COMPARISON OF THE PERFORMANCE CHARACTERISTICS OF PASSIVE AND ACTIVE SUSPENSION SYSTEMS: A RANDOM VIBRATION APPROACH

By

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NOMENCLATURE

c	road roughness coefficient
ζ	damping ratio
C_s	suspension damping
δ	deflection
E[ ]	expected value
H( )	frequency response
j	$\sqrt{-1}$
J	performance index
k	Kalman feedback gain
K_s	suspension stiffness
K_u	tire stiffness
M_s	sprung mass (body)
M_u	unsprung mass (tire)
p	weighting factor
p( )	probability density function
P[ ]	probability
q_1	weighting factor
q_2	weighting factor
R( )	autocorrelation
σ	RMS value
S( )	power spectral density
t	time
u	control force

v	frequency
V	vehicle velocity
w	circular frequency
x_o	road excitation
x_1	axle displacement
x_2	body displacement
y_1	suspension deflection
y_2	tire deflection

CHAPTER I

INTRODUCTION

The basic function of a good suspension system is the isolation of the vehicle body from the effects of the road excitation. Conventionally the problem has been solved by using passive elements. These include combinations of springs and shocks (dampers). These elements are passive in the sense that they do not require an energy source. The control of vibration is effected by the storage or dissipation of energy. The ride quality is obviously a function of the available suspension workspace. Various other factors come into play, one of which is the road holding quality of the suspension. The suspension, while isolating the vehicle body from the excitation, should not cause the tire to lose contact with the road. Other criteria may be incorporated into the discussion, but ride quality, road holding, and available suspension workspace remain primary.

"Soft" suspensions make for better ride quality and are desirable in luxury vehicles. This is at the cost of poorer handling (related to road holding). On the other hand sports-cars require excellent handling, especially on turns and thus have "stiff" suspensions. Hence suspension design involves a trade-off between a comfortable ride and good road handling. The available suspension workspace is the crucial constraint in the above equation. In some vehicles such as formula-one race-cars excellent body isolation has to be maintained, with typically small suspension workspaces, while simultaneously controlling the axle displacements to ensure good handling.

Suspensions fall into three major categories. On one end of the spectrum are the passive suspensions mentioned earlier. However since a passive suspension involves a spring of fixed stiffness and a fixed damper, it's performance varies with the road surface encountered. No passive suspension can be designed for all the various types of road surfaces. It is difficult to design a passive suspension that is "soft" to the road undulation, while is simultaneously "hard" to external forces. The passive suspension is simple to realize physically, but is limited performance-wise.

Active suspensions, on the other hand involve some kind of force actuator (hydraulic, electromechanical, pneumatic, or magnetic) and various measuring and sensing devices (accelerometers, force transducers, and potentiometers). The force produced is a function of a number of measured variables. This ability to modulate forces in response to many variables leads to better performance. Such a design appears to be attractive, but is not without it's drawbacks. Most importantly, though a lot of work has been done on active suspensions primarily using optimal control theory, the improvement in performance that can be achieved has yet to be quantified. Most active suspension designs discussed in the literature require complete knowledge of all state variables. The physical realization of active suspension systems is an interesting engineering problem in it's own right. However it is apparent that the design will be complicated and costly. There is also a substantial energy requirement. Such a system would make sense if it's performance gains offset the additional cost. Various attempts have been made to simplify the control law by considering limited feedback information. Vibration absorbers have been considered as a means of reducing the axle displacements. Such a design would also result in a significant saving of energy.

Semi-active suspensions are derived from active suspensions. The primary purpose of this variety of suspension system is to circumvent the use of a power source to effect the vibration isolation. A semi-active suspension has some active force element. This force element produces a force which is again a function of the state variables, as long as such a force is dissipative. When such a force no longer opposes the relative motion of the ends of the damper, the semi-active suspension shuts off. Physical realizations of such a system might involve a variable rate damper. An "intelligent" shock absorber may be employed where a valve controls the flow of fluid based on a control law. Some of the performance gains of fully active systems may be realized by such an arrangement. It should be noted that such devices are inherently non-linear.

Traditionally the analysis of suspension systems include simulations (analog or digital) of the system model (time domain), frequency response characteristics (frequency-domain), and a study of the eigen-values (poles) of the system. The effect of a change in suspension parameters on the suspension performance has been extensively researched, using the above three techniques. Various types of road excitations are considered. Isolated instances such as potholes have been worked into the analysis. Car models such as the full-car, half-car, and the quarter-car have been used to determine the performance characteristics. Most analyses use different criteria as the basis for comparison, but the ones mentioned earlier are the most popular.

The idea of describing the road as a continuous random excitation is not new. Various types of power spectral densities (PSD) have been suggested depending on the car model. These are based on actual road profile measurements. A lot of literature deals solely with the description of the road roughness. In most cases the road displacement is described by a type of integrated white noise PSD function. Thus if the road is considered as a velocity input, it is a white noise excitation. In the case of a half-car model, the speed-dependent delay between the excitation experienced by the front and the rear wheels is of importance. If a full-car model is considered, the correlation between the two road profiles exciting the model (roll mode) comes into play, in addition to the delay. The quarter-car model, with only a bounce mode, is most commonly used if the primary purpose of the study is to compare the performance of various suspension systems. Root-mean-square (RMS) values of the criteria chosen are compared, and conclusions drawn about the suspension performance.

It is the intention of this study to use the above random vibration approach in order to compare the fully active suspension with the conventional passive suspension system. The active suspension discussed by Thompson (1976) is compared to the spring-shock suspension system. In the case of passive suspension systems there are essentially two variables, the stiffness and the damping. Initially a comparison of the transient response characteristics is made, the eigen-values of the two systems are obtained, and the frequency response functions evaluated. The problem is then extended to the RMS values of certain specific criteria. Next the relative performance of the two suspensions for the same workspace is examined. It is desirable that the suspension does not bottom during it's operation. That is to say that the suspension should only rarely encounter the "bump-stop" position. Though it is unlikely that the suspension will fail if it bottoms just once, it is important enough to know before hand the probability of such an occurence. To the designer, the RMS values of the suspension deflection gives some indication as to the probability of encountering the "bump-stop" position. However to be able to predict the same quantitatively is more desirable.

A quarter-car model is chosen (two degrees of freedom). The criteria singled out to form the basis of comparison are the body acceleration (ride quality), the suspension deflection, and the tire deflection (road holding). The road displacement is assumed to have a PSD of the integrated white noise type. The vehicle is assumed to be traversing the road surface at a fixed velocity. The road excitation is assumed stationary and Gaussian in nature, and the vehicle model linear. The response of the vehicle variables to the random excitation is random in nature too. The above simplifying assumptions imply that the response of the vehicle model is stationary and Gaussian too. The vehicle model is chosen such that it is conducive to analysis, while at the same time yields meaningful results. RMS values for the passive suspension are discussed for varying stiffnesses and damping factors. These values are compared to similar values obtained for the active suspension design presented by Thompson (1976). This is the fully active variety of suspension requiring the complete knowledge of all the state variables. The problem is then extended to the first-passage time probabilities of the suspension deflection for the two types of suspensions. The level crossing values are calculated first. These require the second order statistics of the suspension deflection and it's derivative process.

Thompson (1976) predicts significant performance gains for the active suspension. These gains are evident upon comparing the RMS values and the first-passage times. These estimates of the first-passage times may be refined further using better approximations than the one based on the Poisson crossing assumption. There exists a possibility that a study of this kind will lead to an analysis of the probability of the "failure" of different kinds of suspension systems, similar to the probabilistic analysis of the phenomenon of fatigue.

CHAPTER II

A LITERATURE REVIEW

Passive and active suspension systems for automobiles have been studied extensively, while few predictions of the capabilities of semi-active systems have been made. Apart from various analyses of suspension systems per se, related topics pertinent to this study have been developed by various researchers. Chief among these are the description of the road excitation and specific theoretical problems in the realm of random vibration, namely first-passage time probabilities.

Passive Suspension Systems

Wambold (1983) presents a very lucid tutorial, introducing the effects of road roughness on dynamic vehicle models. A passive suspension system is outlined for both the quarter-car and the half-car models. A quarter-truck model is also presented to compare the effects of road excitation on a truck model as opposed to a car model. Passive systems are used in most literature which outline active and semi-active systems as a basis for comparison of the relative performance characteristics. The design choices available for passive systems vis à vis other suspension systems are discussed at length in Sharp and Hassan (1986), and Chalasani (1986).

Early work in passive suspensions includes a discussion by Thompson (1969-1970) about optimum damping in a randomly excited non-linear suspension. Unsymmetrical damping provides better isolation from large bumps with moderate increases in the mean square values representative of ride quality and road holding. Thompson (1973) expands on the topic of optimum suspension design. The excitations considered include both isolated bumps of varying length and continuous random excitation. The model considered is the half-car model with both bounce and pitch. The effect of variations in the ratio of front to rear spring stiffness, and the inertia coupling ratios, on the ride quality and road holding are studied.

Active Suspension Systems

Optimal active suspension systems are designed using stochastic optimal control theory. Linear full state feedback is assumed, though the case of incomplete state variable information is discussed in some of the literature. The infinite time case is considered, resulting in time-invariant Kalman feedback gains for a controllable system. The resulting active suspensions, though useful for the purposes of analysis, are difficult to realize physically. Quadratic performance indices are used in the analyses, which are formulated to include sprung body acceleration (indicative of the ride quality), suspension working space, and tire deflection (which represents road holding).

Wilson et al (1986) review some linear stochastic control theory relevant to the design of active suspension systems subjected to integrated or filtered white noise. Conditions of observability and controllability are discussed. Limited state feedback problem formulations which avoid potential numerical problems in deriving optimal control laws are outlined. Thompson (1976) considers one such active suspension system subjected to the integrated white noise excitation of the road displacement. It is concluded, by comparing the transient characteristics and mean square values, that the performance of the active suspension is significantly better. Chalasani (1986) shows that if practical design limitations are considered, a 20% improvement in the sprung-mass isolation is realistic, other things being more or less equal. When a more complete car model is considered (full-car model, seven degrees of freedom), the same improvement is estimated to be 15%. Sharp and Hassan (1986) make some attempt to quantify the relative performance characteristics of various suspension systems.

Thompson (1984) outlines a practical simplification to his earlier active suspension design. This results in a simplified feedback structure. The effect of the weighting factors on the system eigen-values is discussed. Thompson and Pearce (1979) outline an active suspension using a half-car model. The speed-dependent time delay between the road disturbance inputs in shown not to effect the optimal control law which is linear and time invariant. A practical realization of this design is discussed. Thompson et al (1984) show how the active suspension for the quarter-car model may be modified using dynamic vibration absorbers applied to the axles. A method for computing feedback gains and vibration absorber spring and damper rates is given. If the active suspension is of the electrohydraulic type, a significant power saving is predicted.

Karnopp (1985) offers an insight into the active suspension of the type in the literature given above. Two different performance indices are used. In one case body isolation is traded off against contact force variation and in the other against suspension deflection. Symmetric root locus sketches are used to show the kind of system that results in each case. The conclusion arrived at is simply that active suspensions designed according to different criteria are not necessarily comparable. Further it may not even be appropriate to compare an optimal active suspension with a conventional passive spring-shock suspension, unless all aspects of suspension performance are considered simultaneously.

Semi-active Suspension Systems

Semi-active suspension systems are an attractive alternative to fully active suspensions. Their performance gains however have not yet been fully quantified. Semi-active suspensions outlined in literature, have some of the advantages of active suspensions and are easier to realize physically. They also result in a saving in energy.

Karnopp et al (1974) describe a type of force generator which can respond to a general feedback signal and control vibration. It however does not require the power supply of a servo mechanism. Computer simulation studies presented, suggest that performance comparable to a fully active system may be achieved. Physical embodiments of the concept are discussed and compared to hardware used in active and passive vibration control systems. Margolis (1982) discusses the expected response of a quarter-car model with an active or semi-active system to non-ideal feedback information. In most practical applications, state variable information is incomplete or has to be signal processed in some manner prior to their use in the control algorithm. Sharp and Hassan (1987) consider a type of semi-active suspension consisting of a spring of fixed stiffness and an active damper. The active damper is considered to be solely an energy dissipator, producing a force that is a linear combination of the state variables as long as such a force opposes the relative motion of the ends of the damper. Systems based on two alternative forms of a control law are studied. Performance of the semi-active suspension is comparable to the fully active suspension under moderate road conditions, but this is not the case on average main road or motorway conditions.

Random Vibration Approach

Virchis and Robson (1971) study the response of a simplified vehicle model accelerating across a random surface. It is shown that for practically occurring values of forward acceleration, mean-square response differs little from that with zero acceleration. Yadav and Nigam (1978) describe the response of a vehicle moving with variable velocity and subjected to a non-stationary road excitation. Response statistics, obtained by using the Monte Carlo method for the non-linear model, and the time domain and evolutionary spectra for the linear model, are compared. Crandall (1970) studies the classical problem of the first crossing probabilities of the linear oscillator. A variety of analytical approximations for the probability density are described and compared with results obtained by simulation and numerical methods. First crossing probabilities for envelopes of the oscillator response are also presented.

Description of Road Roughness

Various mathematical models to describe the road roughness are proposed in literature. Sayers (1986) puts forth an empirical model for random road inputs that accounts for both pitch-plane and roll motions. The Parkhilovskii assumption (that the roll and vertical components of the road excitation are uncorrelated) is shown to agree with measured data. The road model is formulated as a summation of white noise sources and can be used with a variety of analytical methods. Values of the coefficients used in the model are calculated and presented for various road sites and road surfaces. Dodds and Robson (1973) show that typical road surfaces may be considered as realizations of homogeneous and isotropic two dimensional Gaussian random processes. Complete description of any such process is given by a single autocorrelation function evaluated from any longitudinal track. Heath (1987) presents formulae involving single integrations which express the cross spectrum of two parallel road profiles in terms of a single track. The parallel vehicle tracks are again considered to be homogeneous and isotropic random processes. The paper looks at the relationship between the spectra of the two road profiles, which is of greater interest than that between their correlation functions.

CHAPTER III

PRELIMINARIES

Car Models

A quarter-car model is used to model the passive and the active suspension systems (Fig. 1, Fig. 2, Appendix B). In the model, the portion of the sprung-mass corresponding to one corner of the vehicle, and the wheel at one corner, is considered. The passive suspension is modeled as a spring and a shock (damper) in parallel. Both the spring and the damper are assumed to be linear. In the case of the active suspension, the spring and damper of the passive suspension are replaced by an active force element. This force actuator is modeled as an ideal force element that has an infinitely small response time. Optimal control theory is used to formulate a linear full-state feedback control law to relate the actuation force to the state variables. The tire is represented as a spring. Since the damping in the tire is typically very small, it is neglected in the analysis. It is assumed that the tire behaves as a point-contact follower that is in contact with the road at all times.

Road Excitation

Roads have the appearance of random signals, and techniques for describing stochastic signals have proven to be suitable as descriptions of road profiles. Road measurements have shown that, except at very low frequencies, the road profile (vertical displacement of the road surface) can be modeled as an integrated white noise input. Hence the vertical velocity at the tire road interface is modeled as a white noise input (Sayers, 1986). The approximating power spectral density (PSD) function of the road elevation is of the form:

$$S_{x_0}(v) = c/(2\pi v)^2 \tag{1}$$

where x_0 is the elevation of the road, S_{x_0} is the PSD of the longitudinal road profile, v is the wave number (v = 1/wavelength), and c is a roughness scaling factor. The PSD function of the derivative of the road profile (slope) is obtained by multiplying by a factor of $(2\pi v)^2$, to yield the following.

$$S_{\dot{x}_0}(v) = c \tag{2}$$

where $\dot{x}_0$ is the longitudinal slope of the road. This implies that the road slope may be described as white noise in the spatial sense, with a constant PSD amplitude of c. This spatial PSD function can be transformed into a temporal PSD function, using the relations:

$$f = v.V \tag{3}$$

and

$$S_{x_0}(f) = S_{x_0}(v)/V \tag{4}$$

where f is the temporal frequency (Hz) and V is the vehicle speed (m/s). The PSD functions can thus be expressed in terms of circular frequency $w = 2\pi f$ (rad/s).

$$S_{x_0}(w) = cV/w^2 \tag{5}$$

and

$$S_{\dot{x}_0}(w) = w^2 S_{x_0}(w) = cV \tag{6}$$

Realistically speaking, if the PSD function for the road displacement is used in a frequency-domain analysis, a lower frequency cut-off value will have to be applied. Eq. 5 indicates that an integrated white noise signal, contains infinite power. A cut-off frequency at the low end makes sense in the real life situation. Very low frequencies are beyond the response of most highway vehicles. There are however unique advantages in using either of the PSD functions (eq. 5 or eq. 6) to represent the input road signal. All roads smooth or rough, are represented by a single parameter c. Secondly, in the case of a linear system, the mean square value of any output signal of interest is simply related to the integral square value of the corresponding output signal due to a unit step input (provided the integral converges).

Virchis and Robson (1971) have indicated that for practically occurring values of forward acceleration, the mean square response differs little from that with zero acceleration. A simple zero acceleration analysis has a wide applicability. A typical constant velocity is chosen for the vehicle speed. Thus, in terms of the circular frequency, the PSD for the road velocity may be considered to be a white noise input (eq. 6). The random road input is assumed to be stationary in nature. This insures that the statistical properties of the input do not change with time.

Performance Criteria

The principal performance criteria of interest in the comparison of active and passive suspension systems are as follows:

1. Vibration isolation, or ride quality
2. Suspension travel
3. Road holding.

From the vehicle model (Fig. 1, Fig. 2), we can identify the variables representative of the above criteria.

1. The vertical acceleration of the sprung-mass ($\ddot{x}_2$)
2. The deflection of the suspension (x_2-x_1)
3. The deflection of the tire (x_1-x_0).

Since the excitation to the vehicle model is random in nature, the response is random too. Statistics such as the root-mean-square (RMS) values, and the first-passage time probabilities of the above variables are used as a measure of the ride quality, rattlespace requirement, and road holding. These statistics are used as the basis of comparison of the active and passive suspension systems.

Typical Parameter Values

The following numerical data for a conventional automobile front suspension used in Thompson (1976) is assumed in the analysis.

M_u = 28.58 kg	K_u = 155900 N/m
M_s = 288.9 kg	K_s = 19960 N/m
C_s = 1861 Ns/m	
δ_w = 6.3 cm	δ_t = 2.0 cm

The tire static deflection δ_t is derived from the tire radial stiffness K_u and the weight supported. The maximum allowable wheel excursion represents the limit of free travel from the point of fully laden equilibrium to the bump stop position. The available free travel in the direction of rebound is generally greater. In order to investigate the performance of passive suspensions for the entire range from "soft" (K_u/K_s = 20.0) to "stiff" (K_u/K_s = 5.0), the value of K_s is varied from 7795 N/m to 31180 N/m.

The vehicle is assumed to travel over the road at a fixed speed.

$V = 25$ m/s

A typical value for the road roughness coefficient is chosen from Sayers (1986).

$c = 5 \times 10^{-5}$ rad m

System Equations

The following system equations may be written by inspection (Fig. 1, Fig. 2). The passive system is represented by the following.

$$M_s\ddot{x}_2 + C_s(\dot{x}_2-\dot{x}_1) + K_s(x_2-x_1) = 0$$
$$M_u\ddot{x}_1 - C_s(\dot{x}_2-\dot{x}_1) - K_s(x_2-x_1) + K_u(x_1-x_0) = 0 \qquad (7)$$

The active system is represented by the following.

$$M_s\ddot{x}_2 - u = 0$$
$$M_u\ddot{x}_1 + u + K_u(x_1-x_0) = 0 \qquad (8)$$

CHAPTER IV

LINEAR OPTIMAL ACTIVE SUSPENSION

Integral Square Simplification

As mentioned earlier, one of the advantages of using an integrated white noise type of PSD for the road displacement input or a white noise PSD for the road velocity input, is that it facilitates the use of the integral square simplification (Thompson, 1973). In the case of a linear system, the mean square value of any output signal of interest is simply related to the integral square value of the corresponding output signal due to a unit step input (provided the integral converges). Denote the following.

$y_r(t)$ = output for random excitation

$y_s(t)$ = corresponding output for a unit step input

The mean square response to the random signal is given as follows.

$$E[y_r^2(t)] = cV \int_0^\infty y_s^2(t)dt \tag{9}$$

Thus for a fixed vehicle velocity, a system that is optimal for a unit step input (in the sense of minimizing the integral on the right hand side of eq. 9), will also be optimal for the random road input given by eq. 5 or eq. 6. Given below is a justification for the integral square simplification. Define the following transfer functions.

$H_{x_o}(s)$ = transfer function for a displacement input x_o

$H_{\dot{x}_o}(s)$ = transfer function for a velocity input $\dot{x}_o$

They are related as follows.

$$H_{x_o}(s) = s.H_{\dot{x}_o}(s) \qquad (10)$$

Transfer functions are developed later in the analysis assuming a white noise velocity road input. Hence the transfer functions assuming a velocity input is used in the following derivation, in the interest of consistency. Expressing the input-output relations in the frequency domain,

$$X_o(s) = (1/s) \qquad (11)$$

$$Y_m(s) = H_{x_o}(s)X_o(s) \qquad (12)$$

Substitute eq. 10 and eq. 11 in eq. 12.

$$Y_m(s) = H_{\dot{x}_o}(s) \qquad (13)$$

Use Parseval's theorem (Crandall and Mark, 1963; Nigam, 1983).

$$\int_{-\infty}^{\infty} y_m^2(t)dt = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} Y_m(s)Y_m(-s)ds \qquad (14)$$

Substitute eq. 13 in eq. 14 and note the following

(i) for $t < 0$, $y_m(t) = 0$, (ii) $s = jw$.

$$\int_{0}^{\infty} y_m^{2}(t)dt = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} H_{\dot{x}_o}(s)H_{\dot{x}_o}(-s)ds \qquad (15)$$

The mean square value of $y_x(t)$ is given by

$$E[y_x^{2}(t)] = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} H_{\dot{x}_o}(s)H_{\dot{x}_o}(-s)S_{\dot{x}_o}(s)ds \qquad (16)$$

Substitute eq. 6 in eq. 16.

$$E[y_x^{2}(t)] = \frac{cV}{2\pi j}\int_{-j\infty}^{j\infty} H_{\dot{x}_o}(s)H_{\dot{x}_o}(-s)ds \qquad (17)$$

The comparison of eq. 15 and eq. 17, establishes the relation given by eq. 9.

Optimal Active Suspension

Thompson (1976) presents a active suspension design which is used in the following analysis. The vehicle model is developed as follows. The following state variables (Fig. 2) x_1, x_2, $x_3 = \dot{x}_1$, and $x_4 = \dot{x}_2$ are assumed to be all zero initially. The output vector y has two components, $y_1 = x_1$ and $y_2 = x_2$. The following state variable equations are derived from eq. 8.

$$\dot{x}_1 = x_3$$

$$\dot{x}_2 = x_4$$

$$\dot{x}_3 = K_u(x_o - x_1) - u/M_u$$

$$\dot{x}_4 = u/M_s \qquad (18)$$

In the light of the integral square simplification, the system will be optimized for $x_o = 1$ (a unit step). The control force u is assumed to be applied equally to both the axle and the body.

The state and output equations may be expressed in matrix form.

$$[\dot{x}] = [A][x] + [b_1]u + [b_2]x_o$$

$$[y] = [C][x] \qquad (19)$$

where x_o is the disturbance input and $y = [\ x_1\ x_2\ ]^T$ is the output. The system matrix [A], the output matrix [C], and the coefficient vectors $[b_1]$ and $[b_2]$ are given by,

$$[A] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -K_u/M_u & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[C] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$[b_1] = [\ 0 \quad 0 \quad -1/M_u \quad 1/M_s\ ]^T$$

$$[b_2] = [\ 0 \quad 0 \quad K_u/M_u \quad 0\ ]^T \qquad (20)$$

The initial conditions are $[x(0)] = [\ 0\ 0\ 0\ 0\ ]^T$ and the input disturbance is $x_o = U(t)$ (a unit step). We require zero steady state following errors. The desired output vector is given as,

$$[y'] = [\ 1\ 1\ ]^T x_o \qquad (21)$$

Thus the desired axle (x_1) and body (x_2) responses are both unit steps.

A quadratic type of performance index is chosen. It is a weighted sum of the integral square values of the variables representative of the criteria chosen for the comparison.

$$J = \frac{1}{2}\int_0^{\infty} [pu^2 + q_1(x_1-x_0)^2 + q_2(x_2-x_1)^2]dt \qquad (22)$$

where p, q_1, and q_2 are numerical constants. The body force u, the dynamic tire deflection (x_1-x_0), and the suspension deflection (x_2-x_1) are included in the performance index. The body force u is proportional to the vertical acceleration of the body (eqs. 18), which in turn is indicative of the ride quality.

There is no loss in generality if one of the numerical constants, q_2, is set to unity. For random signals of the Gaussian type, an approximate value for q_1 may be arrived at by comparing the values of δ_w and δ_t. To prevent bottoming of the suspension and loss of tire contact with the road atleast 99.7% of the time, the necessary conditions are that ($3\sigma_w < \delta_w$) and ($3\sigma_t < \delta_t$), where σ_w and σ_t are the RMS values of the wheel travel (suspension deflection) and the dynamic tire deflection, respectively. Thus an estimate for q_1 is

$$q_1 = (\delta_w/\delta_t)^2 \qquad (23)$$

This establishes q_1 = 10. The determination of a suitable value for p requires a one dimensional search which may be based on the transient responses of the resulting active suspensions for each value of p. Using eq. 21, the performance may be expressed in matrix form.

$$J = \frac{1}{2}\int_0^\infty \{pu^2 + ([y]-[y'])^T[Q_2]([y]-[y'])\}dt \qquad (24)$$

where $[Q_2]$ is given by,

$$[Q_2] = \begin{bmatrix} q_1+q_2 & -q_2 \\ -q_2 & q_2 \end{bmatrix} \qquad (25)$$

The problem is to find the optimal control u^* for the system given by the eqs. 19 such that the output $[y]$ tracks the desired output $[y']$ and minimizes the performance index (eq. 24) simultaneously. This is an optimal tracking problem. The easiest way to approach the problem is to reduce it to an equivalent linear regulator problem.

The application of a unit step $x_o = U(t)$, instantaneously compresses the tire spring and establishes a new level for the road surface with respect to which we can define new state variables.

$$x_1 = x_1 - x_o$$

$$x_2 = x_2 - x_2$$

$$x_3 = x_3 = \dot{x}_1$$

$$x_4 = x_4 = \dot{x}_2$$

Using [x] and [y] to denote the new state and output vectors, the initial conditions will be given by $[x(0)] = [\ -1\ -1\ 0\ 0\ ]^T$, and system equations will be modified as follows.

$$[\dot{x}] = [A][x] + [b_1]u$$

$$[y] = [C][x] \tag{26}$$

The output vectors in the two coordinate systems are related by $[y] = [y]-[y']$. With the new variables the disturbance x_0 has been eliminated (eqs. 25) and the problem has been reduced to that of a linear regulator with initial conditions. The performance index in terms of the new coordinates is given by,

$$J = \frac{1}{2}\int_0^\infty \{pu^2 + [x]^T[Q][x]\}dt \tag{27}$$

where $[Q] = [C]^T[Q_2][C]$, and from eqs. 20 and eq. 25,

$$[Q] = \begin{bmatrix} q_1+q_2 & -q_2 & 0 & 0 \\ -q_2 & q_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{28}$$

Optimal Control Theory

Certain aspects of optimal control theory pertinent to the foregoing discussion is presented below. This exclusively involves the evaluation of the Kalman feedback gains in the case of the linear regulator. The objective of optimal control theory is to determine the control signals that will cause a process to satisfy the physical constraints and at the same time minimize (or maximize) some performance criterion (Kirk, 1970). The linear regulator system forms an important class of optimal control problems. The control law can be found as a linear time-varying function of the system states. The control law however becomes time-invariant under certain conditions. The following results are primarily due to R. E. Kalman.

Consider a system described in general by the state equations.

$$[\dot{x}(t)] = [A(t)][x(t)] + [B(t)][u(t)] \tag{29}$$

The coefficients of the various matrices may be time varying. The performance measure to be minimized may be expressed in terms of general matrices.

$$J = \frac{1}{2}[x(t_f)]^T[H][x(t_f)] + \frac{1}{2}\int_{t_0}^{t_f}\{[x(t)]^T[Q(t)][x(t)] + [u(t)]^T[R(t)][u(t)]\}dt \tag{30}$$

The final time t_f is fixed, [H] and [Q] are symmetric positive semi-definite matrices, and [R] is a real symmetric positive definite matrix. The states and the control are assumed to be unbounded, and $x(t_f)$, the state at the final time is assumed to be free. Physically speaking we desire to maintain the state vector as close to the origin as possible without an excessive expenditure of control effort.

By formulating the Hamiltonian (Kirk, 1970), and using the conditions for optimality it may be shown that the optimal control is given as follows.

$$u^*(t) = - [R(t)][B(t)]^T[P(t)][x(t)] \qquad (31)$$

This indicates that the optimal control is a time-varying, linear combination of the system states. The matrix [P(t)] depends on t_f. The optimal control may be expressed in terms of the Kalman feedback gains.

$$u^*(t) = [K(t)][x(t)] \qquad (32)$$

Even if the system is fixed, the feedback gain matrix [K(t)] is time-varying in general. In addition, measurements of all of the state variables must be available to implement the optimal control . To determine the feedback gain matrix [K], we have to formulate the transition matrix. However if the order of the system is large, this is a time consuming task. Also, if any of the matrices involved are time-varying, a numerical procedure will have to be resorted to. There is an approach that is more attractive. It can be shown that matrix [P(t)] (eq. 31) satisfies the following matrix differential equation

$$[\dot{P}(t)] = - [P(t)][A(t)] - [A(t)]^T[P(t)] - [Q(t)] + [P(t)][B(t)][R(t)]^{-1}[B(t)]^T[P(t)] \qquad (33)$$

with the boundary condition $[P(t_f)] = [H]$.

The above equation is known as the Ricatti Equation. $[P(t)]$ in general is an (n x n) matrix, making the above a system of n^2 differential equations. However $[P(t)]$ may be shown to be symmetric. Hence only $n(n+1)/2$ equations need be solved. The equations can be numerically integrated. It should be noted that the above equations evolve backwards in time. So the integration should start at time t_f (final time) and proceed backwards to the initial time t_o. $[P(t)]$ is stored for every time step, and thus the feedback gains, and hence the optimal control may be determined for any given time step (eq. 31). There is however a special case. If the process has to be controlled for an infinite duration, Kalman has shown that if (i) the plant is completely controllable, (ii) [A], [B], [Q], and [R] are constant matrices, and (iii) [H] = [0] (no weightage on the final state, see eq. 30), then $[P(t)]$ tends to [P] (a constant matrix), as t_f tends to ∞. Thus if the above conditions are satisfied, the Kalman feedback gains are time-invariant. To determine the [P] matrix for this special case, we either integrate the Ricatti equation backwards in time until a steady state solution is reached, or solve the following set of nonlinear algebraic equations.

$$0 = -[P][A] - [A]^T[P] - [Q] + [P][B][R]^{-1}[B]^T[P] \qquad (34)$$

Determination of the Kalman Feedback Gains

It is easier to simulate the Ricatti differential equation (eq. 33) and obtain the steady state elements of [P] rather than solve the nonlinear algebraic Ricatti equation (eq. 34). The simulation package Parasol-II is used to integrate the matrix Ricatti equation backwards, with the given boundary conditions, and obtain the Kalman feedback gains (Appendix C).

Comparing eqs. 26, 27, 29, and 30, we may identify the following equivalent matrices in our case.

$[H] = [0]$ (no weightage on the final state)

$[R] = [p]$ (1×1) => $[R]^{-1} = [1/p]$

$[Q] = [Q]$ (4×4)

$[A] = [A]$ (4×4)

$[B] = [b_1]$ (4×1)

[P] is a (4 x 4) matrix and being symmetric, there are ten elements to be determined. Once these elements have been determined, the Kalman feedback gains may be computed. The differential Ricatti equation is formulated as follows.

$$[\dot{P}(t)] = -[P(t)][A] - [A]^T[P(t)] - [Q] + \frac{[P(t)][b_1][b_1]^T[P(t)]}{p} \qquad (35)$$

This equation is simulated backwards on Parasol-II, with the "initial" condition, $[P(t_f)] = [H] = [0]$. Once the steady state values of [P] are determined the optimal control law is determined as follows.

$$u^* = [K][x] = [\ k_1 \quad k_2 \quad k_3 \quad k_4\ ][x] \tag{36}$$

where

$$[K] = -\frac{[b_1]^T[P]}{p}$$

The values of the Kalman feedback gains obtained using Parasol-II show good agreement with those presented in Thompson (1976). For the purposes of the analysis that follows, the feedback gains presented in Thompson (1976) will be assumed. Table 1 (Appendix A) shows these values for various values of p, with the values of q_1 and q_2 set to constants. Expanding the optimal control law, we have

$$u^* = k_1x_1 + k_2x_2 + k_3x_3 + k_4x_4 \tag{37}$$

Note that x_1 and x_2 are axle and body displacements relative to the road, and x_3 and x_4 are absolute velocities of the axle and the body respectively. Using the original variables defined (Fig. 2), eq. 37 may be expressed as follows.

$$u^* = k_1(x_1-x_2) + k_3(\dot{x}_1-\dot{x}_2) + (k_1+k_2)(x_2-x_0) + (k_3+k_4)\dot{x}_2 \tag{38}$$

Physically speaking, the control force may be realized by an actuator producing a force

$$f = (k_1+k_2)(x_2-x_0) + (k_3+k_4)\dot{x}_2 \tag{39}$$

in parallel with a spring of stiffness $K_e = k_1$ and a damper with a damping coefficient $C_e = k_3$. The required signals to the actuator are the absolute body velocity $\dot{x}_2$, and the body displacement relative to the road, (x_2-x_0). The measurement of these signals may be realized by using an ultrasonic transmitter and receiver to measure the relative body displacement, and the integrated output of a accelerometer mounted on the body to indicate the absolute body acceleration.

CHAPTER V

TRADITIONAL ANALYSES

Transient Response

On the basis of the transient responses of the body and axle displacements, and the body acceleration, the active suspension design resulting from setting $p = 0.8(10^{-9})$ is chosen. These responses are compared with those obtained for a typical passive system with K_s = 19960 N/m and C_s = 1861 Ns/m. These two typical active and passive suspension systems are compared elsewhere in this study. Parasol-II is used to simulate the two systems. The active suspension is designed taking into consideration the integral square simplification (Chapter IV). Thus both systems are excited with a unit step.

These responses are plotted (Fig. 3 - Fig. 8, Appendix B) and it is evident that the body motions of the active suspension are much better controlled and with less overshoot. This is a indication of superior ride quality. The axle response of the active system has a greater overshoot, but the transient is damped out more quickly. The transient response of active systems are dependent on the choice of the weighting factors in the performance index. The relative large axle overshoot of the active system could be reduced by either reducing the value of p or increasing the value of q_1. This in turn would penalize (and thus reduce) the axle response to a greater extent. A trade-off is implicit in the above argument. Doing so would inevitably result in an increased body force and hence acceleration. A better method would be to apply dynamic vibration absorbers to control the axle displacements (Thompson, Davis, and Salzborn, 1984). Table II (Appendix A) compares the transient characteristics of the two systems.

Eigen-values

The determination of the eigenvalues of an undamped system is a simpler task than doing so for a system with viscous damping. Consider the following general system equation,

$$[M][\ddot{x}] + [C][\dot{x}] + [K][x] = [F(t)] \qquad (40)$$

where [M], [C], and [K] are the mass, damping and stiffness matrices respectively, and [F(t)] is the forcing function. We consider the homogenous equation to determine the eigen-values. The same modal vectors that diagonalize the system for the undamped case do not diagonalize the the [C] matrix in general. There is a special case in which it does so, and that is when proportional damping occurs. If the [C] matrix can be expressed as a linear combination of the [M] and [K] matrices, proportional damping is said to occur.

$$[C] = \alpha[M] + \beta[K] \tag{41}$$

where α and β are constants. The system of equations may be written as,

$$[M][\ddot{x}] + \{\alpha[M]+\beta[K]\}[\dot{x}] + [K][x] = [0] \tag{42}$$

considering the homogenous case. By premultiplying each of the coefficient matrices by the transpose of the modal matrix for the undamped case, and postmultiplying them by the modal matrix, the system of equations can be diagonalized. A method is given below to diagonalize any general second order system of differential equations where the damping may or may not be proportional, and each of the coefficient matrices may or may not be symmetric. This method is due to R. A. Frazer, W. J. Duncan, and A. R. Collar.

Let us say we have a system of n second order equations given as in eq. 40, and consider the homogenous case. The second order system is first reduced to an equivalent system of first order equations. Define a new state vector as follows.

$$[y] = [\ \dot{x}\quad x\ ]^T \tag{43}$$

We can define the original system (eq. 40) as,

$$[\dot{y}] - [H][y] = [0] \tag{44}$$

where [H] is given by

$$-[H] = \left[\begin{array}{c|c} -[M]^{-1}[C] & -[M]^{-1}[K] \\ \hline -[I] & [0] \end{array} \right] \tag{45}$$

[I] is the identity matrix, and the system of equations given by eq. 44 is (2n x 2n). The eigen-values are obtained by assuming a solution of the form $[y] = [\phi].e^{\Omega t}$, where $[\phi]$ is a (2n x 1) modal vector with complex elements, and Ω is a complex number. Substituting the assumed solution in eq. 44 yields a system of homogenous algegraic equations.

$$\{\Omega[I] - [H]\} = [0] \tag{46}$$

The eigenvalues are the roots of the characteristic equation,

$$\det |\Omega[I] - [H]| = 0 \tag{47}$$

[H] is a square (2n x 2n) matrix, hence there are 2n eigen-values which are necessarily complex conjugates. A modal vector $[\phi]_i$ is found by substituting a particular eigen-value Ω_i in eq. 46 and solving the resulting homogenous algebraic equations. These are complex conjugates too. Being complex, phase information is included in the modal vector itself. The modal matrix is square (2n x 2n) and is a linear combination of the modal vectors resulting from each eigen-value.

$$[\phi'] = [\ \ [\phi]_1\ [\phi]_2\ \ldots\ldots\ [\phi]_{2n}\ \ ] \qquad (48)$$

The system of equations given by eq. 44 is diagonalized as follows. Premultiply and postmultiply the coefficient matrices by $[\phi']^{-1}$ and $[\phi']$ respectively.

$$[\phi']^{-1}[\phi'][\dot{y}] - [\phi']^{-1}[H][\phi'][y] = [0] \qquad (49)$$

This yields,

$$[I][\dot{y}] - [\Omega_i][y] = [0] \qquad (50)$$

$[\Omega_i]$ is a diagonal matrix, with the eigen values Ω_i on the diagonal. Subroutines (Tse, Morse and Hinkle, 1978) that follow the same procedure and evaluate the eigen-values are given in Appendix D. Driver programs are presented along the necessary subroutines, to evaluate the eigen-values and the modal vectors for the active and passive systems.

The eigen-values are located at $-6.305 \pm j7.625$ and $-26.29 \pm j78.28$ for the active system and at -2.666 ± 7.607 and $-33.12 \pm j68.62$ for the passive system. The location of the dominant poles correspond to damping ratios of ς = 0.637 for the active and ς = 0.330 for the passive systems. This increased damping ratio accounts partly for the reduced body overshoot in the case of the active suspension.

Frequency Response

Evaluation

Inspecting the vehicle models (Fig. 1, Fig. 2), we can identify the following criteria: suspension deflection is given by $y_1=(x_2-x_1)$, tire deflection is given by $y_2=(x_1-x_0)$ and body acceleration is given by $\ddot{x}_2$. We can express the passive system (eq. 7) as

$$M_s(\ddot{y}_1+\ddot{y}_2) + K_sy_1 + C_s\dot{y}_1 = -M_s\ddot{x}_0$$

$$M_u\ddot{y}_2 + K_uy_2 - K_sy_1 - C_s\dot{y}_1 = -M_u\ddot{x}_0 \qquad (51)$$

or alternately as

$$M_s\ddot{x}_2 = -K_sy_1 - C_s\dot{y}_1$$

$$M_u\ddot{x}_1 = -K_uy_1 + K_sy_1 + C_s\dot{y}_1 \qquad (52)$$

The active system (eq. 8), in terms of the new variables, is given by

$$M_s(\ddot{y}_1+\ddot{y}_2) - k_2y_1 - (k_1+k_2)y_2 - k_4\dot{y}_1$$
$$- (k_3+k_4)\dot{y}_2 = -M_s\ddot{x}_0 + (k_3+k_4)\dot{x}_0$$

$$M_u\ddot{y}_2 + k_2y_1 + (K_u+k_1+k_2)y_2 + k_4\dot{y}_1$$
$$+ (k_3+k_4)\dot{y}_2 = -M_u\ddot{x}_0 - (k_3+k_4)\dot{x}_0 \qquad (53)$$

or alternately as

$$M_s\ddot{x}_2 = u$$

$$M_u\ddot{x}_1 = -K_uy_2 - u \qquad (54)$$

The control force u, is given by eq. 37. The road displacement excitation is of the integrated white noise variety. If the road is considered as a velocity excitation, it is a white noise input. The evaluation of the RMS values (for which the frequency response functions are required), is simplified if the excitation is a white noise. Let us make the following substitutions: $x_o=e^{jwt}$, $y_1=H_{y_1}(w)e^{jwt}$, $y_2=H_{y_2}(w)e^{jwt}$, and $\ddot{x}_2=H_{\ddot{x}_2}(w)e^{jwt}$. Substitute these expressions in eqs. 51 and eqs. 53 and solve for $H_{y_1}(w)$ and $H_{y_2}(w)$. Substitute these frequency response functions in eqs. 52 and eqs. 54 and obtain $H_{\ddot{x}_2}(w)$. The frequency response functions for the passive system are given by,

$$H_{y_1}(w) = \frac{1}{D}\{-jwK_uM_s\}$$

$$H_{y_2}(w) = \frac{1}{D}\{M_uM_sjw^3 + (M_u+M_s)C_sw^2 - (M_u+M_s)K_sjw\}$$

$$H_{\ddot{x}_2}(w) = \frac{1}{D}\{jwK_u(K_s+jwC_s)\}$$

$$D = M_sM_uw^4 - (M_s+M_u)C_sjw^3 - \{(M_u+M_s)K_s+K_uM_s\}w^2 + K_uC_sjw + K_uK_s \qquad (55)$$

Similar functions for the active system are as follows.

$$H_{y_1}(w) = \frac{1}{D}\{-jw[M_s(K_u+k_1+k_2)+M_u(k_1+k_2)] - K_u(k_3+k_4)\}$$

$$H_{y_2}(w) = \frac{1}{D}\{jw^3M_uM_s + w^2(M_sk_3-M_uk_4) + jwk_2(M_u+M_s)\}$$

$$H_{\ddot{x}_2}(w) = \frac{1}{D}\{jw^3[M_u(k_1+k_2)] - w^2K_uk_3 - jwk_2K_u\}$$

$$D = M_u M_s w^4 - (M_s k_3 - M_u k_4) j w^3 - [M_s(K_u+k_1)-k_2 M_u] w^2 - k_4 K_u j w - k_2 K_u \quad (56)$$

Analysis

The frequency response plots for the suspension deflection, tire deflection, and the body acceleration are plotted and compared for the typical active and passive suspensions (Fig. 9 - Fig. 11). The transmissibilities at the sprung and unsprung-mass natural frequencies are of special interest. For the passive suspension, the peak at the sprung-mass natural frequency is pronounced for the body acceleration (Fig. 11). Increased damping eliminates this resonant peak. However the transmissibility increases adversely above the sprung-mass natural frequency. Hence it is desirable to employ a damping coefficient just large enough, such that the transmissibility does not deteriorate at the higher frequencies. The active suspension on the other hand exhibits a well damped behaviour at the sprung-mass natural frequency and a lightly damped behaviour at the unsprung-mass natural frequency, much the same as the passive system. Much of the improvement in the ride quality for the active suspension is experienced at the lower frequencies.

The frequency response plots for the suspension deflection are compared in Fig. 9. The active suspension does show some reduction of the peak response at the sprung-mass natural frequency, but it exhibits extremely large transmissibilities at the lower frequencies. This can be explained by an inspection of eqs. 55 and eqs. 56. $H_{x_q}(w=0)=0$ for the passive system, and $H_{x_q}(w=0)=(k_3+k_4)/k_2$ for the active system. Thus the transmissibility does not go to zero for the active suspension, as the frequency is reduced. The active suspension exhibits larger suspension deflections than the passive suspension for very low input frequencies.

Fig. 10 compares the transmissibilities of the tire deflection for the two suspension systems. Improvement in the tire deflection is shown by the active suspension at the sprung-mass natural frequency. At the unsprung-mass natural frequency however, it's transmissibility is higher than that for the passive system. This may be attributed to the passive system having greater damping for the wheel hop mode ($\zeta_u = C_s/2(K_u M_u)^{1/2}$, ζ_u(active)=0.328 compared to ζ_u(passive)=0.441).

CHAPTER VI

ROOT-MEAN-SQUARE VALUES

Evaluation

The frequency response functions evaluated for the chosen criteria are used to compute the RMS values. In the following discussion it will be assumed that the autocorrelation and the PSD are defined by the following pair of equations.

$$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(w)e^{jw\tau}dw$$

$$S(w) = \int_{-\infty}^{\infty} R(\tau)e^{-jw\tau}d\tau \qquad (57)$$

It may be shown (Nigam, 1983; Crandall and Mark, 1963) that for linear time-invariant systems, subjected to stationary excitation, the response is stationary too. The response mean is given by,

$$E[X(t)] = E[f(t)].H(0) \qquad (58)$$

where X(t) is the response, f(t) is the excitation, and H(w) is the frequency response function. The excitation may be assumed to be of zero mean without any loss of generality (E[f(t)] = 0). This implies that the response is of zero mean too. Further it may be shown that,

$$E[X^2(t)] = \sigma_x^2 = R_x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |H_x(w)|^2 S_z(w)dw \quad (59)$$

To compute the RMS values, the road excitation is considered to be a velocity input, and substituting eq. 6 in eq. 59 we have,

$$\sigma_x^2 = \frac{cV}{2\pi}\int_{-\infty}^{\infty} |H_x(w)|^2 dw \quad (60)$$

The variance and hence the RMS values may be calculated in one of two ways. Either the integral given in eq. 60 is integrated numerically or the integral square simplification is made use of. The appropriate system has to be simulated with a step input and the integral on the right hand side of eq. 9 calculated for the appropriate response variable whose RMS value is to be known. Crandall and Mark (1963) give closed form expressions for the integral given in eq. 60 for standard forms of frequency response functions. Those expressions are made use of. The RMS values obtained using the above closed form expressions, were checked with both numerical integration as well as using the integral square simplification. The following formulae are obtained for the RMS values of the chosen criteria for the passive suspension.

$$\sigma_{x_1}^2/(cV) = \frac{0.5(M_s+M_u)}{C_s}$$

$$\sigma_{x_2}^2/(cV) = \frac{0.5}{K_u^2 C_s M_s^2}(M_u+M_s)\{(M_u+M_s)^2 K_s^2 + K_u[C_s^2(M_s+M_u)-2K_s M_u M_s]\} + M_u M_s^2 K_s^2$$

$$\sigma_{\ddot{x}_2}^2/(cV) = \frac{0.5}{M_s^2 C_s}\{(M_s+M_u)K_s^2 + K_u C_s^2\} \qquad (61)$$

Similar formulae are obtained for the active system

$$\sigma_{x_1}^2/(cV) = \frac{0.5}{D}\,\frac{-K_u(k_3+k_4)^2}{k_2}\{\,[M_s(K_u+k_1)-k_2 M_u]\,.\,[M_s k_3 - M_u k_4] + k_4 K_u M_u M_s\,\} + (M_s k_3 - M_u k_4)[M_s(K_u+k_1+k_2) + M_u(k_1+k_2)]$$

$$\sigma_{x_2}^2/(cV) = \frac{0.5}{D}\,k_2^2(M_u+M_s)^2(M_s k_3 - M_u k_4) - k_4 K_u[(M_s k_3 - M_u k_4)^2 - 2k_2(M_u+M_s)M_u M_s] + M_u M_s[k_2 K_u M_s k_3 - k_4 K_u M_s(K_u+k_1)]$$

$$\sigma_{\ddot{x}_2}^2/(cV) = \frac{0.5}{D}\,k_2^2 K_u^2(M_s k_3 - M_u k_4) - K_u k_4[K_u^2 k_3^2 - 2k_2 K_u M_u(k_1+k_2)] + M_u(k_1+k_2)^2[k_2 K_u k_3 - k_4 K_u(K_u+k_1)]$$

$$D = -k_4 K_u\,\{\,[M_s(K_u+k_1)-k_2 M_u][M_s k_3 - M_u k_4] + k_4 K_u M_u M_s\,\} + k_2 K_u(M_s k_3 - M_u k_4)^2 \qquad (62)$$

We require the second order statistics of the derivative process too, to compute the level crossing rate (Chapter VII). The level crossing rate of the suspension deflection is required in the next part of the study. Given below is the method to compute them. As mentioned above, for the assumptions made about the systems and the nature of the excitation, the mean of the response is of zero mean as the excitation is of zero mean. A similar argument may be used to show that if f(t) is of zero mean, then $\dot{X}(t)$ is of zero mean too.

$$\sigma_{\dot{x}}^2 = \frac{cV}{2\pi} \int_{-\infty}^{\infty} w^2 |H_x(w)|^2 dw \tag{63}$$

Eq. 63 may be written as,

$$\sigma_{\dot{x}}^2 = \frac{cV}{2\pi} \int_{-\infty}^{\infty} |jw\ H_x(w)|^2 dw \tag{64}$$

Using the closed form expressions for the integrals, we get the following formulae.

$$\sigma_{\dot{x}_1}^2/(cV) = 0.5 \frac{K_u}{C_s} \tag{65}$$

for the passive system and,

$$\sigma_{\dot{x}_1}^2/(cV) = \frac{0.5}{D} K_u^2 (M_s k_3 - M_u k_4)(k_3+k_4)^2 - k_4 K_u [\ M_s(K_u+k_1+k_2)+M_u(k_1+k_2)\ ]^2 \tag{66}$$

for the active system. D is as defined in eq. 62

Analysis

RMS values for the active system are computed for different values of the weighting factor p. These values are presented in Table III (Appendix A). Similar values are presented for the passive suspension for K_u/K_s=7.8

(Table IV, Appendix A). Various damping coefficients C_s are considered. These values are plotted in Fig. 13 (Appendix B). Table V (Appendix A) compares in brief, the RMS values for the active system ($p=0.8(10)^{-8}$) and the passive system (C_s=1861 Ns/m). Fig. 12 (Appendix B) illustrates the trade-off between the RMS values for the suspension deflection, and the body acceleration for passive suspensions ranging from "stiff" (K_u/K_s=5.0) to "soft" (K_u/K_s=20.0). A reduction in the damping C_s of the suspension yields a reduction in the RMS value of the body acceleration, but this is at the price of increased RMS values for the suspension deflection. The damping cannot be reduced extensively. In the interest of good road holding, the value of the damping ratio in the wheel hop mode should realistically atleast be $\zeta_u = C_s/2(K_uM_u)^{1/2} = 0.2$. This gives a cut-off point at C_s=844.3 Ns/m. Both the active and the passive systems focussed on have ζ_u values greater than 0.2. As the suspension spring is softened, the trade-off curve moves downwards as is expected.

Concentrating on Table III, it is evident that as the weighting factor for the body force (p) increases, so does the suspension working space requirement along with better vibration isolation. However this improvement in vibration isolation is accompanied by higher RMS values for the tire deflection indicating deteriorating road holding. Thus the vibration isolation can only be improved to a point where sufficient road holding qualities exist. A somewhat similar trend is observed in the case of the passive suspension (Table IV). Reduced damping results in better ride quality. The RMS value for the tire deflection decreases and then increases again. This again indicates that the suspension damping cannot be decreased at will. There is a distinct region of C_s values where most passive suspension designs lie (1400 Ns/m - 2600 Ns/m). Luxury cars have their C_s values set at the lower end of this spectrum for better ride quality, while sports-cars have C_s values roughly in the middle of this range where the best road holding results.

Both the active and passive suspension systems, have more or less the same RMS value for the tire deflection (Table V). The resulting vibration isolation for the active system is about 18% better.

CHAPTER VII

FIRST-PASSAGE TIME PROBABILITIES

Evaluation

Crandall and Mark (1963) and Nigam (1983) show that the expected number of crossings of the level x=a with positive slope per unit time , v_a^+ of a random process x(t) is given by,

$$v_a^+ = \int_0^\infty \dot{x}.p(a,\dot{x})d\dot{x} \tag{67}$$

and the frequency of crossing with negative slope, v_a^- by

$$v_a^- = \int_{-\infty}^0 -\dot{x}.p(a,\dot{x})d\dot{x} \tag{68}$$

The above results are due to S. O. Rice. If x(t) is a stationary random process, the frequencies v_a^+ and v_a^- do not vary with time. However in general, the joint probability density $p(x,\dot{x})$ (x(t) and it's derivative process) is not known. We make another assumption here, and that is that the excitation is Gaussian in nature. This is a reasonable assumption since for a linear system with normal or Gaussian excitation, the response random process is of a Gaussian nature too. With this assumption we now know the joint probability distribution of x and $\dot{x}$.

$$p(x,\dot{x}) = \frac{1}{2\pi\sigma_x\sigma_{\dot{x}}} \exp\left[-\tfrac{1}{2} \{(x/\sigma_x)^2 + (\dot{x}/\sigma_{\dot{x}})^2\}\right] \quad (69)$$

Implicit in the above equation is the fact the x and $\dot{x}$ are uncorrelated, that $E[x\dot{x}] = 0$. The joint probability density function $p(x,\dot{x})$ is an even function of $\dot{x}$. Hence $v_a^+ = v_a^-$. Substitute eq. 69 in eq. 67 or eq. 68 and we have,

$$v_a^+ = v_a^- = (1/2\pi)\ (\sigma_{\dot{x}}/\sigma_x)\ \exp[-a^2/(2\sigma_x^2)] \quad (70)$$

The expected number of crossing at level x=0 is obtained from eq. 70. In the case of narrow band random processes where there are discernible "cycles", the equation given below represents it's "frequency".

$$v_0^+ = v_0^- = (1/2\pi)\ (\sigma_{\dot{x}}/\sigma_x) \quad (71)$$

The probability of an occurrence of a peak above x(t)=a is simply the fraction v_a^+/v_0^+. Knowledge of the process and it's derivative process is essential in order to evaluate the level crossing rate. Using the formulae for the first and second order statistics for the suspension deflection, it's level crossing rates are evaluated. An interesting relation exists between the zero level crossing rates and the suspension parameters for the passive suspension. Substituting appropriate statistics from eqs. 61 and eq 65, we have

$$\nu_0^+ = (1/2\pi)\ [K_u/(M_u+M_s)]^{1/2} \tag{72}$$

Thus the zero level crossing rate is independent of how "soft" or "stiff" the suspension is, and is also independent of the suspension damping. We now use the level crossing rates to determine the first-passage times of the suspension deflection.

If it is simplistically assumed that the suspension "fails" the first time the suspension deflection exceeds the "bump stop" position, $a = \delta_w = 0.063$ m, a first passage approach may be applied. δ_w is the maximum allowable wheel travel and it is assumed to be a typical value of 6.3 cm (Thompson, 1976). This is the maximum allowable wheel excursion till the "bump stop" position is encountered. The allowable excursion in the direction of rebound is generally more. If the level a is sufficiently high, a level crossing may be considered to be a rare event and hence independent. This is known as the Poisson crossing assumption. Let N(t) be the number of upcrossings at x=a in the time (0,t]. We are interested in $|y_1(t)| > 0.063$ m. If $y_1(t)$ is stationary, which it is assuming stationary excitation, N(t) is a Poisson process with the arrival rate (Crandall, 1970),

$$2\nu_a^+ = (1/\pi)\ (\sigma_{\dot{x}_1}/\sigma_{x_1})\ \exp[-a^2/(2\sigma_{x_1}^2)] \tag{73}$$

A level crossing rate is multiplied by a factor of two as we are interested in $|y_1| > a$. The probability of n upcrossings in (0,t] is

$$P(n,t) = (2v_a^+t)^n/n! \; . \; \exp[-2v_a^+t] \tag{74}$$

Let T_a be the time taken for the first excursion beyond $|y_1|=a$ to occur from below.

$$P[T_a>t] = P(0,t) = \exp(-2v_a^+t) \tag{75}$$

The probability of atleast one excursion is thus

$$P[T_a \le t] = 1 - \exp(-2v_a^+t) \tag{76}$$

T_a exhibits an exponential distribution whose probability density function is given by,

$$p_{T_a}(t) = 2v_a^+ \exp(-2v_a^+t) \tag{77}$$

with mean = $1/(2v_a^+)$ and variance = $1/(2v_a^+)^2$.

Analysis

The zero level crossing rates and the level crossing rates for the "bump stop" position for the active suspension are listed in Table VI (Appendix A) for various values of p. Similar values are presented for the passive suspension (K_u/K_s=7.8) in Table VII (Appendix A) for different values of damping. The frequency of zero level crossings for the active suspension is around 4.3 Hz, which is slightly higher than that for the passive suspension (3.5 Hz). This is due to higher RMS values for $\dot{y}_1$, the derivative of the suspension deflection, for the active suspension. The frequency of encountering the "bump stop" position is of the order of 10^{-11} to 10^{-7} Hz for both the active and the region of damping where most passive suspensions lie. This means that on an average, the "bump stop" position is encountered once in approximately 10^7 to 10^{11} cycles.

As mentioned above the zero level crossing rate for the passive suspension is independent of the suspension stiffness K_s and the suspension damping C_s. The permissible operating times , for a 99% reliability, for both the active and passive suspension are presented in Tables VIII and IX respectively (Appendix A). This is the permissible operating time such that $P[T_s>t] = 0.99$. During the operation of the suspension for that duration of time, we can be 99% sure that the suspension will not encounter the "bump stop" position and "fail". For the active suspension ($p=0.8(10^{-8})$), $T_{0.99}=406.14$ hr, whereas the same value for the passive suspension ($C_s=1861$ Ns/m, $K_u/K_s=7.8$) is $T_{0.99}=163.72$ hr. The active suspension not only yields better vibration isolation, but is also apparently more reliable.

CHAPTER VIII

CONCLUSIONS AND RECOMMENDATIONS

It is apparent that the active suspension can be designed to have superior vibration isolation than the conventional passive system. However the extent to which the ride quality is improved is constrained by the available suspension workspace as well as the road holding qualities desired for the vehicle.

In the design of the fully active system (Thompson, 1976), the weighting factors in the performance index are extremely important. They may be set to certain values, such that certain specific performance characteristics are enhanced in the suspension system. For example, in the design considered in this study, the weighting factors for the suspension workspace and the tire deflection are set to constants, and the weighting factor p for the body force (and hence the ride quality) is varied. The "optimum" value of p is decided on by studying the transient responses of the various resulting systems. The system resulting from $p=0.8(10^{-9})$ is said to be "optimum". If the resulting transient characteristics are studied (Table II), is may be observed that the body overshoot is significantly less for the active system. The axle overshoot is slightly higher for the active system but is damped out more rapidly. If the designer were more interested in controlling the axle overshoot, the weighting factor q_1 for the tire deflection would be of greater interest. p and q_2 could be set to constants and the factor q_1 could be varied to see which design best controls the axle deflection.

Thus there is no hard and fast rule as to how the weighting factors have to be set. Certain aspects should be considered though. The damping in the wheel hop mode should at least be ς_u=0.2. Any value below this results in extremely poor road holding. Another aspect pointed out by Thompson (1976) is that the values of the various integrals in the performance index (eq. 22) should have roughly the same value for the "optimum" design. The values of the above integrals for the design considered here, have integral values of slightly varying magnitudes, which suggests a need to refine the choice of the weighting factors.

Frequency response plots of the suspension deflection, tire deflection, and the body acceleration, show lower transmissibilities for the active suspension around the sprung-mass natural frequency. This is not the case around the unsprung-mass natural frequency. The body acceleration response plot, shows lower transmissibilities at both frequencies though. This is the reason for the relatively large improvement in the RMS value for the body acceleration (18%). The RMS values for the suspension deflection are of roughly the same magnitude for the two systems, indicating that a better ride quality is achieved for the active system for roughly the same suspension workspace. "Softening" of the passive suspension moves the tradeoff curves downwards (Fig. 12). "Soft" suspensions give a better ride at the cost of poorer road holding. The active system on the other hand can respond to many different variables at once, and thus shows superior performance characteristics over a wide variety of operating conditions.

With respect to the permissible operating times for a 99% reliability, the active suspension is again superior. The passive suspension that shows a similar operating time to that of the active system, is the heavily damped passive system that has extremely poor vibration isolation. It should be pointed out though that the Poisson crossing assumption does not give an accurate quantitative estimate for the first passage time. It is adequate however for the purposes of comparison. The Poisson assumption assumes independent crossings. This is a reasonable assumption provided the crossing level is very large ($a > 5\sigma$). This is so in our case. The Poisson crossing assumption does not take into account the fact that there is a finite probability of failure at t=0 (i.e. $T_a=0$, or $X(0) > a$). There are better assumptions such as the two state Markov crossing assumption which takes into account the above fact (Nigam, 1983). The two state Markov assumption reduces to the Poisson assumption as the crossing level becomes large (a tends to ∞). It is unlikely that the suspension will fail when the "bump stop" position is encountered just once. A cumulative kind of model to determine the relative reliabilities of the two systems makes more sense. This would consider the amount of time spent above the critical level. Once the total amount of time exceeds some prespecified maximum time, the suspension is said to have "failed".

Full state feedback laws are attractive in terms of analysis, but are difficult to realize physically. Active suspension systems designed assuming limited feedback information are more realistic. The active system is more complicated in design and has a sizable energy requirement. Ways to reduce this energy requirement, such as the use of dynamic vibration absorbers along with a simplified feedback control law may be an attractive alternative. Another approach would be the semi-active suspension system. This in turn means working into the analysis, system non-linearities. The performance gains of semi-active systems have yet to be studied fully. If the application is not extremely specialized, semi-active systems may be employed, thus yielding a lot of the performance gains of fully active systems, and requiring little or no energy.

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APPENDIX A

TABLES

TABLE I

KALMAN FEEDBACK GAINS

p	k_1	k_2	k_3	k_4
$0.2(10^{-9})$	161240	-70711	2305.9	-8037.2
$0.4(10^{-9})$	97692	-50000	1814.4	-6142.9
$0.6(10^{-9})$	71725	-40825	1555.0	-5316.9
$0.8(10^{-9})$	57240	-35355	1385.7	-4827.0
$1.0(10^{-9})$	47898	-31623	1263.4	-4492.5
$1.2(10^{-9})$	41333	-28868	1169.4	-4244.6
$1.5(10^{-9})$	34441	-25820	1061.6	-3968.2
$2.0(10^{-9})$	27150	-22361	934.5	-3649.5

TABLE II

COMPARISON OF TRANSIENT CHARACTERISTICS

Transient Characteristic	Active	Passive
Body Overshoot	8.5%	45.3%
Axle Overshoot	28.0%	18.0%
Peak Body Acceleration	252.7 m/s^2	283.8 m/s^2

TABLE III

RMS VALUES FOR THE
ACTIVE SUSPENSION

p	$\frac{\sigma_{y1}}{(cV)^{1/2}}$	$\frac{\sigma_{y2}}{(cV)^{1/2}}$	$\frac{\sigma_{\ddot{x}_2}}{(cV)^{1/2}}$
$0.2(10^{-9})$	0.2475	0.1062	50.77
$0.4(10^{-9})$	0.2575	0.1102	44.68
$0.6(10^{-9})$	0.2668	0.1131	41.46
$0.8(10^{-9})$	0.2752	0.1157	39.24
$1.0(10^{-9})$	0.2827	0.1181	37.53
$1.2(10^{-9})$	0.2895	0.1204	36.15
$1.5(10^{-9})$	0.2986	0.1236	34.47
$2.0(10^{-9})$	0.3116	0.1286	32.35

TABLE IV

RMS VALUES FOR THE PASSIVE SUSPENSION

C_s (Ns/m)	$\frac{\sigma_{y1}}{(cV)^{1/2}}$	$\frac{\sigma_{y2}}{(cV)^{1/2}}$	$\frac{\sigma_{\ddot{y}_2}}{(cV)^{1/2}}$
4000	0.1992	0.1373	62.65
3200	0.2227	0.1288	56.79
2800	0.2381	0.1250	53.87
2600	0.2471	0.1234	52.15
2400	0.2572	0.1220	50.57
2200	0.2686	0.1209	48.98
2000	0.2817	0.1202	47.40
1861	0.2921	0.1201	46.32
1600	0.3150	0.1207	44.36
1400	0.3367	0.1225	43.00
1200	0.3637	0.1258	41.86
1000	0.3984	0.1315	41.13
844.3	0.4347	0.1386	41.07

TABLE V

COMPARISON OF RMS VALUES

	$\frac{\sigma_{v1}}{(cV)^{1/2}}$	$\frac{\sigma_{v2}}{(cV)^{1/2}}$	$\frac{\sigma_{\ddot{y}_2}}{(cV)^{1/2}}$
Active	0.2752	0.1157	39.24
Passive	0.2921	0.1201	46.32

TABLE VI

LEVEL CROSSINGS FOR THE ACTIVE SUSPENSION

ρ	ν_0^+ (Hz)	$\nu_{0.063}^+$ (Hz)
$0.2(10^{-9})$	4.326	$2.380(10^{-11})$
$0.4(10^{-9})$	4.326	$1.710(10^{-10})$
$0.6(10^{-9})$	4.351	$8.972(10^{-10})$
$0.8(10^{-9})$	4.381	$3.437(10^{-9})$
$1.0(10^{-9})$	4.409	$1.040(10^{-8})$
$1.2(10^{-9})$	4.440	$2.582(10^{-8})$
$1.5(10^{-9})$	4.478	$8.363(10^{-8})$
$2.0(10^{-9})$	4.536	$3.628(10^{-7})$

TABLE VII

LEVEL CROSSINGS FOR THE PASSIVE SUSPENSION

C_s (Ns/m)	V_0^+ (Hz)	$V_{0.063}^+$ (Hz)
4000	3.527	$1.488(10^{-17})$
3200	3.527	$4.467(10^{-14})$
2800	3.527	$2.427(10^{-12})$
2600	3.527	$1.796(10^{-11})$
2400	3.527	$1.330(10^{-10})$
2200	3.527	$9.811(10^{-10})$
2000	3.527	$7.265(10^{-9})$
1861	3.527	$2.912(10^{-8})$
1600	3.527	$4.004(10^{-7})$
1400	3.527	$2.961(10^{-6})$
1200	3.527	$2.168(10^{-5})$
1000	3.527	$1.608(10^{-4})$
844.3	3.527	$7.587(10^{-4})$

TABLE VIII

PERMISSIBLE OPERATING TIMES FOR THE ACTIVE SUSPENSION

p	$T_{0.99}$ (hr)
$0.2(10^{-9})$	58650.42
$0.4(10^{-9})$	8163.04
$0.6(10^{-9})$	1555.42
$0.8(10^{-9})$	406.14
$1.0(10^{-9})$	134.22
$1.2(10^{-9})$	54.06
$1.5(10^{-9})$	16.69
$2.0(10^{-9})$	3.85

TABLE IX

PERMISSIBLE OPERATING TIMES
FOR THE PASSIVE SUSPENSION

C_s	$T_{0.99}$ (hr)
4000	$9.380(10^{10})$
3200	$3.125(10^{7})$
2800	$5.750(10^{5})$
2600	77721.60
2400	10495.34
2200	1422.77
2000	192.14
1861	163.72
1600	3.49
1400	$4.714(10^{-1})$
1200	$6.440(10^{-2})$
1000	$8.680(10^{-3})$
844.3	$1.840(10^{-3})$

APPENDIX B

FIGURES

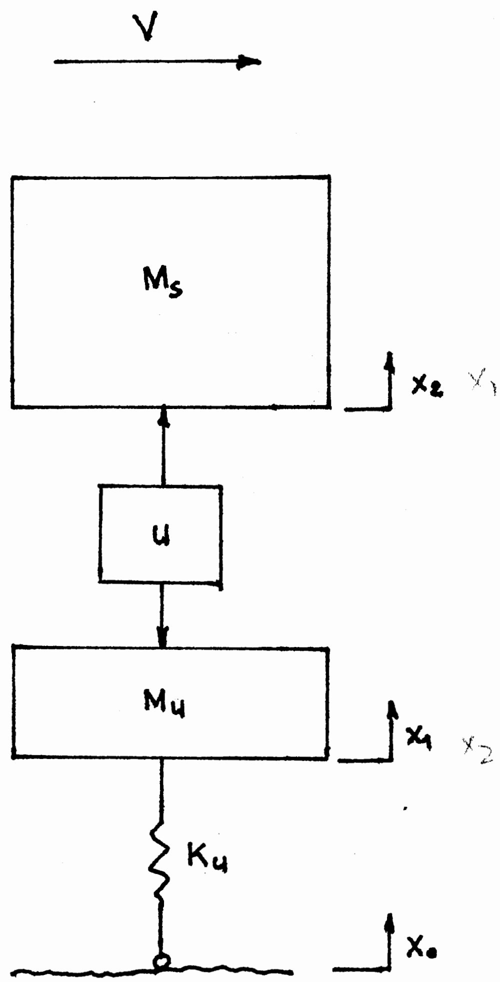

Figure 1. Active Suspension Model

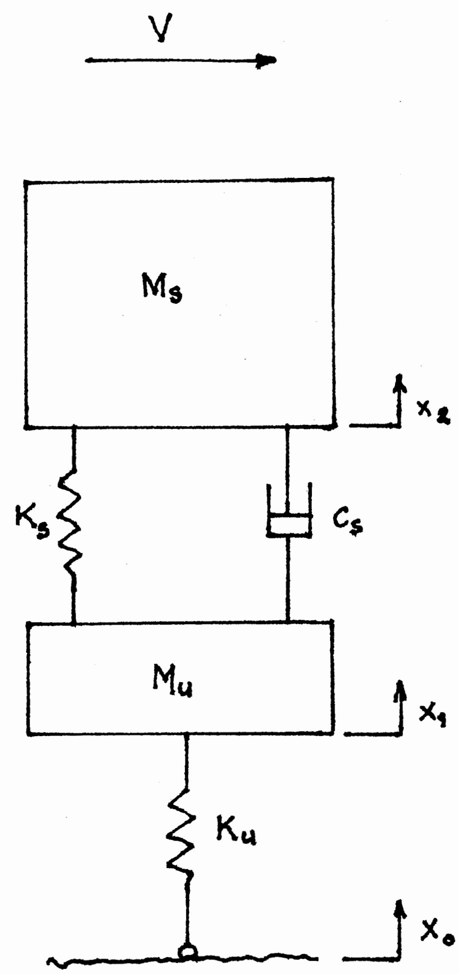

Figure 2. Passive Suspension Model

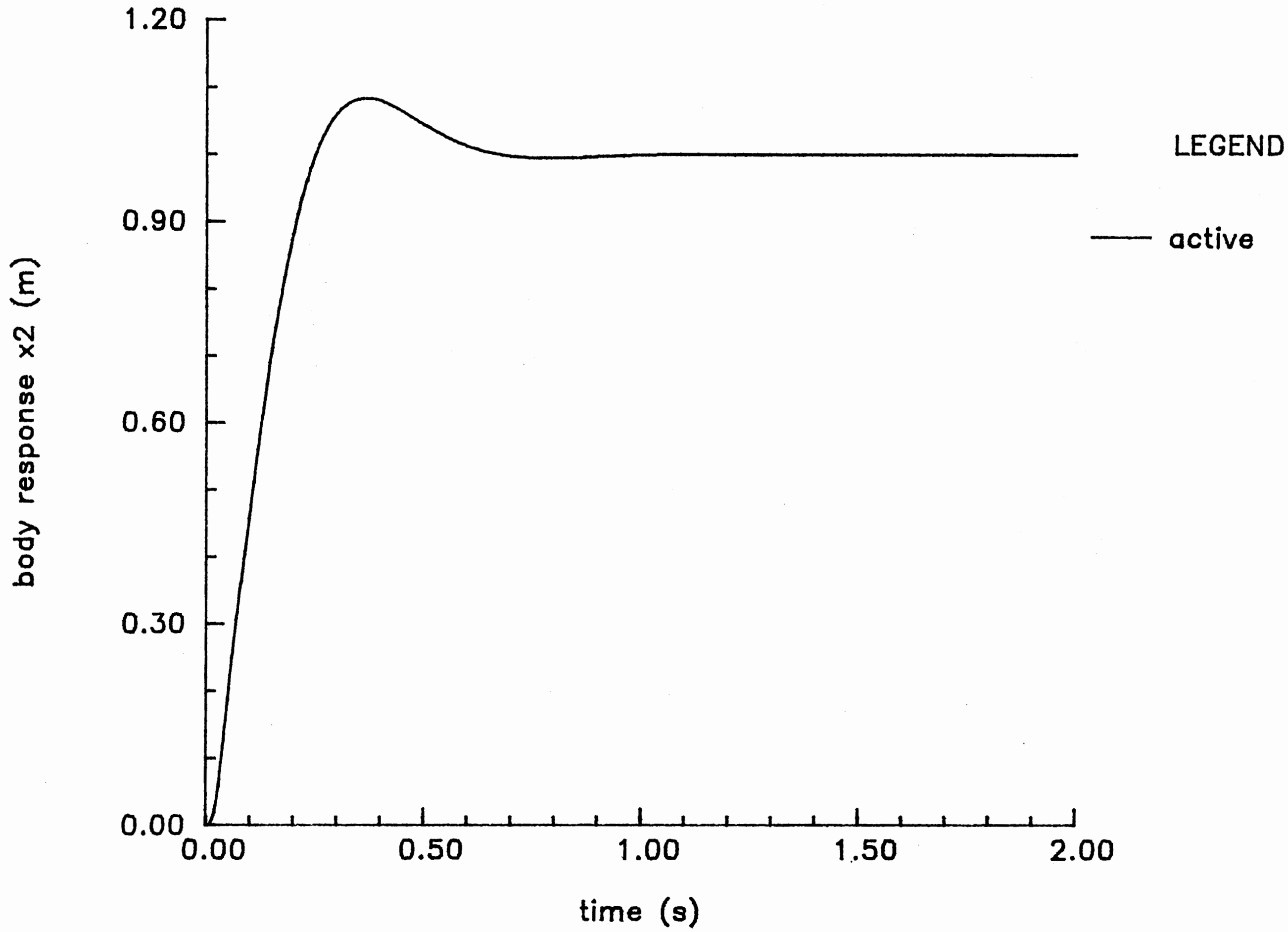

Figure 3. Transient Body Response for the Active Suspension

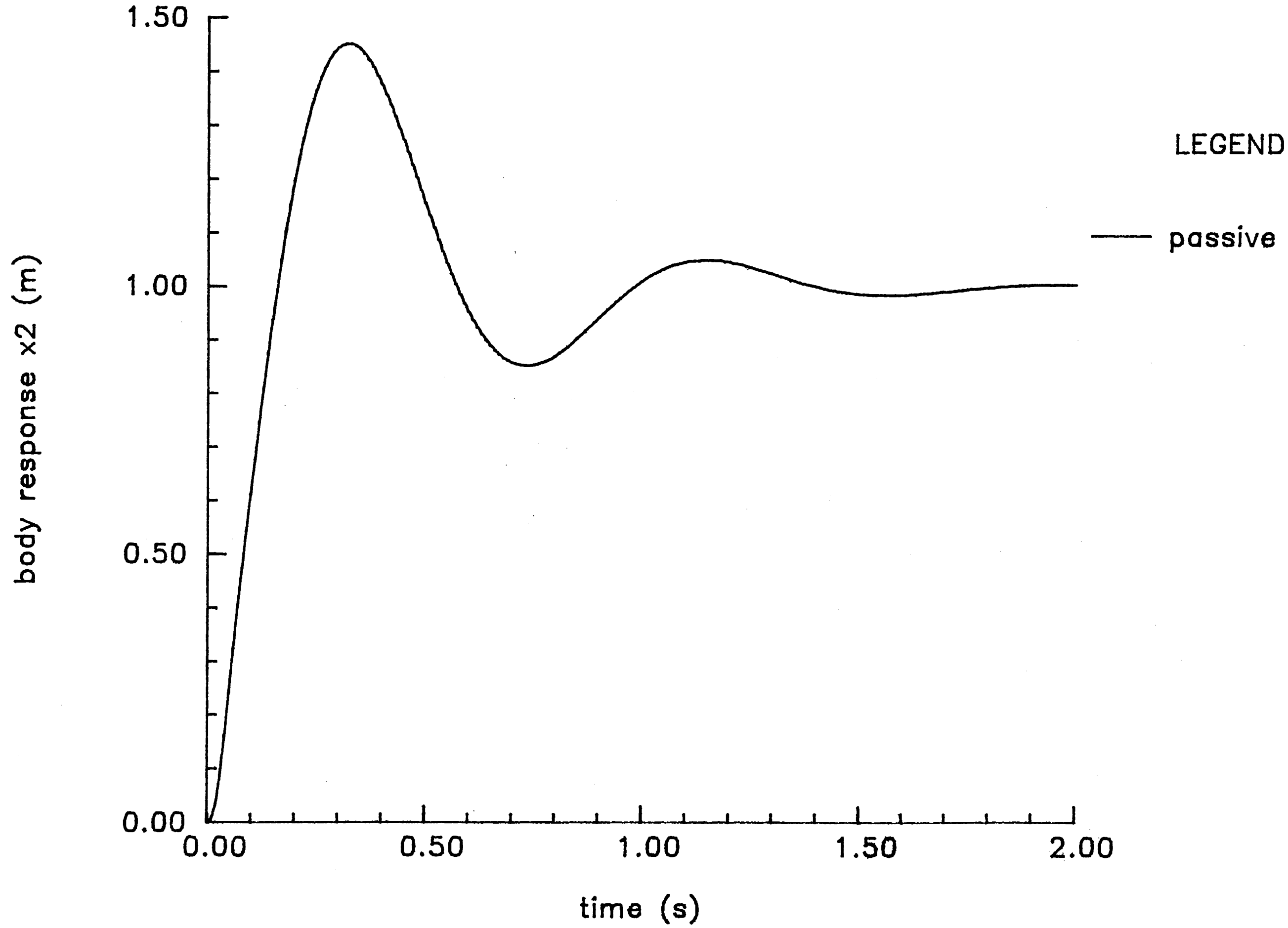

Figure 4. Transient Body Response for the Passive Suspension

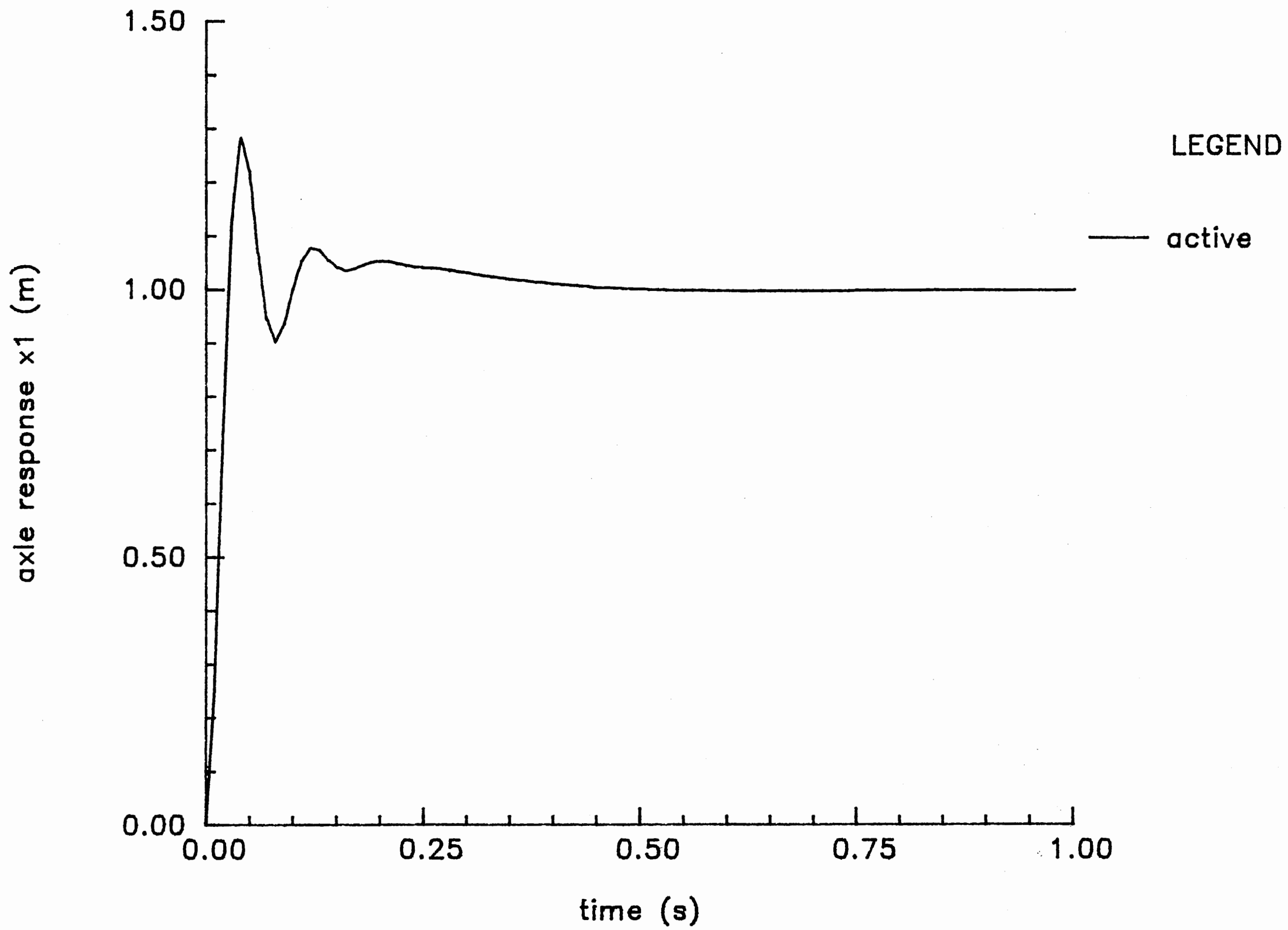

Figure 5. Transient Axle Response for the Active Suspension

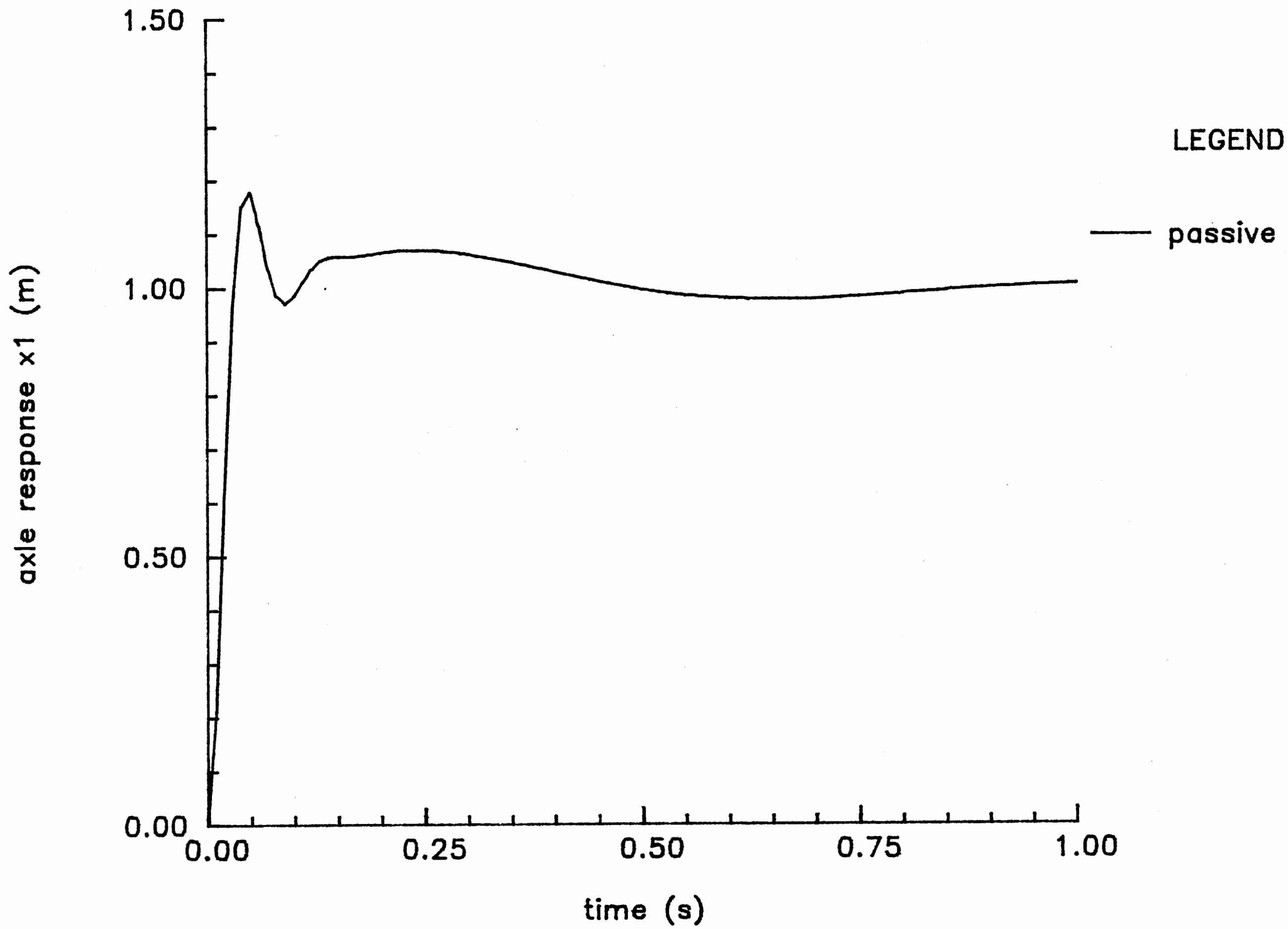

Figure 6. Transient Axle Response for the Passive Suspension

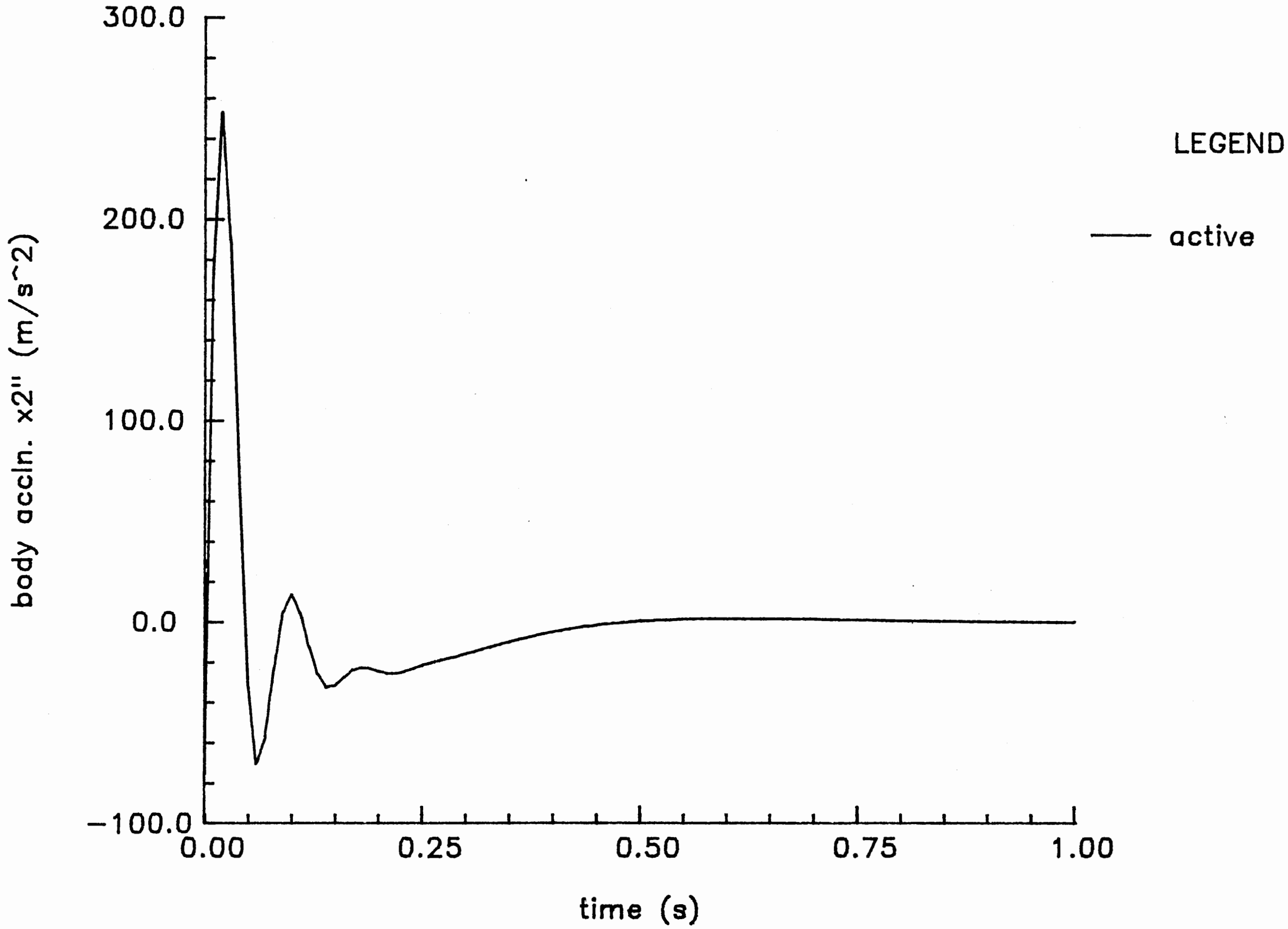

Figure 7. Transient Body Acceleration Response for the Active Suspension

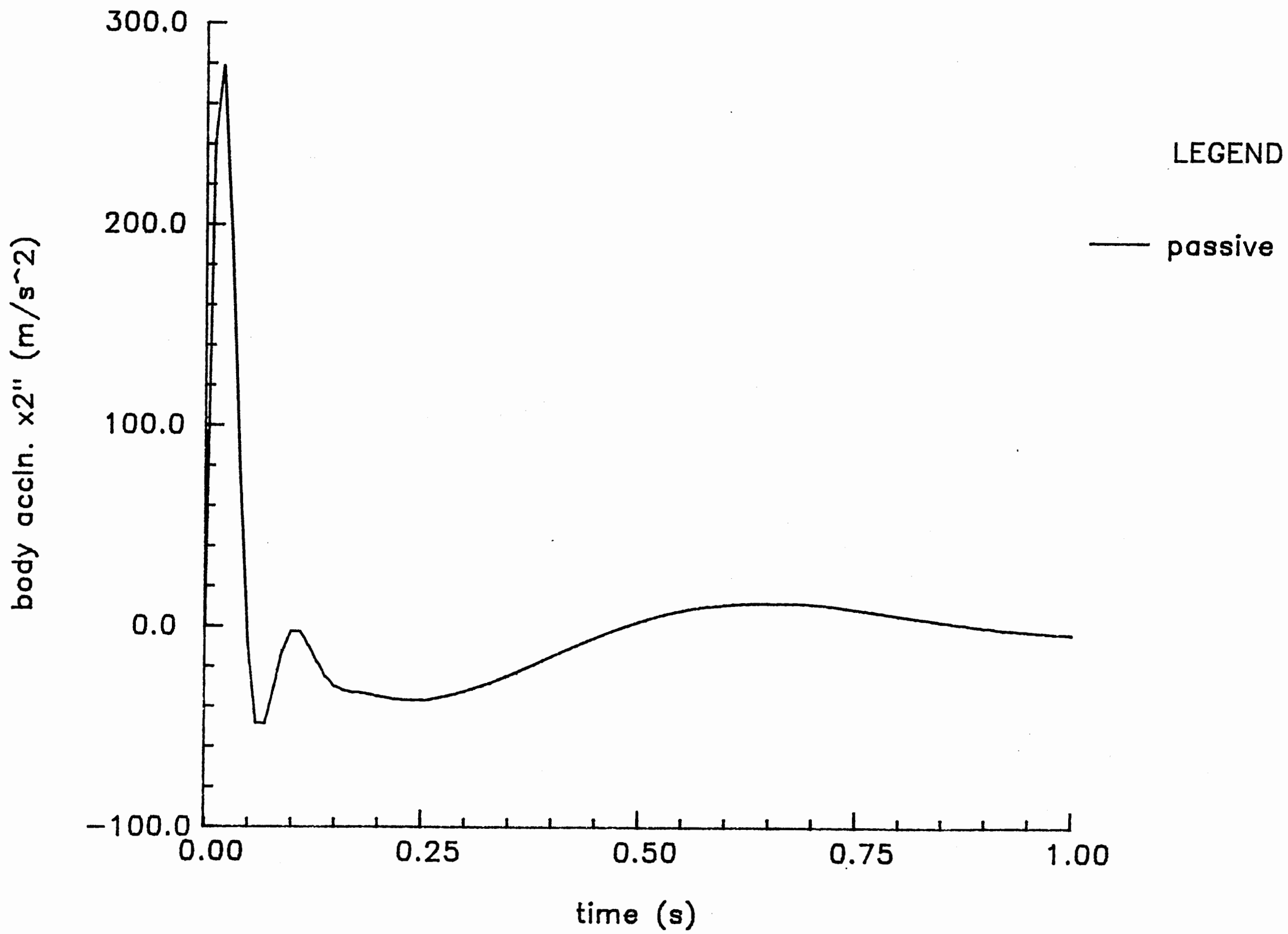

Figure 8. Transient Body Acceleration Response for the Passive Suspension

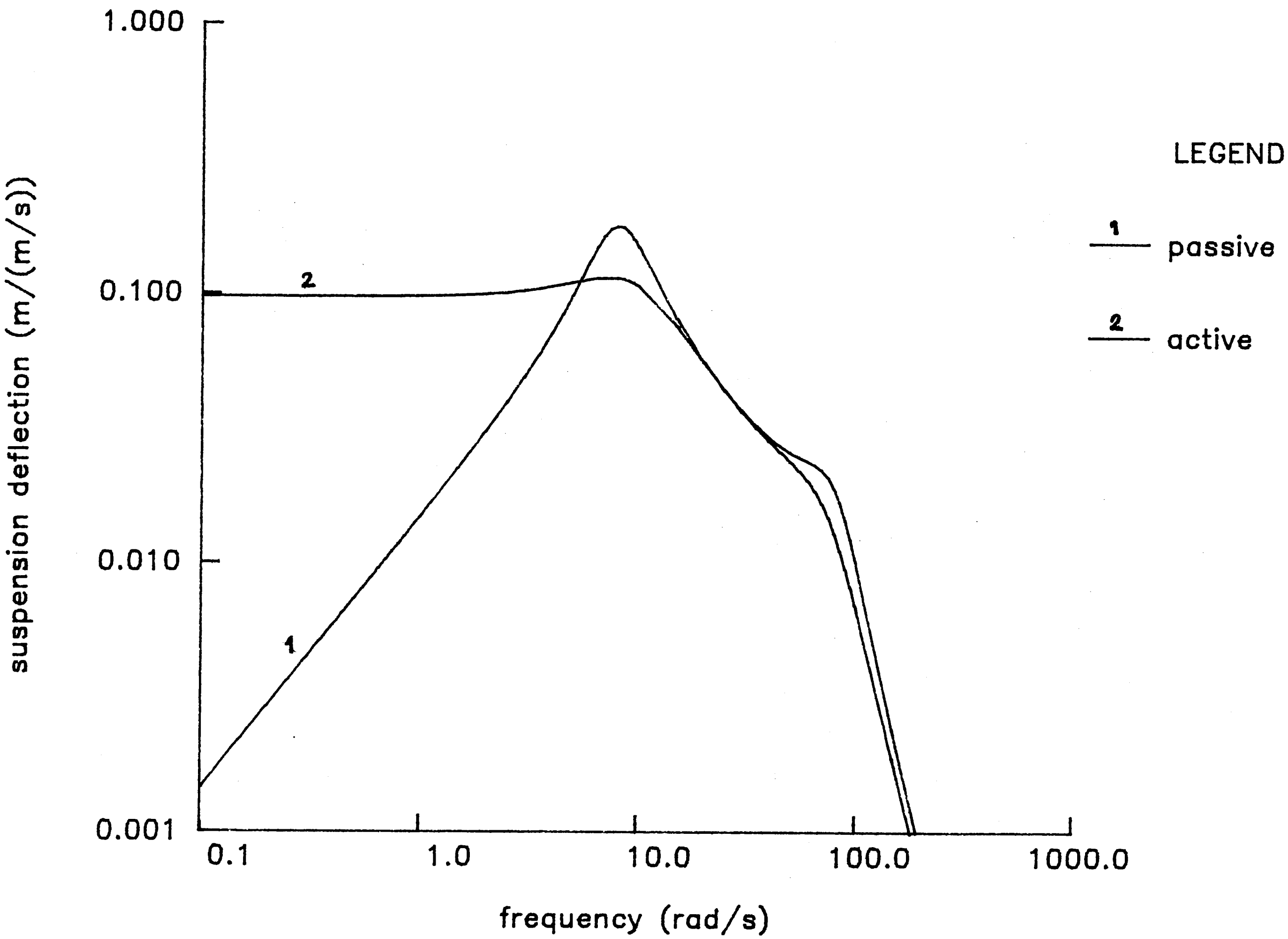

Figure 9. Frequency Response for Suspension Deflection

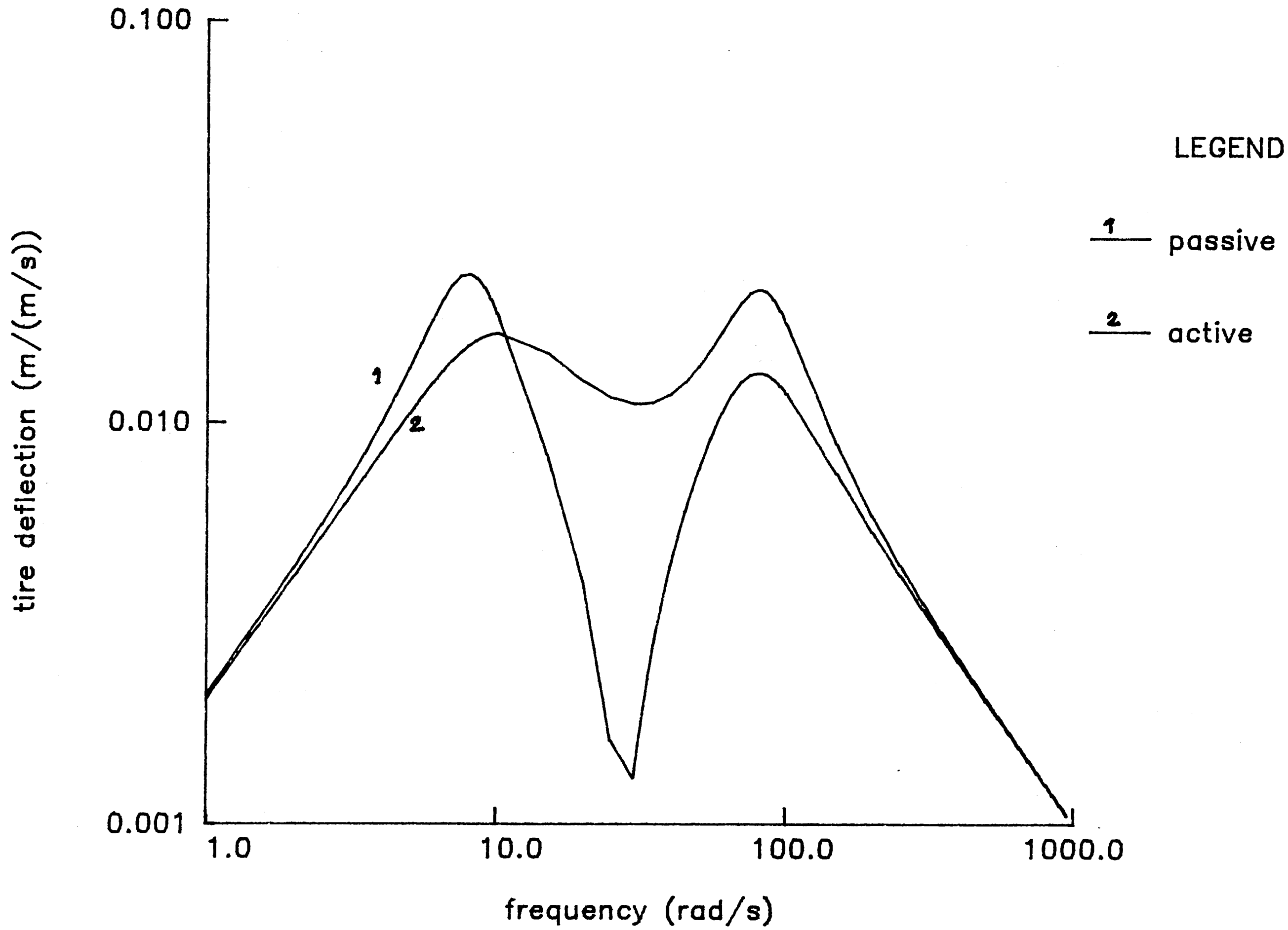

Figure 10. Frequency Response for Tire Deflection

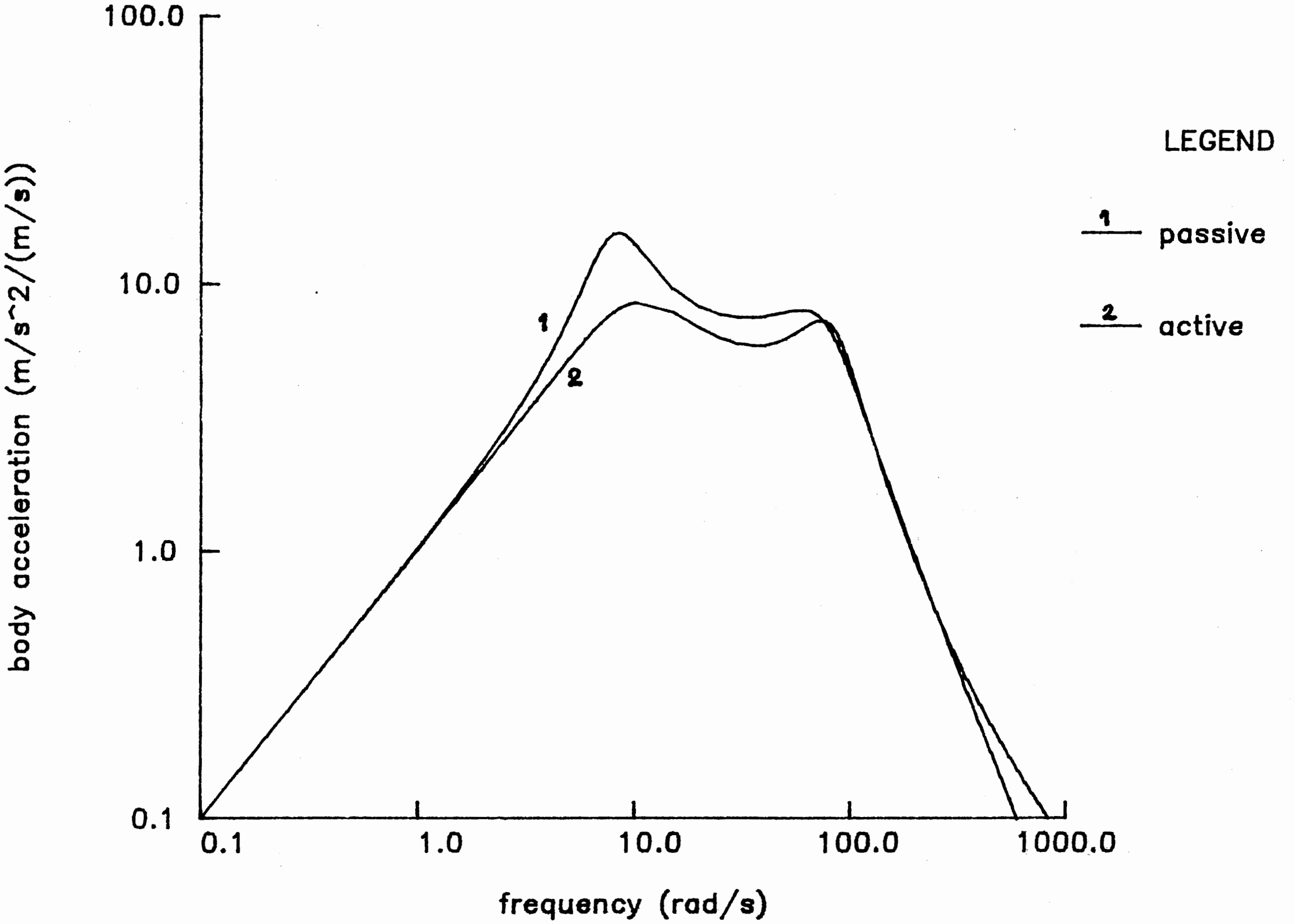

Figure 11. Frequency Response for Body Acceleration

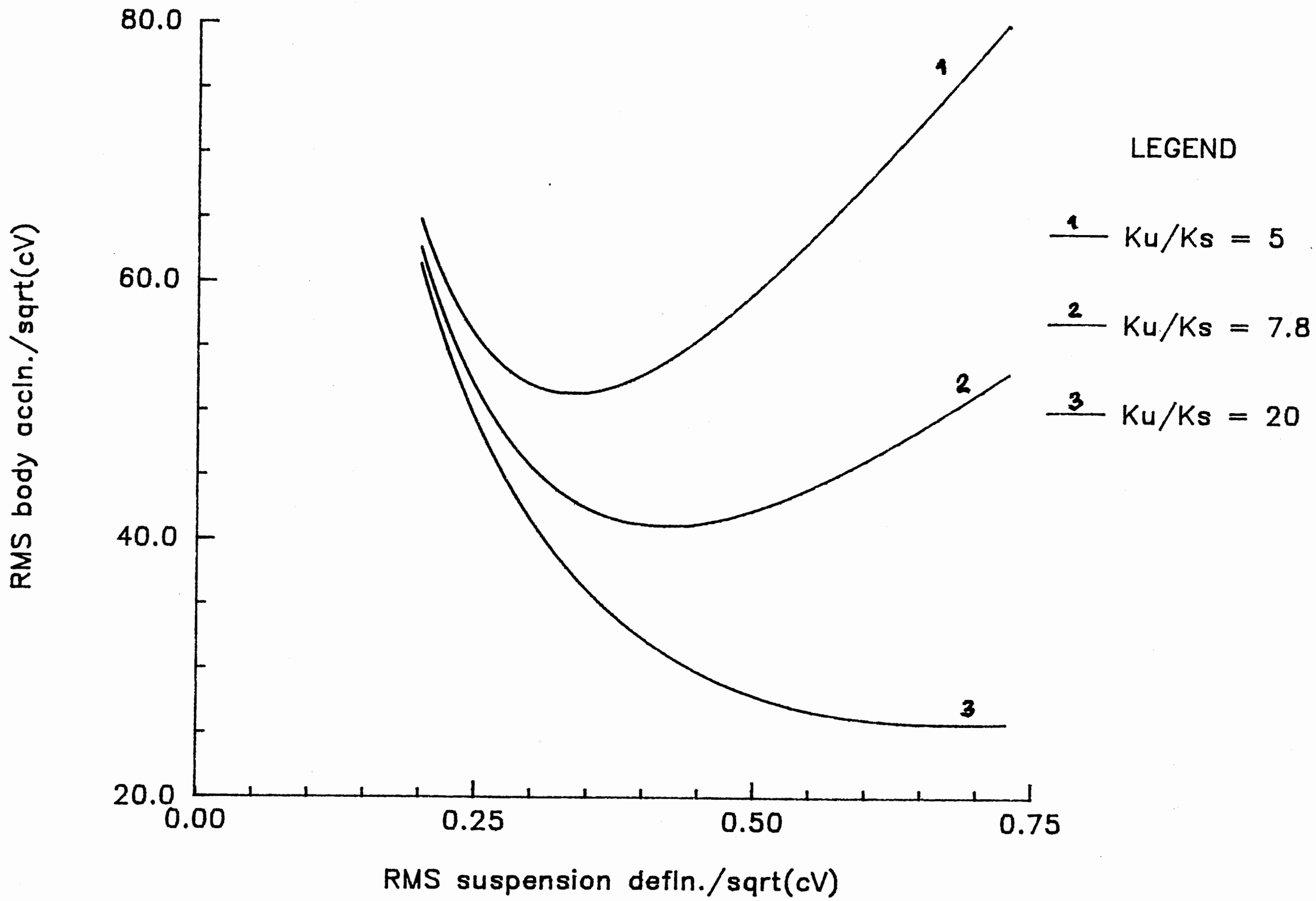

Figure 12. Trade-off between the RMS Values for the Passive Suspension

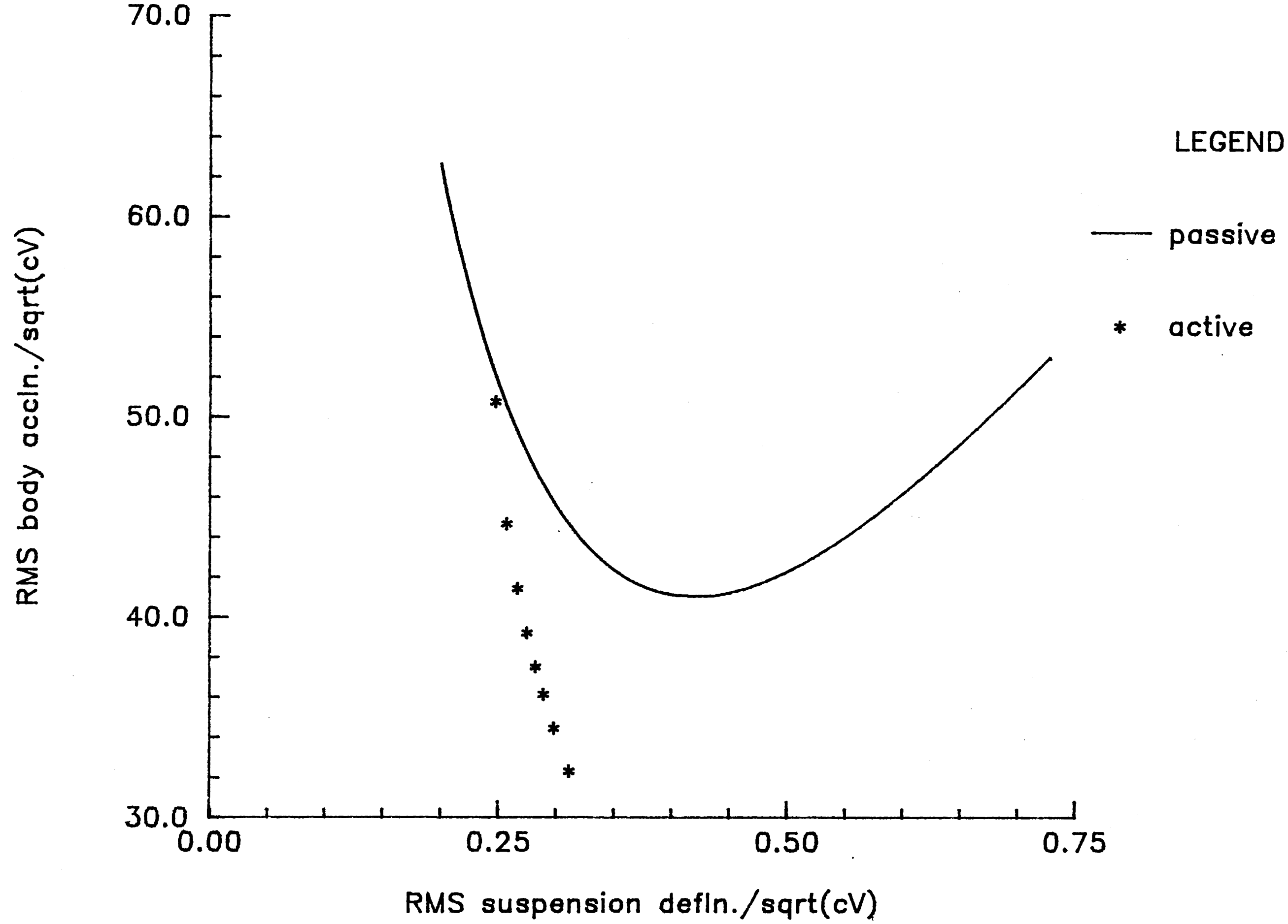

Figure 13. Trade-off between the RMS Values: A Comparison

APPENDIX C

PARASOL PROGRAMS

APPENDIX C

PARASOL PROGRAMS

Four Parasol programs are presented overleaf. Two of them simply simulate the active and passive systems. These programs are pretty straightforward. However, some explanation is in order for the other two programs which evaluate the Ricatti matrix and the Kalman feedback gains.

In our case the Ricatti matrix is of dimension (4 x 4, [P]). We essentially use Parasol to simulate the differential Ricatti equation backwards in time. Since we are considering the infinite time case, the Ricatti matrix is time-invariant (Chapter IV). The simulation starts at time t_f, the final time, and proceeds on to time t_0, the initial time. The "initial" condition is, $[P(t_f)] = [H] = [0]$.

To simulate backwards in time using Parasol, a negative time step has to be established before the simulation is run. Within the Parasol program, the time step is set to a positive value. Just before running the simulation, say with the time interval -step, the statement, -step @.dtp, is typed in. This sets the time step to the required negative value. The simulation is then run as usual, again with a negative time step specified as follows: **-step final_time run**.

The values of [P] (which is symmetric) vary for some time, and then attain constant values as expected. The **final_time** specified has to be a positive value, else an error results. Once [P] attains constant values, the simulation is halted using a break sequence, and the values of the Ricatti matrix noted. A simple matrix multiplication yields the values of the Kalman feedback gains.

```
$dfsb sys
;DIFFERENTIAL RICATTI EQUATION
  #P = P b1 m* R $minv m* b1 $mtr m* P m*
         P A m* m-  A $mtr P m* m-  Q m- $$
endsb
;end of simulation block
;
;initial conditions
  4 4 $mkmz &P $ic
;
;DEFINE MATRICES
;make A matrix
  0        0    1    0
  0        0    0    1
-5454.86   0    0    0
  0        0    0    0
4 4 $mkm @A
;make Q matrix
  11    -1    0    0
  -1     1    0    0
   0     0    0    0
   0     0    0    0
4 4 $mkm @Q
;make b1 matrix
  0    0    -0.035  0.0035
4 1 $mkm @b1
;make R matrix
0.0000000008 1 1 $mkm @R
;
;output
  0.01 @.dtp
  1 8 5 $frmt
  $dffn print .t P 1 1 $mgd  P 2 1 $mgd P 2 2 $mgd
    P 3 1 $mgd P 3 2 $mgd P 3 3 $mgd $$
  ;$dffn print .t P 4 1 $mgd P 4 2 $mgd P 4 3 $mgd
    ;P 4 4 $mgd $$
```

```
;MULTIPLY MATRICES TO GET KALMAN FEEDBACK GAINS
;unit matrix with -1.0
-1.0 1 1 $mkm @mu
;
;R matrix
0.0000000008 1 1 $mkm @R
;
;b1 matrix
0    0    -0.035   0.0035  4 1 $mkm @b1
;
;P matrix (solution of differential Ricatti Equation)
 0.25431    -0.04178    0.00087   -0.00682
-0.03917     0.13580    0.00004    0.01317
 0.00077     0.00011    0.00003    0.00003
-0.00543     0.00922    0.00003    0.00143
4 4 $mkm @P
;
;Matrix Multiplication
mu R $minv m* b1 $mtr m* P m* @K
;
1 10 1 $frmt
K 1 1 $mgd $ptop
K 1 2 $mgd $ptop
K 1 3 $mgd $ptop
K 1 4 $mgd $ptop
```

```
;DIFFERENTIAL EQUATION
;simulation ACTIVE SUSPENSION
$dfsb sys
#x = A x m* b1 u m* m+ $$
u = k $mtr x m* $$
endsb
;
;make A matrix
0        0  1  0
0        0  0  1
-5454.86 0  0  0
0        0  0  0
4 4 $mkm @A
;
;make b1 matrix
0 0 -0.035 0.0035
4 1 $mkm @b1
;
;Kalman feedback gains
57240 -35355 1385.7 -4827.0
4 1 $mkm @k
;
;initial conditions
-1 -1 0 0
4 1 $mkm &x $ic
;
;output
$dffn print .t x 1 1 $mgd x 2 1
  $mgd x 3 1 $mgd x 4 1 $mgd $$
1 7 3 $frmt
```

```
;DIFFERENTIAL EQUATION
;simulation PASSIVE SUSPENSION
$dfsb sys
#x = A x m* b1 u m* m+ $$
u = k $mtr x m* $$
endsb
;
;make A matrix
0         0  1  0
0         0  0  1
-5454.86  0  0  0
0         0  0  0
4 4 $mkm @A
;
;make b1 matrix
0 0 -0.035 0.0035
4 1 $mkm @b1
;
;k matrix
19960.0 -19960.0 1861.0 -1861.0
4 1 $mkm @k
;
;initial conditions
-1 -1 0 0
4 1 $mkm &x $ic
;
;output
$dffn print .t x 1 1 $mgd x 2 1
  $mgd x 3 1 $mgd x 4 1 $mgd $$
1 7 3 $frmt
```

APPENDIX D

EIGEN-VALUE PROGRAMS

APPENDIX D

EIGEN-VALUE PROGRAMS

The FORTRAN programs presented overleaf evaluate the eigen-values and the modal vectors of a general damped, second order, multi-degree of freedom system (Tse, Morse, and Hinkle, 1978). The driver programs and the subroutines are set up in such a way, that they can handle matrices of a maximum dimension of (10 x 10).

The system is specified by the mass [M], damping [C], and the stiffness [K] matrices. The method used is to reduce the system of (n x n) second order equations, to an equivalent system of (2n x 2n) first order equations (Chapter V). As a check, it is verified that the modal vectors do indeed diagonalize the system of equations. The only subroutine called is cmodl(). This in turn requires a host of other subroutines to perform specific tasks such as evaluating the coefficients of the characteristic equation, finding it's roots, and solving the set of equations to evaluate the modes. The eigen-values and the modal vectors are complex in general. It should be noted that the modal vectors, being complex, include phase information in them.

Two sample driver programs (active and passive systems) are included. Data to be specified (in data statements at the top of each driver program) include the above-mentioned matrices, the number of second order equations, the number of iterations, and the accuracy desired. Sample output is included along with each driver program.

```
*** SUBROUTINES TO PERFORM REAL/COMPLEX ***
*** MATRIX MANIPULATIONS ***
*
*
*** COMPLEX MATRIX SUBSTITUTION:  GETS B = A ***
*
       subroutine csubn(a,b,n)
*
       complex*16 a(10,10),b(10,10)
*
       do i=1,n
         do j=1,n
           b(i,j)=a(i,j)
         enddo
       enddo
*
       return
       end
*
*
*** REAL MATRIX SUBSTITUTION:  GETS B = A ***
*
       subroutine subn(a,b,n)
*
       real*8 a(10,10),b(10,10)
*
       do i=1,n
         do j=1,n
           b(i,j)=a(i,j)
         enddo
       enddo
*
       return
       end
*
*
*** COMPLEX MATRIX MULTIPLICATION: ***
*** CALCULATES C = A*B ***
*
       subroutine cmply(a,b,c,n)
*
       complex*16 a(10,10),b(10,10),c(10,10)
*
       do i=1,n
         do j=1,n
           c(i,j)=(0.0,0.0)
           do k=1,n
             c(i,j)=c(i,j)+a(i,k)*b(k,j)
           enddo
         enddo
       enddo
*
       return
```

```
      end
*
*
*** REAL MATRIX MULTIPLICATION: ***
*** CALCULATES C = A*B ***
*
      subroutine mply(a,b,c,n)
*
      real*8 a(10,10),b(10,10),c(10,10)
*
      do i=1,n
        do j=1,n
          c(i,j)=0.0
          do k=1,n
            c(i,j)=c(i,j)+a(i,k)*b(k,j)
          enddo
        enddo
      enddo
*
      return
      end
*
*
*** COMPLEX MATRIX INVERSION: ***
*** FINDS INVERSE (hinvs) OF h ***
*   METHOD:  FADDEEV-LEVERRIER
*   SUBROUTINES REQD:  (1) cmply   (2) csubn
*
      subroutine cinvs(h,hinvs,n)
*
      complex*16 h(10,10),hinvs(10,10)
      complex*16 a(10,10),b(10,10),sum
*
      call csubn(h,a,n)
      nm1=n-1
      do i=1,nm1
        sum=(0.0,0.0)
        do k=1,n
          sum=sum+a(k,k)
        enddo
        sum=sum/i
        do j=1,n
          a(j,j)=a(j,j)-sum
        enddo
        if (i .eq. nm1) then
          call csubn(a,hinvs,n)
        endif
        call cmply(h,a,b,n)
        call csubn(b,a,n)
      enddo
      do i=1,n
        do j=1,n
          hinvs(i,j)=hinvs(i,j)/a(1,1)
        enddo
```

```
      enddo
*
      return
      end
*
*
*** REAL MATRIX INVERSION: ***
*** FINDS INVERSE (hinvs) OF h ***
*   METHOD:  FADDEEV-LEVERRIER
*   SUBROUTINES REQD:  (1) mply   (2) subn
*
      subroutine invs(h,hinvs,n)
*
      real*8 h(10,10),hinvs(10,10)
      real*8 a(10,10),b(10,10),sum
*
      call subn(h,a,n)
      nm1=n-1
      do i=1,nm1
        sum=0.0
        do k=1,n
          sum=sum+a(k,k)
        enddo
        sum=sum/i
        do j=1,n
          a(j,j)=a(j,j)-sum
        enddo
        if (i .eq. nm1) then
          call subn(a,hinvs,n)
        endif
        call mply(h,a,b,n)
        call subn(b,a,n)
      enddo
      do i=1,n
        do j=1,n
          hinvs(i,j)=hinvs(i,j)/a(1,1)
        enddo
      enddo
*
      return
      end
*
*
*** TO FIND CEFFICIENTS OF ***
*** THE CHARACTERISTIC EQUATION ***
*   GIVEN REAL MATRIX-h, SUBROUTINE
*   YIELDS COEFFICIENTS-C's
*   METHOD:  FADDEEV-LEVERRIER
*   SUBROUTINES REQD:  (1) subn  (2) mply
*
      subroutine coeff(h,c,n)
*
      real*8 a(10,10),b(10,10),c(12),h(10,10),sum
*
```

```
      call subn(h,a,n)
      np2=n+2
      c(1)=0.0
      c(np2)=1.0
      do i=1,n
        sum=0.0
        do k=1,n
          sum=sum+a(k,k)
        enddo
        c(np2-i)=-sum/i
        do j=1,n
          a(j,j)=a(j,j)+c(np2-i)
        enddo
        call mply(h,a,b,n)
        call subn(b,a,n)
      enddo
      do i=2,np2
        c(n+4-i)=c(n+4-i)/c(2)
      enddo
*
      return
      end
*
*
*** TO FIND THE COMPLEX ROOTS OF ***
*** THE CHARACTERISTIC EQUATION OF ***
*   POSITIVE DEFINITE SYSTEMS WITH VISCOUS DAMPING
*   ROOTS DISTINCT
*   METHOD:  ITERATIVE
*   GIVEN COEFFICIENT MATRIX-h,
*   SUBROUTINE YIELDS COMPLEX ROOTS
*   SUBROUTINES REQD:  (1) coeff  (2) mply  (3) subn
*
      subroutine croot(h,root,error,niter,n)
*
      real*8 a(2,5),a1,a2,b(12),c(12),da1,da2
      real*8 dbda1(12),dbda2(12),det
      real*8 error,h(10,10),p,q,z
      complex*16 root(2,5)
      data b,dbda1,dbda2,z/36*0.0,0.0/
*
      b(2)=1.0
      call coeff(h,c,n)
      i=n+2
10    l=i/2-1
      a1=0
      a2=0
      do it=1,niter
        do j=3,i
          b(j)=c(j)-a1*b(j-1)-a2*b(j-2)
          dbda1(j)=-a1*dbda1(j-1)-a2*dbda1(j-2)-b(j-1)
          dbda2(j)=-a1*dbda2(j-1)-a2*dbda2(j-2)-b(j-2)
        enddo
        det=dbda1(i-1)*dbda2(i)-dbda1(i)*dbda2(i-1)
```

```
          da1=(-b(i-1)*dbda2(i)+b(i)*dbda2(i-1))/det
          da2=(+b(i-1)*dbda1(i)-b(i)*dbda1(i-1))/det
          a1=a1+da1
          a2=a2+da2
          if ((dabs(da1)-error) .ge. 0.0) then
            goto 40
          endif
          if ((dabs(da2)-error) .ge. 0.0) then
            goto 40
          endif
          do i1=4,i
            c(i1-2)=b(i1-2)
          enddo
          a(1,1)=a1
          a(2,1)=a2
          if (i .eq. 4) then
            goto 11
          endif
          i=i-2
          goto 10
40      enddo
11      nd2=n/2
        do j=1,nd2
          p=a(1,j)**2.0-4.0*a(2,j)
          q=dsqrt(dabs(p))
          if (p .ge. 0.0) then
            goto 12
          endif
          root(1,j)=dcmplx(-a(1,j),q)
          root(2,j)=dcmplx(-a(1,j),-q)
          goto 13
12        root(1,j)=dcmplx(-a(1,j)-q,z)
          root(2,j)=dcmplx(-a(1,j)+q,z)
13        root(1,j)=root(1,j)/(2*a(2,j))
          root(2,j)=root(2,j)/(2*a(2,j))
        enddo
*
        return
        end
*
*
*** SOLUTION OF COMPLEX ALGEBRAIC ***
*** HOMOGENEOUS EQUATIONS ***
*   SUBROUTINES REQD:  (1) cinvs
*                      (2) cmply  (3) csubn
*
        subroutine chomo(a,x,n)
*
        complex*16 a(10,10),b(10,10)
        complex*16 binvs(10,10),x(10),y(10)
        real*8 u,z
        data u,z/1.0,0.0/
*
        x(n)=dcmplx(u,z)
```

```
      nm1=n-1
      do i=1,nm1
        y(i)=-a(i,n)
        do j=1,nm1
          b(i,j)=a(i,j)
        enddo
      enddo
      if (nm1 .eq. 1) then
        binvs(1,1)=1.0/b(1,1)
        goto 10
      endif
      call cinvs(b,binvs,nm1)
10    do i=1,nm1
        x(i)=dcmplx(z,z)
        do j=1,nm1
          x(i)=x(i)+binvs(i,j)*y(j)
        enddo
      enddo
*
      return
      end
*
*
*** CALCULATION OF THE MODAL MATRIX ***
*   POSITIVE DEFINITE SYSTEMS WITH VISCOUS DAMPING
*   SUBROUTINES REQD:  (1) coeff  (2) chomo  (3) croot
*                      (4) cinvs  (5) cmply  (6) csubn
*                      (7) invs   (8) mply   (9) subn
*
      subroutine cmodl(m,c,k,h,u,root,error,niter,n)
*
      real*8 c(10,10),error,h(10,10),h1(10,10),h2(10,10)
      real*8 minvs(10,10),unit(10,10)
      real*8 zero(10,10),k(10,10),m(10,10)
      complex*16 dum(10,10),root(2,5),u(10,10),x(10)
      data unit,zero/200*0.0/
*
      nt2=n*2
      do i=1,n
        unit(i,i)=1.0
      enddo
      call invs(m,minvs,n)
      call mply(minvs,c,h1,n)
      call mply(minvs,k,h2,n)
      do i=1,n
        do j=1,n
          h(i,j)=-h1(i,j)
          h(i,n+j)=-h2(i,j)
          h(n+i,j)=unit(i,j)
          h(n+i,n+j)=zero(i,j)
        enddo
      enddo
      call croot(h,root,error,niter,nt2)
      do jj=1,n
```

```
          do ii=1,2
            do i=1,n
              do j=1,n
                dum(i,j)=m(i,j)*root(ii,jj)**2
     +            +c(i,j)*root(ii,jj)+k(i,j)
              enddo
            enddo
            call chomo(dum,x,n)
            ji=2*(jj-1)+ii
            do l=1,n
              u(n+l,ji)=x(l)
              u(l,ji)=u(n+l,ji)*root(ii,jj)
            enddo
          enddo
        enddo
*
        return
        end
```

```
* DRIVER PROGRAM TO DETERMINE EIGEN-VALUES
* AND MODES OF A DAMPED SYSTEM.
* THE [M], [C], [K] MATRICES NEED NOT BE
* SYMMETRIC.
* THE DAMPING NEED NOT BE PROPORTIONAL.
* THE [M] [C] [K] MATRICES HAVE TO SPECIFIED
* IN THE DATA STATEMENTS
* AT THE TOP OF THE PROGRAM.
* SUBROUTINES REQD: (1) cmodl  (2) coeff  (3) chomo
*                   (4) croot  (5) cinvs  (6) invs
*                   (7) cmply  (8) mply   (9) csubn
*                   (10) subn
*
*
      program eigen
*
      real*8 m(10,10),k(10,10),c(10,10)
      real*8 error,h(10,10)
      complex*16 crt(2,5),u(10,10),buf3(10,10)
      complex*16 buf1(10,10),buf2(10,10)
      data error,niter,n/0.000001,800,2/
* Data for the active suspension system
      data((m(i,j),j=1,2),i=1,2)/28.58,0,0,288.9/
      data((k(i,j),j=1,2),i=1,2)/213140,-35355,
   +    -57240,35355/
      data((c(i,j),j=1,2),i=1,2)/1385.7,-4827,
   +    -1385.7,4827/
*
* Output [M], [C], and [K] matrices
      n2=n*2
      print*, 'M'
      do i=1,n
        print 15, (m(i,j),j=1,n)
      enddo
      print*
      print*, 'C'
      do i=1,n
        print 15, (c(i,j),j=1,n)
      enddo
      print*
      print*, 'K'
      do i=1,n
        print 15,(k(i,j),j=1,n)
      enddo
* Evaluate eigen-values and modes
      call cmodl(m,c,k,h,u,crt,error,niter,n)
* Output [H], eigen-values, and modes
      print*
      print*, 'H'
      do i=1,n2
        print 15, (h(i,j),j=1,n2)
      enddo
      print*
```

```
        print*, 'eigen-values'
        do i=1,n
          do j=1,n
            print 10, crt(i,j)
          enddo
        enddo
        print*
        print*, 'modes'
        do j=1,n2
          print*
          print 5, 'e-value',j
          do i=1,n2
            print 10, u(i,j)
          enddo
        enddo
* Check diagonalization of the [H] matrix
        do i=1,n2
          do j=1,n2
            buf1(i,j)=h(i,j)
          enddo
        enddo
        call cinvs(u,buf2,n2)
        call cmply(buf2,buf1,buf3,n2)
        call cmply(buf3,u,buf1,n2)
        print*
        print*, 'U-1 * H * U (displayed column-wise)'
        print*, 'Eigen-values on diagonal'
        print*
        do j=1,n2
          print 5, 'col',j
          do i=1,n2
            print 10, buf1(i,j)
          enddo
        enddo
*
5       format(1x,a,i5)
10      format(1x,'(',2f20.5,')')
15      format(1x,10f15.5)
*
        stop
        end
```

```
M
       28.58000          .00000
         .00000       288.89999

C
     1385.69995    -4827.00000
    -1385.69995     4827.00000

K
   213140.00000   -35355.00000
   -57240.00000    35355.00000

H
   -48.48495   168.89433   -7457.66272   1237.05389
     4.79647   -16.70820     198.13085   -122.37799
     1.00000      .00000        .00000       .00000
      .00000     1.00000        .00000       .00000

eigen-values
(            -6.30539             7.62482)
(           -26.29119            78.28035)
(            -6.30539            -7.62482)
(           -26.29119           -78.28035)

modes

e-value     1
(            -1.56290             -.89439)
(            -6.30539             7.62482)
(              .03100              .17934)
(             1.00000              .00000)

e-value     2
(            -1.56290              .89439)
(            -6.30539            -7.62482)
(              .03100             -.17934)
(             1.00000              .00000)

e-value     3
(          -784.42557         -1138.82895)
(           -26.29119            78.28035)
(           -10.04899            13.39577)
(             1.00000              .00000)

e-value     4
(          -784.42557          1138.82895)
(           -26.29119           -78.28035)
(           -10.04899           -13.39577)
(             1.00000              .00000)

U-1 * H * U (displayed column-wise)
Eigen-values on diagonal
```

```
col     1
(            -6.30539              7.62482)
(              .00000               .00000)
(              .00000               .00000)
(              .00000               .00000)
col     2
(              .00000               .00000)
(            -6.30539             -7.62482)
(              .00000               .00000)
(              .00000               .00000)
col     3
(              .00000               .00000)
(              .00000               .00000)
(           -26.29119             78.28035)
(              .00000               .00000)
col     4
(              .00000               .00000)
(              .00000               .00000)
(              .00000               .00000)
(           -26.29119            -78.28035)
```

```
* DRIVER PROGRAM TO DETERMINE EIGEN-VALUES
* AND MODES OF A DAMPED SYSTEM.
* THE [M], [C], [K] MATRICES NEED NOT BE
* SYMMETRIC.
* THE DAMPING NEED NOT BE PROPORTIONAL.
* THE [M] [C] [K] MATRICES HAVE TO SPECIFIED
* IN THE DATA STATEMENTS
* AT THE TOP OF THE PROGRAM.
* SUBROUTINES REQD: (1) cmodl  (2) coeff  (3) chomo
*                   (4) croot  (5) cinvs  (6) invs
*                   (7) cmply  (8) mply   (9) csubn
*                   (10) subn
*
*
      program eigen
*
      real*8 m(10,10),k(10,10),c(10,10)
      real*8 error,h(10,10)
      complex*16 crt(2,5),u(10,10),buf1(10,10)
      complex*16 buf2(10,10),buf3(10,10)
      data error,niter,n/0.000001,800,2/
* Data for the passive suspension system
      data((m(i,j),j=1,2),i=1,2)/28.58,0,0,288.9/
      data((k(i,j),j=1,2),i=1,2)/175860,-19960,
   +    -19960,19960/
      data((c(i,j),j=1,2),i=1,2)/1861,-1861,
   +    -1861,1861/
*
* Output [M], [C], and [K] matrices
      n2=n*2
      print*, 'M'
      do i=1,n
        print 15, (m(i,j),j=1,n)
      enddo
      print*
      print*, 'C'
      do i=1,n
        print 15, (c(i,j),j=1,n)
      enddo
      print*
      print*, 'K'
      do i=1,n
        print 15,(k(i,j),j=1,n)
      enddo
* Evaluate eigen-values and modes
      call cmodl(m,c,k,h,u,crt,error,niter,n)
* Output [H], eigen-values, and modes
      print*
      print*, 'H'
      do i=1,n2
        print 15, (h(i,j),j=1,n2)
      enddo
      print*
```

```
        print*, 'eigen-values'
        do i=1,n
          do j=1,n
            print 10, crt(i,j)
          enddo
        enddo
        print*
        print*, 'modes'
        do j=1,n2
          print*
          print 5, 'e-value',j
          do i=1,n2
            print 10, u(i,j)
          enddo
        enddo
* Check diagonalization of the [H] matrix
        do i=1,n2
          do j=1,n2
            buf1(i,j)=h(i,j)
          enddo
        enddo
        call cinvs(u,buf2,n2)
        call cmply(buf2,buf1,buf3,n2)
        call cmply(buf3,u,buf1,n2)
        print*
        print*, 'U-1 * H * U (displayed column-wise)'
        print*, 'Eigen-values on diagonal'
        print*
        do j=1,n2
          print 5, 'col',j
          do i=1,n2
            print 10, buf1(i,j)
          enddo
        enddo
*
5       format(1x,a,i5)
10      format(1x,'(',2f20.5,')')
15      format(1x,10f15.5)
*
        stop
        end
```

```
M
       28.58000         .00000
         .00000      288.89999

C
     1861.00000    -1861.00000
    -1861.00000     1861.00000

K
   175860.00000   -19960.00000
   -19960.00000    19960.00000

H
   -65.11547   65.11547   -6153.25404   698.39048
     6.44168   -6.44168      69.08965   -69.08965
     1.00000     .00000        .00000      .00000
      .00000    1.00000        .00000      .00000

eigen-values
(            -2.65591              7.60674)
(           -33.12266             68.61802)
(            -2.65591             -7.60674)
(           -33.12266            -68.61802)

modes

e-value    1
(             -.83119               .51599)
(            -2.65591              7.60674)
(              .09447               .07629)
(             1.00000               .00000)

e-value    2
(             -.83119              -.51599)
(            -2.65591             -7.60674)
(              .09447              -.07629)
(             1.00000               .00000)

e-value    3
(          -519.91064           -748.76713)
(           -33.12266             68.61802)
(            -5.88370             10.41701)
(             1.00000               .00000)

e-value    4
(          -519.91064            748.76713)
(           -33.12266            -68.61802)
(            -5.88370            -10.41701)
(             1.00000               .00000)

U-1 * H * U (displayed column-wise)
Eigen-values on diagonal
```

```
col     1
(            -2.65591              7.60674)
(              .00000               .00000)
(              .00000               .00000)
(              .00000               .00000)
col     2
(              .00000               .00000)
(            -2.65591             -7.60674)
(              .00000               .00000)
(              .00000               .00000)
col     3
(              .00000               .00000)
(              .00000               .00000)
(           -33.12266             68.61802)
(              .00000               .00000)
col     4
(              .00000               .00000)
(              .00000               .00000)
(              .00000               .00000)
(           -33.12266            -68.61802)
```

VITA

Murari Venkataraman

Candidate for the Degree of

Master of Science

Thesis: COMPARISON OF THE PERFORMANCE CHARACTERISTICS OF PASSIVE AND ACTIVE SUSPENSION SYSTEMS: A RANDOM VIBRATION APPROACH

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