A STUDY OF THE INFINITE FINITE ELEMENTS FOR THE STATIC ANALYSIS

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ELEMENTS FOR THE STATIC

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INTRODUCTION

Background

Although no real physical domains extend to infinity, a number of solutions to continum problems which are assumed to extend to infinity have been produced. Pioneers of elasticity, starting with Lord Kelvin, Boussinesq, Lamb have analyzed various problems of the potential, static and dynamic response in the three dimensional space, and in half space. In engineering models a considerably large medium, as compared to the domain of interest, is often idealised to be an appropriate infinite region.

Many such problems occur in practical life in different fields. Examples arise in structure- soil interaction and structure- fluid interaction where the media bounding the structure extend to infinity. The acoustic radiation problem is another example. In the electrical engineering field, electromagnetic problems for antenna and surveying for mineral deposit are being studied.

One of the first numerical solutions for an unbounded domain was by Richardson (20) who matched his finite difference solution in the region of a dam foundation to the Boussinesq's analytical solution for a point load on a half space. A number of other techniques have been used since in different applications of continum mechanics.

The normal practice in the conventional finite element method is to idealize the unbounded domain by extending the finite element mesh outward to a point where the influence of the exterior domain is negligible. This approach requires an experimentation with several grid sizes and assumed boundary conditions. Results obtained are generally good for static problems but the method is not suitable for many dynamic analyses. Because of computer storage limitations, the number of normal finite elements required to achieve a certain accuracy may be quite large. This will result in a very large number of simultaneous equations which may place a severe limitation on usefulness of the method for practical problems.

The boundary solution procedures have been developed to circumvent the above difficulties and have been used to deal with infinite domain problems. In this method a trial function is selected a priori, and then the boundary conditions are satisfied in a weighted integral sense. Different variations of this method are reported in the literature in different fields.

One of the major disadvantages of the boundary solution procedure is the loss of localized discretization leading to an unbanded system of equations and in some cases nonsymmetric matrices. Also, incorporation of boundary solution procedures into existing finite element programs is cumbersome.

Infinite Finite Elements

The common characteristic in the development of most finite elements is that some quantity for example, potential energy is integrated and minimised over a finite domain. There seems to be no reason why the domain should not be infinite provided the quantity integrated remains finite. An infinite element is a specialized finite element which is of infinite extent in one coordinate direction. This element posses infinite domain with properly selected decaying functions and integration schemes. No matter how the element equations are derived, from a variational principle or directly from governing differential equations some quantity will be integrated throughout the element domain. Therefore this quantity should be bounded and well defined.

Infinite elements were introduced independently by Ungless (11) and by Bettes (1). Ungless used a reciprocal decay and Bettes used an exponential decay in their shape functions. The method was originally applied to infinite half space and potential problems. Subsequently it was applied to unbounded surface wave problems by Bettes and Zienkiewicz (2) and then to a study of coupled hydrodynamic response of concrete gravity dams (3). Chow and Smith (8) have developed periodic infinite elements based on static infinite elements of serendipity family for dealing with multiple wave types encountered in geomechanics problems. Medina (7) describes a parametric infinite element for solving axisymmetric problems under non axisymmetric forcing functions. Lynn and Hadid (6) have used reciprocal decay infinite elements to solve elastic half space problems. Recently Beer (10) has used infinite element for analysis of underground excavation problems in a prestressed infinite medium. Zienkiewicz (21) has proposed a mapped infinite element based on the idea of mapping an infinite region onto a finite one.

These papers describe the necessary basis for the technique and this is found to be often simple and economical. Infinite elements do not destroy the symmetry of equations or their banded structure in the stiffness matrix. No special techniques are needed in their implementation and they can be easily incorporated into existing finite element programs.

Objectives

1. Most of the earlier works which modelled infinite domain problems with infinite elements use only one type of infinite element. To this date no results have been published where all types of infinite elements are used to model one problem. In this work different types of infinite elements will be used to model the problem of an infinite beam on elastic foundation.

2. A comparison of the behavior of different types of

the infinite elements will be made. The effect of various parameters that influence the solution behavior will be discussed in detail.

3. A method for analysing problems where very little is known about the solution behavior will be outlined.

4. A modified infinite element based on mapped infinite element will be proposed.

CHAPTER II

FINITE ELEMENT MODEL OF THE INFINITE BEAM

Background Theory

The problem chosen to analyse the infinite domain behavior is an infinite beam on elastic foundation. There are two reasons for choosing this problem: (1) the availability of analytical solution for comparison with finite and infinite element models, (2) a one dimensional problem is easier to deal, the infinite element is usually constructed by extending the one coordinate direction to infinity.

The analytical solution of an infinite beam on elastic foundation subjected to a concentrated load at the center is available (14). To make use of the symmetry only one half of the beam will be considered. In Figure 1 the origin of coordinate axes (z,x) is located at the centroid of the beam cross section. A concentrated load P is applied to the beam at the origin of the axes. The condition of zero slope is specified to make use of symmetry for extending the solution to the left of the lateral load. The load P causes the beam to deflect, which in turn displaces the elastic foundation. As a result a distributed

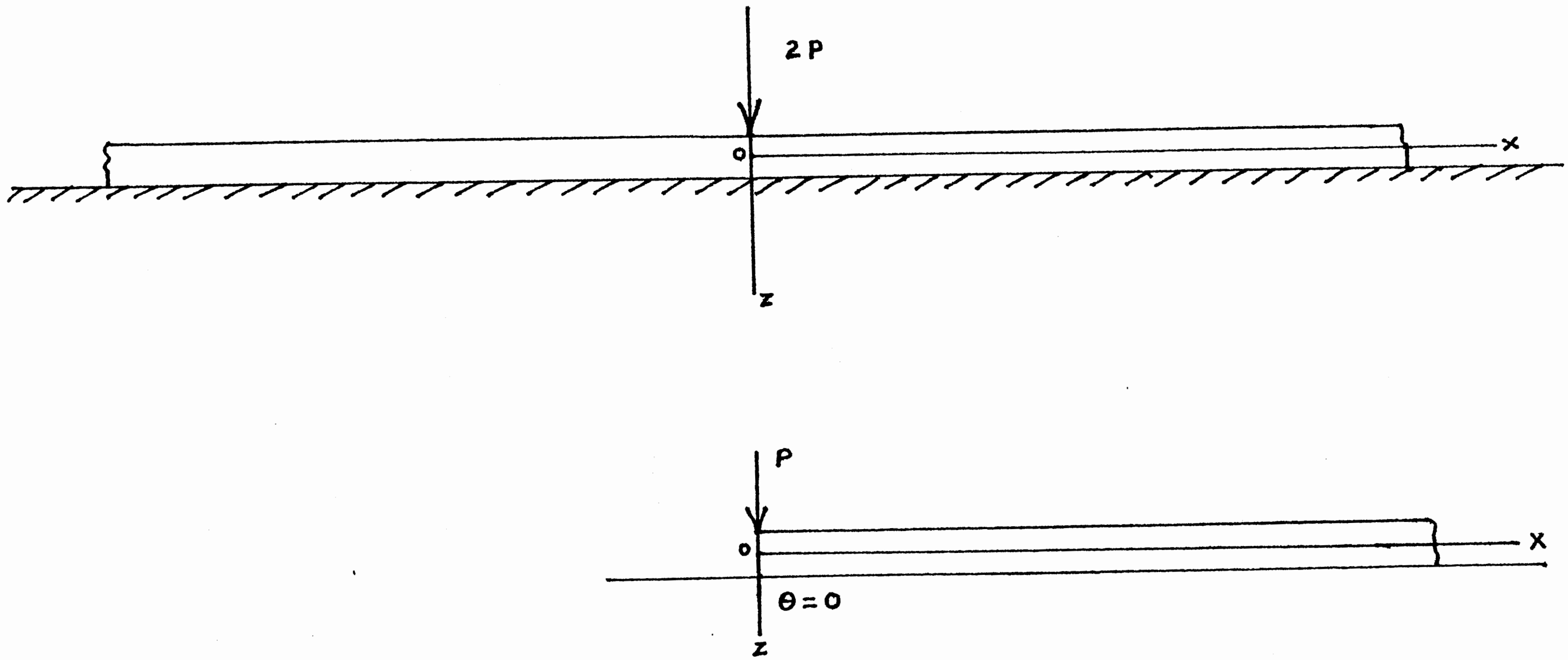

Figure 1. Infinite Beam on Elastic Foundation

force is developed between the beam and the foundation. Thus relative to the beam the resistance of the foundation produces a laterally distributed load "q" (force per unit length) on the beam. In certain regions the deflection of the beam may be negative. Hence, since the beam is assumed to be attached to the foundation, the foundation may in certain areas exert a tensile force on the beam.

The differential equation of bending can be obtained as

$$EI\ d^4w/dx^4 = -q \tag{2.1}$$

For linear elastic foundation the distributed load q is linearly proportional to the deflection w of the beam. Thus

$$q = kw \tag{2.2}$$

where, spring coefficient k may written in the form

$$k = bk_o \tag{2.3}$$

in which b is the beam width and k_o is the elastic spring constant for the foundation.

The general solution of Equation (2.1) may expressed as

$$w = \exp(-\beta x)*(c_1 \sin\beta x + c_2 \cos\beta x) + \exp(-\beta x)*(c_3 \sin\beta x + c_4 \cos\beta x) \tag{2.4}$$

where,

$$\beta = (k/4EI_y)^{1/4}$$

c_1, c_2, c_3, c_4 are constants to be determined from boundary conditions.

The deflections of the beam goes to zero for large values of X. Hence constants c_1 and c_2 must be set equal to zero and the equation for displacement reduces to

$$w = \exp(-\beta x) * (c_3 \sin \beta x + c_4 \cos \beta x) \qquad (2.5)$$

The constants c_3 and c_4 can be evaluated using the boundary conditions. The final equation for the displacement, w, is given by

$$w = (P\beta/k) * (\cos \beta x + \sin \beta x) * \exp(-\beta x) \qquad (2.6)$$

Conventional Finite Elements

The solution of continum problems involves determination of unknown function U such that it satisfies a certain differential equation set

$$A(U) = \left\{ \begin{array}{c} A_1(U) \\ A_2(U) \\ \cdot \\ \cdot \end{array} \right\} = 0 \qquad (2.7)$$

which has to be solved in domain Ω , given in Figure 2, together with the boundary conditions

$$B(U) = \left\{ \begin{matrix} B_1(U) \\ B_2(U) \\ \vdots \end{matrix} \right\} \tag{2.8}$$

on the boundaries Γ of the domain.

The unknown function U is approximated in the finite element method by

$$U \simeq \hat{U} = \sum_{i=1}^{n} N_i a_i = Na \tag{2.9}$$

where, N_i are the shape functions defined in terms of the independent variables (such as coordinates X, Y) and all or some of the parameters a_i are unknown.

The parameters a_i are obtained from a set of equations which have integral forms of the type

$$\int_{\Omega} G_j(\hat{U})\, d\Omega + \int_{\Gamma} g_j(\hat{U})\, d\Gamma = 0 \tag{2.10}$$

where, G_j and g_j are some known functions or operators.

These integral forms lead to an approximation element by element and an assembly of system of equations can be achieved by the use of standard procedures. If G_j and g_j are integrable, we have

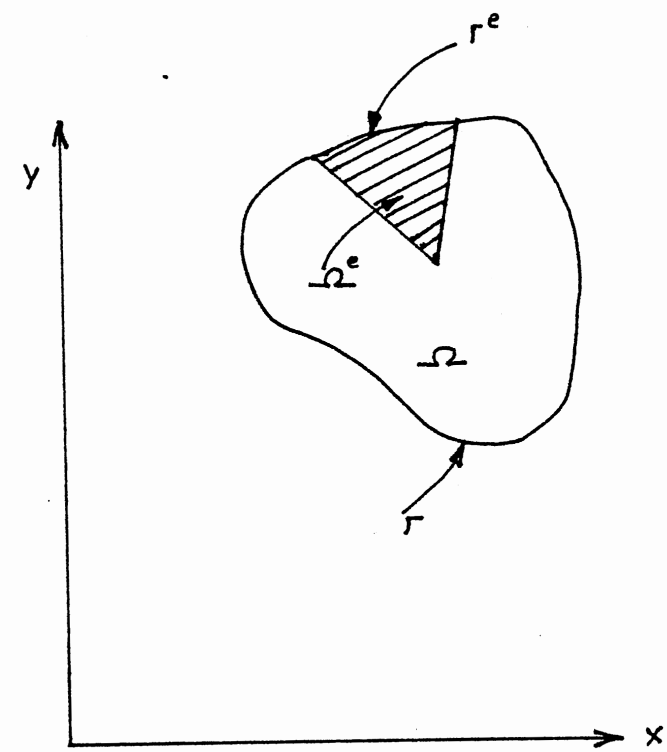

Figure 2. Problem Domain and Boundaries.

$$\int G_j \, d\Omega + \int g_j \, d\Gamma = \sum_{e=1}^{m} \left(\int_{\Omega^e} G_j \, d\Omega + \int_{\Gamma^e} g_j \, d\Gamma \right) \qquad (2.11)$$

where, Ω^e is the domain of each element and Γ^e its part of the boundary.

The trial function N_i is narrowly based. It takes a value of zero everywhere except in elements associated with the node. This leads to a banded set of equations. The difference between various finite element approaches lie in the choice of shape functions and in the manner in which Equations (2.10) are derived. Various procedures like method of weighted residuals, variational principles can be used.

If the differential equations are linear then the approximating equation system will yield a set of linear equations of the form

$$Ka + f = 0 \qquad (2.12)$$

where,

$$K_{ij} = \sum_{e=1}^{m} K_{ij}$$

$$f_i = \sum_{e=1}^{m} f_i^e$$

Finite Element Model

The finite element model of the problem is given in Figure 3. In all results five isoparametric beam elements and one infinite element will be used. The different cases will be analysed by varying the element length of the beam elements. The length of beam elements vary from 10 units to 200 units. The different geometries are given in Figure 5.

Isoparametric Beam Element

The isoparametric beam element is a straight beam element having three nodes as illustrated in the Figure 4. This element can account for shear deformation since energy due to shear as well as bending is considered in the formulation (15). This element is quite versatile and can be used to analyse not only thin beams with negligible shear deformation, but also thick beams and beams of sandwich construction in which shear effects are important. In this work only thin beams are considered.

Two coordinate schemes are used in the element formulation, the global coordinate system (X) and the local coordinate system for element (ξ). The three nodes are at $\xi = -1$, 0, and +1 as shown in Figure 4. Each node i has two displacement degrees of freedom associated with it

w_i the lateral displacement of the beam

$\theta_i = (dw/dx)_i + \phi_i$ the rotation of the normal

Thus the displacements may be listed in the vector

$$\delta^e = \lfloor w_1 , \theta_1 , w_2 , \theta_2 , w_3 , \theta_3 \rfloor \qquad (2.13)$$

The shape functions associated with each node are:

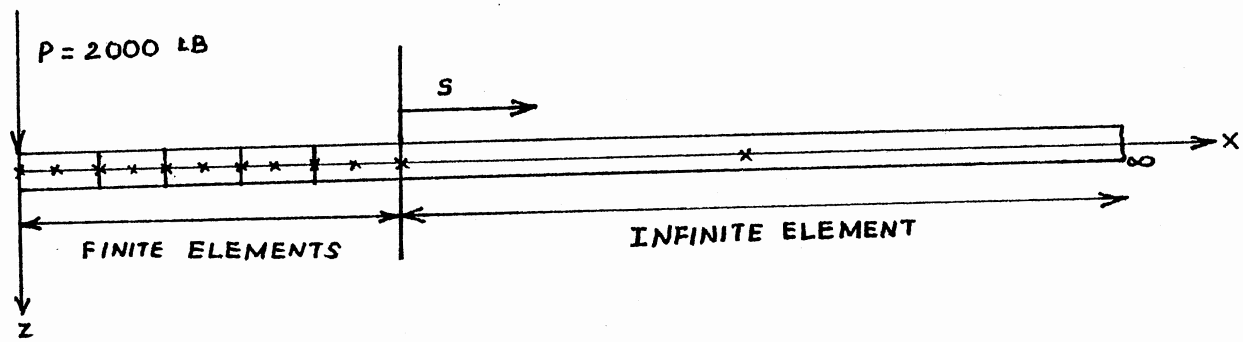

Figure 3. Finite Element Model of the Infinite Beam

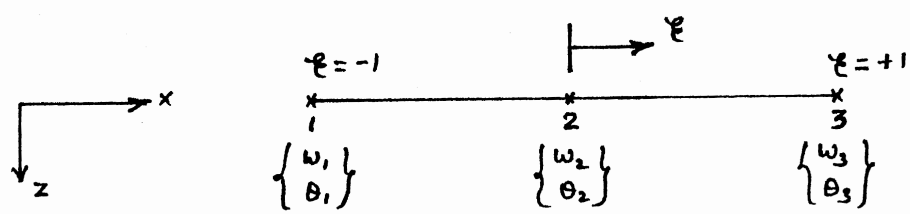

Figure 4. Isoparametric Beam Element

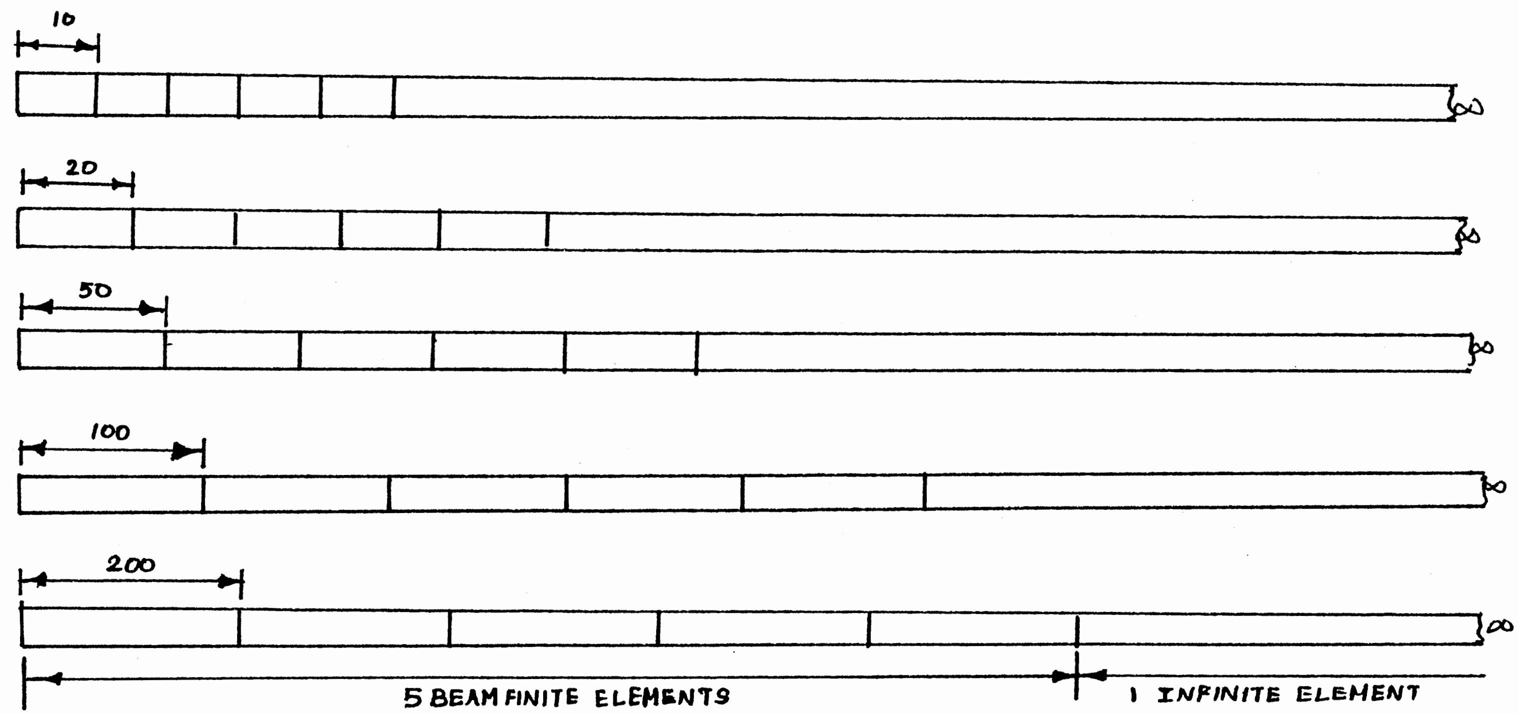

Figure 5. Infinite Element Geometries

$$N_1 = -0.5*(\xi)*(1-\xi)$$

$$N_2 = (1-\xi)*(1+\xi) \qquad (2.14)$$

$$N_3 = 0.5*(\xi)*(1+\xi)$$

The lateral displacement $w(\xi)$ at any point within the element is defined in terms of the shape functions and associated nodal displacements by simple interpolation

$$w(\xi) = N_1(\xi)\ w_1 + N_2(\xi)\ w_2 + N_3(\xi)\ w_3 \qquad (2.15)$$

The rotation at any point $\theta(\xi)$ within the element is defined by

$$\theta(\xi) = N_1(\xi)\ \theta_1 + N_2(\xi)\ \theta_2 + N_3(\xi)\ \theta_3 \qquad (2.16)$$

The X coordinate is defined in a similar way

$$X(\xi) = N_1(\xi)\ X_1 + N_2(\xi)\ X_2 + N_3(\xi)\ X_3 \qquad (2.17)$$

The strains are defined in terms of nodal displacements and derivatives by

$$\begin{bmatrix} \dfrac{\partial\theta}{\partial x} \\ \phi = -\dfrac{\partial\omega}{\partial x}+\theta \end{bmatrix} = \begin{bmatrix} 0 & \partial N_1/\partial x & 0 & -\dfrac{\partial N_2}{\partial x} & 0 & \dfrac{\partial N_3}{\partial x} \\ -\dfrac{\partial N_1}{\partial x} & N_1 & -\dfrac{\partial N_2}{\partial x} & N_2 & -\dfrac{\partial N_3}{\partial x} & N_3 \end{bmatrix} \begin{Bmatrix} \omega_1 \\ \theta_1 \\ \omega_2 \\ \theta_2 \\ \omega_3 \\ \theta_3 \end{Bmatrix}$$

$$\epsilon = \left[B_1 \quad B_2 \quad B_3 \right] \delta^e \qquad (2.18)$$

$$= B\ \delta^e$$

The shape function derivatives in the strain matrix B_i can be calculated from the expression

$$\partial N_i/\partial x = (\partial N_i/\partial \xi)(\partial \xi/\partial x) \qquad (2.19)$$

where, $\partial\xi/\partial x$ is obtained from the jacobian matrix.

The stress strain relation for beam may be written as

$$\begin{bmatrix} M \\ \\ Q \end{bmatrix} = \begin{bmatrix} EI & 0 & d\theta/dx \\ & & \\ 0 & S & \phi \end{bmatrix} \qquad (2.20)$$

where, M is the bending moment and Q is the shear force.

The element stiffness matrix K^e can be evaluated from the energy considerations given in Appendix (A).

$$K^e = \int [B]^T [D] [B] \, dx \qquad (2.21)$$

The stiffness matrix can be evaluated numerically using the Gauss-Legendre quadrature. A submatrix in K^e linking two nodes i and j may be evaluated from the expression

$$K = \int [B_i]^T [D] [B_j] \det J \, d\xi \qquad (2.22)$$

where $\quad dx = \det J \, d\xi$

The distributed lateral load due to spring foundation can be expressed in terms of nodal forces by using virtual work principle. This term was added to stiffness matrix for simplifying the programming. The applied nodal forces and couples may be represented by the vector

$$F^e = [P_1 , C_1 , P_2 , C_2 , P_3 , C_3] \qquad (2.23)$$

For distributed loading "q"

$$P_i = \int N_i \ (-q)\ dx$$
$$= \int N_i \ (-kw)\ dx \qquad (2.24)$$

and $\quad C_i = 0$

The shape functions chosen for the infinite element should realistically model the infinite domain behavior. Also, they should lead to integrations over the element which are finite.

The infinite elements can be constructed as suggested by Bettes (1) by finding the lagrangian polynomials to produce lagrange-type infinite elements. In general it is possible to extend any finite element to infinity. Since finite element shape functions will not be appropriate to describe the behavior of field variables, decay functions are introduced to modify the finite element shape functions. The shape functions will be of the form

$$N_i(s) = f_i(s)\ M_i(s) \qquad (2.25)$$
$$i=1,\ldots n$$

where

$M_i(s)$= Shape functions of the original element

$f_i(s)$= Decay functions

and $\quad f_i(s_i)= 1$

There is no requirements that decay function take any special value at other nodes. The shape functions N_i

should tend to far field value at infinity.

There are different possibilities in the choice of decay functions f_i. For the exponential decay function the decay function f_i is of the form

$$f_i = \exp\left[(s_i - s)/L\right] \tag{2.26}$$

L is an arbitary parameter that determines the severity of decay. The decay function for reciprocal decay is of the form

$$f_i = ((s_i - s_o)/(s - s_o))^N \qquad 2.27$$

where s_o is some origin point. By varying s and n the severity of decay can be changed. In the mapped infinite element the infinite region is mapped onto a finite region and standard shape functions are used to interpolate the unknown functions. The shape functions for different infinite elements will be discussed in detail in the next chapters.

CHAPTER III

IMPLEMENTATION OF MAPPED INFINITE ELEMENTS

Mapped Infinite Element

The infinite region corresponding to infinite element is mapped onto a finite one and standard shape functions are used to interpolate the values of unknown variables. Figure 6 gives mapping of an one dimensional element. The element extending from X_1 to infinity is mapped on to a domain $-1 \leqslant s \leqslant 1$.

The mapping is of the form

$$X = \hat{N}_0(S)\ X_0 + \hat{N}_2(S)\ X_2 \tag{3.1}$$

where X_0 is some arbitary decay origin and,

$$\hat{N}_0(s) = -s/(1-s)$$
$$\hat{N}_2(s) = 1+ (s/1-s)$$

At $s = +1$ $\quad X = (s/1-s)*(X_2-X_0)+X_2 = \infty \qquad (X_2 \neq X_0)$

At $s = 0$ $\quad X = X_2$

At $s = -1$ $\quad X = (X_0/2)+(X_2/2)$

The coordinate X_1 is chosen to be at the outer edge of finite element region. Thus

$$X_1 = (X_0/2)+(X_2/2) \tag{3.2}$$

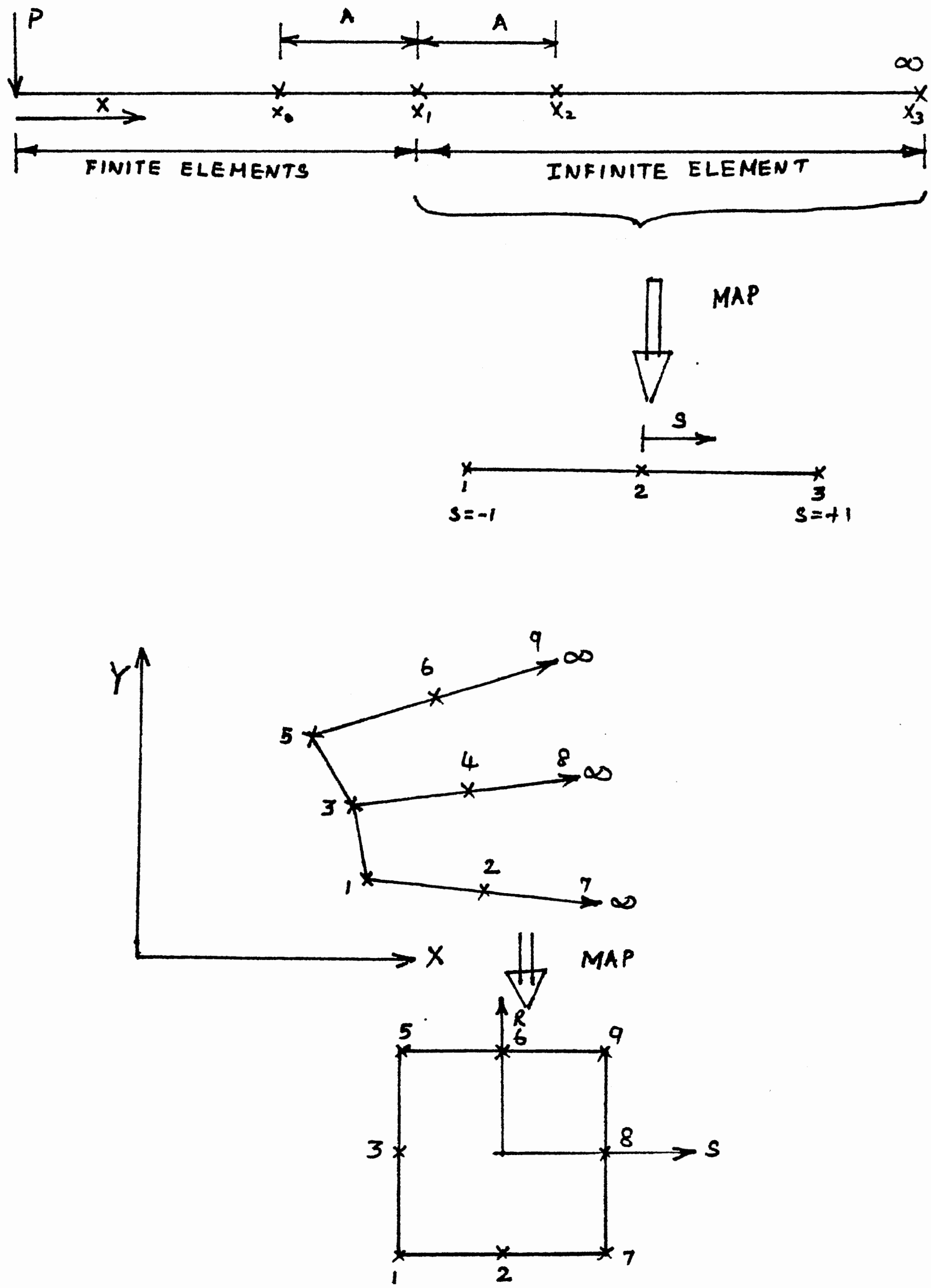

Figure 6. Geometry of the Mapped Infinite Element

Using these relationships the mapping function can be defined as

$$X = (2X_1 - X_2)\ \hat{N}_0(s) + \hat{N}_2(s)\ X_2 \qquad (3.3)$$

An important condition of any mapping used is that

$$\hat{N}_0 + \hat{N}_2 = 1$$

which is here identically satisfied. This is necessary in order to that the mapping does not alter with any changes in origin of the coordinate system. Also this condition is necessary for convergence of solution (13).

Any shift in origin ΔX leads to

$$X' = X_i + \Delta X \qquad (3.4)$$

Substituting Equation (3.4) in Equation (3.3) gives

$$X + \Delta X = (X_0 + \Delta X)\ \hat{N}_0 + (X_2 + \Delta X)\ \hat{N}_2$$

$$\Delta X = \Delta X\ (\hat{N}_0 + \hat{N}_2)$$

which is true only if

$$\hat{N}_0 + \hat{N}_2 = 1$$

The shape functions associated with nodal displacements are given by

$$w(s) = N_1(s)\ w_1 + N_2(s)\ w_2$$

$$\theta(s) = N_1(s)\ \theta_1 + N_2(s)\ \theta_2 \qquad (3.5)$$

where, $\quad N_1(s) = -s/2 + s\ /2$

$N_2(s) = 1 - s^2$

The shape function corresponding to the third node is condensed out as the displacements tend to zero at infinity This automatically imposes the boundary conditions.

It is possible to extend this concept to two or more dimensions. A typical two dimensional element is given in Figure 6 with the mapped versions in s, r domains. One dimensional mapping given by Equation (3.3) can be applied to line through points 1 and 2 such that,

$$\begin{aligned} X &= (2X_1 - X_2)\,\hat{N}_0 + \hat{X}_2\hat{N}_2 \\ Y &= (2Y_1 - Y_2)\,\hat{N}_0 + \hat{Y}_2\hat{N}_2 \end{aligned} \qquad (3.6)$$

The same can be applied to the lines through points 3, 4 and 5, 6 giving the complete mapping for the element

$$\begin{aligned} X = {} & N_1(r) * ((\ 2X_1 - X_2)\ N_0 + X_2\hat{N}_2) + \\ & N_2(r) * ((2X_3 - X_4)\ \hat{N}_0 + X_4\hat{N}_2) + \\ & N_3(r) * ((2X_5 - X_6)\ \hat{N}_0 + X_6\hat{N}_2) \end{aligned}$$

Similarly expressions for Y can be got. The terms $N_1(r)$, $N_2(r)$, $N_3(r)$ are standard polynomial shape functions in the r direction.

Mapped Infinite Element with Reciprocal Decay

The infinite domain is mapped on to the finite domain using the same mapping function as in mapped infinite element. The inverse mapping can be found by solving Equation (3.3) for s yielding

$$s = 1 - (2a\ /\ X - X_0) = 1 - 2a/r \qquad (3.7)$$

where

$$a = X_2 - X_1 \quad \text{and} \quad r = X - X_0$$

There may be a severe decay in solution behavior for some problems. In such problems using interpolation function similar to the one given in Equation (3.5), may lead to some errors. A modification to the shape functions will made in this work for modelling such problems. A decay term $(1/r)^N$ will be added to the shape functions to make the element more versatile.

From Equation (3.7),

$$r = 2a/1-s$$
$$(1/r)^N = (1-s/2a)^N \tag{3.8}$$

The nodal displacements are given by

$$w (s) = N_1(s) \; w_1 + N_2(s) \; w_2$$
$$\theta (s) = N_1(s) \; \theta_1 + N_2(s) \; \theta_2 \tag{3.9}$$

where,

$$N (s) = (-s/2 + s /2)*(1-s/2)^N$$
$$N (s) = (1- s^2)*(1- s)^N$$

The exponent n can be varied to match the severity of the decay of the problem.

Effect of Decay Origin

The location of second node X_2 influences the decay origin. The decay origin X_0 is given by

$$X_0 = 2X_1 - X_2$$

The changes in decay origin change the decay length. By moving the second node farther increases the decay length. This causes a reduction in severity of decay. The effect of decay length was studied on different mesh sizes given in Figure 5. In all cases 5 beam elements and one infinite element was used.

Figure 7 illustrates the effect of decay length on different mesh sizes for the mapped infinite element. For very coarse meshes the effect of decay length on maximum displacement is negligible. Refining the mesh size increases the accuracy of the solution. For an optimum mesh size of beam elements (20 units), there is a small variation with decay length and the results are good. For a beam element length of 10 units (when the finite elements are not extended far enough), there is a wide variation in the displacments for different values of decay length. The displacements are underestimated for small values of decay length due to the overestimation of the stiffness of the infinite element. For large values of decay length, the severity of decay in shape functions is small. Hence, the infinite element behaves like a regular finite element. Imposition of boundary conditions lead to the underestimation of displacements.

Figures 8, 9 and 10 give similar results for mapped infinite element with reciprocal decay for different exponents n. By increasing n, even for large values of decay length, good results are obtained. By making n larger, the severity of decay is also increased. Large values of decay length reduce the severity of decay. Thus the increase of n offsets reduction in decay due to

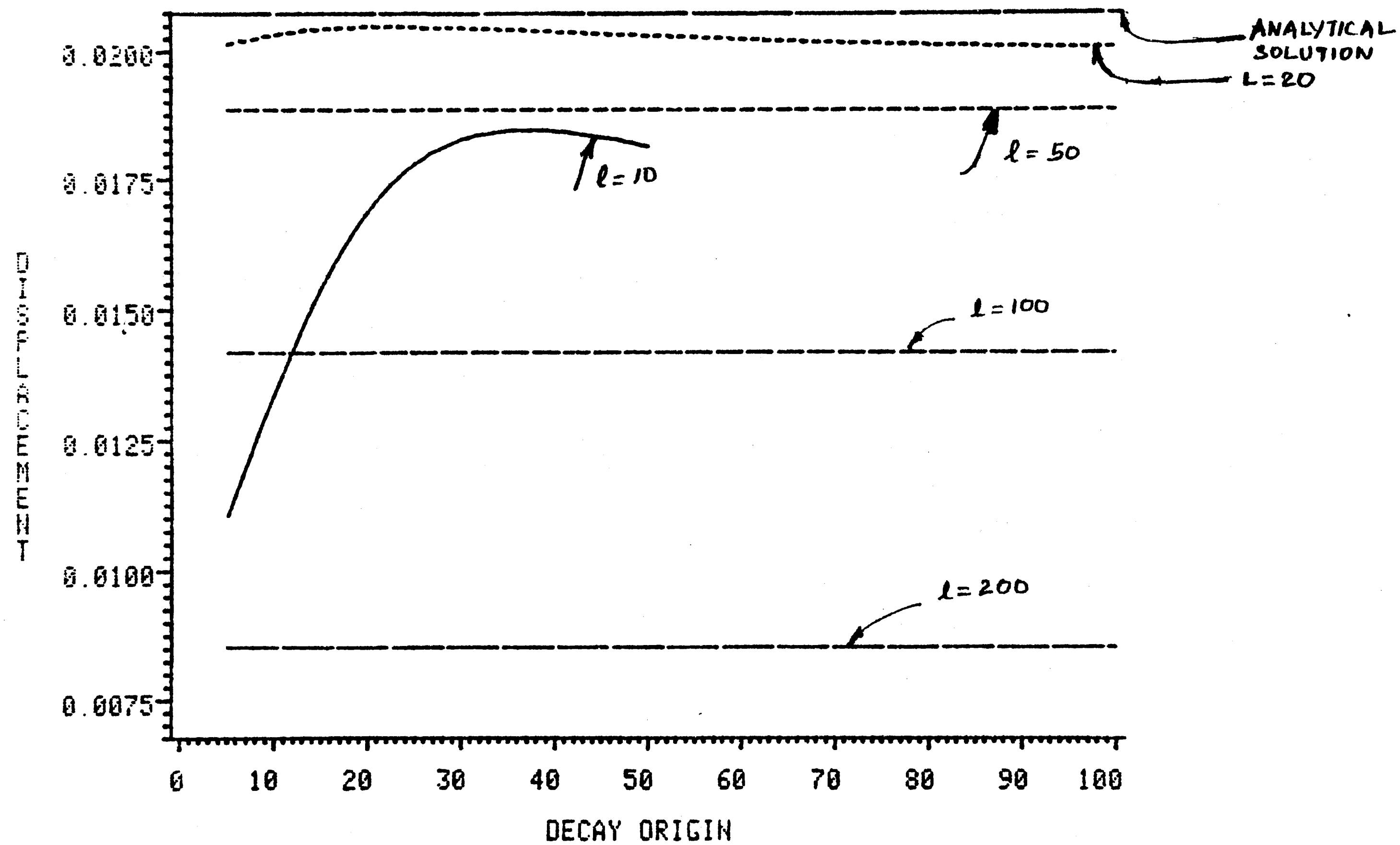

Figure 7. Effect of Decay Origin on Different Element Meshes for the Mapped Infinite Element

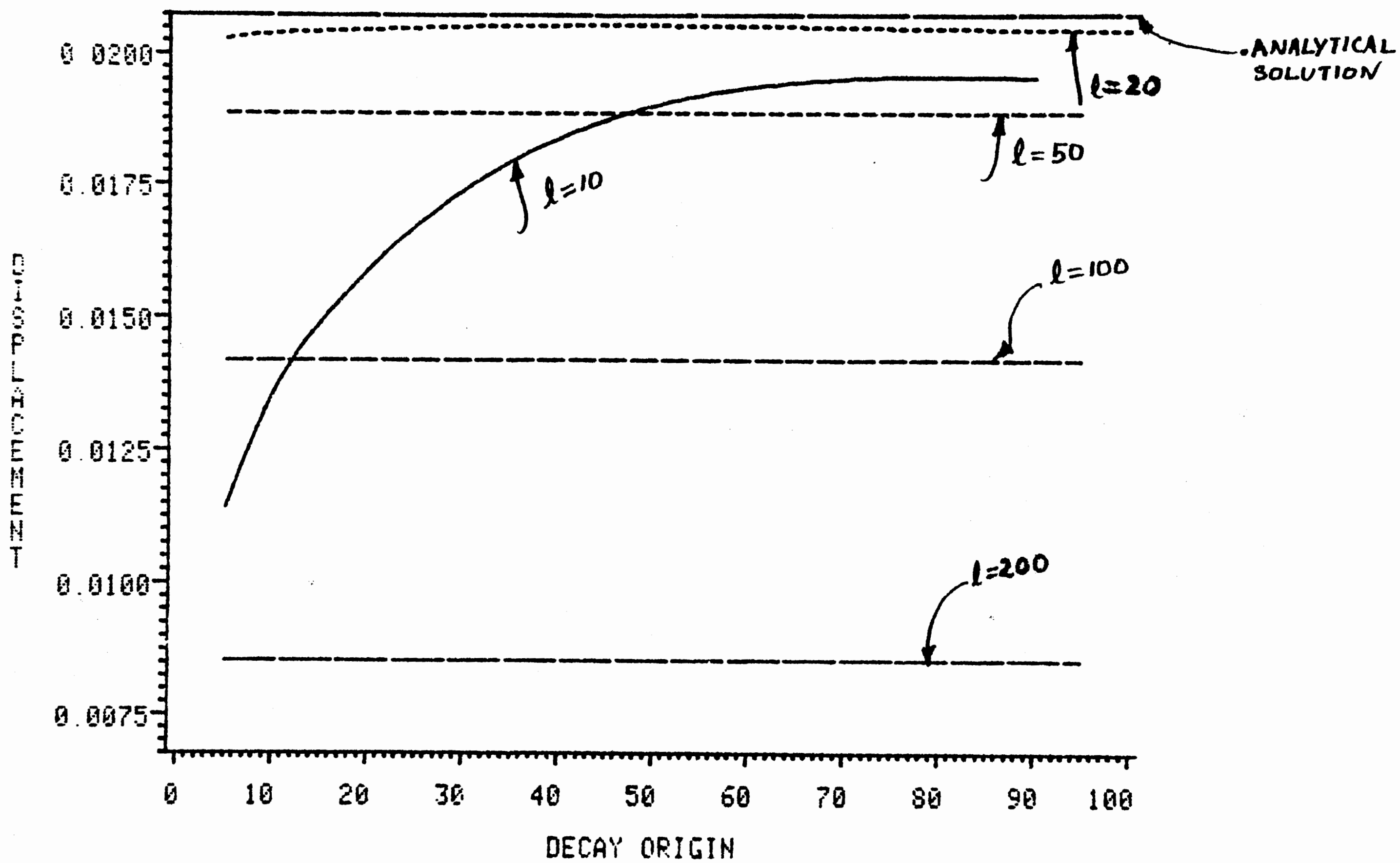

Figure 8. Effect of Decay Origin on Different Element Meshes for Mapped Infinite Element with Reciprocal Decay

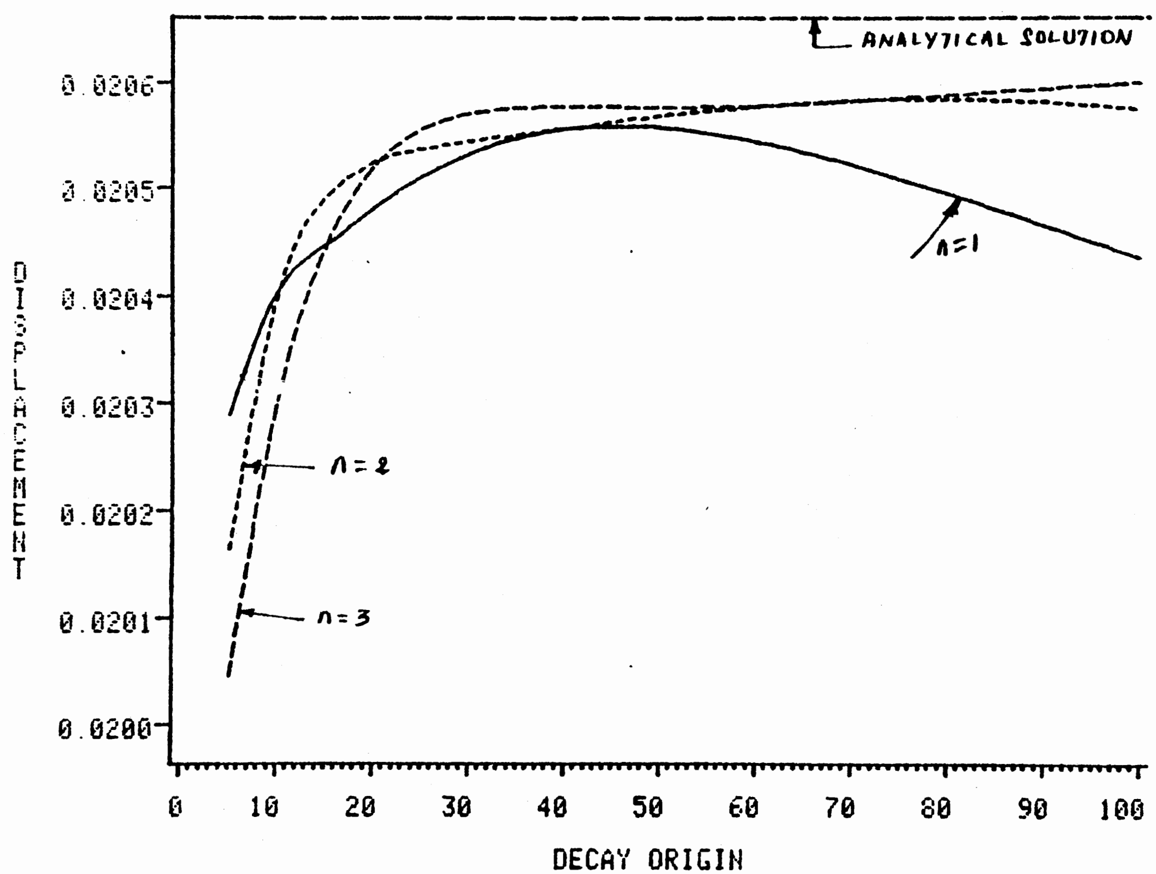

Figure 9. Effect of Decay Origin for Different Exponents in Mapped Infinite Element with Reciprocal Decay (Mesh Size=20)

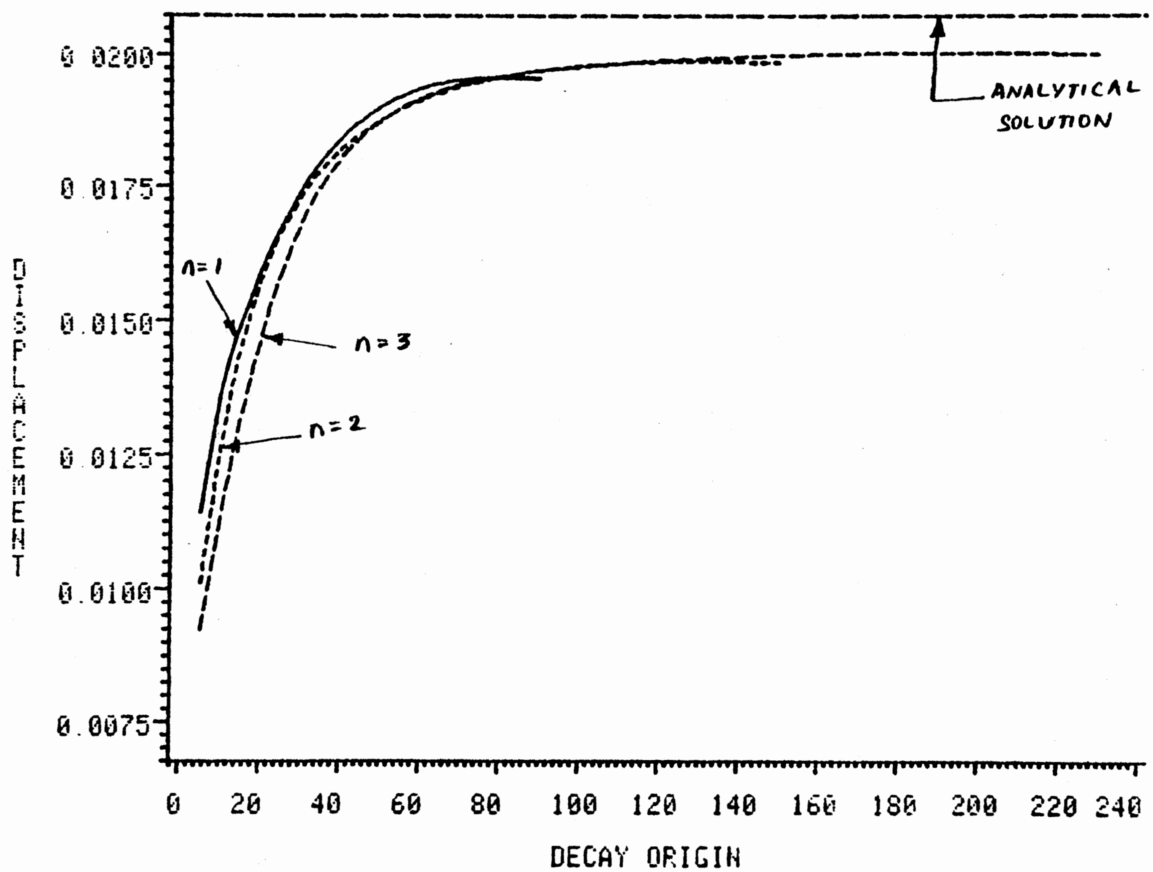

Figure 10. Effect of Decay Origin for Different Exponents in Mapped Infinite Element with Reciprocal Decay (Mesh Size=10)

large values of decay length. If the finite elements are not extended far enough the decay origin and n can affect the result significantly. But a coarse mesh leads to poor results. Hence, an optimum mesh size should be used to model the infinite domain problems.

Comparison of Mapped Infinite Elements

Figure 11 gives the comparison of the two types of mapped infinite elements with finite element model and analytical solution. For the finite element model 6 beam elements was used. The infinite element model had 5 beam elements and one infinite element. The length of beam elements was 10 units. The graphs show that the mapped infinite element with reciprocal decay give better results than the mapped infinite element with no reciprocal decay and the finite element model. The mapped infinite element with no reciprocal decay gives inferior result than the finite element model. The reciprocal decay increases the versatility of the mapped infinite element. For problems with severe decay in the solution behavior, by varying exponent n good results can be obtained.

Doubling the length of beam elements to 20 units increase the accuracy of both finite element and infinite element models. But a still coarser mesh size of beam elements lead to poor results. For very coarse mesh sizes

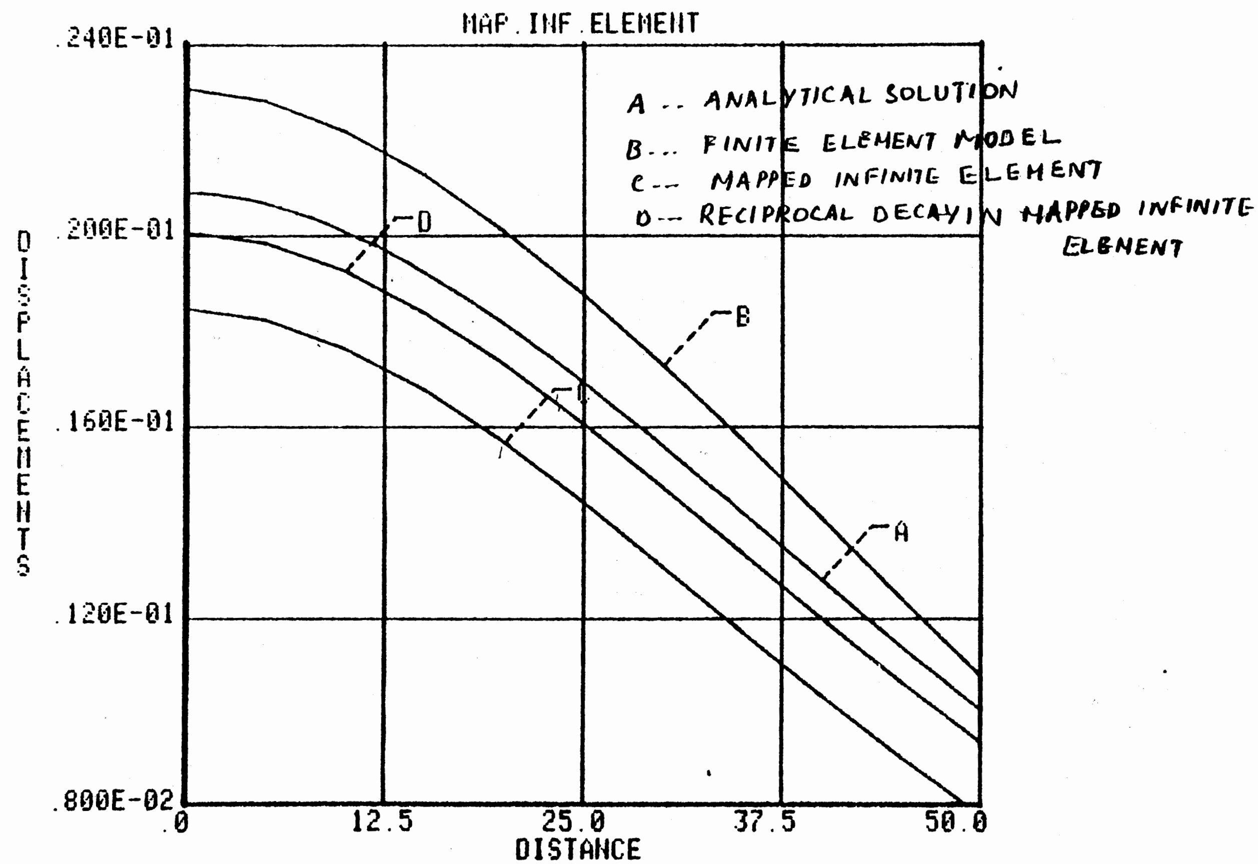

Figure 11. Comparison of Mapped Infinite Element Models

there is no difference in solution behavior between finite element and infinite element models. Thus the infinite element problems cannot be solved by extending the finite elements far and making the mesh size coarse. The infinite elements can be used to find the optimum mesh size. This will reduce the number of elements needed to achieve a certain accuracy. The mapped infinite element is simple in concept and the standard shape functions are used to interpolate the unknown functions. The numerical integration for evaluating the stiffness of the infinite element can be performed by standard Gauss-Legendre scheme. This is an advantage over the other types of infinite elements, which require special schemes for numerical integration.

CHAPTER IV

IMPLEMENTATION OF RECIPROCAL DECAY INFINITE ELEMENT

Reciprocal Decay Infinite Element

The shape functions for the infinite element are defined in terms of a local coordinate system s. The infinite element has three nodes, the third one at infinity. The first two nodes are placed at s = 0 and 50. The shape functions associated with the nodal displacements are given by

$$w(s) = N_1(s)\ w_1 + N_2(s)\ w_2$$
$$\theta(s) = N_1(s)\ \theta_1 + N_2(s)\ \theta_2 \qquad (4.1)$$

where,

$$N_1(s) = ((s_2 - s\ )/(s_2 - s_1))*((s_1 - s_0)/(s - s_0))^N$$
$$N_2(s) = ((s_1 - s\ )/(s_1 - s_2))*((s_2 - s_0)/(s - s_0))^N$$

s_0 is an arbitary decay origin which can be used to change the severity of decay. The exponent n can also be varied to match the severity of decay. A typical shape function is sketched in Figure 12. Small values of decay origin lead to severe decay and for large values of decay origin the decay is very small. Increasing 'n' leads to high

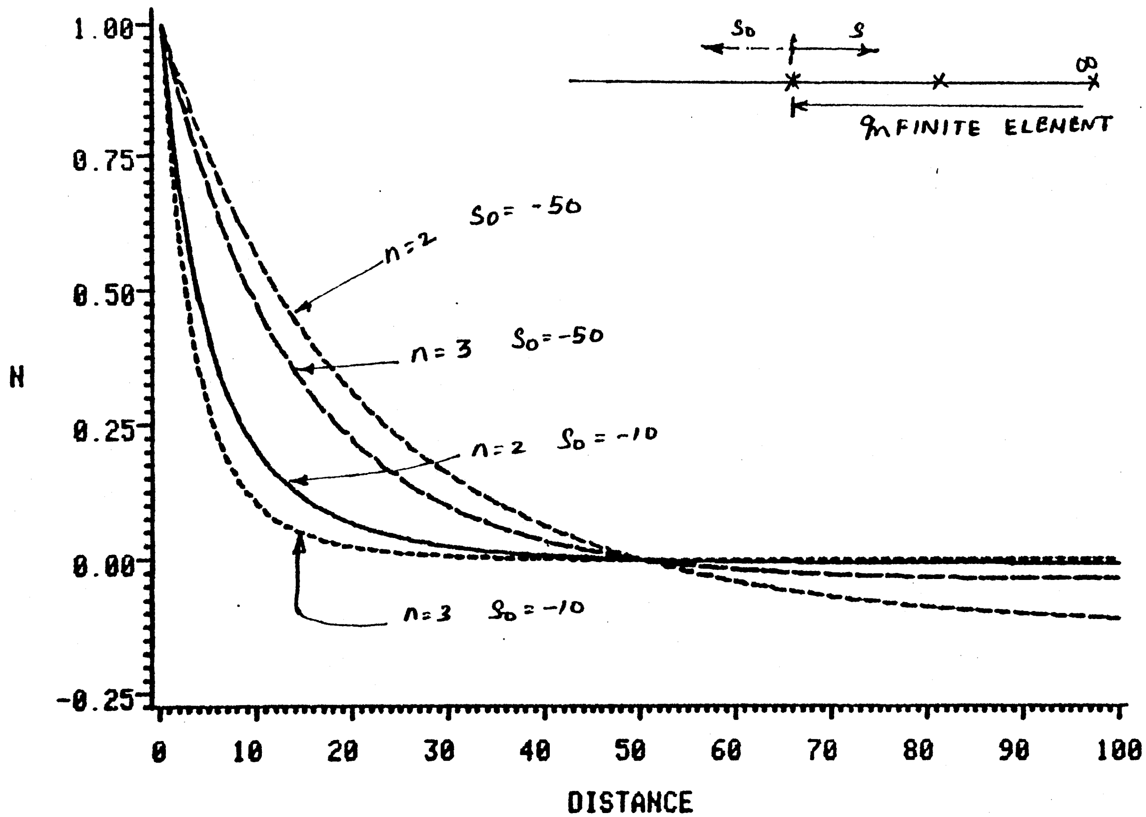

Figure 12. Shape Functions for the Reciprocal Decay Element

decays in the shape functions. The location of second node of the infinite element is arbitary and it was found that its location does not affect the results.

Evaluation of stiffness matrix for the infinite element involves numerical integration of functions of the form

$$\int_{a}^{\infty} f(s)\, ds$$

Two possibilities will be considered. The terms in the stiffness can be evaluated by using Gauss-Laguerre scheme. Alternately, the Gauss-Legendre scheme may be modified for an unbounded range as suggested by Bettes (5).

Numerical Integration

The Gauss-Laguerre integration scheme can be used by multiplying the weights by exp(+s). This scheme is not exact. It was found that using more sampling points led to poor results. When the terms in the stiffness matrix were examined it was found that some of them were unbounded. The use of more sampling points led to an overestimation of the stiffness of the infinite element. Figure 14 gives the effect of sampling points for beam element length of 10 units for different exponents. A total of 6 elements was used, similar to the example on mapped infinite element. An optimum value of decay length was used in the graphs. Using seven to fifteen sampling points generally gave good results. Figure 13 illustrates results for

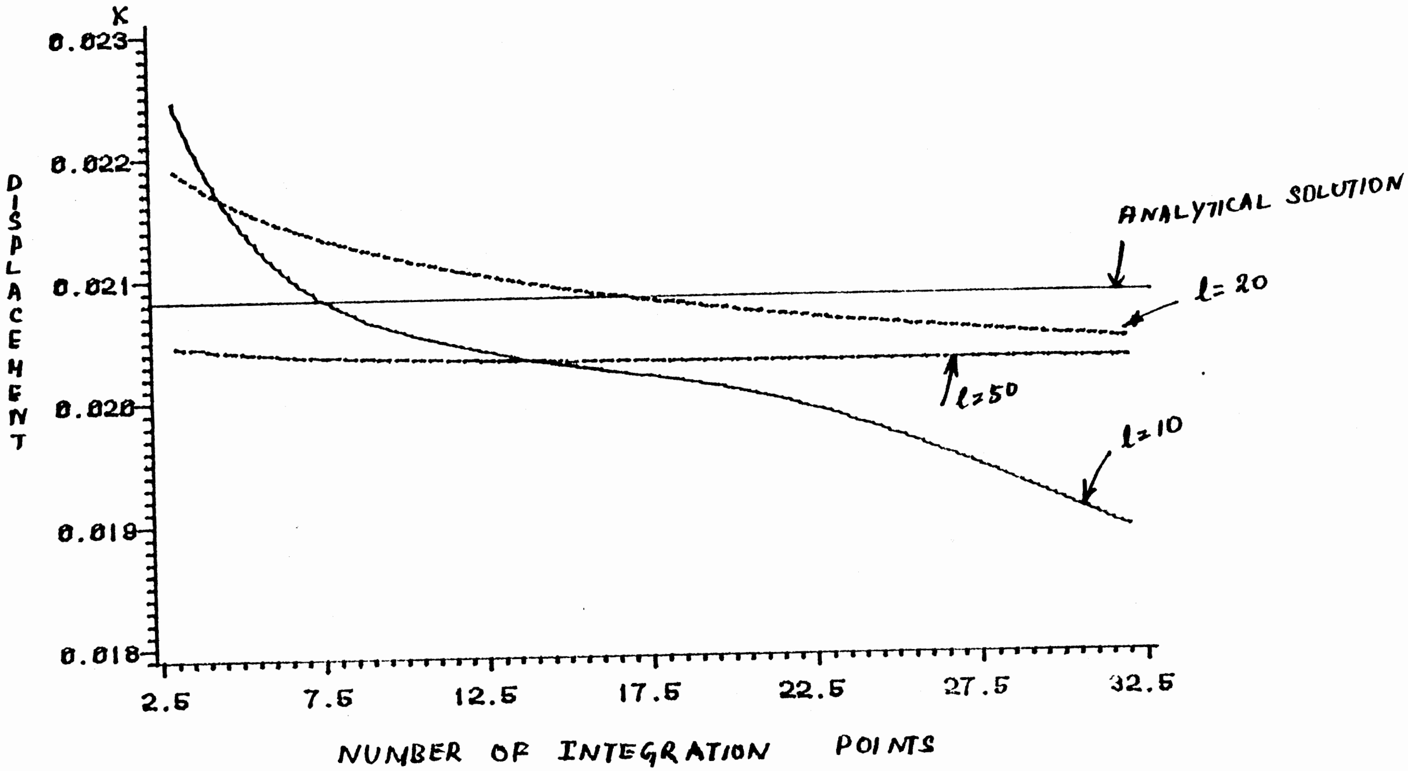

Figure 13. Effect of Gauss-Laguerre Integration Points on the Different Element Mesh Sizes

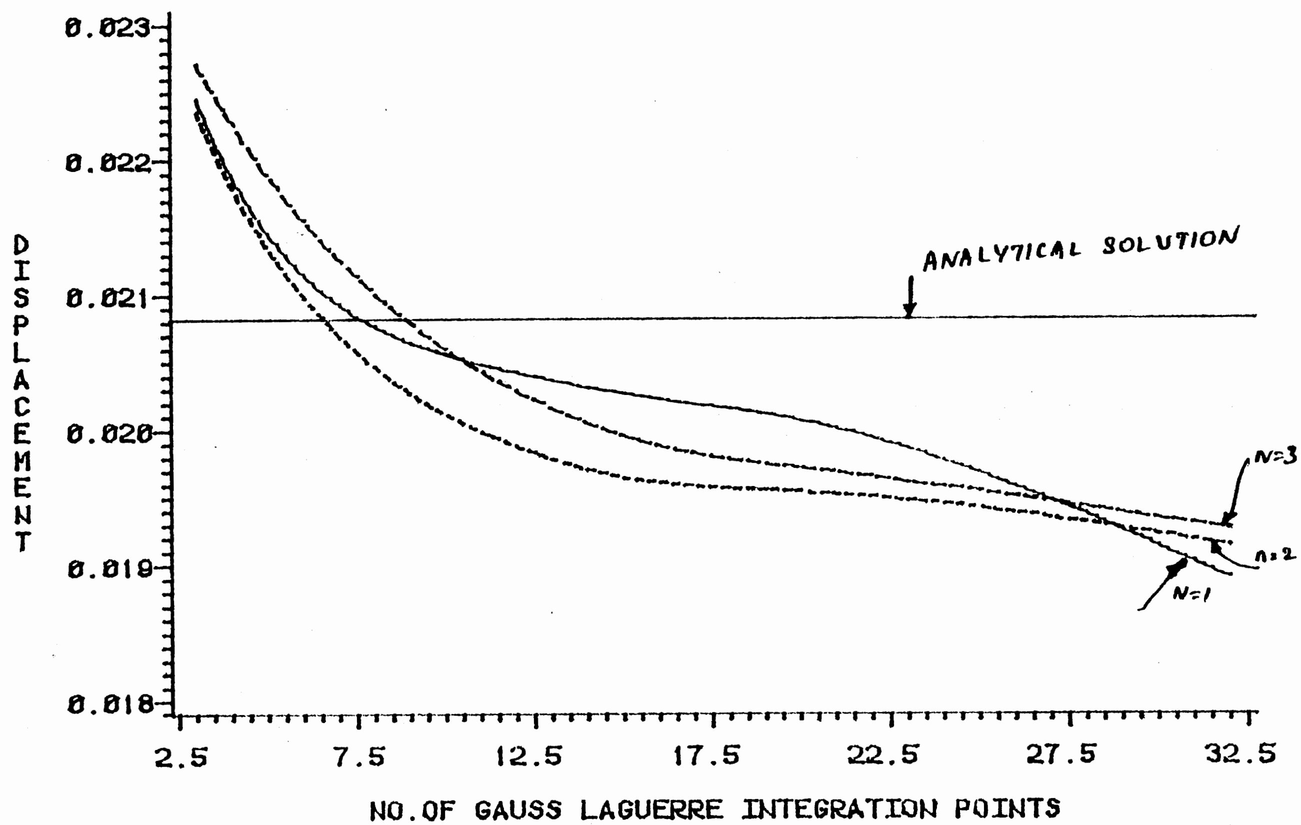

Figure 14. Effect of Gauss-Laguerre Integration Points for Different Exponents

different mesh sizes. The pattern is similiar for all cases. The displacements are underestimated , when more sampling points are used. For coarse meshes the effect is negligible and the results are generally poor. This is due to the poor modelling of the problem.

The second method of performing the numerical integration is by transforming the semi infinite interval to the finite bounds of -1 to +1. This is achieved by a transformation given in Appendix(B). The standard Gauss-Legendre scheme can then be used to evaluate the terms in the stiffness matrix.

This scheme was applied by considering 48 sampling points. Five beam elements and one infinite element were used. The length of the beam elements was 10 units. In Figure 17 the maximum displacement was plotted for different exponents n and for different values of decay origin. Increasing n led to better results. For large values of decay length the infinite element behaves like a regular beam element as the decay is negligible. Imposing the boundary conditions for the node at infinity leads to the underestimation of the displacements. Small values of decay length lead to the overestimation of the stiffness of the infinite element. So an optimum value of the decay origin should be used.

Figure 15 illustrates the solution behavior of the infinite element model. The displacements farther from the load are overestimated. The function used

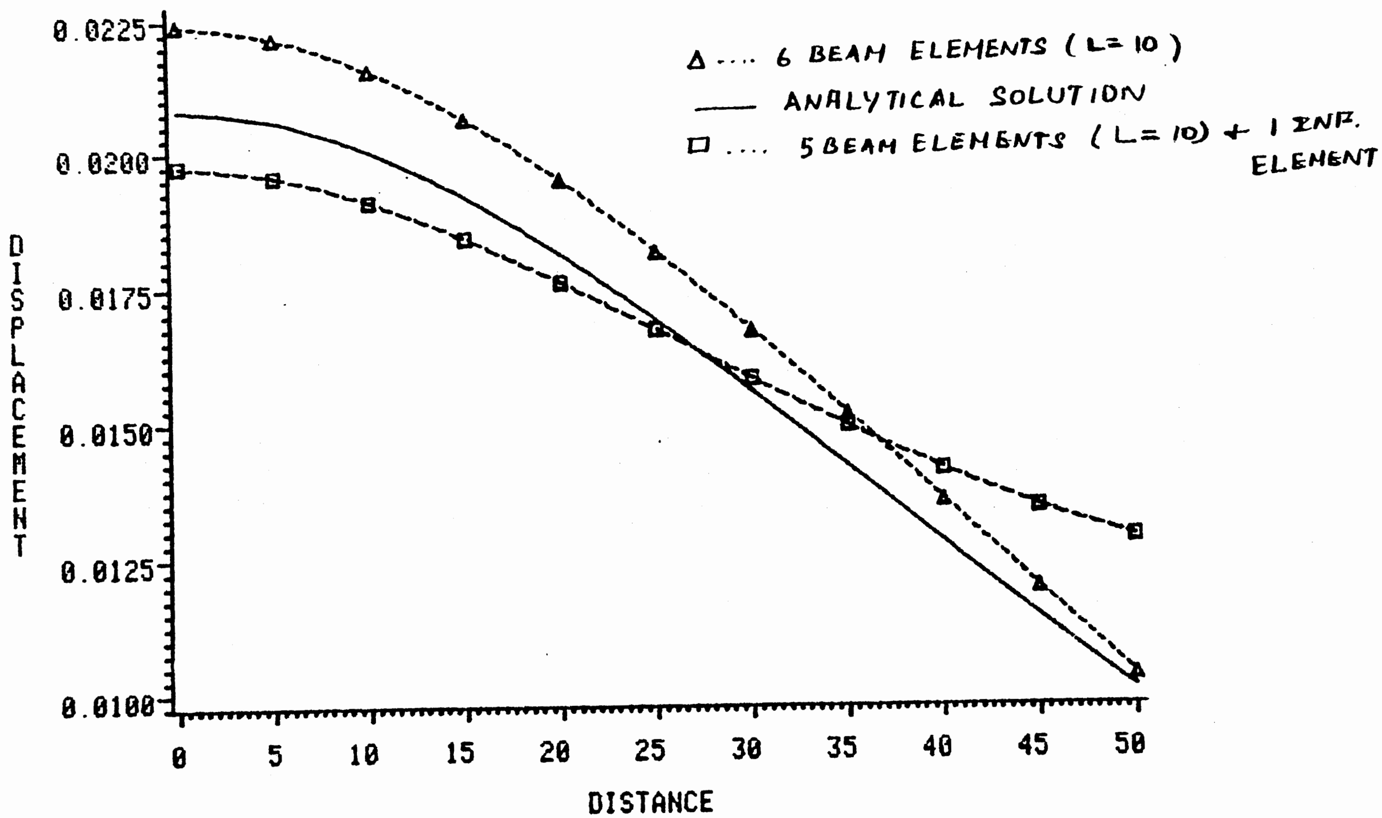

Figure 15. Comparison of Infinite and Finite Element Models for the Reciprocal Decay Infinite Element (Gauss-Legendre Scheme)

for the transformation of the unbounded region to the finite region is shown in Figure 27 . The graph is not smooth and this can be one reason for poor solution behavior. To overcome the problem of the third node at infinity, which makes the terms in the stiffness matrix unbounded, the third node was placed at a large but finite distance. This large interval can be converted to the interval -1 to +1 by a simple transformation. For this range the Gauss-Legendre scheme can be used. The solution behavior is very poor as illustrated in Figure 16.

Effect of Decay Origin

The decay origin can significantly affect the results if the beam elements are not extended far enough. Figures 17 and 18 give the effect of decay origin for beam element length of 10 units. At low values of decay origin and large values of exponent n lead to severe decay in the interpolation functions. The stiffness of the infinite element is overestimated and the displacements are small. For large values of decay origin and small values of n the decay is small and the infinite element behaves like a regular beam element. Imposition of the boundary conditions lead to the underestimation of the displacements. Only for Gauss-Laguerre scheme, the displacements are not underestimated for large values of decay origin. This is

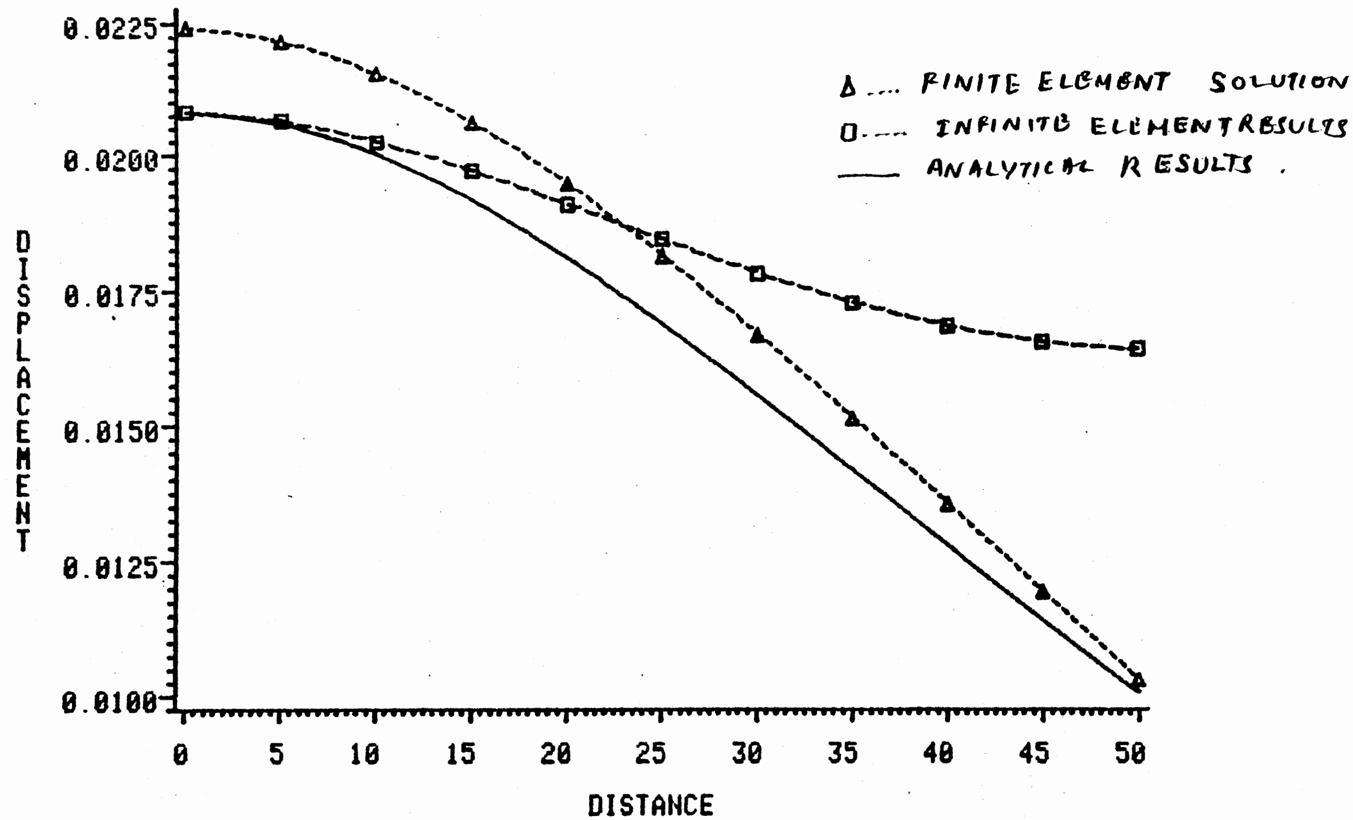

Figure 16. Solution Behavior of the Reciprocal Decay Infinite Element with the Third Node at a Large and Finite Distance

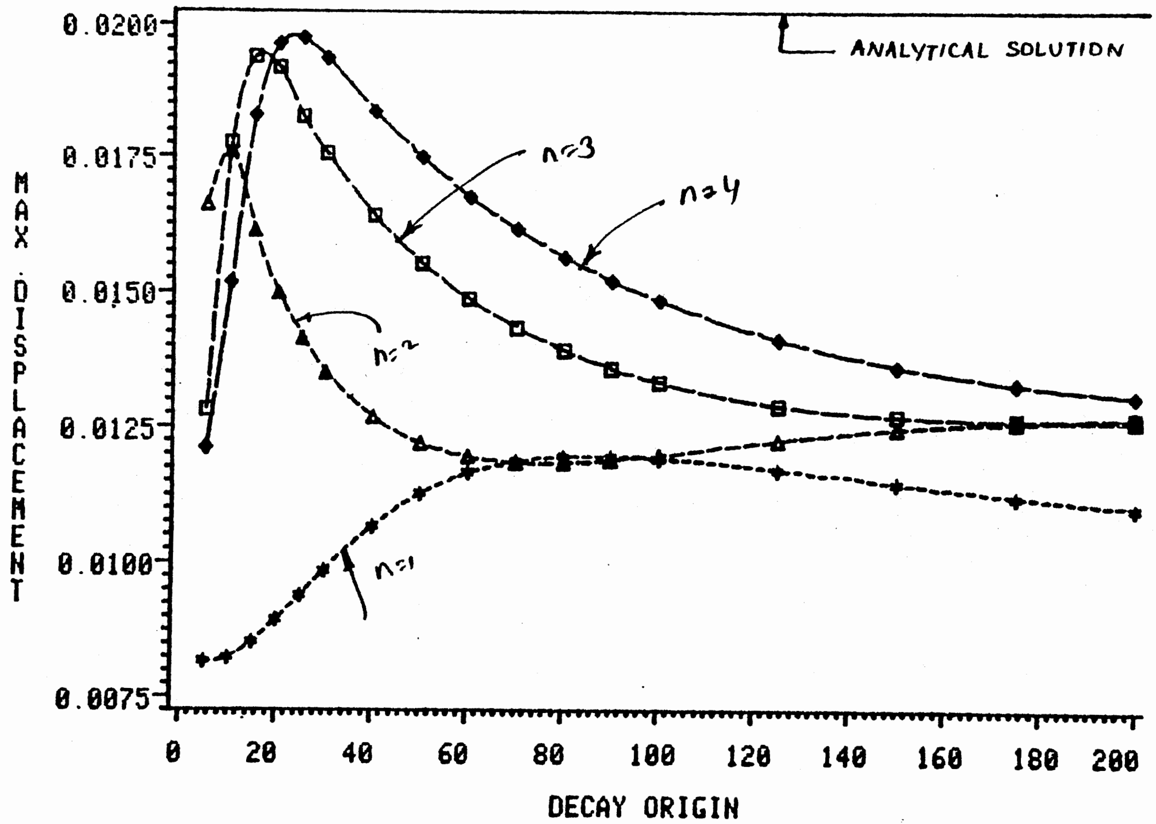

Figure 17. Effect of Decay Origin for Different Exponents (Gauss-Legendre Scheme)

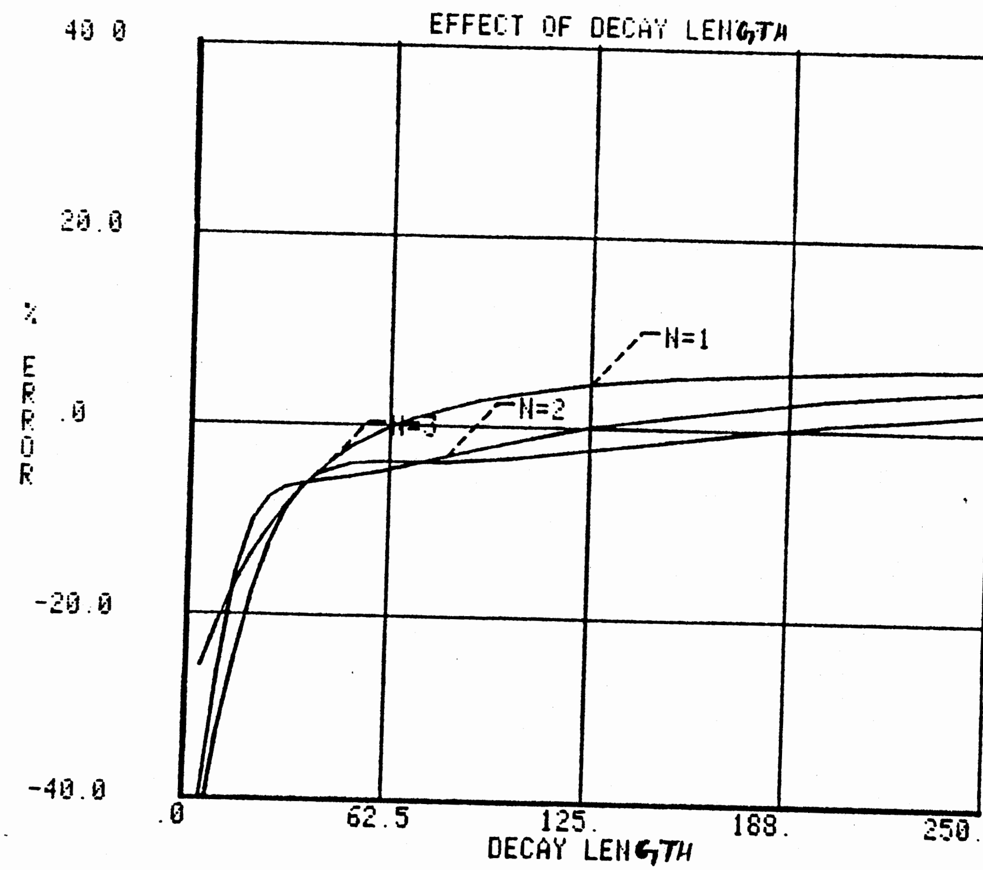

Figure 18. Effect of Decay Origin for Different Exponents (Gauss-Laguerre Scheme)

probaly due to inaccuracies in the stiffness matrix evaluation, but the exact reason is not clear. For very coarse mesh sizes the effect of decay origin is negligible Refining the mesh size leads to better results. For an optimum mesh size of beam elements there is not much variation in the displacements and the results are good. It is very hard to find this mesh size in a conventional finite element analysis of infinite domain problems. By studying the effect of decay parameters on different meshes, it is possible to find this optimum mesh size in an infinite element model. Figure 19 illustrate this behavior.

Comparison of Results

Figures 20 and 21 provide a comparison of finite and infinite element model. The terms in the stiffness matrix were evaluated using Gauss-Laguerre scheme. 15 sampling points were used. In Figure 20 a total of 6 elements was used for finite element and infinite element models. The element length of beam elements was 10 units. The problem was solved first by using six finite elements and then by replacing the last element with infinite element.

The infinite element model gives closer values to the analytical solution than the finite element model. The number of beam elements in the finite element model was doubled to twelve elements and this result is compared with

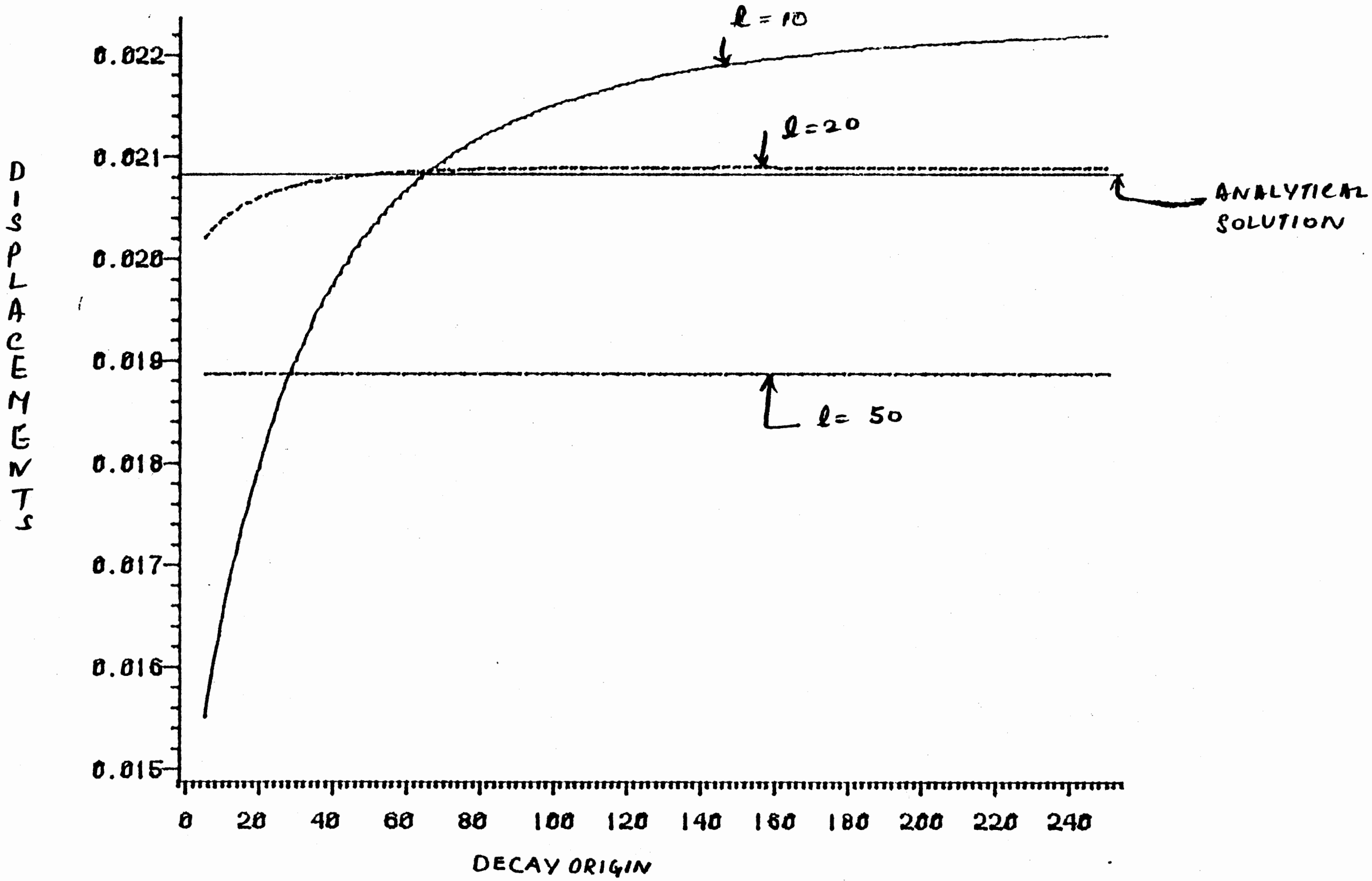

Figure 19. Effect of Decay Origin on Different Element Meshes (Gauss-Laguerre Scheme)

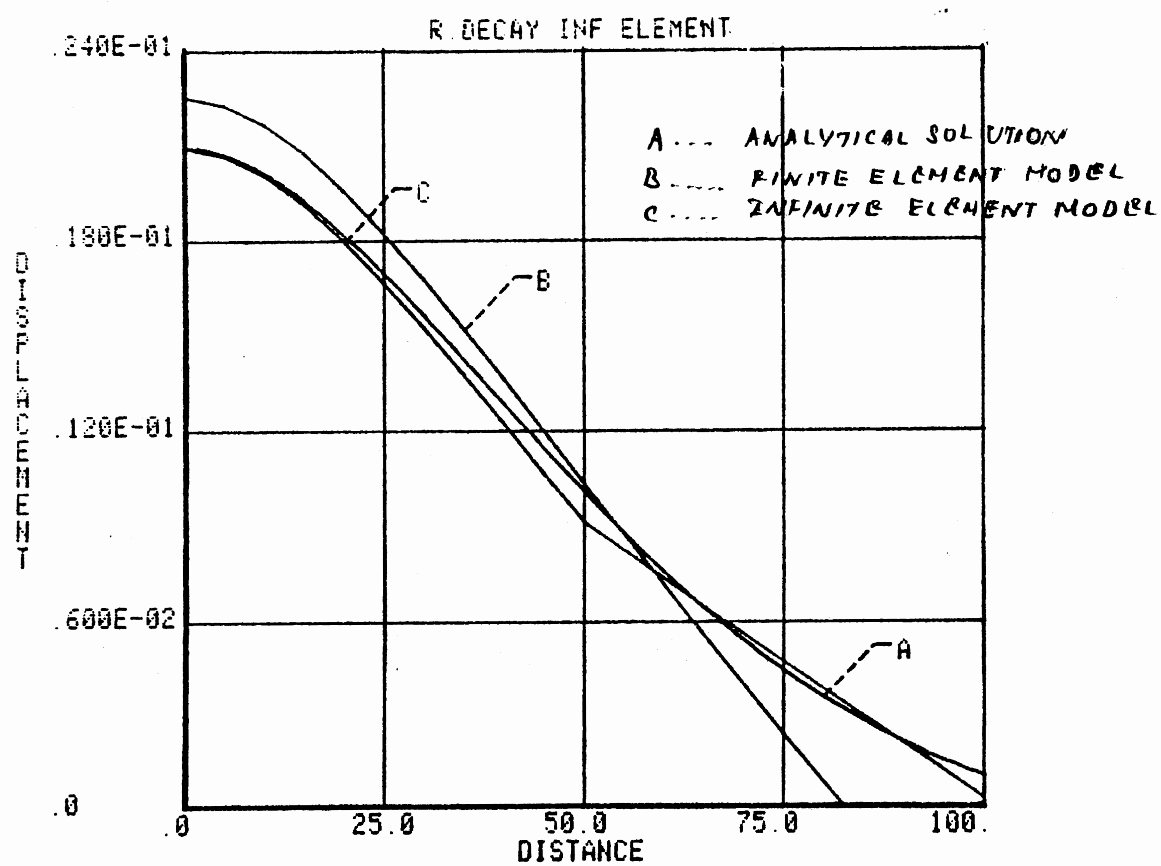

Figure 20. Comparison of the Finite Element (Six Elements) and Reciprocal Decay Infinite Element (Six Elements) Models

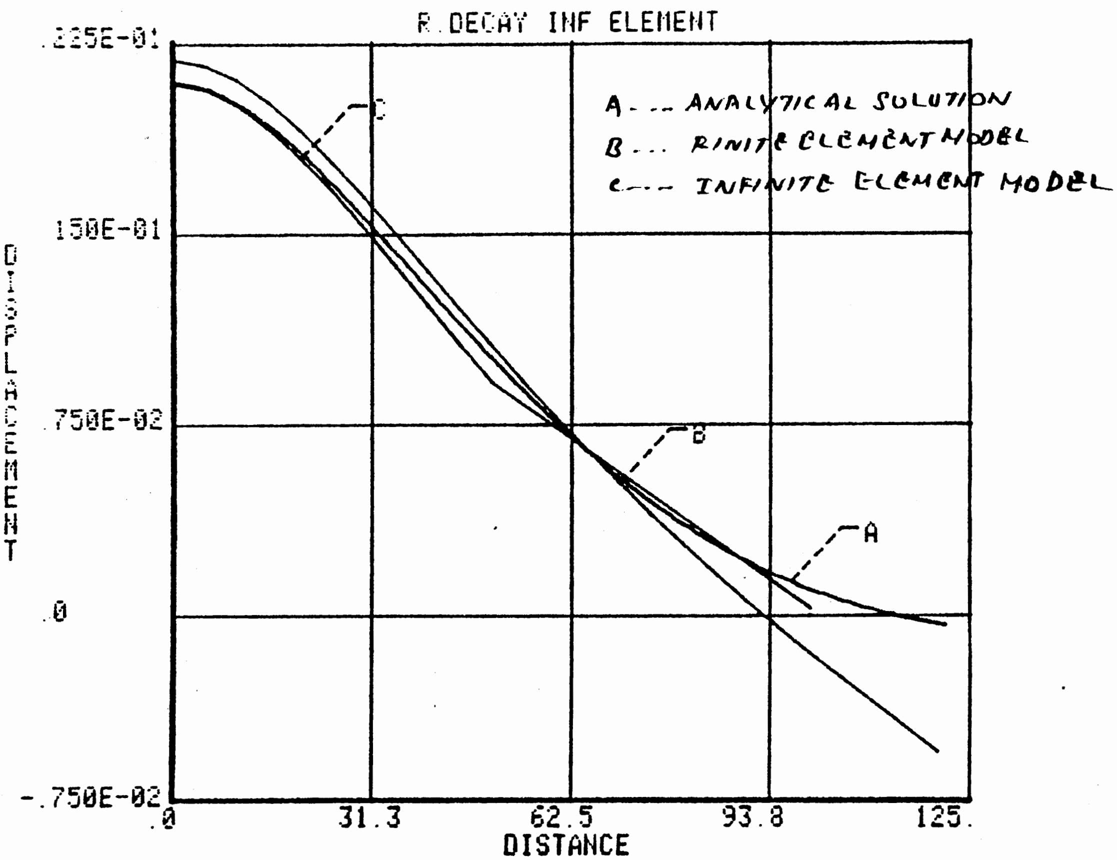

Figure 21. Comparison of The Finite Element (12 Elements) and Reciprocal Decay Infinite Element (Six Elements) Models

the infinite element model having six elements in Figure 21. The infinite element model still gives better results than the finite element model with only half the number of elements. The farfield solution of both the finite element and the infinite element models do not agree with the analytical results. The use of infinite elements can cut down the number of elements required to achieve desired accuracy. Eventhough, the stiffness matrix is not properly defined it is interesting to note that the results obtained are better than the results of the finite element model.

CHAPTER V

IMPLEMENTATION OF EXPONENTIAL DECAY INFINITE ELEMENT

Exponential Decay Infinite Element

The infinite element has three nodes the third one at infinity. The shape functons are defined in terms of the local coordinate system s (Figure 3). The three nodes are placed at s= 0, 50 and ∞. The location of second node is arbitary and it was found that its location does not alter the results.

The shape functions associated with nodal displacement are given by

$$w(s)= N_1(s)\ w_1 + N_2(s)\ w_2$$
$$\theta(s)= N_1(s)\ \theta_1 + N_2(s)\ \theta_2 \qquad (5.1)$$

where,

$$N_1(s)= (\exp(s_1 -s)/L)*(s_2 -s)/(s_2 -s_1)$$
$$N_2(s)= (\exp(s_2 -s)/L)*(s_1 -s)/(s_1 -s_2)$$

The shape function corresponding to the third node at infinity is condensed out as displacements are negligible at a large distance from the application of load.

L is an arbitary decay parameter. There is no fixed mathematical reason for choosing it. It helps to vary the severity of decay. Small values of L can be used when the solution decays fast. A typical shape function is sketched in Figure 22. The terms in the stiffness matrix can be exactly integrated by using the Gauss-Laguerre scheme. This is an advantage over the reciprocal decay infinite element which presents some numerical integration problems.

Effect of Decay Parameter L

The effect of decay parameter was studied by varying 'L' from 2 to 100. Different mesh sizes were used as shown in Figure 5. In all cases 5 beam elements and one infinite element were used. Figure 23 gives a plot of maximum dislacement for different values of L. For coarse meshes the effect of decay parameter L is negligible. The results obtained are poor. Refining the mesh size of beam elements to an optimum value (20 units) gives good results and there is a small variation of displacements for different values of L. For a small mesh size of beam elements there is considerable variation in the displacements.

The displacements are underestimated for small values of L. This is due to overestimation of stiffness of infinite element. For large values of L the decay is small and the infinite element behaves like a beam element. Imposing the artificial boundary condition leads to underestimation

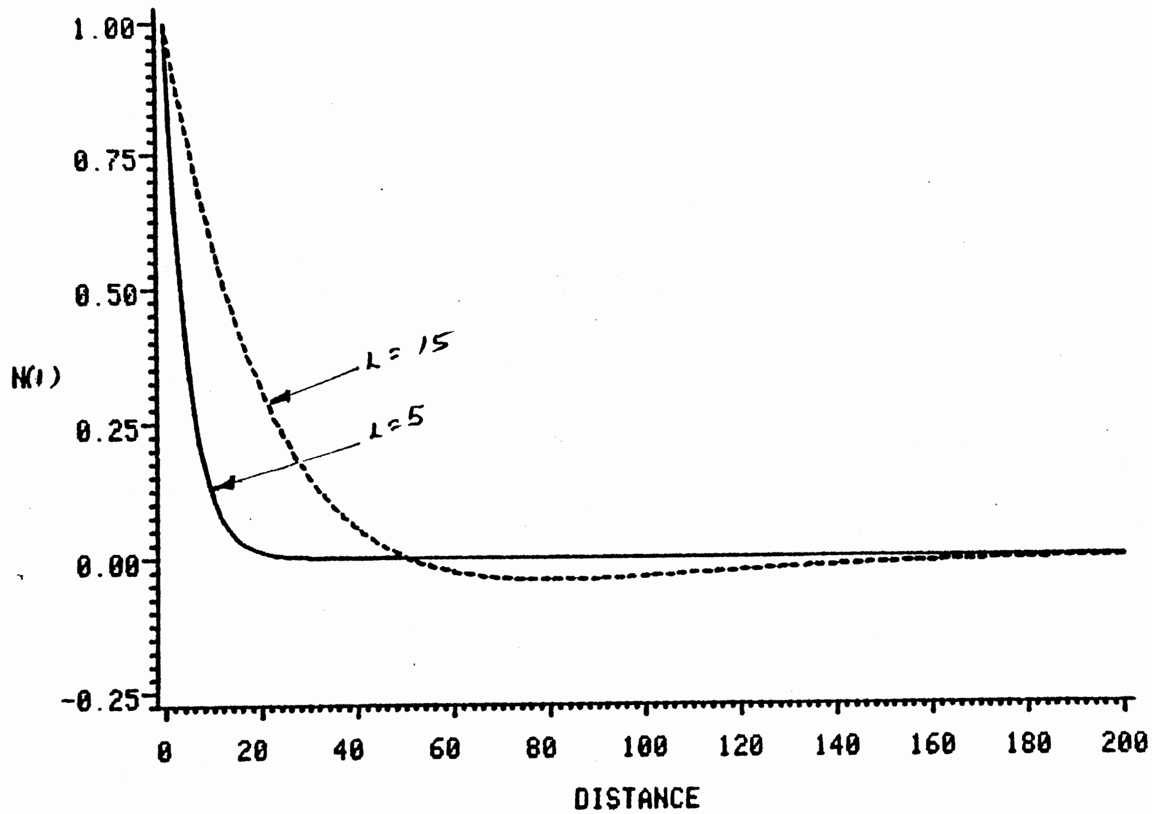

Figure 22. Shape Functions for Exponential Decay Element

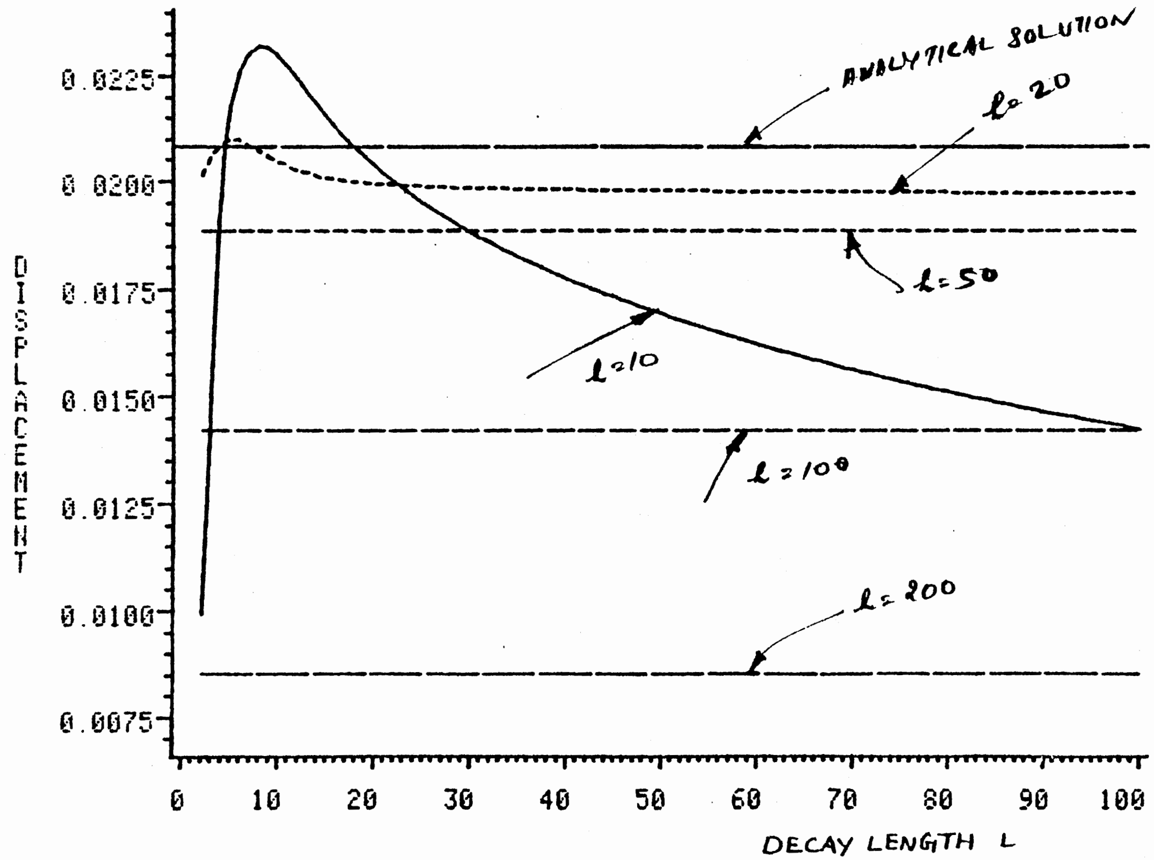

Figure 23. Effect of Decay Parameter 'L' on Different Meshes

of displacements.

Comparison of Results

Figure 24 gives the comparison of finite and infinite element models. An optimum value of L was used. The length of beam elements was 10 units. The infinite element model was constructed by using five beam elements and one infinite element. The infinite element model gives closer results to the analytical solution than the finite element model having six and twelve beam elements. Thus, with only half the number of beam elements, the infinite element gives closer results to the analytical solution. This result is significant since, in complex three dimensional problems the number of regular finite elements needed to achieve a certain accuracy may be quite large and the use of infinite elements can cut down the number of finite elements.

Figure 25 gives the result for doubling the length of beam elements. A total of 6 elements was used in the finite element and infinite element models. Both models give good results. Further increase in lengths of beam elements leads to poor results. The infinite domaim problems cannot be solved by using coarse mesh sizes. An optimum mesh size should be chosen to cut down the number of elements. It is hard to find this optimum mesh size by using the conventional finite

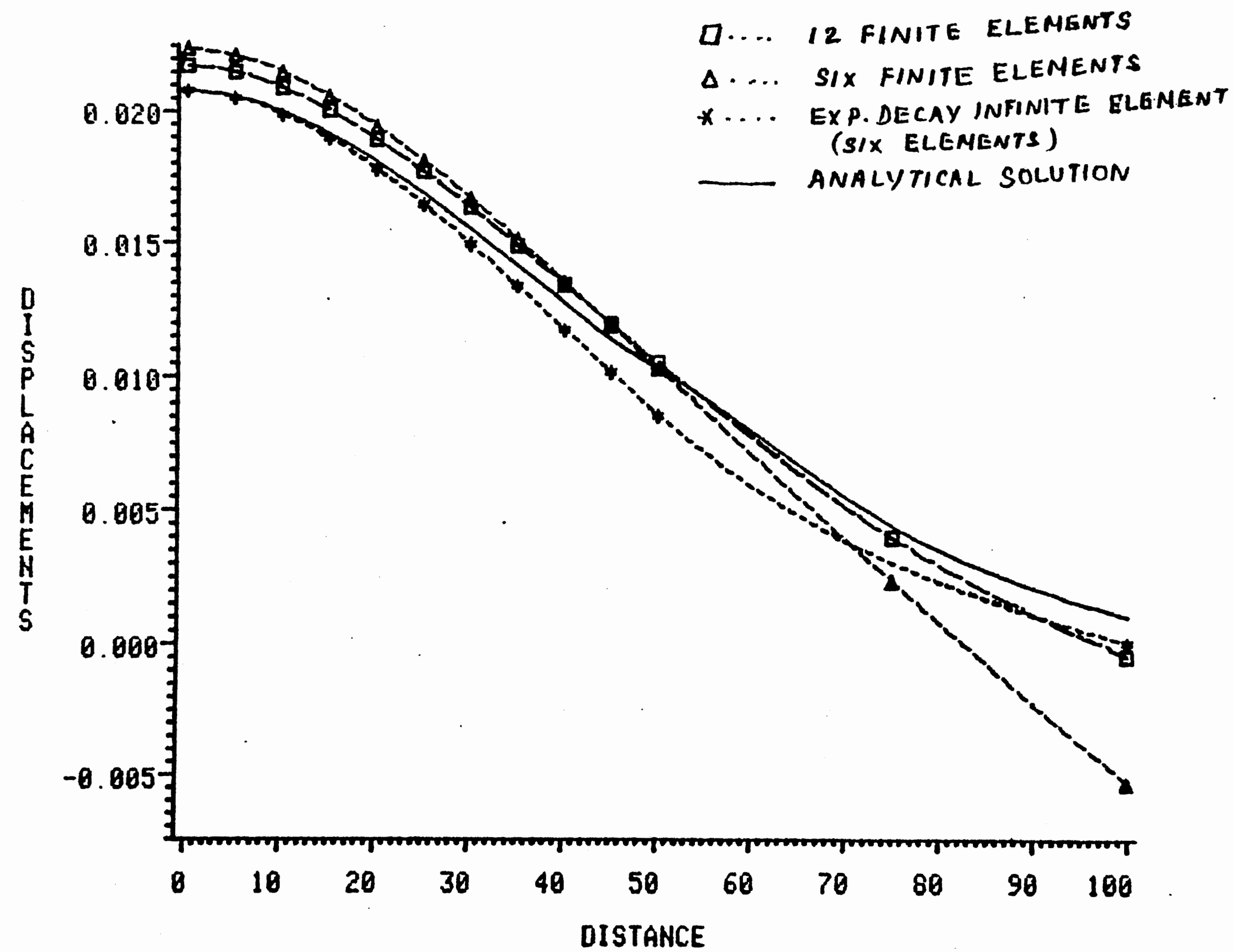

Figure 24. Comparison of the Finite Element and Exponential Decay Infinite Element Models (Mesh Size=10)

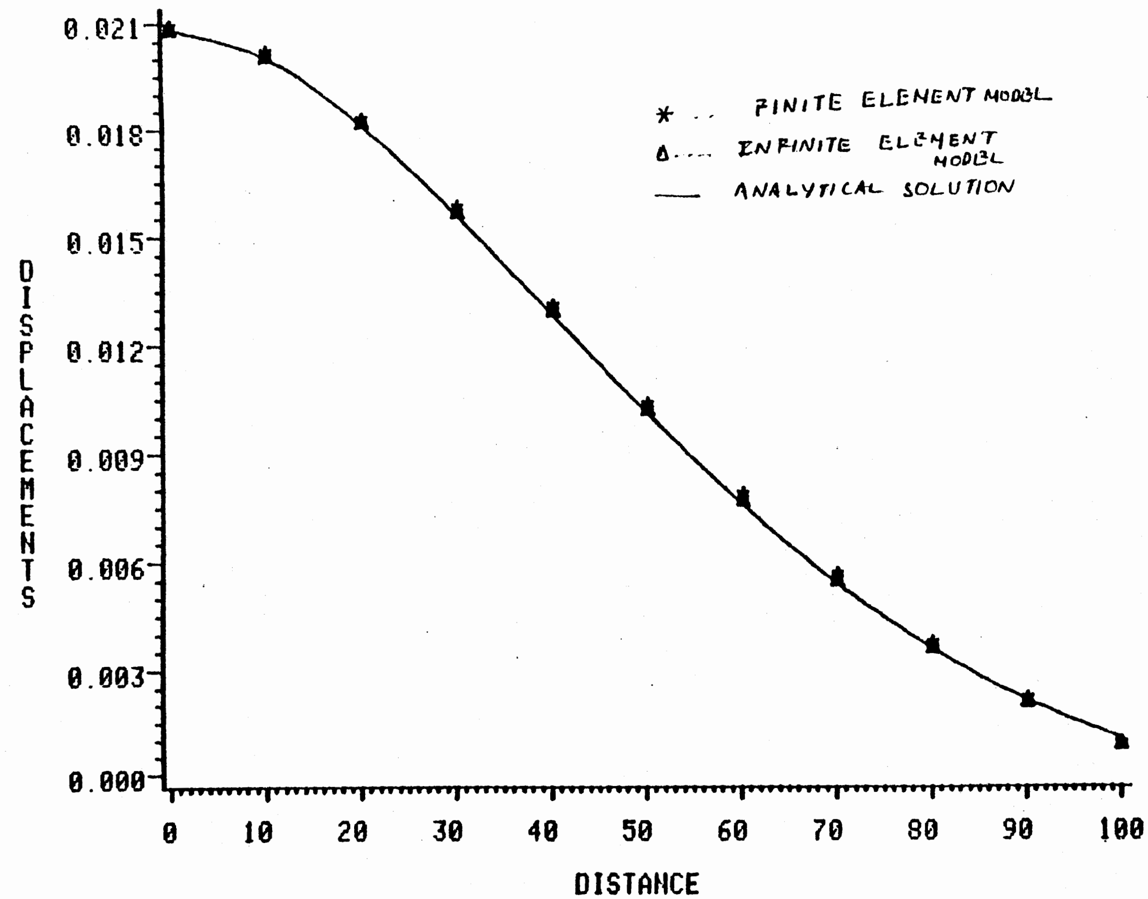

Figure 25. Comparison of the Finite Element and Exponential Decay Infinite Element Models (Mesh Size=20)

elements. By using infinite element and studying the effect of decay parameters on various mesh sizes it is possible to find an optimum mesh size. It should be noted that the infinite elements do not generally give good results in the far field. The infinite elements give the effect of far field on the domain of interest.

Comparison of Different Infinite Elements

Figure 26 provide the comparison of different types of the infinite elements with finite element model. A total of 6 elements was used in all cases. All infinite elements with the exception of mapped infinite element with no reciprocal decay, give better results than the finite element model. In all cases it was found that for small mesh sizes (if the beam elements are not extended far enough), variation of arbitary parameters like decay origin,decay length L lead to considerable changes in solution behavior. For very coarse mesh sizes, the variation of these parameters have no effect and the results are poor. Refining the mesh size from a very coarse size leads to solutions with increasing accuracy. For an optimum mesh size there is a small variation in solution behavior and good results can be obtained.

This behavior of infinite elements can be used for solving complex problems where no analytical solution exist. Trials should be made with different mesh sizes

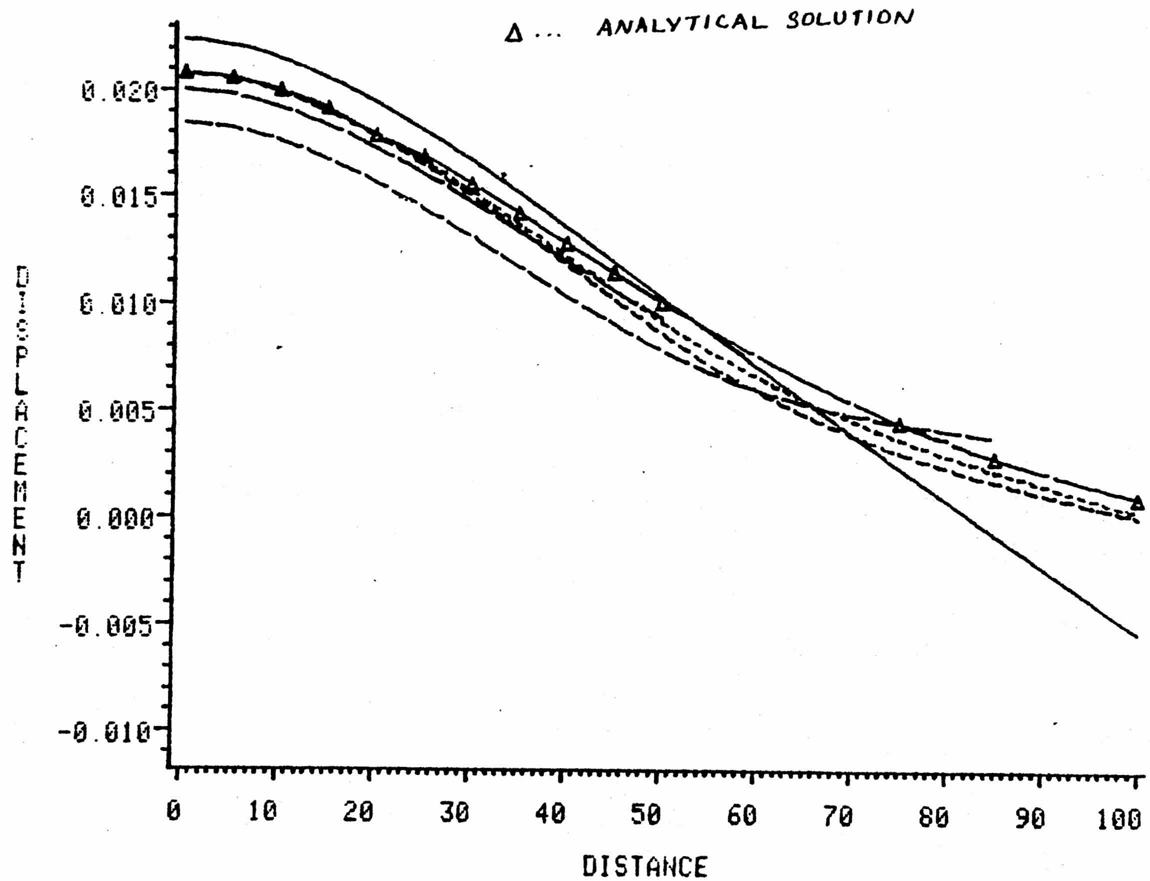

Figure 26. Comparison of the Different Infinite Elements

and decay parameters. An optimum mesh size should then be chosen where there is only a small variation in solution behavior.

The infinite element models generally do not give good results in the far field. The infinite element do not represent the far field behavior. The effect of far field on the domain of interest is modelled by the infinite element. The desired level of accuracy for the infinite domain problem can be achieved with less number of elements in an infinite element model than with finite element model.

The reciprocal decay element has some numerical integration problems. Hence, the stiffness matrix of the infinite element may not be well defined. It is posssible to get good results if small number of sampling points are used in the stiffness matrix evaluation. The mapped infinite element is simple in concept as no special numerical integration schemes are needed. The reciprocal decay in the mapped infinite element makes it more versatile for dealing with problems having severe decay in the solution behavior.

CHAPTER VI

CONCLUSIONS AND RECOMMENDATIONS

The infinite element is an elegant way for analysing infinite domain problems. This method has a number of advantages.

1. This method can be used to model complex problems where little is known about the solution behavior. By studying the effect of decay parameters on different meshes it is possible to arrive at a good solution.

2. The infinite elements are simpler theoretically. They simply appear as a slightly different element type.

3. The infinite elements can be introduced into existing finite element programs simply as addition to the element library. Most of the other techniques like boundary element method, need special procedures. The infinite elements do not destroy either the symmetry of the equations or their banded structure.

4. The infinite element should make the analysis of infinite domain problems more economical by reducing the number of elements used to model regions remote from the domain of interest. This benefit should be most marked in three dimensional problems.

5. This method can be applied to non linear problems

which are difficult to solve by boundary integral methods.

There are some disadvantages in the use of infinite elements. A suitable quadrature formula for a semi infinite interval must be included in the finite element program. The choice of various decay parameters is arbitary and can affect the results if they are not properly chosen. The infinite elements do not represent the true solution behavior of farfield. Thus, the infinite element cannot be used to find good solutions in the far field.

The recommendations for future study are as follows:

1. The effect of decay parameters should be investigated for dynamic analysis.

2. The infinite elements can be applied to the acoustic radiation problems, where finite element solutions have proved to be complex and expensive.

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APPENDIX A

DEVELOPMENT OF STRUCTURAL EQUATIONS

The structural equations can be developed from the energy considerations (15). In any continum problems the actual number of degrees of freedom are infinite and, unless a closed form solution is available an exact analysis is impossible. In the finite element method the continum is divided into a series of elements which are connected at a finite number of points known as nodal points. The governing equilibrium equations can be obtained by minimising the total potential energy of the system. The total potential energy ,π, can be expressed as

$$\pi = 1/2 \int_V [\sigma]^T \epsilon \, dv - \int_V [\delta]^T p \, dv - \int_S [\delta]^T q \, ds \qquad (A.1)$$

where σ and ϵ are the stress and strain vectors, δ the displacements at any point, p the body force per unit volume and q the applied surface tractions. Integrations are taken over the volume V of the structure and loaded surface area, S.

The first term on the right hand side of A.1 represents the internal strain energy and the second and third terms are respectively the work contributions of the body forces and distributed surface loads.

In the finite element displacement method, the displacement is assumed to have unknown values only at the nodal points, so that the variation within the element is described in terms of the nodal values by means of simple interpolation functions. Thus,

$$\delta = N\delta^e \tag{A.2}$$

where, N is the set of interpolation functions termed the shape functions and δ^e is the vector of nodal displacements of the element. The strains within the element can be expressed in terms of the element nodal displacements as

$$\epsilon = B\delta^e \tag{A.3}$$

where B is the strain matrix generally composed of derivatives of the shape functions. Finally the stresses may be related to the strains by the use of an elasticity matrix D, as follows

$$\sigma = D\epsilon \tag{A.4}$$

The total potential energy of the continum will be the sum of the energy contributions of the individual elements. Thus,

$$\pi_e = \int_{V_e} [\delta^e]^T [B]^T [D] [B] \delta^e dv - \int_{V_e} [\delta^e]^T [N]^T p\, dv - \int_{S_e} [\delta^e]^T [N] q\, ds \tag{A.5}$$

where V_e is the element volume and S_e the loaded element surface area. Performance of the minimisation for element e with respect to the nodal displacements δ^e for the element results in

$$\partial\pi_e/\partial\delta^e = \int_{V_e}([B]^T[D][B])\delta^e \, dv - \int_{V_e}[N]^T p \, dv - \int_{S_e}[N]^T q \, ds = K^e\delta^e - F^e \qquad (A.6)$$

where,

$$F^e = \int_{V_e}[N]^T p \, dv + \int_{S_e}[N]^T q \, ds \qquad (A.7)$$

are the equivalent nodal forces for the element, and

$$K^e = \int_{V_e}[B]^T[D][B] \, dV_e$$

The summation of the terms in A.6 over all the elements, when equated to zero, results in a system of equilibrium equations for the complete continum. These equations are then solved by any standard technique to yield the nodal displacements.

APPENDIX B

TRANSFORMATION FOR SEMI-INFINITE NUMERICAL INTEGRATION

The evaluation of stiffness matrix for the reciprocal decay infinite element involves numerical integration of integrals of the form

$$\int_{a}^{\infty} f(s)\ ds \qquad (B.1)$$

This interval can be transformed to the range -1 to +1 by a mapping of the form

$$s = 2/1-t \qquad (B.2)$$

Assuming a=1, the following corresponding points can be identified,

At t=1 s= 2/0 = ∞

At t=-1 s= 2/2 = 1

Thus the integral given by B.1 can written as

$$\int_{a}^{\infty} f(s)\ ds = \int_{-1}^{+1} g(t)\ dt \qquad (B.3)$$

The function g(t) is given by

$$g(t)= f(2/-t +1)*2/(1-t)^2$$

This can be easily programmed by modifying the weights and abscissa of standard Gauss-Legendre formula.

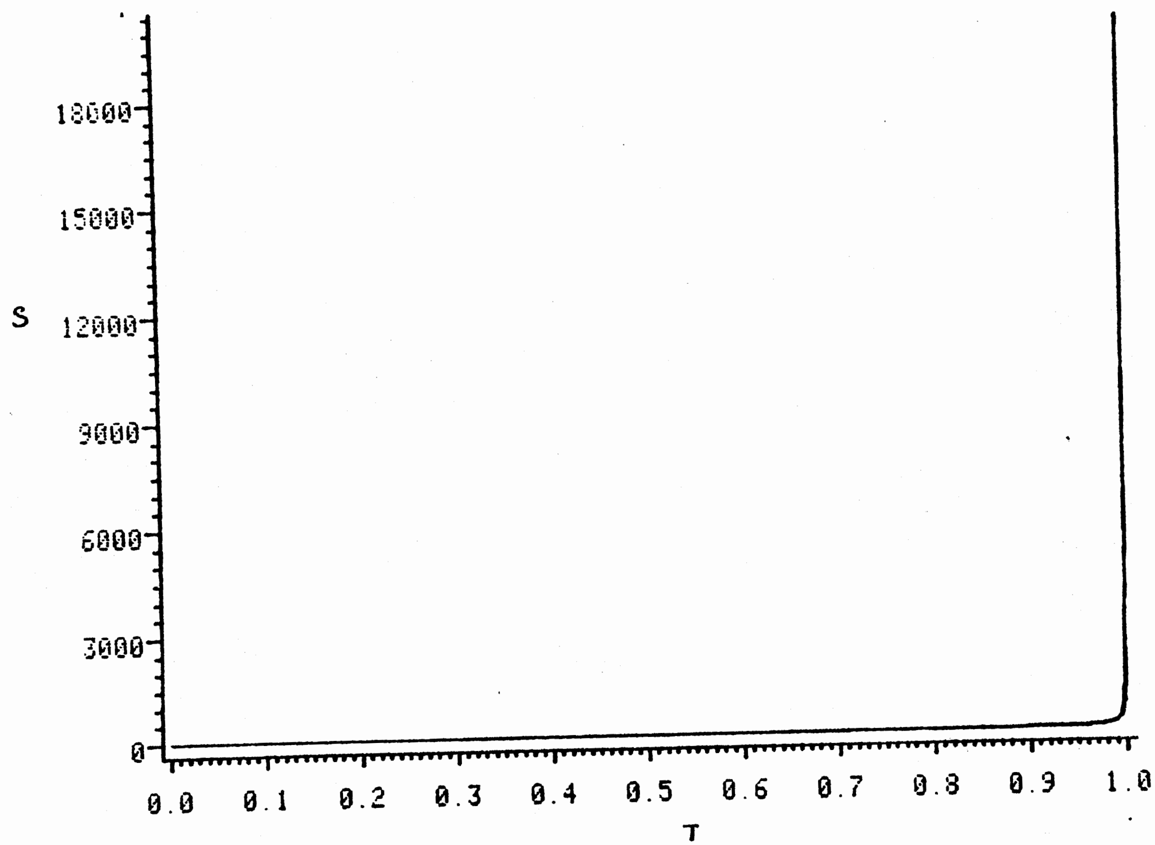

Figure 27. Mapping Function S=2/1-T

The mapping function used for transformation is plotted in Figure 27. It shows a steep climb and is not smooth.

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