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STRUCTURES AND TOPOLOGY

CHAPTER I

INTRODUCTION

From the very beginning, topologies have been defined in terms of neighborhood systems. Yet, even after the publication of Weil's Sur les Espaces à Structure Uniforme [10], no one seems to have noticed that "indexed" neighborhood systems may be used in the most general space as well as in the relatively special uniform space. This is strange for at least two reasons.

First, as will be seen, uniformly indexed systems of neighborhoods - what we shall call "structures" in this dissertation - provide a natural bridge from topological spaces in general to uniform spaces and metric spaces in particular. Indeed, starting in a metric space, point-set topology is reached by traversing this bridge in the reverse direction; when the metric is last seen, it is seen in the form of an indexed system of neighborhoods, viz., the system of open spheres $V_\varepsilon(x)$ of radius ε centered at x which provide the definition of open set. It was this observation, no doubt , that led Weil to formulate the notion of uniformity.

Secondly, much of the work in point-set topology has been concerned with generalizing results from analysis -

results which more often than not require concepts like uniform continuity, Cauchy sequences, totally bounded sets, contraction maps, etc., and these concepts in turn require that one be able to compare the "sizes" of neighborhoods at different points in the spaces concerned. But one does not need uniform spaces - much less metric spaces - to meet this requirement. All of the concepts named and many more can be defined in the most general topological space in terms of neighborhood "structures", as we shall see.

Of course the mere ability to define a concept in an extremely general setting does not carry with it the possibility of obtaining significant results with that concept in that setting. It is one of the theses implicit in this dissertation that not only do metric concepts generalize quite naturally to the most abstract of spaces, but many of the familiar theorems couched in terms of those concepts can be extended to such spaces as well.

In Chapter II the fundamental ideas in this work are formulated and in terms of them we investigate the relationship between certain families of dyadic relations over a set and the resulting topologies for the set. This chapter is something of a footnote to Weil's treatise, for here we are simply pushing his ideas to their natural conclusion.

What we are dealing with are various generalizations of Weil's concept of <u>uniform</u> <u>structure</u>, which in turn may be thought of as a kind of generalized equivalence relation (though Weil did not present it in this way). Given a set X, an equivalence relation over X is a dyadic relation $V \subseteq X \times X$ satisfying

i) $(x,x) \in V$;

ii) $(y,x) \in V$ implies $(x,y) \in V$;

and iii) $(x,y) \in V$ and $(y,z) \in V$ imply $(x,z) \in V$;

for all $x,y,z \in X$. Now substitute for V a family of relations $V_I = [V_a]_{a \in I}$ closed under intersection and satisfying

i') for each $a \in I$, $(x,x) \in V_a$;

ii') for each $a \in I$, there is a $b \in I$ such that $(y,x) \in V_b$ implies $(x,y) \in V_a$;

and iii') for each $a \in I$, there is a $b \in I$ such that $(x,y) \in V_b$ and $(y,z) \in V_b$ imply $(x,y) \in V_a$;

for all $x,y,z \in X$. V_I then is a uniform structure for X. What were equivalence classes before now become the "uniformly indexed neighborhoods" referred to above, viz., the subsets $V_a(x) = [y \in X\colon (x,y) \in V_a]$, with $x \in X$, $a \in I$. A uniformity is a uniform structure to which belongs every relation which contains a member of that structure.

One of Weil's principal results is that topological spaces which admit uniformities are precisely those which are completely regular, which means that for every point x and every closed set F disjoint from x in such a space there is a continuous real-valued function which vanishes at x and is identically 1 on F. In Chapter II we characterize more general kinds of topological spaces in terms of the kinds of "structures" they admit. (Much of Chapter II is taken from the author's <u>Indexed Systems of Neighborhoods for General Topological Spaces</u> [3].)

Another important result proved by Weil is that every uniform space has a completion. In Chapter III we shall show that, in one sense, the theory of uniformities is "minimal" if one wishes to deal only with spaces which do have completions without exception. Somewhat more precisely, what we prove is that regular spaces which have regular completions without exception are already completely regular. This result depends heavily on the particular definition of "Cauchy filter" which we choose to use. It is possible that a different definition would yield a more elegant theory.

Also in Chapter III we consider various kinds of weak "commutative laws" for a family of relations V_I under composition. For example, V_I might satisfy the condition: for each $a \varepsilon I$ and each $b \varepsilon I$, there is a $c \varepsilon I$ such that $V_a V_c \subseteq V_b V_a$. It turns out that conditions like this correspond in a natural way to various kinds of "continuity" of the members of V_I. In terms of one of these definitions of "relational continuity" we give yet another sufficient condition for a space to be completely regular and conjecture that the condition is also necessary.

Finally, in Chapter IV, we extend the metric idea of "contraction map" to apply to transformations of quite general topological spaces which are equipped with "structures" and we prove a fixpoint theorem analogous to the one discussed by Kolmogorov and Fomin in their _Functional Analysis_ [6]. The author owes his interest in the subject treated in Chapter IV to conversations with Dr. J. Mathews who has generalized the Kolmogorov-Fomin fixpoint theorem in a somewhat different direction. [Cf., 7.]

CHAPTER II

STRUCTURES AND TOPOLOGY

In this chapter the main concepts of this dissertation are defined. In terms of these concepts we then investigate the relationship between certain families of dyadic relations over a set and the resulting topologies for the set.

<u>2.1</u>. <u>Basic Definitions</u>. A family $V_I = [V_a]_{a \varepsilon I}$ of dyadic relations over a set X will be called <u>reflexive</u> if each V_a contains the diagonal Δ of $X \times X$, <u>symmetric</u> if for each $a \varepsilon I$ there is a $b \varepsilon I$ such that (the transpose) $V_b^{-1} \subseteq V_a$, and <u>transitive</u> if for each $a \varepsilon I$ there is a $b \varepsilon I$ and a $c \varepsilon I$ such that $V_c V_b \subseteq V_a$. (Composition is denoted by juxtaposition and, for powers, by superscripts.) V_I is <u>locally transitive</u> if for each $x \varepsilon X$, each $a \varepsilon I$, $V_c V_b(x) \subseteq V_a(x)$, where b and c may depend upon x as well as upon a.

If V_I is reflexive and closed under the operation of intersection, it will be called a <u>structure</u> for X. Thus a reflexive symmetric transitive family of dyadic relations, closed under intersection, is a <u>uniform</u> structure for the set concerned. So it is that uniformities are generalized equivalence relations. Here, however, we are concerned with structures which may be much weaker than those which are uniform.

If V_I and U_J are structures for X, the second will be said to be <u>as fine as</u> the first if for each $e \in I$ there is a $d \in J$ such that $U_d \subseteq V_e$. Two structures are <u>equivalent</u> if each is as fine as the other.

By a <u>space</u> we shall mean a set together with an equivalence class of structures for that set. Notation: (X, V_I), where V_I is a representative structure for X. If $Y \subseteq X$ and, for each $a \in I$, $U_a = V_a \cap Y \times Y$, then (Y, U_I) is a <u>subspace</u> of (X, V_I).

Given a map $f:X \to Y: x \to f(x)$ from a set X into a space (Y, V_I), the structure <u>induced by</u> f on X is the family $f^{-1} V_I f = [f^{-1} V_a f]_{a \in I}$. (Clearly $f^{-1} V_I f$ <u>is</u> a structure for X, is symmetric if V_I is symmetric, <u>etc</u>.) If X happens to be already equipped with a structure U_J, and if U_J is as fine as $f^{-1} V_I f$, then f is said to be <u>uniformly continuous</u>. Two spaces are <u>isomorphic</u> if there exists a one-one uniformly continuous mapping of the one onto the other whose inverse is also uniformly continuous.

Each time a concept is defined in terms of the notion of structure, one may ask, Is it invariant with respect to equivalence of structures and with respect to isomorphism of spaces? Unless specifically noted to the contrary, it may be taken for granted that all such concepts defined in this dissertation are indeed well-defined relative to equivalence and isomorphism. The proof of this in each case is easy and will be omitted.

By and large, concepts which can be defined without mentioning particular points in the space are "uniform" (or "global"), and for each of these there is usually a corresponding "local" concept. For example, a structure U_J is <u>(locally) as fine as</u> a structure V_I <u>at</u> some point x of the space if for each $e \varepsilon I$ there is a $d \varepsilon J$ such that $U_d(x) \subseteq V_e(x)$, where d depends perhaps on x as well as on e. Two structures are <u>locally equivalent</u> if each is everywhere locally as fine as the other. A map $f:X \rightarrow Y$ of (X, U_J) into (Y, V_I) is <u>continuous</u> if U_J is locally as fine as $f^{-1}V_I f$. <u>Homeomorphisms</u> between spaces, being one-one bicontinuous maps, are what one might term "local isomorphisms".

The thing that locally equivalent structures have in common, of course, is a topology. Given a space (X, V_I), its <u>topology</u> is the class $\mathfrak{T}$ of subsets $G \subseteq X$ which are <u>open</u> with respect to V_I. That is, for each $x \varepsilon G$ there is an $a \varepsilon I$ such that $V_a(x) \subseteq G$. Clearly $\mathfrak{T}$ satisfies the more usual definition: it is closed under finite intersection and unrestricted union. $\mathfrak{T}$ is the topology for X <u>defined by</u> V_I.

Since locally equivalent structures are not necessarily equivalent (and homeomorphisms need not be isomorphisms), the concept of topological space, viz., a set plus a topology, is less definite than the notion of "space" defined above in terms of structure. Nevertheless, every topological space admits a defining structure, as we shall see.

An <u>open neighborhood structure</u> for a set X is a structure U_I with the property that each subset of the form $U_a(x)$ ($a \varepsilon I$, $x \varepsilon X$) is open in the topology of the space, the adjective "open" being omitted if this property is replaced by

n.s.) given $a \varepsilon I$, there is a $b \varepsilon I$ such that for each $x \varepsilon X$ and each $y \varepsilon U_b(x)$, there is a $c \varepsilon I$ such that $U_c(y) \subseteq U_a(x)$.

(n.s.) may be thought of as a very rudimentary and provincial "triangle inequality".

Given a structure V_I for a set, it is not difficult to see that it may be replaced by an open neighborhood structure U_I which is as fine as V_I and which defines the same topology. If V_I is symmetric or locally transitive, then U_I can be made symmetric or locally transitive. If V_I satisfies (n.s.), then U_I may be chosen equivalent to V_I. Henceforth, unless otherwise noted, we shall assume that all structures treated are neighborhood structures.

<u>2.2</u>. <u>General Topological Spaces</u>. First we show that every topological space admits a "uniformly indexed" system of open neighborhoods. The postulates for open neighborhood structures are recapitulated in

<u>Theorem 2.2.1</u>. The pair $(X, \mathfrak{T})$, where X is a set

and $\mathfrak{T}$ is a family of subsets, is a topological space if and only if there exists a family V_I of dyadic relations over X satisfying

i) for each $x \varepsilon X$, $x \varepsilon \bigcap_{a \varepsilon I} V_a(x)$;

ii) to each pair $a \varepsilon I$, $b \varepsilon I$, there corresponds some $c \varepsilon I$ such that for all $x \varepsilon X$, $V_c(x) = V_a(x) \cap V_b(x)$;

iii) given $y \varepsilon V_a(x)$, there is a $b \varepsilon I$ such that $V_b(y) \subseteq V_a(x)$;

and iv) $G \varepsilon \mathfrak{T}$ if and only if for each $x \varepsilon G$ there is an $a \varepsilon I$ such that $V_a(x) \subseteq G$.

The proof of sufficiency is standard and is omitted. Assume, then, that a topological space $(X, \mathfrak{T})$ is given and define for each $G \varepsilon \mathfrak{T}$, the subset $V_G = [X \times G] \cup [(X - G) \times X]$ of $X \times X$. Let I be the class of all finite subfamilies of $\mathfrak{T}$ and define, for each $a \varepsilon I$, $V_a = \bigcap_{G \varepsilon a} V_G$. Then V_I is easily seen to meet all the requirements of the theorem.

The usefulness of this theorem is limited by the fact that the structure V_I cannot in general be made symmetric. The reason for this can be found in the fact that the V_a of Theorem 2.2.1 are not necessarily neighborhoods of the diagonal in the product topology. (See Theorem 2.3.1.)

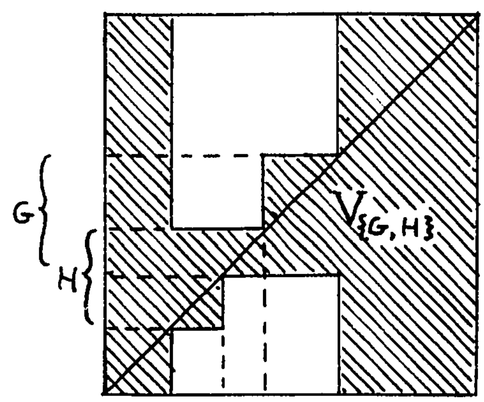

Fig. 1

2.3. R_0-spaces. The next question then is, For which topological spaces can the V_a be chosen as neighborhoods of the diagonal? It turns out that only a very mild "regularity" condition is needed. Since it seems that the spaces which satisfy this condition have not appeared explicitly in the literature before, several characterizations will be given in the next theorem.

In a topological space $(X, \mathfrak{T})$ let A and B be subsets of X. A is said to be separated from B by an open set G if $A \subseteq G$ and $G \cap B = \emptyset$. This is the case if and only if $A \cap \bar{B} = \emptyset$. A and B are entirely separated if they are separated by disjoint open sets. We shall refer to singletons as "points". Thus a T_1-space is a topological space in which distinct points are separated from one another.

Theorem 2.3.1. The following statements about a topological space $(X, \mathfrak{T})$ are equivalent:

a) closed sets are separated from the points that they exclude;

b) every open set contains the closure of each of its points;

c) $\mathfrak{T}$ is defined by a structure V_I for X, such that each V_a is an open neighborhood of the diagonal;

d) $\mathfrak{T}$ is defined by a symmetric open neighborhood structure;

e) for all $x \in X$, $y \in X$, either $[\bar{x}] \cap [\bar{y}] = \emptyset$ or $[\bar{y}] = [\bar{x}]$;

f) $\mathfrak{T}$ is isomorphic (lattice-theoretically) to the topology of a T_1-space.

An R_0-space, by definition, is one which satisfies any, hence all, of these conditions.

Proof: (a) implies (b): If G is an open set containing x, then $X - G$ is a closed set excluding it. Hence $(X - G) \cap [\bar{x}] = \emptyset$, i.e., $[\bar{x}] \subseteq G$.

(b) implies (c): Let $[V_a]_{a \varepsilon I}$ be the family of all open neighborhoods (in the product topology) of the diagonal. Property (i) of Theorem 2.2.1 is equivalent to $\Delta \subseteq \bigcap_{a \varepsilon I} V_a$. (ii) is satisfied with $V_c = V_a \cap V_b$. Now let $G \varepsilon \mathfrak{T}$ and $x \varepsilon G$ be given. Define $U = [X \times G] \cup [(X - [\bar{x}]) \times X]$. Since $[\bar{x}] \subseteq G$, U is an open neighborhood of Δ. Hence $U = V_a$, for some $a \varepsilon I$, and $V_a(x) = G$. Thus (iv) holds. Then so does (iii), since $V_a(x)$ is open in X whenever V_a is open in $X \times X$.

(c) implies (d): Symmetry of U_J is implied by the statement that $U_a^{-1} = U_a$, for all $a \varepsilon J$. Now if V_a is an open neighborhood of Δ, then so are $V_a \cap V_a^{-1}$ and $V_a \cup V_a^{-1}$. Since $V_a \cap V_a^{-1} \subseteq V_a \subseteq V_a \cup V_a^{-1}$, the family V_I of open neighborhoods of Δ is equivalent to the family U_J consisting of all symmetric open neighborhoods formed in this manner.

(d) implies (e): If $z \varepsilon [\bar{x}] \cap [\bar{y}]$, then $[\bar{z}] \subseteq [\bar{x}] \cap [\bar{y}]$. Also, for all $a \varepsilon I$, $x \varepsilon U_a(z)$, whence $z \varepsilon U_a(x)$. Then $x \varepsilon [\bar{z}]$, and so $[\bar{x}] \subseteq [\bar{z}] \subseteq [\bar{y}]$. Similarly, $[\bar{y}] \subseteq [\bar{x}]$. In summary, if $[\bar{x}] \cap [\bar{y}] \neq \emptyset$, then $[\bar{y}] = [\bar{x}]$.

(e) implies (f): For each $G \varepsilon \mathfrak{T}$, define $G^* = [x^* \subseteq X: x^* = [\bar{x}]$, for some $x \varepsilon G]$. Let $\mathfrak{T}^*$ be the family of all such sets. Then $(X^*, \mathfrak{T}^*)$ is a T_1-space and $G \to G^*$ is a one-one mapping of $\mathfrak{T}$ onto $\mathfrak{T}^*$ which preserves unions and intersections. The proof of the latter depends only on the properties of open sets and the fact that if $x \varepsilon X - G$, with $G \varepsilon \mathfrak{T}$, then $[\bar{x}] \subseteq X - G$, which is true in any space. For the proof that $\mathfrak{T}^*$ is a T_1-topology, note that if $[\bar{x}] \neq [\bar{y}]$, then $[\bar{x}] \subseteq X - [\bar{y}]$, whence $[\bar{x}] \varepsilon (X - [\bar{y}])^*$. Similarly, $[\bar{y}] \varepsilon (X - [\bar{x}])^*$, so that $[\bar{x}]$ and $[\bar{y}]$ are separated from each other in $(X^*, \mathfrak{T}^*)$.

(f) implies (a): Suppose that $\mathfrak{T}^*$ is the T_1-topology to which $\mathfrak{T}$ is isomorphic. Clearly, the corresponding lattices Z and Z^* of closed sets are also isomorphic. Now the lattice of closed sets for a T_1-space is characterized by the fact that each of its members is a join of atoms, i.e., of members which cover the null-element. Let $[A_a]_{a \varepsilon J}$ be the atoms of Z. Then for each $x \varepsilon X$, $[\bar{x}] = \bigcup_{a \varepsilon J_x} A_a$, for some $J_x \subseteq J$. Then $x \varepsilon A_x$, for some $x \varepsilon J_x$, and since A_x is closed, $[\bar{x}] \subseteq A_x$. But, since A_x is an atom (and $[\bar{x}] \neq \emptyset$), $[\bar{x}] = A_x$. (On the other hand, each atom, being a minimal nonvoid closed set, must be the closure of some point in X.) Thus for each $y \varepsilon X$ and each $F = \bigcup_{x \varepsilon F} [\bar{x}]$ in Z, either $y \varepsilon F$ or $F \cap [\bar{y}] = \emptyset$, and so closed sets are separated from the points that they exclude.

This closes a cycle of implication through the statements to be shown equivalent; the proof is complete.

Corollary. T_1-spaces are precisely those which are both R_0 and T_0. (A T_0-space is one in which at least one of each pair of distinct points is separated from the other.)

Proof: Statement (f) asserts that every T_1-space is an R_0-space. On the other hand, (e) asserts that $x \in [\bar{y}]$ implies $y \in [\bar{x}]$, and in a T_0-space this can happen only when $y = x$. Thus in an R_0-space which is also T_0, distinct points are separated from each other.

Just as there are many examples of T_0-spaces which are not T_1, there are many examples of R_0-spaces which are not even T_0. (Consider, for instance, the points of the plane with topology given by the pseudo-metric d: $d((x,y), (x',y')) = |x - x'|$.)

In a metric space the closure of a set is the set of all points zero-distant from it. It is perhaps surprising that a useful analogue of this statement can be formulated, not only for uniform spaces, but for spaces which, in a sense, are no stronger than a T_0-space:

Theorem 2.3.2. $(X,\mathcal{T})$ is an R_0-space if and only if is defined by a structure V_I such that, for each $A \subseteq X$,

$$\bar{A} = \bigcap_{a \in I} V_a(A)$$

(where $V_a(A) = \bigcup_{x \in A} V_a(x)$).

Proof: Using a structure V_I for X which defines $\mathcal{T}$

in accordance with Theorem 2.3.1 (d), the proof of $\bar{A} = \bigcap_{a\varepsilon I} V_a(A)$ proceeds exactly as it does in the theory of uniformities. Conversely, if for each $x \varepsilon X$, $[\bar{x}] = \bigcap_{a\varepsilon I} V_a(x)$, then every open set contains the closure of each of its points and so, by Theorem 2.3.1 (b), $(X,\mathfrak{J})$ is an R_o-space.

2.4. Interlude: Trennungsaxiome and regularity. Statement (f) of Theorem 2.3.1 suggests the question, What topologies are isomorphic to the topologies of T_o-spaces? The answer is, Every topology is isomorphic to a T_o-topology! The T_o-separation axiom, while restricting the space, imposes no condition at all on the lattice of open sets. (This may be shown by using the isomorphism induced by the mapping $x \rightarrow [\bar{x}]$ which occurs in the proof of Theorem 2.3.1.) Continuation of this line of questioning leads to a simple classification scheme for axioms of separation and regularity. It is offered here as a natural extension of a remark made by Kelley [5, p.130] on pseudo-metrics.

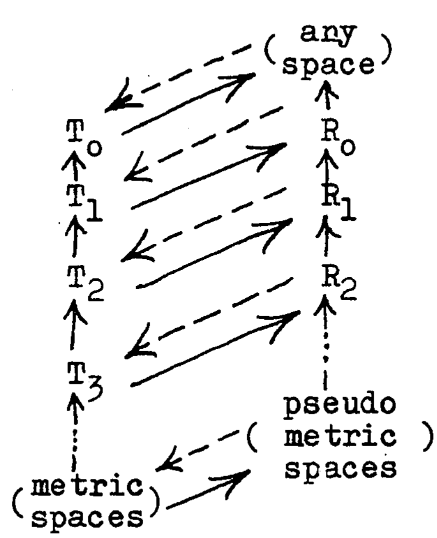

Fig. 2

The scheme is diagrammed in Fig. 2. Each T_k is a separation axiom (this is the standard notation) and each R_k is what we are calling, somewhat arbitrarily, an "axiom of regularity". The R_o-spaces were characterized in Theorem 2.3.1. R_1-spaces are those in which the closures of points are entirely separated wherever they do not coincide, and R_2-spaces are those which are properly called <u>regular</u> - those in which points and closed sets are entirely separated wherever the former are not contained in the latter. Each solid arrow represents an implication (absence represents absence), and each fractured arrow represents the existence of isomorphisms between topologies. Finally, each separation axiom is defined as the conjunction of two weaker axioms: $T_k = R_{k-1} \wedge T_{k-1} = R_{k-1} \wedge T_0$. (But the usual definition of "normality" must be modified slightly if R_3 is to be the axiom for normal spaces.

<u>2.5</u>. R_1-<u>spaces</u>. These seem to have appeared for the first time in Fig. 2. Now and then one finds theorems in the literature which begin with the premise, "If $(X,\mathfrak{T})$ is either Hausdorff or regular," and which could be both simplified and strengthened by replacing the phrase "either Hausdorff or regular" by "R_1-space". As an example, the following is the strengthened version of a theorem quoted by Kelley [5, p.146, <u>quo vide</u> for definitions].

<u>Theorem 2.5.1</u>. Every locally compact R_1-space has a compact local base at each of its points.

Proof: Let V_I define the topology of the locally compact R_1-space $(X,\mathfrak{T})$. Given $V_a(x)$, let W be a compact neighborhood of x. Since $W - V_a(x)$ is closed in W, it is compact. Now, for each $y \in W - V_a(x)$, we have $[\bar{y}] \cap [\bar{x}] = \emptyset$.

This is true and the following is possible since the space is R_1. Choose b and c so that $V_b(x) \cap V_c(y) = \emptyset$. Choose from the covering $[V_c(y)]_{y \varepsilon W - V_a(x)}$ a finite subcovering $[V_{c_i}(y_i)]_{i=1}^{n}$. Then $[\bigcap_{i=1}^{n} V_{b_i}(x)] \cap G = \emptyset$, where $G = \bigcup_{i=1}^{n} V_{c_i}(y_i)$. Hence $X - G$ is a closed neighborhood of x, and $U = W \cap [X - G]$ is a compact neighborhood of x. Since $W - V_a(x) \subseteq G$, $X - G \subseteq X - [W - V_a(x)]$, whence $U \subseteq W \cap (X - [W - V_a(x)]) = W \cap V_a(x) \subseteq V_a(x)$. Thus we have found a compact neighborhood of x contained in the given neighborhood $V_a(x)$.

We have not yet characterized R_1-spaces in terms of the structures they admit. This is done in

<u>Theorem 2.5.2</u>. An R_o-space $(X, \mathfrak{T})$ is R_1 if and only if its topology is defined by a symmetric structure V_I such that, for every $x \varepsilon X$,

$$\bigcap_{b \varepsilon I} V_b^2(x) = \bigcap_{a \varepsilon I} V_a(x).$$

Proof: First suppose that $(X, \mathfrak{T})$ is R_1. Since then it is also R_o, $\mathfrak{T}$ is defined by a symmetric structure V_I. In fact, we may take each $V_a (a \varepsilon I)$ to be symmetric. If $y \varepsilon X - \bigcap_{a \varepsilon I} V_a(x)$, then $[\bar{x}] \cap [\bar{y}] = \emptyset$, whence there is a $b \varepsilon I$ such that $V_b(x) \cap V_b(y) = \emptyset$. Then $y \varepsilon X - V_b(V_b(x))$. That is, $y \not\varepsilon \bigcap_{a \varepsilon I} V_a(x)$ implies $y \varepsilon \bigcup_{b \varepsilon I} [X - V_b^2(x)] = X - \bigcap_{b \varepsilon I} V_b^2(x)$. Thus $\bigcap_{b \varepsilon I} V_b^2(x) \subseteq \bigcap_{a \varepsilon I} V_a(x)$. The reverse inclusion is immediate. Conversely, if the V_a are symmetric $(a \varepsilon I)$ and V_I satisfies the above condition, then the space is R_1. For if $V_b(x) \cap V_b(y) \neq \emptyset$ for every $b \varepsilon I$, then $x \varepsilon V_b^2(y)$ for every $b \varepsilon I$. Thus $x \varepsilon \bigcap_{b \varepsilon I}$

$V_b(V_b(y)) \subseteq \bigcap_{a \varepsilon I} V_a(y) = [\bar{y}]$. That is, $[\bar{x}] = [\bar{y}]$.

Corollary. In an R_1-space with topology defined by a structure satisfying the conditions of the theorem, $\bigcap_{a \varepsilon I} \overline{V_a(x)} = \bigcap_{a \varepsilon I} V_a(x)$.

2.6. Regular spaces. Topological spaces were created by banishing the metric from metric spaces. The problem of metrization of topological spaces is essentially the problem of describing in non-metrical terms - in fact, in terms of topological invariants - the amount of "structure" a space must have in order to re-admit a metric. But the question may be turned around: What metric-like properties does a space with a given "amount" of topological structure have?

One virtue of neighborhood structures is that they often show very clearly which metric or pseudo-metric properties are restored with each step toward metrizability. Regular spaces provide a good example of this. Since they are but a half-step away from completely regular spaces, the following result should not be too unexpected:

Theorem 2.6.1. A topological space $(X, \mathfrak{T})$ is regular if and only if $\mathfrak{T}$ is defined by a symmetric structure V_I which satisfies the "local triangle inequality":

v) given $x \varepsilon X$ and $a \varepsilon I$, there is a $b \varepsilon I$, such that if $z \varepsilon V_b(x)$ and $y \varepsilon V_b(z)$, then $y \varepsilon V_a(x)$.

That is, regular spaces are those whose topologies are defined by symmetric structures which are locally transitive.

Proof: We use the fact that a space is regular if and only if each neighborhood of a point contains a closed neighborhood of that point.

Now if $(X, \mathfrak{I})$ is regular, then a forteriori it is an R_0-space and so we may choose V_I to be the family of all open neighborhoods of Δ in the product topology. Given $V_a(x)$ from this system, choose a closed neighborhood U of x contained in $V_a(x)$. Choose a second closed neighborhood V satisfying $x \varepsilon V \subseteq U^o$ (the interior). Define $W = [(X - U) \times X] \cup [(X - V) \times V_a(x)] \cup [X \times U^o]$. (See Fig. 3.) Then W contains Δ and is open, whence $W = V_b$, for some $b \varepsilon I$. Moreover, $V_b(V_b(x)) = V_b(U^o) = V_a(x)$, as desired.

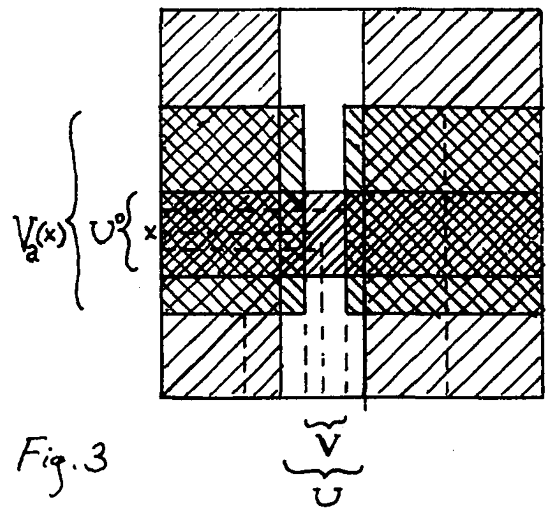

Fig. 3

Conversely, suppose V_I symmetric and locally transitive. Then given any neighborhood U of x, there exists an $a \varepsilon I$ and a $b \varepsilon I$, such that $x \varepsilon V_b(x) \subseteq \overline{V_b(x)} = \bigcap_{c \varepsilon I} V_c(V_b(x)) \subseteq V_b(V_b(x)) \subseteq V_a(x) \subseteq U$, and so the space is regular.

The fact that the b in Property (vi) depends in general not only on a, but on x, makes (vi) a local property. If, for each a, the b could be chosen "uniformly," i.e., independently of x, then of course we would be in a uniform space.

Locally transitive structures whose members are symmetric will be called <u>regular</u>. (By the method indicated in the proof of "(c) implies (d)" of Theorem 2.3.1, any symmetric structure for a set may be replaced by an equivalent structure each of whose members is symmetric.)

CHAPTER III

COMPLETENESS; RELATIONAL CONTINUITY

What we are aiming at in the first section of this chapter is the result that, roughly speaking, regular spaces which have regular completions without exception are already completely regular. In the remainder of the chapter some questions on "relational continuity" will be treated.

3.1. Completeness. By a Cauchy filter in a space (X,V_I) we shall mean a filter $\mathcal{P}$ which contains "small sets". That is, for each $a \varepsilon I$ there is a $p_a \varepsilon X$ such that $V_a(p_a) \varepsilon \mathcal{P}$. This is one of the more usual formulations of the Cauchy property.

To get the same results with nets as are obtainable with filters, one must work with nets $x_D = [x_n]_{n\varepsilon D}$ which satisfy: For each $a \varepsilon I$, there is a $p_a \varepsilon X$ and an $n(a) \varepsilon D$ such that for all $n \geq n(a)$, $x_n \varepsilon V_a(p_a)$. These might be termed semiregular in deference to the fact that Cauchy nets are sometimes called "regular". The latter are defined by the condition that for each $a \varepsilon I$ there be an $n(a) \varepsilon D$ such that $m \geq n(a)$ and $n \geq n(a)$ imply $x_m \varepsilon V_a(x_n)$.

(If V_I is uniform, the two concepts coincide.)

Semiregular nets are not necessarily Cauchy. In fact, even convergent nets (which are always semiregular) may not be Cauchy if the structure is not symmetric. On the other hand, semiregular nets and Cauchy filters are intimately related through the points $[p_a]_{a\varepsilon I}$ which, if I is directed by defining $a \geq b$ to mean $V_a \subseteq V_b$, also form a net. (Dragnet might be an appropriate name, since it "drags" the given net or filter along with it.) By means of this relationship it is easily shown that spaces in which semiregular nets always converge are precisely those spaces in which Cauchy filters always converge. Such spaces are complete.

Thus completeness may be studied in any space whatsoever. One fundamental result, well-known from the study of uniform spaces, is

Theorem 3.1.1. Any closed subspace of a complete space is complete (re the relativization of the given structure). Conversely, in a Hausdorff (T_2) space, complete subspaces are always closed.

Proof: Standard.

We shall say that a structure V_I for X is finite if for each $a \varepsilon I$ there is a finite subset $F_a \subseteq X$ such that $X \subseteq V_a(F_a)$. The space (X,V_I), when V_I is finite, is said to be totally bounded. It happens that every topological space admits a finite structure:

Theorem 3.1.2. There is a finite structure defining the topology of any given space. If the space is R_0 (or R_2, or completely regular), then that structure may be chosen also to be symmetric (and locally transitive, and transitive).

Proof: For spaces in general, the construction used in the proof of Theorem 2.2.1 is easily seen to yield finite structures. Given an R_0-space $(X, \mathfrak{T})$, then, let I be the family of all finite open coverings. For each $a \in I$, define $V_a = \bigcup_{G \varepsilon a} G \times G$. Clearly V_I is reflexive and symmetric. Moreover, $V_a \cap V_b = [\bigcup_{G \varepsilon a} G \times G] \cap [\bigcup_{H \varepsilon b} H \times H] = \bigcup_{G,H} ([G \times G] \cap [H \times H]) = \bigcup_{G \cap H} ([G \cap H] \times [G \cap H]) = V_c$, where c is the finite open covering whose members are of the form $G \cap H$ with $G \varepsilon a$ and $H \varepsilon b$. Thus V_I is closed under intersection. Now given $G \varepsilon \mathfrak{T}$ and $x \varepsilon G$, let $a = [G, X - [\bar{x}]]$. Then $a \varepsilon I$ and $V_a(x) = G$. So V_I defines $\mathfrak{T}$. Finally, (X, V_I) is totally bounded: Given $a \varepsilon I$, let F_a be a selection of points, one from each member of a. Then $X = V_a(F_a)$ and F_a is finite. Now if $(X, \mathfrak{T})$ happens to be regular, an argument similar to the one used in the proof of Theorem 2.6.1 will show that V_I is locally transitive. Proofs of existence of finite uniform structures for completely regular spaces are well-known.

As in the theory of uniformities, total boundedness may be characterized in terms of the existence of semiregular nets:

Theorem 3.1.3. A space is totally bounded if and only if each net (filter) has a semiregular subnet (Cauchy superfilter).

Proof: Standard, using universal subnets (or ultrafilters).

As usual, a topological space is said to be compact if each of its open coverings has a finite subcovering. Totally bounded spaces are "precompact" in the following sense:

Theorem 3.1.4. A space has a compact topology if and only if it is totally bounded and complete.

Proof: Compact spaces (X,V_I) are totally bounded since each covering of the form $[V_a(x)]_{x\varepsilon X}$ has a finite subcovering. (If V_I is not an open neighborhood structure, choose an equivalent structure which is.) Since every filter has a limit point, compact spaces are also complete. Now, conversely, suppose (X,V_I) is totally bounded and complete. By Theorem 3.1.3, every net in X has a semiregular subnet which, by completeness, converges. It is well-known that the existence of convergent subnets without exception is equivalent to compactness.

For compact Hausdorff spaces, not only are all uniformities equivalent - a standard result - but indeed all regular structures are equivalent:

Theorem 3.1.5. All regular structures defining a compact R_1-topology are equivalent; all are uniform.

Proof: First we show that if V_I is a regular structure defining a compact R_1-topology for X, then $\overline{V}_I = [\overline{V}_a]_{a\varepsilon I}$ (closure taken in the product topology) generates an equivalent structure. We need two auxiliary results:

Lemma A. Let $(X,\mathfrak{J})$ be defined by a symmetric structure V_I. For each $R \subseteq X \times X$, $\overline{R} = \bigcap_{a\varepsilon I} V_a R V_a$.

Lemma B. If V_I is a regular structure, then $\bigcap_{b\varepsilon I} V_b^3 = \bigcap_{a\varepsilon I} V_a$.

The first may be proven just as it is in the theory of uniformities; the second may be shown using a modification of the proof of Theorem 2.5.2.

Now, given $a \varepsilon I$, we have $\bigcap_{b\varepsilon I} \overline{V}_b = \bigcap_{b\varepsilon I} \bigcap_{c\varepsilon I} V_c V_b V_c \subseteq \bigcap_{c\varepsilon I} V_c^3 = \bigcap_{b\varepsilon I} V_b \subseteq V_a$. That is, $X - V_a \subseteq \bigcup_{b\varepsilon I} [X - \overline{V}_b]$, so that V_a together with $[X - \overline{V}_b]_{b\varepsilon I}$ form an open covering of the compact set $X \times X$. The existence of a finite subcovering gives us $\bigcap_{i=1}^{n} \overline{V}_{b_i} \subseteq V_a$, for some choice of $b_1, \ldots, b_n$. Thus the structure generated by $\overline{V}_I$ (by taking finite intersections) is as fine as the structure V_I. But then it is clear that the two are in fact equivalent, as was to be shown.

To finish the proof of Theorem 3.1.5, one may now proceed as in the proof of the corresponding theorem for uniformities, showing that all regular structures are equivalent to the structure consisting of the family of all neighborhoods of Δ in $X \times X$ and that this family is in fact a uniformity. The crucial ingredient in the standard proof is the existence of a structure defining $\mathcal{T}$ all of whose members are closed, and this is provided by $\overline{V}_I$.

<u>Theorem 3.1.6</u>. (Gál) Let $\overline{X}$ have a regular topology and let X be dense in $\overline{X}$. If every net in X has a cluster point in $\overline{X}$, then $\overline{X}$ is compact.

Proof: By Theorem 3.1.3, X is totally bounded <u>re</u> any structure V_I defining its topology, since every net in X, having as it does a subnet convergent in $\overline{X}$, has a semi-regular subnet. Let V_I be a regular structure. We shall show that each net x_D in $\overline{X}$ has a cluster point. For each $(a,n) \in I{\times}D$, choose $x_{a,n} \in V_a(x_n) \cap X$. If $I{\times}D$ is directed by letting $(a,n) \geq (b,m)$ mean that $n \geq m$ and $V_a \subseteq V_b$, then $x_{I\times D}$ is a net in X and so has a cluster point p in $\overline{X}$. Now given $a \in I$, let $b \in I$ be such that $V_b^2(p) \subseteq V_a(p)$. Given $n \in D$, let $m \geq n$ and $c \geq b$ be such that $x_{c,m} \in V_b(p)$. Then since also $x_{c,m} \in V_c(x_m) \subseteq V_b(x_m)$, we have $x_m \in V_b^2(p) \subseteq V_a(p)$. Thus p is a cluster point for x_D. We conclude (using the cluster-points-for-nets characterization of compactness) that $\overline{X}$ must be compact, q.e.d.

<u>Remark</u>: The use of structures simplifies and illuminates the proof of Gál's theorem considerably. (Cf., 4.)

Now we are ready to consider the question of completions for regular spaces. By a <u>completion</u> of a space (X,V_I) we mean a complete space which contains an isomorphic replica of (X,V_I) as a dense subspace. Our result is a negative one: It says that there can be no general procedure for constructing regular completions of spaces which do not have uniform structures.

Theorem 3.1.7. Let $(X,\mathfrak{T})$ be a regular topological space. If for each regular structure V_I defining $\mathfrak{T}$ there is a regular completion of (X,V_I), then $(X,\mathfrak{T})$ is completely regular.

Proof: Given a regular space $(X,\mathfrak{T})$ there is, by Theorem 3.1.2, a finite regular structure U_I defining $\mathfrak{T}$. Let $(\bar{X},V_I)$ be a completion of (X,U_I) whose topology is regular. Since (X,U_I) is totally bounded, each of its nets has a semiregular subnet (Theorem 3.1.3) and each such subnet converges to a point in $\bar{X}$. That is, each net in X has a cluster point in $\bar{X}$. Then by Theorem 3.1.6, $(\bar{X},V_I)$ has a compact topology. It is well known that subspaces of compact Hausdorff spaces are completely regular. Since we have shown that $(X,\mathfrak{T})$ is such a space, our proof is done.

Remark: Cohen [2] has found a way of completing spaces whose structures seem to be intermediate between those which are regular and those which are uniform. He has not shown, however (nor has the present author), that the topologies of these spaces are not necessarily completely regular. Counterexamples abound, of course, to show that spaces which admit regular structures need not admit uniformities, so Theorem 3.1.7 is not without some significance.

3.2. Relational Continuity. If $f:X \to X$ is a transformation of a space (X,V_I) into itself, it is not difficult to see that f is continuous (i.e., V_I is locally as fine as $f^{-1}V_I f$) if and only if at each $x \in X$ it is possible to find a $c \in I$ for each $b \in I$ so that $f(V_c(x)) \subseteq V_b(f(x))$. Extending this property to dyadic relations, we shall say that a relation $R \subseteq X \times X$ is right-continuous (re V_I) if for each $x \in X$, each $b \in I$, there is a $c \in I$ such that $RV_c(x) \subseteq V_bR(x)$. R is left-continuous if, given $x \in X$, $b \in I$, there is a $c \in I$ such that $V_cR(x) \subseteq RV_b(x)$. (A function $f:X \to X$ is left-continuous as a relation if and only if it is an "open map", i.e., transforms open sets into open sets.) Uniform left- and right-continuity are defined by making the usual modifications in the above definitions. For symmetric relations in a space with a symmetric structure it turns out that uniform left- and uniform right-continuity are equivalent concepts. (To see this, use the fact that if R, S, and T are symmetric relations, then $RT \subseteq SR$ and $TR \subseteq RS$ are equivalent statements.)

If every member of a structure is both [uniformly] left- and right-continuous relative to that structure, then the structure is said to be [uniformly] smooth. The problem of constructing smooth structures for a space is a distant cousin of the problem, proposed by Menger [8] and solved by Bing [1], of constructing convex metrics for certain metric spaces. (Cf., 1 or 8 for definitions.) For if a space has a convex metric, then it has a uniform structure V_I (defined by that metric) which is uniformly smooth. In fact, such structures are even <u>commutative</u>: $V_bV_a = V_aV_b$, for all $a, b \in I$. The problem of determining in general which spaces admit smooth, uniformly smooth, or commutative structures is still open. We can show, however, that spaces with regular structures which are smooth and closed under composition must be completely regular.

<u>Lemma</u>: Let V_I be a smooth regular structure for X and assume that V_I is closed under composition. Then for each $x \in X$, $a \in I$, and $b \in I$, there is a $c \in I$ such that $V_cV_cV_a(x) \subseteq V_bV_a(x)$.

Proof: Given $a, b \in I$ and $x \in X$, choose $k, l, m, n \in I$ so that at x: $V_aV_k \subseteq V_bV_a$, $V_lV_l \subseteq V_k$, $V_m(V_aV_l) \subseteq (V_aV_l)V_l$, and $V_nV_a \subseteq V_aV_l$. Let $V_c = V_m \cap V_n$. Then, at x, $V_cV_cV_a \subseteq V_cV_nV_a \subseteq V_cV_aV_l \subseteq V_m(V_aV_l) \subseteq V_aV_lV_l \subseteq V_aV_k \subseteq V_bV_a$, as desired.

<u>Theorem 3.2.1</u>. If V_I is a smooth regular structure for a set X and is closed under composition, then the topology of the space (X, V_I) is completely regular.

Proof: We wish to show that for each open $G \subseteq X$ and each $x \in G$ there is a continuous real valued function $f: X \to$ [Reals] such that $f(x) = 0$ and $f(X - G) = 1$. First choose $a \in I$ so that $V_a(x) \subseteq G$ and define inductively the sequence $a_o, a_1, a_2, \ldots$ in I as follows: Put $a_o = a$. Assuming that a_m has been defined for all $m < n$, let $W_J^{(n)}$ be the set of all compositions of the form $V_{a_{n_t}} V_{a_{n_{t-1}}} \cdots V_{a_{n_1}} V_{a_{n_o}}$, where $0 \leq n_o < n_1 < \ldots < n_t < n$. For each $b \varepsilon J$, let $a_n(b)$ be such that $V_{a_n(b)} V_{a_n(b)} W_b^{(n)}(x) \subseteq V_{a_{n-1}} W_b^{(n)}(x)$, using the above lemma. Then define $V_{a_n} = \bigcap_{b \varepsilon J} V_{a_n}(b)$, which is possible, since J is finite. The desired function f may now be defined in terms of these V_{a_n} $(n = 0,1,2,\ldots)$. The reader is referred to Weil [10, p.13] or to Nöbling [9, p.193] for details.

Comment: By replacing the phrase "smooth regular structure" in the premise of the lemma and of the theorem with "uniformity", and modifying the proof of the lemma accordingly, one obtains Weil's classical result (and its proof) that spaces which admit uniformities are completely regular. The converse of that result is also true. But whether or not completely regular spaces always admit smooth regular structures - or for that matter, smooth uniformities - still remains to be decided. Our conjecture is that they do.

CHAPTER IV

FIXPOINT THEOREM FOR CONTRACTIONS OF A WELL-CHAINED SPACE

If a map $A:X \to X: x \to Ax$ in a complete metric space (X,d) is such that, for some $c<1$, $d(Ax,Ay) \leq cd(x,y)$ holds for all $x, y \in X$, then the equation $Ax = x$ has in X a unique solution. The usefulness of this "principle of contraction mappings" in analysis has been well illustrated by Kolmogorov and Fomin in [5, pp.43-51].

Presented with an elegant little theorem of great applicability such as this, one is moved to wonder, How far can it be pushed? Can the concept of "contraction" be made meaningful in more general spaces and, if so, does the quoted fixpoint theorem have a significant generalization? For regular developable spaces an affirmative answer has been provided by Dr. J. Mathews [7]. This final chapter offers a fixpoint theorem for well-chained spaces with transitive structure. The definitions proposed in the first section and the lemmata leading to Theorem 4.2.2 are formulated in as general a manner as seemed compatible with naturalness. It may be that they are of interest independently of the particular fixpoint theorem presented here.

<u>4.1.</u> <u>Definitions</u>. A space (X, V_I) is said to be <u>well-chained from</u> $x \in X$ <u>to</u> $y \in X$ if for each $a \in I$ there is a natural number k such that $y \in V_a^k(x)$. A <u>well-chained</u> space is one which is well-chained between each pair of its points. This is simply a generalization of the corresponding concept for metric spaces.

For a metric space (X,d), the <u>structure defined by</u> d is the family of relations of the form $V_a = [(x,y) \in X \times X; d(x,y) < a]$, for $a > 0$ real. Such a structure is of course uniform.

Now a map $A: X \to X: x \to Ax$ is called an r/s-<u>map</u> relative to a structure V_I for X, r and s being natural numbers, if for every $a \in I$,

$$V_a^s \subseteq A^{-1} V_a^r A \,,$$

or equivalently,

$$A V_a^s \subseteq V_a^r A \,,$$

which is to say: if two points of X are no more than an a-chain of length s apart, then their images under A will be no more than an a-chain of length r apart. If $r < s$, the space "contracts" more or less uniformly under such a map. Notice that these properties are <u>not</u> invariant with respect to equivalence of structures.

An <u>eventual contraction</u> of a space (X, V_I) is a map $A: X \to X$ some iteration of which, say A^n, is an r/s-map with $r < s$, relative to some V_I-equivalent structure for X.

These definitions were suggested by the familiar notion of a contraction map in a metric space (X,d). A map $A:X \to X$ is an <u>eventual metric contraction</u> of (X,d) if, for some metric d^* equivalent to d, some number m, and some real $c < 1$, $d^*(A^m x, A^m y) \leq cd^*(x,y)$, for all $x, y \in X$. (d and d^* are equivalent if there exist constants $a \neq 0$ and b such that $ad \leq d^* \leq bd$; equivalent metrics define equivalent structures.)

<u>4.2</u>. <u>Results</u>. First we note that eventual contractions do indeed generalize the concept of an eventual metric contraction:

<u>Theorem 4.2.1</u>. Every eventual metric contraction is an eventual contraction of the space whose structure is defined by the given metric.

Proof: Let $A:X \to X$ be an eventual metric contraction of (X,d) and let d^* be the metric required by the above definition. It is not difficult to see that if n is chosen large enough (viz., $\geq -\log 2^m/\log c$), then $d^*(A^n x, A^n y) \leq \frac{1}{2} d^*(x,y)$, for all $x, y \in X$. Given $a < 0$, suppose $y \in V_a^2(x)$. I.e., $d^*(x,z) < a$ and $d^*(z,y) < a$, for some $z \in X$. Then $d^*(A^n x, A^n z) < \frac{a}{2}$ and $d^*(A^n z, A^n y) < \frac{a}{2}$, whence $A^n y \in V_{\frac{a}{2}}^2(A^n x) \subseteq V_a(A^n x)$. Thus A^n is a 1/2-map relative to a structure for X equivalent to that defined by d, and so A is an eventual contraction of the space.

The converse of Theorem 4.2.1 is false, as can be seen by this

Example: Let (X_i, d_i) be a metric space with $d_i \leq 1$, for $i = 1,2$, and suppose $X_1 \cap X_2 = \emptyset$. Define (X,d) by taking $X = X_1 \cup X_2$ and $d = d_i$ on $X_i \times X_i (i = 1,2)$, $d = 1$ on $X_1 \times X_2 \cup X_2 \times X_1$. Choose $x_1 \varepsilon X_1$, $x_2 \varepsilon X_2$, and define $A: X \to X$ by

$$Ax = \begin{cases} x_2, & \text{if } x \varepsilon X_1. \\ x_1, & \text{if } x \varepsilon X_2. \end{cases}$$

Then A is an r/s-map for every pair r,s (relative to the structure defined by d), yet for every n and every metric d^* for X we have $d^*(A^n x_1, A^n x_2) = d^*(x_1, x_2) \neq 0$, so that A is not an eventual metric contraction.

The following lemmas lead to a fixpoint theorem.

Lemma A. With respect to locally transitive structures, r/s-maps are continuous. If either $r = 1$ or the structure is transitive, an r/s-map is uniformly continuous.

Proof: Straightforward, using $A(V_b(x)) \subseteq AV_b^s(x) \subseteq V_b^r(Ax) \subseteq V_a(Ax)$, for b properly chosen.

Lemma B. An r/s-map (relative to some structure) is also an mr/ms-map relative to the same structure, for every $m = 1,2, \ldots$. Likewise, for $s' \leq s$ and $r' \geq r$, r/s-maps are also r'/s'-maps.

Proof: If $V_a^s \subseteq A^{-1}V_a^r A$, then $V_a^{ms} \subseteq A^{-1}V_a^r AA^{-1}V_a^r A \ldots A^{-1}V_a^r A$ (m times) $= A^{-1}V_a^{mr}A$, since $AA^{-1} = \Delta$. Thus r/s-maps are mr/ms-maps. The second assertion is also immediate.

Remark: The "cancellation law" converse of Lemma B is not valid in general.

Lemma C. The n^{th} iteration of an r/s-map is an r^n/s^n-map (re the same structure).

Proof: By induction on n. Let $A:X \to X$ be an r/s-map relative to V_I. Basis: For $n=1$ the assertion holds by hypothesis. Induction Step: Suppose that A^n is an r^n/s^n-map. Then A^n is also an sr^n/s^{n+1}-map, by Lemma B, and A is also an $r^{n+1}/r^n s$-map:

$$V_a^{s^{n+1}} \subseteq A^{-n} V_a^{sr^n} A^n \subseteq A^{-n-1} V_a^{r^{n+1}} A^{n+1},$$

for each $a \in I$, whence A^{n+1} is an r^{n+1}/s^{n+1}-map.

Lemma D. If $r < s$ and A is an r/s-map relative to a [locally] transitive structure V_I, then there exists a structure equivalent to [defining the same topology as] V_I relative to which some iteration of A is a 1/2-map.

Proof: Choose n so that $s^n \geq 2r^n$ and define U_I by putting $U_a = V_a^{r^n}$, for each $a \in I$. It is easily seen that U_I and V_I are equivalent if the latter is transitive (and that they define the same topology if the latter is locally transitive). But for each $a \in I$, $U_a^2 \subseteq V_a^{s^n} \subseteq A^{-n} V_a^{r^n} A^n = A^{-n} U_a A^n$, by Lemma C, whence A^n is a 1/2-map relative to U_I.

Lemma E. Let A be an r/s-map of X relative to V_I. Suppose that, for some $x \in X$, the space is well-chained from x to Ax. If $r < s$, then for each $a \in I$ there is an n such that, for all $m = 1, 2, \ldots$, $A^m x \in V_a^n(x)$.

Proof: By induction on m. Let k be such that $Ax \in V_a^k(x)$ and put $n = ks$. Basis: $m = 1$: $Ax \in V_a^k(x) \subseteq V_a^n(x)$. Induction step: Suppose $A^m x \in V_a^n(x)$. Then by Lemma B,

$$A^{m+1}x \in AV_a^{sk}(x) \subseteq AA^{-1}V_a^{rk}\,A(x) = V_a^{rk}(Ax),$$

and by definition of k and the fact that $r < s$,

$$V_a^{rk}(Ax) \subseteq V_a^{rk}V_a^k(x) \subseteq V_a^{sk}(x).$$

Thus $A^{m+1}x \in V_a^n(x)$.

<u>Lemma F</u>. In a well-chained space, the successive images of a given point under iterations of a 1/2-map form a Cauchy sequence.

Proof: Let (X, V_I) be well-chained and suppose that A is a 1/2-map relative to V_I. We wish to show that, for every $a \in I$, there is an n such that if $m \geq n$ then $A^m x \in V_a(A^n x)$, $x \in X$ being given. Let N be, by Lemma E, large enough so that all images of x under iterations of A belong to $V_a^N(x)$, and choose $n \geq \log_2 N$. Then, if $m > n$, we have

$$A^{m-n}x \in V_a^N(x) \subseteq V_a^{2^n}(x) \subseteq A^{-n}V_aA^n(x),$$

by Lemma C (since $1^n = 1$). That is, $A^m x \in A^nA^{-n}V_aA^n(x) = V_a(A^n x)$, as desired.

<u>Theorem 4.2.2</u>. Every eventual contraction of a sequentially complete well-chained T_o-space with transitive structure has a unique fixpoint.

Proof: If A is an eventual contraction of (X, V_I), then some iteration of A is an r/s-map, with $r < s$, relative to a structure equivalent to V_I. Since V_I and all equivalent structures are transitive, Lemma D tells us that a further iteration of A, say $A^n = B$, is a 1/2-map relative to a V_I-equivalent structure, say U_J. Since the space is well-chained, Lemma F gives us a Cauchy sequence $x, Bx, B^2x, \ldots$ $(x \in X)$. But by completeness, this sequence converges to some point $p \in X$. Then, B being continuous (Lemma A), we have $Bp = B \lim_{m\to\infty} B^m x = \lim_{m\to\infty} B^{m+1}x = p$, so p is a fixpoint for B. It is B's only fixpoint. For suppose that $Bq = q$. Since the space is well-chained, for each $a \in J$ there is a k large enough so that $(p,q) \in U_a^{2^k} \subseteq B^{-k}U_aB^k$ (using Lemma C and remembering that B is a 1/2-map re U_J). But then $(p,q) = (B^kp, B^kq) \in U_a$. Thus, for every $a \in J$, $q \in U_a(p)$ and, in a like manner, $p \in U_a(q)$. Since the space is T_0, this implies $q = p$. Now we can show that p is a fixpoint for A: Since $BAp = A^{n+1}p = ABp = Ap$ and since B has but one fixpoint, $Ap = p$. Finally, p is the only fixpoint for A because it is the only one for $B = A^n$.

Comment: The example following Theorem 4.2.1 shows that the assumption that the space be well-chained is essential in Theorem 4.2.2. It is crucial not only in the proof of existence of a fixpoint but also in the proof of uniqueness (as can be seen by interchanging x_1 and x_2 in the definition of A in the example). So Theorem 4.2.2 is not quite a true generalization of the fixpoint theorem for metric contractions quoted in the opening paragraph of this chapter. Finally, note that $A:X \to X$ need not be assumed continuous, as was done in [6, p.50].

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