THE INTEGRAL GRADED HOMOLOGY OF REEB CHORD COMPLEX OF LEGENDRIAN KNOTS IN $\mathbb{R}^3$

By

XIAOWEI YANG

Bachelor of Science in Mathematics
Baoji University of Arts and Sciences
Baoji, Shannxi, P.R.China
1999

Master of Science in Mathematics
Southwest Jiaotong University
Chengdu, Sichuan, P.R.China
2003

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Dissertation Approved:

Dr. Weiping Li

Dissertation Advisor

Dr. Robert Myers

Dr. Alan Noell

Dr. Mindy McCann

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Name: Xiaowei Yang

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Title of Study: THE INTEGRAL GRADED HOMOLOGY OF REEB CHORD COMPLEX OF LEGENDRIAN KNOTS IN $\mathbb{R}^3$

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The main work of this thesis concerns the classification of Legendrian knots up to Legendrian isotopy in $\mathbb{R}^3$ with standard contact structure. In the thesis, we construct a topological invariant of Legendrian knots: the integral graded homology of Reeb chord complex of Legendrian knots. The invariant is a $\mathbb{Z}$-graded vector space over $\mathbb{Z}_2$.

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CHAPTER 1

Introduction

Legendrian knot theory is the theory of knots in contact topological and geometric field. Legendrian knot theory plays an important role in low dimensional topology since Legendrian knots reveal the topology and geometry of the underlying contact manifold. For instance, Legendrian knots are used to detect overtwistedness of contact structures [13], to detect topological properties of knots [22] and to distinguish contact structures [21]. The main work of this thesis is to construct a topological invariant of Legendrian knots: the integral graded homology $CH_*(L)$ of Reeb chord complex of Legendrian knots in contact 3-dimensional manifold $\mathbb{R}^3$ with standard contact structure.

The theory of Legendrian knots was first introduced by Dmitry Fuchs and Serge Tabachnikov [18] in 1990's, since then many mathematicians like Y. Eliashberg [8], J. Etnyre, K. Honda [12], L. Ng [25] and Y. Chekanov [3], have worked on this topic. Recently, this topic has been active and many important results have been published.

A Legendrian knot in $\mathbb{R}^3$ is a smooth knot which is tangent to a standard contact structure ξ_{std} on $\mathbb{R}^3$ everywhere. The standard contact structure ξ_{std} on $\mathbb{R}^3$ is a completely non-integrable 2-plane field given by the kernel of the 1-form $dz - ydx$. Many important constructions and results in low-dimensional topology depend on our ability to distinguish Legendrian knots. Thus the main question of Legendrian knot theory is to classify Legendrian knots up to Legendrian isotopy.

Two Legendrian knots are Legendrian isotopic or equivalent if one can be deformed into the other through Legendrian knots. Generally, we would like to classify Legendrian knots by constructing invariants associated to a Legendrian knot that remain unchanged when we deform the knot. For topological knots, there are many invariants (see [2, 30] for example), but for Legendrian knots, very few invariants have been found so far even though many mathematicians have worked on this topic.

It has been known for some time that there are three classical invariants of Legendrian knots :*Topological knot type* $k(L)$, *Thurston-Bennequin number* $tb(L)$, and *Rotation number* $r(L)$.

Legendrian knots have been classified for a few knot types by classical invariants, including Legendrian unknots [8], Legendrian torus knots and Legendrian figure knots [12]. For a relatively long time, we do not know whether the classical invariants can determine Legendrian knots completely up to Legendrian isotopy. In [3], Chekanov found a new invariant: *stable type* of differential graded algebra (DGA), where the algebra is $\mathbb{Z}_{2r(L)}$ graded over $\mathbb{Z}_2$ and generated by the crossings of the diagram of the Legendrian knot and the differential comes from counting certain immersed polygons whose edges lie in the diagram of the knot and whose vertices lie at the crossings. He proved that the DGA $(\mathcal{A}_{Z_2}, \partial)$ associated to a Legendrian knot changes by stable tame isomorphisms under Legendrian isotopy, moreover the contact homology $CH_*(L) = \dfrac{\ker\partial}{\mathrm{Im}\partial}$ is unchanged under Legendrian isotopy.

This new invariant can show that there exist Legendrian knots (5_2) which have the same classical invariants, but are not Legendrian isotopic (see [3]). So the classical invariants do not completely determine the Legendrian isotopy type of Legendrian knots. After Chekanov, J.Etnyre, L.Ng and J.Sabloff [14] extended Chekanov's DGA to a DGA over $\mathbb{Z}[t, t^{-1}]$ with a full $\mathbb{Z}$ grading. Even though we have known these non-classical new invariants and three classical invariants, but the question of the classification of Legendrian knots is still widely open.

Based on Etnyre, Ng and Sabloff's work [14] and Eliashberg, Givental and Hofer's work [9], in this thesis, we have found a new topological invariant of Legendrian knots: the integral graded homology $CH_*(L)$ of Reeb chord complex of Legendrian knots, which is a $\mathbb{Z}$-graded vector space over $\mathbb{Z}_2$ and is obtained from a chain complex not from DGA. It is different from Chekanov's invariant.

Let L be a generic Legendrian knot, to L we want to assign a $\mathbb{Z}$-graded vector space and to define a differential ∂_n on the vector space. In our approach, first we grade Reeb chords -i.e., Reeb flow lines that begin and end on Legendrian knot L by the Maslov index [29], then we define a chain complex (C_n, ∂_n) (called Reeb chord complex of Legendrian knots), where C_n is a vector space over $\mathbb{Z}_2$ generated by Reeb chords with Maslov index n and ∂_n comes from counting J-holomorphic curves in the symplectization of $(\mathbb{R}^3, \xi_{std})$ with two Reeb chords as asymptotes so that the difference between indices is one. Finally, we define the homology $CH_*(L)$ of Reeb chord complex of Legendrian knots. We have the following results by the ideas of Floer homology [17], PDE and complex analysis techniques.

Proposition 3.3.1 *Under the assumption $r(L) \neq 0$, the differential $\partial_n : C_n \longrightarrow C_{n-1}$ satisfies $\partial_{n-1} \circ \partial_n = 0$ and $\{(C_n, \partial_n)\}_{n\in\mathbb{Z}}$ is a chain complex.*

Then we obtain the following main theorem of the thesis.

Theorem 4.0.1 *Under the assumption $r(L) \neq 0$, the homology $CH_n(L) = \dfrac{\mathrm{Ker}\partial_n}{\mathrm{Im}\partial_{n+1}}$ of Reeb chord complex is a topological invariant of Legendrian knots for all $n \in \mathbb{Z}$.*

The rest of the thesis is divided into three chapters. Chapter 2 surveys the basic notions of contact topology, especially the Legendrian knot theory and outlines results concerning invariants of Legendrian knots.

In Chapter 3, first, we define the grading of Reeb chords and show that the grading is well-defined in Section 3.1. Then, we describe the moduli space of J-holomorphic disks and some important properties of J-holomorphic disks. Especially, we discuss a version of Gromov compactness of the moduli space in Section 3.2. Finally, we define

Reeb chord complex of Legendrian knots and prove that the Reeb chord complex is a chain complex (i.e. the proposition 3.3.1) in Section 3.3. The chapter ends up with the description of the homology $CH_*(L)$ of Reeb chord complex of Legendrian knot L.

In Chapter 4, we prove the invariance of the homology $CH_*(L)$, i.e. Theorem 4.0.1. The proof consists of three steps which are shown in three sections 4.1, 4.2 and 4.3 respectively.

CHAPTER 2

Preliminaries

In this chapter, we first outline the basic notions of contact topology and Legendrian knots. The second section surveys known results about invariants of Legendrian knots. Especially, the Chekanov's invariant: *stable type* of differential graded algebra and its extension. (For more detailed discussion, see [11].)

2.1 The standard contact structure on $\mathbb{R}^3$

Let M be an orientable 3-manifold. A *contact structure* ξ on M is a completely non-integrable tangent 2-plane field. Locally ξ is represented as the kernel of a 1-form α. We know that non-integrability condition is equivalent to $\alpha \wedge d\alpha \not\equiv 0$. Hence ξ is a contact structure if and only if $\alpha \wedge d\alpha \not\equiv 0$.

The simplest example of contact structure on $\mathbb{R}^3$ is given by the kernel of 1-form $\alpha = dz - ydx$, this is called *standard contact structure* on $\mathbb{R}^3$ and denoted by ξ_{std}. There are many other contact structures on $\mathbb{R}^3$, but Darboux's theorem says that all of them locally look like the standard one. Note that $\xi_{std} = \ker(dz - ydx) = \operatorname{span}\{\partial_y, \partial_x + y\partial_z\}$. The plane field ξ_{std} is indicated in Figure 2.1.

A *Legendrian knot* L in $(\mathbb{R}^3, \xi_{std})$ is an embedded S^1 that is tangent to ξ_{std} everywhere. If we parametrize L:

$$\gamma(t) : S^1 \longrightarrow \mathbb{R}^3 : t \longmapsto (x(t), y(t), z(t))$$

Then the fact that L is tangent to ξ_{std} can be expressed by $\gamma'(t) \in \xi_{std}$ or since $\xi_{std} = \ker(dz - ydx)$, we have

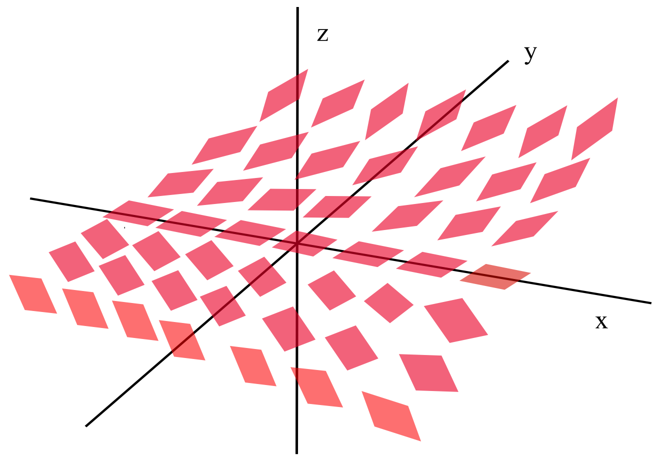


Figure 2.1: The standard contact structure ξ_{std} on $\mathbb{R}^3$

$$z'(t) - y(t)x'(t) = 0. \tag{2.1}$$

Moreover, we will assume that $\gamma(t)$ is a smooth function and an immersion in $\mathbb{R}^3$. Usually, there are two ways to visualize a Legendrian knot in $(\mathbb{R}^3, \xi_{std})$ via projecting L onto a plane.

- The *front projection* $\sigma : S^1 \longrightarrow \mathbb{R}^2 : t \longmapsto (x(t), z(t))$, i.e. we project L onto xz-plane.
- The *Lagrangian projection* $\Pi : S^1 \longrightarrow \mathbb{R}^2 : t \longmapsto (x(t), y(t))$, i.e. we project L onto xy-plane.

Note that (2.1) implies that $z'(t) = y(t)x'(t)$. Hence we can always recover the y coordinate of $\gamma(t)$ from the front projection σ by rewriting (2.1) as

$$y(t) = \frac{z'(t)}{x'(t)} \qquad \text{if } x'(t) \text{ is non-zero.}$$

Thus anytime $x'(t) = 0$ so $z'(t)$ must be. But this implies that L has no vertical tangencies. Hence there must be some cusps for any Legendrian knot, in this case $y(t_0) = \lim_{t \longrightarrow t_0} \frac{z'(t)}{x'(t)}$. It is interesting to note that our cusps must be "semi-cubic parabolas " i.e. after a change of coordinates $z(\theta) = 3\theta^3$ and $x(\theta) = 2\theta^2$ (see [11]).

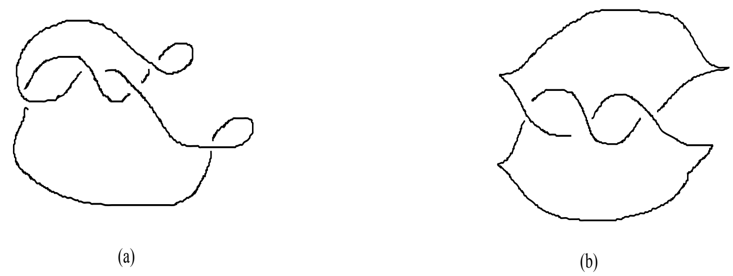


Figure 2.2: The Lagrangian projection (a) and the front projection (b) of Legendrian right trefoil knot.

2.2 Classical and DGA invariants of Legendrian knots

As we mentioned early, the fundamental problem in Legendrian knot theory is to determine whether two Legendrian knots are Legendrian isotopic or not. Also note that Legendrian knots are plentiful: any smooth knot can be approximated by a Legendrian knot continuously. Two Legendrian knots L_0 and L_1 are *Legendrian isotopic* if there is a continuous family L_t, $t \in [0,1]$, of Legendrian knots starting at L_0 and ending at L_1.

We would like to classify Legendrain knots by constructing invariants, there are three so called classical invariants. Obviously, *underlying topological knot type* $K(L)$ is an invariant of Legendrian knot since any Legendrian isotopy between Legendrian knots is also a topological isotopy of underlying knots.

The second classical invariant is the *Thurston-Bennequin number* $tb(L)$, which measures the twisting of contact planes around the knot L. More precisely, we push L out a small distance along z-axis, then we obtain a new Legendrian knot, denote by $\hat{L}$. Define $tb(L)$ to be the linking number $lk(L, \hat{L})$ of L and $\hat{L}$. In the front projection σ, there is a formula to compute Thurston-Bennequin number:

$$tb(L) = \text{writhe}(\sigma(L)) - \text{number of right cusps in } \sigma(L).$$

The last classical invariant, the *rotation number* $r(L)$ is defined for oriented Legendrian knots. It measures the twisting of the tangent direction to L inside ξ_{std}. More precisely, let Σ be a Seifert surface for L, if we trivialize ξ_{std} over Σ, then $r(L)$ is the winding number of the oriented tangent direction to L with respect to this trivialization. Accordingly, in the front projection,

$$r(L) = \frac{1}{2}(D_c - U_c).$$

Where D_c is the number of down cusps and U_c is the number of up cusps. Also note that $r(L)$ depends on the orientation of L, but $tb(L)$ is independent of the choice of orientation of L. There is an important relation between Thurston-Bennequin number $tb(L)$ and rotation number $r(L)$ as follows.

Theorem 2.2.1 (**Bennequin [11]**) *Let L be a Legendrian knot in* $(\mathbb{R}^3, \xi_{std})$. *Let* Σ *be a Seifert surface for L. Then* $tb(L) + |r(L)| \leq -\chi(\Sigma)$.

Legendrian knots have been classified for a few knot types by classical invariants, including Legendrian unknots [8], Legendrian torus knots and Legendrian figure knots [12]. For a relatively long time, we do not know whether the classical invariants can determine Legendrian knots completely up to Legendrian isotopy. To answer this question, Chekanov found a new invariant: *stable type* of differential graded algebra DGA.(see [3])

The algebra . Let $\mathcal{A}_{Z_2}$ be the $\mathbb{Z}_{2r(L)}$ graded free associative unital algebra over $\mathbb{Z}_2$ and generated by the set C of double points in $\Pi(L)$ of the Legendrian knot. We assume C is a finite set (since we assume L is generic).

The grading. To each crossing $c \in C$ there are two points c^+ and c^- in L that project to c. We let c^+ be the point with large z coordinate. Choose a map $\gamma_c : [0,1] \longrightarrow L$ that parametrizes an arc running from c^+ to c^-. Consider the Lagrangian projection $\Pi(\gamma_c)$ and denote the rotation number $r(\Pi(\gamma_c))$, now define

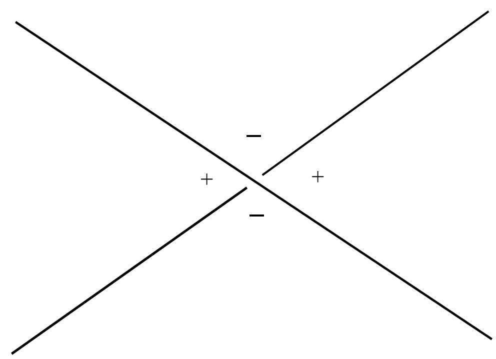


Figure 2.3: Sections near a crossing in $\Pi(L)$

the grading on c to be

$$|c| = 2r(\Pi(\gamma_c)) - \frac{1}{2} \qquad \mod 2r(L)$$

The differential . The differential comes from counting certain immersed polygons whose edges lie in the $\Pi(L)$ and whose vertices lie at the crossings. More precisely, The neighbourhood of a crossing c in $\mathbb{R}^2$ is divided into four sections by $\Pi(L)$. Two are labelled positive (+) and two are labelled negative (−) (see Figure 2.3). Let P_{k+1} be a $k+1$ sided polygon with vertices labelled counter-clockwise $v_0, ..., v_k$ and set

$$\mathcal{M}^a_{b_1 \dots b_k} = \left\{ u : (P_{k+1}, \partial P_{k+1}) \mapsto (\mathbb{R}^2, \Pi(L)) | u(v_0) = a, u(v_i) = b_i, i = 1, ..., k \right\}$$

where u maps a small neighbourhood of v_0 to a positive (+) section and maps each v_i to a negative (−) section in $\Pi(L)$.

We can now define

$$\partial a = \sum_{b_1 \dots b_k} (\#_{Z_2} \mathcal{M}^a_{b_1 \dots b_k}) b_1 ... b_k$$

Chekanov [3] shows that the differential above is well-defined and $\partial^2 = 0$. So we have a differential graded algebra $(\mathcal{A}_{\mathbb{Z}_2}(a_1, ...a_n), \partial)$, note that this is not an invariant. To get an invariant, Chekanov defines the *stable tame isomorphism class* of $\mathcal{A}_{\mathbb{Z}_2}$. First, we define a special algebra $\varepsilon_i = \mathcal{A}_{\mathbb{Z}_2}(e_1, e_2)$ by setting $|e_1| = i, |e_2| = i - 1, \partial e_1 = e_2, \partial e_2 = 0$.

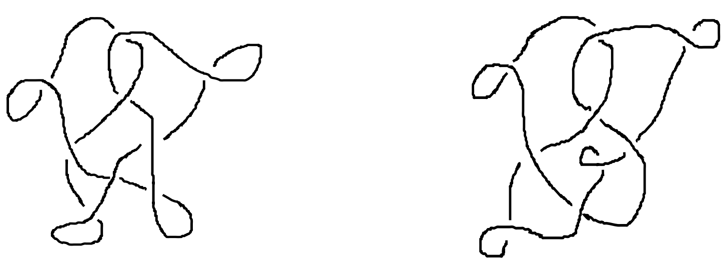
Definition 2.2.1 ([14]) *The degree i stabilization* $S_i(\mathcal{A}_{\mathbb{Z}_2}(a_1, ...a_n))$ *of* $\mathcal{A}_{\mathbb{Z}_2}(a_1, ...a_n)$ *is defined to be* $\mathcal{A}_{\mathbb{Z}_2}(a_1, ...a_n, e_1^i, e_2^i)$*. The grading and differential are inherited from both* $\mathcal{A}_{\mathbb{Z}_2}$ *and* ε_i*. Two algebras* $\mathcal{A}_{\mathbb{Z}_2}$ *and* $\mathcal{A}'_{\mathbb{Z}_2}$ *are stable tame isomorphic if there exist two sequences of stabilizations* $S_{i_1}, ...S_{i_n}$ *and* $S_{j_1}, ...S_{j_m}$ *and a tame isomorphism*

$$\phi : S_{i_n}(...S_{i_1}(\mathcal{A}_{\mathbb{Z}_2})...) \longrightarrow S_{j_m}(...S_{j_1}(\mathcal{A}'_{\mathbb{Z}_2})...).$$

Two differential algebras $(\mathcal{A}_{\mathbb{Z}_2}, \partial)$ *and* $(\mathcal{A}'_{\mathbb{Z}_2}, \partial')$ *are stable tame isomorphic if there is a stable tame isomorphism from* $\mathcal{A}_{\mathbb{Z}_2}$ *to* $\mathcal{A}'_{\mathbb{Z}_2}$ *that is also a chain map.*

Theorem 2.2.2 (Chekanov [3]) *The DGA* $(\mathcal{A}_{Z_2}, \partial)$ *associated to a Legendrian knot changes by stable tame isomorphisms under Legendrian isotopy, moreover the contact homology* $CH_*(L) = \dfrac{ker\partial}{Im\partial}$ *is unchanged under Legendrian isotopy.*

This new invariant can show that there exist a pair of Legendrian 5_2 knots which have the same classical invariants ($tb = 1$ and $r = 0$), but are not Legendrian isotopic (see Figure 2.4). So the classical invariants do not completely determine the Legendrian isotopy type of Legendrian knots.



(1) (2)

Figure 2.4: The two non-Legendrian isotopic Legendrian 5_2 knots .

After Chekanov, J.Etnyre , L.Ng and J.Sabloff [14] extended Chekanov's stable type of DGA invariant to a DGA over $\mathbb{Z}[t, t^{-1}]$ with a full $\mathbb{Z}$ grading. Their set-up is very similar to Chekanov's DGA.

The algebra. $\mathcal{A}(a_1,...a_n)$ is the $\mathbb{Z}$-graded free associative unital algebra over $\mathbb{Z}[t,t^{-1}]$ and generated by the set C of double points in $\Pi(L)$.

The grading . The grading for t is defined to be $2r(L)$. To grade a generator a_i, we first specify a capping path γ_{a_i} in L which begins at the undercrossing of a_i and follows the orientation of L , and ends when it reaches the overcrossing of a_i. Define the grading of a_i to be

$$|a_i| = -2r(\Pi(\gamma_{a_i})) - \frac{1}{2}$$

In addition, define *the sign of a crossing* a_i to be $\text{sgn}a_i = (-1)^{|a_i|}$. To each sector Q in the neighbourhood of a crossing a_i, we associate a sign ε_{Q,a_i}, called the orientation sign.

The differential. The differential ∂ comes from counting immersed disks in $\Pi(L)$. Let $D^2_* = D^2 \setminus \{x, y_1...y_n\}$, where $\{x, y_1...y_n\} \subset \partial D^2$. Fix a homology class $A \in H_1(L) = \mathbb{Z}$ and define $\triangle^A(a;b_1....b_n)$ to be the space of all orientation-preserving immersions $f : (D^2_*, \partial D^2_*) \longmapsto (\mathbb{R}^2, \Pi(L))$ (up to reparametrization) that satisfy:

- The homology class $[(\Pi(L))^{-1}(\text{Im}(f|_{\partial D^2}) \cup \gamma_a \cup -\gamma_{b_1} ... \cup -\gamma_{b_n})]$ is A.

- The map f maps the boundary punctures to the crossings and covers either one or three sections with majority of sections positive at the crossing a and negative at all other crossings b_i.

Formally, we define the dimension of $\triangle^A(a;b_1....b_n)$ to be

$$\dim(\triangle^A(a;b_1....b_n)) = |a| - \sum |b_i| + 2n(A)r(L) - 1$$

Eventually, we define the differential to be :

$$\partial a = \sum_{\dim(\triangle^A(a;b_1....b_n))=0} \epsilon(a;b_1...b_n)t^{-n(A)}b_1...b_n$$

where $\epsilon(a;b_1...b_n)$ is the product of orientation signs corresponding to sectors of $a, b_1, ..., b_n$.

J. Etnyre, L. Ng and J. Sabloff [14] show that the differential is well-defined and $\partial^2 = 0$. Moreover the stable type of the algebra $\mathcal{A}(a_1, ...a_n)$ is an invariant of Legendrian knots.

Note that if we set $t = 1$, modulo the coefficient by two and the grading by $2r(L)$, then we can recover Chekanov's DGA. Even though we have known these non-classical invariants and three classical invariants, we may classify some Legendrian knots (see [4], for example). But the question of the classification of Legendrian knots is still widely open.

CHAPTER 3

The Integral Graded Homology of Reeb Chord Complex

In this chapter, we define a new topological invariant of Legendrian knots: the integral graded homology $CH_*(L)$ of Reeb chord complex of Legendrian knots. This is a $\mathbb{Z}$-graded vector space over $\mathbb{Z}_2$ and the invariance will be shown in next chapter. To this end, we first define the grading of each Reeb chord by Maslov index [29]. It turns out that we can grade the set of Reeb chords by integers. Then we discuss some properties of J-holomorphic disks and the moduli space of J-holomorphic disks in our situation. Finally, we define a chain complex (C_n, ∂_n) called Reeb chord complex, where C_n is a vector space over $\mathbb{Z}_2$ generated by the set of Reeb chords with Maslov index n and the differential ∂_n comes from counting certain J-holomorphic curves with Lagrangian boundary and asymptotic properties. We end this chapter by defining the integral graded homology $CH_*(L)$ of Reeb chord complex of Legendrian knots.

3.1 The grading of Reeb chords

In this section, we grade Reeb chords by Maslov index, the Reeb chords of a Legendrian knot in $(\mathbb{R}^3, \xi_{std})$ are Reeb flow lines for the Reeb vector field R_α starting and ending on a Legendrian knot. Recall that the standard contact structure ξ_{std} is the kernel of a 1-form $\alpha = dz - ydx$, so ξ_{std}= span $\{\partial_y, \partial_x + y\partial_z\}$. The approach in this section is analogous to [14].

Definition 3.1.1 (**[19]**) *A Reeb vector field R_α of $(\mathbb{R}^3, \xi_{std})$ is a vector field such that $d\alpha(R_\alpha, -) \equiv 0$ and $\alpha(R_\alpha) \equiv 1$.*

From this definition, we can see that the Reeb vector field in our situation is parallel to the z-direction ∂_z. In fact, with respect to the basis $\{\partial_x, \partial_y, \partial_z\}$, we suppose that $R_\alpha = (a, b, c)$, then $d\alpha = dx \wedge dy$, the condition $d\alpha(R_\alpha, -) \equiv 0$ is equivalent to

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \Longrightarrow \begin{pmatrix} b \\ -a \\ 0 \end{pmatrix} = 0 \Longrightarrow a = b = 0$$

Combining this with $\alpha(R_\alpha) \equiv 1$, then we have that $R_\alpha = (0, 0, 1) = \partial_z$. Hence, all the Reeb flow lines are vertical lines parallel to z-axis.(see Figure 3.1)

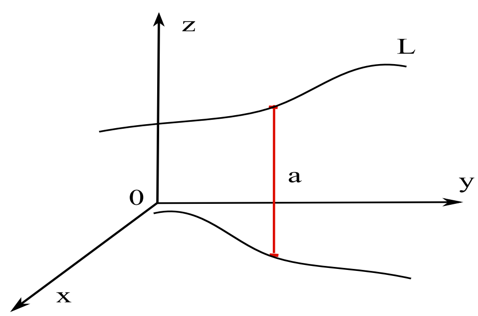


Figure 3.1: $R_\alpha = (0, 0, 1) = \partial_z$. All Reeb chords are parallel to z-axis starting and ending on Legendrian knot L

Given a generic Legendrian knot L, we parametrize L :

$$\gamma(t) : S^1 \longrightarrow \mathbb{R}^3 \quad \text{by} \quad t \longmapsto (x(t), y(t), z(t)).$$

For a Reeb chord $a(t) : [0, 1] \longrightarrow \mathbb{R}^3$, i.e. $\dot{a}(t) = R_\alpha$ and $a(0)$, $a(1) \in L$, to define the index of a Reeb chord $a(t)$, we choose a path γ_a inside Legendrian knot L that connects $a(1)$ to $a(0)$, then there is an orientable surface F such that $\partial F = \gamma_a(t) \cup a(t)$. We know that any orientable two plane bundle is trivial over a surface with boundary (see [19]). Hence, $\xi|_{\partial F}$ is trivial, so we can choose a symplectic trivialization of ξ_{std} over ∂F:

$$\partial F \times \mathbb{R}^2 \overset{\Phi}{\cong} \xi|_{\partial F} \qquad \text{such that} \qquad \Phi^* d\alpha = \omega_0$$

where Φ is a diffeomorphism, ω_0 is the standard symplectic form of $\mathbb{R}^2$. We define a line bundle $\Lambda(t)$ over $\partial F = \gamma_a(t) \cup a(t)$ as follows.

$$\Lambda(t) = \begin{cases} T_{\gamma_a(t)}L, & \textit{over } \gamma_a(t). \\ \\ D\Psi_\alpha(t)(T_{a(0)}L), & \textit{over } a(t). \end{cases}$$

where $\{\Psi_\alpha(t)\}$ is a one-parameter group of diffeomorphisms induced by Reeb vector field R_α. Using the trivialization, $\Lambda(t)$ may be viewed as a path of Lagrangian subspaces of a fixed symplectic space $(\mathbb{R}^2, \omega_0)$ in the following sense. In fact, for $\forall X, Y \in TL$, we have :

$$d\alpha(X,Y) = X\alpha(Y) - Y\alpha(X) - \alpha[X,Y] = 0.$$

It follows that $T_{\gamma_a(t)}L \subset \left(\xi|_{\gamma_a(t)}, d\alpha|_{\gamma_a(t)}\right)$ is Lagrangian for any point in $\gamma_a(t)$. For Reeb vector field R_α, by Cartan formula, we have $L_{R_\alpha} d\alpha = i_{R_\alpha} d\alpha + d i_{R_\alpha}\alpha = 0$ and

$$\begin{aligned} \frac{d}{dt}\Psi_\alpha^*(t)(d\alpha) &= \lim_{h\to 0} \frac{\Psi_\alpha^*(t+h)(d\alpha) - \Psi_\alpha^*(t)(d\alpha)}{h} \\ &= \Psi_\alpha^*(t)\left(\lim_{h\to 0} \frac{\Psi_\alpha^*(h)(d\alpha) - d\alpha}{h}\right) \\ &= \Psi_\alpha^*(t)(L_{R_\alpha} d\alpha) \\ &= 0. \end{aligned}$$

This shows that $\Psi_\alpha(t)$ preserves the symplectic form $d\alpha$, i.e. $\Psi_\alpha^*(t)(d\alpha) = d\alpha$. Thus, $D\Psi_\alpha(t)$ maps $T_{a(0)}L$ to Lagrangian spaces. Actually,

$$\Psi_\alpha(t) : \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \qquad \text{given by} \qquad (x,y,z) \longmapsto (x,y,z+t).$$

So $D\Phi_\alpha(t) = \mathrm{Id}$ with respect to $\{\partial_x, \partial_y, \partial_z\}$. Hence, by using above trivialization, we may view the line bundle $\Lambda(t)$ over ∂F as a path of Lagrangian subspaces of the symplectic space $(\mathbb{R}^2,\ \omega_0)$. In fact, if we choose a specific trivialization :

$$\Phi : \partial F \times \mathbb{R}^2 \longrightarrow \xi|_{\partial F} \qquad \text{given by} \qquad (q,\zeta) \longmapsto (q,\zeta).$$

Then we may identify $\Lambda(t)$ with the following Lagrangian path $\bar{\Lambda}(t)$ in $(\mathbb{R}^2, \omega_0)$:

$$\bar{\Lambda}(t) = \{(\lambda x'(t), \lambda y'(t)) | \lambda \in \mathbb{R}, 0 \leq t \leq 1\} \subset (\mathbb{R}^2, \omega_0).$$

For the line bundle $\Lambda(t)$ over $\partial F = \gamma_a(t) \cup a(t)$, there are two choices of $\gamma_a(t)$ inside L for the Reeb chord $a(t)$, so we need to specify one of the choices of $\gamma_a(t)$. In order to do so, here we put the following conditions on the path $\bar{\Lambda}(t)$ as follows:

$$\begin{cases} x'(0) > 0, \\ y(t) \neq y(0), \qquad 0 < t < 1. \end{cases} \tag{3.1}$$

These conditions guarantee that $\gamma_a(t)$ is specified uniquely inside L. Actually, the first condition gives an orientation of $\gamma_a(t)$ and the second guarantees that $\gamma_a(t)$ connecting $a(1)$ and $a(0)$ inside L goes around these endpoints only once. Based on above discussion, for a Reeb chord $a(t)$, we define the Maslov index $\mu(\bar{\Lambda}(t))$ of the Lagrangian path $\bar{\Lambda}(t)$ as the grading of Reeb chord $a(t)$.

$$|a(t)| := \mu(\bar{\Lambda}(t)) = \sum_{\substack{x'(t)=0, \\ t \in (0,1)}} -\text{sign}\,\{x''(t) y'(t)\} \tag{3.2}$$

Proposition 3.1.1 *The grading of Reeb chords defined above is well-defined under the condition 3.1.*

Proof. First, note that $x'(t) = 0$ corresponds to the cusps of legendrian knot L. Cusps of a generic Legendrian knot are isolated and there are only finitely many cusps, hence this is a finite sum. At $x'(t) = 0$, after a change of coordinates, each cusp can be semi-cubic parabola [11], i.e. $z(\theta) = 3\theta^3, x(\theta) = 2\theta^2$, so we have $x''(\theta) \neq 0$. It follows that $x''(t) \neq 0$. Also note that $\gamma(t)$ is an immersion, so whenever $x'(t) = 0$, we always have $y'(t) \neq 0$. It follows that $x''(t)y'(t) \neq 0$. Hence the grading of Reeb chords is an integer.

The grading $\mu(\bar{\Lambda}(t))$ of Reeb chord $a(t)$ is independent of the choice of surface F. For any such two orientable surfaces F_1 and F_2, let $\bar{\Lambda}_1(t)$ and $\bar{\Lambda}_2(t)$ be corresponding Lagrangian paths, then F_1 and F_2 induce opposite orientations on the boundary, i.e. $\partial F_1 = -\partial F_2$, if we parametrize $\partial F_1 = \gamma_a(t) \cup a(t)$, then $\partial F_2 = \gamma_a(1-t) \cup a(1-t)$. For ∂F_1 and ∂F_2, they have the same cusps. Under the condition(3.1), $x''(t)y'(t)$ has opposite sign at each $x'(t) = 0$ with respect to ∂F_1 and ∂F_2. Therefore we have that $\mu(\bar{\Lambda_1}(t)) - \mu(\bar{\Lambda_2}(t)) = 0$. i.e. $\mu(\bar{\Lambda_1}(t)) = \mu(\bar{\Lambda_2}(t))$.

Finally, the grading $\mu(\bar{\Lambda}(t))$ of Reeb chord $a(t)$ is also independent of the choice of the trivialization Φ of $\xi|_{\partial F}$. In fact, given a different trivialization Ψ of $\xi|_{\partial F}$, by the property of naturality of Maslov index (see [29]), that is : $\mu(\Psi(\bar{\Lambda}(t))) = \mu(\bar{\Lambda}(t))$. Hence we have shown that the grading of Reeb chords is independent of the choices of surface F and the trivializaton Φ of $\xi|_{\partial F}$. So the grading of Reeb chord is well-defined. ■

By the formula $r(L) = \frac{1}{2}(D_c - U_c)$, we can see that $\mu(\bar{\Lambda}(t)) = 2r(\gamma_a)$. In fact, each upward cusp corresponds to $x''(t)y'(t) > 0$ and each downward cusp corresponds to $x''(t)y'(t) < 0$, hence $\sum_{\substack{x'(t)=0,\\ t\in(0,1)}} -\text{sign}\,\{x''(t)y'(t)\} = D_c - U_c = 2r(\gamma_a)$. (see Figure 3.2)

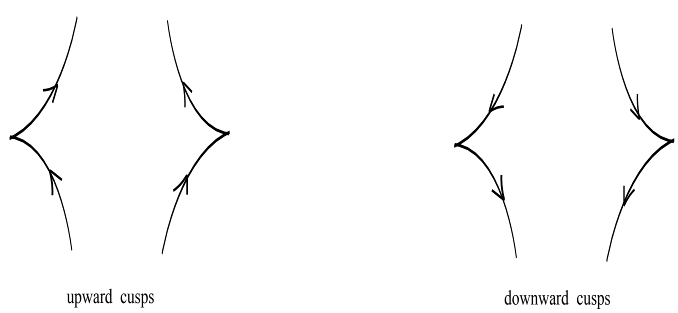


Figure 3.2: Upward cusps correspond to $x''(t)y'(t) > 0$ and downward cusps correspond to $x''(t)y'(t) < 0$.

3.2 The moduli space of J-holomorphic curves

In this section, we consider the set of J-holomorphic curves [20, 24] satisfying the prescribed boundary and asymptotic conditions. For $(\mathbb{R}^3, \xi_{std}) = (\mathbb{R}^3, \mathrm{Ker}(dz - ydx))$, we first consider the symplectization $(\mathbb{R}^3 \times \mathbb{R}, \omega = d(e^t\alpha))$ of $(\mathbb{R}^3, \xi_{std})$, where t stands for $\mathbb{R}$-direction. Note that $(\mathbb{R}^3 \times \mathbb{R}, \omega)$ is a 4-dimensional symplectic manifold with symplectic form $\omega = d(e^t\alpha) = e^t(dt \wedge \alpha + d\alpha)$. Given a Legendrian knot L, then $L \times \mathbb{R}$ is a Lagrangian submanifold of $(\mathbb{R}^3 \times \mathbb{R}, \omega)$, i.e. $\omega\mid_{L\times\mathbb{R}} \equiv 0$. In fact, for any X, $Y \in T(L \times \mathbb{R}) \cong TL \times T\mathbb{R}$, we have that :

$$\omega(X, Y) = d(e^t\alpha)(X, Y) = X(e^t\alpha(Y)) - Y(e^t\alpha(X)) - e^t\alpha[X, Y] = 0$$

Under the symplectization, there is a relation shown as follows.

$$\text{Legendrian} \quad L \subset (\mathbb{R}^3, \xi_{std}) \longrightarrow \text{Lagrangian} \;\; L \times \mathbb{R} \subset (\mathbb{R}^3 \times \mathbb{R}, \omega)$$

For this relation, we have the following .

Proposition 3.2.1 *If L_1 and L_2 are Legendrian isotopic in $(\mathbb{R}^3, \xi_{std})$, then $L_1 \times \mathbb{R} \cong L_2 \times \mathbb{R}$ are Lagrangian isotopic in $(\mathbb{R}^3 \times \mathbb{R}, \omega)$.*

Proof. Recall that if L_1 and L_2 are Legendrian isotopic, then there is an ambient isotopy $\Phi_s : (\mathbb{R}^3, \xi_{std}) \longrightarrow (\mathbb{R}^3, \xi_{std})$ satisfying the following.

(1). $\{\Phi_s\}$ is a family of contactomorphisms, i.e. $\Phi_s^*(\alpha) = f_s\alpha$. Where $f_s : \mathbb{R}^3 \longrightarrow \mathbb{R}^+$ is a suitable smooth family of smooth functions.

(2). $\Phi_0 = \mathrm{Id}$, $\quad \Phi_1(L_1) = L_2 \quad$ and $\quad \Phi_s(L_1)$ are Legendrian for all s

Now, we define a map on $\mathbb{R}^3 \times \mathbb{R}$ as follows.

$$\hat{\Phi}_s \quad : \mathbb{R}^3 \times \mathbb{R} \longrightarrow \mathbb{R}^3 \times \mathbb{R}$$

$$(x, y, z, t) \longmapsto (\Phi_s(x, y, z), t + \log(1/f_s))$$

then $\hat{\Phi}_s$ is a family of diffeomorphisms of $\mathbb{R}^3 \times \mathbb{R}$. To see this, suppose $(\Phi_s(x,y,z), t + \log(1/f_s)) = (\tilde{x}, \tilde{y}, \tilde{z}, T)$, we have that $D\hat{\Phi}_s = \begin{pmatrix} D\Phi_s & 0 \\ A & 1 \end{pmatrix}$ Where $A = (T_x, T_y, T_z)$, it follows that $\det(D\hat{\Phi}_s) \neq 0$, also $\hat{\Phi}_s$ is surjective. Hence $\hat{\Phi}_s$ is a diffeomorphism for each s. In addition,

$$\begin{aligned} \hat{\Phi}_s^*(\omega) &= \hat{\Phi}_s^*(de^T\alpha) = d\hat{\Phi}_s^*(e^T\alpha) \\ &= d(e^{t+\log(1/f_s)}\Phi_s^*(\alpha)) \\ &= d(e^{t+\log(1/f_s)}f_s\alpha) \\ &= d(e^t\alpha) \\ &= \omega. \end{aligned}$$

It follows that $\hat{\Phi}_s$ is a symplectomorphism for each s. Hence $\hat{\Phi}_s$ is Lagrangian isotopic.

■

Definition 3.2.1 (**[24]**) *A J-holomorphic curve in an almost complex manifold* $(\mathbb{R}^3 \times \mathbb{R}, J)$ *is a smooth map*

$$f : (\Sigma, j) \longrightarrow (\mathbb{R}^3 \times \mathbb{R}, J)$$

satisfying Cauchy-Riemann equation : $\bar{\partial}_J f = \frac{1}{2}(df + J \circ df \circ j) = 0$. *Where* Σ *is a Riemann surface with complex structure* j.

Here, we choose the Riemann surface to be the unit disk $\mathbb{D}$ with two punctures on the boundary, i.e. $\mathbb{D} \setminus \{p, q\}$, the complex structure induced from $\mathbb{C}$, i.e. $j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.(By Uniformization theorem of Riemann surface, for unit disk there is only one complex structure up to conformal equivalence) In this thesis, the J-holomorphic curves we are interested in are maps

$$f : (\mathbb{D} \setminus \{p, q\}, \partial\mathbb{D} \setminus \{p, q\}) \longrightarrow (\mathbb{R}^3 \times \mathbb{R}, L \times \mathbb{R})$$

such that $f(\partial\mathbb{D} \setminus \{p, q\}) \subset L \times \mathbb{R}$ and satisfy certain asymptotic conditions near the punctures. In this case, J-holomorphic curves are also called J-holomorphic disks. In order to describe such J-holomorphic disks, we parametrize $\mathbb{D} \setminus \{p, q\}$ by using the following conformal map

$$\begin{aligned} \Phi \quad : \mathbb{D} \setminus \{p, q\} \quad &\longrightarrow \mathbb{R} \times [0, 1] \\ z \longmapsto \quad &\frac{1}{\pi} \log \frac{1+z}{1-z} i \end{aligned}$$

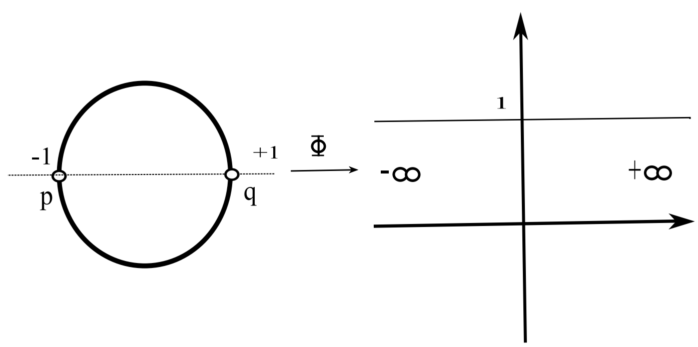


Figure 3.3: Conformal mapping between $\mathbb{D} \setminus \{p, q\}$ and $\mathbb{R} \times [0, 1]$

Remark 3.2.1 *By Riemann mapping theorem, such a conformal map exists. (See Figure 3.3)*

For convenience, we denote $\Sigma = \mathbb{R} \times [0, 1]$. It is easy to see that Φ satisfies the following properties.

(a) $\Phi|_{\partial\mathbb{D} \setminus \{p,q\}} = \partial\Sigma = \mathbb{R} \times \{0\} \cup \mathbb{R} \times \{1\}$.

(b) $\lim_{z \to p} \Phi(z) = -\infty$ and $\lim_{z \to q} \Phi(z) = +\infty$

(c) Φ maps lower-half boundary of $\partial\mathbb{D} \setminus \{p, q\}$ onto x-axis and maps upper-half boundary of $\partial\mathbb{D} \setminus \{p, q\}$ onto horizontal line $y = 1$.

From now on, we identify $\mathbb{D} \setminus \{p, q\}$ with $\Sigma = \mathbb{R} \times [0,1]$. We choose coordinates (s,t) of $\Sigma = \mathbb{R} \times [0,1]$ and (X,Y,Z,T) of $\mathbb{R}^3 \times \mathbb{R}$.

Remark 3.2.2 *If we choose coordinators as above, then for a J-holomorphic disk $\tilde{u}$, the Cauchy-Riemann equation $\bar{\partial}_J \tilde{u} = 0 \Leftrightarrow \frac{\partial \tilde{u}}{\partial s} + J(\tilde{u}(s,t))\frac{\partial \tilde{u}}{\partial t} = 0$.*

Given two Reeb chords $p(t)$ and $q(t)$, we consider the following set of J-holomorphic curves.

Definition 3.2.2 *The moduli space $\mathcal{M}(p,q)$ is the set of smooth maps :*

$$\begin{aligned} \tilde{u}(s,t) = (u,T): \quad (\Sigma, \partial\Sigma) &\longrightarrow (\mathbb{R}^3 \times \mathbb{R}, L \times \mathbb{R}) \\ (s,t) &\longmapsto (X,Y,Z,T) \end{aligned}$$

satisfying the following conditions.

(a) $\tilde{u}$ is J-holomoprhic, i.e. $\frac{\partial \tilde{u}}{\partial s} + J(\tilde{u}(s,t))\frac{\partial \tilde{u}}{\partial t} = 0$,

(b) $\tilde{u}$ has finite energy: $E(\tilde{u}) = \int_\Sigma \tilde{u}^* d(e^T\alpha) < \infty$,

(c) $\tilde{u}$ satisfies the Lagrangian boundary : $\tilde{u}|_{\partial\Sigma} \subset L \times \mathbb{R}$. i.e. $\tilde{u}(s,0), \tilde{u}(s,1) \subset L \times \mathbb{R}$,

(d) Asymptotic properties :

$$\lim_{s \to -\infty} u(s,t) = p(t) \qquad \text{and} \qquad \lim_{s \to -\infty} T(s,t) = +\infty$$
$$\lim_{s \to +\infty} u(s,t) = q(t) \qquad \text{and} \qquad \lim_{s \to +\infty} T(s,t) = -\infty.$$

In above definition, we say $p(t)$ positive and $q(t)$ negative. The finite energy strip condition (b) is needed here to guarantee the convergence of u at infinity (see [1]) and to obtain compactness of the moduli space in the sense of Gromov compactness [20] (Also note that $u(\mathbb{D})$ is contained in a compact region $K \subset \mathbb{R}^3$ by condition (d)).

We know that for any symplectic manifold (M, ω) there is an almost complex structure J compatible with ω, i.e. $\exists(1,1)$-tensor J such that

(a) $\qquad J \in \mathrm{End}(TM) \qquad \text{and} \qquad J^2 = -\mathrm{Id}.$

(b) $\omega(\cdot, J\cdot)$ defines a Riemannian metric on M.

Now, $(M, \omega) = (\mathbb{R}^3 \times \mathbb{R}, de^T\alpha)$, we choose a basis $\{\partial_X, \partial_Y, \partial_Z, \partial_T\}$ for $T(\mathbb{R}^3 \times \mathbb{R})$ and an almost complex structure J on $\mathbb{R}^3 \times \mathbb{R}$ so that the matrix representation of such a J with respect to this basis is given as follows.

$$J = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -Y & 0 & 1 \\ Y & 0 & -1 & 0 \end{pmatrix}$$

Proposition 3.2.2 *Above J is an almost complex structure on $\mathbb{R}^3 \times \mathbb{R}$ and compatible with $\omega = d(e^T\alpha)$. J is also invariant under translation along T-direction.*

Proof. It is easy to verify that $J^2 = -\mathrm{Id}$ and J is smooth, so J is an almost complex structure on $\mathbb{R}^3 \times \mathbb{R}$. Note that the symplectic form $\omega = d(e^T\alpha) = e^T(dT \wedge dZ - YdT \wedge dX + dX \wedge dY)$, so it has the following matrix representation with respect to the basis $\{\partial_X, \partial_Y, \partial_Z, \partial_T\}$:

$$\omega = \begin{pmatrix} 0 & e^T & 0 & Ye^T \\ -e^T & 0 & 0 & 0 \\ 0 & 0 & 0 & -e^T \\ -Ye^T & 0 & e^T & 0 \end{pmatrix}$$

Now, in order to check compatibility, it suffices to check that $\omega(w, Jv)$ defines a Riemann metric on $\mathbb{R}^3 \times \mathbb{R}$ for any $v, w \in T(\mathbb{R}^3 \times \mathbb{R})$. Actually, the condition is equivalent to the condition that the following matrix is positive definite.

$$\begin{pmatrix} 0 & e^T & 0 & Ye^T \\ -e^T & 0 & 0 & 0 \\ 0 & 0 & 0 & -e^T \\ -Ye^T & 0 & e^T & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -Y & 0 & 1 \\ Y & 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} e^T + Y^2e^T & 0 & -Ye^T & 0 \\ 0 & e^T & 0 & 0 \\ -Ye^T & 0 & e^T & 0 \\ 0 & 0 & 0 & e^T \end{pmatrix}$$

It is easy to see that above symmetric matrix on the right hand is positive definite. Hence J is compatible with ω. Since J is independent of T, so it is invariant under T-translation. ∎

Thus, every J-holomorphic disk is a part of vertical invariant family. So we consider the space $\widehat{\mathcal{M}}(p,q) := \mathcal{M}(p,q)/\mathbb{R}$.

Proposition 3.2.3 *With above complex structure J and coordinates, a smooth map $\tilde{u} = (X,Y,Z,T)$ is a J-holomorphic disk if and only if $\tilde{u}$ is the solution of the following equation system:*

$$\begin{cases} X_s - Y_t &= 0 \\ Y_s + X_t &= 0 \\ Z_s + T_t &= YY_t \\ T_s - Z_t &= -YX_t \end{cases} \tag{3.3}$$

Proof. If $\tilde{u} = (X,Y,Z,T)$ is a J-holomorphic disk, then it satisfies Cauchy-Riemann equation: $\dfrac{\partial \tilde{u}}{\partial s} = -J(\tilde{u}(s,t))\dfrac{\partial \tilde{u}}{\partial t}$. Now if we choose the almost complex structure J as above, then we have the following

$$\begin{pmatrix} X_s \\ Y_s \\ Z_s \\ T_s \end{pmatrix} = -\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -Y & 0 & 1 \\ Y & 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} X_t \\ Y_t \\ Z_t \\ T_t \end{pmatrix} = \begin{pmatrix} Y_t \\ -X_t \\ YY_t - T_t \\ -YX_t + Z_t \end{pmatrix}$$

This is the equation system (3.3). Thus the proposition follows. ∎

Therefore, the moduli space $\widehat{\mathcal{M}}(p,q)$ is the set of the solutions to equation system (3.3) satisfying Lagrangian boundary condition and asymptotic properties at Reeb chords $p(t)$ and $q(t)$ as described in the definition 3.2.2. The important feature of the moduli space $\widehat{\mathcal{M}}(p,q)$ is that it is a finite dimensional smooth manifold and can be compactified. Especially, if the difference between indices of Reeb chords p and q is one, then the moduli space we consider is compact.

Lemma 3.2.1 **([9])** *The moduli space* $\widehat{\mathcal{M}}(p,q)$ *is a manifold of dimension* $\dim\widehat{\mathcal{M}}(p,q) = |p| - |q| - 1$.

The moduli space we consider here may not be compact, but it can be compactified by adding to it broken J-holomorphic curves. This is similar to convergence to broken trajectories in Floer homology [16, 17]. In order to compactify the moduli space, we prove a version of the Gromov compactness [20]. To this end, we first define broken J-holomorphic curve. For convenience, we denote $\mathbb{D}_j^* = \mathbb{D} \setminus \{p^j, q^j\}$.

Definition 3.2.3 *A broken curve* $\tilde{u} = (u,T) = ((u^1,T^1),(u^2,T^2),\cdots,(u^N,T^N))$ *is union of J-holomorphic disks,* $\tilde{u}^j = (u^j,T^j) : (\mathbb{D}_j^*, \partial\mathbb{D}_j^*) \longrightarrow (\mathbb{R}^3 \times \mathbb{R}, L \times \mathbb{R})$*. To each* p^j *with* $j \geq 2$ *is associated a puncture* q^l *for some* $l \neq j$ *such that* $\lim_{z\to p^j} u^j(z) = \lim_{z\to q^l} u^l(z)$.

Remark 3.2.3 *In fact, the domain of a broken curve* $\tilde{u}$ *is a connected, simply-connected union of unit disks* $\mathbb{D}_1^* \cup \mathbb{D}_2^* \cdots \cup \mathbb{D}_N^*$ *by identifying* p^j *and* q^l *as in above definition. It can be parametrized by a smooth curve* $\tilde{v} : (\mathbb{D}^*, \partial\mathbb{D}^*) \longrightarrow (\mathbb{R}^3 \times \mathbb{R}, L \times \mathbb{R})$. *(see Figure 3.3)*

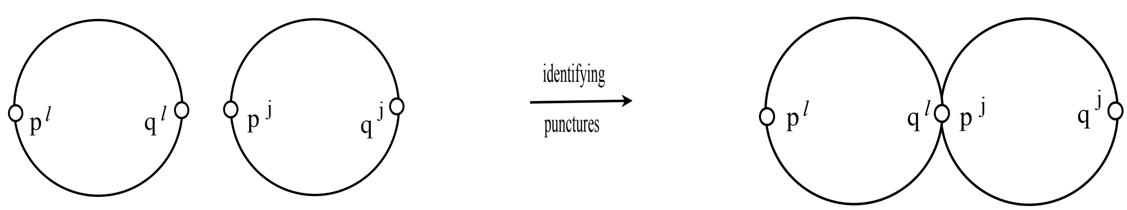


Figure 3.4: The domain of a broken J-holomorphic curve

Definition 3.2.4 *A sequence of J-holomorphic disks* $\tilde{u}_n = (u^n, T^n)$ *converges to a broken curve* $\tilde{u} = (u,T) = ((u^1,T^1),(u^2,T^2),\cdots,(u^N,T^N))$ *if the following holds.*

(1) For each $k \leq N$, there exists a sequence $\phi_n^k : \mathbb{D}_k^* \longrightarrow \mathbb{D}_k^*$ of conformal maps and a finite set $X^k \subset \mathbb{D}_k^*$ such that $\tilde{u}_n \circ \phi_n^k$ converges to $\tilde{u}^k$ uniformly in the C^∞-topology on compact subsets of $\mathbb{D}_k^* \setminus X^k$.

(2) There is a sequence $f_n : \mathbb{D}^* \longrightarrow \mathbb{D}^*$ of diffeomorphisms such that $\tilde{u}_n \circ f_n$ converges to $\tilde{v}$ (the parametrization of $\tilde{u}$) in the C^0-topology.

Remark 3.2.4 *For the second convergence in above definition, $\tilde{u}$ may be parametrized as in remark 3.2.3. The definition of broken curves and convergence here are the same as those given in* [5, 7]. *The finite set $X^k \subset \mathbb{D}_k^*$ is the set of singularities of J-holomorphic disks.* (see [20, 26])

Proposition 3.2.4 *Any sequence $\tilde{u}_n$ in $\widehat{\mathcal{M}}(p, q)$ has a subsequence converging to a broken holomorphic curve $\tilde{u} = (\tilde{u}^1, \tilde{u}^2, \cdots, \tilde{u}^N)$ in the sense of definition 3.2.4.*

Before giving the proof of the statement, we recall some facts about sequence $\tilde{u}_n$ in $\widehat{\mathcal{M}}(p, q)$, Let $\tilde{u}_n = (X^n, Y^n, Z^n, T^n)$, then the coordinates of $\tilde{u}_n$ satisfy the equation

$$\begin{cases} X_s^n - Y_t^n &= 0 \qquad (3.4.1) \\ Y_s^n + X_t^n &= 0 \qquad (3.4.2) \\ Z_s^n + T_t^n &= Y^n Y_t^n \qquad (3.4.3) \\ T_s^n - Z_t^n &= -Y^n X_t^n \qquad (3.4.4) \end{cases} \tag{3.4}$$

by the proposition 3.2.3. If we denote by $\bar{u}_n = (X^n, Y^n)$ the first two components of $\tilde{u}_n$, then the energy of $\bar{u}_n$ is also finite. Since $\tilde{u}$ has finite energy: $E(\tilde{u}_n) = \int_\Sigma \tilde{u}_n^* d(e^T \alpha) < \infty$, it follows that $\bar{u}_n$ has finite energy: $E(\bar{u}_n) = \int_\Sigma \bar{u}_n^* d(\alpha) < \infty$. Also we have to note that $\tilde{u}_n$ satisfies the Lagrangian boundary condition, hence $\bar{u}_n = (X^n, Y^n)$ has to satisfies the same boundary conditions. By previous section, we know this condition is x and y coordinates of Legendrian knot L, i.e. $\bar{u}_n|_{\partial\Sigma} \subset \Pi(L)$. Now, if we know the convergence of the sequence of $\bar{u}_n$, then by the equation system

(3.4) and boundary conditions, we can know the convergence of $\tilde{u}_n$. Note that for Reeb chords $p(t)$ and $q(t)$, they correspond to crossings of Lagrangian projection of Legendrian knot L, we denote these crossings by p and q .

To prove this proposition, we need the following lemma given in [5]. Here we also need some notations, by above discussion, we may have a new moduli space in terms of $\bar{u}_n$, denote by $\mathbb{M}(p,q)$. We also call the element of $\mathbb{M}(p,q)$ holomorphic disk.

Definition 3.2.5 *The moduli space* $\mathbb{M}(p,q)$ *is the set of smooth maps :*

$$\begin{aligned} \bar{u}(s,t): \quad (\Sigma, \partial\Sigma) &\longrightarrow (\mathbb{C}, \Pi(L)) \\ (s,t) \quad &\longmapsto (X,Y) \end{aligned}$$

Satisfying the following conditions.

(a) $\bar{u}$ *is holomorphic,*

(b) $\bar{u}$ *has finite energy:* $E(\bar{u}) = \int_\Sigma \bar{u}^* d(\alpha) < \infty$,

(c) $\bar{u}$ *satisfies the Lagrangian boundary :* $\bar{u}|_{\partial\Sigma} \subset \Pi(L)$. *i.e.* $\bar{u}(s,0), \bar{u}(s,1) \subset \Pi(L)$,

(d) Asymptotic properties :

$$\lim_{s\to-\infty} \bar{u}(s,t) = p \quad \textit{and} \quad \lim_{s\to+\infty} \bar{u}(s,t) = q.$$

Lemma 3.2.2 ([5]) *Assume* $\bar{u}_n \in \mathbb{M}(p,q)$ *is a sequence of holomorphic disks, then there exists a subsequence* $\bar{u}_{n_j}$ *converging to a broken curve* $\bar{u}$ *in the sense of definition 3.2.4.*

With this lemma and above discussion, now we prove proposition 3.2.4.

Proof. of **Proposition 3.2.4**: given any sequence $\tilde{u}_n \in \widehat{\mathcal{M}}(p,q)$, we denote by $\tilde{u}_n = (X^n, Y^n, Z^n, T^n)$, then we have an associated sequence $\bar{u}_n = (X^n, Y^n)$, by the equation (3.4) and the definition of the moduli space $\widehat{\mathcal{M}}(p,q)$, we have that $\bar{u}_n = (X^n, Y^n) \in \mathbb{M}(p,q)$. By lemma 3.2.2, there exists a subsequence $\bar{u}_{n_j} = (X^n_j, Y^n_j)$

converging to a broken curve $\bar{u}$ in the sense of definition 3.2.4. Now, we can obtain a subsequence $\tilde{u}_{n_j} \in \widehat{\mathcal{M}}(p, q)$ by solving equation (3.4) with the Lagrangian boundary conditions of $\tilde{u}_n$. Since the solution to equation(3.4) is unique by PDE, hence we have the subsequence $\tilde{u}_{n_j} = (X_j^n, Y_j^n, Z_j^n, T_j^n) \in \widehat{\mathcal{M}}(p, q)$ such that $\bar{u}_{n_j} = (X_j^n, Y_j^n)$ converging to a broken curve $\bar{u} = (x, y)$. By the definition 3.2.4, if the first convergence holds, i.e. for the broken curve $\bar{u} = (\bar{u}_1, \bar{u}_2, \dots \bar{u}_N)$, for each $k \leq N$, there exists a sequence $\phi_{n_j}^k : \mathbb{D}_k^* \longrightarrow \mathbb{D}_k^*$ of conformal maps and a finite set $X^k \subset \mathbb{D}_k^*$ such that $\bar{u}_{n_j} \circ \phi_{n_j}^k$ converges to $\bar{u}_k$ uniformly in the C^∞-topology on compact subsets of $\mathbb{D}_k^* \setminus X^k$. Here $\bar{u}_k \in \mathbb{M}(p^k, q^k)$, hence we can use $\bar{u}_k$ to construct a broken curve $\tilde{u}_k$ as follows uniquely. First, we take X and Y coordinates of $\tilde{u}_k$ to be $\bar{u}_k$, then we solve the equation (3.4) to obtain Z and T coordinates, thus we have $\tilde{u}_k = (x, y, Z, T)$. Also note that $\bar{u}_{n_j} \circ \phi_{n_j}^k$ converges to $\bar{u}_k$ in the C^∞-topology, so $\tilde{u}_{n_j} \circ \phi_{n_j}^k = (X_j^n \circ \phi_{n_j}^k, Y_j^n \circ \phi_{n_j}^k, Z_j^n \circ \phi_{n_j}^k, T_j^n \circ \phi_{n_j}^k)$ must converge to $\tilde{u}_k = (x, y, Z, T)$ by the same way. ∎

Proposition 3.2.5 *If $r(L) \neq 0$ and $|p| = |q| + 2$, then the moduli space $\widehat{\mathcal{M}}(p, q)$ is one dimensional compact manifold up to broken J-holomorphic disks.*

Proof. In [28] proves that there are no bubbling-off disks and spheres under the assumption $r(L) \neq 0$ (i.e. the minimal Maslov number $\Sigma(L) \geq 2$). So any sequence of J-holomorphic disks in $\widehat{\mathcal{M}}(p, q)$ has a subsequence converging to a broken J-holomorphic disk in the sense of the first convergence in definition 3.2.4. Hence the closure of $\widehat{\mathcal{M}}(p, q)$ is compact. By lemma 3.2.1, in this case $\dim \widehat{\mathcal{M}}(p, q) = 1$. ∎

3.3 The differential and Reeb chord complex

Based on previous sections, we are in a position to define the homology of a generic Legendrian knot L. Throughout the thesis, we assume that there are only finitely many Reeb chords for the given Legendrian knot L as usual.

Let C_n be the vector space over $\mathbb{Z}_2$ generated by the set of Reeb chords with grading n, that is

$$C_n = \left\{ \sum_{i=1}^{k} n_i a_i \,|n_i \in \mathbb{Z}_2, |a_i| = n \right\}$$

where the set $\{a_1, a_2 \ldots a_{k-1}, a_k\}$ is all Reeb chords of L with grading n. For any generator $a \in C_n$, we define the differential $\partial_n : C_n \longrightarrow C_{n-1}$ as follows.

$$\partial_n a = \sum_{|b|=n-1} \#_{\mathbb{Z}_2} \widehat{\mathcal{M}}(a,b) b \tag{3.5}$$

We then extend ∂_n by linearity to C_n. Where $\#_{\mathbb{Z}_2}\widehat{\mathcal{M}}(a,b)$ is the number of points in $\widehat{\mathcal{M}}(a,b)$ modulo 2, and where the sum ranges over all generators $b \in C_{n-1}$.

We will show that the differential ∂ is well-defined and (C_*, ∂_*) is a chain complex, then we define the homology $CH_*(L)$.

Proposition 3.3.1 *Under the assumption $r(L) \neq 0$, the differential $\partial_n : C_n \longrightarrow C_{n-1}$ satisfies $\partial_{n-1} \circ \partial_n = 0$ and $\{(C_n, \partial_n)\}_{n \in \mathbb{Z}}$ is a chain complex*

Proof. : First, by assumption, there are only finitely many Reeb chords and J-holomorphic curves in $\widehat{\mathcal{M}}(a,b)$ are non-parametrized (i.e. Two J-holomorphic curves f, g in $\widehat{\mathcal{M}}(a,b)$ are equivalent if there is a conformal map $\phi : \mathbb{D} \setminus \{a,b\} \longrightarrow \mathbb{D} \setminus \{a,b\}$ such that $f = g \circ \phi$). By the proposition 3.2.4 and lemma 3.2.1, we have that $\widehat{\mathcal{M}}(a,b)$ is a compact zero-dimensional manifold. Hence $|\widehat{\mathcal{M}}(a,b)|$ is finite. So the differential ∂ is well-defined. By (3.5), we have

$$\begin{aligned} \partial_{n-1}\partial_n(a) &= \partial_{n-1}\left(\sum_{|b|=n-1} \#_{\mathbb{Z}_2}\widehat{\mathcal{M}}(a,b) b \right) \\ &= \sum_{|c|=n-2} \left(\sum_{|b|=n-1} \#_{\mathbb{Z}_2}\widehat{\mathcal{M}}(a,b) \cdot \#_{\mathbb{Z}_2}\widehat{\mathcal{M}}(b,c) \right) c \end{aligned} \tag{3.6}$$

Given two Reeb chords , **a** of index n and **c** of index $n-2$, we will show that the sum $\sum_{|c|=n-2} \sum_{|b|=n-1} \#_{\mathbb{Z}_2}\widehat{\mathcal{M}}(a,b) \cdot \#_{\mathbb{Z}_2}\widehat{\mathcal{M}}(b,c)$ is zero, then It follows that $\partial^2 = 0$. Showing this has two steps.

Step 1. Gluing of J-holomorphic curves. Given a pair of J-holomorphic curves :

$$\tilde{u} = (u, T) = (x, y, z, T) \in \widehat{\mathcal{M}}(a, b)$$
$$\tilde{v} = (v, \tilde{T}) = (\tilde{x}, \tilde{y}, \tilde{z}, \tilde{T}) \in \widehat{\mathcal{M}}(b, c)$$

Then $\lim_{s \to +\infty} u(s,t) = b(t) = \lim_{s \to -\infty} v(s,t)$. Also, note that both $u(s,t)|_{\partial\Sigma} \subset L$ and $v(s,t)|_{\partial\Sigma} \subset L$ are smooth away from punctures and $\lim_{s \to +\infty} u(s,i) = b(i) = \lim_{s \to -\infty} v(s,i)$ where $i = 0, 1$. So $u(s,i)$ and $v(s,i)$ lie on the same strand of L near $b(i)$. Now, choose ρ sufficiently large, let α: $\mathbb{R} \longrightarrow [0,1]$ be a cut-off function such that

$$\alpha(s)|_{s \leq \rho} \equiv 1 \qquad \alpha(s)|_{s \geq \rho+1} \equiv 0.$$

Similarly, let β: $\mathbb{R} \longrightarrow [0,1]$ be a cut-off function such that

$$\beta(s)|_{s \leq -\rho-1} \equiv 0 \qquad \beta(s)|_{s \geq -\rho} \equiv 1.$$

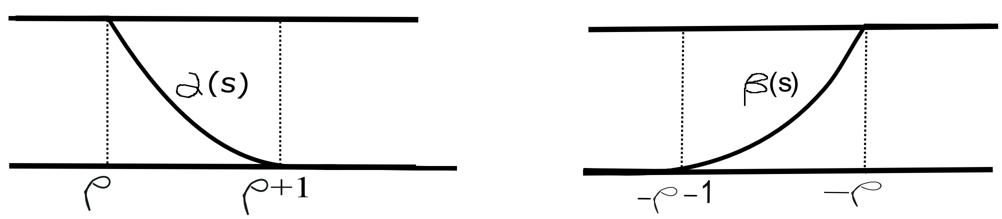


Figure 3.5: Two cut-off functions $\alpha(s)$ and $\beta(s)$

We glue $\tilde{u}$ and $\tilde{v}$ together to obtain a one-parameter family of approximate solutions W_ρ of equation 3.3 as follows.

$$W_\rho = \tilde{u} \#_\rho \tilde{v} = \begin{cases} \tilde{u}(s,t), & s \leq \rho \\ \alpha(s)\tilde{u}(s,t) + \beta(s - 2\rho - 1)\tilde{v}(-s,t), & \rho \leq s \leq \rho + 1 \\ \tilde{v}(s - 2\rho - 2, t). & s \geq \rho + 1 \end{cases}$$

By the conditions on u and v, we have the following two cases. See Figure 3.6

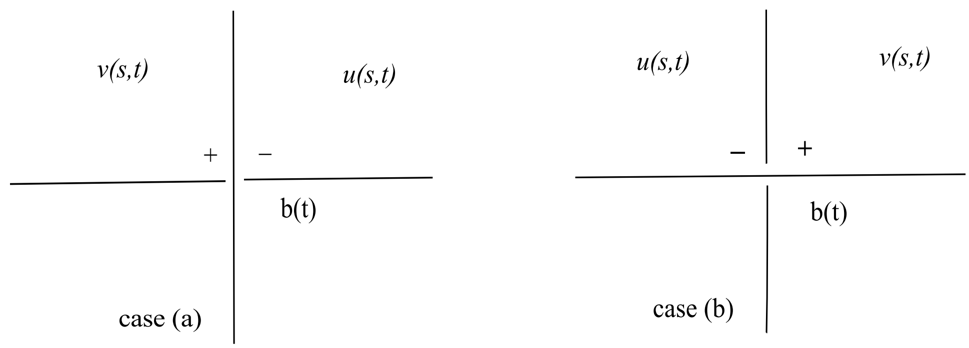


Figure 3.6: There are two cases (a) and (b) for the gluing

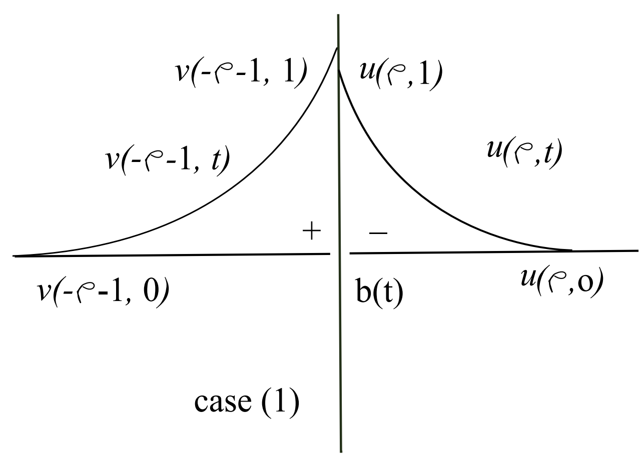


Figure 3.7: One of the cases for the case (a)

For case (a), there are three cases in terms of the endpoints of $u(s, 1)$ and $v(-s, 1)$ over $[\rho, \rho + 1]$ (See Figure 3.7 and Figure 3.8). For case (b), there are also three cases in terms of the endpoints of $u(s, 0)$ and $v(-s, 0)$ over $[\rho, \rho + 1]$. Here, we only prove for case (1), one of the cases of the case (a).(See Figure 3.7.) The proof of the other two cases is the same, the only difference is the endpoints of $u(s, 1)$ and $v(-s, 1)$ over $[\rho, \rho + 1]$. Based on W_ρ, we will construct a family of J-holomorphic curves $M_\rho = W_\rho + \tilde{\eta}_\rho \in \widehat{\mathcal{M}}(a, c)$ for some $\tilde{\eta}_\rho$. Where

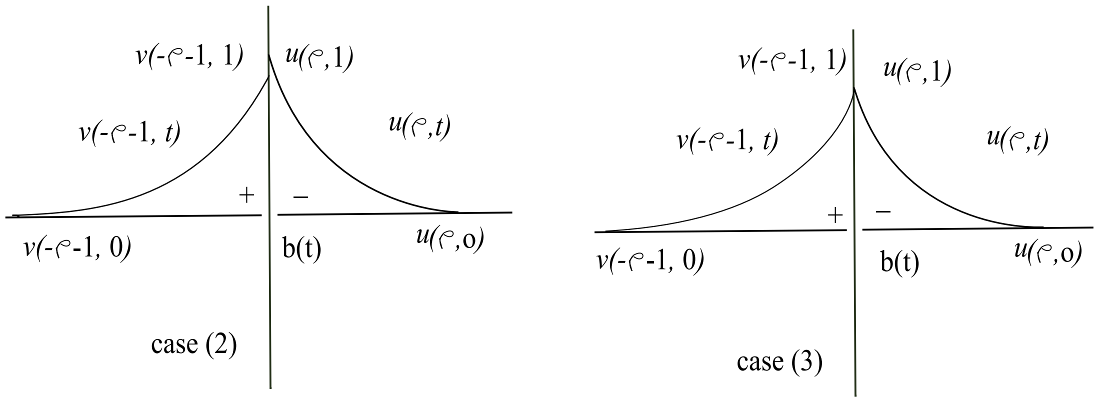


Figure 3.8: The other two cases for the case (a)

$$\tilde{\eta}_\rho = (\eta_\rho, \bar{T}) \quad : \quad \Sigma \quad \longrightarrow \mathbb{R}^3 \times \mathbb{R}$$

is a smooth map such that

(a) $\bar{\partial}_J(W_\rho + \tilde{\eta}_\rho) = 0$

(b) $\tilde{\eta}_\rho|_{\Sigma \setminus \Omega} = 0$, where $\Omega = [\rho, \rho+1] \times [0,1]$

(c) η_ρ satisfies the following boundary conditions on Ω

$$\begin{cases} \alpha(s)u(s,0) + \beta(s-2\rho-1)v(-s,0) + \eta_\rho(s,0) = \gamma_0(s) \\ \alpha(s)u(s,1) + \beta(s-2\rho-1)v(-s,1) + \eta_\rho(s,1) = \gamma_1(s) \\ \eta_\rho|_{\{\rho,\rho+1\}\times[0,1]} = 0. \end{cases}$$

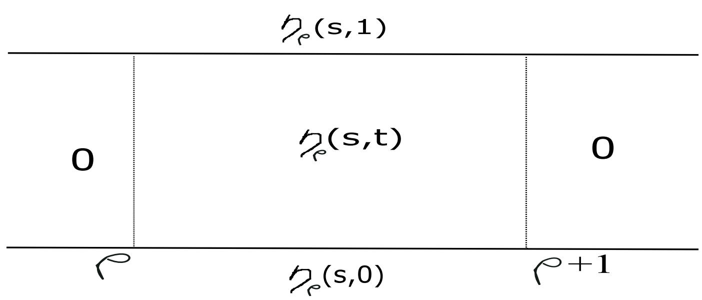


Figure 3.9: $\tilde{\eta}_\rho = 0$ on $\Sigma \setminus \Omega$ and η_ρ satisfies the boundary conditions on Ω.

Where $\gamma_0(s)$ and $\gamma_1(s)$ are two reparametrizations of Legendrian knot L of pieces of $[v(-\rho-1,1), b(1)]$ and $[v(-\rho-1,0), u(\rho,0)]$ over $[\rho,\rho+1]$ respectively. If such an $\tilde{\eta}_\rho$ exists uniquely and only depends on ρ, then $M_\rho = W_\rho + \eta_\rho \in \widehat{\mathcal{M}}(a,c)$. So we have a family of solutions of (3.1). This shows that ends of moduli space $\widehat{\mathcal{M}}(a,c)$ correspond to pairs of connecting J-holomorphic curves $\tilde{u} \in \widehat{\mathcal{M}}(a,b)$ and $\tilde{v} \in \widehat{\mathcal{M}}(b,c)$, Also $\widehat{\mathcal{M}}(a,c)$ is a compact one-dimensional manifold. Hence one obtains

$$\begin{aligned}\partial_{n-1}\partial_n a &= \sum_{|c|=n-2}\left(\sum_{|b|=n-1} \#_{\mathbb{Z}_2}\widehat{\mathcal{M}}(a,b)\cdot \#_{\mathbb{Z}_2}\widehat{\mathcal{M}}(b,c)\right) c \\ &= 0\end{aligned}$$

Step 2. The existence of $\tilde{\eta}_\rho$. The rest of the proof is to show the existence of such an $\tilde{\eta}_\rho$. Suppose that

$$\begin{aligned}\tilde{\eta}_\rho = (\eta_\rho, \bar{T}) \quad : \Sigma \quad &\longrightarrow \mathbb{R}^3\times\mathbb{R} \\ (s,t) \longmapsto \quad &(\bar{x},\bar{y},\bar{z},\bar{T})\end{aligned}$$

If we set $M_\rho = (X, Y, Z, \mathfrak{T})$, then $M_\rho = W_\rho + \tilde{\eta}_\rho$ has coordinates as follows.

$$\begin{cases} X = \alpha(s)x(s,t) + \beta(s-2\rho-1)\tilde{x}(-s,t) + \bar{x}(s,t) \\ Y = \alpha(s)y(s,t) + \beta(s-2\rho-1)\tilde{y}(-s,t) + \bar{y}(s,t) \\ Z = \alpha(s)z(s,t) + \beta(s-2\rho-1)\tilde{z}(-s,t) + \bar{z}(s,t) \\ \mathfrak{T} = \alpha(s)T(s,t) + \beta(s-2\rho-1)\tilde{T}(-s,t) + \bar{T}(s,t) \end{cases} \tag{3.7}$$

Note that $\tilde{\eta}_\rho$ satisfies the equation $\bar{\partial}_J(W_\rho + \tilde{\eta}_\rho) = 0$, It is equivalent to the following.

$$\begin{cases} X_s - Y_t \quad = 0 \qquad\qquad (3.8.1) \\ Y_s + X_t \quad = 0 \qquad\qquad (3.8.2) \\ Z_s + \mathfrak{T}_t \quad = YY_t \qquad\quad (3.8.3) \\ \mathfrak{T}_s - Z_t \quad = -YX_t \qquad (3.8.4) \end{cases} \tag{3.8}$$

Combining with (3.7) then (3.8.1) and (3.8.2) are equivalent to the following equations in terms of $\bar{x}$ and $\bar{y}$ coordinates of $\tilde{\eta}_\rho$ over Ω

$$\begin{cases} \bar{x}_s - \bar{y}_t = -(\alpha' x + \alpha x_s + \beta' \tilde{x} - \beta \tilde{x}_s - \alpha y_t - \beta \tilde{y}_t) = h_1 \\ \bar{y}_s + \bar{x}_t = -(\alpha' y + \alpha y_s + \beta' \tilde{y} - \beta \tilde{y}_s + \alpha x_t + \beta \tilde{x}_t) = h_2 \end{cases} \tag{3.9}$$

Based on these equations, we can derive Possion equation for $\bar{x}$ as follows

$$\begin{cases} \triangle \bar{x} = f_1 & \text{on } \Omega \\ \bar{x}|_{\partial\Omega} = g_1 & \text{on } \partial\Omega \end{cases} \tag{3.10}$$

Where $f_1 = \partial_s h_1 - \partial_t h_2$ and g_1 is a function over $\partial\Omega$ given by the condition (c) of $\tilde{\eta}_\rho$ Then, by [15] we know that there is a unique solution for $\bar{x}$ over Ω with the prescribed boundary condition.

Similarly, we can find $\bar{y}$ and $\bar{z}$ coordinates of $\tilde{\eta}_\rho$ by solving Possion equations respectively

$$\begin{cases} \triangle \bar{y} = f_2 & \text{in } \Omega \\ \bar{y}|_{\partial\Omega} = g_2 & \text{on } \partial\Omega \end{cases}$$

and

$$\begin{cases} \triangle \bar{z} = f_3 & \text{in } \Omega \\ \bar{z}|_{\partial\Omega} = g_3 & \text{on } \partial\Omega \end{cases}$$

Where f_i and g_i $(i = 2, 3)$ are obtained by the same way as (3.10). We have found coordinates of $\eta_\rho = (\bar{x}, \bar{y}, \bar{z})$. By the equations (3.8.3) and (3.8.4), using Poincare lemma, we can find $\bar{T}$ uniquely up to adding a constant, it is fine with moduli space $\widehat{\mathcal{M}}(a, c)$ since we modulo $\mathbb{R}$-direction. So far we have found $\tilde{\eta}_\rho = (\bar{x}, \bar{y}, \bar{z}, \bar{T})$ uniquely and $\tilde{\eta}_\rho$ only depends on ρ. Therefore, this shows that (C_n, ∂_n) is a chain complex. This completes the proof. ■

We call the chain complex (C_n, ∂_n) Reeb chord complex of Legendrian knots. Now, we are in a position to define the integral graded homology of Reeb chord complex as follows

$$CH_n(L) = \frac{\mathrm{Ker}\partial_n}{\mathrm{Im}\partial_{n+1}} \tag{3.11}$$

We call $CH_*(L)$ the integral graded homology of Reeb chord complex of Legendrian knot L. This is a $\mathbb{Z}-$graded vector space over $\mathbb{Z}_2$. We will prove that the homology $CH_*(L)$ is a topological invariant of Legendrian knots in the next chapter.

CHAPTER 4

Invariance of The Integral Graded Homology of Reeb Chord Complex of Legendrian Knots

The aim for this chapter is to prove the following main theorem of the thesis.

Theorem 4.0.1 *Under the assumption $r(L) \neq 0$, the homology $CH_n(L) = \dfrac{\text{Ker}\partial_n}{\text{Im}\partial_{n+1}}$ of Reeb chord complex is a topological invariant of Legendrian knots for all $n \in \mathbb{Z}$.*

The proof is based on the following three sections, it turns out that $CH_*(L)$ is unchanged under the Legendrian Reidemeister moves. This shows that the homology $CH_*(L)$ is a Legendrian invariant.

Throughout the chapter, we choose a small neighbourhood of Reeb chords such that all Legendrian Reidemeister moves take place in this small neighbourhood. Meanwhile, we use L' to stand for the Legendrian knot L after we perform one of Legendrian Reidemeister moves and we use $\tilde{\partial}$ and ∂ to represent the corresponding differentials. For convenience, we use $\mathcal{M}(L)$ and $\mathcal{M}(L')$ to denote the set of all J-holomorphic disks corresponding to L and L' respectively.

4.1 Invariance of Move I

In this section, we prove that the homology $CH_*(L)$ is unchanged under Legendrian Reidemeister move I (See Figure 4.1). We show that there is a one-to-one correspondence between the set of J-holomorphic disks in corresponding moduli spaces before and after move I, this means that $\tilde{\partial} = \partial$. Since there are no new-born Reeb chords in move I, we number all Reeb chords in the same way for both knots L and L'.

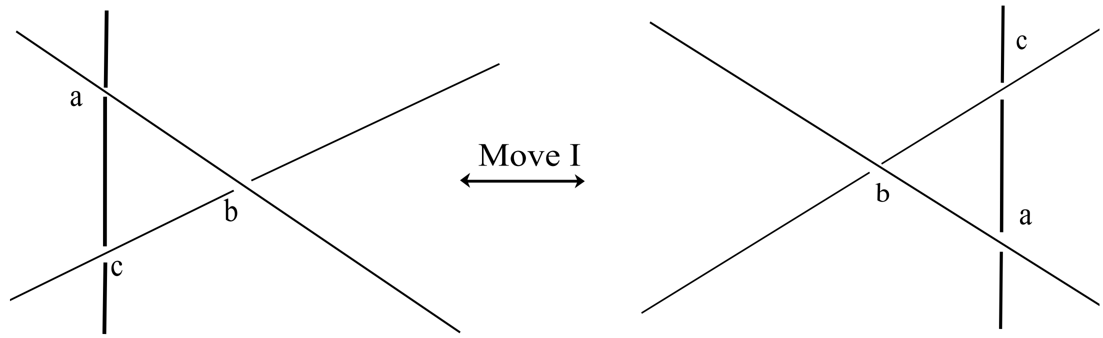


Figure 4.1: Legendrian Reidemeister Move I

Proposition 4.1.1 *The homology $CH_*(L)$ is unchanged under Legendrian Reidemeister move* I.

Proof. First, let's look at a J-holomorphic disk with the Reeb chord **a** as one of asymptotes and covers the corner shown in Figure 4.2

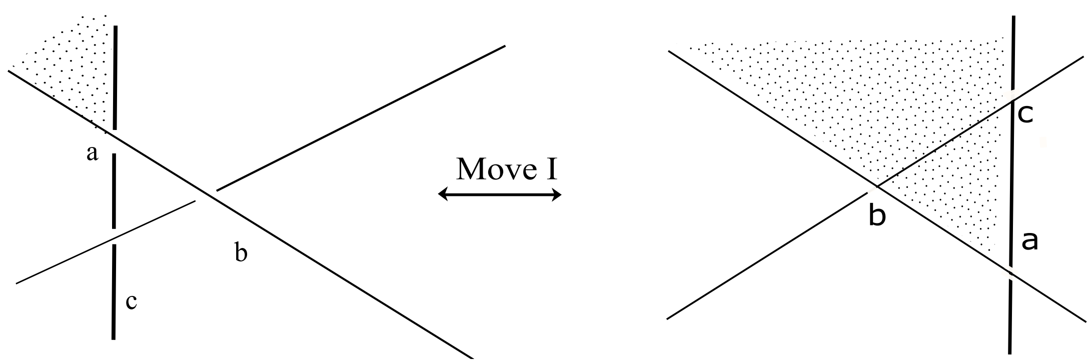


Figure 4.2: Correspondence between J-holomorphic disks with one of asymptotes at **a** in Legendrian Reidemeister Move I

If there is a J-holomorphic disk with Reeb chord **a** as one of asymptotes shown in the first picture of Figure 4.2, then after we perform move I, then there is a corresponding J-holomorphic disk with Reeb chord **a** as one of asymptotes shown in the second picture of Figure 4.2. Conversely, this is also true. Thus there is a one-to-one correspondence between J-holomorphic disks in this case. Similarly, when the segment $[a, c]$ is not the boundary of J-holomorphic disks, we can show that there is a one-to-one correspondence between J-holomorphic disks before and after Legendrian Reidemeister move I.

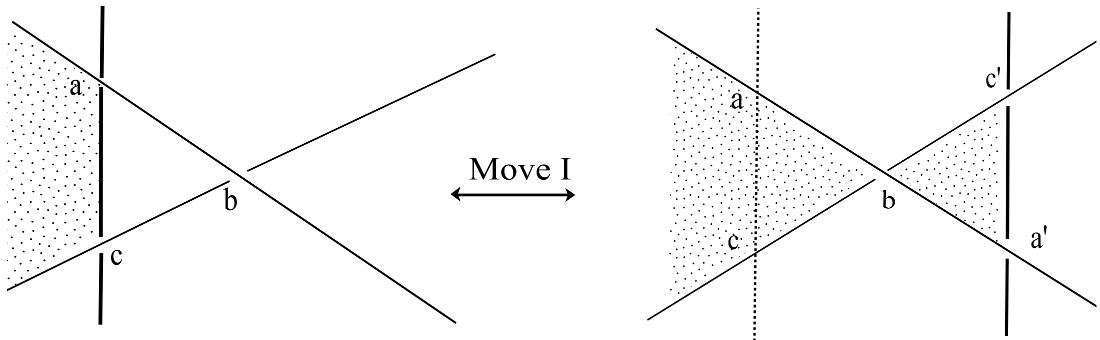


Figure 4.3: Correspondence between J-holomorphic disks with asymptotes at **a** and **c** in Legendrian Reidemeister Move I

If the segment $[a, c]$ is the boundary of a J-holomorphic disk (see Figure 4.3), we will also show that there is a one-to-one correspondence between J-holomorphic disks before and after Legendrian Reidemeister move I. Suppose that there is a J-holomorphic disk u with $[a, c]$ as one of boundaries and **a**, **c** as asymptotes shown in the first picture of Figure 4.3, then after we perform Legendrian Reidemeister move I, we shall show that there is a corresponding J-holomorphic disk v shown in the second picture of Figure 4.3. In fact, to obtain such a J-holomorphic disk v from u, It suffices to show there is a J-holomorphic disk as shown in the Figure 4.4, we glue this disk with u along $[a, c]$, then we obtain the corresponding J-holomorphic disk v. Thus we show that there is a one-to-one correspondence between J-holomorphic disks before and after Reidemeister move I. In order to show the existence of such a J-holomorphic disk, let's look at the following local model. We choose a small cube with edge ε (see the first picture of Figure 4.4).

Let $Q = [0, \varepsilon] \times [0, \varepsilon]$ and we parametrize Q by (s, t). For this local model, we have two concrete J-holomorphic disks as follows.

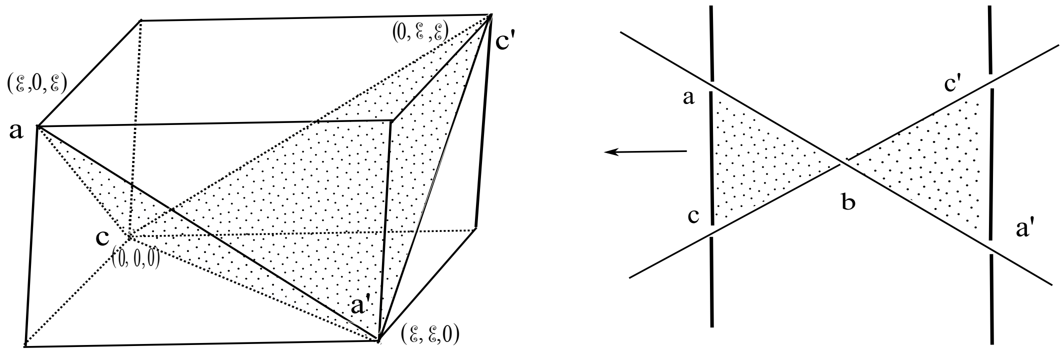


Figure 4.4: Local picture for the existence of a J-holomorphic disk in move I

$$\text{If } s \leq t\ , \qquad W_1(s,t) = \begin{cases} x = s \\ y = t \\ z = t - s \\ T = \dfrac{1}{2}t^2 + t + s \end{cases} \tag{4.1}$$

$$\text{If } s \geq t\ , \qquad W_2(s,t) = \begin{cases} x = s \\ y = t \\ z = s - t \\ T = \dfrac{1}{2}t^2 - t - s \end{cases} \tag{4.2}$$

It is easy to verify that both W_1 and W_2 satisfy equations (3.1). $\overline{W}_1$ satisfies boundary condition given by the boundary of triangle $(a'c'c)$ and $\overline{W_2}$ satisfies boundary condition given by boundary of triangle (caa'). Where $\overline{W}_i$ are (x, y, z) coordinates of W_i $(i = 1, 2)$ in $\mathbb{R}^3$. We also notice that $\overline{W}_1$ and $\overline{W}_2$ meet at edge cc'. In order to obtain a J-holomorphic disk based on this local model. First, choose $\delta > 0$ sufficiently small, let

$$\Omega = \{(s,t) \in Q \mid 0 \le s \le \delta, \quad 0 \le t \le s+\delta\}$$
$$\cup \{(s,t) \in Q \mid \delta \le s \le \varepsilon-\delta, \quad s-\delta \le t \le s+\delta\}$$
$$\cup \{(s,t) \in Q \mid \varepsilon-\delta \le s \le \varepsilon, \quad s-\delta \le t \le \varepsilon\}$$

and

$$A = \{(s,t) \in Q \mid 0 \le s \le \varepsilon-\delta, \quad \delta+s \le t \le \varepsilon\}$$
$$B = \{(s,t) \in Q \mid \delta \le s \le \varepsilon, \quad 0 \le t \le s-\delta\}$$

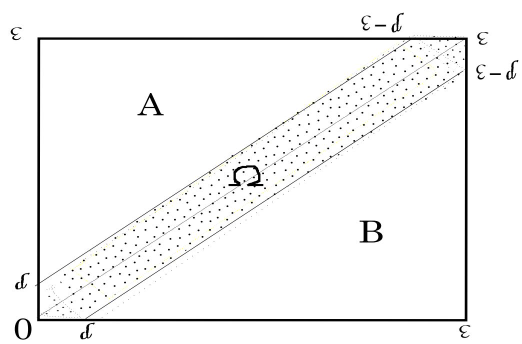


Figure 4.5: The subsets Ω, A and B of $[0,\varepsilon] \times [0,\varepsilon]$

Then we define a map W_δ: $[0,\varepsilon] \times [0,\varepsilon] \longrightarrow \mathbb{R}^3 \times \mathbb{R}$ as follows

$$W_\delta(\zeta) = \begin{cases} W_1(\zeta) & \zeta \in A \\ \alpha(\zeta)W_1(\zeta) + \beta(\zeta)W_2(\zeta) & \zeta \in \Omega \\ W_2(\zeta) & \zeta \in B \end{cases} \tag{4.3}$$

Where $\zeta = (s,t) \in Q$, α and β are two cut-off functions on Q such that

- $\alpha(\zeta)|_A = 1 \quad \text{and} \quad \alpha(\zeta)|_B = 0$
- $\beta(\zeta)|_A = 0 \quad \text{and} \quad \beta(\zeta)|_B = 1$

Based on W_δ, we will construct a J-holomorphic disk $M_\delta = W_\delta + \tilde{\eta}_\delta$ for some $\tilde{\eta}_\delta$. Where

$$\tilde{\eta}_\delta = (\eta_\delta, \tilde{T}) : \qquad Q \longrightarrow \mathbb{R}^3 \times \mathbb{R}$$

is a smooth map such that

(a) $\bar{\partial}_J(W_\delta + \tilde{\eta}_\delta) = 0 \qquad \text{on } Q = [0,\varepsilon] \times [0,\varepsilon]$

(b) $\tilde{\eta}_\delta|_{Q\setminus\Omega} = 0$

(c) η_δ on ∂Q satisfies the following boundary conditions.

$$(\overline{W}_\delta + \eta_\delta)|_{\partial Q} = \begin{cases} (s, \varepsilon, \varepsilon - s), & 0 \le s \le \varepsilon \\ (0, t, t), & 0 \le t \le \varepsilon \\ (s, 0, s), & 0 \le s \le \varepsilon \\ (\varepsilon, t, \varepsilon - t). & 0 \le t \le \varepsilon \end{cases}$$

Where $\overline{W}_\delta$ is (x, y, z) coordinates of W_δ. For convenience, we suppose $M_\delta = (X, Y, Z, T)$ and $\tilde{\eta}_\delta = (\tilde{x}, \tilde{y}, \tilde{z}, \tilde{T})$, then by above conditions on $\tilde{\eta}_\delta$, we will show that there exists such an $\tilde{\eta}_\delta$ satisfying all above conditions and $\tilde{\eta}_\delta$ only depends on parameter δ. Thus we obtain a J-holomorphic disk M_δ. By (4.1), (4.2) and (4.3), we have

$$M_\delta = W_\delta + \tilde{\eta}_\delta = \begin{cases} (s, t, t - s, \frac{1}{2}t^2 + t + s), & (s,t) \in A \\ (X(s,t), Y(s,t), Z(s,t), T(s,t)), & (s,t) \in \Omega \\ (s, t, s - t, \frac{1}{2}t^2 - t - s), & (s,t) \in B \end{cases}$$

Where X, Y, Z and T are smooth functions on Ω and are given as follows.

$$\begin{cases} X = \alpha(s,t)s + \beta(s,t)s + \tilde{x}(s,t) \\ Y = \alpha(s,t)t + \beta(s,t)t + \tilde{y}(s,t) \\ Z = (\alpha(s,t) - \beta(s,t))(t-s) + \tilde{z}(s,t) \\ T = \frac{1}{2}(\alpha(s,t) + \beta(s,t))t^2 + (\alpha(s,t) - \beta(s,t))(t+s) + \tilde{T}(s,t). \end{cases} \tag{4.4}$$

Now, combine (4.4) with the condition (a), we can derive the equation given below that η_δ has to satisfy.

$$\begin{cases} \tilde{x}_s - \tilde{y}_t = (\beta_t + \alpha_t)t - (\beta_s + \alpha_s)s, & (4.5.1) \\ \tilde{y}_s + \tilde{x}_t = -(\beta_t + \alpha_t)s - (\beta_s + \alpha_s)t, & (4.5.2) \\ \tilde{z}_s + \tilde{T}_t = \frac{1}{2}t^2(\alpha_t + \beta_t)(2\alpha + 2\beta - 1) - (\beta_s - \alpha_s + \alpha_t - \beta_t)s & \\ +(\alpha^2 + \beta^2 + \tilde{y}_t\alpha + \tilde{y}_t\beta + \tilde{y}\alpha_t + \tilde{y}\beta_t - \alpha_s + \beta_s - \alpha_t - \beta_t - \alpha - \beta)t & \\ +\tilde{y}(\alpha + \beta + \tilde{y}_t), & (4.5.3) \\ \tilde{T}_s - \tilde{z}_t = -\frac{1}{2}(\alpha_s + \beta_s)t^2 - (\alpha_s - \beta_s - \alpha_t + \beta_t + \tilde{x}_t\alpha + \tilde{x}_t\beta)t - \tilde{y}\tilde{x}_t & \\ -(\alpha_s + \beta_s - \alpha_t + \beta_t + \tilde{y}\alpha_t + \tilde{y}\beta_t)s - (\alpha + \beta)(\beta_t + \alpha_t)ts. & (4.5.4) \end{cases} \tag{4.5}$$

By (4.5.1) and (4.5.2), it follows that $\tilde{x}$ satisfies Possion equation with prescribed boundary condition given in (c).

$$\begin{cases} \triangle \tilde{x} = f_1 \quad \text{in } \Omega \\ \tilde{x}|_{\partial\Omega} = g_1 \quad \text{on } \partial\Omega \end{cases} \tag{4.6}$$

Where $f_1 = -2(\beta_s + \alpha_s) - (\beta_{ss} - \beta_{tt} + \alpha_{ss} - \alpha_{tt})s$ and g_1 is a piecewise smooth map on $\partial\Omega$. By the condition (c), we have

$$g_1 = \begin{cases} \varepsilon(1-\alpha(\varepsilon,t)-\beta(\varepsilon,t)) & s=\varepsilon, \quad \varepsilon-\delta \le t \le \varepsilon \\ s(1-\alpha(s,\varepsilon)-\beta(s,\varepsilon)) & t=\varepsilon, \quad \varepsilon-\delta \le s \le \varepsilon \\ -s(\alpha(s,s+\delta)+\beta(s,s+\delta)) & t=s+\delta, \quad 0 \le s \le \varepsilon-\delta \\ 0 & s=0, \quad 0 \le s \le \delta \\ s(1-\alpha(s,0)-\beta(s,0)) & t=0, \quad 0 \le s \le \delta \\ -s(\alpha(s,s-\delta)+\beta(s,s-\delta)) & t=s-\delta, \quad \delta \le s \le \varepsilon \end{cases}$$

For the equation (4.5), since Ω is simple connected, we know that there is a unique solution for $\tilde{x}$ over Ω.

Similarly, we can find $\tilde{y}$ and $\tilde{z}$ coordinates of $\tilde{\eta}_\rho$ by solving the following Possion equations respectively

$$\begin{cases} \triangle\tilde{y} = f_2 & \text{in } \Omega \\ \tilde{y}|_{\partial\Omega} = g_2 & \text{on } \partial\Omega \end{cases}$$

and

$$\begin{cases} \triangle\tilde{z} = f_3 & \text{in } \Omega \\ \tilde{z}|_{\partial\Omega} = g_3 & \text{on } \partial\Omega \end{cases}$$

Where f_i and g_i $(i = 2,3)$ are obtained by the same way as (4.5). We have found coordinates of $\eta_\delta = (\tilde{x}, \tilde{y}, \tilde{z})$. By the equations (4.4.3) and (4.4.4), using Poincare lemma, we can find $\tilde{T}$ uniquely up to adding a constant, it is fine since we modulo $\mathbb{R}$-direction. So far we have found $\tilde{\eta}_\delta = (\tilde{x}, \tilde{y}, \tilde{z}, \tilde{T})$ uniquely and $\tilde{\eta}_\delta$ only depends on δ. Thus we obtain a smooth J-holomorphic disk M_δ. This completes the proof. ■

4.2 Invariance of Move II

In this section, we prove that the homology $CH_*(L)$ is unchanged under Legendrian Reidemeister move II. The proof is similar to the proof of move I. We also claim that $\partial = \tilde{\partial}$. As in move I, since there is no new-born Reeb chords in move II, so we number all Reeb chords the same before and after move II (see Figure 4.6).

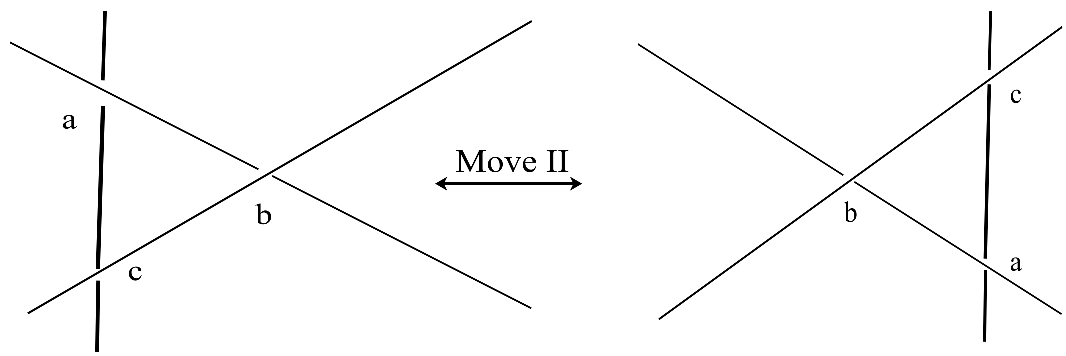


Figure 4.6: Legendrian Reidemeister Move II

In move II, we notice that the strand of L containing $a(1)$ is under the strand of L containing $c(1)$. In move I, the strand of L containing $a(1)$ is over the strand of L containing $c(1)$. In fact, this difference will not affect the fact that $\partial = \tilde{\partial}$.

Proposition 4.2.1 *The homology $CH_*(L)$ is unchanged under Legendrian Reidemeister move* II.

Proof. First, let's look at one of possibilities, if there is a J-holomorphic disk with Reeb chord **c** as one of asymptotes and boundary as shown in the first picture of Figure 4.7, then after we perform move II, there is a corresponding J-holomorphic disk with the same asymptote at Reeb chord **c** as shown in the second picture of Figure 4.7. Conversely, this correspondence also holds. Similarly, when $[a, c]$ is not the boundary of a J-holomorphic disk, we can show that there is a one-to-one correspondence between the set of J-holomorphic disks before and after move II.

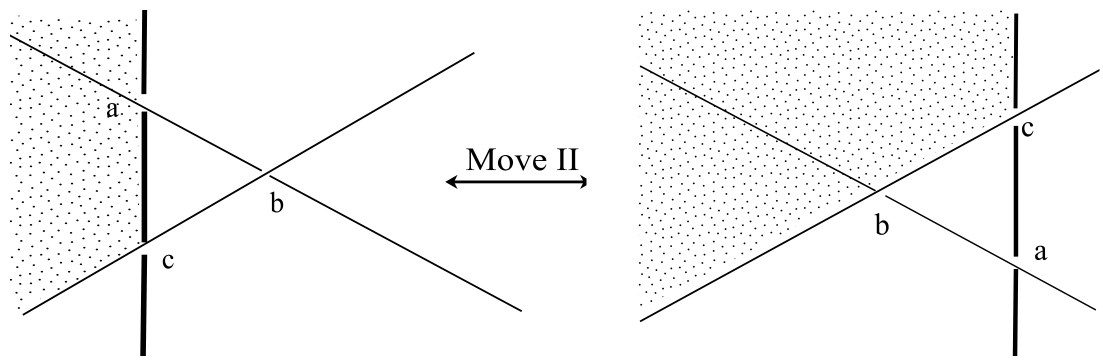


Figure 4.7: Correspondence between J-holomorphic disks with one of asymptotes at **c** in Legendrian Reidemeister Move II

If there is a J-holomorphic disk with $[a, c]$ as one of boundaries shown in Figure 4.8, then we can carry out the same proof as in proposition 4.1.1 to show that there is also a one-to-one correspondence between J-holomorphic disks. Thus, we prove that there is a one-to-one correspondence between the set of J-holomorphic disks before and after move II. Hence $\partial = \tilde{\partial}$. ■

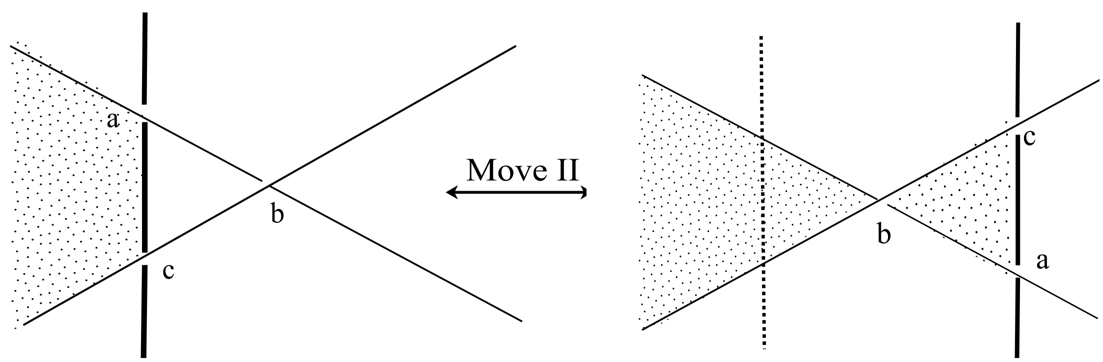


Figure 4.8: Correspondence between J-holomorphic disks with asymptotes at **a** and **c** in Legendrian Reidemeister Move II

4.3 Invariance of Move III

In this section, we prove that the homology $CH_*(L)$ is unchanged under Legendrian Reidemeister move III.

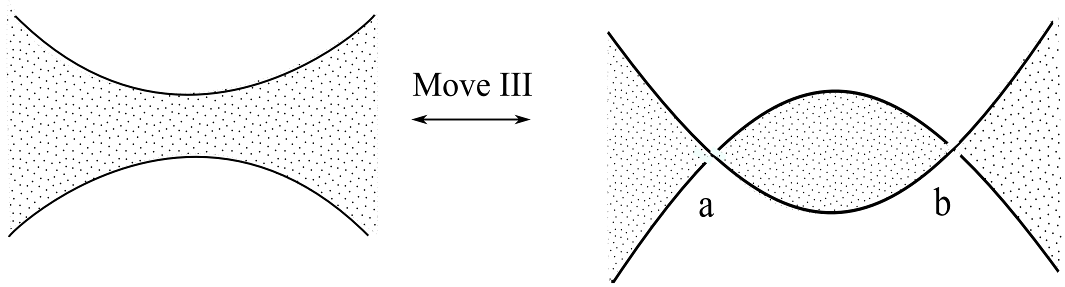


Figure 4.9: Legendrian Reidemeister move III

First, we notice that if we perform move III, then there are two new-born Reeb chords $\mathbf{a}$ and $\mathbf{b}$. We assume that $|\mathbf{a}| = k$ and $|\mathbf{b}| = k-1$, after Reidemeister move III, denote its homology by $\widehat{CH_*(L)}$ and boundary operator by $\tilde{\partial}_n$, then let's consider the following chain complex:

$$\cdots \to C_{k+1} \xrightarrow{\tilde{\partial}_{k+1}} C_k \oplus \langle \mathbf{a} \rangle \xrightarrow{\tilde{\partial}_k} C_{k-1} \oplus \langle \mathbf{b} \rangle \xrightarrow{\tilde{\partial}_{k-1}} C_{k-2} \to \dots . \tag{4.7}$$

It is easy to see that for the boundary operator: $\tilde{\partial}_n = \partial_n$ if $n \leq k-2$ or $n \geq k+2$, Since the rest part of Legendrian knot is the same as before if we perform Reidemeister move III. Hence, there is no need to prove $\widehat{CH_n(L)} = CH_n(L)$ when $n \leq k-2$ or $n \geq k+2$. In the rest part of this section, we only need to prove that $\widehat{CH_n(L)} = CH_n(L)$ when $n = k+1, k, k-1$. For convenience, we may choose bases $\{\lambda_1, \lambda_2, ...\lambda_n\}$ for C_{k+1}, $\{\beta_1, \beta_2, ...\beta_m, \mathbf{a}\}$ for $C_k \oplus \langle \mathbf{a} \rangle$, $\{\eta_1, \eta_2 ... \eta_s, \mathbf{b}+\eta\}$ for $C_{k-1} \oplus \langle \mathbf{b} \rangle$ and $\{\gamma_1, \gamma_2, ...\gamma_t\}$ for C_{k-2} respectively. Here we also assume that $\tilde{\partial}_k \mathbf{a} = \mathbf{b} + \eta$, where $\eta \in C_{k-1}$.

If there is no risk of any confusion, we still use the same notations $\tilde{\partial}_n$ and ∂_n for the matrix representations of $\tilde{\partial}_n$ and ∂_n with respect to corresponding bases. Then we have:

$$\tilde{\partial}_{k+1} = \begin{bmatrix} \partial_{k+1} \\ B_{1,n} \end{bmatrix} \qquad \text{and} \qquad \tilde{\partial}_k = \begin{bmatrix} \partial_k & 0 \\ B_{1,m} & 1 \end{bmatrix} \tag{4.8}$$

Where $B_{1,n}$ is an $1 \times n$ matrix whose entry $b_{1,i}$, $(i = 1, 2 \ldots, n)$ is the coefficient of $\mathbf{a}$ in the linear combination of $\tilde{\partial}_{k+1}\lambda_i$ with respect to the basis $\{\beta_1, \beta_2, \ldots, \beta_m, \mathbf{a}\}$ for $C_k \oplus \langle \mathbf{a} \rangle$, similarly, $B_{1,m}$ is an $1 \times m$ matrix whose entry $b_{1,j}(j = 1, 2, ...m)$ is the coefficient of $\mathbf{b} + \eta$ in the linear combination of $\tilde{\partial}_k \beta_j$ with respect to the basis $\{\eta_1, \eta_2 ... \eta_s, \mathbf{b} + \eta\}$ for $C_{k-1} \oplus \langle \mathbf{b} \rangle$. Note that $\partial_k \circ \partial_{k+1} = \tilde{\partial}_k \circ \tilde{\partial}_{k+1} = 0$, therefore, we have

$$\begin{bmatrix} \partial_k & 0 \\ B_{1,m} & 1 \end{bmatrix} \begin{bmatrix} \partial_{k+1} \\ B_{1,n} \end{bmatrix} = \begin{bmatrix} \partial_k \circ \partial_{k+1} \\ B_{1,m} \circ \partial_{k+1} + B_{1,n} \end{bmatrix} = 0 \tag{4.9}$$

It follows that :

$$B_{1,m} \circ \partial_{k+1} = B_{1,n} \tag{4.10}$$

With respect to the bases $\{\lambda_1, \lambda_2, ...\lambda_n\}$ for C_{k+1}, and $\{\beta_1, \beta_2, ...\beta_m, \mathbf{a}\}$ for $C_k \oplus \langle \mathbf{a} \rangle$, the matrix representation of $\mathrm{Im}\tilde{\partial}_{k+1}$ can be given as follows:

$$\begin{aligned} \mathrm{Im}\tilde{\partial}_{k+1} &= \left\{ \begin{bmatrix} \partial_{k+1} X \\ B_{1,n} X \end{bmatrix} \middle| \forall X \in C_{k+1} \right\} \\ &\overset{by(\,4.9)}{=} \left\{ \begin{bmatrix} \partial_{k+1} X \\ B_{1,m} \circ \partial_{k+1} X \end{bmatrix} \middle| \forall X \in C_{k+1} \right\} \\ &= \left\{ \begin{bmatrix} Y \\ B_{1,m} Y \end{bmatrix} \middle| Y = \partial_{k+1} X \in \mathrm{Im}\partial_{k+1}, \forall X \in C_{k+1} \right\} \end{aligned} \tag{4.11}$$

Similarly, with respect to the bases $\{\eta_1, \eta_2, \ldots, \eta_s, \mathbf{b} + \eta\}$ and $\{\beta_1, \beta_2, ...\beta_m, \mathbf{a}\}$ the matrix representation of Ker $\tilde{\partial}_k$ and Im$\tilde{\partial}_k$ can be given as follows:

$$\begin{aligned}\mathrm{Ker}\tilde{\partial}_k &= \left\{ \begin{bmatrix} X \\ x \end{bmatrix} \in C_k \oplus \langle \mathbf{a} \rangle \,\middle|\, \partial_k X = 0, B_{1,m}X = x \right\} \\ &= \left\{ \begin{bmatrix} X \\ x \end{bmatrix} \in C_k \oplus \mathbf{a} \,\middle|\, X \in \mathrm{Ker}\ \partial_k, B_{1,m}X = x \right\} \end{aligned} \tag{4.12}$$

$$\begin{aligned}\mathrm{Im}\ \tilde{\partial}_k &= \left\{ \begin{bmatrix} \partial_k & 0 \\ B_{1,m} & 1 \end{bmatrix} \begin{bmatrix} X \\ x \end{bmatrix} \,\middle\|\, \begin{bmatrix} X \\ x \end{bmatrix} \in C_k \oplus \langle \mathbf{a} \rangle \right\} \\ &= \left\{ \begin{bmatrix} \partial_k X \\ B_{1,m}X + x \end{bmatrix} \,\middle\|\, \begin{bmatrix} X \\ x \end{bmatrix} \in C_k \oplus \langle \mathbf{a} \rangle \right\} \end{aligned} \tag{4.13}$$

Now, we are ready to prove one of the main lemmas of this section.

Lemma 4.3.1 $\widehat{CH_k(L)} \cong CH_k(L)$.

Proof. First, we define a map :

$$\varphi \quad : \mathrm{Ker}\partial_k \longrightarrow \mathrm{Ker}\tilde{\partial}_k$$

$$X \longrightarrow \begin{bmatrix} X \\ B_{1,m}X \end{bmatrix}$$

Then, we will show that φ is isomorphic. In fact, φ has the following properties.

(a) For any $X_1, X_2 \in \mathrm{Ker}\partial_k$, we have

$$\begin{aligned}\varphi(X_1 + X_2) &= \begin{bmatrix} X_1 + X_2 \\ B_{1,m}(X_1 + X_2) \end{bmatrix} \\ &\overset{def}{=} \begin{bmatrix} X_1 \\ B_{1,m} \end{bmatrix} + \begin{bmatrix} X_2 \\ B_{1,m}X_2 \end{bmatrix} \\ &= \varphi(X_1) + \varphi(X_2).\end{aligned}$$

This shows that φ is a homorphism between two vector spaces Ker ∂_k and Ker$\tilde{\partial}_k$.

(b) If $\varphi(X) = 0 \implies \begin{bmatrix} X \\ B_{1,m}X \end{bmatrix} = 0 \implies X = 0$, this implies that φ is injective.

(c) $\forall \beta \in \mathrm{Ker}\tilde{\partial}_k$, by (4.12), it follows that β is of the form $\begin{bmatrix} X \\ B_{1,m}X \end{bmatrix}$ for some $X \in \mathrm{Ker}\partial_k$. Hence, φ is surjective.

Therefore, φ is an isomorphism. We also note that

$$\mathrm{Im}\tilde{\partial}_{k+1} = \left\{ \begin{bmatrix} Y \\ B_{1,m}Y \end{bmatrix} \middle| Y = \partial_{k+1}X \in \mathrm{Im}\partial_{k+1}, \forall X \in C_{k+1} \right\} \tag{4.14}$$

Then,

$$\varphi|_{\mathrm{Im}\partial_{k+1}} = \mathrm{Im}\tilde{\partial}_{k+1}$$

Thus, we have that $\widehat{CH_k}(L) = \dfrac{\mathrm{Ker}\tilde{\partial}_k}{\mathrm{Im}\tilde{\partial}_{k+1}} \cong \dfrac{\mathrm{Ker}\partial_k}{\mathrm{Im}\partial_{k+1}} = CH_k(L)$, this proves the lemma.

■

Lemma 4.3.2 $\widehat{CH_{k+1}}(L) \cong CH_{k+1}(L)$.

Proof. Note that :

$$\begin{aligned}
\mathrm{Ker}\tilde{\partial}_{k+1} &= \left\{ X \in C_{k+1} \middle| \begin{bmatrix} \partial_{k+1}X \\ B_{1,n}X \end{bmatrix} = 0 \right\} \\
&\overset{by(\ 4.11)}{=} \left\{ X \in C_{k+1} \middle| \begin{bmatrix} \partial_{k+1}X \\ B_{1,m}\partial_{k+1}X \end{bmatrix} = 0 \right\} \\
&\Leftrightarrow \quad X \in \mathrm{Ker}\partial_{k+1}
\end{aligned}$$

Hence, $\mathrm{Ker}\tilde{\partial}_{K+1} = \mathrm{Ker}\partial_{k+1}$. Obviously, $\mathrm{Im}\tilde{\partial}_{k+2} = \mathrm{Im}\partial_{k+2}$. So, we have that $\widehat{CH_{k+1}}(L) = \dfrac{\mathrm{Ker}\tilde{\partial}_{k+1}}{\mathrm{Im}\tilde{\partial}_{k+2}} \cong \dfrac{\mathrm{Ker}\partial_{k+1}}{\mathrm{Im}\partial_{k+2}} = CH_{k+1}(L)$, this proves the lemma. ■

Finally, we only need to check for $n = k - 1$. By chain complex (4.7) and the bases we choose for $C_k \oplus \langle \mathbf{a} \rangle$,$C_{k-1} \oplus \langle \mathbf{b} \rangle$ and C_{k-2} as before, also we notice that

$C_{k-1} \oplus \langle \mathbf{b} \rangle = C_{k-1} \oplus \langle \mathbf{b} + \eta \rangle$, with respect to those bases, then the matrix representation of $\tilde{\partial}_{k-1}$ is as follows :

$$\tilde{\partial}_{k-1} = \begin{bmatrix} \partial_k & 0 \end{bmatrix} \tag{4.15}$$

Since $\tilde{\partial}_{k-1}|_{C_{k-1}} = \partial_k$ and $\tilde{\partial}_{k-1}(\mathbf{b} + \eta) = 0$ and the first move does not affect other Reeb chords. Hence, by (4.13)

$$\begin{aligned} \mathrm{Im}\tilde{\partial}_k &= \left\{ \begin{bmatrix} \partial_k & 0 \\ B_{1,m} & 1 \end{bmatrix} \begin{bmatrix} Y_m \\ y \end{bmatrix} \middle| \begin{bmatrix} Y_m \\ y \end{bmatrix} \in C_k \oplus \langle \mathbf{a} \rangle \right\} \\ &= \left\{ \begin{bmatrix} \partial_k Y_m \\ B_{1,m} Y + y \end{bmatrix} \middle| \begin{bmatrix} Y_m \\ y \end{bmatrix} \in C_k \oplus \langle \mathbf{a} \rangle \right\} \end{aligned} \tag{4.16}$$

and by (4.12), we have

$$\begin{aligned} \mathrm{Ker}\tilde{\partial}_{k-1} &= \left\{ \begin{bmatrix} X_s \\ x \end{bmatrix} \in C_{k-1} \oplus \langle \mathbf{b} + \eta \rangle \middle| \begin{bmatrix} \partial_{k-1} & 0 \end{bmatrix} \begin{bmatrix} X_s \\ x \end{bmatrix} = 0 \right\} \\ &= \left\{ \begin{bmatrix} X_s \\ x \end{bmatrix} \in C_{k-1} \oplus \langle \mathbf{b} + \eta \rangle \middle| \partial_{k-1} X_s = 0 \right\} \\ &= \left\{ \begin{bmatrix} X_s \\ x \end{bmatrix} \in C_{k-1} \oplus \langle \mathbf{b} + \eta \rangle \middle| X_s \in \mathrm{Ker}\partial_{k-1} \right\} \end{aligned} \tag{4.17}$$

Lemma 4.3.3 *$\widehat{CH_{k-1}(L)} \cong CH_{k-1}(L)$.*

Proof. By (4.17), we know that any element in $\mathrm{Ker}\tilde{\partial}_{k-1}$ is of the form $\begin{bmatrix} X_s \\ x \end{bmatrix}$, where $X_s \in \mathrm{Ker}\partial_{k-1}$. So we define a map:

$$\Phi \quad : \mathrm{Ker}\tilde{\partial}_{k-1} \longrightarrow \mathrm{Ker}\partial_{k-1}$$

$$\begin{bmatrix} X_s \\ x \end{bmatrix} \longrightarrow X_s$$

then, Φ satisfies the following conditions.

(a). Φ is a homomorphism. Since for any $\begin{bmatrix} X_s \\ x \end{bmatrix}, \begin{bmatrix} \tilde{X}_s \\ \tilde{x} \end{bmatrix} \in \mathrm{Ker}\tilde{\partial}_{k-1}$,

$$\Phi\left(\begin{bmatrix} X_s \\ x \end{bmatrix} + \begin{bmatrix} \tilde{X}_s \\ \tilde{x} \end{bmatrix}\right) \stackrel{def}{=} X_s + \tilde{X}_s = \Phi\left(\begin{bmatrix} X_s \\ x \end{bmatrix}\right) + \Phi\left(\begin{bmatrix} \tilde{X}_s \\ \tilde{x} \end{bmatrix}\right).$$

(b).

$$I =: \ \mathrm{Ker}\Phi \stackrel{def}{=} \left\{ \begin{bmatrix} 0 \\ x \end{bmatrix} \middle| \ x \in Z_2 \right\}$$

This is a subspace of $\mathrm{Ker}\tilde{\partial}_{k-1}$. Since $\Phi\left(\begin{bmatrix} X_s \\ x \end{bmatrix}\right) = 0 \Rightarrow X_s = 0$

(c). By definition, it is easy to see that Φ is subjective. Hence,

$$\mathrm{Ker}\tilde{\partial}_{k-1}/I \cong \mathrm{Ker}\partial_{k-1}.$$

(d). By (4.14), we have

$$\Phi(\mathrm{Im}\tilde{\partial}_k) = \mathrm{Im}\partial_k$$

In fact, for $\forall \begin{bmatrix} \partial_k Y_m \\ B_{1,m}Y_m + y \end{bmatrix} \in \mathrm{Im}\tilde{\partial}_k$, then its image is $\partial_k Y_m (\in \mathrm{Im}\partial_k)$ under the map Φ. Since the Φ is subjective, this gives us $\Phi(\mathrm{Im}\tilde{\partial}_k) = \mathrm{Im}\partial_k$.

(e). $\mathrm{Ker}\Phi \subset \mathrm{Im}\tilde{\partial}_k$. By (b), we may choose $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathrm{Ker}\Phi$, then by (4.8), it is easy to

obtain that $\tilde{\partial}_k \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathrm{Im}\tilde{\partial}_k$.

Thus, by (c),(d) and (e), we have

$$\mathrm{Ker}\tilde{\partial}_{k-1}/I/\mathrm{Im}\tilde{\partial}_k/I \cong \frac{\mathrm{Ker}\partial_{k-1}}{\mathrm{Im}\partial_k}$$

Hence,

$$\widehat{CH_{k-1}}(L) = \frac{\mathrm{Ker}\tilde{\partial}_{k-1}}{\mathrm{Im}\tilde{\partial}_k} \cong \frac{\mathrm{Ker}\partial_{k-1}}{\mathrm{Im}\partial_k} = CH_{k-1}(L)$$

■

By above three lemmas, we conclude

Proposition 4.3.1 *The homology $CH_*(L)$ is unchanged under Legendrian Reidemeister move* III.

4.4 The proof of invariance

Theorem 4.1 *Under assumption $r(L) \neq 0$, the homology $CH_n(L) = \dfrac{\mathrm{Ker}\partial_n}{\mathrm{Im}\partial_{n+1}}$ of Reeb chord complex is a topological invariant of Legendrian knots for all $n \in \mathbb{Z}$.*

Proof. The invariance of the homology $CH_*(L)$ follows from propositions 4.1.1, 4.2.1 and 4.3.1. ■

Hence we have shown that the homology $CH_*(L)$ of Reeb chord complex is unchanged under Legendrian Reidemeister moves, thus it is an invariant of Legendrian knots. It is a $\mathbb{Z}-$ graded vector space over $\mathbb{Z}_2$.

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VITA

Xiaowei Yang

Candidate for the Degree of

Doctor of Philosophy

Dissertation: THE INTEGRAL GRADED HOMOLOGY OF REEB CHORD COMPLEX OF LEGENDRIAN KNOTS IN $\mathbb{R}^3$

Major Field: Mathematics

Biographical:

Education:

Completed the requirements for the Doctor of Philosophy in Mathematics at Oklahoma State University, Stillwater, Oklahoma in July, 2014.

Completed the requirements for the Master of Science in Mathematics at Southwest Jiaotong University, Chengdu, Sichuan, P.R.China in 2003.

Completed the requirements for the Bachelor of Science in Mathematics at Baoji University of Arts and Sciences, Baoji, Shannxi, P.R.China in 1999.

Experience:

Teaching Assistant in the department of Mathematics at Oklahoma State University, Aug, 2007- Aug, 2014.

Lecturer in the department of Mathematics at Southwest Jiaotong University, Aug, 2003- July, 2007.

Professional Memberships:

American Mathematics Society.

Chinese Mathematics Society.