest combination of bounds. For example, if the optimal sample interval for λ_1 is given by

$$1.2 \leq \tau \leq 5.0$$

and the optimal sample interval for λ_2 is given by

$$2.0 \leq \tau \leq 7.0$$

then the overall optimal sample interval will be given by

$$2.0 \leq \tau \leq 5.0 \quad . \quad (5.21)$$

Relations (5.18) to (5.21) are important, for they define symbols and terms used throughout the remainder of the chapter, and in all of the remaining figures.

1. Proper Data Length

It is desired to extract as much information as possible from the data set defined in (5.16). For a given sample period τ , the information in the data set should increase as M increases. However, for $\text{Re}(\lambda_1) < 0$

(as is the case for stable systems), as M approaches infinity, z_M approaches 0. Thus, with noisy data, or with measurements of a finite precision, there will be some finite value of M beyond which there is no information in z_M .

Consider the data set given by

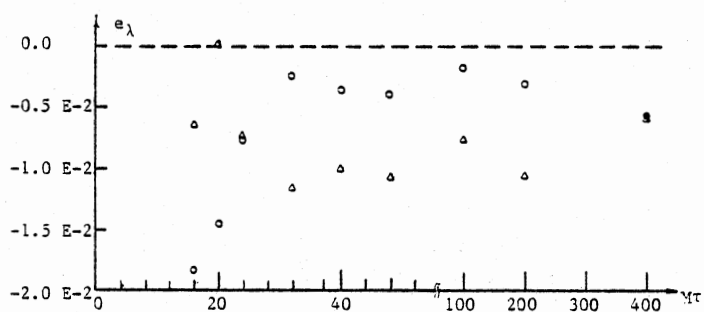
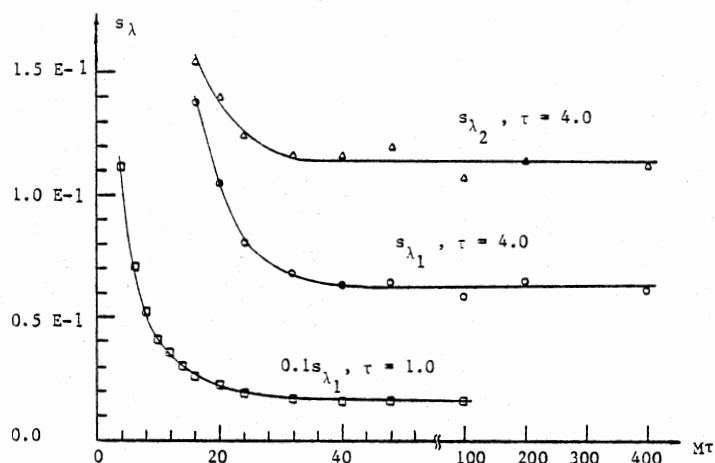
$$z_k = 0.252e^{-0.062k\tau} - 0.252e^{-0.402k\tau} + v_k, \quad k = 0, 1, \dots, M-1. \quad (5.22)$$

Let v_k be uniformly distributed with width $w = 0.01$, and let $\tau = 4.0$.

Figure 5 plots e_{λ_i} and s_{λ_i} against the data length $M\tau$. It is especially apparent from the s_{λ_i} curves that there is no more information to be gained from data lengths beyond about $M\tau = 30$. There appears to be significant information however at $M\tau = 20$. Table VII shows the values of the fast and slow components of z_k for $M\tau = 20$ and $M\tau = 30$. Since the noise distribution is uniform with width $w = 0.01$, it is apparent that the fast component (λ_2) is effectively zero for either $M\tau = 20$ or $M\tau = 30$. For the slow component (λ_1), information appears to be present at $M\tau = 30$, but this information is apparently of such poor quality that it does not improve the Prony solution.

The graph of s_{λ_1} for a sample period $\tau = 1.0$ shows similar shape, but with the "knee" of the curve shifted from about $M = 30$ to $M = 20$. Thus, the position of the knee is dependent on τ . This is probably due to the accuracy of the solution being dependent on τ , with the optimal choice for τ moving the "knee" of the curve further to the right than any other choice for τ . In the following section it is shown that $\tau = 4.0$ is a much better sample period than $\tau = 1.0$.

In order to gain as much information as possible, yet without wasting computation time, a data length of $M\tau = 40.0$ is generally used for the



Legend: $\circ, \square \hat{\lambda}_1 = -0.062$; $\triangle \hat{\lambda}_2 = -0.402$

Data Function: $z_k = 0.252e^{-0.062k\tau} - 0.252e^{-0.402k\tau} + v_k$,
 $k = 0, 1, \dots, M-1$ (5.22)

$\gamma = M-2$, $\phi=3$, 400 Monte Carlo Trials

$v_k =$ Uniform Noise, $w = 0.01$

Figure 5. Choice of Proper Data Length

TABLE VII

COMPONENT VALUES
FOR FIGURE 5

Component	$M\tau = 20$	$M\tau = 30$
$.252e^{-0.062 M\tau}$	0.073	0.039
$.252e^{-0.402 M\tau}$	8.1 E-5	1.5 E-6

remaining simulations to be discussed.

2. Optimal Sample Period

Example 1: Consider the data function given by

$$z_k = 0.252e^{-0.402k\tau} + v_k, \quad k = 0, 1, \dots, M-1 \quad (5.23)$$

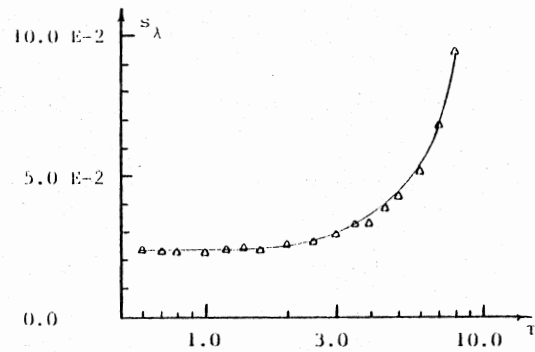
For Figure 6(a) the v_k are uniformly distributed with width $w = 0.01$. No attempt to find the lower bound for τ was made. It is apparent that s_λ is much greater than e_λ , and therefore the optimal sample interval is defined by s_λ alone as in inequality (5.20). Table X (at the end of the chapter) shows that this optimal sample interval is given by

$$? \leq \tau \leq 5.0.$$

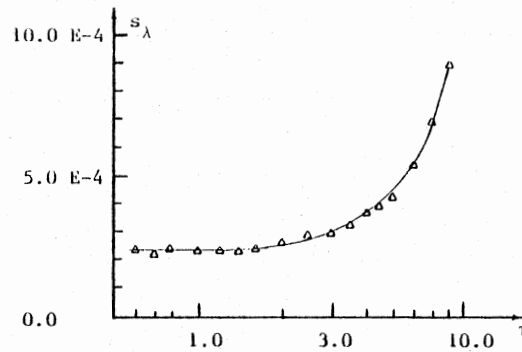
The e_λ values should be randomly distributed, but they show some structure for unknown reasons. For Figure 6(b) the v_k are uniformly distributed with width $w = 0.0001$. It is readily apparent that the shape of the s_λ curve is practically identical to that of Figure 6(a). This is an important result, for it indicates that until the noise strength becomes too great, the shape of the standard deviation curve is independent of the noise strength.

If the noise strength becomes great enough, then it is to be expected that the characteristics which cause the shape of the s_λ curve will no longer dominate, and the curve will take a different shape. This is apparent from Figure 6(c), where the width of the uniform noise is $w = 0.1$. Table X shows that inequality (5.10) is violated for $\tau = 4.0$. The shape of the e_λ curve is due to this inequality being violated.

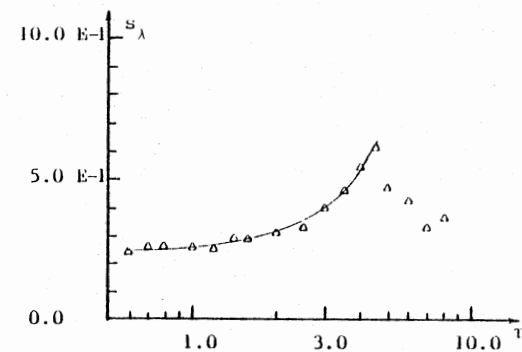
It is suspected that the shape of the standard deviation curves in Figures 6(a) and 6(b) are due to the statistical nature of sampled data



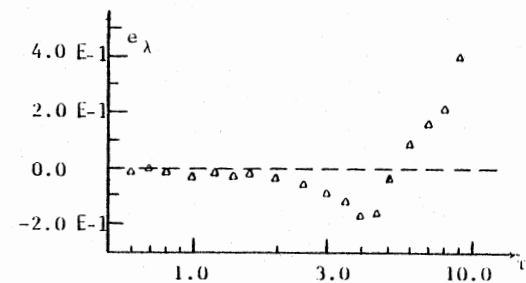
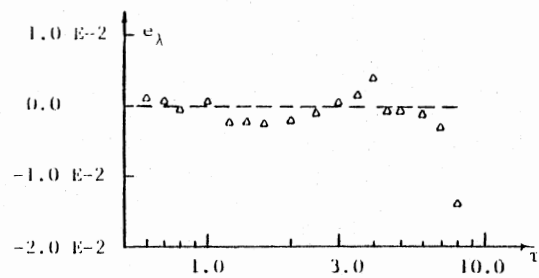
(a) $w = 0.01$



(b) $w = 0.0001$



(c) $w = 0.1$



$$\text{Data Function: } z_k = 0.252e^{-0.402k\tau} + v_k, \quad k = 0, 1, \dots, M-1 \quad (5.23)$$

$M = 40.0$; $\gamma = M-1$; $\psi = 2$; 400 Monte Carlo trials

$v_k = \text{Uniform Noise}$

Figure 6. Choice of Optimal τ for an Example with $N=1$

from an exponential function. Consider why this might be true. For a given value of k , z_k will be taken later in time as τ increases. But for a decaying exponential function, this implies that z_k is smaller in value. Since the noise power is constant with time, the signal to noise ratio of the sampled data decreases as τ increases. The shape of the curves in Figure 6 could be due to this statistical property of sampled data. However, the shape of the curves could also be due to the statistics of processes within Prony's method.

Example 2: Consider once again the data function given by (5.23) and repeated here:

$$z_k = 0.252e^{-0.062k\tau} - 0.252e^{-0.402k\tau} + v_k, \quad k = 0, 1, \dots, M-1. \quad (5.24)$$

Let v_k be uniformly distributed with width 0.01. Figure 7(a) and (b) shows the normalized mean error and standard deviation for λ_1 , λ_2 , p_1 , and p_2 . Table X shows that the overall optimal sample interval is given by

$$1.4 \leq \tau \leq 5.0. \quad (5.25)$$

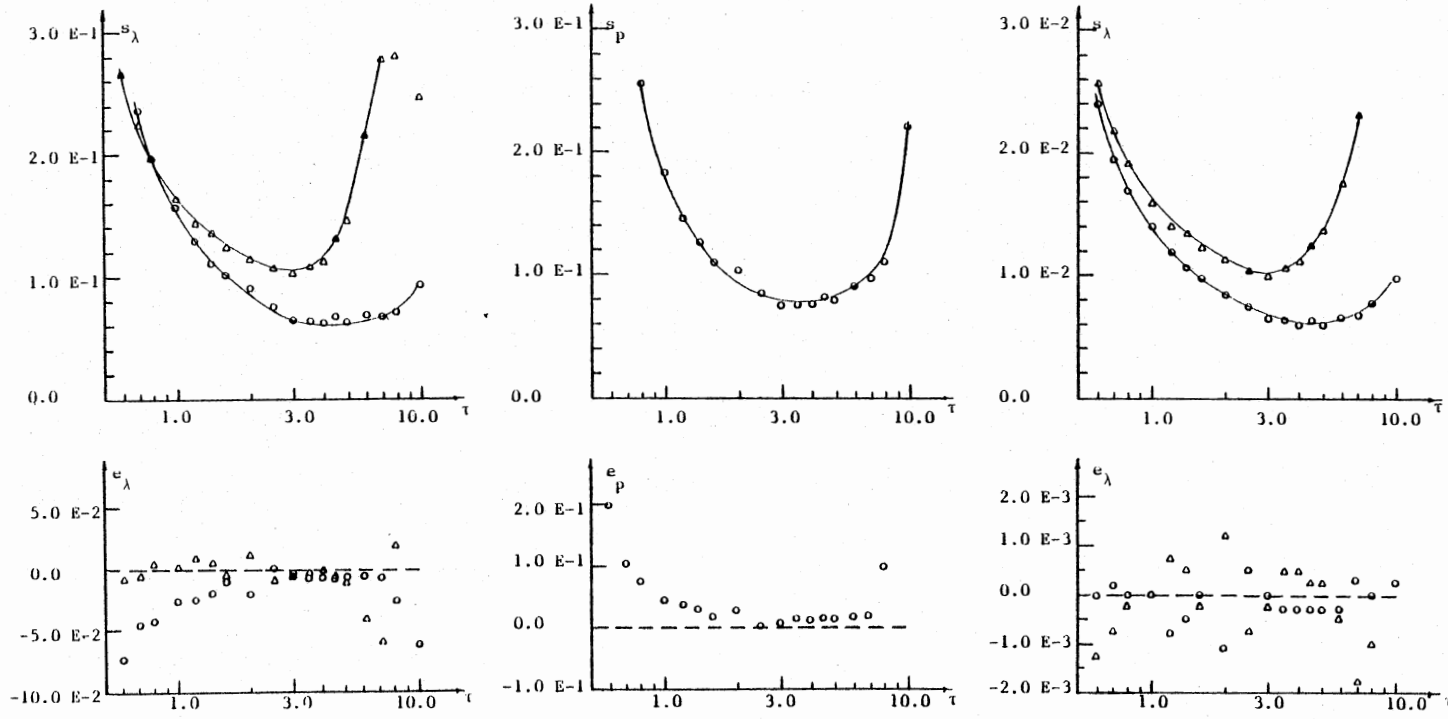
This interval violates the bounds given by (5.10) as

$$0.32 \leq \tau \leq 3.3.$$

But the data function (5.24) is very similar to that given in Table VI (a), and for that function it was noted that inequality (5.15) gives the proper bounds with $\delta = 2$. This would give an interval of

$$0.32 \leq \tau \leq 4.9. \quad (5.26)$$

Thus, (5.25) and (5.26) are in near agreement. Figure 7(a) also shows that values for e_{λ_2} are dramatically larger for $\tau > 5.0$. The τ_0^+ bound for λ_2 is much more strict than for λ_1 . This is reasonable, since as τ increases, the exponential term associated with λ_N becomes inaccurate



(a) $w = 0.01$; Statistics for λ

(b) $w = 0.01$; Statistics for p_1
(Statistics for p_2 almost identical)

(c) $w = 0.001$; Statistics for λ

Legend: $\circ \hat{\lambda}_1 = -0.062, \hat{p}_1 = 1.0; \triangle \hat{\lambda}_2 = -0.402.$

Data Function: $z_k = 0.252e^{-0.062k\tau} - 0.252e^{-0.402k\tau} + v_k, k = 0, 1, \dots, M-1$ (5.24)

$M\tau = 40.0; \gamma = M-2; \psi = 3; 400$ Monte Carlo trials

$v_k =$ Uniform Noise

Figure 7. Choice of Optimal τ for an Example with $N=2$

more quickly.

For $\tau \geq 7$, some of the computed values for the λ_i were negative or complex. Since this would result in complex λ_i , these values were not used and did not contribute to the statistics. As τ increased, the number of unused values increased. It is of note that this difficulty occurred outside the bounds given by (5.15).

The optimal sample interval for p_i is given from Figure 7(b) and inequality (5.20) as

$$1.2 \leq \tau \leq 8.0 \quad (5.27)$$

This interval is less strict than that observed for the λ_i , and therefore contributes nothing to the overall optimal sample interval. This behavior is typical of the other data functions also, therefore, the s_p and e_p curves are not shown in the remaining figures. It is of note, however, that the e_p curve has a definite structure, and that in the optimal sample interval the estimates for p_i have a bias of about 0.003 from Figure 7(b) and equation (5.19).

Figure 7(c) repeats 7(a), excepting that the noise width is given by $w = 0.001$. Once again it is noted that the shape of the standard deviation curve is independent of the noise strength.

For $\tau \geq 3$, the shape of the curve for s_{λ_2} is about the same as that from Figure 6(a). The fast exponential dominantly effects the statistics of the Prony solution for the larger values of τ .

Figure 8 shows the statistics for Prony's solution for τ for the function

$$z_k = 0.252e^{-0.062k\tau} + v_k, \quad k = 0, 1, \dots, M-1 \quad (5.28)$$

A comparison of Figures 6 and 7 would indicate that even for the smaller

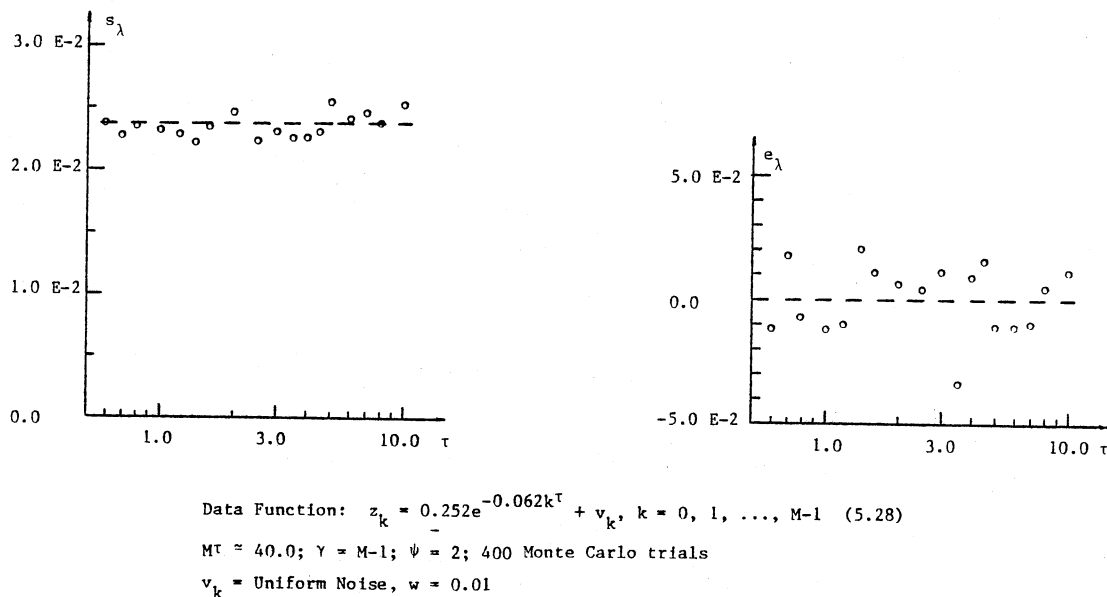


Figure 8. Choice of Optimal τ for a Second Example with $N=1$

values of τ , the slower exponential does not by itself dominantly affect the statistics shown in Figure 7. Rather, the statistics in Figure 7 are due to the interaction of the fast and slow exponentials.

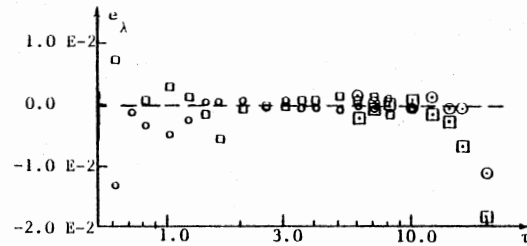
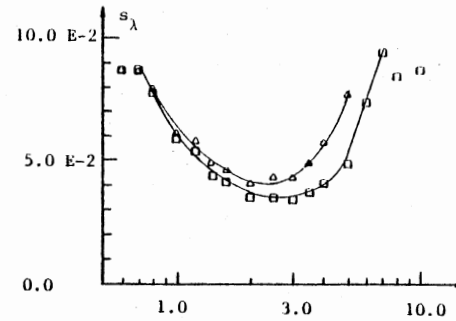
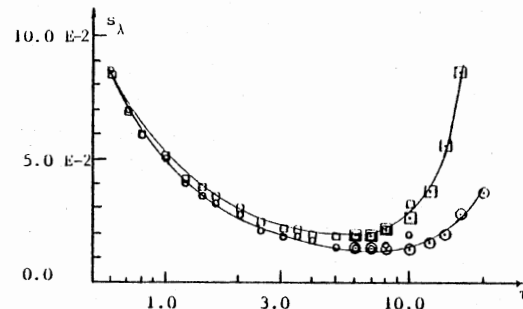
Example 3: As λ_N decreases, one would expect τ_0^+ to increase. This is in fact what happens. Figure 9(a) shows the normalized mean error and standard deviation for the Prony solution of λ_i for the data function

$$z_k = 0.252e^{-0.062k\tau} - 0.252e^{-0.200k\tau} + v_k, k = 0, 1, \dots, M-1 \quad (5.29)$$

The optimal sample period is given in Table X as

$$2.5 \leq \tau \leq 12.0 \quad (5.30)$$

A comparison of Figure 9(a) and Figure 7(a) shows the same general shape for the standard deviation curves, but with the right-hand boundary shifted to the right.



(a) Data Function:

$$z_k = 0.252e^{-0.062k\tau} - 0.252e^{-0.200k\tau} + v_k,$$

$$k = 0, 1, \dots, M-1$$

Legend: $\circ \lambda_1 = -0.062, M_1 = 40.0$

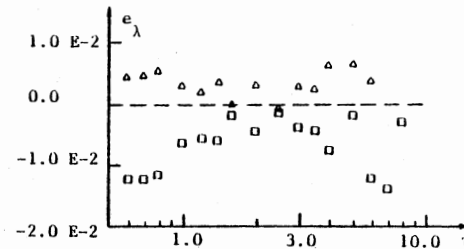
$\square \lambda_1 = -0.062, M_1 = 150.0$

$\triangle \lambda_2 = -0.200, M_2 = 40.0$

$\diamond \lambda_2 = -0.200, M_2 = 150.0$

$\gamma = M-2, \psi = 2, 400$ Monte Carlo trials

$v_k =$ Uniform Noise, $w = 0.001$



(b) Data Function:

$$z_k = 0.252e^{-0.200k\tau} - 0.252e^{-0.402k\tau} + v_k,$$

$$k = 0, 1, \dots, M-1$$

Legend: $\circ \lambda_1 = -0.200, M_1 = 40.0$

$\triangle \lambda_2 = -0.402, M_2 = 40.0$

$\gamma = M-2, \psi = 2, 400$ Monte Carlo trials

$v_k =$ Uniform Noise, $w = 0.001$

Figure 9. Choice of Optimal τ for More Examples with $N=2$

Figure 9(b) gives the same information for the data function

$$z_k = 0.252e^{-0.200k\tau} - 0.252e^{-0.402k\tau} + v_k, \quad k = 0, 1, \dots, M-1 \quad (5.31)$$

One might expect τ_0^+ to be unchanged from Figure 7(c) and data function (5.24) since λ_2 is unchanged. This is essentially the case, even though the shape of the standard deviation curve is considerably altered. This change in the shape of the standard deviation curve is due to λ_1 and λ_2 being less widely separated. The optimal sample interval is given in Table X as

$$1.0 \leq \tau \leq 5.0. \quad (5.32)$$

A comparison of the values of e_λ and s_λ in Figures 7(c) and 9(b) shows that for the λ_i more widely separated, the Prony solution is more accurate. The same observation applies to comparing 7(c) with 9(a).

Example 4: Consider the data function given by

$$z_k = 0.252e^{-0.062k\tau} + 0.252e^{-0.200k\tau} - 0.252e^{-0.402k\tau} + v_k, \quad k = 0, 1, \dots, M-1 \quad (5.33)$$

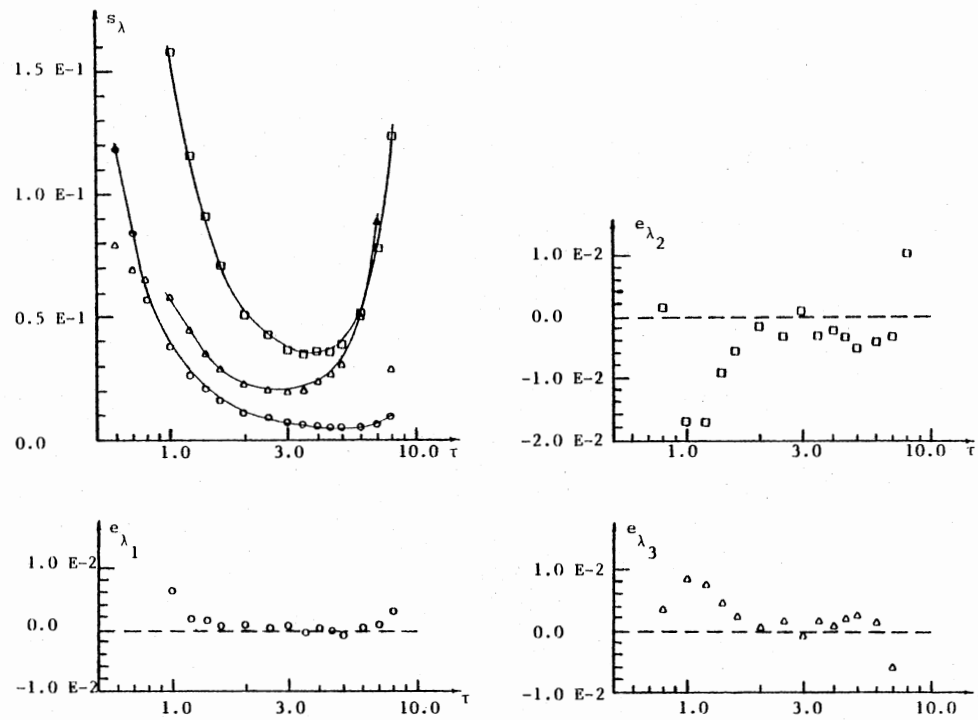
The normalized mean error and standard deviation for the Prony solution values of the λ_i are plotted in Figure 10 against τ . Table X gives the optimal sample interval as

$$2.5 \leq \tau \leq 5.0. \quad (5.34)$$

Once again, τ_0^+ seems to depend primarily on λ_N . (Compare the optimal sample intervals in Table X for figures 6(a,b), 7(a,c) and 9(b) where λ_N is the same.) The lower edge τ_0^- is the same as that given for Figure 9(a) for data function (5.29). Thus, τ_0^- seems to depend primarily on the interaction of the λ_i , and is a function of the separation of the λ_i .

Example 5: The function given by

$$x(t) = 0.252e^{-0.062t} - 0.252e^{-0.402t} \quad (5.35)$$



Legend: $\circ \hat{\lambda}_1 = -0.062$; $\square \hat{\lambda}_2 = -0.200$; $\triangle \hat{\lambda}_3 = -0.402$

Data Function: $z_k = 0.252e^{-0.062} + 0.252e^{-0.200} - 0.252e^{-0.402} + v_k, k = 0, 1, \dots, M-1$ (5.33)

$M = 40.0$; $\gamma = M-3$; $\psi = 4$; 400 Monte Carlo trials

$v_k = \text{Uniform Noise}, w = 0.0001$

Figure 10. Choice of Optimal τ For an Example with $N=3$

has occurred repeatedly in this research. This function is of special interest because it represents an actual ecological system.

Batch experiments with radiolabelled methyl parathion were performed at the Athens Environmental Research Laboratory. Eight flasks were inoculated with algal biomass obtained from sustained (control) microcosms which had never been exposed to the pesticide methyl parathion. In this experiment, entitled Flask Study 4, treatment number 2 involved a single dose of methyl parathion introduced into the water compartment of a flask containing a living algal mat and sediments. Measurements of the levels of methyl parathion and its degradation products (units: nCi/ml of ^{14}C) in the water were made with a variable sample period starting at a daily rate. The measured data values are given in Table VIII. A curve fit based upon nonlinear regression for the entire data set (experiment designation EXT) yielded equation (5.35) as obtained by EPA scientists using a standard on-line computer program. This data set provides an interesting practical test for Prony's method.

Prony's method was applied to these same data points to obtain a fitting function of the form

$$z(t) = p_1 e^{\lambda_1 t} + p_2 e^{\lambda_2 t} \quad (5.36)$$

Sample periods of $\tau = 1, 2, 3,$ and 4 days were used. Missing data in the data set were provided by interpolation as indicated in Table VIII. Prony's solution was formulated according to Algorithm 4.3, but without step 3. The results of the calculations are given in Table IX.

Several things are of note from Table IX. The solutions for sample periods of $\tau = 2$ and $\tau = 3$ are more accurate than those from the other sample periods. From the simulation of Example 2 (equation 5.24), and

TABLE VIII
EXT DATA SET

Day	Data	Day	Data	Day	Data
0	0.000	8	0.129	16	0.091**
1	0.064	9	0.123**	17	0.083
2	0.107	10	0.118	18	0.080**
3	0.138	11	0.124*	19	0.077*
4	0.146*	12	0.110**	20	0.072**
5	0.148**	13	0.098	21	0.068*
6	0.151	14	0.098**	22	0.069**
7	0.139	15	0.099*	23	0.070

*Interpolation by Equation (5.35). **Interpolation by Geometric Mean

TABLE IX
PRONY'S SOLUTION FOR EXT DATA SET

Item	Equation (5.35)	$\tau = 1$	$\tau = 2$	$\tau = 3$	$\tau = 4$
λ_1	-0.062	-0.080	-0.067	-0.061	-0.047
λ_2	-0.402	-0.311	-0.377	-0.468	-0.716
p_1	0.252	0.317	0.258	0.234	0.190
p_2	-0.252	-0.312	-0.257	-0.233	-0.190
M	---	24	12	8	6
γ	---	22	10	6	4
ψ	---	3	3	3	3
d_1	---	0.882	0.598	0.468	0.370
d_2	---	0.103	0.118	0.126	0.118
d_3	---	0.014	0.010	0.011	0.0032
$\sqrt{\gamma}\sigma_v^*$	---	0.13 E-2	0.91 E-3	0.71 E-3	0.58 E-3

* $\sigma_v = 0.000289$ is assumed.

the curves of Figure 7, one would expect the solution at $\tau = 4$ to be almost as good as the solution at $\tau = 2$. Such is not the case, but the reason may be attributed to the sparseness of the data.

The d_3 values of Table IX would indicate a noise value with standard deviation of about $\sigma_v = 0.003$. This is computed from equation (4.72) where $s_{K+1,K+1} = d_3$ and $m = \gamma$. For example, for $\tau = 1$,

$$d_3 \approx \sqrt{\gamma} \sigma_v \quad (5.37)$$

$$\sigma_v = 0.014/\sqrt{22} \approx 0.003. \quad (5.38)$$

The values in data set EXT were measured to 3 significant digits. This would indicate uniform noise of width $w = 0.001$, which corresponds to standard deviation of $\sigma_v = 0.000289$. Comparison with (5.38) indicates that a significant amount of noise is present in the data which cannot be attributed to rounding error of the measurement device.

D. Summary

Table X summarizes much of the information from the simulation of various data functions. The overall optimal sample interval for each of the functions examined is within the bounds predicted by inequalities (5.10) or (5.15), with $\delta = 2$ in (5.15). The data function associated with Figure 7(a) is an exception but even here the interval is very nearly within the bound.

The upper edge (τ_0^+) of the overall sample interval is determined differently for two basic cases. The first case is illustrated by Figure 6(c), where the noise is of great enough strength to become the predominant effect on τ_0^+ . In this case, the upper bound for τ provided by the sample theory (inequalities 5.10 or 5.15) will also be a rough

estimate for τ_0^+ . The second case is illustrated by the remaining figures, where the value for τ_0^+ is essentially independent of the noise strength. In these figures, τ_0^+ seems to be determined primarily by λ_N , and the value for τ_0^+ is given by

$$\tau_0^+ \approx 2.2/\lambda_N. \quad (5.40)$$

Before any quantitative relationship of the nature of (5.40) is stated with any degree of certainty, a great deal more research is needed simulating many other functions. The dependence of τ_0^+ on the separation of the λ_i especially needs attention.

The lower edge (τ_0^-) of the overall sample interval is likewise determined by two similar cases. In the first case the noise is of great enough strength to have the predominant affect on τ_0^- . This research did not produce any examples of this nature. The second case is illustrated by all the examples in this chapter, where τ_0^- is essentially independent of the noise strength. In these examples, τ_0^- seems to be determined primarily by the interaction of the λ_i , and is a function of the separation of the λ_i . No reasonable quantitative relationship for determining τ_0^- was found.

Even if a relationship similar to (5.40) cannot be shown to be true in general, a given data function can be simulated on a computer, and its optimal sample interval can be observed. This of course assumes that there is prior knowledge of the data function. This knowledge is not typically available when one is attempting to estimate a data function or identify a system. Nevertheless, once a reasonable estimate is available, an optimal sample period may be determined.

This procedure was essentially followed with the EXT data set. Data

function (5.24) was available as an estimate of the data set. This data function showed an optimal sample interval of

$$1.4 \leq \tau \leq 5.0.$$

When Prony's method was applied to the EXT data set, $\tau = 1$ produced poor results, and $\tau = 2.0$ and 3.0 produced good results as expected. However, $\tau = 4.0$ produced bad results. Thus, this data set showed basic agreement with what was expected from the simulation, except at $\tau = 4.0$.

TABLE X
OPTIMAL SAMPLE PERIOD AND SAMPLE BOUNDS
FOR VARIOUS DATA FUNCTIONS

Data Function	Figure	i	Optimal τ for λ_1^*		Optimal τ overall *		Bounds from:		
			τ_0^-	τ_0^+	τ_0^-	τ_0^+	τ^-	τ^+	τ^+
5.23	6(a)	1	?	5.0	?	5.0	0.050	9.8	--
5.23	6(b)	1	?	5.0	?	5.0	4.9 E-4	21.2	--
5.23	6(c)	1	?	3.5	?	3.5	0.55	4.0	--
5.24	7(a)	1	1.4	10.0	1.4	5.0	0.32	3.3	4.9
		2	0.8	5.0					
5.24	7(c)	1	1.2	10.0	1.2	6.0	3.2 E-2	5.2	7.7
		2	0.8	6.0					
5.29	9(a)	1	2.5	16.0	2.5	12.0	3.2 E-2	10.4	15.6
		2	1.6	12.0					
5.31	9(b)	1	1.0	5.0	1.0	5.0	9.9 E-3	5.2	7.7
		2	0.8	5.0					
5.33	10	1	2.5	8.0	2.5	5.0	3.2 E-3	4.2	5.3
		2	1.6	6.5					
		3	1.4	5.5					

* Values are not interpolated. Only actual data points are used.

** $\varepsilon = w/2$

CHAPTER VI

CONCLUSION AND RECOMMENDATIONS

Two questions have been addressed by this research. First, given the noisy sampled output of a system, can the number of exponential modes in the data be determined? Second, what is the optimal sample period for a given system? For both questions, Prony's method of fitting data to an exponential curve have been examined.

Chapter IV answered the first question by demonstrating that the problem of determining the number of exponential modes in the data reduces to that of determining the rank of matrix (4.2). Various matrix factorizations which can produce an estimate of the rank of a matrix were surveyed. The singular value factorization was found to give the best estimate of the rank. Numerical examples verified this and also showed that Prony's method produces a better estimate of the fitting function when the singular value factorization is used than when any of the other factorizations are used.

Chapter V answered the second question by developing a sampling theory which gives an upper and lower bound on the sample period. Several numerical examples were simulated on the computer. For these examples, the optimal interval for the sample period was found to be consistent with the bounds specified by the sampling theory. In addition, where the noise is not the dominant effect on this optimal sample interval, the upper bound appears to be determined primarily by the fast exponential

mode, and the lower bound appears to be a function of the separation of the exponential modes.

The conclusion from Chapter IV should be obvious. When a least squares approach to Prony's solution is desired, the singular value factorization should be used. If it is used, it provides information regarding the rank of the matrix, it provides information regarding the standard deviation of the noise in the data, and it provides the least squares estimate for the fitting function.

The conclusions from Chapter V cannot be stated so definitely. It is difficult to quantitatively predict the optimal sample interval other than by computer simulation of the data function. This requires some prior knowledge of the function being investigated.

Several areas of further research could be explored. These are listed below.

1. Example 2 of Section IV.F.2 demonstrated that the singular value factorization as incorporated in Algorithm 4.3 produces a better fitting function than does the QR factorization. The reason for this is unknown. It is also possible that this is true only for certain functions. These are areas of further research.

2. It was stated that the shape of the standard deviation curves in Figure 6 could be due to the statistical nature of sampled data from an exponential function. The same comment could be made regarding the standard deviation curves in the figures following Figure 6. If this is in fact the case, it has important implications for all sampling methods. To investigate this would require simulations similar to that done in this research, but using curve fitting techniques other than Prony's method. For example, non-linear regression analysis could be used. If

the results obtained are similar to those obtained in this research, then it could be concluded that the shape of the standard deviation curves is independent of the fitting method, and therefore due to the statistical nature of the sampled data.

3. In this research the relative sizes of the p_i were constant; that is, $p_1 = p_2 = \dots = p_N$. If instead $p_1 > p_2 > p_N$, then it is to be expected that the statistics of the solution would be dominated by the exponential mode associated with p_1 , and that the exponential mode associated with p_N would be more difficult to extract from the data.

4. To develop a greater ability to predict the optimal sample period for a given data function requires a great deal more research. Numerous data functions would need to be simulated, and the results tested on actual data sets.

5. Householder (1949) demonstrated that Prony's method is not a true least-squares fit to the data. He also suggested an iterative technique with Prony's method to obtain the true least-squares fit. The extent to which this technique improves Prony's method is an area for further research. The technique could also have an effect on the optimal sample period.

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