A THEORETICAL STUDY OF GRAVITATIONAL COLLAPSE FOR A QUARK STAR

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CHAPTER I

INTRODUCTION

The idea that a massive object, such as a star, might undergo a collapse due to the mutual gravitational attraction of its component parts may well have existed long before 1930. Both Newton's Law of Universal Gravitation and its successor, the General Theory of Relativity of Einstein admitted such a possibility. However, astronomical and common physical evidence seemed to indicate that other forces such as those arising from thermal expansion would always prevent such a gravitational collapse. Thus study in this area of gravitational theory was nearly non-existent before 1930 as it was thought an unfruitful pursuit. Interestingly, the same astronomical and astrophysical considerations were eventually to give importance to such studies. Prior to 1915, most stellar research was confined to astronomical observations of the physical characteristics of stars such as shown in a Hertzsprung-Russell diagram, and to examine the structure of stellar objects in terms of classical concepts of thermo-dynamics, fluid mechanics and Newtonian gravitation.

However, in about 1915 a new class of stellar objects was observed, the white dwarfs, whose properties could not be explained by means of concepts existing at the time. It was not until 1926 that Fowler, employing the newly developed Fermi-Dirac quantum statistics, described the composition of white dwarfs, stating that their average densities of

10^5 to 10^8 gm/cm^3, as compared with the average density of our Sun, a little over that of water, 1 gm/cm^3, gave rise to a new state of matter. Such matter was composed of completely ionized atoms surrounded by a degenerate electron gas and the internal pressure was due to this quantum degeneracy. With this premise, Chandrasekhar[1] showed that the radius of a white dwarf was inversely proportional to the cube root of its mass, thus indicating that all white dwarfs would reach stable configurations.

Shortly however, he and, independently, Landau discovered that a modification to the original premise was necessary, and this led to the important result that there existed an upper limit of $1.4M_{\odot}$, or 1.4 Solar masses, to the amount of mass which could be supported by the electron degeneracy pressure against gravity. With this introduction of the "Chandrasekhar Limit" came the realization that certain stars might not undergo quasistatic evolution but instead experience a catastrophic gravitational collapse. Thus employing Einstein's gravitation theory as the most complete theory available, research began on the possibility of such an event. During the early years appeared many classic papers such as those of Tolman, Oppenheimer and Snyder, and Oppenheimer and Volkoff[2,3,4]. While Tolman dealt with several general static solutions of the gravitational field equations, Oppenheimer and Snyder examined the continued collapse of a spherical collection of mass experiencing no internal pressure, that is a free fall. Oppenheimer and Volkoff considered the behavior of a new type of matter, a further stage in the collapse scenario. An object of this type consisted of matter similar to that of white dwarfs which had been further compressed, forcing most electrons to enter the nuclei of the ionized atoms and causing most protons within to change to neutrons. Such a neutron star would experience an internal pressure due to the degeneracy of the neutrons. Similarly to the case of the white dwarf, it was shown that there was again an upper limit to the mass of a neutron star, this time of $.7M_{\odot}$, the "Oppenheimer-Volkoff" limit. In addition to these results further concepts were introduced in the mid to late 1930's, such as the possibility that a sufficiently massive object might collapse until even radiation would be unable to escape its intensely concentrated gravitational field. Also a connection was suggested that the triggering mechanism of a supernova might be related to the gravitational collapse of a star, and that the remnants of such an explosion might form a neutron star.

However, it must be noted that these papers employed certain assumptions such as specifically simple equations of state, relations between density and pressure within a star, and dealing primarily with the stability of hydrostatic solutions. At this point further research into the hydrodynamic collapse of stellar objects was barred by the lack of knowledge of the internal structure, if any, of neutrons without which much further evolution of a star could not be adequately described, and also by the need to develop methods of examining general relativistic hydrodynamic collapse. In the 1960's with the analytical calculations of Misner and Sharp[5] as a background, May and White[6] employed a numerical approach to examine dynamic collapse with known equations of state. In this period also arose the quark model of strongly interacting particles such as the neutron, the hadrons, from which in the mid to late 1970's it became possible to examine properties of quark matter such as the nature of a phase change from neutron to quarks and importantly to provide possible equations of state. It thus becomes relevant to consider a star in an advanced state of collapse consisting of predominantly quark matter. It is this point at which our work will enter, examining certain features of the general relativistic hydrodynamic behavior of such an object.

Chapter II develops the form of the Einstein equations employed, and their treatment as hydrodynamic equations. Chapter III then details the numerical integration procedure used to solve these equations. Finally Chapter IV explains the application of this method to a quark star, with results and concluding remarks.

CHAPTER II

THE GENERAL RELATIVISTIC HYDRODYNAMIC EQUATIONS

Available Formalism

The theory of differential geometry introduces a general form for the element of arc length on a manifold as

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$$

where we employ the Einstein summation convention; that is, repeated Greek indices are to be summed over. The components $g_{\alpha\beta}$ of the metric tensor generate a covariant tensor field which describes the geometry of the space.

Einstein's General Theory of Relativity employs this concept of a metric to represent gravitation as a manifestation of the curvature of space-time, thus taking x^0 as the time coordinate. The presence of mass-energy acts as a source of this curvature in a fashion which Einstein expressed in the tensor equation[7]

$$G^\alpha{}_\beta \equiv R^\alpha{}_\beta - \frac{1}{2} g^\alpha{}_\beta R = \frac{8\pi G}{c^4} T^\alpha{}_\beta$$

where the source term on the right side contains the energy momentum tensor components $T^\alpha{}_\beta$. The remaining quantities may be determined from the metric tensor according to the following relations: The curvature scalar

$$R = R^{\alpha}{}_{\alpha},$$

the Ricci tensor

$$R^{\alpha}{}_{\beta} = g^{\alpha\lambda}R_{\lambda\beta} = g^{\alpha\lambda}R^{\gamma}{}_{\lambda\gamma\beta},$$

and the Riemann curvature tensor

$$R^{\alpha}{}_{\beta\gamma\delta} = \Gamma^{\alpha}{}_{\beta\delta,\gamma} - \Gamma^{\alpha}{}_{\beta\gamma,\delta} + \Gamma^{\alpha}{}_{\lambda\gamma}\Gamma^{\lambda}{}_{\beta\delta} - \Gamma^{\alpha}{}_{\lambda\delta}\Gamma^{\lambda}{}_{\beta\gamma},$$

involving the Christoffel symbol of the second kind

$$\Gamma^{\alpha}{}_{\beta\gamma} = \frac{1}{2}\, g^{\alpha\lambda}[g_{\lambda\beta,\gamma} + g_{\lambda\gamma,\beta} - g_{\beta\gamma,\lambda}].$$

We note that by way of notation, a comma in the subscript denotes partial differentiation while a semicolon will be taken to denote covariant differentiation. G is the Universal Gravitation constant and c is the speed of light in vacuo.

Additionally if we consider only mass configurations behaving as perfect fluids with proper number density of baryons n, negligable temperature, and with the introduction of comoving coordinates such that the proper fluid 4-velocity takes the form

$$u^{\alpha} = \begin{cases} u^0, & \alpha=0 \\ 0, & \alpha\neq 0 \end{cases}$$

which also must satisfy the standard condition for any 4-velocity

$$u^{\alpha}u_{\alpha} = g_{\alpha\lambda}u^{\alpha}u^{\lambda} = c^2,$$

we may write the energy momentum tensor as[8]

$$T^{\alpha}{}_{\beta} = \rho w g_{\beta\lambda} u^{\alpha} u^{\lambda} - g^{\alpha}{}_{\beta} p.$$

We define ρ as the proper mass density, ε as the proper non-gravitational internal energy per unit mass, p as the proper pressure, and

$$w = 1 + \varepsilon/c^2 + p/\rho c^2$$

as the proper relativistic enthalpy. Thus conservation of baryon number is given by

$$(nu^{\alpha})_{;\alpha} = 0.$$

Also conservation of energy momentum is given as

$$T^{\alpha}{}_{\beta;\alpha} = 0.$$

The Einstein Equations

We will confine ourselves to a spherically symmetric dynamic mass configuration such that a general line element is given by[9]

$$ds^2 = a^2 c^2 dt^2 - b^2 d\mu^2 - R^2 d\Omega^2$$

where $d\Omega^2 = d\theta^r + \sin^2\theta\, d\phi^2$ is the element of solid angle on a sphere. The functions a, b and R are functions only of t and the radial coordinate μ. We note that R is a function which gives the correct proper area of $4\pi R^2$ for a sphere of radius μ, but may not necessarily be interpreted as a radius of such a sphere itself.

This metric tensor gives $u^0 = a^{-1}c$ from which we may write the energy momentum tensor components as

$$T^0{}_0 = \rho c^2(1 + \varepsilon/c^2),$$

$$T^1{}_1 = T^2{}_2 = T^3{}_3 = - P, \qquad (2\text{-}1)$$

and

$$T^\alpha{}_\beta = 0, \qquad \alpha \neq \beta.$$

With these relations the Einstein equations take the form

$$4\pi G\rho(1 + \varepsilon/c^2)R^2R' = \frac{c^2}{2}\left[R + \frac{R\dot{R}^2}{a^2c^2} - \frac{RR'^2}{b^2}\right]'$$

$$\equiv Gm', \qquad (2\text{-}2)$$

$$\frac{4\pi G}{c^2} PR^2\dot{R} = - \frac{c^2}{2}\left[R + \frac{R\dot{R}^2}{a^2c^2} - \frac{RR'^2}{b^2}\right]^{\cdot}$$

$$\equiv G\dot{m}, \qquad (2\text{-}3)$$

$$4\pi G\rho wR^3 = c^2\left[R + \frac{R\dot{R}^2}{a^2c^2} - \frac{RR'^2}{b^2}\right] + \frac{R^3c}{ab}\left[\left(\frac{ca'}{b}\right)' - \left(\frac{\dot{b}}{ac}\right)^{\cdot}\right],$$

and

$$\dot{R}' - \frac{a'\dot{R}}{a} - \frac{\dot{b}R'}{b} = 0. \qquad (2\text{-}4)$$

At this point we have felt it convenient to avoid confusion by representing differentiation with respect to μ by a prime and with respect to t by a dot. We have introduced the function m which represents the total gravitational mass interior to a sphere of radius μ , or

$$m = 4\pi \int_0^\mu \rho(1 + \varepsilon/c^2)R^2R'\, d\mu.$$

The integrand represents the total mass-energy density times the proper volume of a spherical shell between μ and $\mu + d\mu$.

These expressions along with an equation of state of the form

$$P = P[\rho(1 + \varepsilon/c^2)]$$

would in general allow solutions to be obtained for all functions. However, as the equation of state will usually be given as $P(\rho,\varepsilon)$, a further dynamic relation would be necessary. Such a situation arises in classical hydrodynamic problems, and thus we are led to examine the conservation of energy and momentum explicitly from this viewpoint. We will find that such examination will fulfill another requirement, that of fixing the radial scale.

The Hydrodynamic Approach

If we assume that all particles have identical masses, conservation of baryon number becomes

$$(\rho u^\alpha)_{;\alpha} = \frac{1}{c}\left(\frac{\rho}{a}\right)^\cdot + \Gamma^\alpha{}_{0\alpha}\left(\frac{\rho}{a}\right) = (\rho R^2 b)^\cdot = 0 \tag{2-5}$$

with the obvious result

$$\rho R^2 b = f(\mu).$$

For convenience we choose $f(\mu) = (4\pi)^{-1}$ which fixes the scale such that, if we write

$$4\pi R^2 \rho b = 1,$$

we have

$$\mu = 4\pi \int_0^\mu \rho R^2 b \, d\mu.$$

We realize that the coordinate then defines the proper mass contained within a sphere of radius μ. Such a choice of scale is commonly employed in classical hydrodynamics, and having mass as a "Lagrangian" coordinate insures conservation of mass.

Conservation of energy and momentum gives the relations for a comoving reference frame

$$[\rho(1 + \varepsilon/c^2)]^{\cdot} + (\frac{\dot{b}}{b} + \frac{2\dot{R}}{R})\rho w = 0$$

or by employing (2-4)

$$\dot{\varepsilon} = -P(\frac{1}{\rho})^{\cdot}$$

which is simply the first law of thermodynamics in the case of no heat flow or entropy generation, and

$$\frac{a'}{a} + \frac{P'}{\rho c^2 w} = 0. \qquad (2\text{-}6)$$

With a hydrodynamic approach in mind we make the following useful definitions:

$$\Gamma = \frac{R'}{b} = 4\pi\rho R^2 R',$$

and

$$U = \frac{\dot{R}}{a}.$$

Equation (2-1) then becomes

$$\Gamma^2 = 1 + \frac{U^2}{c^2} - \frac{2mG}{Rc^2}, \qquad (2\text{-}7)$$

and by taking the partial derivative with respect to t and employing the definition of Γ, we arrive at

$$\dot{U} = -a[\frac{2\pi R^2 P'\Gamma}{w} + \frac{mG}{R^2} - \frac{2mG}{Rc^2}].$$

From equation (2-6) we find

$$\frac{(aw)'}{aw} = [\varepsilon' + P(\frac{1}{\rho})']/wc^2$$

and finally with (2-4), conservation of mass becomes

$$\frac{(\rho R^2)'}{\rho R^2} = -a(\frac{U}{R})' \ .$$

Thus our task will be to obtain solutions to the general relativistic hydrodynamic equations

$$\dot{U} = -a[4\pi R^2 \Gamma P'/w + \frac{mG}{R^2} + \frac{4\pi G}{c^2} PR], \tag{2-8}$$

$$\dot{R} = aU, \tag{2-9}$$

$$\frac{(\rho R^2)^{\cdot}}{\rho R^2} = -a(\frac{U}{R})', \tag{2-10}$$

$$\dot{\varepsilon} = -P(\frac{1}{\rho})^{\cdot}, \tag{2-11}$$

$$\frac{(aw)'}{aw} = [\varepsilon' + P(\frac{1}{\rho})']/wc^2, \tag{2-12}$$

$$m = 4\pi \int_0^\mu \rho(1 + \varepsilon/c^2)R^2 R' \, d\mu, \tag{2-13}$$

$$\Gamma = 4\pi\rho R^2 R', \tag{2-14}$$

$$P = P(\varepsilon,\rho), \tag{2-15}$$

and

$$w = 1 + \varepsilon/c^2 + P/\rho c^2, \tag{2-16}$$

subject to certain boundary and initial conditions. As initial conditions we will assume that proper mass and internal energy densities are uniform throughout the configuration and that the object is uniformly at rest, or $U(\mu,t=0) = 0$. For boundary conditions we take $U(\mu=0,t) = R(\mu=0,t) = P(\mu=\mu_{max},t) = 0$ to be logical choices. In order that the proper time be equal to the clock time of an observer at the outer boundary, we choose $a(\mu=\mu_{max},t) = 1$. This fixes the scale of the time coordinate. Additionally, from (2-7) we see that $\Gamma(\mu=0,t) = +1$ is a reasonable choice if not to allow ρ and ε to be singular at the origin.

CHAPTER III

A NUMERICAL METHOD OF SOLUTION

In principle, the set of equations (2-8) through (2-16) may be solved analytically for suitable equations of state. Generally however, such occasions are rare due to the complex non-linear nature of Einstein's equations. We will therefore introduce a numerical integration method based on a finite difference approximation to the relativistic hydrodynamic equations developed by May and White.[10]

Introduction of the Approximation

The definition of the derivative is given as

$$\frac{d}{dx} f(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

over the domain $a \leqq x \leqq b$. We now introduce the forward finite difference approximation

$$\frac{d}{dx} f(x) \approx \frac{f_{m+1} - f_m}{\Delta x},$$

$$f(a) \approx f_0,$$

and

$$f(b) \approx f_N$$

where by dividing the interval into N segments

$$\Delta x = \frac{b - a}{N}.$$

Naturally we would expect this approximation to only be accurate for $x \ll (b-a)$, that is for sufficiently large N. Similarly we may approximate an integral as a summation.

The Finite Difference Equations

We begin by defining our intervals and notation. Radially our mass configuration will be divided into spherical shell labelled by particlar integer values of the subscript j. The origin will be denoted by 1 and the outer boundary by J. R, m, μ, and a are given at shell boundaries. The quantities ΔR, ρ, ε, Δm, $\Delta\mu$, and w are the naturally defined over volumes such as the zone between the j and j-1 shells, which we denote as the $j - 1/2$ zone. We will advance from one time, t^n, to a later time, t^{n+1}, by the interval $\Delta t^{n+1/2}$, the initial time being t^0. Although most functions may be given a value at some particular time, U must be defined over a time interval as it is essentially a change in distance per change in time. Often in our difference equations a function naturally defined as a zone quantity may be given in terms of another function which is not, thus requiring some form of interpolation. Different forms of interpolation are given in Appendix B.

In order to follow the time development of the mass configuration we will employ a predictor relation of the form

$$F^{n+1}_{\text{Predicted}} = F^n + \frac{\Delta t^{n+1/2}}{2\Delta t^{n+1/2}} (F^n - F^{n-1})$$

which is based on the assumption that F is an approximately linearly varying function in time. By applying this predictor for certain quantities we may determine all others, then return and correct the predicted values. We will find it useful to predict on ε and a.

During this hydrodynamic evolution we may encounter the generation of compression or shock waves as one shell accelerates relative to an adjacent shell. As our energy-monmentum tensor and the resulting hydrodynamic equations do not account for the energy of such waves we will introduce an artificial viscosity term Q which will enable the shock energy to be transformed into ε, the internal energy. The development of this term will be detailed in Appendix C.

Boundary conditions will be expressed as in Chapter II with the exception of a. We will find it convenient to make use of the condition $a_J = a_{J-1} = 1/w_{J-1/2}$. We now present the essential difference equations:

$$U_j^{n+1/2} = U_j^{n-1/2} - \Delta t^n a_j^n [4\pi (R_j^n)^2 \frac{\Gamma_j^n}{w_j^n} \frac{(P_{j+1/2}^n + Q_{j+1/2}^{n-1/2} - P_{j-1/2}^n - Q_{j-1/2}^{n-1/2})}{\Delta\mu j}$$

$$+ \frac{Gm_j^n}{(R_j^n)^2} + \frac{4\pi G}{c^2} (P + Q)_j^n R_j^n],$$

$$\Delta R_{j-1/2}^{n+1} = \Delta R_{j-1/2}^n + \Delta t^{n+1/2} [a_j^{n+1/2} U_j^{n+1/2} - a_{j-1}^{n+1/2} U_{j-1}^{n+1/2}],$$

$$\rho_{j-1/2}^{n+1/2} = \rho_{j-1/2}^n \frac{(R^2)_{j-1/2}^n}{(R^2)_{j-1/2}^{n+1}} \exp[-a_{j-1/2}^{n+1/2} (\frac{U'}{R'})_{j-1/2}^{n+1/2} \Delta t^{n+1/2}]$$

$$(\frac{U'}{R'})_{j-1/2}^{n+1/2} = \frac{U_j^{n+1/2} - U_{j-1}^{n+1/2}}{\Delta R_{j-1/2}^{n+1/2}},$$

$$\varepsilon_{j-1/2}^{n+1} = \varepsilon_{j-1/2}^n - (P_{j-1/2}^{n+1} + Q_{j-1/2}^{n+1/2})(1/\rho_{j-1/2}^{n+1} - 1/\rho_{j-1\,2}^n),$$

$$a^{n+1}_{j-1/2} = a^{n+1}_{j+1/2} \frac{w^{n+1}_{j+1/2}}{w^{n+1}_{j-1/2}} \exp[\varepsilon^{n+1}_{j+1/2} - \varepsilon^{n+1}_{j-1/2}$$

$$+ (P + Q)^{n+1}_{j}(1/\rho^{n+1}_{j+1/2} - 1/\rho^{n+1}_{j-1/2})/w^{n+1}_{j}\ c^2],$$

$$m^{n+1}_{j} = \sum_{k=2}^{j} \Delta m^{n+1}_{k-1/2},$$

$$\Delta m^{n+1}_{j-1/2} = 4\pi\rho^{n+1}_{j-1/2}(1 + \varepsilon^{n+1}_{j-1/2}/c^2)(R^2)^{n+1}_{j-1/2}\Delta R^{n+1}_{j-1/2},$$

$$\Gamma^{n+1}_{j-1/2} = \frac{4\pi\rho^{n+1}_{j-1/2}(R^2)^{n+1}_{j-1/2}\Delta R^{n+1}_{j-1/2}}{\Delta\mu_{j-1/2}},$$

$$P^{n}_{j-1/2} = P(\rho^{n}_{j-1/2}, \varepsilon^{n}_{j-1/2}),$$

and

$$w^{n}_{j-1/2} = 1 + \varepsilon^{n}_{j-1/2}/c^2 + P^{n}_{j-1/2}/\rho^{n}_{j-1/2}c^2.$$

Tests of the Computer Code

We begin by noting that by programming in Fortran 66 it becomes necessary to slightly modify the radial and time notations as this language does not allow half integer array labels. Thus radially we denote shells by odd integers, from the origin being 1, to the outer boundary, JJ+1, and the zones by even integers from 2 to JJ. The number of shells is therefore JJ/2. For the time notation, we choose quantities at the "current" time t^n to be represented by the array time index 3. t^{n+1} is given by 5, t^{n-1} by 1, and the values at

$\Delta t^{n-1/2}$ and $\Delta t^{n+1/2}$ as 2 and 4 respectively. In order to reduce the amount of memory needed, at the end of each time cycle, values for 5 are reassigned to 3, etc. to begin the next cycle. With this the array's dimension employed to index time is reduced to 5.

Convenient units are kilometers, solar masses (1.8 x 10^{30}kg), and seconds. Thus our only input constants are

$$G = 1.2 \times 10^{11} \frac{km^3}{M.s^2},$$

and

$$c^2 = 9 \times 10^{10} \ km^2/s^2.$$

Prior to examining the quark star it is useful to test the code by applying it to several cases whose solutions are known. Unless we encounter an equation of state which gives a negative pressure, the limiting case for a gravitational collapse is provided by a free-fall for which $P = 0$, $\varepsilon = 0$. Equations (2-8) through (2-16) reduce to

$$\dot{U} = - \frac{mG}{R^2}$$

$$\dot{R} = U,$$

and

$$\frac{(\rho R^2)^{\cdot}}{(\rho R^2)} = - \frac{U'}{R'},$$

equivalent to the Newtonian equations of free fall. Also from (2-3) we find

$$\dot{m} = 0$$

Integrating these equations we find for initially uniform mass density

$$t\left(\frac{8\pi G\rho_o}{3}\right)^{1/2} = \chi^{1/2}(1-\chi)^{1/2} + \sin^{-1}(1-\chi)^{1/2}$$

where

$$\chi = \frac{R(\mu,t)}{R(\mu,o)} .$$

The total time of collapse is therefore

$$\tau_c = \left(\frac{3}{8\pi G\rho_o}\right)^{1/2} .$$

This will be useful as an estimate of the collapse time in determining the initial time step.

We illustrate both the analytical and numerical results in Figure 1. The computed results show the expected quantitative behavior, but with times for each value of χ which differ from those of the theory by a constant amount on the order of the program's initial time step size. Adjusting the numerical results by such an amount, we then find the agreement with theory to be excellent.

We also examine the case of the equation of state

$$P = \frac{2}{3}\rho\varepsilon$$

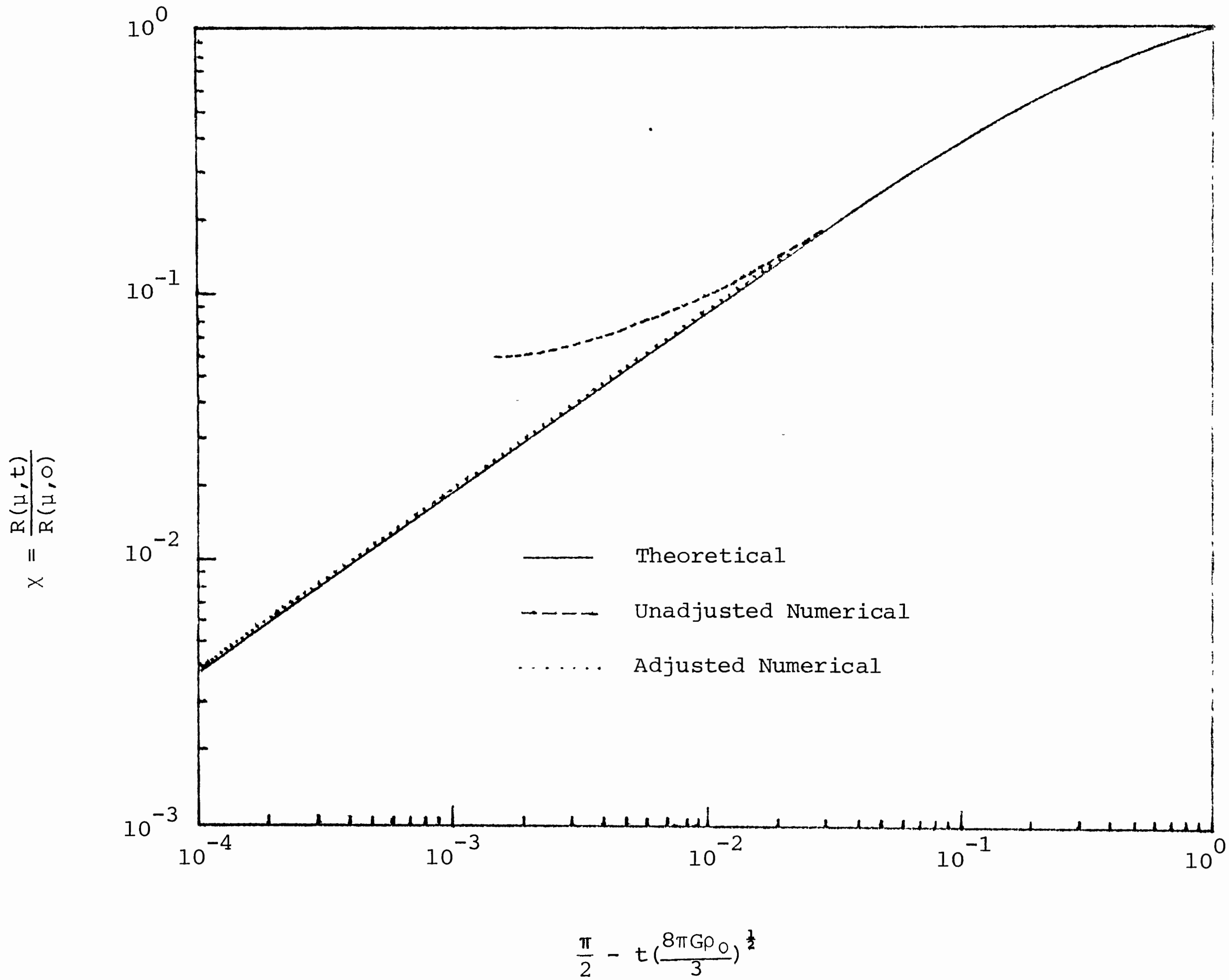

Figure 1. Free Fall of a 21 $M_{\odot}$ Object, P = 0.

a $\gamma = \frac{5}{3}$ gas relation. Initial conditions employed were

$$\rho_o = 6.8 \times 10^{-12} \frac{M_\odot}{km^3}$$

and

$$\varepsilon_o = 1.72 \times 10^6 \frac{km^2}{s^2} .$$

Figure 2 shows our numerical results and those of May and White, as well as the limiting curve of the free fall. We, again find excellent agreement with their results as our results show both that different mass shells collapse by different percentages, and that for the given initial conditions a "bounce" occurs at $t \approx .38$ sec. However, we find a larger bounce for the $\frac{\mu}{\mu_{max}} = 10^{-6}$ shell. We feel that this effect is due to the interpolation of P employed in the computer code, which erroneously gives a value of zero for the pressure at the origin. In order to avoid this failure, we artificially assign P(1,3) = P(2,3), feeling that this is a logical choice to avoid a discontinuity at the innermost zone. Nevertheless, there are many possible choices available, such as extrapolating P(1,3) from P(2,3) and P(3,3) etc. by perhaps some power law in the radius. Physical intuition would indicate the pressure at the origin to be greater than in the inner zone. The work of May and White does not detail their choice for P at the origin, and as the differences between the data become more pronounced as we approach the center of the configuration, we believe that same are due to their choice for P(1,3).

Having thus tested the program in known elementary cases we now proceed to examine the case of a quark star.

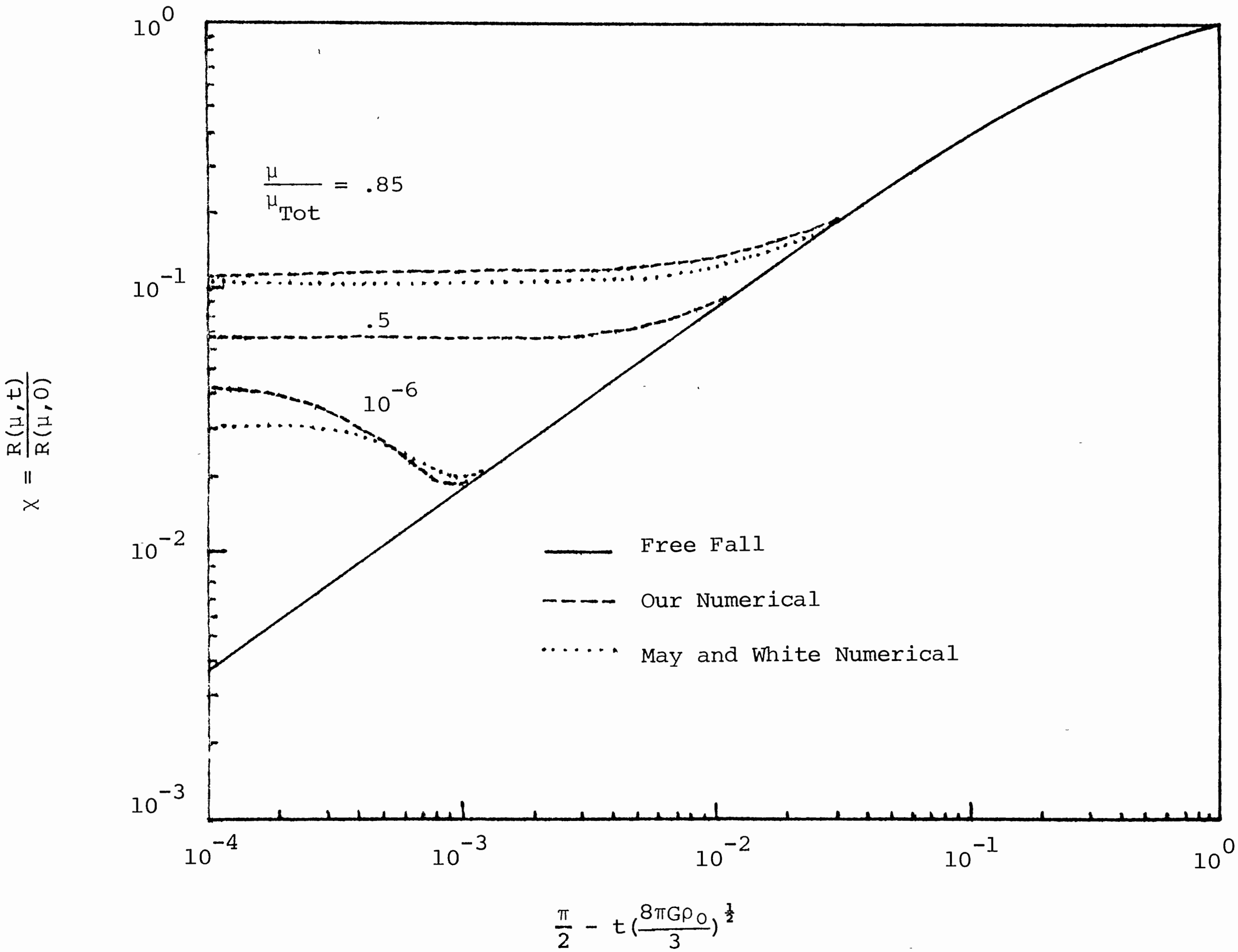

Figure 2. Hydrodyanmic Evolution of a 21 $M_{\odot}$ Object, $P = \frac{2}{3}\rho\varepsilon$.

CHAPTER IV

THE QUARK STAR

Survey of Equations of State

A quark star may be considered to form following the neutron matter epoch in the life of a stellar object where a neutron is considered a bound state of two down and one up quarks. Hence we begin by displaying the equation of state for a neutron star employed by Oppenheimer and Volkoff in their classic paper. Considering neutron matter as a cold Fermi gas, a parameterized form of the equation of state is

$$\rho = K(\sinh t - t),$$

$$P = \frac{1}{3}K\,(\sinh t - 8\sinh\frac{t}{2} + 3t),$$

$$K = \frac{\pi m^4 c^5}{4\,h^3},$$

and

$$t = 4\,\log\{\frac{p_f}{mc} + [1 + (\frac{p_f}{mc})^2]^{1/2}\}$$

where m is the neutron rest mass and p_f is the limiting Fermi momentum, given in terms of the proper number density n by

$$n = \frac{8\pi}{3h^3}\,p^3{}_f.$$

The small number of electrons necessary to prevent beta decay of neutrons does not appreciably contribute to the internal pressure. Such an equation of state ignores the possibility of the neutron matter existing in several phases within such an object. Indeed it is thought that for a typical neutron star with a 10 km radius there exists a solid crust of about 1 km thickness surrounding a fluid interior. However, the behavior of matter during such phase changes is not well understood and thus many dynamic calculations neglect the effect of such a phase change. We will follow this pattern, assuming a phase change from all states of neutron matter to a single state of quark matter to have occured prior to the density at which our examination begins. A general discussion of the structure of neutron stars can be found in Baym and Pethick[11], and Weinberg.[12]

The first derivation of an equation of state for quark matter is due to Itoh[13], who employed para-statistics to obtain

$$P = \frac{3\pi m^4 c^5}{h^3} f(\chi),$$

$$\rho = 3nmc^2 + \frac{3\pi m^4 c^5}{h^3} g(\chi),$$

$$f(\chi) = \chi(2\chi^2 - 3)(\chi^2 + 1)^{1/2} + 3 \sinh^{-1}\chi,$$

$$g(\chi) = 8\chi^3[(\chi^2 + 1)^{1/2} - 1] - f(\chi)$$

where m is a quark mass, n is the baryon number density, and χ, the ratio of the limiting Fermi momentum to mc is given by

$$n = \frac{8\pi m^3 c^3}{h^3} \chi^3.$$

This result includes the condition that in order that the degeneracy energy be minimized, half of the down quarks will transmute into strange quarks so that $n = n_u = n_d = n_s$. We note that this equation of state somewhat resembles that of Oppenheimer and Volkoff. However, as quantum chromodynamics (QCD) has gained acceptance as the candidate theory of strong interactions Itoh's approach appears to be naive, failing to include such effects as quark confinement in such a way that the interaction between particles becomes weak for sufficiently small separations, leaving free quarks.

A more relevant approach is that of Freedman and McLerran.[14] Taking the masses of both up and down quarks as zero, they obtain an equation of state which at sufficiently high number densities simply takes the form

$$P = \frac{1}{3}\tilde{\varepsilon},$$

$$\tilde{\varepsilon} = \frac{9}{4}\pi^{2/3} n^{4/3}$$

where $\tilde{\varepsilon}$ is the total energy density. At low quark matter densities however, the relation becomes more complex

$$P = n^2 \frac{d}{dn}\left(\frac{\tilde{\varepsilon}}{n}\right),$$

$$\tilde{\varepsilon} = \frac{3}{4}\sum_i \mu_i n_i + (.204\ \alpha_c^{\ 2}) \sum_i \mu_i^{\ 4},$$

$$n_i = \frac{1}{\pi^2}\mu_i^{\ 3}\ (1 - 2.55\ \alpha_c - 3.24\ \alpha_c^{\ 2}\ \ln \alpha_c - 5.74\ \alpha_c^{\ 2})$$

where the μ_i are the quantum thermodynamic potentials of each species of quarks and α_c is the chromodynamic structure or strong coupling constant. As α_c itself is dependent in a complex fashion on $\tilde{\varepsilon}$, these latter expressions are far from simple to evaluate. Fortunately Freedman and McLerran provide an alternate expression for the equation of state.

Introducing the MIT bag model, which deals with the confinement problem by constraining quarks to be contained within a spherical "bag", phenomonological matching between quark matter and nuclear matter leads to

$$P = \frac{1}{3}\tilde{\tilde{\varepsilon}},$$

$$\tilde{\tilde{\varepsilon}} = \tilde{\varepsilon} - 4B,$$

where B the "bag constant", essentially the thermodynamic potential of the vacuum, and fits between theory and the spectroscopy of light hadrons give B a value of $56 \mathrm{MeV/fm}^3$. As natural units are employed in the derivation, the mass scale is fixed such that the unit of length is one fermi, or 10^{-13} cm. The introduction of this "bag constant" causes P to increase more slowly as a function of $\rho\varepsilon$ at near nuclear densities than at higher densities. This is called a "softening" of the equation of state.

The importance of this effect to match with neutron matter at hadronic density can be seen by looking at the form of the Oppenheimer-Volkoff equation of state. This equation also shows the softening effect; however in this case the magnitude of this effect is dependent on ρ where it is constant in the quark star relation.

In order to apply the Freedman and McLerran equation of state to our program we will find it necessary to exercise some care. Their model is dependent on zero quark mass while our routine introduces the proper mass density ρ which would strictly vanish in this case. However, we may still employ ρ as before if we define

$$\tilde{\tilde{\varepsilon}} = \rho\varepsilon.$$

Also as we work in solar mass, km, s units, it will be necessary to convert quantities from natural units. We obtain

$$\varepsilon_{\odot} = \frac{1}{\rho_{\odot}} [(4.2 \times 10^6) \frac{9}{4} \pi^{2/3} n_{\text{nat.}}^{4/3} - 4 B_{\odot}],$$

and

$$\rho_{\odot} = (9.28 \times 10^{-4}) n_{\text{nat.}}$$

Taking these relations to generate initial values of ρ and ε for an initial value of n, we will then allow all quantities to change according to the hydrodynamic equations.

Results

We find approximately 2.1 $M_\odot$ to be the mass limit for an object initially consisting entirely of quark matter and obeying the Freedman and McLerran equation of state. For masses larger than this amount, a state of continued collapse is reached in which the object falls within its Schwarzchild radius

$$R_s = \frac{2mG}{c^2}.$$

For a 2.2 $M_\odot$ object, this is 5.87 km. If we examine the gravitational red shift of light, given by

$$z \equiv \frac{\Delta\lambda}{\lambda} = [1 - \frac{2mG}{Rc^2}]^{-1/2} - 1$$

we see that $z \to \infty$ as $R \to R_s$ and thus such an object is cut off from its surroundings as a black hole or "frozen" star. This event takes place in finite proper time, well approximated by τ_c of a free fall. We have $\tau_c \approx 1.15 \times 10^{-4}$ sec. Any observer outside the configuration will observe this collapse to take an infinite amount of time.[15]

The appearance of the 2.1 $M_\odot$ limit is independent of the initial "radial" velocity for the cases studied here, $U < .2c$. In order for stability to be reached, the internal energy and therefore the pressure must rise sufficiently to halt collapse before R approaches R_{se} for any shell, and a non-zero U in the examined range merely serves to change the elapsed time from inception of the collapse in the program to the appearance of a bounce. Following this bounce, the configuration will collapse and bounce again, eventually damping its motion to reach a static state. During this process strong shocks are propagated throughout the object, as shown in Figure 3, which details density fluctuations at several times for a 2.1 $M_{\odot}$ quark star. The fact that densities in some regions drops below quark matter density would seem to indicate that a phase change to neutron matter takes place. Indeed, this effect gives rise to negative, or attractive pressures indicating quark confinement difficulties. However this is simply due to our choice of equation of state which does not take into effect transitions to nuclear matter. Nevertheless these negative pressures suggest the appearance of a neutron matter crust surrounding a quark matter core. For 2.1 $M_{\odot}$ such a quark core may extend to 4 km within the 14 km static object. Under these conditions the name "quark" star seems a misnomer.

The outer few shells show an interesting effect for the quark star. In all continued collapse cases, these shells collapse from the first time step while they expand initially for stable masses even though inner shells begin collapsing. This would appear to be a convenient general criterion for determining a mass limit, but it does not provide a trustworthy test as in the $P = \frac{2}{3}\rho\varepsilon$ case of Chapter III this behavior is not present.

Concluding Remarks

The apparent occurance of phase changes serves to stress the importance of more complete knowledge of the equation of state and its behavior around nuclear density. In this regard more exact examinations of the astrophysics must wait on future developments from particle physics.

A great deal of research has been conducted into supernova creation

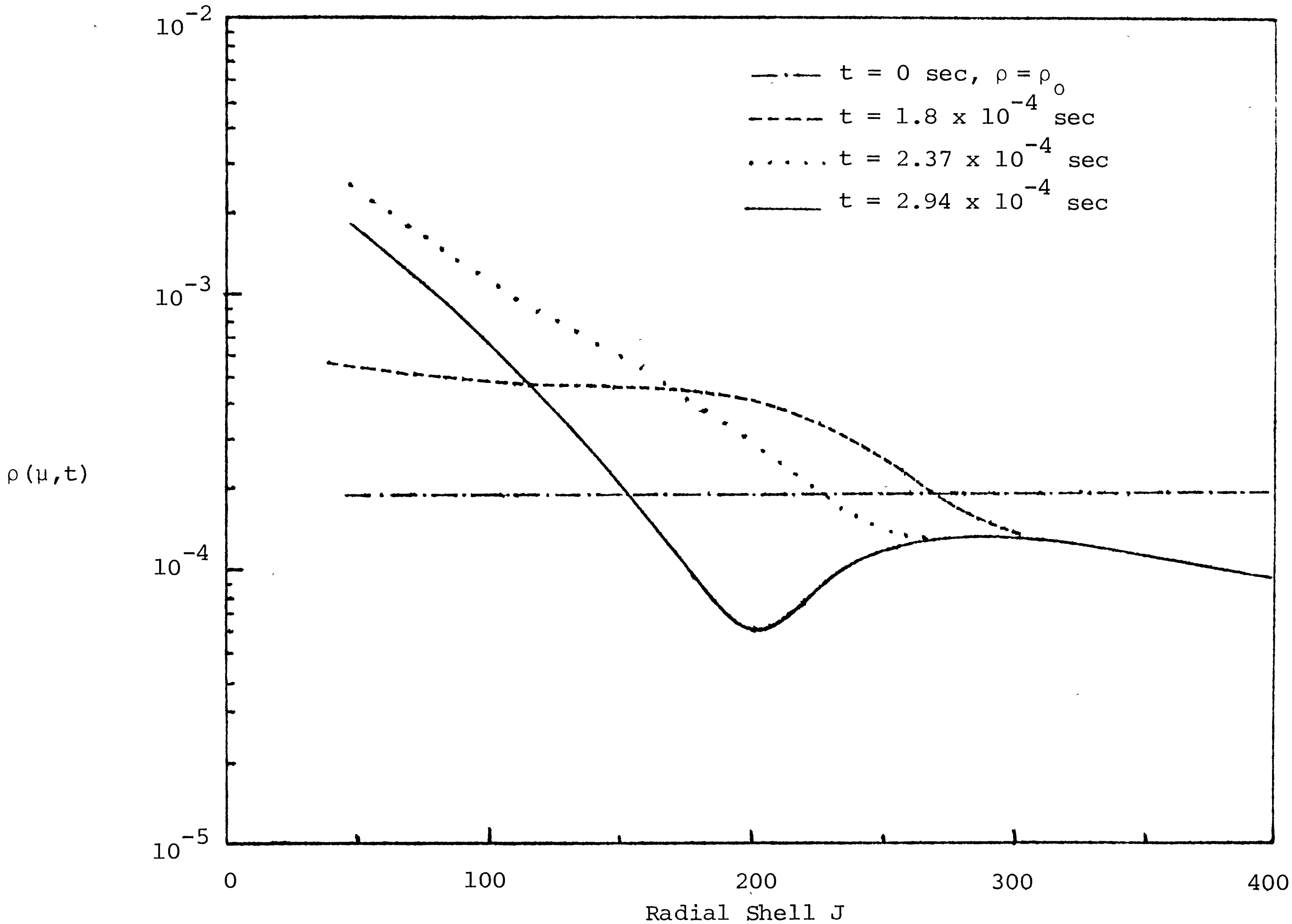

Figure 3. Density Fluctuations During Hydrodynamic Evolution of a 2.1 $M_\odot$ Quark Star.

by allowing an object to collapse on a core which is governed by an extremely stiff equation of state. It is felt that with the inclusion of such effects as neutrino deposition sufficient mass would be ejected by the ensuing shock waves to account for the appearance of a supernova. The core would perhaps remain as a neutron star. In our examination we have looked at an object composed initially of only quark matter; as this does not appear to describe the final static configuration we would expect an entirely quark star to occur only immediately following some extreme compression. Thus a complete picture of a quark star would need to include consideration of supernova generation. The existence of primordial collections of quarks remaining from the early universe would seem to be somewhat precluded by the internal structure of quark configurations; sufficient mass to prevent a transition to neutron matter would merely provide a black hole.

Finally, a further source of development is the numerical scheme itself. In examining a continued collapse the program does not function effectively to allow the generation of useful information. Near its Schwarzchild radius the rate of collapse approaches c and thus the time step control ceases to increment the time. However as discussed by Shapiro[16] and Teukolsky, the May and White scheme gives rise to non-physical results near R_S. This is due to the fact that the metric employed is not the most general.

The most general line element contains a $d\mu\, dt$ term which may be transformed away by a change of coordinates. Shapiro and Teukolsky employ this metric to obtain a different numerical routine which allows examination of hydrodynamic collapse up to and even past the Schwarzchild radius. Thus a more stringent analysis of quark star evolution could be obtained by their method. Despite this development, the May and White scheme remains the basic model for relativistic hydrodynamic numerical calculations.

BIBLIOGRAPHY

1. S. Chandrasekhar, "An Introduction to the Study of Stellar Structure", (Dover Publications, USA, 1957), p. 431.

2. R. C. Tolman, Phys. Rev. 55, 364 (1939).

3. J. R. Oppenheimer and H. Snyder, Phys. Rev. 56, 455 (1939).

4. J. R. Oppenheimer and G. M. Volkoff, Phys. Rev. 55, 374 (1939).

5. C. W. Misner and D. H. Sharp, Phys. Rev. 136, Number 2, 571 (1964).

6. M. M. May and R. H. White, Phys. Rev. 141, Number 4, 1232 (1966).

7. C. W. Misner, K. S. Thorne and J. A. Wheeler, "Gravitation", W. H. Freeman and Company, San Francisco, 1973), Chapter 17.

8. R. C. Tolman, "Relativity, Thermodynamics and Cosmology", (Clarendon Press, Oxford, 1934), p. 215.

9. R. C. Tolman, "Relativity, Theromodynamics and Cosmology", (Clarendon Press, Oxford, 1934), p. 239.

10. M. M. May and R. H. White, Meth. Comp. Phys. 7, 219 (1967).

11. G. Baym and C. Pethick, Ann. Rev. Astron. Astrophys. 17, 415 (1979).

12. S. Weinberg, "Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity", (John Wiley & Sons, New York, 1972), p. 317.

13. N. Itoh, Prog. Theor. Phys. 44, 291 (1970).

14. B. Freedman and L. McLerran, Phys. Rev. D 17, Number 4, 1109 (1978).

15. J. V. Narlikar, "General Relativity and Cosmology", (MacMillan, New York, 1979), p. 156.

16. S. L. Shapiro and S. A. Teukolsky, Astrophys. J. 234, 177 (1979).

17. H. Dingle, Proc. N.A.S. 19, 559 (1933).

18. C. W. Misner, K. S. Thorne and J. A. Wheeler, "Gravitation", (W. H. Freeman and Company, San Francisco, 1973), p. 348.

19. L. D. Landau and E. M. Lifshitz, "Fluid Mechanics", (Addison-Wesley, Reading, Mass., 1959), p. 245.

20. A. D. Booth, "Numerical Methods",(Academic Press, New York, 1955), p. 111.

21. J. Von Neumann and R. D. Richtmyer, J. App. Phys. 21, 232 (1950).

APPENDICES

APPENDIX A

DERIVATION OF THE EINSTEIN EQUATIONS

The most common method at one time for determining the Einstein equations from the metric coefficients was due to Dingle[17]. Given a line element in the form

$$ds^2 = -A(dx^1)^2 - B(dx^2)^2 - C(dx^3)^2 + D(dx^4)^2$$

where x^4 is identical to our time coordinate x^0 and A, B, C and D are generally functions of all the coordinates, Dingle provided formulas for the Einstein equations. Although the use of such relations is less tedious than calculating all Christoffel symbols, Riemann tensor components, and Ricci tensor components, this method is by no means algebraically simple.

With the increased use of differential forms in general relativity, there arose the method of rotation 1-forms for calculating the Einstein equations[18]. We will employ this procedure as it is both more elegant and more modern.

Rotation 1-Forms

Taking our line element to be

$$ds^2 = a^2c^2dt^2 - b^2d\mu^2 - R^2d\Omega^2$$

we define the orthonormal tetrad

$$\omega^0 = ac\ dt,$$

$$\omega^1 = b\ d\mu,$$

$$\omega^2 = R\ d\theta,$$

and

$$\omega^3 = R\sin\theta\ d\phi$$

with which the metric takes the Minkowski form

$$g_{\alpha\beta} = \begin{cases} 1 & \alpha=\beta=0 \\ -1 & \alpha=\beta=1,2,3 \\ 0 & \alpha\neq\beta. \end{cases}$$

Taking the total exterior derivative of each component of our tetrad find

$$d\omega^0 = \frac{a'}{ab}\,\omega^1\wedge\omega^0,$$

$$d\omega^1 = \frac{\dot{b}}{acb}\,\omega^0\wedge\omega^1,$$

$$d\omega^2 = \frac{\dot{R}}{acR}\,\omega^0\wedge\omega^2 + \frac{R'}{bR}\,\omega^1\wedge\omega^2, \tag{A-1}$$

and

$$d\omega^3 = \frac{\dot{R}}{acR}\,\omega^0\wedge\omega^3 + \frac{R'}{bR}\,\omega^1\wedge\omega^3 + \frac{\cos\theta}{R}\,\omega^2\wedge\omega^3,$$

where $\wedge$ denotes a wedge, or exterior product.

Now the affine connections $\omega^\alpha{}_\beta$ are defined by the relation

$$d\omega^\alpha = -\omega^\alpha{}_\beta\wedge\omega^\beta, \tag{A-2}$$

and additionally constrained by

$$dg_{\alpha\beta} = \omega_{\alpha\beta} + \omega_{\beta\alpha}.$$

This latter equation gives for our case

$$\omega_{\alpha\beta} = -\omega_{\beta\alpha}$$

If we now expand the equations given by (A-2), for example with $\alpha=0$

$$d\omega^0 = -\omega^0{}_1 \wedge \omega^1 - \omega^0{}_2 \wedge \omega^2 - \omega^0{}_3 \wedge \omega^3,$$

and compare with the relations (A-1), we arrive at

$$\omega^0{}_1 = \omega^1{}_0 = \frac{a'}{ab}\,\omega^0 + \frac{\dot{b}}{acb}\,\omega^1,$$

$$\omega^0{}_2 = \omega^2{}_0 = \frac{\dot{R}}{acR}\,\omega^2,$$

$$\omega^0{}_3 = \omega^3{}_0 = \frac{\dot{R}}{acR}\,\omega^3,$$

$$\omega^1{}_2 = -\omega^2{}_1 = -\frac{R'}{bR}\,\omega^2,$$

$$\omega^1{}_3 = -\omega^3{}_1 = -\frac{R'}{bR}\,\omega^3,$$

and

$$\omega^2{}_3 = -\omega^3{}_1 = -\frac{R'}{bR}\,\omega^3.$$

The curvature 2-forms defined as

$$\mathcal{R}^\alpha{}_\beta = d\omega^\alpha{}_\beta + \omega^\alpha{}_\lambda \wedge \omega^\lambda{}_\beta$$

then give the expressions

$$\mathcal{R}^0{}_1 = -\mathcal{R}^{01} = \left[\frac{\ddot{b}}{a^2c^2b} - \frac{\dot{b}\dot{a}}{a^3c^2b} - \frac{a''}{ab^2}\,\frac{a'b'}{ab^3}\right]\omega^0 \wedge \omega^1,$$

$$\mathcal{R}^0{}_2 = -\mathcal{R}^{02} = \left[\frac{\ddot{R}}{a^2c^2R} - \frac{\mathring{R}a'}{a^3c^2R} - \frac{a'R'}{ab^2R}\right]\omega^1\wedge\omega^2$$

$$+ \left[\frac{\mathring{R}'}{acbR} - \frac{\mathring{R}a'}{a^2cbR} - \frac{\dot{b}R'}{acb^2R}\right]\omega^1\wedge\omega^2,$$

$$\mathcal{R}^0{}_3 = -\mathcal{R}^{03} = \left[\frac{\ddot{R}}{a^2c^2R} - \frac{\mathring{R}\mathring{a}}{a^3c^2R} - \frac{a'R'}{ab^2R}\right]\omega^0\wedge\omega^3$$

$$+ \left[\frac{\mathring{R}'}{acbR} - \frac{\mathring{R}a'}{a^2cbR} - \frac{\dot{b}R'}{acb^2R}\right]\omega^1\wedge\omega^3, \qquad \text{(A-3)}$$

$$\mathcal{R}^1{}_2 = -\mathcal{R}^{12} = \left[\frac{a'\mathring{R}}{a^2bcR} - \frac{\mathring{R}'}{acbR} + \frac{R'\dot{b}}{acb^2R}\right]\omega^0\wedge\omega^2$$

$$+ \left[\frac{\dot{b}\mathring{R}}{a^2c^2bR} - \frac{R''}{b^2R} + \frac{R'b'}{b^3R}\right]\omega^1\wedge\omega^2,$$

$$\mathcal{R}^1{}_3 = -\mathcal{R}^{13} = \left[\frac{a'\mathring{R}}{a^2bcR} - \frac{\mathring{R}'}{acbR} + \frac{R'\dot{b}}{acb^2R}\right]\omega^0\wedge\omega^3$$

$$+ \left[\frac{\dot{b}\mathring{R}}{a^2c^2bR} - \frac{R''}{b^2R} + \frac{R'b'}{b^3R}\right]\omega^1\wedge\omega^3,$$

and

$$\mathcal{R}^2{}_3 = -\mathcal{R}^{23} = \left[\frac{1}{R^2} + \frac{\mathring{R}^2}{a^2c^2R^2} - \frac{R'^2}{b^2R^2}\right]\omega^2\wedge\omega^3.$$

An alternate definition of these curvature 2-forms is given by

$$\mathcal{R}^{\alpha\beta} = \frac{1}{2}R^{\alpha\beta}{}_{\gamma\delta}\,\omega^\gamma\wedge\omega^\delta$$

where the $R^{\alpha\beta}{}_{\gamma\delta}$ are components of the Riemann tensor, and by expanding this form and comparing the results with equations (A-3), we arrive at

$$R^{01}{}_{01} = -\left[\frac{\ddot{b}}{a^2c^2b} - \frac{\dot{b}\mathring{a}}{a^3c^2b} - \frac{a''}{ab^2} + \frac{a'b'}{ab^3}\right] \equiv A,$$

$$R^{02}{}_{02} = R^{03}{}_{03} = -\left[\frac{\ddot{R}}{a^2c^2R} - \frac{\dot{R}\dot{a}}{a^3c^2R} - \frac{a'R'}{ab^2R}\right] \equiv B,$$

$$R^{02}{}_{12} = R^{03}{}_{13} = -\left[\frac{\dot{R}'}{acbR} - \frac{\dot{R}a'}{a^2cbR} - \frac{\dot{b}R'}{acb^2R}\right] \equiv C,$$

$$R^{23}{}_{23} = -\left[\frac{1}{R^2} + \frac{\dot{R}^2}{a^2c^2R^2} - \frac{R'^2}{b^2R^2}\right] \equiv D,$$

and

$$R^{12}{}_{12} = R^{13}{}_{13} = -\left[\frac{\dot{b}\dot{R}}{a^2c^2bR} - \frac{R''}{b^2R} + \frac{R'b'}{b^3R}\right] \equiv E.$$

All other components vanish.

The components of the curvature tensor are then related, with use of the symmetry property $R^{\alpha\beta}{}_{\gamma\delta} = R^{\beta\alpha}{}_{\delta\gamma}$,

$$R^0{}_0 = A + 2B,$$

$$R^0{}_1 = 2C,$$

$$R^1{}_1 = A + 2E,$$

and

$$R^2{}_2 = R^3{}_3 = B + D + E.$$

All other components vanish.

This gives the curvature scalar as

$$R = R^\alpha{}_\alpha = 2A + 4B + 2D + 4E;$$

finally the Einstein equations are given by

$$G^\alpha{}_\beta = R^\alpha{}_\beta - \frac{1}{2}\, g^\alpha{}_\beta R = \frac{8\pi G}{c^4}\, T^\alpha{}_\beta$$

or

$$G^0{}_0 = -(D + 2E),$$

$$G^0{}_1 = 2C,$$

$$G^1{}_1 = -(D + 2B),$$

and

$$G^2{}_2 = G^3{}_3 = -(A + B + E).$$

All other components vanish.

A particularly useful equation is

$$G^0{}_1 = 2C = -2\left[\frac{\dot{R}'}{acbR} - \frac{\dot{R}a'}{a^2cbR} - \frac{\dot{b}R'}{acb^2R}\right] = \frac{8\pi G}{c^4} T^0{}_1$$

which vanishes as our energy-momentun tensor satisfies (2-1). Reducing this form we have

$$\dot{R}' = \frac{\dot{R}a'}{a} + \frac{\dot{b}R'}{b} .$$

If we now employ this simplifying relation, (2-1), and perform some algebra, we arrive at the form of the Einstein equations found in Chapter II:

$$4\pi G\rho(1 + \varepsilon/c^2)R^2R' = \frac{c^2}{2}\left[R + \frac{R\dot{R}^2}{a^2c^2} - \frac{RR'^2}{b^2}\right]',$$

$$\frac{4\pi G}{c^2} PR^2\dot{R} = -\frac{c^2}{2}\left[R + \frac{R\dot{R}^2}{a^2c^2} - \frac{RR'^2}{b^2}\right]^{\bullet},$$

$$4\pi G\rho wR^3 = c^2\left[R + \frac{R\dot{R}^2}{a^2c^2} - \frac{RR'^2}{b^2}\right] + \frac{R^3c}{ab}\left[\left(\frac{ca'}{b}\right)' - \left(\frac{\dot{b}}{ac}\right)^{\bullet}\right],$$

and

$$\dot{R}' - \frac{a'\dot{R}}{a} - \frac{\dot{b}R'}{b} = 0.$$

APPENDIX B

ADDITIONAL DIFFERENCING CONSIDERATIONS

Interpolation Relations

Some difference equations will include the quantity $(R^2)^n_{j-1/2}$. As R and thus R^2 are naturally defined at shell boundaries, we will interpolate by examining the volume between the j and j-1 shells. This volume is given by

$$V^n_{j-1/2} = \frac{4\pi}{3}\,[(R^n_j)^3 - (R^n_{j-1})^3].$$

Now for small $\Delta R^n_{j-1/2}$, we may approximate this by

$$V^n_{j-1/2} = 4\pi(R^2)^n_{j-1/2}\,\Delta R^n_{j-1/2}$$

where we take $4\pi(R^2)^n_{j-1/2}$ to be the proper surface area of a sphere of radius which lies in the $j-1/2$ zone. Thus we have

$$(R^2)^n_{j-1/2} = \frac{1}{3}\,\frac{[(R^n_j)^3 - (R^n_{j-1})^3]}{\Delta R^n_{j-1/2}}$$

$$= \frac{1}{3}\,\frac{(R^n_j - R^n_{j-1})[(R^n_j)^2 + R^n_j R^n_{j-1} + (R^n_{j-1})^2]}{R^n_j - R^n_{j-1}}$$

$$= \frac{1}{3}\,[(R^n_j)^2 + R^n_j R^n_{j-1} + (R^n_{j-1})^2]$$

The radial spacing $\Delta\mu$, which remains constant due to the comoving coordinates, will be interpolated according to time interval, which we take to be from t^n to t^{n+1}. We see that in the differencing notation

$$\Delta\mu_j = \frac{1}{2}\left[\Delta\mu_{j-1/2} + \Delta\mu_{j+1/2}\right]$$

which takes into account the case of unequal mass zoning. Quantities which are essentially energies per unit mass will be treated according to

$$w_j^n \Delta\mu_j = \frac{1}{2}\left[w_{j-1/2}^n \Delta\mu_{j-1/2} + w_{j+1/2}^n \Delta\mu_{j+1/2}\right]$$

or

$$w_j^n = \frac{1}{2\Delta\mu_j}\left[w_{j-1/2}^n \Delta\mu_{j-1/2} + w_{j+1/2}^n \Delta\mu_{j+1/2}\right]$$

which indicates that the energy over the fictitious $\Delta\mu_j$ zone is the sum of the energies in the half zones on either side of the j shell. It will also be convenient to employ this form for (aw) as it deals effectively with a slowly changing scalar quantity, which we assume (aw) to be.

Pressure and the artificial viscosity term Q representing essentially non-slowly varying vector quantities are interpolated linearly according to

$$P_j^n = \frac{1}{2\Delta\mu j}\left[P_{j+1/2}^n \Delta\mu_{j-1/2} + P_{j-1/2}^n \Delta\mu_{j+1/2}\right].$$

Time intervals will be interpolated by

$$\Delta t^n = \frac{1}{2}\left[\Delta t^{n-1/2} + \Delta t^{n+1/2}\right].$$

Time Step Control

As hydrodynamic collapse progresses, some physical quantities may change their values by large percentages from one time cycle to the next, thus endangering the reliability of the numerical scheme. Therefore we employ a procedure which decreases the time step size whenever change in the functions exceeds some percentage. We find it sufficient to base this control on ρ and ε, employing the relations

$$\Delta t_\rho = \min[A\ \rho^{n+1}_{j-1/2} \frac{\Delta t^{n+1/2}}{|\rho^{n+1}_{j-1/2} - \rho^{n}_{j-1/2}|},\ \text{all } j],$$

and

$$\Delta t_\varepsilon = \min[B\ \varepsilon^{n+1}_{j-1/2} \frac{\Delta t^{n+1/2}}{|\varepsilon^{n+1}_{j-1/2} - \varepsilon^{n}_{j-1/2}|},\ \text{all } j],$$

where A and B are given precentages.

We also note that analogous to the classical Courant-Friedrickson-Lewy stability condition[20] we must require that the speed of sound through the configuration be less than the proper mesh velocity, or

$$c_s < \frac{(\frac{\Delta R}{b})}{(\frac{\Delta t}{a})} = \frac{\frac{\Delta\mu}{4\pi\rho R^2}}{\frac{\Delta t}{a}} .$$

in order to insure numerical stability of our solution. The speed of sound is given by[19]

$$c_s = \sqrt{\frac{\partial P}{\partial e}},$$

with

$$e = \rho(1 + \varepsilon/c^2),$$

the total proper mass-energy density.

In order to assure compliance with this condition we must define

$$\Delta t_c = \min[D \frac{\Delta\mu_{j-1/2}\, a^{n+1}_{j-1/2}}{\rho^{n+1}_{j-1/2}\,(R^2)^{n+1}_{j-1/2}\,(C_s)^{n+1}_{j-1/2}}, \text{ all } j].$$

where D is some given allowable percentage.

Time steps are therefore chosen according to

$$\Delta t^{n+\frac{3}{2}} = \min[\Delta t_\rho, \Delta t_\varepsilon, \Delta t_c, \Delta t_{input}, 1.2\, \Delta t^{n+1/2}].$$

APPENDIX C

SHOCKS AND ARTIFICIAL VISCOSITY

In the analysis of gravitational collapse the equations of fluid flow play a central role. However it is well known that while investigating the flow of a compressible fluid by solving the equations using stepwise numerical procedures a complication arises due to the presence of shocks. The shocks manifest themselves mathematically as surfaces on which density, fluid velocity, temperature, entropy and the like have discontinuities. These require boundary conditions, the so-called 'jump conditions', connecting the values of these quantities on the two sides of each surface. These were first derived by Rankine and Hugoniot using the fundamental principles of conservation of mass flow, conservation of momentum and conservation of energy. The application of these to the current problem is, however, complicated by the fact that the shock surfaces are in motion relative to the network of points in space-time used for the numerical work.

Von Neumann and Richtmeyer[21] showed a way out of this difficulty. It is well known in fluid mechanics that the effect of dissipative mechanisms like viscosity on the shocks is to weaken and smear the shocks out so that the mathematical surfaces of discontinuity are replaced by thin layers in which pressure, density, temperature etc vary rapidly but continuously. Von Neumann and Richtmeyer utilized this fact and introduced an 'artificial viscosity' term into the fluid equations such that the shocks are spread over several zones of the differencing mesh, the jump conditions are automatically satisifed and the entire calculation can be carried out as though there were no shocks at all. The essential effect of the damping from a physical point of view is to convert the kinetic energy into internal energy. The important point in their method is the introduction of a suitable scalar stress which is added to the pressure term in the equations. They derive the correct expression for this 'artificial viscosity' which will achieve the desired objective. Their general equations are

$$\rho_o \dot{U} = - (P + Q)' ,$$

$$\dot{\varepsilon} + (P + Q) \dot{V} = 0, \tag{C-1}$$

and

$$\rho_o \dot{V} = U' .$$

Here ρ_o is the initial density, V the specific volume U the fluid velocity and ε the internal energy per unit mass. The connection between ε, p, V is established by a suitable equation of state.

We will consider shock waves to be spherically symmetric radially travelling waves such that across the shock front all dynamic and thermodynamic are given by expressions of the form

$$f(\mu,t) = f(\mu - Mt) \tag{C-2}$$

for $\mu_o < \mu < \mu_o$.

In other words after travelling a long distance we have a steady-state plane shock and the different dynamical quantities of interest depend on μ and t only through the combination

$$\mu - Mt.$$

We employ subscripts b and a to denote behind and ahead of the shock front. A wave incident on a mass shell μ is equivalent to a wave incident on a 2 dimensional plane provided a shock travels with sufficient speed. In such 'slab symmetry' we label such a plane by μ which is taken to be the $\frac{\mu}{4\pi R^2}$, or the proper mass per unit area of the slab. We also find from (C-2) that M may be interpreted as the mass flux through the slab. Employing this picture in the special relativistic limit for which G = 0, equations (2-8) through (2-16) reduce to

$$[U(1 + \varepsilon/c^2)]^{\cdot} = - (aP\Gamma)',$$

$$[\Gamma c^2(1 + \varepsilon/c^2)]^{\cdot} = - (aPU)',$$

$$(\Gamma V)' = (aU)',$$

$$\Gamma = (1 + U^2/c^2)^{1/2} = \rho R',$$

$$\dot{\varepsilon} = - P\dot{V},$$

and

$$\frac{(aw)'}{aw} = (\varepsilon' + PV')/wc^2. \tag{C-3}$$

We have introduced the specific volume $V = \frac{1}{\rho}$. Also it is useful to note that if we take $v\Gamma = U$ then we find

$$\Gamma = (1 - \frac{v^2}{c^2})^{-1/2} \tag{C-4}$$

which is identical to the Special Relativistic gamma factor. Equation (C-4) simply states that U is the comoving rate of change of the surface area function as mentioned in Chapter II.

Following Von Neumann and Richtmeyer, if we now introduce the artificial viscosity term Q such that $P \rightarrow P + Q$, $Q > 0$ for $\mu_o < \mu < \mu_o$

and 0 elsewhere, in (C-2) then equations (C-3) become the Hugoniot jump relations across the shock.

$$MU(1 + \varepsilon/c^2) - a\Gamma(P + Q) = c_1,$$

$$M\Gamma c^2(1 + \varepsilon/c^2) - aU(P + Q) = c_2,$$

and

$$M\Gamma V + aU = c_3. \qquad \text{(C-5)}$$

Examination of the slab symmetric form of (2-11) shows that

$$\varepsilon' + (P + Q)V' = 0$$

and thus we find

$$aw = c_4.$$

These expressions are usually written in fluid mechanical analysis of shocks as the jump relations in the notation

$$[\eta] \equiv \eta_a - \eta_b = 0.$$

At this point it is important to note that the explicit form of the artificial viscosity term naturally depends on the assumed equation of state. We now stipulate for this development the equation of state

$$P = (\gamma - 1)\ \varepsilon/V.$$

The choice of this is contingent on the requirement that the analytical form of the term Q ensures that the transition between points μ_a and μ_b is affected smoothly and extends over a few zones. Thus from

$$Q = \frac{(\gamma + 1)}{2V} M^2\ (V_a - V)(V - V_b)$$

with Q assuming the form

$$Q = k^2\ \Delta\mu^2\ M^2\ \frac{V'^2}{V}$$

we have

$$V = \frac{1}{2}(V_a + V_b) + \frac{1}{2}(V_a - V_b)\sin[(\frac{\gamma+1}{2})^{1/2}\frac{(\mu - Mt + \mu_o)}{k\Delta\mu}]$$

where μ_o is a constant of integration. Therefore

$$\mu_a - \mu_b = k\Delta\mu\ \pi(\frac{2}{\gamma+1})^{1/2}. \tag{C-6}$$

Examination of this relation shows that for suitable choices of $\Delta\mu$, k and γ, the shock is spread over several zones thus converting the shock's mechanical energy into internal energy.

The spherically symmetric general relativistic problem for arbitrary pressure although extremely complex is analogous to this discussion. Here we simply give the form of the artificial viscosity term employed in our numerical scheme as suggested by the earlier work of May and White

$$Q = k^2\rho\ (\frac{\Delta\mu}{R^2})^2\ \frac{[(R^2U)']^2}{\Gamma},\quad \dot{\rho} > 0,$$

$$= 0 \qquad\qquad \dot{\rho} \leqq 0.$$

APPENDIX D

THE COMPUTER CODE

```
C          THIS PROGRAM CALCULATES AND PRINTS VALUES FOR PHYSICAL
C          QUANTITIES AT TIME INTERVALS DURING THE HYDRODYNAMIC
C          EVOLUTION OF A STELLAR OBJECT.  AS THIS PROGRAM IS BASED
C          ON THE NUMERICAL TECHNIQUE OF MAY AND WHITE, REFERENCE
C          10, WE WILL GIVE THE EQUATION NUMBERS FROM THEIR ARTICLE
C          WHERE THEY OCCUR IN OUR PROGRAM.
C
C          WE DEFINE THE FOLLOWING QUANTITIES:
C          JSTEPS=NUMBER OF RADIAN ZONES
C          JJ=OUTER ZONE LABEL
C          JL=OUTER BOUNDARY LABEL
C          JM=NEXT TO OUTERMOST ZONE
C          JN=RADIAL PRINT PARAMETER
C          NN=NUMBER OF TIME STEPS TO BE CALCULATED
C          NNN=NUMBER OF TIME STEPS COMPLETED
C          T=ELAPSED TIME AFTER NNN TIME STEPS
C          NAA=TIME PRINT PARAMETER
C          EMTOT=TOTAL INITIAL MASS OF THE OBJECT
C          RHOAVE=UNIFORM INITIAL PROPER MASS DENSITY
C          EPSAVE=UNIFORM INITIAL SPECIFIC PROPER ENERGY
C          G=UNIVERSAL GRAVITATION CONSTANT
C          CSQ=SQUARE OF SPEED OF LIGHT
C          DELTTT=INITIAL TIME STEP SIZE
C          DELTT=120% OF PREVIOUS TIME STEP. SERVES TO INCREASE
C           STEP SIZE WHEN QUANTITIES VARY SLOWLY.
C          DELTC=TIME STEP ACCORDING TO ALLOWED PERCENT OF SPEED
C           OF SOUND, PC
C          DELTE=TIME STEP ACCORDING TO ALLOWED PERCENT CHANGE
C           IN EPS, PE
C          DELTR=TIME STEP ACCORDING TO ALLOWED PERCENT CHANGE
C           IN RHO, PR
C          DELRR=INITIAL RADIAL STEP SIZE
C          X=Y=Z=ARBITRARY REAL NUMBERS EMPLOYED TO FIND MINIMUM
C           VALUES AMONG DELTC, DELTE, DELTR, DELTT, AND DELTTT.
C
      IMPLICITREAL*8(A-H,O-Z)
      DIMENSION DELT(5),R(402,5),DELR(402,5),RSQ(402,5)
     C,DELEM(402,5),RHO(402,5),EPS(402,5),EM(402,5),
     CGA(402,5),U(402,5),DELMU(402),P(402,5),W(402,5)
     C,FF(402),A(402,5),PQ(402,5),Q(402,5),AA(402,5),
     CEEPS(402,5),UR(402,5),PP(402,5),DELTE(402),
     CDELTR(402),F(402),CS(402,5),DELTC(402)
      JSTEPS=200
      JJ=JSTEPS*2
      JL=JJ+1
      JM=JJ-2
      JN=50
      AJJ=JSTEPS
```

```
      NN=100000
      NAA=25
      EMTOT=1.D0
      RHOAVE=1.86D-4
      EPSAVE=1.198D11
      G=1.2D11
      CSQ=9.D10
      X=10.D0
      Y=X
      Z=X
      PC=.2D0
      PE=.02D0
      PR=.02D0
      NA=NAA
      FOURPI=4.D0*3.14159265358979D0
      DELRR=(3.D0*EMTOT/FOURPI/RHOAVE)**(1.D0/3.D0)/AJJ
      DELTTT=1.D-6
      DELT(2)=DELTTT
      DELT(4)=DELTTT
C
C           THIS SECTION SERVES TO DETERMINE INITIAL VALUES FOR ALL
C           FUNCTIONS FROM THE INITIAL AND BOUNDARY CONDITIONS.
C
      DO 10 J=2,JJ,2
      DELR(J,3)=DELRR
      RHO(J,3)=RHOAVE
      EPS(J,3)=EPSAVE
10    CONTINUE
      DO 20 J=2,JJ,2
      I1=J+1
      I2=J-1
C          (103)
      R(I1,3)=R(I2,3)+DELR(J,3)
C          (104)
      RSQ(J,3)=(R(I2,3)**2+R(I2,3)*R(I1,3)+R(I1,3)**2)/3.D0
C          (105)
      DELEM(J,3)=FOURPI*RHO(J,3)*(1.D0+EPS(J,3)/CSQ)*RSQ(J,3)
     C*DELR(J,3)
C          (106)
      EM(I1,3)=EM(I2,3)+DELEM(J,3)
C          (107)
      IF(J.EQ.2) GA(2,3)=DSQRT(1.D0-G/CSQ*EM(3,3)/R(3,3))
      IF(J.GT.2) GA(J,3)=DSQRT(1.D0-G/CSQ*(EM(I1,3)/
     CR(I1,3)+EM(I2,3)/R(I2,3)))
C          (108)
      DELMU(J)=FOURPI*RHO(J,3)*RSQ(J,3)*DELR(J,3)/GA(J,3)
C          (110)
      P(J,3)=2.D0/3.D0*RHO(J,3)*EPS(J,3)-4.D0/3.D0*4.9778D6
C          (112)
      W(J,3)=1.D0+(EPS(J,3)+P(J,3)/RHO(J,3))/CSQ
20    CONTINUE
      DO 50 J=2,JJ,2
      I1=J+1
      I3=J+2
C          (109)
      IF(J.EQ.2) DELMU(1)=DELMU(2)/2.D0
      DELMU(I1)=(DELMU(J)+DELMU(I3))/2.D0
C          (113)
      IF(J.EQ.2) W(1,3)=W(2,3)
      W(I1,3)=(W(J,3)*DELMU(J)+W(I3,3)*DELMU(I3))/2.D0/DELMU(I1)
C          (114)
      P(I1,3)=(P(I3,3)*DELMU(J)+P(J,3)*DELMU(I3))/2.D0/DELMU(I1)
```

```
50    CONTINUE
C          (115)
      A(JL,3)=1.D0/W(JL,3)
      A(JJ,3)=A(JL,3)
      DO 60 J=2,JM,2
      I1=J+1
      I3=J+2
C          (117)
      FF(I1)=(EPS(I3,3)-EPS(J,3)+P(I1,3)*(1.D0/RHO(I3,3)-1.D0/
     CRHO(J,3)))/W(I1,3)/CSQ
60    CONTINUE
      DO 70 J=2,JM,2
      M1=JJ-J
      M2=JJ-J+2
      M3=JJ-J+1
C          (116)
      A(M1,3)=A(M2,3)*W(M2,3)/W(M1,3)/(1.D0+FF(M3)+FF(M3)**2
     C/2.D0)
70    CONTINUE
      DO 80 J=2,JM,2
      I1=J+1
      I3=J+2
C          (118)
      IF(J.EQ.2) A(1,3)=A(2,3)
      A(I1,3)=(A(J,3)*W(J,3)+A(I3,3)*W(I3,3))/2.D0/W(I1,3)
80    CONTINUE
C
C          THIS SECTION CALCULATES THE TIME EVOLVED VALUES OF
C          ALL FUNCTIONS.
C
      DO 1000 N=1,NN,1
C          (119)
      DELT(3)=(DELT(2)+DELT(4))/2.D0
      IF(N.GT.1) T=DELT(2)+T
      NNN=N-1
      NA=NA+1
      DO 90 J=2,JJ,2
      I1=J+1
      I2=J-1
      I3=J+2
C          (120)
      IF(J.EQ.2) PQ(1,3)=P(2,3)+Q(2,2)
      PQ(I1,3)=((P(I3,3)+Q(I3,2))*DELMU(J)+(P(J,3)+Q(J,2))*
     CDELMU(I3))/2.D0/DELMU(I1)
C          (122)
      IF(J.EQ.2) GA(1,3)=1.D0
      GA(I1,3)=(GA(J,3)*DELMU(J)+GA(I3,3)*DELMU(I3))/2.D0
     C/DELMU(I1)
C          (123)
      U(I1,4)=U(I1,2)-DELT(3)*A(I1,3)*(FOURPI*R(I1,3)**2*
     CGA(I1,3)/W(I1,3)*(P(I3,3)+Q(I3,2)-P(J,3)-Q(J,2))/DELMU(I1)
     C+G*EM(I1,3)/R(I1,3)**2+FOURPI*G/CSQ*PQ(I1,3)*R(I1,3))
      IF(N.GT.1) GO TO 95
C          (124)
      AA(J,4)=A(J,3)
C          (125)
      IF(J.EQ.2) AA(1,4)=A(1,3)
      AA(I1,4)=A(I1,3)
C          (126)
      EEPS(J,4)=EPS(J,3)
      GO TO 96
C          (124)
```

```
95     AA(J,4)=A(J,3)+DELT(4)/2.D0/DELT(2)*(A(J,3)-A(J,1))
C           (125)
       IF(J.EQ.2) AA(1,4)=A(1,3)+DELT(4)/2.D0/DELT(2)*(A(1,3)
      C-A(1,1))
       AA(I1,4)=A(I1,3)+DELT(4)/2.D0/DELT(2)*(A(I1,3)-A(I1,1))
C           (126)
       EEPS(J,4)=EPS(J,3)+DELT(4)/2.D0/DELT(2)*(EPS(J,3)-EPS(J,1))
C           (127)
96     DELR(J,5)=DELR(J,3)+DELT(4)*(AA(I1,4)*U(I1,4)-AA(I2,4)*
      CU(I2,4))
C           (128)
       R(I1,5)=R(I2,5)+DELR(J,5)
C           (129)
       DELR(J,4)=(DELR(J,5)+DELR(J,3))/2.D0
C           (130)
       R(I1,4)=(R(I1,5)+R(I1,3))/2.D0
C           (131)
       RSQ(J,5)=(R(I2,5)**2+R(I2,5)*R(I1,5)+R(I1,5)**2)/3.D0
       GAMABS=DABS(GA(J,3))
       IF(GAMABS.LT.1.D-3) GO TO 110
       IF(DELR(J,4).EQ.0.D0) GO TO 110
C           (132)
       UR(J,4)=(U(I1,4)-U(I2,4))/DELR(J,4)
       GO TO 120
C           (133)
110    IF(J.EQ.2) UR(2,4)=(FOURPI*CSQ*RHO(2,3)*RSQ(2,3)*(GA(3,3)
      C-GA(1,3))/DELMU(2)+FOURPI*G*RHO(2,3)*(1.D0+EPS(2,3)/CSQ)*
      CDSQRT(RSQ(2,3))-G/2.D0*EM(3,3)/R(3,3)**2)/(U(3,4)+
      CU(1,4))
       IF(J.GT.2) UR(J,4)=(FOURPI*CSQ*RHO(J,3)*RSQ(J,3)*(GA(I1,3)
      C-GA(I2,3))/DELMU(J)+FOURPI*G*RHO(J,3)*(1.D0+EPS(J,3)/CSQ)*
      CDSQRT(RSQ(J,3))-G/2.D0*(EM(I1,3)/R(I1,3)**2+EM(I2,3)/
      CR(I2,3)**2))/(U(I1,4)+U(I2,4))
120    CONTINUE
C           (135)
       F(J)=AA(J,4)*UR(J,4)*DELT(4)
C           (134)
       RHO(J,5)=RHO(J,3)*RSQ(J,3)/RSQ(J,5)/(1.D0+F(J)+F(J)**
      C2/2.D0)
C           (136)
       RHO(J,4)=(RHO(J,5)+RHO(J,3))/2.D0
C           (137)
       IF(RHO(J,5).GT.RHO(J,3)) Q(J,4)=2.D0*RHO(J,4)*(R(I1,4)**2
      C*U(I1,4)-R(I2,4)**2*U(I2,4))**2/GA(J,3)/RSQ(J,3)**2
C           (138)
       PP(J,4)=2.D0/3.D0*RHO(J,4)*EEPS(J,4)-4.D0/3.D0*4.9778D6
C           (139)
       EPS(J,5)=EPS(J,3)-(PP(J,4)+Q(J,4))*(1.D0/RHO(J,5)-1.D0/
      CRHO(J,3))
C           (110)
       P(J,5)=2.D0/3.D0*RHO(J,5)*EPS(J,5)-4.D0/3.D0*4.9778D6
C           (140)
       W(J,5)=1.D0+(EPS(J,5)+(P(J,5)+Q(J,4))/RHO(J,5))/CSQ
90     CONTINUE
       DO 130 J=2,JJ,2
       I1=J+1
       I3=J+2
C           (120)
       PQ(I1,5)=((P(I3,5)+Q(I3,4))*DELMU(J)+(P(J,5)+Q(J,4))*
      CDELMU(I3))/2.D0/DELMU(I1)
C           (121)
       IF(J.EQ.2) W(1,5)=W(2,5)
```

```
      W(I1,5)=(W(J,5)*DELMU(J)+W(I3,5)*DELMU(I3))/2.D0/DELMU(I1)
130   CONTINUE
C          (141)
      A(JL,5)=1.D0/W(JL,5)
      A(JJ,5)=A(JL,5)
      DO 140 J=2,JM,2
      I1=J+1
      I3=J+2
C          (143)
      FF(I1)=(EPS(I3,5)-EPS(J,5)+PQ(I1,5)*(1.D0/RHO(I3,5)
     C-1.D0/RHO(J,5)))/W(I1,5)/CSQ
140   CONTINUE
      DO 150 J=2,JM,2
      M1=JJ-J
      M2=JJ-J+2
      M3=JJ-J+1
C          (142)
      A(M1,5)=A(M2,5)*W(M2,5)/W(M1,5)/(1.D0+FF(M3)+FF(M3)**
     C2/2.D0)
150   CONTINUE
      DO 160 J=2,JM,2
      I1=J+1
      I3=J+2
C          (144)
      IF(J.EQ.2) A(1,5)=A(2,5)
      A(I1,5)=(A(I3,5)*W(I3,5)+A(J,5)*W(J,5))/2.D0/W(I1,5)
160   CONTINUE
      DO 170 J=2,JJ,2
      I1=J+1
C          (148)
      GA(J,5)=FOURPI*RHO(J,5)*RSQ(J,5)*DELR(J,5)/DELMU(J)
C          (149)
      DELEM(J,5)=GA(J,5)*(1.D0+EPS(J,5)/CSQ)*DELMU(J)
C          (150)
      EM(I1,5)=EM(I2,5)+DELEM(J,5)
      ABSEPS=DABS(EPS(J,5)-EPS(J,3))
      ABSRHO=DABS(RHO(J,5)-RHO(J,3))
C          (152)
      DELTE(J)=PE*EPS(J,5)*DELT(4)/ABSEPS
C          (153)
      DELTR(J)=PR*RHO(J,5)*DELT(4)/ABSRHO
C          (160)
      CS(J,5)=DSQRT(2.D0/3.D0*EPS(J,5)/(1.D0+EPS(J,5)/CSQ))
C          (154)
      DELTC(J)=PC*DELMU(J)/RHO(J,5)/RSQ(J,5)*A(J,5)/CS(J,5)
170   CONTINUE
      DO 190 J=2,JJ,2
      X=DMIN1(X,DELTE(J))
      Y=DMIN1(Y,DELTR(J))
      Z=DMIN1(Z,DELTC(J))
190   CONTINUE
      DELTT=1.2D0*DELT(4)
      X=DMIN1(X,Y)
      X=DMIN1(X,Z)
      X=DMIN1(X,DELTT)
      X=DMIN1(X,DELTTT)
C
C          THIS SECTION PRINTS NNN,T,R,U,A,GA,RHO,EPS,P AND Q AT
C          SPECIFIED INTERVALS.
C
      IF(NA.LT.NAA) GO TO 200
      WRITE(6,2000)NNN
```

```
      WRITE(6,3000)T
      DO 210 J=JN,JJ,JN
      I1=J+1
      WRITE(6,12000)J
      ZZ=R(I1,3)
      WRITE(6,4000)ZZ
      ZZ=U(I1,2)
      WRITE(6,5000)ZZ
      ZZ=A(J,3)
      WRITE(6,6000)ZZ
      ZZ=GA(J,3)
      WRITE(6,7000)ZZ
      ZZ=RHO(J,3)
      WRITE(6,8000)ZZ
      ZZ=EPS(J,3)
      WRITE(6,9000)ZZ
      ZZ=P(J,3)
      WRITE(6,10000)ZZ
      ZZ=Q(J,2)
      WRITE(6,11000)ZZ
210   CONTINUE
      NA=0
200   CONTINUE
      DELT(2)=DELT(4)
      DELT(4)=X
      X=10.D0
      Y=X
      Z=X
C
C          THIS SECTION REASSIGNS QUANTITIES TO PREVIOUS TIME
C          DESIGNATIONS IN ORDER TO BEGIN THE NEXT TIME CYCLE.
C
      DO 220 J=2,JJ,2
      I1=J+1
      P(J,3)=P(J,5)
      Q(J,2)=Q(J,4)
      GA(J,3)=GA(J,5)
      GA(I1,3)=GA(I1,5)
      U(I1,2)=U(I1,4)
      A(I1,1)=A(I1,3)
      R(I1,3)=R(I1,5)
      R(I1,5)=0.D0
      W(I1,3)=W(I1,5)
      EM(I1,3)=EM(I1,5)
      EM(I1,5)=0.D0
      A(J,1)=A(J,3)
      EPS(J,1)=EPS(J,3)
      A(J,3)=A(J,5)
      A(I1,3)=A(I1,5)
      EPS(J,3)=EPS(J,5)
      DELR(J,3)=DELR(J,5)
      RSQ(J,3)=RSQ(J,5)
      RHO(J,3)=RHO(J,5)
220   CONTINUE
      A(1,1)=A(1,3)
      A(1,3)=A(2,5)
      GA(1,3)=GA(1,5)
1000  CONTINUE
2000  FORMAT(1X,4H N= ,I10)
3000  FORMAT(1X,4H T= ,1D24.16)
4000  FORMAT(10X,4H R= ,1D24.16)
5000  FORMAT(10X,4H U= ,1D24.16)
```

```
6000  FORMAT(10X,4H A= ,1D24.16)
7000  FORMAT(10X,5H GA= ,1D24.16)
8000  FORMAT(10X,6H RHO= ,1D24.16)
9000  FORMAT(10X,6H EPS= ,1D24.16)
10000 FORMAT(10X,4H P= ,1D24.16)
11000 FORMAT(10X,4H Q= ,1D24.16)
11500 FORMAT(10X,4H M= ,1D24.16)
11600 FORMAT(10X,5H RS= ,1D24.16)
12000 FORMAT(6X,4H J= ,I10)
      STOP
      END
```

VITA

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Master of Science

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