POTENTIAL THEORY OF THE FARTHEST DISTANCE FUNCTION

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POTENTIAL THEORY OF THE FARTHEST DISTANCE FUNCTION

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Let $E \subset \mathbb{R}^N$, $N \geq 2$. Then the farthest distance function of E is given by

$$d_E(x) = \sup_{y \in E} |x - y|, \text{ for } x \in \mathbb{R}^N.$$

The farthest distance function has been studied extensively in analysis, approximation theory, optimization theory, and computational geometry. The farthest distance function can be expressed via the potential of a unique probability measure σ_E, an application that arose in the study of reverse triangle inequalities for polynomials. Specifically, if $E \subset \mathbb{R}^2$ then

$$\log d_E = \int \log |z - t| \, \mathrm{d}\sigma_E(t)$$

and if $E \subset \mathbb{R}^N$, $N \geq 3$, then

$$d_E^{2-N} = \int |z - t|^{2-N} \, \mathrm{d}\sigma_E(t).$$

The properties and density of the measure σ_E in the planar case $E \subset \mathbb{R}^2$ was considered by Laugesen and Pritsker. Their work was continued by Gardiner and Netuka.

The present work considers the properties of d_E and σ_E for $E \subset \mathbb{R}^N$ for $N \geq 2$ with a primary emphasis on $N \geq 3$. In addition to proving that d_E^{2-N} is the Newtonian potential of a unique probability measure, where $E \subset \mathbb{R}^N$, $N \geq 3$, we prove results on the continuity and differentiability of d_E. We provide methods to compute σ_E, prove results about the support and density of σ_E, and consider an open conjecture of Laugesen and Pritsker regarding the quantity $\sigma_E(E)$. Finally, we extend the potential representation of d_E to a certain range of Riesz potentials. Connections to various applications and explicit examples are provided throughout.

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CHAPTER 1

Introduction

Let $E \subset \mathbb{R}^N$, $N \geq 2$. Then the *farthest distance function of* E is given by

$$d_E(x) = \sup_{y \in E} |x - y|, \text{ for } x \in \mathbb{R}^N.$$

If E is bounded, we define the *farthest point map of* E by

$$y_E(x) = \Big\{ y \in \overline{E} : d_E(x) = |x - y| \Big\}$$

where $\overline{E}$ is the closure of E.

Early work on these functions was done in 1940 by Jessen who gave conditions for concluding that the map y_E is single-valued on the complement of E [36]. Further research characterizing the sets on which y_E is single-valued was performed by Motzkin, Strauss, and Valentine in 1953 [50] and Fitzpatrick in 1980 [22]. Fitzpatrick also connected properties of the map y_E to properties of the farthest distance function d_E. Their work is discussed in Section 2.1.

These functions can be defined on more general spaces than the Euclidean spaces we are using. In fact, Motzkin, Strauss, and Valentine chose to situate E in a metric space, while Fitzpatrick chose a Banach space. Thus these functions can appear in wide-ranging fields. The farthest point and farthest distance maps have been studied extensively in analysis, approximation theory, optimization theory, and computational geometry. We will discuss these connections in Chapters 2 and 4.

We are particularly interested in the potential theory of the farthest distance function. The farthest distance function can be expressed via potentials. This representation first appeared in 1994 [12] in connection with the study of norms of products of polynomials.

Let $P(z) = \prod_{i=1}^{n}(z - \alpha_i)$ be a monic polynomial with complex coefficients and let $\|P\|_E$ be the uniform (sup) norm on E. Then for an arbitrary set of polynomials P_j, $j = 1, \ldots, m$, it is clear by the triangle inequality that

$$\left\| \prod_{j=1}^{m} P_j \right\|_E \leq \prod_{j=1}^{m} \|P_j\|_E.$$

A reverse inequality requires a multiplicative constant and there was a great deal of work finding constants M depending only on E so that

$$\prod_{j=1}^{m} \|P_j\|_E \leq M^n \left\| \prod_{j=1}^{m} P_j \right\|_E \tag{1.1}$$

where n is the degree of $\prod_{j=1}^{m} P_j$.

Kneser [39] found the first sharp constant M. Let $E = [-1, 1]$ and consider only two factors so that $m = 2$. Then (1.1) holds with multiplicative constant

$$M = 2^{(n-1)/n} \prod_{k=1}^{\deg P_1} \left(1 + \cos \frac{2k-1}{2n}\pi\right)^{\frac{1}{n}} \prod_{k=1}^{\deg P_2} \left(1 + \cos \frac{2k-1}{2n}\pi\right)^{1/n}. \tag{1.2}$$

The Chebyshev polynomial shows that this constant is sharp. A weaker result was previously given by Aumann [3]. Borwein [10] provided an alternative proof for this constant. He showed further that on $E = [-1, 1]$, (1.1) holds for any number of factors m with multiplicative constant

$$M = 2^{(n-1)/n} \prod_{k=1}^{\left[\frac{n}{2}\right]} \left(1 + \cos \frac{2k-1}{2n}\pi\right)^{2/n}. \tag{1.3}$$

Another series of such constants were found for $E = B$, the closed unit disk. Mahler [45], building on a weaker result by Gelfond [26, p. 135], showed that (1.1) holds for

$$M = 2. \tag{1.4}$$

While the base 2 cannot be decreased, Kroó and Pritsker [42] showed that for $m \le n$, we can use $M = 2^{(n-1)/n}$. Furthermore, Boyd [12, 11] expressed the multiplicative constant as a function of the number of factors m and found

$$M = \exp\left(\frac{m}{\pi}\int_0^{\pi/m} \log\left(2\cos\frac{t}{2}\right)\right).$$

This constant is asymptotically best possible for each fixed m as $n \to \infty$.

The work of Boyd relied on a representation of the farthest distance function via a logarithmic potential. For each P_j, choose $c_j \in B$ such that $\|P_j\|_B = P_j(c_j)$. Let $S \subset B$ be the set of points c_j. By taking the logarithm of $\prod_{j=1}^m \|P_j(z)\|_E$, it can be shown that

$$\sum_{j=1}^{m} \log \|P_j(z)\|_E \le \sum_{\alpha} \log d_S(\alpha)$$

where the summation takes place over all zeros α of P, counted with multiplicity. Using Jensen's formula, Boyd showed that for a finite set $S \subset \mathbb{C}$, there is a measure σ_S such that $\log d_S$ is the logarithmic potential of σ_S and is given by

$$\log d_S(z) = \int \log|z-t|\,\mathrm{d}\sigma_S(t).$$

Boyd then used this representation in computing the constant M.

Pritsker [53] extended Boyd's logarithmic potential representation. For any bounded set $E \subset \mathbb{C}$ containing at least two points there exists a unique positive unit Borel measure σ_E such that

$$\log d_E(z) = \int \log|z-t|\,\mathrm{d}\sigma_E(t).$$

He used this representation to conclude that if such a set E has positive capacity then a sharp multiplicative constant in (1.1) is given by

$$M_E = \frac{\exp\left(\int \log d_E(z)\,\mathrm{d}\mu_E(z)\right)}{\mathrm{cap}(E)} \tag{1.5}$$

where $\mathrm{cap}(E)$ is the capacity of E, μ_E is its equilibrium measure, and d_E is its farthest distance function. This constant generalizes several previous results. We can calculate that for the segment $[-1, 1]$ we have $M_{[-1,1]} \approx 3.20991$ which is the asymptotic version of Borwein's constant from (1.3) since

$$2^{(n-1)/n} \prod_{k=1}^{[n/2]} \left(1 + \cos \frac{2k-1}{2n}\pi\right)^{2/n} \approx 3.20991$$

as $n \to \infty$. For the closed unit disk we find $M_B = 2$, which is the constant given by Mahler (1.4). Furthermore, Pritsker and Ruscheweyh [57, 55] showed that M_B is a lower bound on M_E for any compact E with positive capacity. They conjectured the $M_{[-1,1]}$ is an upperbound for all non-degenerate continua. Shortly afterward, Baernstein, Laugesen, and Pritsker [5] showed $M_{[-1,1]}$ is an upper bound for centrally symmetric continua. The assumption that E has positive capacity is vital. For example, if E is a finite set, then no inequality of the form (1.1) is possible for any number of factors $m \geq 2$. If E is countable, then the constant M could grow arbitrarily fast as m grows large.

Laugesen and Pritsker later considered the properties and density of the measure σ_E in $\mathbb{C}$ [44]. Their work was continued by Gardiner and Netuka who proved the open conjecture of Laugesen and Pritsker that $\sigma_E(E) \leq 1/2$ [25]. These results will be discussed in detail in Chapter 3.

The main focus of this dissertation will be on examining the farthest distance function for sets $E \subset \mathbb{R}^N$ where $N \geq 3$. We will extend previous results for the farthest distance function in the plane and consider applications.

We will begin by discussing the farthest distance function in Chapter 2. We begin by computing some examples and developing the properties of d_E in Section 2.1. Theorem 2.1 is an extension of a previous result of Fitzgerald which connects the smoothness of d_E to the smoothness of y_E. We consider the special case of polytopes in Section 2.2 which also includes applications to computational geometry. Section 2.3 provides an introduction to some basic potential theory results. The main result of this section is Theorem 2.6 which expresses d_E^{2-N} as the Newtonian potential of a representing measure σ_E. Specifically,

$$d_E^{2-N}(x) = \int |x-y|^{2-N} d\sigma_E(y)$$

for some unique probability measure σ_E.

In Chapter 3, we discuss the properties of the representing measure σ_E. We will compute the measure for specific examples in Sections 3.1 and 3.2. We will also give a purely geometric formulation of the representing measure in the case that E is a polytope in Theorem 3.1. This will lead us to a sharp bound on the sum of certain angles for polygons [65]. In Section 3.3, we will continue considering examples and give a characterization of the support and its complement in Propositions 3.7 and 3.8. Finally, we will consider an open conjecture of Laugesen and Pritsker [44] regarding the quantity $\sigma_E(E)$ in Section 3.4. Proposition 3.13 offers a proof of the conjecture in the special case of bodies of constant width.

In Chapter 4, we consider Riesz potentials and applications to polarization inequalities. In Section 4.1, we present an overview of α-superharmonic functions and prove Theorem 4.2, which is an extension of Theorem 2.6 to Riesz potentials. We use this theorem to obtain reverse triangle inequalities for Riesz potentials in Theorem 4.4 of Section 4.2. This representation can be applied to polarization inequalities [20, 30] which is considered in Section 4.3. This chapter is based on joint work with Pritsker and Saff [59] which is built on previous work by Pritsker and Saff [58].

CHAPTER 2

Farthest Distance Function

In this chapter we develop the properties of the farthest distance function d_E. We will begin with examples of d_E for some sets E, and establish basic properties of d_E in Section 2.1. We will also consider the differentiability of d_E and the behavior of d_{E_n} as the sets E_n converge in a suitable sense. We specifically address the case where E is a polytope, such as a polygon or a polyhedron, in Section 2.2. We will particularly be interested in the so-called farthest point Voronoi cells of the vertices of E.

We will then turn our attention to expressing d_E via potentials by expanding on previous work in the plane. After showing that d_E^{2-N} is superharmonic we will apply the Riesz Decomposition Theorem to express d_E^{2-N} as a Newtonian potential in Section 2.3. All but the simplest proofs may be found in Section 2.4.

2.1 Properties of the Farthest Distance Function

We have defined the farthest distance function

$$d_E(x) = \sup_{t \in E} |x - t|$$

for any set $E \subseteq \mathbb{R}^N$. However, if E is unbounded, we have $d_E \equiv +\infty$. We eliminate this case by assuming E is bounded and we denote its boundary by ∂E. We begin our discussion with a simple example. We will then extend our results to more complicated examples by considering the behavior of d_E under various operations on E.

Example 2.1 (Ball). *Let B be the unit ball in $\mathbb{R}^N$ centered at the origin and let $x \in \mathbb{R}^N$. If x is the origin, then all points on the surface of the ball are simultaneously farthest and $d_E(0) = 1$. Otherwise, consider the line passing through x and the origin. This line intersects ∂B in two points, one of which is the nearest point and one of which is the farthest. Thus $d_B(x) = |x| + 1$. Similarly, we can consider the farthest distance function of any ball. Denote the ball centered at x_0 of radius $R > 0$ by $B(x_0, R)$. Then $d_{B(x_0,R)}(x) = |x - x_0| + R$.*

Notice that in Example 2.1, the farthest point always lays on the surface of the ball. This is a special case of a more general fact.

Proposition 2.1. *Let $\overline{E}$ denote the closure of E and $\operatorname{conv}(E)$ denote the convex hull of E. Then*

$$d_E = d_{\overline{E}} = d_{\partial E} = d_{\operatorname{conv}(E)}.$$

As a result, when considering d_E we may assume E is compact and convex whenever convenient. We already assumed that E is bounded to avoid the case where $d_E \equiv +\infty$. If $E = y$ is a singleton, then $d_E(y) = 0$. We will often require that E contain at least two points in order to avoid this case. If E contains at least two points then d_E is bounded below by $\operatorname{diam}(E)/2$ where $\operatorname{diam}(E) > 0$ is the diameter of E. When we particularly wish to consider the farthest distance function of a single point $y \in \mathbb{R}^N$, we will denote it by d_y.

We can apply Proposition 2.1 to find the farthest distance function for new sets.

Example 2.2 (Hot Dog). *Let $E = \overline{B}(-a,1) \cup \overline{B}(a,1) \subset \mathbb{R}^N$, be the union of two unit balls, outlined in Figure 2.1 in dashed gray. Let $H = \operatorname{conv}(E)$, outlined in Figure 2.1 in solid black. Since $d_H = d_E$, we obtain a piecewise defined function*

$$d_H(x) = \max\left\{d_{B(-a,1)}(x), d_{B(a,1)}(x)\right\}.$$

The value of d_H in two half-spaces is shown in Figure 2.1. Notice that on the hyperplane between the half-spaces, shown as a vertical dashed line, we have $d_{B(-a,1)}(x) = d_{B(a,1)}(x)$.

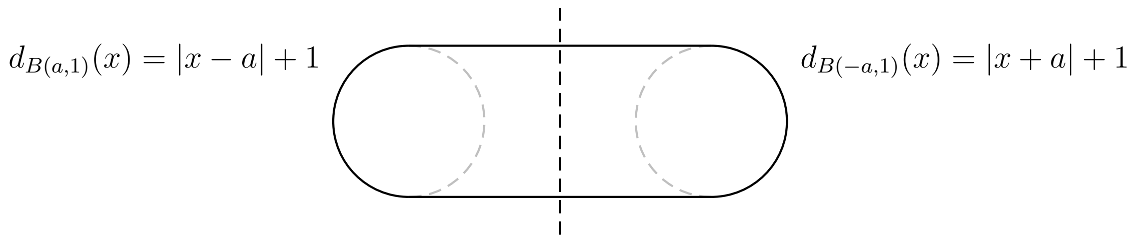

Figure 2.1: Farthest distance function for a hot dog

The farthest distance function behaves particularly well under some standard transformations of E. We denote the scaling of E by $\lambda E = \{\lambda y : y \in E\}$.

Proposition 2.2. *The farthest distance function d_E is invariant under rigid motions and homogeneous under scaling. In other words, $d_{TE}(Tx) = d_E(x)$ for any rigid transformation T and $d_{\lambda E}(\lambda x) = \lambda d_E(x)$ for all $\lambda > 0$.*

A useful way to extend examples in the plane to $\mathbb{R}^3$ is by using bodies of rotation. We express a point $x \in \mathbb{R}^3$ in cylindrical coordinates as $x = (r, h, \theta)$ with $r \geq 0$ being radius, h being height, and $\theta \in [0, 2\pi)$ being the angle. If a set E lies in the (r, h)-plane and either E lies on the $r \geq 0$ side of the plane or E is symmetric about the h axis, then we may rotate E about the h axis to obtain a body of rotation $\mathcal{E} \subset \mathbb{R}^3$. If $x = (r_x, h_x, \theta_x) \in \mathbb{R}^3$ and $y = (r_y, h_y, \theta_y) \in \partial\mathcal{E}$ then

$$|x-y|^2 = (r_x^2 + h_x^2) + (r_y^2 + h_y^2) - 2r_x r_y(\cos\theta_x \cos\theta_y + \sin\theta_x \sin\theta_y) - 2h_x h_y.$$

Since finding $d_{\mathcal{E}}(x)$ requires maximizing $|x-y|$ over $y \in \partial\mathcal{E}$ and r_y and h_y are independent of θ_y, it is clear that $\theta_y = \theta_x + \pi$. Hence,

$$d_{\mathcal{E}}(r, h, \theta) = d_E(-r, h) \tag{2.1}$$

where d_E is the farthest distance function in the plane.

Example 2.3 (Torus)**.** *Consider the ball $B = B(x, 1) \subset \mathbb{R}^2$, where $x = (a, 0)$ and $a > 1$. We may rotate B around the vertical axis to obtain a torus $\mathcal{T}$. Applying* (2.1) *and Example 2.1, we conclude that*

$$d_{\mathcal{T}}(r, h, \theta) = d_B(-r, h) = \big|(-r, h) - x\big| + 1 = \big|(r, h) + (a, 0)\big| + 1.$$

Having established some examples and basic properties, we now turn our attention to continuity and differentiability properties of d_E. Our first result is a consequence of the triangle inequality.

Proposition 2.3. *The farthest distance function d_E is non-negative and Lipschitz continuous with constant 1.*

Proof. That d_E is non-negative is clear. Choose $x, y \in \mathbb{R}^N$. There exists $t \in \partial E$ such that $d_E(x) = |x - t|$. Applying the triangle inequality we have $d_E(x) = |x - t| \leq |x - y| + |y - t| \leq |x - y| + d_E(y)$. This argument is symmetric and we obtain $|d_E(x) - d_E(y)| \leq |x - y|$. ◻

Applying Rademacher's Theorem, we conclude that d_E is differentiable almost everywhere. It is not possible for d_E to be differentiable in all of $\mathbb{R}^N$. For example, notice that $d_{B(0,1)}$ is not differentiable at the center of the ball 0.

Recall that we defined the farthest point map for a compact set E by

$$y_E(x) = \big\{y \in E : |x - y| = d_E(x)\big\}.$$

If E is a singleton y then it is clear that d_y is not differentiable at y. If a compact set E has at least two points, then d_E is differentiable at a point $x \in \mathbb{R}^N$ if and only if y_E is single-valued at x [24, Corollary 2.5]. There always exists at least one point $x \in \text{conv}(E)$ which does not have a unique farthest point in E [24, Theorem 2.1] and hence d_E cannot be differentiable everywhere. Sketches of these facts are provided in Section 2.4.

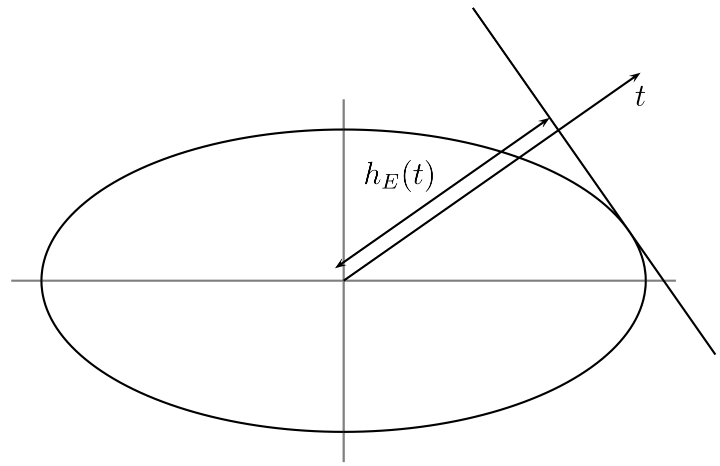

Figure 2.2: Support function and line of an ellipse

There are many results about uniqueness of farthest points. That the differentiability of d_E at x implies y_E is single-valued was reported by Zhivkov [67] and a proof was later published by Fitzpatrick [22, Theorem 2.3]. Motzkin, Strauss, and Valentine [50] characterized the planar sets on which y_E is single-valued.

While we assume $E \subset \mathbb{R}^N$, the work of Fitzpatrick in [22] merely assumed that E was a subset of a Banach space, while Motzkin, Strauss, and Valentine in [50] situated E in a metric space. Recent results on the number of farthest points have focused on spaces such as Banach spaces, Hilbert spaces, and metric spaces, see [16, 17, 19, 6, 35].

We will now discuss some properties of hypersurfaces in $\mathbb{R}^N$. The *support function* of E is the function $h_E : \mathbb{R}^N \to \mathbb{R}$ given by

$$h_E(t) = \sup_{y \in E} y \cdot t$$

where $y \cdot t$ denotes the usual dot product given by $y \cdot t = \sum_{i=1}^N y_i t_i$. The function h_E is positively homogeneous, meaning $h(\lambda t) = \lambda h(t)$ for all $\lambda \geq 0$, and subadditive, meaning $h_E(s+t) \leq h_E(s) + h_E(t)$ [28, Theorem 4.3]. In fact, any function with these two properties is the support function of a unique compact convex set.

We can picture the meaning of the support function most easily for unit vectors in $\mathbb{R}^2$. Think of the line whose unit normal vector is t. It divides the plane into two half-planes. Orient them so that t points toward the positive half-plane. Place the line so that it intersects ∂E and E lies entirely on the closed negative half-plane, as shown in Figure 2.2. Then $h_E(t)$ is the signed distance from this line to the origin. This line is called a support line, or, in higher dimensions, a support hyperplane. Since h_E is positively homogeneous, it is determined entirely by its values for unit vectors.

The support function behaves nicely under various transformations of E, including translation, rotation, Minkowski addition, and scaling. We denote the Minkowski addition of sets X and Y by

$$X + Y := \{x + y : x \in X, y \in Y\}.$$

For example, the hot dog set H from Example 2.2 may be written as $H = [-a, a] + \overline{B}(0, 1)$ where $[-a, a]$ is a line segment. We may now denote translation by a vector x by $E + x$. We also establish notation for expressing rotation.

We may express a point $x \in \mathbb{R}^N$ using rectangular or hyperspherical coordinates. In rectangular coordinates we will write $x = (x_1, \ldots, x_N)$. In hyperspherical coordinates we will write $x = (\rho, \phi_1, \ldots, \phi_{N-1})$ where $\rho \in [0, +\infty)$ is $|x|$, and the angles given are generally (but not necessarily) restricted to $\phi_k \in [0, \pi]$ for $k = 1, \ldots, N-2$ and $\phi_{N-1} \in [0, 2\pi)$. The two coordinate systems are related by the following equations:

$$\begin{aligned}
x_1 &= \rho\cos(\phi_1), \\
x_2 &= \rho\sin(\phi_1)\cos(\phi_2), \\
x_3 &= \rho\sin(\phi_1)\sin(\phi_2)\cos(\phi_3), \\
x_{N-1} &= \rho\sin(\phi_1)\cdots\sin(\phi_{N-2})\cos(\phi_{N-1}), \text{ and} \\
x_N &= \rho\sin(\phi_1)\cdots\sin(\phi_{N-2})\sin(\phi_{N-1}).
\end{aligned}$$

Hyperspherical coordinates are a higher dimensional generalization of the usual polar and spherical coordinates. The ordering of the rectangular coordinates in this convention differs from the usual order for spherical coordinates. We will write $u = (0, \theta_1, \ldots, \theta_{N-1})$ to refer to an angle in $\mathbb{R}^N$. Let $\mathcal{U}^N$ be the set of all such angles in $\mathbb{R}^N$. While an angle $u \in \mathcal{U}^N$ is not actually a point in $\mathbb{R}^N$, this notation is convenient. For example, we may add or subtract two angles coordinate-wise. Furthermore, we may denote the rotation of x about the origin by the angle u as

$$x + u = (\rho, \phi_1 + \theta_1, \ldots, \phi_{N-1} + \theta_{N-1})$$

and the rotation of a set E by $E + u$. Note that this is different from the usual addition of points using rectangular coordinates. We will denote the unit vector in the direction u by $\vec{u} = (1, \theta_1, \ldots, \theta_{N-1})$ where $\vec{u}$ is expressed here in hyperspherical coordinates where the first coordinate is $|\vec{u}|$.

Now we can discuss the behavior of h_E under transformations of E. We have $h_{E+x}(t) = h_E(t) + x \cdot t$ for any $x \in \mathbb{R}^N$, $h_{E+u}(t+u) = h_E(t)$ for any $u \in \mathcal{U}^N$, $h_{E+F}(t) = h_E(t) + h_F(t)$, and $h_{\lambda E}(t) = \lambda h_E(t)$ for $\lambda \geq 0$ [49, Theorems 3.4.2 and 3.4.4].

We also wish to consider the curvature of E. We will often refer to curvature and to radii of curvature. If κ is a curvature, then $1/\kappa$ is a radius of curvature. Assume the hypersurface ∂E is at least C^2. Then the *Gauss map of* ∂E is given by $N : \partial E \to \partial B(0,1)$ where $N(y)$ is the unit normal vector to ∂E at y. The *shape operator* is the differential of the Gauss map, given by $S(y) = \nabla N(y)$. The eigenvalues of the shape operator are the principal curvatures of the surface. Its determinant is called the Gaussian curvature (or Gauss-Kronecker curvature). For a more complete discussion of curvatures, see [18, p. 129] and [52, Chapter 4].

For example, if E lies in the plane and ∂E is C^2 in a neighborhood of y, then the surface may be parametrized in a neighborhood of y by $(t, \phi(t))$ where $\phi : \mathbb{R} \to \mathbb{R}$ is C^2. Then the Gauss map in this neighborhood is given by $N(t) = (\phi'(t), 1)/|(\phi'(t), 1)|$

and the shape operator is given by

$$S(t) = \frac{\phi''(t)}{\left(\phi'(t)^2 + 1\right)^{3/2}}$$

which is the curvature at $(t, \phi(t))$. If $E \subset \mathbb{R}^3$, the shape operator is sometimes referred to as the second fundamental form, denoted II, of the surface ∂E. However, most authors distinguish between the two as the second fundamental form is a bilinear form arising from the shape operator.

We can now offer necessary and sufficient conditions for higher-order differentiability of the farthest distance function d_E.

Theorem 2.1. *Let E be a convex set containing at least two points and let $U \subset \mathbb{R}^N$ be an open set where y_E is single-valued. Then y_E is continuous and d_E is differentiable. Furthermore, $d_E \in C^{k+1}(U)$ if and only if $y_E \in C^k(U)$.*

Theorem 2.2. *Let $x \in \mathbb{R}^N$ such that $y_E(x)$ is single-valued. Assume that all principal curvatures of ∂E at $y_E(x)$ exist, are finite, and are not equal to $1/d_E(x)$, meaning x does not lie on the evolute of ∂E. Then $y_E \in C^k$ in a neighborhood of x if one of the following holds:*

1. *The surface ∂E is C^{k+1} in a neighborhood of $y_E(x)$.*

2. *The support function h_E is C^{k+2} on a neighborhood of $y_E(x) - x$.*

The conditions given in Theorem 2.2 are sufficient but not necessary. If $y_E(U)$ is a single vertex, the boundary ∂E and support function h_E are certainly not at all smooth, but y_E and d_E are both C^∞. The condition that $d_E(x)$ not be a radius of curvature excludes the case when ∂E at $y_E(x)$ nearly approximates a disk with center x.

We will now consider the properties of d_E and d_{E_n} for sets E_n that approximate E in a suitable sense. We begin with a uniqueness statement.

Proposition 2.4. *Let E and F be compact sets. Then $d_E = d_F$ if and only if* $\operatorname{conv}(E) = \operatorname{conv}(F)$*. In fact,*

$$\operatorname{conv}(E) = \bigcap_{x \in \mathbb{R}^N} \overline{B}\left(x, d_E(x)\right).$$

Thus the farthest distance function uniquely determines a compact convex set.

The proof uses a given farthest distance function to construct a unique compact convex set. In fact, any positive real valued function f on $\mathbb{R}^N$ can be used to construct a unique set by calculating $\bigcap_{x\in\mathbb{R}^N} \overline{B}\left(x, f(x)\right)$. If this set is nonempty, then it is compact and convex. However, there is no guarantee that f is its farthest distance function. For example, the functions $f(x) = |x|^2 + 5/4$ and $g(x) = 100|x| + 1$ both generate the unit ball $B(0,1)$ using this construction although neither is a farthest distance function.

An immediate consequence of the construction used in the proof of Proposition 2.4 is containment of a set based on the majorization of its farthest distance function.

Corollary 2.1. *Let E and F be compact convex sets. Then $d_E \le d_F$ if and only if $E \subset F$.*

Proof. It is clear that if $E \subset F$ then $d_E \le d_F$. For the opposite conclusion, assume $d_E \le d_F$. By Proposition 2.4, we know that $E = \bigcap_{x\in\mathbb{R}^N} B\left(x, d_E(x)\right)$ and $F = \bigcap_{x\in\mathbb{R}^N} B\left(x, d_F(x)\right)$. Since $B\left(x, d_E(x)\right) \subset B\left(x, d_F(x)\right)$ for each $x \in \mathbb{R}^N$, it follows that $E \subset F$. $\square$

More generally, uniform convergence of farthest distance functions is equivalent to convergence of sets under the Hausdorff metric D_H. The Hausdorff distance between two sets X and Y is defined as

$$D_H(X,Y) := \inf\left\{\epsilon \ge 0 : X \subset Y + B(0,\epsilon) \text{ and } Y \subset X + B(0,\epsilon)\right\}.$$

The set $X + B(0,\epsilon)$ is often referred to as the parallel body of X at distance ϵ. Equivalently, we may write

$$D_H(X,Y) = \max\left\{\inf_{x\in X}\sup_{y\in Y}|x-y|, \inf_{y\in Y}\sup_{x\in X}|x-y|\right\}.$$

The Hausdorff distance is not a metric on bounded subsets of $\mathbb{R}^N$. For example, for any open set U we have $D_H(U,\overline{U}) = 0$ even though $U \neq \overline{U}$. However, it is a metric on compact convex subsets of $\mathbb{R}^N$ [28, p. 84].

Lemma 2.1. *Let E_n be a sequence of bounded convex sets. The Hausdorff distances $D_H(E,E_n)$ converge to zero as $n\to\infty$ if and only if the farthest distance functions d_{E_n} converge uniformly to d_E as $n\to\infty$.*

Here, uniform convergence refers to convergence in the uniform (or sup) norm. We make a short digression here to provide an alternative characterization of d_E. For a point $x\in\mathbb{R}^N$, we have

$$D_H(x,E) = \max\left\{\sup_{y\in E}|x-y|, \inf_{y\in E}|x-y|\right\}$$

and thus $d_E(x) = D_H(x,E)$. Using a formula relating Hausdorff distance to the support function [49, Theorem 3.1.10], we can express d_E in terms of the support function of E by

$$d_E(x) = \sup_{|t|=1}\left|h_E(t) - x\cdot t\right|. \tag{2.2}$$

Thus we may use any positively homogeneous and subadditive function to construct the farthest distance function of some compact, convex set.

Given a set E, we wish to establish some specific sequences of sets E_n such that d_{E_n} has desirable properties and Lemma 2.1 applies. We will examine polytopes more carefully in Section 2.2, but for now we simply establish a useful convergence property.

Proposition 2.5. *Let $\{x_i\}\subset\partial E$ be a dense sequence of points and let E_n be the convex hull of $\{x_i\}_{i=1}^n$. Then d_{E_n} form a monotone increasing sequence of functions which converge uniformly to d_E as $n\to\infty$.*

At the other end of the spectrum, we also wish to approximate E with particularly smooth sets. We can do this via convolution of the support function with a smoothing kernel. Recall that h_E is determined by its values on unit vectors. Thus we can consider a support function that is defined only on directions. We define the *restricted support function* $p_E : \mathcal{U}^N \to \mathbb{R}$ by

$$p_E(u) = \sup_{y \in E} y \cdot \vec{u} = h_E(\vec{u}).$$

A related function is the *width function*, $w_E : \mathcal{U}^N \to \mathbb{R}$, given by

$$w_E(u) = p_E(u) + p_E(u + \psi)$$

where $\psi = (0, \pi, 0, \ldots, 0) \in \mathcal{U}^N$ plays a role similar to the role of π in polar coordinates. In the plane, the width $w_E(u)$ is the distance between parallel support lines.

We use the smoothing kernels on $\mathcal{U}^N \subset \mathbb{R}^{N-1}$ given in [2, p. 69] as

$$\gamma_n(u) = n^{N-1}\gamma(1 - n^2|u|^2),$$

where $|u|$ is calculated as the usual norm in $\mathbb{R}^{N-1}$,

$$\gamma(t) = \begin{cases} C_{N-1}e^{-1/t} & t > 0 \\ 0 & t \leq 0 \end{cases},$$

and C_{N-1} is a normalizing constant chosen so that

$$\omega_{N-1} \int_0^1 t^{N-2}\gamma(1 - t^2)\,\mathrm{d}t = 1.$$

Each kernel γ_n is supported on $B(0, 1/n) \subset \mathbb{R}^{N-1}$. Further, given any continuous function f, the convolution

$$(\gamma_n * f)(t) = \int \gamma_n(u - t)f(u)\,\mathrm{d}u = \int \gamma_n(u)f(t + u)\,\mathrm{d}u$$

converges uniformly to f as $n \to \infty$.

Lemma 2.2. *If $E \subset \mathbb{R}^N$ is a compact convex set, then $\gamma_n * p_E$ is the restricted support function of a compact convex set.*

Proposition 2.6. *Let E be a compact convex set which is the closure of a domain in $\mathbb{R}^N$. There is an increasing sequence of compact convex sets $E_n \subset E$ with C^2 boundaries such that d_{E_n} converge uniformly to d_E. Furthermore, the sets E_n have the following properties:*

1. *All principle curvatures exist and are finite for each E_n.*

2. *The restricted support functions p_{E_n} are C^∞.*

It follows from Proposition 2.6 that at any point x which is not a center of curvature of ∂E at $y_E(x)$, the farthest distance function d_E can be approximated by a sequence of farthest distance functions d_{E_n} which are C^∞ in a neighborhood of x.

2.2 Farthest Distance Function for Polytopes

The farthest distance functions of polytopes have some special properties and are studied extensively in computational geometry. We begin with an example of the simplest polytope, the line segment.

Example 2.4 (Line Segment)**.** *Let $L = [-a, a]$ be a line segment in $\mathbb{R}^N$ centered at the origin. Then $d_L(x) = \max\{|x-a|, |x+a|\}$. As shown in Figure 2.3, d_L breaks $\mathbb{R}^N$ into two half spaces.*

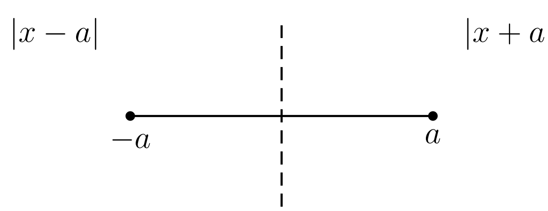

Figure 2.3: Farthest distance function for the segment L

The example of the segment is an important one that we will return to repeatedly. For any polytope P, the farthest distance function d_P behaves locally like d_L almost everywhere. We consider the triangle to illustrate.

Example 2.5 (Regular Triangle)**.** *Let T be the regular triangle in $\mathbb{R}^2$ with vertices a, b, c. Its farthest distance function is given by $d_T(z) = \max\{|z-a|, |z-b|, |z-c|\}$. As in the case of the segment, d_T a is piecewise defined function that breaks the plane into three regions as shown in Figure 2.4. Locally $d_T = d_L$ almost everywhere where L is chosen as an appropriate side of the triangle. The only exception is the centroid of the triangle where $d_T(z) = |z-a| = |z-b| = |z-c|$.*

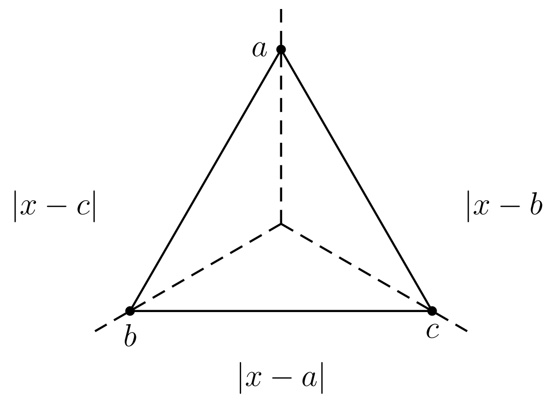

Figure 2.4: Farthest distance function for the regular triangle T

This example is typical for polytopes. Let P be a convex polytope in $\mathbb{R}^N$ and let $\mathcal{V}(P)$ denote the outer vertices of P, or the set of vertices which cannot be expressed as a convex combination of other vertices. We define F_v, the *farthest point Voronoi cell of* v, as the set of points $x \in \mathbb{R}^N$ such that $d_P(x) = |x-v|$. In Example 2.5, there were three farthest point Voronoi cells. The farthest point Voronoi cells are convex and unbounded [15, p. 164, 165], they cover the space so that $\bigcup_{v\in\mathcal{V}(P)} F_v = \mathbb{R}^N$, and the interiors of the cells are disjoint. The boundaries of the cells lie on hyperplanes. Specifically, if $F_v \cap F_w$ is nonempty, then it is a closed convex subset of the hyperplane which is the perpendicular bisector of the segment $[v, w]$.

The function

$$d_P(x) = d_{\mathcal{V}(P)}(x) = \max_{v \in \mathcal{V}(P)} |x - v|$$

is piecewise defined on the farthest point Voronoi cells. Furthermore, almost every $x \in \mathbb{R}^N$ has an open neighborhood where $d_P = d_L$ for an appropriate choice of L. The set of points where this is not the case consists of the intersections of three of more farthest point Voronoi cells and is contained in a finite intersection of $N-2$-dimensional hyperplanes. We give two more examples, one in the plane and one in space.

Example 2.6 (Parallelogram)**.** *Let P be the parallelogram shown in Figure 2.5 with vertices a, b, c, and d. The farthest point Voronoi cells are shown. They intersect on the dashed lines which lie along the perpendicular bisectors of the segments $[a,b]$, $[b,d]$, $[d,c]$, $[c,a]$, and $[a,d]$. Note that, unlike in the triangle example, three cells intersect at two distinct points.*

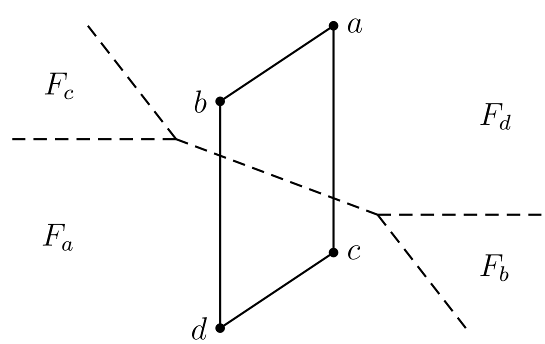

Figure 2.5: Farthest point Voronoi cells for a parallelogram

Example 2.7 (Regular Tetrahedron)**.** *Let T be the regular tetrahedron in $\mathbb{R}^3$. Consider just two vertices, v and w, and the edge between them. The plane of points equidistant from v and w, which is the perpendicular bisector of the segment $[v,w]$, passes through the centroid of the tetrahedron and contains the opposite edge of the tetrahedron. This plane is the boundary of a half-space containing w but not con-*

taining v. The farthest point Voronoi cell F_v is found by intersecting three such half-spaces, one for each edge adjacent to v. Their intersection is a 3-sided unbounded pyramid whose apex is the centroid of the tetrahedron and which contains the face of the tetrahedron opposite from v, as shown in Figure 2.6.

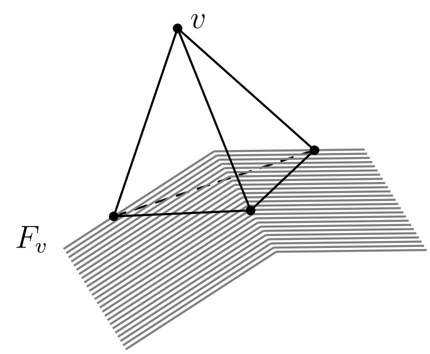

Figure 2.6: The farthest point Voronoi cell for vertex v of T

As in the plane, $d_T = d_L$ locally almost everywhere for an appropriate choice of the line segment L. This is not true where three or more vertices are farthest. This includes the centroid, where all four vertices are equidistant, and portions of the lines passing through the centroids of each face of the tetrahedron.

We refer to the collection of farthest point Voronoi cells and their boundary components as a farthest point Voronoi diagram. The farthest point Voronoi diagrams, and particularly methods to efficiently compute them, are a topic of interest in computational geometry, see [7]. Generalizations that are of current research interest include order-K Voronoi diagrams [1], farthest colored Voronoi diagrams [68], and farthest line or polygon Voronoi diagrams [4, 14].

In the plane, the farthest point Voronoi diagram for a polygon may be computed in $O(n \log n)$ time where n is the number of vertices [64]. It can be computed in linear time as well, but doing so requires a recursive algorithm that involves large constants [46, 47]. For any $v, w \in \mathcal{V}(P)$, the boundary segment $F_v \cap F_w$ is a (possibly unbounded) line segment. The number of such segments is $O(n)$ [15, Theorem 7.14], [24, Chapter III]. The lower limit is actually n, as seen in the case of the regular triangle. The upper limit is $2n - 3$ as seen in the case of the parallelogram.

We complete this section by returning to the regular triangle and considering a related set which combines the example of the regular triangle with our first example in Section 2.1, the ball. We are particularly interested in this set because it is a body of constant width, meaning its width function is constant.

Example 2.8 (Reuleaux Triangle)**.** *Let $T \subset \mathbb{R}^2$ be the regular triangle with vertices a, b, and c from Example 2.5, drawn in gray in Figure 2.7. Assume T has side length 1. Then we form the Reuleaux triangle by intersecting the three disks $B(a, 1)$, $B(b, 1)$, and $B(c, 1)$. The result R is outlined with a solid black line in Figure 2.7. The width function $w_R \equiv 1$ since any two distinct parallel support lines will intersect R at a vertex v and on the boundary of the ball $B(v, 1)$. The farthest distance function d_R is piecewise defined on six domains in $\mathbb{R}^2$ and their boundaries. The dashed black lines in the figure separate the domains. We observe that in three domains, we have $d_R(x) = d_T(x) = |x - v|$, while in the remaining three domains, we have $d_R(x) = d_{B(v,1)}(x) = |x - v| + 1$ where v is one of the vertices of the triangle: a, b, or c.*

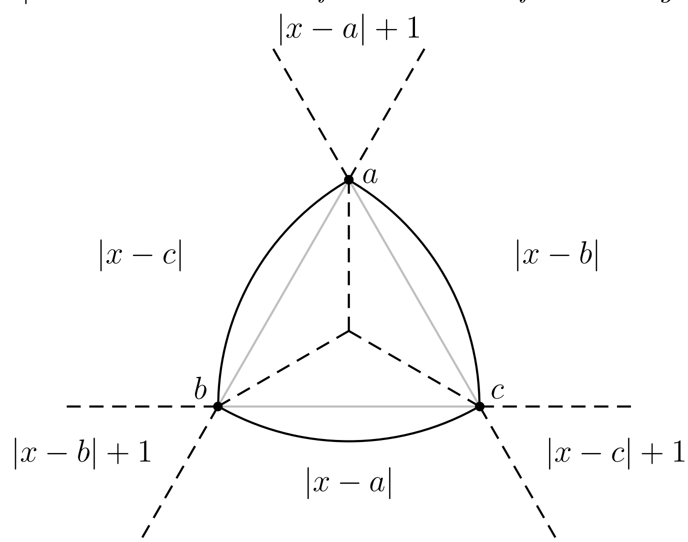

Figure 2.7: Farthest distance function for the Reuleaux triangle R

2.3 Representation via Newtonian Potentials

We now turn our attention to showing that d_E can be expressed via potentials. This section focuses on Newtonian potentials in $\mathbb{R}^N$, $N \geq 3$, and closely follows similar work done on logarithmic potentials in $\mathbb{R}^2$ by Boyd [12], Pritsker [53], and Laugesen and Pritsker [44]. We will consider Riesz potentials in Chapter 4.

We begin with some basic definitions and theorems regarding superharmonic functions drawn primarily from texts by Armitage and Gardiner [2] and Landkof [43]. In $\mathbb{R}^N$, $N \geq 2$, harmonic and superharmonic functions fill roles analogous to straight lines and concave functions in $\mathbb{R}$. Let $D \subset \mathbb{R}^N$ be a domain, meaning an open connected set.

Definition 2.1. *A real-valued function u on a domain D is* lower semicontinuous *(l.s.c.) if it takes values in $(-\infty, +\infty]$ and the set $\{x|u(x) > a\}$ is open in D for all real numbers a.*

For example, the function u given by $u(0) = 0$ and $u(x) = 1$ for all other $x \in \mathbb{R}^N$ is l.s.c. For convenience we exclude functions which are identically $+\infty$. An equivalent definition is that a function $u : G \to (-\infty, +\infty]$ is l.s.c. if $u(x) \leq \liminf_{t\to x} u(t)$ for every $x \in D$ [2, Theorem 3.1.3 (i)].

We say u is upper semicontinuous (u.s.c.) if $-u$ is l.s.c. Clearly, u is continuous if it is both u.s.c. and l.s.c. In fact, we can often approximate l.s.c. functions by continuous ones.

Proposition 2.7. *Let $u : D \to (-\infty, +\infty]$ be a l.s.c. function on a domain D which is bounded below. Then there exists an increasing sequence of continuous functions whose pointwise limit is u [2, Lemma 3.2.1].*

Just as convex functions 'sit below' and concave functions 'sit above' straight lines, subharmonic functions 'sit below' and superharmonic functions 'sit above' harmonic functions.

Definition 2.2. *A l.s.c. function u on a domain D is* superharmonic *if for every open ball B such that $\overline{B} \subset D$ and for every function h that is continuous on $\overline{B}$ and harmonic on B with the property that $h(x) \leq u(x)$ on ∂B, then we have $h(x) \leq u(x)$ on B.*

A function u is said to be subharmonic if $-u$ is superharmonic. Just as harmonic functions satisfy the Mean Value Property and Laplace's Equation, superharmonic functions satisfy a Mean Value Inequality and a Laplace Inequality. We introduce some notation before stating these properties.

For a domain D and a superharmonic function $u \in C^2(D)$, let Δu denote the Laplacian of u on D. Let

$$\omega_N := \frac{2\pi^{N/2}}{\Gamma(N/2)}$$

denote the surface area of a unit sphere in $\mathbb{R}^N$ [43, p. 18]. We denote the normalized surface area measure on the sphere of radius r by $\mathrm{d}S/(\omega_N r^{N-1})$ where $\mathrm{d}S$ is the surface area element. We define

$$\mathcal{M}(u; x, r) := \int_{\partial B(x,r)} u(t) \frac{\mathrm{d}S(t)}{\omega_N r^{N-1}}$$

as the average of the function u over a sphere centered at x of radius r.

Proposition 2.8. *[Mean Value Inequality] A l.s.c. function u on a domain D is superharmonic if and only if*

$$u(x) \geq \mathcal{M}(u; x, r) \tag{2.3}$$

for every center $x \in D$ and every radius r such that $\overline{B(x,r)} \subset D$ [2, Theorem 3.1.3 (ii)].

Proposition 2.9. *[Laplace Inequality] A function $u \in C^2(D)$ is superharmonic in D if and only if $\Delta u \leq 0$ everywhere in D [32, p. 40].*

Proposition 2.10. *[Minimum Principle] Let u be a superharmonic function on a domain $D \subset \mathbb{R}^N$. If u attains a global minimum on D then u is constant [43, p. 55].*

We now turn our attention back to the farthest distance function and our goal of expressing it via potentials. If $E \subset \mathbb{R}^2$, then it has been shown that $-\log d_E$ is superharmonic [53, Lemma 5.1]. In higher dimensions, we will consider d_E^{2-N}.

Example 2.9 (Ball)**.** *Consider the ball $B = B(0,R)$ in $\mathbb{R}^N$, $N \geq 3$. Recall that $d_B(x) = |x| + R$. In hyperspherical coordinates, we write this as $d_B(x) = \rho + R$. Note that $d_B \in C^2\left(\mathbb{R}^N \setminus 0\right)$ and so we shall consider d_B on this domain and at the origin separately.*

We will use the Laplace Inequality to show that d_B^{2-N} is superharmonic in $\mathbb{R}^N \setminus 0$. For a real-valued function f depending only on ρ, the Laplacian is given in hyperspherical coordinates by [34, p. 10]:

$$\Delta f = \frac{\partial^2 f}{\partial \rho^2} + \frac{\partial f}{\partial \rho}\frac{N-1}{\rho}.$$

We calculate that in $\mathbb{R}^N \setminus 0$,

$$\begin{aligned}
\Delta d_B^{2-N} &= (2-N)(1-N)(\rho+R)^{-N} + (2-N)(\rho+R)^{1-N}\frac{N-1}{\rho} \\
&= (N-2)(N-1)(\rho+R)^{-N}\left(1 - \frac{\rho+R}{\rho}\right) \\
&= (N-2)(N-1)(\rho+R)^{-N}\left(\frac{-R}{\rho}\right)
\end{aligned}$$

which is negative for all $\rho > 0$. By the Laplace Inequality, Proposition 2.9, d_B^{2-N} is superharmonic in $\mathbb{R}^N \setminus 0$.

At the origin, for any $r > 0$ we have

$$d_B^{2-N}(0) = R^{2-N} > (r+R)^{2-N} = \mathcal{M}(d_B^{2-N}; 0, r)$$

and hence by the Mean Value Inequality, Proposition 2.8, d_B^{2-N} is superharmonic at the origin. It follows that d_B^{2-N} is superharmonic in $\mathbb{R}^N$.

We now consider arbitrary sets $E \subset \mathbb{R}^N$, $N \geq 3$. We begin by expressing d_E^{2-N} as

$$d_E^{2-N}(x) = \left(\sup_{y \in E} |x-y|\right)^{2-N} = \inf_{y \in E} |x-y|^{2-N}$$

where $d_y^{2-N}(x) = |x-y|^{2-N}$ is the Newtonian kernel. Since $d_y^{2-N}(x) \in C^2(\mathbb{R}^N \setminus y)$, we may calculate the Laplacian to show that it is superharmonic. In rectangular coordinates,

$$d_y^{2-N}(x) = \left[\sum_{i=1}^{N} (x_i - y_i)^2\right]^{1-N/2}$$

and the partial derivatives are given by

$$\begin{aligned}
\frac{\partial d_y^{2-N}}{\partial x_i} &= (2-N) d_y^{-N} (x_i - y_i) \\
\frac{\partial^2 d_y^{2-N}}{\partial x_i^2} &= (2-N) d_y^{-N} \left(\frac{-N(x_i - y_i)^2}{d_y^2} + 1 \right).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\Delta d_y^{2-N} &= \sum_{i=1}^{N} \frac{\partial^2 d_y^{2-N}}{\partial x_i^2} \\
&= (2-N) d_y^{-N} \left(\frac{-N d_y^2}{d_y^2} + N \right) \\
&= 0
\end{aligned}$$

and d_y^{2-N} is harmonic except at y. At the point y, we have $d_y^{2-N}(y) = \infty$ and thus by the Mean Value Inequality, Proposition 2.8, d_y^{2-N} is superharmonic at y and in all of $\mathbb{R}^N$. We have now expressed $d_E^{2-N} = \inf_{y \in E} d_y^{2-N}$ as an infimum of superharmonic functions.

Theorem 2.3. *Let K be a compact topological space and let $v : D \times K \to [-\infty, +\infty)$ be a function such that*

1. *v is lower semicontinuous on $D \times K$ and*

2. *$x \to v(x, y)$ is superharmonic in D for each $y \in K$.*

Then the function u given by $u(x) = \inf_{y \in K} v(x, y)$ is superharmonic on D.

Since by Proposition 2.1 we may assume E is compact whenever convenient, it follows that d_E^{2-N} is superharmonic for any bounded set $E \subset \mathbb{R}^N$, $N \geq 3$. We've already seen that $d_y^{2-N}(x) = |x-y|^{2-N}$ is harmonic in $\mathbb{R}^N \setminus y$. In a sense, this is the only case in which d_E^{2-N} is harmonic.

Proposition 2.11. *Let D be a domain in $\mathbb{R}^N$, $N \geq 3$. Then d_E^{2-N} is harmonic in D if and only if there exists a point $y \in \partial E \setminus D$ such that $d_E(x) = d_y(x)$ for all $x \in D$.*

Proposition 2.11 also holds in the plane $\mathbb{R}^2$ for the superharmonic function $(-\log d_E)$ [44, Proposition 2.1].

Example 2.10 (Line Segment). *Let $L = [-a, a] \subset \mathbb{R}^N$, $N \geq 3$, be the line segment we considered in Example 2.4. The function $d_L^{2-N} = \min\left\{|x-a|^{2-N}, |x+a|^{2-N}\right\}$ is harmonic almost everywhere, the exception being the hyperplane which is the perpendicular bisector of the segment L where $|x-a| = |x+a|$.*

The Riesz Decomposition Theorem states that any superharmonic function is the sum of a potential and a harmonic function. The theorem is stated here for Newtonian potentials in $\mathbb{R}^N$, $N \geq 3$. This theorem also holds in $\mathbb{R}^2$ with the logarithmic potential in place of the Newtonian one.

Theorem 2.4 (Riesz Decomposition Theorem). *Let u be a superharmonic function in a domain $D \subset \mathbb{R}^N$, $N \geq 3$. Then there exists a unique positive Borel measure μ on D such that for every compact set $K \subset D$,*

$$u(x) = \int_K |x-y|^{2-N} \, \mathrm{d}\mu(y) + h_K(x)$$

where h_K is a harmonic function on the interior of K.

We call the measure μ the *representing measure* of u. We will denote the Newtonian potential of μ by

$$U^\mu(x) := \int |x-y|^{2-N} \, \mathrm{d}\mu(y).$$

There are many published proofs of this important theorem. All proofs choose μ to be the Laplacian of u, in some suitable sense, and then show that the desired properties hold. Most proofs rely on a distributional Laplacian to construct μ since u need not even be continuous, c.f. Landkof [43], Hayman and Kennedy [32], and Armitage and Gardiner [2]. Another approach, used by Helms [34], is to approximate u by sufficiently smooth functions for which the Laplacian does exist.

An immediate consequence of the proof is that the harmonic function h_K must be the greatest harmonic minorant of u on K. See [2, Section 3.4] for a discussion of harmonic minorants. If $D = \mathbb{R}^N$ and u is bounded below then the greatest harmonic minorant is a constant.

Theorem 2.5. *Let u be a non-negative superharmonic function in $\mathbb{R}^N$ and let μ be its representing measure. Then*

$$u(x) = \int_{\mathbb{R}^N} |x-y|^{2-N}\, \mathrm{d}\mu(y) + c = U^{\mu}(x) + c$$

where c is the largest non-negative constant bounding u from below [2, p. 105-106].

Recall that for a bounded set $E \subset \mathbb{R}^N$, the function d_E^{2-N} is superharmonic in $\mathbb{R}^N$ and hence the Riesz Representation Theorem applies. Clearly, $d_E(x)$ is always nonnegative and grows linearly as $|x| \to \infty$. Thus $d_E^{2-N}(x)$ is positive everywhere and arbitrarily close to zero in a neighborhood of infinity. Theorem 2.5 applies and the constant given must be zero. It follows that d_E^{2-N} is the Newtonian potential of some measure.

So far we have only required that $E \subset \mathbb{R}^N$ be bounded in order to avoid the case where $d_E \equiv +\infty$. We must now also assume that E contains at least two points to avoid another degenerate case where $d_E(y) = 0$ for some $y \in \mathbb{R}^N$.

Theorem 2.6. *Let E be a bounded subset of $\mathbb{R}^N$, $N \geq 3$, consisting of at least two points. Then there exists a unique positive unit Borel measure σ_E with unbounded support on $\mathbb{R}^N$ such that*

$$d_E^{2-N}(x) = \int_{\mathbb{R}^N} |x-y|^{2-N}\, \mathrm{d}\sigma_E(y) = U^{\sigma_E}(x). \tag{2.4}$$

This theorem was first proved for bounded subsets E in the plane consisting of at least two points. In that case the theorem states that $\log d_E$ is the logarithmic potential of σ_E, a unique positive unit Borel measure with unbounded support. It was proved for finite sets, such as the set of vertices of a polygon, by Boyd [12, p. 450] and later for general compact sets by Pritsker [53, p. 3980]. Alternative proofs are presented in [44] and [25].

We will explore the properties of the representing measure σ_E in Chapter 3. Many of those results are straightforward generalizations of previous results in the plane while other results are new. While the discussion will focus on the higher dimensional case, this is simply for convenience. Unless otherwise noted, the results and proofs are given for $E \subset \mathbb{R}^N$, $N \geq 2$. We will consider analogues of Theorem 2.6 for Riesz potentials in Chapter 4.

2.4 Proofs

Proof of Proposition 2.1. Let $x \in \mathbb{R}^N$. Since $d_E(x) = \sup_{y\in E} |x - y|$, it is clear that $d_E = d_{\overline{E}}$.

The closure $\overline{E}$ is compact and hence there exists a point $y \in \overline{E}$ such that $d_E(x) = |x - y|$. If y is in the interior of E then there exists $\epsilon > 0$ such that $B(y, \epsilon) \subset E$ and hence $d_E(x) \geq |x - y| + \epsilon$ which is a contradiction. Thus $y \in \partial E$ and it follows that $d_{\overline{E}} = d_{\partial E}$.

We turn to the final equality. It is clear that $d_E \leq d_{\text{conv}(E)}$ since $E \subset \text{conv}(E)$. Without loss of generality, we may assume E is closed and hence so is $\text{conv}(E)$. Let $x \in \mathbb{R}^N$ and choose $y \in \text{conv}(E)$ such that $d_E(x) = |x - y|$. We express y as a convex combination of points in E. Let $y_1, \ldots, y_m \in E$ and $t_1, \ldots t_m \in [0, 1]$ with $\sum_{j=1}^m t_j = 1$ so that $y = \sum_{j=1}^m t_j y_j$. Then we have

$$d_{\text{conv}(E)}(x) = \left| x - \sum_{j=1}^m t_j y_j \right| \leq \sum_{j=1}^m t_j |x - y_j| \leq d_E(x)$$

and hence $d_E = d_{\mathrm{conv}(E)}$. $\square$

Proof of Proposition 2.2. Since rigid motions preserve distances, the first claim is clear. We turn our attention to scaling. Let $\lambda > 0$. Then

$$\begin{aligned} d_{\lambda E}(\lambda x) &= \max_{y \in E} |\lambda x - \lambda y| \\ &= \lambda \max_{y \in E} |x - y| \\ &= \lambda d_E\left(x\right), \end{aligned}$$

and the second claim holds. $\square$

Sketch of proof that y_E is not single-valued on all of conv(E). If each point x in a compact convex set E has a unique farthest point then the map $x \to y_E(x)$ gives a continuous mapping. By Brouwer's fixed point theorem this map has a fixed point. Since E has at least two points, this is a contradiction. Thus at least one point in E has two distinct farthest points. $\square$

Proof of Theorem 2.1. In the case that $k = 0$, this statement was proved under more general conditions in [22, Corollary 3.6]. The proof given here uses the same ideas. For ease in computation, we define $f(x) = d_E^2(x)$. Since $d_E(x) > 0$, it will suffice to show that $f \in C^{k+1}(U)$. Let $\vec{u} \in \partial B(0,1)$ and $r > 0$. We compute

$$\begin{aligned} &\limsup_{r \to 0} \frac{1}{r}\left(d_E^2(x) - d_E^2(x + r\vec{u}) - r\vec{u} \cdot \Big(-2\left(x - y_E(x)\right)\Big)\right) \\ &\qquad \leq \limsup_{r \to 0} \frac{1}{r}\left(\left|x - y_E(x)\right|^2 - \left|x - y_E(x) + r\vec{u}\right|^2 + 2r\vec{u} \cdot \left(x - y_E(x)\right)\right). \end{aligned}$$

Since

$$\left|x - y_E(x) + r\vec{u}\right|^2 = \left(\left|x - y_E(x)\right|^2 + 2r\vec{u} \cdot \left(x - y_E(x)\right) + r^2\right),$$

we conclude that

$$\limsup_{r \to 0} \frac{1}{r}\left(d_E^2(x) - d_E^2(x + r\vec{u}) - r\vec{u} \cdot \Big(-2\left(x - y_E(x)\right)\Big)\right) \leq \limsup_{r \to 0} \frac{-r^2}{r} = 0.$$

Since this inequality holds for $\vec{u}$ in any direction, it follows that

$$\lim_{r\to 0}\frac{1}{r}\left(d_E^2(x)-d_E^2(x+r\vec{u})-r\vec{u}\cdot\left(-2\left(x-y_E(x)\right)\right)\right)=0$$

and hence

$$\nabla f(x)=-2\left(x-y_E(x)\right).$$

Thus $f\in C^{k+1}(U)$ if and only if $y_E\in C^k(U)$. $\square$

Proof of Theorem 2.2. Let x^0 be point in $\mathbb{R}^N$. In the case $k=0$ there is nothing to prove since we already assume that y_E is single-valued on a neighborhood of x and hence continuous in that neighborhood by Theorem 2.1.

Let $k\geq 1$. We assume ∂E is C^{k+1} in a neighborhood of $y_E(x^0)$ for some $x^0\in U$ and show y_E is C^k in a neighborhood of x^0. This was already shown for $E\subset\mathbb{R}^2$ in [24, Theorem 2.6]. A similar statement for the nearest distance function was shown in [40, Theorem 3]. We mimic the approaches of those proofs here.

We create a sufficiently smooth parametrization of a neighborhood of $y_E(x^0)$. Denote the coordinates of $\mathbb{R}^N$ by $(x_1,\ldots,x_N)$ and $x^0=(x_1^0,\ldots,x_N^0)$. Translating and rotating if necessary, we assume that $y_E(x^0)=0$, E lies in the closed upper half-space given by $x_N\geq 0$, and the $x_N=0$ hyperplane is tangent to ∂E at 0. It follows that $x^0=(0,\ldots,0,d_E(x^0))$. For a sufficiently small neighborhood $V\subset\partial E$ of 0, there is a C^{k+1} map $\phi:(-\epsilon,\epsilon)^{N-1}\to\mathbb{R}$ such that $\big(t,\phi(t)\big)=\big(t_1,\ldots,t_{N-1},\phi(t_1,\ldots,t_{N-1})\big)$ is a parametrization of V. Furthermore, we may assume that $\phi(0)=0$, and $(\partial\phi/\partial t_i)(0)=0$ for each $i=0,\ldots,N-1$.

Let W be the inverse image of V under y_E, which is an open neighborhood of x^0. Let $t:W\to(-\epsilon,\epsilon)^{N-1}$ be the map such that $y_E(x)=\big(t(x),\phi\circ t(x)\big)$. Then $t(x)=\big(t_1(x),\ldots,t_{N-1}(x)\big)$ can be found by maximizing $\big|(t,\phi(t))-x\big|^2$. For each $i=0,\ldots,N-1$ we set

$$F_i(x,t)=(t_i-x_i)+\frac{\partial\phi}{\partial t_i}(t)(\phi(t)-x_N).$$

We wish to apply the Implicit Function Theorem [41, Theorem 3.3.1] to $F : W \times (-\epsilon, \epsilon)^{N-1} \to \mathbb{R}^{N-1}$ where $F(x,t) = \big(F_i(x,t)\big)$. Since $F(x^0, 0) = 0$, it remains to show that the Jacobian matrix $[\partial F_i/\partial t_j](x^0, 0)$ is invertible.

Calculating the partial derivatives we obtain

$$\frac{\partial F_i}{\partial t_j}(x,t) = \frac{\partial \phi}{\partial t_i}(t)\frac{\partial \phi}{\partial t_j}(t) + \frac{\partial^2 \phi}{\partial t_i \partial t_j}(t)\left(\phi(t) - x_N\right)$$

for each $i \neq j$ and

$$\frac{\partial F_i}{\partial t_i}(x,t) = 1 + \frac{\partial \phi}{\partial t_i}^2(t) + \frac{\partial^2 \phi}{\partial t_i^2}(t)\left(\phi(t) - x_N\right).$$

Thus at $(x^0, 0)$, we have

$$\left[\frac{\partial F_i}{\partial t_j}\right](x^0,0) = I - d_E(x^0)H_\phi(0) = -d_E(x^0)\left(H_\phi(0) - \frac{1}{d_E(x^0)}I\right)$$

where H_ϕ is the Hessian matrix of ϕ.

The Gauss map of V is given by $N(t) = v(t)/|v(t)|$ where $N(t)$ is the normal vector at the point $(t, \phi(t))$ and

$$v(t) = \left(\frac{\partial \phi}{\partial t_1}(t), \ldots, \frac{\partial \phi}{\partial t_{N-1}}(t), -1\right).$$

Thus the shape operator is given by $S(t) = H_\phi(t)$. Since no principal curvature at 0 is $1/d_E(x^0)$, it follows that $1/d_E(x^0)$ is not an eigenvalue of $H_\phi(0)$ and hence the Jacobian of F has nonzero determinant at 0. We conclude by the Implicit Function Theorem that $t \in C^k(W)$ and hence $y_E \in C^k(W)$ as desired.

Now we assume that h_E is C^{k+2} in a neighborhood of $y_E(x^0) - x^0$ and show that ∂E is C^{k+1} in a neighborhood of $y_E(x^0)$. Let $U \subset \mathcal{U}^N$ be a neighborhood of the direction of the vector $y_E(x^0) - x^0$ such that h_E is C^{k+2} on vectors $\vec{u}$ with $u \in U$. Let $V \subset \partial E$ be the set of points $y \in \partial E$ such that $h_E(\vec{u}) = y \cdot \vec{u}$ for some $u \in U$. Since E is strictly convex at $y_E(x^0)$, V is a neighborhood of $y_E(x^0)$. We will show V has a parametrization which is C^{k+1}. For later computational convenience, we assume that $\vec{u}$ has no zero rectangular coordinates for all $u \in \mathcal{U}^N$.

Let $F(\xi, u) = \xi \cdot \vec{u} - h_E(\vec{u})$ for $\xi \in \mathbb{R}^N$ and $u \in \mathcal{U}^N$. Then the hyperplanes of support for ∂E in V are given by the equations $F(\xi, u) = 0$ and $(\partial F/\partial \theta_i)(\xi, u) = 0$ for each i where $u = (0, \theta_1, \ldots, \theta_{N-1})$. Their envelope is V. Thus, solving these equations for the point ξ will yield a parametrization of V. If such a solution exists, the parametrization will depend on $u \in \mathcal{U}^N$ and be in terms of the first partial derivatives of h_E, hence it will be C^{k+1}.

We will show that such a solution exists by forming a matrix of the equations and showing that its determinant is nonzero. The equations can be written as

$$\xi \cdot \vec{u} = h_E(\vec{u})$$

and

$$\xi \cdot \frac{\partial \vec{u}}{\partial \theta_i} = \frac{\partial h_E(\vec{u})}{\partial \theta_i}$$

for each i. Recall that in rectangular coordinates the vector $\vec{u}$ is given by

$$\vec{u} = (\cos\theta_1, \sin\theta_1\cos\theta_2, \ldots, \sin\theta_1 \cdots \sin\theta_{N-2}\cos\theta_{N-1}, \sin\theta_1 \cdots \sin\theta_{N-2}\sin\theta_{N-1}).$$

We choose the N rectangular coordinates of ξ as the variables and form a matrix of the left hand sides of the equation. This yields the N by N matrix

$$\begin{bmatrix} u_1 & u_2 & u_3 & u_4 & \cdots & u_{N-1} & u_N \\ -\tan\theta_1 u_1 & \cot\theta_1 u_2 & \cot\theta_1 u_3 & \cot\theta_1 u_4 & \cdots & \cot\theta_1 u_{N-1} & \cot\theta_1 u_N \\ 0 & -\tan\theta_2 u_2 & \cot\theta_2 u_3 & \cot\theta_2 u_4 & \cdots & \cot\theta_2 u_{N-1} & \cot\theta_2 u_N \\ 0 & 0 & -\tan\theta_3 u_3 & \cot\theta_3 u_4 & \cdots & \cot\theta_3 u_{N-1} & \cot\theta_3 u_N \\ 0 & 0 & 0 & -\tan\theta_4 u_4 & \cdots & \cot\theta_4 u_{N-1} & \cot\theta_4 u_N \\ \vdots & \vdots & \vdots & & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -\tan\theta_{N-1} u_{N-1} & \cot\theta_{N-1} u_N \end{bmatrix}.$$

Since each component of $\vec{u}$ is nonzero, this matrix has nonzero determinant if the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ -\tan\theta_1 & \cot\theta_1 & \cot\theta_1 & \cot\theta_1 & \cdots & \cot\theta_1 & \cot\theta_1 \\ 0 & -\tan\theta_2 & \cot\theta_2 & \cot\theta_2 & \cdots & \cot\theta_2 & \cot\theta_2 \\ 0 & 0 & -\tan\theta_3 & \cot\theta_3 & \cdots & \cot\theta_3 & \cot\theta_3 \\ 0 & 0 & 0 & -\tan\theta_4 & \cdots & \cot\theta_4 & \cot\theta_4 \\ \vdots & \vdots & \vdots & & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -\tan\theta_{N-1} & \cot\theta_{N-1} \end{bmatrix}.$$

has nonzero determinant. We compute $\det(A)$ by expanding the minors using the first column. This yields

$$\det(A) = (\tan\theta_1 + \cot\theta_1) \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ -\tan\theta_2 & \cot\theta_2 & \cot\theta_2 & \cdots & \cot\theta_2 & \cot\theta_2 \\ 0 & -\tan\theta_3 & \cot\theta_3 & \cdots & \cot\theta_3 & \cot\theta_3 \\ 0 & 0 & -\tan\theta_4 & \cdots & \cot\theta_4 & \cot\theta_4 \\ \vdots & \vdots & & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\tan\theta_{N-1} & \cot\theta_{N-1} \end{vmatrix}.$$

Repeating this process we find

$$\det(A) = \prod_{i=1}^{N-1} (\tan\theta_i + \cot\theta_i)$$

which is nonzero.

$\square$

Proof of Proposition 2.4. This proof follows one given in [44, Prop 2.7] for the planar case. We begin by showing that

$$\operatorname{conv}(E) = \bigcap_{x\in\mathbb{R}^N} \overline{B}\left(x, d_E(x)\right).$$

It is clear that $\operatorname{conv}(E) \subset \bigcap_{x\in\mathbb{R}^N} \overline{B}\left(x, d_E(x)\right)$ since $\operatorname{conv}(E) \subset \overline{B}\left(x, d_E(x)\right)$ for every $x \in \mathbb{R}^N$. For the opposite inclusion we proceed by contradiction. Suppose

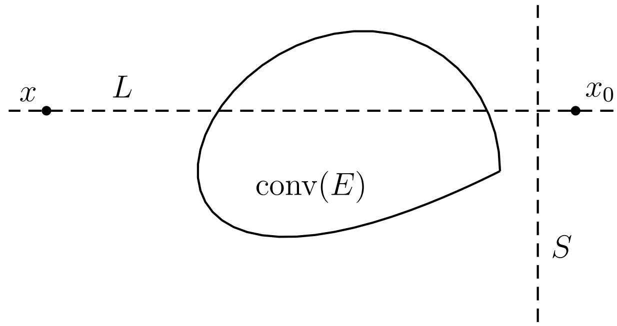

Figure 2.8: A convex set separated from a point by a hyperplane

there exists $x_0 \in \bigcap_{x\in\mathbb{R}^N} \overline{B}\left(x, d_E(x)\right)$ such that $x_0 \notin \mathrm{conv}(E)$. Since $\mathrm{conv}(E)$ is compact and convex, there exists a hyperplane S separating x_0 and $\mathrm{conv}(E)$, as shown in Figure 2.8. Let L be the line perpendicular to S passing through x_0. There exists $x \in L$ on the other side of S from x_0 with $|x|$ sufficiently large so that $\mathrm{conv}(E) \subset B\left(x, |x - x_0|\right)$. It follows that $d_E(x) < |x - x_0|$. But this contradicts the assumption that $x_0 \in \overline{B}\left(x, d_E(x)\right)$.

If $d_E = d_F$ then we have shown by this construction that $\mathrm{conv}(E) = \mathrm{conv}(F)$. On the other hand, if $\mathrm{conv}(E) = \mathrm{conv}(F)$ then Proposition 2.1 shows us that $d_E = d_F$. $\square$

Proof of Lemma 2.1. Begin by assuming $D_H(E, E_n)$ converges to 0. For any $\epsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that $D_H(E, E_n) < \epsilon$ for $n \geq n_0$. Since $E \subset E_n + B(0, \epsilon)$ and $E_n \subset E + B(0, \epsilon)$ it follows that $|d_{E_n} - d_E| < \epsilon$. Thus the functions d_{E_n} converge uniformly to d_E.

Now assume the farthest distance functions d_{E_n} converge uniformly to d_E. For any $\epsilon/2 > 0$, there exists an $n_0 \in \mathbb{N}$ such that $|d_{E_n} - d_E| < \epsilon/2$ for all $n \geq n_0$. In other words, $d_{E_n} < d_E + \epsilon/2$. Since $d_{E+B(0,\epsilon/2)} = d_E + \epsilon/2$, it follows from Corollary 2.1 that $E_n \subset E + B(0, \epsilon/2)$. Similarly, $E \subset E_n + B(0, \epsilon/2)$ and hence $D_H(E, E_n) \leq \epsilon/2 < \epsilon$. $\square$

Proof of Proposition 2.5. It is clear that the sets E_n form a monotone increasing sequence of subsets of E and thus d_{E_n} form a monotone increasing sequence of functions.

By compactness, for any $\epsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that $\partial E \subset \bigcup_{i=0}^{n} B(x_i, \epsilon)$ for all $n \geq n_0$. Thus, $E \subset E_n + B(0, \epsilon)$ for all $n \geq n_0$ and $D_H(E, E_n)$ tends to zero as $n \to \infty$. Applying Lemma 2.1, the conclusion follows. $\square$

Proof of Lemma 2.2. This proof is adapted from one given in [25, Lemma 1]. Consider the function

$$f(x) = \int \gamma_n(u) h_{E-u}(x)\, \mathrm{d}u.$$

Since h_E is subadditive and positively homogeneous of degree 1, the function f is as well. Hence it is the extended support function of a compact convex set [27, p. 20]. Since $h_{E-u}(x) = h_E(x+u)$, it follows that $f(x) = (\gamma_n * p_E)(v)$ for $x \in S^{N-1}$ and $\vec{v} = x$. $\square$

Proof of Proposition 2.6. From Lemma 2.2, for each $n \in \mathbb{N}$, there is a compact convex set L_n whose support function is $\gamma_n * p_E$. Denote the parallel body of L_n by $F_n = L_n + \overline{B}(0, 1/n)$. The sets F_n are compact, convex, smooth, and all radii of curvature are at least $1/n$. This sequence is not necessarily increasing. We will choose a subsequence and perform a suitable dilation to complete the proof. Assume, without loss of generality, that the origin lies in E and that $p_E(u) \geq 1$ for all $u \in \mathcal{U}^N$. Let m be an integer and let $\epsilon_m = 1/(2^{m+2} - 3)$. Then there exists F_{n_m} such that $|p_m - p_E| \leq \epsilon_m$ where $p_m = \gamma_{n_m} * p_E + 1/n_m$ is the support function of F_{n_m}. Let $\lambda_m = 1/(2^{m+2}\epsilon_m)$. The function $\lambda_m p_m$ is the support function of a set that is compact, convex, smooth, and has positive radii of curvature. We call it E_m and form a new sequence of sets $\{E_m\}_{m=1}^{\infty}$. The support functions $\lambda_m p_m$ still converge uniformly to p_E. It remains only to show that the sets E_m are increasing. We will show that

$$(1 - 2^{-m})p_E \leq \lambda_m p_m \leq (1 - 2^{-m-1})p_E.$$

It will then follow that $p_m \leq p_{m+1} \leq p_E$ and hence $E_m \subset E_{m+1} \subset E$.

First, note that

$$\lambda_m = \frac{1}{2^{m+2}\epsilon_m} = \frac{1 - 2^{-m}}{1 - \epsilon_m} = \frac{1 - 2^{-m-1}}{1 + \epsilon_m}.$$

We begin with the upper bound, calculating that

$$\begin{aligned}
(1-2^{-m-1})p_E - \lambda_m p_m &\geq (1-2^{-m-1})p_E - \frac{1-2^{-m-1}}{1+\epsilon_m}(p_E+\epsilon_m) \\
&\geq (1-2^{-m-1})\left(p_E\left(1-\frac{1}{1+\epsilon_m}\right)-\frac{\epsilon_m}{1+\epsilon_m}\right) \\
&\geq (1-2^{-m-1})\left(1-\frac{1}{1+\epsilon_m}-\frac{\epsilon_m}{1+\epsilon_m}\right) \\
&= 0
\end{aligned}$$

where the first step used the fact that $p_m \leq p_E + \epsilon_m$ and the third step used the facts that $1 - 1/(1+\epsilon_m) > 0$ and $p_E \geq 1$. Next we prove the lower bound by calculating that

$$\begin{aligned}
\lambda_m p_m - (1-2^{-m})p_E &\geq \frac{1-2^{-m}}{1-\epsilon_m}(p_E-\epsilon_m) - (1-2^{-m})p_E \\
&\geq (1-2^{-m})\left(p_E\left(\frac{1}{1-\epsilon_m}-1\right)-\frac{\epsilon_m}{1-\epsilon_m}\right) \\
&\geq (1-2^{-m})\left(\frac{1}{1-\epsilon_m}-1-\frac{\epsilon_m}{1-\epsilon_m}\right) \\
&= 0
\end{aligned}$$

where the first step used the fact that $p_m \geq p_E - \epsilon_m$ and the third step used the facts that $1/(1-\epsilon_m) - 1 > 0$ and $p_E \geq 1$. ☐

Proof of Theorem 2.3. This proof is adapted from one given in [60, p. 38] for a similar statement in $\mathbb{R}^2$. We begin by showing $u : D \to [-\infty, +\infty)$ is lower semi-continuous. Choose $a \in \mathbb{R}$. If the set $\{x \in D : u(x) > a\}$ is nonempty, let w be an element. We will find an open neighborhood of w in the set $\{x \in D : u(x) > a\}$. Since $u(w) = \inf_{y \in K} v(w,y) > a$, we have for each $k \in K$, that $v(w,k) > a$. The function v is lower semi-continuous, so for each $k \in K$ there exists a neighborhood of $(w,k) \in D \times K$ where v is greater than a. For each k, choose $B(w, r_k) \subset D$, with $r_k > 0$, and $N_k \subset K$ such that $v(x,y) > a$ for each $(x,y) \in B(w,r_k) \times N_k$. The neighborhoods N_k form an open cover of K. Choose a finite subcover $N_{k_1}, \ldots, N_{k_n}$.

Let $r = \min\{r_{k_1}, \ldots, r_{k_n}\} > 0$. Since $v(x, y) > a$ for all $(x, y) \in B(w, r) \times K$ we have $u(x) > a$ for $x \in B(w, r)$. Thus $B(w, r) \subset \{x \in D : u(x) > a\}$ and u is lower semi-continuous.

We now show u is superharmonic by showing it satisfies the Mean Value Inequality. Let $w \in D$ and choose $r > 0$ such that $\overline{B}(w, r) \subset D$. The function v is superharmonic and thus satisfies the Mean Value Inequality. For each $k \in K$, we have

$$v(w, k) \geq \frac{1}{\omega_N r^{N-1}} \int_{\partial B(w,r)} v(t, k)\, \mathrm{d}t \geq \frac{1}{\omega_N r^{N-1}} \int_{\partial B(w,r)} u(t)\, \mathrm{d}S(t).$$

By taking the infimum of the left hand side over K we find u also satisfies the Mean Value Inequality. $\square$

Proof of Proposition 2.11. This proof is adapted from one given in [44, Proposition 2.1] for a similar statement for $-\log d_E$ in $\mathbb{R}^2$. If $d_E(x) = d_y(x)$ everywhere in D for some $y \in \partial E \setminus D$, then we already showed that d_E^{2-N} is harmonic in D by calculating the Laplacian. Now assume d_E^{2-N} is harmonic in D. Choose $x_0 \in D$ and $y \in \partial E$ such that $d_E(x_0) = |x_0 - y|$. Define u on D by $u(x) = |x - y|^{2-N} - d_E^{2-N}(x)$. As the sum of a superharmonic function, $|x - y|^{2-N}$, and a harmonic function, $-d_E^{2-N}(x)$, u is superharmonic [43, p. 55]. It is also non-negative since $d_E(x) \geq |x - y|$ for every $x \in D$. Since $u(x_0) = 0$, we apply the Minimum Principle to see that $u \equiv 0$. Thus $d_E(x) = |x - y|$ in D. Since d_E^{2-N} is continuous in D, it follows that $y \notin D$. $\square$

Proof of Theorem 2.6. Applying the Riesz Decomposition Theorem and Theorem 2.5 we previously concluded that d_E^{2-N} is the Newtonian potential of a unique positive Borel measure, denoted σ_E. It remains to show that σ_E is unit measure with unbounded support.

We may assume that the origin is in E since d_E is translation invariant, see Proposition 3.2. Consider the ball $B(0, R)$ of radius $R > \mathrm{diam}(E)$ about the origin. We average

$$d_E^{2-N}(x) = \int_{\mathbb{R}^N} |x - y|^{2-N}\, \mathrm{d}\sigma_E(y)$$

with respect to the surface area measure on $B(0,R)$, and obtain

$$\int_{\partial B(0,R)} d_E^{2-N}(x)\,\mathrm{d}S(x) = \int_{\partial B(0,R)} \int_{\mathbb{R}^N} |x-y|^{2-N}\,\mathrm{d}\sigma_E(y)\,\mathrm{d}S(x). \tag{2.5}$$

For ease in notation, we denote the common value by $M(R)$.

Consider the left hand side of (2.5). We know that on $\partial B(0,R)$ the farthest distance function is bounded above and below by $R \le d_E(x) \le R + \mathrm{diam}(E)$. Thus

$$\big(R + \mathrm{diam}(E)\big)^{2-N} \le M(R) \le R^{2-N}. \tag{2.6}$$

On the other hand, we may apply Tonelli's Theorem to the right hand side of (2.5) to obtain

$$M(R) = \int_{\mathbb{R}^N} \int_{\partial B(0,R)} |x-y|^{2-N}\,\mathrm{d}S(x)\,\mathrm{d}\sigma_E(y).$$

It is a standard fact [2, p. 100] that

$$\int_{\partial B(0,R)} |x-y|^{2-N} \frac{\mathrm{d}S(x)}{\omega_N R^{N-1}} = \begin{cases} R^{2-N} & \text{if } |y| \le R, \\ |y|^{2-N} & \text{if } |y| > R. \end{cases}$$

Thus we have

$$M(R) = R^{2-N}\sigma_E\big(B(0,R)\big) + \int_{\mathbb{R}^N \setminus B(0,R)} |y|^{2-N}\,\mathrm{d}\sigma_E(y).$$

Since $0 < |y|^{2-N} < R^{2-N}$ for $|y| > R$, we obtain

$$R^{2-N}\sigma_E\big(B(0,R)\big) \le M(R) \le R^{2-N}\sigma_E(\mathbb{R}^N). \tag{2.7}$$

Combining inequalities (2.6) and (2.7) we find

$$R^{2-N}\sigma_E\big(B(0,R)\big) \le R^{2-N}$$

and

$$\big(R + \mathrm{diam}(E)\big)^{2-N} \le R^{2-N}\sigma_E(\mathbb{R}^N).$$

Hence

$$\sigma_E\big(B(0,R)\big) \le 1$$

and

$$\sigma_E(\mathbb{R}^N) \geq \left(\frac{R}{R + \operatorname{diam}(E)} \right)^{N-2}$$

for every $R \geq \operatorname{diam}(E)$. By letting $R \to \infty$, we conclude $\sigma_E(\mathbb{R}^N) = 1$.

It remains to show that σ_E has unbounded support. Suppose not, by way of contradiction. Then for sufficiently large R, $\operatorname{supp} \sigma_E \subset B(0, R)$. It follows that d_E^{2-N} is harmonic in $\mathbb{R}^N \setminus \overline{B}(0, R)$. By Proposition 2.11 there exists a point $y \in \partial E$ such that $d_E = |x - y|$ for all $x \in \mathbb{R}^N \setminus \overline{B}(0, R)$. Since E contains at least one other point inside $B(0, R)$ we reach a contradiction. $\square$

CHAPTER 3

Representing Measure

In Chapter 2, we explored the properties of the farthest distance function d_E for bounded sets $E \subset \mathbb{R}^N$. Our main result was Theorem 2.6 which stated that d_E can be expressed via the potential of a representing measure σ_E. We will now consider this representing measure. Throughout this chapter, unless otherwise noted, we will assume that E is a bounded set in $\mathbb{R}^N$, $N \geq 2$, consisting of at least two points.

We have focused on the representing measure for the Newtonian potentials d_E^{2-N} where $E \subset \mathbb{R}^N$, $N \geq 3$. However, recall that for $E \subset \mathbb{R}^2$, the function $\log d_E$ may be expressed as the logarithmic potential of a representing measure σ_E such that $\log d_P(x) = \int \log |x - y| \, \mathrm{d}\sigma_P(y)$ [53, p. 3980]. Through some abuse of terminology we will refer to both of these measures simply as the representing measure of the set E where the potential used depends on the dimension of the space in which we choose E to be situated. Notice that the dimension is not intrinsic to E. We may choose to situation the line segment L, for example, in any $\mathbb{R}^N$, $N \geq 2$ and thus we must be explicit about the ambient space we are considering.

In Section 3.1, we discuss properties of representing measures and calculate examples of σ_E for certain sets E. We consider the special case of polytopes in Section 3.2, where we give a completely geometric formulation of the representing measure. In Section 3.3, we will explore the support and density of the representing measure. We are particularly interested in an open conjecture of Laugesen and Pritsker [44] regarding the quantity $\sigma_E(E)$. We will consider this conjecture in Section 3.4. Proofs may be found in Section 3.5.

3.1 Calculating the Representing Measure

In Section 2.3, we used the Riesz Decomposition Theorem to prove the existence of a unique positive unit Borel measure σ_E such that d_E can be expressed via a potential of σ_E. The proof of the Riesz Decomposition Theorem is constructive and we will use this construction to calculate σ_E for a selection of sets E. We will also consider methods of approximating the representing measure when we cannot calculate an explicit formula.

For any function u superharmonic on a domain D, the proof of the Riesz Decomposition Theorem constructs a measure μ on D such that u may be expressed as the sum of its potential and a harmonic function. The representing measure μ is the unique measure on D such that

$$\frac{1}{(2-N)\omega_N}\int_D u\Delta f\,\mathrm{d}V = \int_D f\,\mathrm{d}\mu \tag{3.1}$$

for all $f \in C_0^\infty(D)$ the space of real valued, infinitely differentiable functions on D with compact support. If u is sufficiently nice, then μ is simply the Laplacian of u.

Proposition 3.1. *If $u \in C^2(D)$ then its representing measure in D is*

$$\mathrm{d}\mu = \frac{1}{(2-N)\omega_N}\Delta u\,\mathrm{d}V \tag{3.2}$$

where $\omega_N = 2\pi^{N/2}/\Gamma(N/2)$ denotes the surface area of the unit ball in $\mathbb{R}^N$ [32, p. 113].

In the plane, the formula for the representing measure of a superharmonic function u is given by $\mathrm{d}\mu = -\Delta u/(2\pi)$ [44, Example 1.2].

Example 3.1 (The N-Ball)**.** *Consider the ball $B = B(0,R)$ in $\mathbb{R}^N$, $N \geq 3$. We calculated in Example 2.9 that*

$$\Delta d_B^{2-N} = -\frac{R(N-2)(N-1)}{\rho(\rho+R)^N}.$$

Applying Proposition 3.1 and noting that $\mathrm{d}V = \omega_N \rho^{N-1}\,\mathrm{d}\rho$ *in hyperspherical coordinates, we obtain*

$$\mathrm{d}\sigma_B = R(N-1)\frac{\rho^{N-2}}{(\rho+R)^N}\,\mathrm{d}\rho \tag{3.3}$$

in $\mathbb{R}^N \setminus 0$*, where* 0 *denotes the origin. Since* d_E^{2-N} *is a finite potential, the origin must have zero mass. Thus we may use this formula to calculate the mass of the measure in* $B(0,r)$ *as*

$$\sigma_B\left(B(0,r)\right) = R(N-1)\int_0^r \frac{\rho^{N-2}}{(\rho+R)^N}\,\mathrm{d}\rho.$$

It follows that $\sigma_B\left(\mathbb{R}^N\right) = \lim_{r\to\infty}\sigma_B\left(B(0,r)\right) = 1$ *and* $\sigma_B(B) = \sigma_B\left(B(0,R)\right) = 2^{1-N}$.

Equation (3.3) also holds for the representing measure σ_B of $B = B(0,R) \subset \mathbb{R}^2$ [44, Example 1.2].

Notice that the quantity $\sigma_B(B)$ does not depend on the radius R. In fact, the representing measure is invariant under scaling, in addition to rigid motions. Thus we may choose any size or orientation we find convenient when considering examples.

Proposition 3.2. *The representing measure* σ_E *is invariant under rigid motions and scaling. Specifically, if* $D \subset \mathbb{R}^N$*, then* $\sigma_{TE}(TD) = \sigma_E(D)$ *for any rigid transformation* T *and* $d_{\lambda E}(\lambda D) = \sigma_E(D)$ *for all* $\lambda > 0$*.*

The calculation of the representing measure is local and thus we will often separately calculate σ_E on domains of $\mathbb{R}^N$ and their boundaries.

Example 3.2 (Hot Dog)**.** *We recall the hot dog from Example 2.2. It was the convex hull of two balls given by* $H = \operatorname{conv}\left(\overline{B}(-a,1) \cup \overline{B}(a,1)\right)$*. For convenience we choose* a *to lie on the* x_1 *coordinate axis. Let* $h_+ = \left\{x \in \mathbb{R}^N : x_1 > 0\right\}$ *and* $h_- = \left\{x \in \mathbb{R}^N : x_1 < 0\right\}$ *be two open half spaces. Then* $d_H(x) = d_{B(-a,1)}(x)$ *for* $x \in h_+$ *and* $d_H(x) = d_{B(a,1)}(x)$ *for* $x \in h_-$*.*

Consider $d_{B(a,1)}$ *on* $\mathbb{R}^N$*. Since* $\sigma_{B(a,1)}$ *is invariant under translation, we may apply Example 3.1. The representing measure* $\sigma_{B(a,1)}$ *is a unit measure supported on all of*

$\mathbb{R}^N$ *and symmetric about the point* a*. In particular, if* h *is a half-space containing a hemisphere of* $\overline{B}(a,1)$*, then* $\sigma_{B(a,1)}(h) = 1/2$ *and if* $D \subset \mathbb{R}^N$ *is a domain, then* $\sigma_{B(a,1)}(D) > 0$*. It follows that* $\sigma_{B(a,1)}(h_+) > 1/2$ *and hence* $\sigma_H(h_-) = \sigma_{B(a,1)}(h_-) < 1/2$*. Similarly* $\sigma_H(h_+) = \sigma_{B(-a,1)}(h_+) < 1/2$*.*

Denote the $x_1 = 0$ *hyperplane by* S*, which is also* $\mathbb{R}^N \setminus (h_+ \cup h_-)$*. We already calculated that* $\sigma_H(h_+ \cup h_-) < 1$*. Since* σ_H *is a unit measure, it follows that* $\sigma_H(S) > 0$*. No other hyperplane has positive mass.*

Example 3.3 (Torus)**.** *Let* $\mathcal{T} \subset \mathbb{R}^3$ *be the torus from Example 2.3. Recall that*

$$d_{\mathcal{T}}(r,h,\theta) = \big|(r,h) + (a,0)\big| + 1$$

for $r \geq 0$*. The function* $d_{\mathcal{T}}^{-1}$ *is* C^2 *for* $r > 0$*. In cylindrical coordinates, the Laplacian of a rotationally invariant function* f *is given by*

$$\Delta f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{\partial^2 f}{\partial h^2} = \frac{1}{r}\frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial r^2} + \frac{\partial^2 f}{\partial h^2}$$

and $dV = r\,\mathrm{d}\theta\,\mathrm{d}h\,\mathrm{d}r$ *[69, p. 443]. We compute that*

$$\Delta d_{\mathcal{T}}^{-1}(r,h,\theta) = \frac{-a}{r\,\big|(r,h)+(a,0)\big|\,d_{\mathcal{T}}^2(r,h,\theta)}$$

and thus

$$\mathrm{d}\sigma_{\mathcal{T}} = \frac{a}{4\pi}\,\big|(r,h)+(a,0)\big|^{-1}\,d_{\mathcal{T}}^{-2}(r,h,\theta)\,\mathrm{d}\theta\,\mathrm{d}h\,\mathrm{d}r$$

for $r > 0$*. Since the potential* $d_{\mathcal{T}}^{-1}$ *is finite and the horizontal axis where* $r = 0$ *is a polar set, see [34, Example 4.2.8], its mass must be zero and so this formula holds on all of* $\mathbb{R}^3$*.*

Let E be a strictly convex set with strictly positive radii of curvature and C^2 boundary. Recall from Theorem 2.2 that if $x \in \mathbb{R}^N$ is not a center of curvature of ∂E and $y_E(x)$ is single valued, then d_E, and hence d_E^{2-N}, is C^2 in a neighborhood of x. Thus if we consider subsets of the plane which exclude centers of curvature, we can compute σ_E using Proposition 3.1.

We begin by providing a parametrization of ∂E in terms of its restricted support function. Since E is strictly convex, for each $u \in \mathcal{U}^N$ there is a unique point $\gamma_E(u) \in E$ such that $p_E(u) = \gamma_E(u) \cdot \vec{u}$. Furthermore, every point $y \in \partial E$ is $\gamma_E(u)$ for some $u \in \mathcal{U}^N$. Thus, γ_E is a parametrization of ∂E which we call the *support parametrization*.

The parametrization γ_E can be expressed in terms of p_E and its first partial derivatives which necessarily exist since E is strictly convex. This is most easily done in the plane where

$$\gamma_E(u) = p_E(u)\vec{u} + \frac{\partial p_E}{\partial \theta}(u)\frac{\partial \vec{u}}{\partial \theta} \tag{3.4}$$

where $u = (0, \theta)$ and $\vec{u} = (\cos\theta, \sin\theta)$ is given in rectangular coordinates.

Example 3.4 (Ellipse)**.** *Let E be the ellipse parametrized by $(a\cos\psi, b\sin\psi)$. Its support function is given by $p_E(u) = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta}$ and its support parametrization is*

$$\gamma_E(u) = \frac{1}{p_E(u)}\begin{pmatrix} a^2\cos\theta \\ b^2\sin\theta \end{pmatrix}.$$

Example 3.5 (Reuleaux Triangle)**.** *Let R be the planar Realeaux Triangle we considered in Example 2.8 with vertices placed at $\left(1/\sqrt{3}\right)e^{i\phi_v}$ where $\phi_a = \pi/2$, $\phi_b = 7\pi/6$, and $\phi_c = 11\pi/6$. Then its support function is piecewise defined much like its farthest distance function. Specifically,*

$$p_R(u) = \begin{cases} 1 + \frac{1}{\sqrt{3}}\cos(\theta - \phi_v) & \left|\theta - (\phi_v + \pi)\right| \le \frac{\pi}{6} \\ \frac{1}{\sqrt{3}}\cos(\theta - \phi_v) & |\theta - \phi_v| \le \frac{\pi}{6} \end{cases}$$

where $u = (0, \theta)$ and these cases only overlap at the endpoints as shown in Figure 3.1. The support function has continuous derivative

$$\frac{\partial p_E}{\partial \theta}(u) = -\frac{1}{\sqrt{3}}\begin{cases} \sin(\theta - \phi_v) & \left|\theta - (\phi_v + \pi)\right| \le \frac{\pi}{6} \\ \sin(\theta - \phi_v) & |\theta - \phi_v| \le \frac{\pi}{6} \end{cases}$$

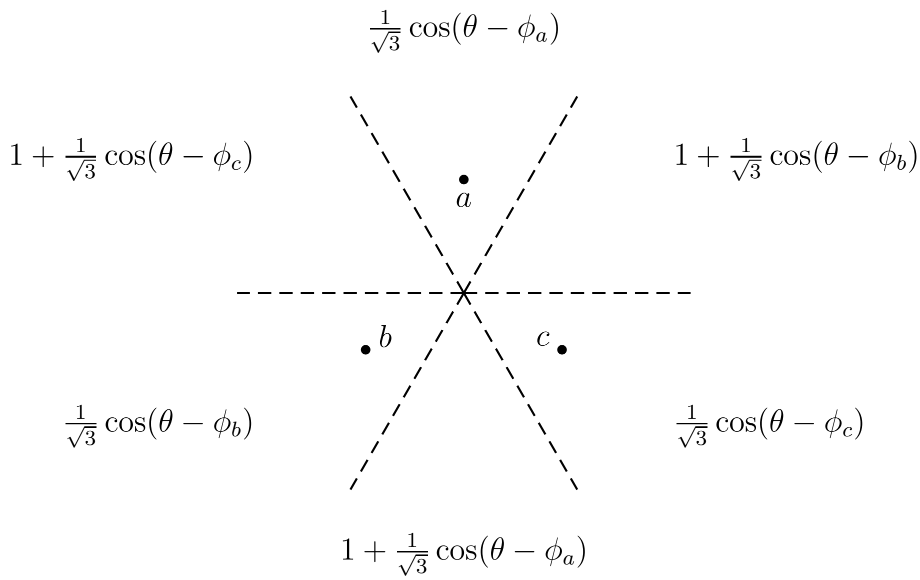

Figure 3.1: Restricted support function for the Reuleaux triangle R

and hence we have the support parametrization

$$\gamma_R(u) = \begin{cases} \vec{u} + \frac{1}{\sqrt{3}} e^{i\phi_v} & \left|\theta - (\phi_v + \pi)\right| \leq \frac{\pi}{6} \\ \frac{1}{\sqrt{3}} e^{i\phi_v} & |\theta - \phi_v| \leq \frac{\pi}{6}. \end{cases}$$

In $\mathbb{R}^3$ the support parametrization is given by

$$\gamma_E(u) = p_E(u)\vec{u} + \frac{\partial p_E}{\partial \theta_1}(u)\frac{\partial \vec{u}}{\partial \theta_1} + \frac{\partial p_E}{\partial \theta_2}(u)\frac{\partial \vec{u}}{\partial \theta_2}\csc^2\theta_1 \tag{3.5}$$

where $u = (0, \theta_1, \theta_2)$ and $\vec{u} = (\cos\theta_1, \sin\theta_1\cos\theta_2, \sin\theta_1\sin\theta_2)$ in rectangular coordinates. For example, the support parametrization of a standard ellipsoid E with semiaxes a, b, and c is given by

$$\gamma_E(u) = \frac{1}{p_E(u)} \begin{pmatrix} a^2\cos\theta_1 \\ b^2\sin\theta_1\cos\theta_2 \\ c^2\sin\theta_1\sin\theta_2 \end{pmatrix}$$

where $p_E(u) = (a^2\cos^2\theta_1 + b^2\sin^2\theta_1\cos^2\theta_2 + c^2\sin^2\theta_1\sin^2\theta_2)^{1/2}$. A proof of (3.5) is given in Section 3.5 and a similar computation can be done in higher dimensions.

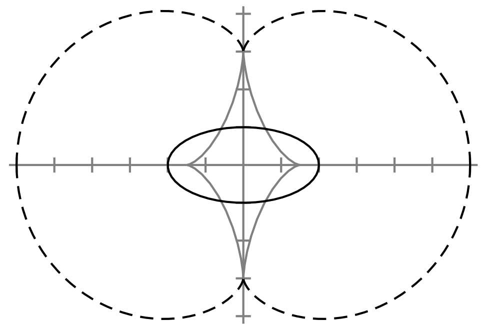

Figure 3.2: The unbounded set $C_2(E)$ is a subset of the complement of the ellipse E

We say a strictly convex set is *c-round*, for some $c \geq 1$, if for every $u \in \mathcal{U}^N$ we have

$$E \subset \overline{B}\left(\gamma_E(u) - cw_E(u)\vec{u}, cw_E(u)\right).$$

For example, the ball and Reuleaux triangles are both 1-round.

Let E be a c-round set with C^2 restricted support function p_E. For any $b \geq c$, we define

$$C_b(E) = \left\{\gamma_E(u) - \rho\vec{u} : u \in \mathcal{U}^N \text{ and } \rho > bw_E(u)\right\}$$

which is an open set contained in the complement of E. The shape of this set may be unexpected. If B is a ball, then w_B is constant and $\mathbb{R}^N \setminus C_b(B)$ is simply a parallel body. However, that need not happen and in fact the set $\mathbb{R}^N \setminus C_b(E)$ need not even be convex.

Example 3.6 (Ellipse)**.** *Let E be the ellipse considered in Example 3.4 with semiaxes $a = 2$ and $b = 1$. Since E is symmetric about the origin, $w_E = 2p_E$ and we can compute that E is 2-round. Figure 3.2 illustrates the boundary of $C_2(E)$ in dashed lines. The evolute, or set of centers of curvature of E, is drawn in gray. The points $(0, \pm 4)$ are both centers of curvature on the boundary of $C_2(E)$. All other centers of curvature are strictly inside $\mathbb{R}^2 \setminus C_2(E)$.*

Consider the map $F : (bw_E(u), \infty) \times \mathcal{U}^N \to C_b(E)$ given by

$$F(\rho, u) = \gamma_E(u) - \rho\vec{u}.$$

Since

$$E \subset \overline{B}\left(\gamma_E(u) - cw_E(u)\vec{u}, cw_E(u)\right) \subset \overline{B}\left(F(\rho, u), \rho\right)$$

it follows that $d_E\left(F(\rho, u)\right) = \rho$, $y_E\left(F(\rho, u)\right) = \gamma_E(u)$, and all radii of curvature at $\gamma_E(u)$ are strictly less than $bw_E(u)$. Thus, by Theorem 2.2, d_E is C^2 in $C_b(E)$. Furthermore, the map F is one-to-one and hence invertible. Thus we may compute Δd_E^{2-N} in $C_b(E)$ using the Inverse Function Theorem. This was originally done in the plane where

$$\mathrm{d}\sigma_E\left(F(\rho, u)\right) = \frac{1}{2\pi}\frac{r_E(u)}{\rho^2}\,\mathrm{d}\rho\,\mathrm{d}u \tag{3.6}$$

where $r_E(u)$ is the radius of curvature at the point $\gamma_E(u)$ [25, Lemma 3]. In $\mathbb{R}^3$ a similar calculation yields

$$\mathrm{d}\sigma_E\left(F(\rho, u)\right) = \frac{\sin\theta_1}{2\pi K_E(u)} \cdot \frac{H_E(u)\rho - 1}{\rho^3}\,\mathrm{d}\rho\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2 \tag{3.7}$$

where $K_E(u)$ and $H_E(u)$ are the Gaussian and mean curvatures respectively at the point $\gamma_E(u) \in \partial E$. Proofs of Equations (3.6) and (3.7) are given in Section 3.5.

In the next section we will calculate σ_P in the case P is a polytope. However, in other cases a direct computation of σ_E may be impractical. A version of the following lemma was given in [63, p. 85] for $\mathbb{R}^2$. It provides a way to calculate the mass of the representing measure in a neighborhood without finding a formula for the measure. We extend it to higher dimensions.

Proposition 3.3. *Let u be a finite superharmonic function on an open set $D \subset \mathbb{R}^N$, $N \geq 3$, and μ its representing measure. For almost every r such that $B(w, r) \subset D$, we have*

$$\mu\left(B(w, r)\right) = \frac{r^{N-1}}{N-2}\lim_{\delta\to 0}\frac{\mathcal{M}(u; w, r) - \mathcal{M}(u; w, r+\delta)}{\delta}. \tag{3.8}$$

Another method for calculating σ_E when E is ill-behaved is to approximate E by simpler sets whose representing measure is known.

Proposition 3.4. *Let $\{E_n\}$ be a monotone sequence of sets in $\mathbb{R}^N$, $N \geq 3$, such that d_{E_n} converges uniformly to d_E as $n \to \infty$. Then $\sigma_{E_n} \to \sigma_E$ weak* as $n \to \infty$.*

In particular, we may use Proposition 2.2 to construct a sequence of sets E_n such that σ_{E_n} can be computed by calculating the Laplacian of $d_{E_n}^{2-N}$ and the resulting measures converge weak* to σ_E.

In the planar case, there is a result similar to Proposition 3.4 which is sufficient for such purposes. If d_{E_n} are monotonically increasing to d_E, then

$$\liminf_{n\to\infty} \sigma_{E_n}(V) \geq \sigma_E(V)$$

for any open set V [25, Lemma 2].

3.2 Representing Measure for Polytopes

We now turn our attention to polytopes, such as polygons and polyhedra. Let P be a polytope. Recall from Section 2.2 that $\mathcal{V}(P)$ denotes the set of outer vertices of P and F_v denotes the farthest point Voronoi cell of $v \in \mathcal{V}(P)$. Since $d_P(x) = |x - v|$ on the interior of F_v, it follows by Proposition 2.11 that d_P^{2-N} is harmonic on the interior of each farthest point Voronoi cell. Applying Proposition 3.1 we find that the representing measure σ_P must be supported entirely on the boundaries of the farthest point Voronoi cells.

The boundaries of the farthest point Voronoi cells consist of subsets of hyperplanes and their intersections. We present a lemma that can be used to calculate the representing measure of a superharmonic function in such a case.

Lemma 3.1. *Suppose a Lipschitz continuous function u is superharmonic everywhere on a domain $D \subset \mathbb{R}^N$, $N \geq 3$, and harmonic except on a hyperplane S. Then its representing measure is given by*

$$\mathrm{d}\mu = \frac{1}{(2-N)\omega_N}\left(\frac{\partial u}{\partial \boldsymbol{n}_+} + \frac{\partial u}{\partial \boldsymbol{n}_-}\right)\mathrm{d}S \tag{3.9}$$

where $\boldsymbol{n}_\pm$ are the normal vectors to S in the positive and negative directions.

We may use this lemma to calculate a formula for σ_P when P is a polytope. This lemma does not apply to the hot dog in Example 3.2. In that case we found that for a particular hyperplane S, we had $\sigma_H(S) > 0$. However, this formula will not apply because d_H^{2-N} is not harmonic anywhere on $\mathbb{R}^N \setminus S$.

Example 3.7 (Line Segment)**.** *Let $L = [-a, a]$ be the line segment from Example 2.4, positioned along the x_1 coordinate axis. Then the farthest distance function is given by*

$$d_L(x) = \left[\left(|x_1| + a \right)^2 + x_2^2 + \cdots + x_N^2 \right]^{1/2}.$$

Let S be the $x_1 = 0$ hyperplane. While d_L^{2-N} is not differentiable on S, $d_L = d_a$ for $x_1 < 0$ and $d_L = d_{-a}$ for $x_1 > 0$ and hence d_L^{2-N} is harmonic in $\mathbb{R}^N \setminus S$. Choose the positive normal vector $\boldsymbol{n}_+$ to S to be in the positive x_1 direction. Then

$$\frac{\partial d_L^{2-N}}{\partial \boldsymbol{n}_+} = \frac{\partial d_L^{2-N}}{\partial \boldsymbol{n}_-} = \frac{(2-N)a}{d_L^N},$$

and applying Lemma 3.1 we find

$$\mathrm{d}\sigma_L = \frac{2a}{\omega_N} d_L^{-N} \, \mathrm{d}S \tag{3.10}$$

where $\mathrm{d}S$ is the area differential on the hyperplane S.

Let P be a polytope. Choose vertices $v, w \in \mathcal{V}(P)$ and let F_v and F_w be their corresponding farthest point Voronoi cells. For any x on the $N-1$ dimensional interior of $F_v \cap F_w$ there exists a neighborhood where $d_P = d_L$ where L is the line segment $[v, w]$. Thus the representing measure on the $N-1$ dimensional interior of $F_v \cap F_w$ is given by

$$\mathrm{d}\sigma_P = \frac{|v-w|}{\omega_N} d_P^{-N} \, \mathrm{d}S. \tag{3.11}$$

Note that $F_v \cap F_w$ may be empty. For example, in the parallelogram from Example 2.6 we had $F_b \cap F_c = \emptyset$. The intersections between hyperplanes consist of the sets $F_u \cap F_v \cap F_w$ for $u, v, w \in \mathcal{V}(P)$. These are subsets of a countable number of $N-2$

dimensional hyperplanes. Since σ_P is a finite potential, their mass must be zero. Thus the formula given in (3.11) is valid on all of $F_v \cap F_w$. The representing measure σ_P for a polygon $P \subset \mathbb{R}^2$ is given by the same formula [44, Example 1.3].

We can give a simpler, and purely geometric, formulation for σ_P. We begin by considering a polygon $P \subset \mathbb{R}^2$. Assume that $v, w \in \mathcal{V}(P)$ lie on the horizontal x-axis and are symmetric about the origin, so that $v = a$ and $w = -a$ for some $a > 0$. Then $F_v \cap F_w$ lies on the vertical y-axis. Let S be the segment of $F_v \cap F_w$ between y_1 and y_2.

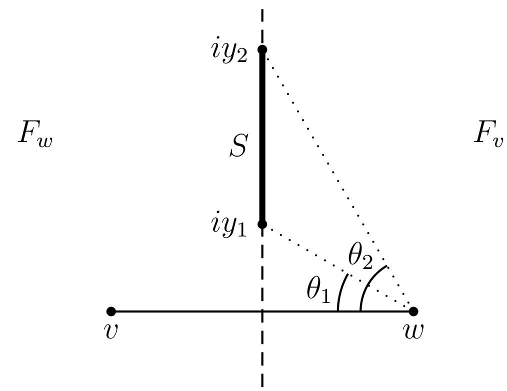

Figure 3.3: The angle subtended by S, a segment of the support of σ_P, at vertex w

Applying (3.11), we calculate

$$\begin{aligned} \sigma_P(S) &= \frac{a}{\pi} \int_{y_1}^{y_2} (a^2 + y^2)^{-1} \, \mathrm{d}y \\ &= \frac{1}{\pi} \left[\tan^{-1}(y/a) \right]_{y_1}^{y_2} \\ &= \frac{1}{\pi} (\theta_2 - \theta_1) \end{aligned} \tag{3.12}$$

where θ_1 and θ_2 are as shown in Figure 3.3.

The quantity $\theta_2 - \theta_1$ is the angle that S subtends at both v and w. For a set $S \subset \mathbb{R}^N$, we denote the angle that S subtends at $v \in \mathbb{R}^N$ by $\Omega_v(S)$. Thus, for polygons in the plane, we may rewrite (3.11) as

$$\sigma_P(S) = \frac{1}{\pi} \Omega_v(S) = \frac{1}{2\pi} \left(\Omega_v(S) + \Omega_w(S) \right).$$

In fact, this equation holds for all polytopes.

In higher dimensions $\Omega_v(S)$ refers to solid angles. Consider a connected set in the plane, $S \subset \mathbb{R}^2$. To find the angle subtended by S at a point v, we first find S', the projection of S onto $B(v,1)$. Then $\Omega_v(S)$ is the length of S'. Similarly, if $S \subset \mathbb{R}^N$ is connected and v is a point, we may project S onto $B(v,1)$ and calculate its $N-1$ dimensional area. This quantity will be denoted by $\Omega_v(S)$. For example, if S is a hyperplane in $\mathbb{R}^N$ and v is any point not on that plane, then the projection of S onto $B(v,1)$ is an open hemisphere and $\Omega_v(S) = \omega_N/2$.

Theorem 3.1. *Let $P \subset \mathbb{R}^N$ be a compact polytope consisting of at least two points. Let $v, w \in \mathcal{V}(P)$ be outer vertices of P and let $S \subset F_v \cap F_w$. Then*

$$\sigma_P(S) = \frac{2}{\omega_N}\ \Omega_v(S) = \frac{1}{\omega_N}\ \left(\Omega_v(S) + \Omega_w(S)\right). \tag{3.13}$$

Furthermore, if $D \subset \mathbb{R}^N$, then

$$\sigma_P(D) = \frac{1}{\omega_N} \sum_{v \in \mathcal{V}(P)} \Omega_v(\partial F_v \cap D). \tag{3.14}$$

Example 3.8 (Regular Triangle)**.** *Returning to the regular triangle from Example 2.5, we see that $\Omega_a(F_a \cap T) = \pi/3$ and by symmetry $\sum_{v \in \mathcal{V}(T)} \Omega_v(F_v \cap T) = \pi$. Applying* (3.14)*, we find $\sigma_T(T) = 1/2$.*

Recall from Example 3.1 that in the plane, $\sigma_B(B) = 1/2$ for B a disk. We now see that $\sigma_T(T) = 1/2$ as well. This is not a coincidence. We will consider these examples in greater depth in Section 3.4. However, we can provide a geometric proof that covers these special cases.

Proposition 3.5. *Let P be a polygon inscribed in a closed disk B. Then $\sigma_P(B) = 1/2$.*

It follows that $\sigma_P(P) \leq 1/2$. If P is a regular polygon with an odd number of vertices, then $\operatorname{supp}(\sigma_P) \cap B = \operatorname{supp}(\sigma_P) \cap P$ and hence $\sigma_P(P) = 1/2$. This result was previously proved in [44, Theorem 2.5].

There is no analogue to Proposition 3.5 in higher dimensions. Recall from Example 2.1 that $\sigma_B(B) = 2^{1-N}$ for $B \subset \mathbb{R}^N$ and thus in $\mathbb{R}^3$ we have $\sigma_B(B) = 1/4$. We give a counterexample to show that $\sigma_P(B)$ need not be $\sigma_B(B)$ for polyhedra P inscribed in the ball.

Example 3.9 (Tetrahedron)**.** *Let T be the regular tetrahedron inscribed in the unit ball in $\mathbb{R}^3$. Recall that for each vertex v, the farthest point Voronoi cell F_v is an unbounded pyramid containing the face of T opposite from v. Thus the solid angle of $\partial F_v \cap T$ at v is the same as the solid angle of T at v. Consider the projection of T onto $B(v, 1)$. It forms a spherical triangle. The angles of the triangle are the dihedral angles of T. On a unit sphere, the area of a triangle is the sum of its angles minus π [48, p. 196]. Since the dihedral angles of the regular tetrahedron have measure* arccos(1/3)*, we conclude that $\Omega_v(F_v \cap T) = \Omega_v(T) = 3 \arccos \frac{1}{3} - \pi$. Applying* (3.14) *to the four vertices of T, we have*

$$\sigma_T(T) = \frac{1}{4\pi}\, 4 \left(3 \arccos \frac{1}{3} - \pi \right) \approx 0.175479656 < 1/4.$$

Furthermore, if B is the unit ball in $\mathbb{R}^N$ in which T is inscribed, then we may numerically compute that $\sigma_T(B) \approx 0.2685 > 1/4$.

We can use (3.11) to compute the measure $\sigma_P(P)$ for random polyhedra in $\mathbb{R}^3$. Given a number of vertices n, a computer program[1] randomly chooses n points, calculates their convex hull P, and then calculates $\sigma_P(P)$. This process was repeated one million times for selected small values of n and the largest computed value of $\sigma_P(P)$ was reported. Those values are summarized in Table 3.1.

Note that for $n = 4$, the computer program clearly did not find the optimal tetrahedron since $0.16115 < \sigma_T(T)$ where T is the regular tetrahedron. While it appears that the regular tetrahedron is the extremal case for $n = 4$, the other extremal cases

[1]The code, written in Matlab, is available at

http://www.math.okstate.edu/~wcopley/FDFPolyhedra/.

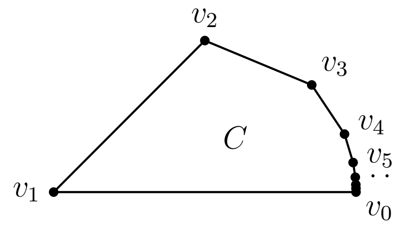

Figure 3.4: The closure of the convex hull of a countable sequence of points

are difficult to guess. For example, among polyhedra with 6 vertices inscribed in the sphere, the regular octahedron has maximum volume and minimum discrete energy [66]. However, $\sigma_P(P) \approx 0.1475$ for the regular octahedron which is not maximal. Which polyhedra achieve the maximum measure possible for a given number of vertices remains an intriguing mystery.

n	$\sigma_P(P)$
4	0.16115
6	0.17241
8	0.18786
12	0.20096
20	0.2229

Table 3.1: Maximum value of $\sigma_P(P)$ found for randomly generated polyhedra

We consider one last example. The following is not a polytope, but its representing measure can be computed using the same formulas.

Example 3.10 (The Closed Convex Hull of a Countable Set)**.** *We define a sequence of angles*

$$(\theta_n)_{n=0}^{\infty} = \left(\frac{\pi}{2^{n-1}}\right)_{n=0}^{\infty} = \left(2\pi, \pi, \frac{\pi}{2}, \frac{\pi}{4}, \ldots\right)$$

and set $v_n = (\cos\theta_n, \sin\theta_n)$. Let $C \subset \mathbb{R}^2$ be the closure of the convex hull of the points v_n, as shown in Figure 3.4.

While C is not a polytope, for each $x \in \mathbb{R}^2$ there is some n such that $d_C(x) =$

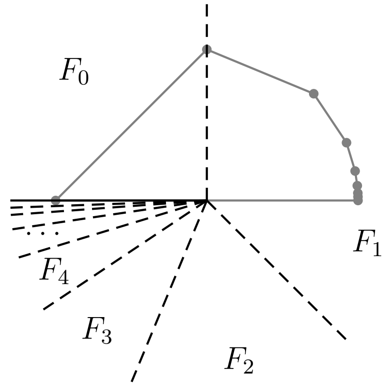

Figure 3.5: The farthest point Voronoi cells for C

$|x - v_n|$ *and thus we can define the farthest point Voronoi cells as we did for polytopes. Let* F_n *be the farthest point Voronoi cell for* v_n *as shown in Figure 3.5. The shared boundaries between cells are drawn as dashed rays from the origin. On each of these rays we may calculate the representing measure using* (3.11)*. Notice that the ray* R *from* $(0,0)$ *through* $(-1,0)$*, drawn in Figure 3.5 as a solid line, is part of the boundary of the cell* F_0 *but is not on the boundary of any other cell and hence* (3.11) *does not apply. The remaining rays are* $F_n \cap F_{n+1}$ *for some* $n \geq 0$*. We can compute that the angle subtended by each ray is*

$$\Omega_{v_n}(F_n \cap F_{n+1}) = \frac{\theta_n - \theta_{n+1}}{2} = \frac{\pi}{2^{n+1}}$$

and thus the measure on $\mathbb{R}^N \setminus R$ *is given by*

$$\begin{aligned} \sigma_C\left(\mathbb{R}^N \setminus R\right) &= \frac{1}{2\pi} \sum_{n=0}^{\infty} \Omega_{v_n}(\partial F_n) \\ &= \frac{1}{\pi} \sum_{n=0}^{\infty} \Omega_{v_n}\left(F_n \cap F_{n+1}\right) \\ &= 1. \end{aligned}$$

It follows that $\sigma_C(R) = 0$*.*

3.3 Support and Density of the Representing Measure

We have calculated the representing measure for several example sets. These examples have provided a picture of what the support of the representing measure may be. For a set $E \subset \mathbb{R}^N$, we will consider the support of the representing measure σ_E as well as what sets must have measure zero.

Our first result applies to the ball, the hot dog, and the torus examples from Section 3.1, all of which have reasonably smooth boundaries.

Proposition 3.6. *If ∂E is C^1-smooth, then* $\operatorname{supp}(\sigma_E) = \mathbb{R}^N$.

We also saw that in the case of polytopes, the representing measure is supported entirely on hyperplanes. In this section we will consider additional examples to illustrate the wide range of shapes the support of the representing measure may take. For ease in illustrating, these examples are presented in the plane. However, similar examples may be constructed in higher dimensions.

Example 3.11. *[Reuleaux Triangle] Let T be the regular triangle and let R be its associated Reuleaux triangle as discussed in Example 2.8. We calculated d_R in Example 2.8, as shown in Figure 2.7. Notice that $\log d_R$ is piecewise C^2, and hence we may calculate σ_R on the various domains where $\Delta \log d_R$ exists. Unlike in the case of the hot dog in Example 3.2, it will also be easy to find a formula for the measure on the boundaries between those domains.*

First, we consider the domains where $d_R = d_v$ for v an appropriate vertex of T. In that case, $\log d_R$ is harmonic [44, Theorem 1.1] and by Proposition 3.1 we have $\sigma_R = 0$ on those domains. In the remaining domains we have $d_R = d_{B(v,1)}$ for v an appropriate vertex of T and hence $\sigma_R = \sigma_{B(v,1)}$. The Reuleaux triangle is drawn in gray in Figure 3.6 and each of the six domains is labeled with the value of σ_T on that domain.

Let A be the domain where $\sigma_R = \sigma_{B(a,1)}$. Note that the boundary of this domain consists of rays that form an angle of $\pi/3$ at a. Since the unit measure $\sigma_{B(a,1)}$ is rotationally symmetric about a, it follows that $\sigma_R(A) = 1/6$. The domains where $\sigma_R = \sigma_{B(v,1)}$ for a vertex v are rotationally symmetric about the origin and hence each has measure $1/6$, for a total measure of $1/2$ on those domains.

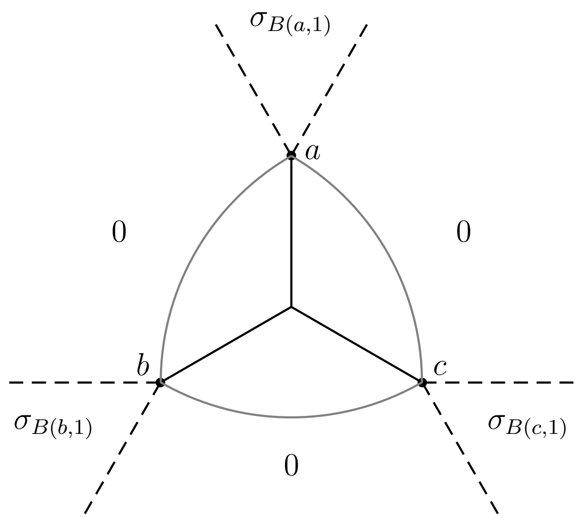

Figure 3.6: Representing measure σ_R for the Reuleaux triangle R on domains in $\mathbb{R}^2$

It remains to calculate the measure on the boundaries between the domains. Those boundaries are drawn in Figure 3.6 with solid or dashed lines. Let S be the solid lines which denote the boundaries between domains where $d_R = d_v$ for some vertex v. Let D be the dashed lines which denote the boundaries of the domains where $d_R = d_{B(v,1)}$ for some vertex v.

Note that inside R, we have $d_R = d_T$. Thus, $\sigma_R(S) = \sigma_T(S) = 1/2$ as we calculated in Example 3.8. Since we already found that the measure of the domains where $d_R = d_{B(y,1)}$ is $1/2$, we calculate that $\sigma_R(\mathbb{R}^N \setminus D) = 1$ and thus $\sigma_R(D) = 0$.

We are now prepared to characterize the shape of the complement of $\mathrm{supp}(\sigma_E)$.

Proposition 3.7. *The complement of the support of the representing measure, $\mathbb{R}^N \setminus$ $\mathrm{supp}(\sigma_E)$, is either empty or the union of at most countably many disjoint domains.*

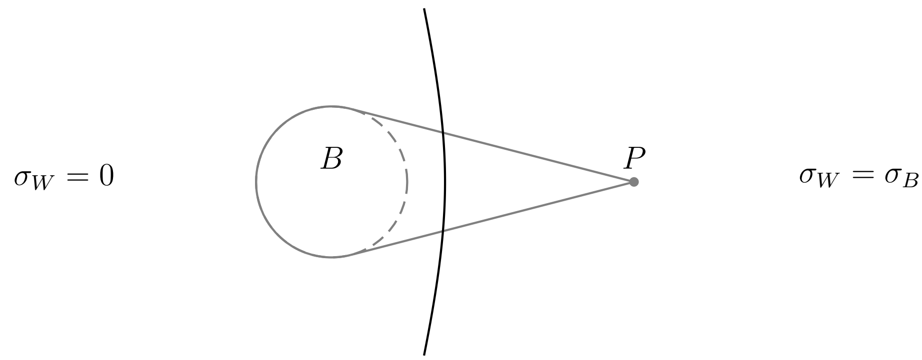

Figure 3.7: Representing measure σ_W for the water drop W on domains in $\mathbb{R}^2$

These domains are precisely the domains where the farthest point function y_E is constant. The domains are unbounded and convex.

We already saw in Example 3.10 that the number of such convex disjoint domains in $\mathbb{R}^N \setminus \operatorname{supp}(\sigma_E)$ need not be finite. The closure of the domains need not be farthest point Voronoi cells, as the following example demonstrates.

Example 3.12 (Water Drop)**.** *Let $W \subset \mathbb{R}^2$ be the convex hull of $B = \overline{B}\big((-2,0),1\big)$ and $P = (2,0)$, drawn in gray in Figure 3.7.*

We can easily see that to the right of the water drop, we must have $d_W = d_B$ and hence $\sigma_W = \sigma_B$. Similarly, on the left of the drop, it is clear the $d_W = d_P$ and hence $\sigma_W = \sigma_P = 0$. Since $\sigma_W = \sigma_B$ on a strict subset of $\mathbb{R}^2$, it follows that the boundary between these two domains has positive mass. We are interested in the shape of that boundary.

We wish to find the set $z \in \mathbb{R}^2$ such that $d_B(z) = d_P(z)$. Let $z = (x, y)$. Then

$$d_B(z) = \sqrt{(x+2)^2 + y^2} + 1, \text{ and}$$
$$d_P(z) = \sqrt{(x-2)^2 + y^2}.$$

It is clear that for $x > 0$, and in fact for $x > -1/2$, that $d_B > d_P$. So we may assume $x \leq -1/2$. Since d_B and d_P are both positive we may square both sides without adding *extraneous solutions and we find*

$$\begin{aligned} d_B^2(z) &= d_P^2(z) \\ (x+2)^2 + y^2 + 1 + 2\sqrt{(x+2)^2+y^2} &= (x-2)^2 + y^2 \\ \sqrt{(-x+2)^2+y^2} &= -4x - \frac{1}{2}. \end{aligned}$$

Noting that $-4x - 1/2 > 0$ *since* $x \leq -1/2$, *we square both sides again, finding*

$$x^2 + 4x + 4 + y^2 = 16x^2 + 4x + \frac{1}{4}.$$

We may rewrite this in the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where $a = 1/2$ *and* $b = \sqrt{15}/2$. *This is the equation of a hyperbola. Since we know* $x \leq 0$, *we are interested in only one branch of the hyperbola, which is drawn in black in Figure 3.7. The hyperbola has asymptotes* $y = \pm\sqrt{15}x$. *These asymptotes are the perpendicular bisectors of the two straight line segments in* ∂W.

We now consider the shape of the support. We have seen that the support may be made up of disjoint domains with their boundaries as well as hyperplane segments. In fact, these are the only possibilities.

Proposition 3.8. *The support of the representing measure* $\mathrm{supp}(\sigma_E)$ *is simply connected and is the union of at most countably many sets of the following types:*

- *disjoint unbounded domains,* U_i,
- *their boundaries,* ∂U_i, *and*
- *closed convex subsets of hyperplanes where the measure is given by Theorem 3.1.*

Furthermore, the intersections between the domain boundaries and hyperplane subsets have zero mass.

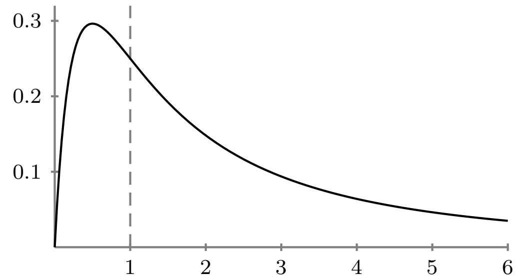

Figure 3.8: The density of σ_B in $\mathbb{R}^3$

In addition to the shape of the support, we wish to consider the density of the measure. We begin with simple upper and lower bounds on suitable balls.

Proposition 3.9. *Let $B(x,r) \subset \mathbb{R}^N$ be a ball. Then*

$$\sigma_E\left(B(x,r)\right) \leq \left(\frac{r}{d_E(x)-r}\right)^{N-1}$$

for sufficiently small r.

Proposition 3.10. *Let $B(x,r) \subset \mathbb{R}^N$ be a ball such that $x \in E$. Then*

$$\sigma_E\left(B(x,r)\right) \geq \left(1-\frac{\operatorname{diam}(E)}{r}\right)\left(1-\frac{\operatorname{diam}(E)}{r+\operatorname{diam}(E)}\right)^{N-1}$$

for sufficiently large r.

It is clear from these propositions as well as our work on angles in Section 3.2 that the representing measure σ_E must be concentrated near E. We consider a few examples for which the density can be explicitly calculated.

Example 3.13 (Ball)**.** *Let B be the ball we considered in Example 3.1. If B is the unit ball in $\mathbb{R}^3$, then the measure on a ball of radius r is given by $\sigma_B\left(B(0,r)\right) = 2\int_0^r \rho(\rho+1)^{-3}d\rho$. Figure 3.8 shows the density of the measure by plotting the integrand $2\rho(\rho+1)^{-3}$ with respect to ρ. The dashed line indicates the surface of the ball B. The maximum occurs at $\rho = \frac{1}{2}$.*

Example 3.14 (Line Segment)**.** *Let L be the line segment from Example 3.7. If L is in $\mathbb{R}^3$ and $a = 1$, then the measure is supported on the plane which is the perpendicular*

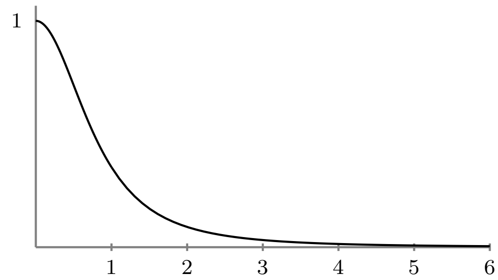

Figure 3.9: The density of σ_L in $\mathbb{R}^3$

bisector of L *and is given by* $\mathrm{d}\sigma_L = d_L^{-3}\,\mathrm{d}S/(2\pi)$. *Thus the measure on a ball of radius* r *is given by*

$$\sigma_L\big(B(0,r)\big) = \frac{1}{2\pi}\int_0^r\int_0^{2\pi} d_L^{-3}\,\mathrm{d}\theta\,\mathrm{d}\rho = \int_0^r \big(1+\rho^2\big)^{-3/2}\,\mathrm{d}\rho.$$

Figure 3.9 shows the density of the measure by plotting the integrand $\big(1+\rho^2\big)^{-3/2}$ *with respect to* ρ.

In view of these results, as well as Proposition 3.5, it is natural to inquire about the quantity $\sigma_E(E)$, at least in the case that E in convex. We will consider this topic in detail in the next section.

3.4 The Mass of the Representing Measure on E

We begin our discussion of $\sigma_E(E)$ with a straightfoward lower bound.

Proposition 3.11. *Suppose* E *is the closure of a domain. Then* $\sigma_E(E) > 0$.

The requirement that E is the closure of a domain is necessary. Consider the line segment L. Since $\mathrm{supp}(\sigma_L)$ is the perpendicular bisector of L it follows that $\mathrm{supp}(\sigma_L) \cap L$ is a single point. Since σ_L has a finite potential and a single point is a polar set [2, Example 5.1.2], $\sigma_L(L) = 0$.

We will be more interested in finding an upper bound for $\sigma_E(E)$. Laugesen and Pritsker considered this in [44]. Their work primarily focused on the planar case $E \subset \mathbb{R}^2$ where they conjectured that $\sigma_E(E) \le 1/2$. They proved the conjecture for the special case of a polygon inscribed in a disk. A similar proof, written with a geometric emphasis, is given for Proposition 3.5. Gardiner and Netuka established the full strength of the conjecture in the plane in [25], where they also characterized the sets for which the upper bound is achieved. Before presenting their results, we establish some notation.

Let E^* denote the *farthest point envelope* of E which is defined by

$$E^* = \bigcap_{x \in E} \overline{B}\left(x, d_E(x)\right).$$

Notice that this is similar to the construction we used in Proposition 2.4, except that the intersection is only over E rather than all points in $\mathbb{R}^N$. It is clear that $E \subset E^*$. We have already seen an example of a farthest point envelope. If T is a regular triangle then T^* is a Reuleaux triangle. As another example, if $L = [-1, 1]$ is a line segment and $B = B(0, 1)$, then $L^* = B^* = B$. It is not necessarily the case that $E^* = \mathrm{conv}(E)^*$. For example, $\{-1, 1\}^*$ is a lens shape that contains B.

Theorem 3.2 (Theorem 1 and 2 given by Gardiner and Netuka in [25])**.** *If $E \subset \mathbb{R}^2$ is not a single point, then $\sigma_E(E) \leq 1/2$. Furthermore, equality is achieved if and only if $F = \mathrm{conv}(E)^*$ is a body of constant width and $\sigma_F(F) = \sigma_E(E)$.*

A recent manuscript by Kawohl, Nitsch and Sweers [37] provides a slightly different proof of Theorem 3.2 in the case that ∂E is C^2 and the curvature of ∂E is bounded below by some positive constant. Much like Gardiner and Netuka's proof, this proof relies on selecting a suitable parametrization of ∂E^*. An additional result of their proof is that if ∂E is C^1 and $\sigma_E(E) = 1/2$, then E is of constant width. This excludes sets, such as the regular triangle, which do not have smooth boundaries.

Since $d_E = d_{\mathrm{conv}(E)}$ and $E \subset \mathrm{conv}(E)$, it follows that $\sigma_E(E) \leq \sigma_{\mathrm{conv}(E)}\big(\mathrm{conv}(E)\big)$. Thus we may assume E is convex for convenience, without weakening the results. Throughout this section we will assume that $E \subset \mathbb{R}^N$ is a compact convex set consisting of at least two points.

We calculated in Example 3.11 that $\sigma_R(R) = \sigma_T(T)$ where T is the regular triangle and R, the Reuleaux triangle, is a body of constant width. Hence it follows that $\sigma_T(T) = 1/2$. In fact, equality will hold for all bodies of constant width [25, Thm 2] and all regular polygons with an odd number of vertices [44, Thm 2.6].

We can combine (3.14) with Theorem 3.2 to easily obtain an unexpected result about sums of angles. Proving this result geometrically appears to be nontrivial.

Corollary 3.1. *Let P be a compact convex polygon. Then*

$$\sum_{v \in \mathcal{V}(P)} \Omega_v(F_v \cap P) \leq \pi.$$

The proof of Gardiner and Netuka of Theorem 3.2 used several useful properties of E^*.

Lemma 3.2. *The farthest point envelope E^* is 1-round.*

Since E^* is strictly convex, it follows that p_{E^*} is differentiable. However, it need not be C^2. The Reuleaux triangle we considered in Example 3.5 is an example of such a farthest point envelope. However, we can approximate E^* by smooth sets by applying Proposition 2.6. Furthermore, these sets are c-round for c arbitrarily close to 1.

Proposition 3.12. *Let E be a compact convex set and let $c > 1$. Then there exists an increasing sequence of compact convex sets $E_n \subset E^*$ such that d_{E_n} converge uniformly to d_{E^*} and hence σ_{E_n} converge weak* to σ_{E^*}. Furthermore, the sets E_n have the following properties:*

1. *The restricted support functions p_{E_n} are C^∞.*

2. *All radii of curvature are strictly positive for each E_n.*

3. *Each set E_n is c-round.*

Recall that if a set E is c-round and smooth, then the representing measure of E on a suitable subset of the complement, denoted $C_b(E)$ where $b > c$, can be computed using Proposition 3.1. In the plane it is given by (3.6) and in $\mathbb{R}^3$ it is given by (3.7). If c is arbitrarily close to 1 then $C_b(E)$ can be chosen to be arbitrarily close to the complement of E. Thus Gardiner and Netuka computed the measure σ_{E^*} on the complement of $E^* \subset \mathbb{R}^2$ and concluded that $\sigma_{E^*}(E^*) \leq 1/2$ [25, Lemma 6]. A similar computation in higher dimensions presents difficulties as the curvatures become more complicated. Gardiner and Netuka then showed that $\sigma_E \leq \sigma_{E^*}$ on E [25, Lemma 7]. Since $E \subset E^*$, the conclusion immediately follows.

Laugesen and Pritsker made a conjecture on the mass $\sigma_E(E)$ for E in higher dimensions as well.

Conjecture 3.1 (Conjecture given by Laugesen and Pritsker in [44])**.** *For $E \subset \mathbb{R}^N$, $N \geq 3$, the mass $\sigma_E(E)$ is bounded above by 2^{1-N}. Furthermore, equality is achieved if only if E is a closed ball.*

We already calculated that $\sigma_B(B) = 2^{1-N}$ for any ball $B \subset \mathbb{R}^N$ in Example 3.1 and so it is clear that equality is achieved in that case. We consider some examples in R^3. We calculated $\sigma_T(T)$ for T the regular tetrahedron in Example 3.9 and found that it is strictly less than the extreme value of $1/4$. In fact, we found the same for millions of randomly generated polyhedra as shown in Table 3.1.

Next we wish to consider an example of a body of constant width. By rotating the Reuleaux triangle about one of its axes of symmetry we can generate a body of constant width in $\mathbb{R}^3$ [13].

Example 3.15. *[Rotated Reuleaux Triangle]*

Recall that we calculated the farthest distance function d_R for the Reuleaux triangle R in Example 2.8. We also calculated σ_R in Example 3.11.

We will be using cylindrical coordinates (r, h, θ) with r being radius, h being height, and θ being the angle. Let R lie in the r, h-plane with the vertices a, b, and c as given

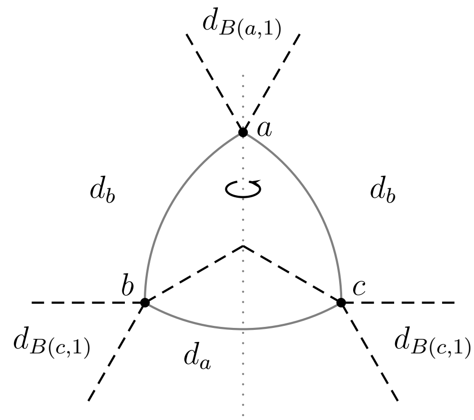

Figure 3.10: Farthest distance function for the rotated Reuleaux triangle $\mathcal{R}$

in Example 3.11. Rotate R about the h axis to generate $\mathcal{R}$, a body of constant width in $\mathbb{R}^3$. We will calculate $\sigma_{\mathcal{R}}(\mathcal{R})$.

As we found in Examples 2.3 and 3.3 of the torus, the farthest distance function of such a body of rotation is rotationally invariant. In particular, by (2.1), we have $d_{\mathcal{R}}(r,h,\theta) = d_R(r,h)$ which is shown in Figure 2.7.

Let $x = (r,h,\theta) \in \mathbb{R}^3$ be a point such that $r > 0$. Then the farthest distance function $d_{\mathcal{R}}$ is given in Figure 3.10. The diagram is a cross section so that the upper region is a cone and symmetric regions drawn on the left and right of the h axis are a single region. Thus $d_{\mathcal{R}}$ is piecewise defined on four regions.

While we know the value of $d_{\mathcal{R}}$ in all of the regions, only two intersect $\mathcal{R}$ and hence we only need to calculate $\sigma_{\mathcal{R}}$ on those two regions. Denote by A the region where $d_{\mathcal{R}}(r,h,\theta) = d_a(r,h)$. Since a remained constant during rotation, we can apply Propositions 3.1 and 2.11 to conclude that $\sigma_{\mathcal{R}}(A) = 0$.

Let B be the region where $d_{\mathcal{R}}(r,h,\theta) = d_b(r,h)$. Formulas for the Laplacian and volume differential in cylidrical coordinates were given in Example 3.3. Applying them, we can calculate that on B, the measure $\sigma_{\mathcal{R}}$ is given by

$$\mathrm{d}\sigma_{\mathcal{R}} = \frac{1}{(2-N)\omega_N}\Delta d_{\mathcal{R}}^{2-N}\,\mathrm{d}V = \frac{1}{8\pi}d_{\mathcal{R}}^{-3}\,\mathrm{d}\theta\,\mathrm{d}h\,\mathrm{d}r.$$

The set $B \cap \mathcal{R}$ consists of points (r,h,θ) such that $r > 0$ and the planar point

(r,h) lies above the line segment $[0,c]$ and below the curve $\partial B(c,1)\cap R$. Without loss of generality, we choose the vertices of R to be

$$\begin{aligned} a &= \left(0, \frac{1}{\sqrt{3}}\right), \\ b &= \left(-\frac{1}{2}, -\frac{1}{2\sqrt{3}}\right), \text{ and} \\ c &= \left(\frac{1}{2}, -\frac{1}{2\sqrt{3}}\right). \end{aligned}$$

Then these curves are given by

$$\begin{aligned} [0,c] &= \left\{(r,h) \mid r = -\sqrt{3}h\right\} \\ \partial B(c,1)\cap R &= \left\{(r,h) \mid \left(r-\frac{1}{2}\right)^2 + \left(h + \frac{1}{2\sqrt{3}}\right)^2 = 1, 0 \le r \le \frac{1}{2}, \frac{-1}{2\sqrt{3}} \le h \le \frac{1}{\sqrt{3}}\right\}. \end{aligned}$$

Thus we calculate the measure of B by integrating

$$\sigma_{\mathcal{R}}(B) = \int_0^{1/2} \int_{-r/\sqrt{3}}^{-1/2\sqrt{3}+\sqrt{-r^2-r+3/4}} \int_0^{2\pi} \frac{1}{8\pi} d_R^{-3}\, \mathrm{d}\theta\, \mathrm{d}h\, \mathrm{d}r$$

and we find that $\sigma_{\mathcal{R}}(\mathcal{R}) = \sigma_{\mathcal{R}}(B) \approx .104225\ldots < \frac{1}{4}$.

In the plane, the bodies of constant width achieve the extreme value $\sigma_E(E) = 1/2$. Conjecture 3.1 states that the equality $\sigma_E(E) = 2^{1-N}$ for $E \subset \mathbb{R}^N$ is achieved only for balls. The rotated Reuleaux triangle provided an example in support of this part of the conjecture. We can also prove it in the special case of C^2 smooth bodies of constant width. Notice that the rotated Reuleaux triangle is not C^2 smooth and so the result of Example 3.15 is not implied by the following proposition.

Proposition 3.13. *Let $E \subset \mathbb{R}^N$, $N \ge 3$, be a body of constant width with C^2 boundary. Then $\sigma_E(E) \le 2^{1-N}$ with equality if and only if E is a closed ball.*

The examples of the tetrahedron and rotated Reuleaux triangle, the data in Table 3.1, and Proposition 3.13 provide strong evidence in support of Conjecture 3.1. The recent manuscript by Kawohl, Nitsch and Sweers [37] provides further results in this direction. They prove a more general version of Proposition 3.13 which does not require E to have a C^2 boundary. They also offer a proof of Conjecture 3.1 in the case of centrally symmetric sets. A centrally symmetric set is a set, such as the ellipsoid, for which $x \in E$ implies $-x \in E$.

3.5 Proofs

Proof of Proposition 3.2. This proof is adapted from [44, Proof of Theorem 2.5] which proved a similar result in the planar case. Recall from Proposition 2.2 that d_E is invariant under rigid transformations, with $d_{TE}(Tx) = d_E(x)$ for any rigid transformation T. It follows that σ_E is also invariant under rigid transformations.

The function d_E is homogeneous under scaling so that $d_{\lambda E}(\lambda x) = \lambda d_E(x)$ for $\lambda > 0$. We compute

$$\begin{aligned}\int |\lambda x - y|^{2-N} \,\mathrm{d}\sigma_{\lambda E}(y) &= d_{\lambda E}^{2-N}(\lambda x) \\ &= \lambda^{2-N} d_E^{2-N}(x) \\ &= \lambda^{2-N} \int |x - y|^{2-N} \,\mathrm{d}\sigma_E(y) \\ &= \int |\lambda x - \lambda y|^{2-N} \,\mathrm{d}\sigma_E(y).\end{aligned}$$

Thus $\mathrm{d}\sigma_{\lambda E}(\lambda y) = \mathrm{d}\sigma_E(y)$ and the conclusion follows. $\square$

Proof of Equation (3.5). To make the following proof easier to read we will let p be the restricted support function p_E and γ be the support parametrization γ_E. If we also let $u = (0, \phi, \theta)$ then we can denote the partial derivatives of functions using subscripts. We will use the same notation in the proof of (3.7).

For a set E, we can express γ in terms of p. The planes of support for E are given by the equations $F(\xi, u) = 0$, $F_\phi(\xi, u) = 0$, and $F_\theta(\xi, u) = 0$ where $F(\xi, u) = \xi \cdot \vec{u} - p(u)$. This parametrized set of planes has an envelope and that envelope is the surface ∂E.

We begin finding the parametrization γ by calculating the derivatives of F. Let $\xi = (\xi_1, \xi_2, \xi_3)$. The derivatives of $\vec{u}$ are given by

$$\vec{u}_\phi = \begin{pmatrix} -\sin\phi \\ \cos\phi\cos\theta \\ \cos\phi\sin\theta \end{pmatrix}.$$

and

$$\vec{u}_\theta = \begin{pmatrix} 0 \\ -\sin\phi\sin\theta \\ \sin\phi\cos\theta \end{pmatrix}.$$

and so we have the system of equations

$$\begin{aligned} F(\xi, u) &= \xi_1\cos\phi + \xi_2\sin\phi\cos\theta + \xi_3\sin\phi\sin\theta - p(u) = 0 \\ F_\phi(\xi, u) &= \xi_1\sin\phi + \xi_2\cos\phi\cos\theta + \xi_3\cos\phi\sin\theta - p_\phi(u) = 0 \\ F_\theta(\xi, u) &= -\xi_2\sin\phi\sin\theta + \xi_3\sin\phi\cos\theta - p_\theta(u) = 0. \end{aligned}$$

Solving these equations yields

$$\xi = \begin{pmatrix} \cos\phi\left(p(u) + p_\phi(u)\cot\phi\right) \\ \sin\phi\cos\theta\left(p(u) + p_\phi(u)\cot\phi - p_\theta(u)\csc^2\phi\tan\theta\right) \\ \sin\phi\sin\theta\left(p(u) + p_\phi(u)\cot\phi + p_\theta(u)\csc^2\phi\cot\theta\right) \end{pmatrix}$$

as desired. □

Proof of Equation (3.6). The following computation was done in [25, Lemma 3]. The map F is given by

$$\begin{aligned} F(\rho, u) &= \left(p_E(u) - \rho\right)\vec{u} + \frac{\partial p_E}{\partial\theta}(u)\frac{\partial\vec{u}}{\partial\theta} \\ &= \begin{pmatrix} \left(p_E(u) - \rho\right)\cos\theta - \frac{\partial p_E}{\partial\theta}(u)\sin\theta \\ \left(p_E(u) - \rho\right)\sin\theta + \frac{\partial p_E}{\partial\theta}(u)\cos\theta \end{pmatrix} \end{aligned}$$

for $\rho > cw_E(u)$. We will denote the rectangular coordinates by (x_1, x_2) as usual. Denote the radius of curvature of E at the point $\gamma_E(u)$ by $r_E(u)$. It is given by $r_E(u) = p_E(u) - p_E''(u)$. Then the derivative matrix of F is

$$\begin{pmatrix} -\cos\theta & \big(\rho - r_E(u)\big)\sin\theta \\ -\sin\theta & -\big(\rho - r_E(u)\big)\cos\theta \end{pmatrix}$$

which has determinant $\rho - r_E(u)$ and inverse

$$\frac{1}{\rho - r_E(u)} \begin{pmatrix} -\big(\rho - r_E(u)\big)\cos\theta & -\big(\rho - r_E(u)\big)\sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}.$$

Since $d_E\big(F(\rho, u)\big) = \rho$, it follows from the Inverse Function Theorem and chain rule that

$$\frac{\partial \log d_E}{\partial x_1}\big(F(\rho,\theta)\big) = \frac{\partial}{\partial \rho}\log\rho \frac{\partial \rho}{\partial x_1}\big(F(\rho,\theta)\big) = \frac{-\cos\theta}{\rho}$$

and, by the chain rule again,

$$\frac{\partial^2 \log d_E}{\partial x_1^2}\big(F(\rho,\theta)\big) = \frac{\sin^2\theta}{\rho\big(\rho - r_E(u)\big)} - \frac{\cos\theta}{\rho}.$$

Similarly,

$$\frac{\partial^2 \log d_E}{\partial x_2^2}\big(F(\rho,\theta)\big) = \frac{\cos^2\theta}{\rho\big(\rho - r_E(u)\big)} - \frac{\sin\theta}{\rho}$$

and hence

$$\Delta \log d_E\big(F(\rho,\theta)\big) = \frac{r_E(u)}{\rho^2\big(\rho - r_E(u)\big)}.$$

Since $(-\log d_E)$ is superharmonic, we apply [44, Example 1.2] and write σ_E as

$$\mathrm{d}\sigma_E\big(F(\rho,u)\big) = \frac{1}{2\pi}\frac{r_E(u)}{\rho^2\big(\rho - r_E(u)\big)}\,\mathrm{d}x_1\,\mathrm{d}x_2.$$

Using the Jacobian determinant we find

$$\mathrm{d}\sigma_E\big(F(\rho,u)\big) = \frac{1}{2\pi}\frac{r_E(u)}{\rho^2}\,\mathrm{d}\rho\,\mathrm{d}u. \tag{3.15}$$

$\square$

Proof of Equation (3.7). We use the same notation we introduced in the proof of (3.5). We must begin by expressing the curvature of ∂E in terms of the support p. In $\mathbb{R}^3$, curvature is expressed via the fundamental forms. The first fundamental form for E, denoted I, is the inner product on the tangent space, so that $I(v,w) = \langle v, w\rangle = v^T I w$ where v and w are given in terms of γ_ϕ and γ_θ. It is given by the symmetric matrix

$$I = \begin{bmatrix} \gamma_\theta \cdot \gamma_\theta & \gamma_\theta \cdot \gamma_\phi \\ \gamma_\theta \cdot \gamma_\phi & \gamma_\phi \cdot \gamma_\phi \end{bmatrix}.$$

It describes the metric properties of the surface including length and area differentials and is positive definite. The second fundamental form is given by

$$II = \begin{bmatrix} \gamma_{\theta\theta} \cdot n & \gamma_{\theta\phi} \cdot n \\ \gamma_{\theta\phi} \cdot n & \gamma_{\phi\phi} \cdot n \end{bmatrix}$$

where n is the unit normal vector to E, given by

$$n = \pm \frac{\gamma_\theta \times \gamma_\phi}{|\gamma_\theta \times \gamma_\phi|}.$$

We can compute the fundamental forms of E in terms of p. First, we calcuate

$$n = \pm \begin{pmatrix} \cos\phi \\ \sin\phi\cos\theta \\ \sin\phi\sin\theta \end{pmatrix}.$$

and choose $n = -\vec{u}$. Computing I and II, we obtain

$$I = \begin{bmatrix} e^2 \csc^2\phi + f^2 & f(e\csc^2\phi + g) \\ f(e\csc^2\phi + g) & g^2 + f^2\csc^2\phi \end{bmatrix}.$$

and

$$II = \begin{bmatrix} e & f \\ f & g \end{bmatrix}.$$

where

$$
\begin{aligned}
e =& \sin^2 \phi (p + p_\phi \cot \phi + p_{\theta\theta} \csc^2 \phi) \\
f =& p_{\theta\phi} - p_\theta \cot \phi \\
g =& p + p_{\phi\phi}
\end{aligned}
$$

are the coefficients of the second fundamental form. The matrix $I^{-1}II$ has eigenvalues (and hence ∂E has principal curvatures)

$$
\kappa_{1,2} = \frac{1}{2} \left(e + g \sin^2 \phi \pm \sqrt{(e + g \sin^2 \phi)^2 + 4 f^2 \sin^2 \phi} \right).
$$

The Gaussian and mean curvatures are given by

$$
\begin{aligned}
K =& \frac{\sin^2 \phi}{eg - f^2} \\
H =& \frac{1}{2} \frac{e + g \sin^2 \phi}{eg - f^2}.
\end{aligned}
$$

where each curvature depends on the direction u and is the curvature of ∂E at the point $\gamma(u)$. Also note that $e \csc^2 \phi + g = 2H/K$.

We can now compute the Jacobian of $F(\rho, u) = F\left(\rho, (0, \phi, \theta)\right)$ which is

$$
\begin{aligned}
J &= \begin{pmatrix} F_\rho & F_\phi & F_\theta \end{pmatrix} \\
&= - \begin{pmatrix} \vec{u} & (\rho - g)\vec{u}_\phi - f \csc^2 \phi \, \vec{u}_\theta & (\rho - e \csc^2 \phi)\vec{u}_\theta - f \vec{u}_\phi \end{pmatrix}
\end{aligned}
$$

It has determinant

$$
\begin{aligned}
d =& \sin \phi (\rho - g)(\rho - e \csc^2 \phi) - f^2 \csc \phi \\
=& \sin \phi \left(\rho^2 - \rho \frac{2H}{K} + \frac{1}{K} \right) \\
=& \frac{\sin \phi}{K} \left(K\rho^2 - 2H\rho + 1 \right).
\end{aligned}
$$

Since $H^2 \geq K$ and we are working over $C_b(E)$, the determinant is a positive real number. Thus the Jacobian has inverse

$$J^{-1} = \begin{vmatrix} \frac{\partial \rho}{\partial F} \\ \frac{\partial \phi}{\partial F} \\ \frac{\partial \theta}{\partial F} \end{vmatrix}$$

$$= \frac{-\csc\phi}{d} \begin{vmatrix} d \sin\phi \ \vec{u} \\ (\rho \sin^2\phi - e) \ \vec{u}_\phi + f\vec{u}_\theta \\ (\rho - g)\vec{u}_\theta + f\vec{u}_\phi \end{vmatrix}.$$

We apply the chain rule and inverse function theorem to find $\Delta\rho^{-1}$ which is

$$\begin{aligned} \Delta\rho^{-1} =& \frac{\partial^2}{\partial x_1^2}\rho^{-1} + \frac{\partial^2}{\partial x_2^2}\rho^{-1} + \frac{\partial^2}{\partial x_3^2}\rho^{-1} \\ =& -\frac{\sin\phi}{d}\left(\frac{e\csc^2\phi + g}{\rho^2} - \frac{2(eg - f^2)\csc^2\phi}{\rho^3}\right) \\ =& -\frac{1}{d} \cdot \frac{2\sin\phi}{K} \cdot \frac{H\rho - 1}{\rho^3}. \end{aligned}$$

After a change of variables, we have that on $C_b(E)$ that

$$\mathrm{d}\sigma_E = \frac{\sin\phi}{2\pi K} \cdot \frac{H\rho - 1}{\rho^3} \, \mathrm{d}\rho \, \mathrm{d}\phi \, \mathrm{d}\theta.$$

□

Proof of Proposition 3.3. Recall that by the Riesz Decomposition Theorem there exists a function h which is harmonic in D such that $u(x) = U^\mu(x) + h(x)$ in D. We average u over the sphere $\partial B(w,r)$ to find

$$\begin{aligned} \mathcal{M}(u; w, r) &= \frac{1}{\omega_N r^{N-1}} \int_{\partial B(w,r)} \big(U^\mu(x) + h(x)\big) \, \mathrm{d}x \\ &= \frac{1}{\omega_N r^{N-1}} \int_{\partial B(w,r)} \int_D |x-y|^{2-N} \, \mathrm{d}\mu(y) \, \mathrm{d}x + \frac{1}{\omega_N r^{N-1}} \int_{\partial B(w,r)} h(x) \, \mathrm{d}x \\ &= \int_D \frac{1}{\omega_N r^{N-1}} \int_{\partial B(w,r)} |x-y|^{2-N} \, \mathrm{d}x \, \mathrm{d}\mu(y) + h(w) \\ &= \int_{B(w,r)} r^{2-N} \, \mathrm{d}\mu(y) + \int_{D \setminus B(w,r)} |y-w|^{2-N} \, \mathrm{d}\mu(y) + h(w). \end{aligned}$$

where we applied the Fubini-Tonelli Theorem and Mean Value Property on the third step and [2, p. 100] on the fourth.

The difference between the mean on the ball of radius r and the mean on a slightly larger ball of radius $r+\delta$, $\delta > 0$, can be calculated as

$$\begin{aligned}\mathcal{M}(u;w,r) - \mathcal{M}(u;w,r+\delta) &= \int_{B(w,r)} \left(r^{2-N} - (r+\delta)^{2-N}\right) \mathrm{d}\mu(y) \\ &\quad + \int_{B(w,r+\delta)\setminus B(w,r)} \left(|y-w|^{2-N} - (r+\delta)^{2-N}\right) \mathrm{d}\mu(y) \\ &\leq \int_{B(w,r+\delta)} \left(r^{2-N} - (r+\delta)^{2-N}\right) \mathrm{d}\mu(y) \\ &= \int_{B(w,r+\delta)} \left(\frac{\delta(N-2)}{r(r+\delta)^{N-2}} + O(\delta^2)\right) \mathrm{d}\mu(y).\end{aligned}$$

Taking the limit as $\delta \to 0^+$, we find

$$\lim_{\delta\to 0^+} \frac{\mathcal{M}(u;w,r) - \mathcal{M}(u;w,r+\delta)}{\delta} \leq \frac{N-2}{r^{N-1}}\, \mu\left(\overline{B}(w,r)\right).$$

Similarly, for $-\delta < 0$, we have

$$\lim_{\delta\to 0^+} \frac{\mathcal{M}(u;w,r) - \mathcal{M}(u;w,r-\delta)}{-\delta} \geq \frac{N-2}{r^{N-1}}\, \mu\left(B(w,r)\right).$$

Note that $\mathcal{M}(u;w,r)$ is monotonic and bounded for finite values of r. Thus $\mu\left(\partial B(w,r)\right) = 0$ and the limit as $\delta \to 0$ exists for all r, with, at most, countably many exceptions. $\square$

Proof of Proposition 3.4. We first prove the lemma for $E \subset \mathbb{R}^N$, $N \geq 3$. If the sets E_n are monotone increasing, it follows that the functions d_{E_n} are increasing. Hence their negative powers $d_{E_n}^{2-N}$ form a monotone decreasing sequence of potentials. Since $\lim_{n\to\infty} d_{E_n}^{2-N} = d_E^{2-N}$, the conclusion follows from [43, Theorem 3.10, p. 192]. On the other hand, if the sets E_n are monotone decreasing, then $d_{E_n}^{2-N}$ forms a monotone increasing sequence of potentials. Furthermore, the sequence is majorized by the potential d_E^{2-N} and so the conclusion follows from [43, Theorem 3.9, p. 192]. $\square$

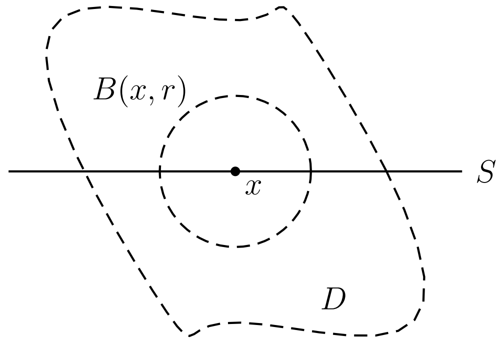

Figure 3.11: A domain intersecting a hyperplane

Proof of Lemma 3.1. This proof follows the proof given in [63, p. 92] for a similar statement in the plane. We assume without loss of generality that $D \cap S$ is connected. Since u is harmonic except on S, the support of μ is contained in S. We will calculate μ on $D \cap S$. Choose $x \in D \cap S$ and $r > 0$ such that $\overline{B(x,r)} \subset D$, as shown in Figure 3.11.

We break D into two pieces by cutting along S. Choose a positive side of S and denote the normal to S in that direction by $\boldsymbol{n}_+$ and the normal in the other direction as $\boldsymbol{n}_-$. The open subsets of D and $B(x,r)$ that lie on the positive and negative sides of S will be denoted with the appropriate subscript.

Let $v(x,t) = |x-t|^{2-N}$. Notice that for a fixed x, the function $v(x,\cdot)$ is harmonic wherever $t \neq x$ and, in particular, on a neighborhood of the compact set $\overline{D_+} \setminus B(x,r)_+$. Further, u is harmonic outside a neighborhood of S. Thus we may choose a compact set which has a neighborhood where both u and v are harmonic. Let $K_\delta = \left(\overline{D_+} \setminus B(x,r)_+\right) \setminus N_\delta(S)$, where $N_\delta(S)$ is a small δ-neighborhood of the hyperplane, as shown in Figure 3.12.

We may use Green's Identity [32, p. 22] to calculate the integral on K_δ with respect to t as

$$\int_{K_\delta} (v\Delta u - u\Delta v)\,\mathrm{d}V = -\int_{\partial K_\delta} \left(v\frac{\partial u}{\partial \boldsymbol{n}} - u\frac{\partial v}{\partial \boldsymbol{n}} \right) \mathrm{d}S$$

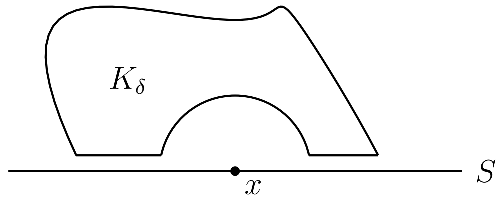

Figure 3.12: The compact set K_δ

where $\boldsymbol{n}$ denotes the inward normal derivative on $D \setminus B(x,r)$ in all expressions. Since u and v are harmonic on a neighborhood of K_δ, the left hand side is zero. Since u is Lipschitz, the derivatives $\partial u/\partial \boldsymbol{n}$ on the right hand side are bounded and will converge as $\delta \to 0^+$ a.e. on S. Thus

$$-\int_{\partial(D_+ \setminus B(x,r)_+)} \left(v\frac{\partial u}{\partial \boldsymbol{n}} - u\frac{\partial v}{\partial \boldsymbol{n}} \right) \mathrm{d}S = 0.$$

We break the left hand side of this expression apart into the component that lies on S and the two components that do not, obtaining

$$\int_{\partial(D_+ \setminus B(x,r)_+)\setminus S} \left(u\frac{\partial v}{\partial \boldsymbol{n}} - v\frac{\partial u}{\partial \boldsymbol{n}} \right) \mathrm{d}S + \int_{\partial(D_+ \setminus B(x,r)_+)\cap S} \left(u\frac{\partial v}{\partial \boldsymbol{n}_+} - v\frac{\partial u}{\partial \boldsymbol{n}_+} \right) \mathrm{d}S = 0.$$

Similarly we may find on the negative side of S that

$$\int_{\partial(D_- \setminus B(x,r)_-)\setminus S} \left(u\frac{\partial v}{\partial \boldsymbol{n}} - v\frac{\partial u}{\partial \boldsymbol{n}} \right) \mathrm{d}S + \int_{\partial(D_- \setminus B(x,r)_-)\cap S} \left(u\frac{\partial v}{\partial \boldsymbol{n}_-} - v\frac{\partial u}{\partial \boldsymbol{n}_-} \right) \mathrm{d}S = 0.$$

Adding the expressions for the positive and negative side of S together, and noting that the normal derivatives of v along $S \setminus B(x,r)$ will cancel each other, we find

$$\begin{aligned}\int_{(D\setminus B(x,r))\cap S} v \left(\frac{\partial u}{\partial \boldsymbol{n}_+} + \frac{\partial u}{\partial \boldsymbol{n}_-} \right) \mathrm{d}S &= \int_{\partial D} \left(u\frac{\partial v}{\partial \boldsymbol{n}} - v\frac{\partial u}{\partial \boldsymbol{n}} \right) \mathrm{d}S \\ &\quad + \int_{\partial B(x,r)} \left(u\frac{\partial v}{\partial \boldsymbol{n}} - v\frac{\partial u}{\partial \boldsymbol{n}} \right) \mathrm{d}S.\end{aligned}$$

In a neighborhood of ∂D, u does not depend on the choice of x while v is harmonic as a function of x. Therefore the first integral on the right is harmonic as a function of x. We call it $h(x)$, modulo an appropriate constant. The second integral on the right can be written as

$$\int_{\partial B(x,r)} \left(u\,\frac{\partial v}{\partial \boldsymbol{n}} - v\frac{\partial u}{\partial \boldsymbol{n}} \right) \mathrm{d}S = \int_{\partial B(x,r)} u\frac{2-N}{r^{N-1}}\,\mathrm{d}S - \int_{\partial B(x,r)} \frac{1}{r^{N-2}}\frac{\partial u}{\partial \boldsymbol{n}}\,\mathrm{d}S.$$

As r tends to zero, the continuity of u and boundedness of $\partial u/\partial \boldsymbol{n}$ imply that

$$\lim_{r\to 0}\int_{\partial B(x,r)}\left(\frac{2-N}{r^{N-1}}u-\frac{1}{r^{N-2}}\frac{\partial u}{\partial \boldsymbol{n}}\right)\mathrm{d}S=(2-N)\omega_N u(x).$$

Therefore,

$$\begin{aligned}u(x)&=\frac{1}{(2-N)\omega_N}\int_{D\cap S} v\left(\frac{\partial u}{\partial \boldsymbol{n}_+}+\frac{\partial u}{\partial \boldsymbol{n}_-}\right)\mathrm{d}S+h(x)\\&=\frac{1}{(2-N)\omega_N}\int_{D\cap S}|x-t|^{2-N}\left(\frac{\partial u}{\partial \boldsymbol{n}_+}+\frac{\partial u}{\partial \boldsymbol{n}_-}\right)\mathrm{d}S+h(x)\end{aligned}$$

and the conclusion follows from the Riesz Decomposition Theorem □

Proof of Theorem 3.1. The following proof is presented for polyhedra $P\subset\mathbb{R}^3$. The proof in higher dimensions, using hyperspherical coordinates in place of spherical, is similar hence we omit it.

Assume v and w lie on the z-axis at $|v-w|/2$ and $-|v-w|/2$ respectively. Applying (3.11) we have

$$\mathrm{d}\sigma_P(\xi)=\frac{|v-w|}{4\pi}\; d_P^{-3}(\xi)\,\mathrm{d}\xi=\frac{|v|}{2\pi}\; d_P^{-3}(\xi)\,\mathrm{d}x\,\mathrm{d}y$$

for $\xi\in F_v\cap F_w$.

The angle $\Omega_v(S)$ can be found by expressing S in spherical coordinates centered at v so that $\xi\in S$ is written as $v+\rho(\cos\theta\sin\phi,\sin\theta\sin\phi,\cos\phi)$ where $\phi\in[0,\pi]$ and $\theta\in[0,2\pi)$. Then $\Omega_t(S)=\int_S\sin\phi\,\mathrm{d}\theta\,\mathrm{d}\phi$, the usual surface area integral in spherical coordinates.

Let $\xi\in S$ have coordinates $(x,y,0)$. It forms angles θ and ϕ as shown in Figure 3.13.

Then $d_P(\xi)=|v|\sec(\pi-\phi)=-|v|\sec\phi$ and we have

$$x=d_P(\xi)\sin(\pi-\phi)\cos\theta=-|v|\tan\phi\cos\theta$$

$$y=d_P(\xi)\sin(\pi-\phi)\sin\theta=-|v|\tan\phi\sin\theta.$$

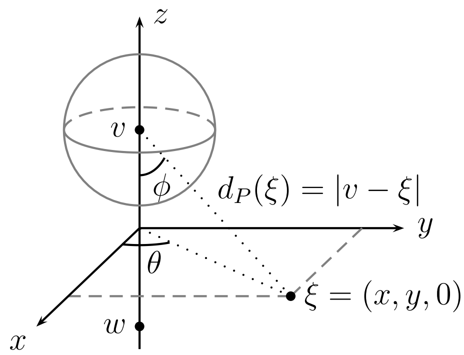

Figure 3.13: The point ξ in rectangular coordinates relative to the origin and spherical coordinates relative to v

We calculate the Jacobian by

$$\left|\frac{\partial(x,y)}{\partial(\theta,\phi)}\right| = |v|^2 \begin{vmatrix} \tan\phi\sin\theta & -\sec^2\phi\cos\theta \\ -\tan\phi\cos\theta & -\sec^2\phi\sin\theta \end{vmatrix}$$
$$= -\left(|v|^2\sec^3\phi\right)\sin\phi.$$

We use this result to perform a change of variable and obtain

$$\begin{aligned} d\sigma_P &= \frac{|v|}{2\pi}\ d_P^{-3}\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{|v|}{2\pi}\ (|v|\sec\phi)^{-3}\left(|v|^2\sec^3\phi\right)\sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta \\ &= \frac{1}{2\pi}\ \sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta. \end{aligned}$$

Thus

$$\sigma_P(S) = \frac{2}{4\pi}\Omega_v(S) = \frac{1}{4\pi}\left(\Omega_v(S) + \Omega_w(S)\right).$$

For each vertex $v \in \mathcal{V}(P)$, the set $\partial F_v \cap D$ is a collection of subsets of a finite number of hyperplanes and their intersections. As F_v is the intersection of halfspaces disjoint from v, the angles subtended at v by each part of the hyperplanes cannot

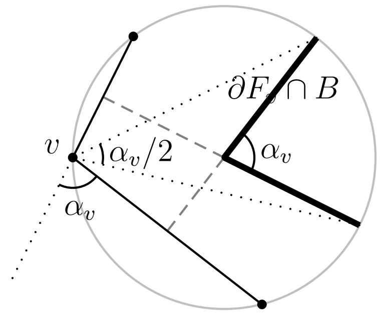

Figure 3.14: The angle $\Omega_v(\partial F_v \cap B)$ is shown with dotted lines from v to the endpoints of $\partial F_v \cap B$, which is shown in bold

overlap. Taking a sum over all $v \in \mathcal{V}(P)$ we find

$$\begin{aligned}\sigma_P(D) &= \frac{1}{2} \sum_{v \in \mathcal{V}(P)} \sigma_P(\partial F_v \cap D) \\ &= \frac{1}{\omega_N} \sum_{v \in \mathcal{V}(P)} \Omega_v(\partial F_v \cap D)\end{aligned}$$

where the $1/2$ arises because each planar part in the support of σ_P lies on the boundaries of two farthest point Voronoi cells. $\square$

Proof of Proposition 3.5. Let B be a disk and P an inscribed polygon. For $v \in \mathcal{V}(P)$ let α_v be the external angle at v. The farthest point Voronoi cell F_v is the intersection of all halfplanes $\{z : \ |z - v| \geq |z - w|\}$ for $w \in \mathcal{V}(P)$. Since P is inscribed all edges are chords. The boundary of each of these halfplanes is a perpendicular bisector of a chord which must pass through the center of the disk B. Thus F_v is the intersection of the two halfplanes corresponding to the edges adjacent to v, as shown in Figure 3.14.

The perpendicular bisectors of the adjacent edges form an angle of measure α_v at the center of the disk. The inscribed angle in a disk is half the central angle so

$\Omega_v(\partial F_v \cap B) = \alpha_v/2$. Applying Theorem 3.1, we calculate

$$\begin{aligned}\sigma_P(B) &= \frac{1}{2\pi}\sum_{v\in\mathcal{V}(P)} \Omega_v(\partial F_v \cap B)\\ &= \frac{1}{2\pi}\sum_{v\in\mathcal{V}(P)} \frac{\alpha_v}{2}.\end{aligned}$$

The sum of exterior angles of a polygon is always 2π and so $\sigma_P(B) = 1/2$. $\square$

Proof of Proposition 3.6. If not, then there exists a domain D such that $\mathrm{supp}(\sigma_E) \cap D = \emptyset$. By Proposition 2.11, there exists $t \in \partial E \setminus D$ such that $d_E(x) = |x - t|$ on D. This implies that the line segment $[x, t]$ is orthogonal to ∂E at t for every $x \in D$, which is impossible. $\square$

Proof of Proposition 3.7. If $\mathbb{R}^N \setminus \mathrm{supp}(\sigma_E)$ is nonempty, then it is necessarily the union of at most countably many disjoint domains. It follows from Propositions 2.11 and 3.1 that the farthest point function y_E is constant on each domain.

Recall that for a polytope, we defined F_y to be the farthest point Voronoi cell for y, or the set where $d_E = d_y$. For an arbitrary set E and point $y \in \partial E$, we define D_y to be the open set where $d_E = d_y$. We will show each D_y is convex and unbounded.

Let x_1 and x_2 be in D_y and let x be a point on the segment $[x_1, x_2]$. Then we have

$$E \subset \overline{B}\left(x_1, |x_1 - y|\right) \cap \overline{B}\left(x_2, |x_2 - y|\right) \subset \overline{B}\left(x, |x - y|\right).$$

Hence $d_E(x) = |x - y|$ and $x \in D_y$.

Let $x_0 \in D_y$ and let L be the line passing through x_0 and y. For each $x \in L$ such that $|x - y| > |x_0 - y|$ we have $E \subset \overline{B}\left(x_0, |x_0 - y|\right) \subset \overline{B}\left(x, |x - y|\right)$ and thus $d_E(x) = |x - y|$. The same argument holds for every point in a neighborhood of x_0. It follows that an unbounded neighborhood is contained in D_y. $\square$

Proof of Proposition 3.8. Applying Proposition 3.7, we conclude that $\mathrm{supp}(\sigma_E)$ must be simply connected. We will now consider subsets of $\mathrm{supp}(\sigma_E)$. Let U be the interior of $\mathrm{supp}(\sigma_E)$.

As an open set, U consists of at most countably many domains. Let U_i be a domain in the interior of $\mathrm{supp}(\sigma_E)$. Since y_E is single-valued almost everywhere, there exists $x_0 \in U_i$ such that y_E is single-valued in a neighhborhood of x_0. Let $\vec{n} = (x_0 - y_E(x_0))/|x_0 - y_E(x_0)|$ be the vector in the direction from $y_E(x_0)$ to x_0. If ∂E is sufficiently smooth at $y_E(x_0)$, this is the inward normal vector. Consider the ray $x_0 + t\vec{n}$ for $t \geq 0$. For each x on that ray, $E \subset \overline{B}(x_0, |x_0 - y_E(x_0)|) \subset \overline{B}(x, |x - y_E(x_0)|)$ and hence $y_E(x) = y_E(x_0)$. Now assume that U_i is bounded. Then there exists some $t > 0$ such that $x = x_0 + t\vec{n}$ is in the complement of $\mathrm{supp}(\sigma_E)$. By Proposition 3.7, there is a neighborhood of x such that y_E is constant and must be $y_E(x_0)$. Since this argument applies to any point in a neighborhood of x_0, we reach a contradiction. Thus U_i is unbounded.

Since $\mathrm{supp}(\sigma_E)$ is closed, it follows that $\overline{U} \subset \mathrm{supp}(\sigma_E)$.

Finally we consider $\mathrm{supp}(\sigma_E) \setminus \overline{U}$. This consists of intersections between the boundaries of domains in the complement of $\mathrm{supp}(\sigma_E)$. Let S be the intersection of the boundaries of two such domains D_1 and D_2. Recall that y_E is constant on each of these domains and let $y_E \equiv y_1$ on D_1 and let $y_E \equiv y_2$ on D_2. It follows that for $x \in S$ we have $d_E(x) = |x - y_1| = |x - y_2|$ and hence S is a subset of a hyperplane. Since S forms a boundary between two convex sets, it must also be convex.

The intersection between hyerplane subsets and the boundaries of domains in $\mathrm{supp}(\sigma_E)$ consists of countably many $N-2$ dimensional curves which are polar sets in $\mathbb{R}^N$. See [32, Section 5.4] or [2] for a more complete discussion of Hausdorff measure and capacity. Since the potential of σ_E is finite, this boundary must have zero mass and the final claim follows. $\square$

Proof of Proposition 3.9. This result is a generalization of [44, Theorem 2.2] which applies to the planar case. We will apply Proposition 3.3 to estimate the measure of the ball $B(w,r)$. In particular,

$$\sigma_E\left(B(x,r)\right) = \frac{r^{N-1}}{N-2}\lim_{\delta\to 0}\frac{\mathcal{M}(d_E^{2-N};x,r)-\mathcal{M}(d_E^{2-N};x,r+\delta)}{\delta}.$$

Because d_E is Lipschitz 1 we have

$$d_E\left(x+(r+\delta)t\right) \le d_E(x+rt)+\delta$$

for every $t\in B(0,1)$ and sufficiently small $\delta>0$. Thus

$$\begin{aligned} d_E^{2-N}(x+rt)-d_E^{2-N}\left(x+(r+\delta)t\right) &\le d_E^{2-N}(x+rt)-\left(d_E(x+rt)+\delta\right)^{2-N} \\ &= \delta(N-2)d_E^{-1}(x+rt)\left(d_E(x+rt)+\delta\right)^{2-N}+O(\delta^2). \end{aligned}$$

Using this calculation and the Dominated Convergence Theorem we find

$$\begin{aligned} \sigma_E\left(B(w,r)\right) &\le \frac{r^{N-1}}{\omega_N}\lim_{\delta\to 0^+}\int_{\partial B(0,1)}\Big(d_E^{-1}(x+rt)\left(d_E(x+rt)+\delta\right)^{2-N}+O(\delta)\Big)\,\mathrm{d}t \\ &= \frac{r^{N-1}}{\omega_N}\int_{\partial B(0,1)}\lim_{\delta\to 0^+}\Big(d_E^{-1}(x+rt)\left(d_E(x+rt)+\delta\right)^{2-N}+O(\delta)\Big)\,\mathrm{d}t \\ &= \frac{r^{N-1}}{\omega_N}\int_{\partial B(0,1)} d_E^{1-N}(x+rt)\,\mathrm{d}t. \end{aligned}$$

If $r<d_E(x)$, we may estimate $d_E(x+rt)$ from below by $d_E(x)-r$ and the conclusion follows. □

Proof of Proposition 3.10. This result is a generalization of [44, Theorem 2.2] which applies to the planar case. Assume $r>\operatorname{diam}(E)$.

For each $t\in\partial B(0,1)$ and $y\in E$, the three points x, $x+rt$, and y form a triangle, as shown in Figure 3.15. Denote the angle at the vertex $x+rt$ by $\alpha(y,t)$. Let β be the angle at y. Then by the Law of Sines we have

$$\sin\alpha(y,t) = |x-y|\frac{\sin\beta}{|x+rt-x|} \le \frac{\operatorname{diam}(E)}{r}.$$

By the Law of Cosines we have

$$\begin{aligned} 2|x+rt-x||x+rt-y|\cos\alpha(y,t) &= |x+rt-x|^2+|x+rt-y|^2-|x-y|^2 \\ &\ge r^2-\operatorname{diam}(E)^2 \\ &> 0. \end{aligned}$$

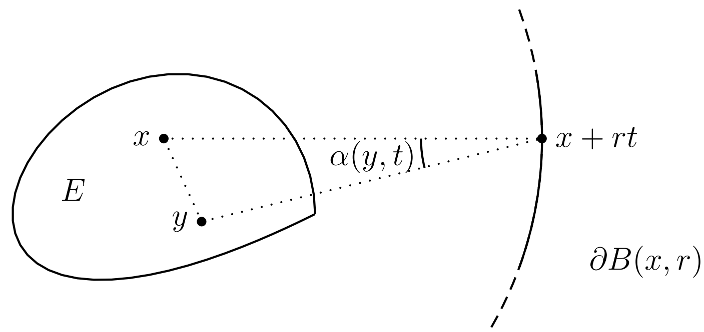

Figure 3.15: The angle $\alpha(y,t)$ at the point $x+rt$.

Hence, $\cos\alpha(y,t) > 0$ and $\alpha(y,t) < \pi/2$. Since these conditions hold for every y in the compact set E and t in the compact set $\partial B(0,1)$, we have

$$\alpha = \max_{y,t} \alpha(y,t)$$

such that $0 < \alpha < \pi/2$ and

$$\sin\alpha \leq \frac{\operatorname{diam}(E)}{r}. \tag{3.16}$$

For each $t \in \partial B(0,1)$, let $y_t \in E$ be its farthest point in E so that $d_E(x+rt) = |x+rt-y_t|$. Then

$$d_E\big(x+(r+\delta)t\big) - d_E(x+rt) \geq \big|x+(r+\delta)t-y_t\big| - |x+rt-y_t| \tag{3.17}$$

$$\geq \delta\cos\alpha + O(\delta^2). \tag{3.18}$$

and so

$$\begin{aligned} d_E^{2-N}(x+rt) - d_E^{2-N}\big(x+(r+\delta)t\big) &\geq |x+rt-y_t|^{2-N} - \big|x+(r+\delta)t-y_t\big|^{2-N} \\ &\geq (N-2)\delta\cos\alpha\ d_E^{1-N}(x+rt) + O(\delta^2). \end{aligned}$$

Applying Proposition 3.3, we find

$$\begin{aligned} \sigma_E\big(B(w,r)\big) &\geq \frac{r^{N-1}}{\omega_N} \lim_{\delta\to 0^+} \int_{\partial B(0,1)} \Big(\cos\alpha\ d_E^{1-N}(x+rt) + O(\delta)\Big)\,\mathrm{d}t \\ &\geq r^{N-1}\,(1-\sin\alpha)\,d_E^{1-N}(x+rt) \\ &\geq r^{N-1}\left(1-\frac{\operatorname{diam}(E)}{r}\right)\big(r+\operatorname{diam}(E)\big)^{1-N} \\ &= \left(1-\frac{\operatorname{diam}(E)}{r}\right)\left(\frac{r}{r+\operatorname{diam}(E)}\right)^{N-1}. \end{aligned}$$

$\square$

Proof of Proposition 3.11. Let D be a domain such that $\overline{D} = E$. Suppose by way of contradiction that $\sigma_E(D) = 0$. Then d_E^{2-N} is harmonic in D and so by Proposition 2.11 there exists $y \in \partial D$ such that $d_E(x) = |x - y|$ for all $x \in D$. Since E is not a singleton, the function d_E is bounded below by $\mathrm{diam}(E)/2$. But $|x - y|$ is arbitrarily close to zero in a neighborhood of y which is a contradiction. $\square$

Proof of Lemma 3.2. The set E^* is the intersection of a compact family of strictly convex sets and hence is also strictly convex. Set $u \in \mathcal{U}^N$ and let $y \in \partial E^*$ be the point $\gamma_{E^*}(u)$. Let w be the width $w_{E^*}(u)$. We wish to show that $E^* \subseteq \overline{B}(y - w\vec{u}, w)$. Let h be a hyperplane passing through y and parallel to $\vec{u}$. Choose one side of h to be positive and the other negative. Then define two closed subsets of E, denoted by

$$E^*_+ = \{x \in E^* : x \in h \text{ or } x \text{ lies on the positive side of } h\}$$

and

$$E^*_- = \{x \in E^* : x \in h \text{ or } x \text{ lies on the negative side of } h\}.$$

Similarly, we will use the subscripts $+$ and $-$ to denote closed subsets lying on the positive or negative sides of h, respectively.

Let $F = \{x \in E : |x - y| = d_E(x)\}$. Clearly F is nonempty, since $y \in \partial E^*$ and hence on the boundary of at least one of the balls $\overline{B}(x, d_E(x))$ of which E^* is an intersection.

Now suppose, for the sake of contradiction, that $F_- = \emptyset$. Then F_+ and h can be separated by a new hyperplane h', parallel to h and lying on its positive side. Set the positive and negative sides of h' in the same direction as h. Let

$$A = \bigcap_{\{x \in E : x \text{ on the negative side of } h'\}} \overline{B}(x, d_E(x)).$$

Either E lies entirely on the positive side of h' and hence $A = \emptyset$, or $E^* \subset A$ and y is in the interior of A. Either way, when we notice that y lies on the negative side of h'

and

$$E^* = A \cap \left(\bigcap_{\{x \in E : x \text{ on the positive side of } h'\}} \overline{B}(x, d_E(x)) \right),$$

we reach the contradictory conclusion that the strictly convex set E^* contains a non-trivial portion of the hyperplane perpendicular to h and passing through y. Hence $F_- \neq \emptyset$.

Choose $x \in F_-$. Notice that $E^* \subseteq \overline{B}(x, d_E(x)) = \overline{B}(x, |x-y|)$. Consider the hyperplane perpendicular to h and passing through $y - w\vec{u}$ and let z be the point in that hyperplane that lies on the line passing through y and x. It follows that $E^* \subseteq \overline{B}(z, |z-y|)$. Since z is on the negative side of h, it follows that $E^*_+ \subseteq \overline{B}(y - w\vec{u}, w)$. This can be seen by choosing $t \in E^*_+$ and considering the triangle with vertices t, $y - w\vec{u}$, and z. Since the angle formed at $y - w\vec{u}$ is at least $\pi/2$, the Law of Cosines implies that $|z-t|^2 \geq |(y - w\vec{u}) - t|^2$.

Similar reasoning may be applied to F_+ and E^*_- to reach the desired inclusion. $\square$

Proof of Lemma 3.12. This proof follows the proof given in [25, Lemma 5] which proves this result in the planar case. We will focus on $E \subset \mathbb{R}^N$, $N \geq 3$. Recall from the proof of Proposition 2.6 that the sets $F_n = \gamma_n * p_{E^*} + \overline{B}(0, 1/n)$ have most of the desired properties. We will show these sets are c-round and then dilate them to form an increasing sequence.

Let $1 < b < c$. As w_{E^*} is uniformly continuous and $w_{F_n} \to w_{E^*}$ uniformly, we may choose $n_0 \in \mathbb{N}$ sufficiently large to ensure that

$$w_{E^*}(u_1) \leq b w_{E^*}(u_2) \text{ whenever } |u_1 - u_2| \leq 1/n_0 \tag{3.19}$$

and

$$b w_{E^*} + 1/n \leq c w_{F_n} \text{ whenever } n \geq n_0. \tag{3.20}$$

Choose $u_0 \in \mathcal{U}^N$. Since E^* is 1-round we may apply (3.19) to find

$$\begin{aligned} E^* &\subseteq \bigcap_{u \in \mathcal{U}^N} \overline{B}(\gamma_{E^*}(u) - w_{E^*}(u)\vec{u}, w_{E^*}(u)) \\ &\subseteq \bigcap_{\{t \in \mathcal{U}^N : |t-u_0| \le 1/n\}} \overline{B}(\gamma_{E^*}(t) - bw_{E^*}(u_0)\vec{t}, bw_{E^*}(u_0)). \end{aligned}$$

We write $B_t = \overline{B}(\gamma_{E^*}(t) - bw_{E^*}(u_0)\vec{t}, bw_{E^*}(u_0))$. Then

$$\begin{aligned} p_{F_n}(u) &= \int \gamma_n(u-v) p_{E^*}(v)\,\mathrm{d}v + 1/n && (3.21) \\ &= \int \gamma_n(u_0 - t) p_{E^*}(u - u_0 + t)\,\mathrm{d}t + 1/n && (3.22) \\ &\le \int \gamma_n(u_0 - t) p_{B_t}(u - u_0 + t)\,\mathrm{d}t + 1/n && (3.23) \end{aligned}$$

since $E^* \subseteq B_t$. Further, we have equality when $u = u_0$ since $\gamma_{E^*}(t) = \gamma_{B_t}(t)$.

Recall that $p_{B_t}(u - u_0 + t) = p_{B_t + u_0 - t}(u)$. Let $z_t \in \mathbb{R}^N$ be the point such that $B_t + u_0 - t = \overline{B}(z_t, bw_{E^*}(u_0))$. The support function for a closed ball is given by

$$p_{\overline{B}(x,R)}(u) = x \cdot \vec{u} + R.$$

Thus we may rewrite (3.21) as

$$\begin{aligned} p_{F_n}(u) &\le \int \gamma_n(u_0 - t)(\vec{u} \cdot z_t)\,\mathrm{d}t + bw_{E^*}(u_0) + 1/n \\ &= \vec{u} \cdot \left(\int \gamma_n(u_0 - t) z_t\,\mathrm{d}t \right) + bw_{E^*}(u_0) + 1/n \end{aligned}$$

with equality when $u = u_0$. Thus F_n is contained in a ball of radius $bw_{E^*}(u_0) + 1/n$. More specifically,

$$\begin{aligned} F_n &\subseteq \overline{B}\left(\gamma_{F_n}(u_0) - \left(bw_{E^*}(u_0) + 1/n\right)\vec{u_0}, bw_{E^*}(u_0) + 1/n\right) \\ &\subseteq \overline{B}\left(\gamma_{F_n}(u_0) - cw_{E^*}(u_0)\vec{u_0}, cw_{E^*}(u_0)\right) \end{aligned}$$

for $n \ge n_0$ and thus F_n is c-round.

By using a suitable dilation centered at a point in E^* we obtain an increasing sequence E_n with all the desired properties. See the proof of Proposition 2.6 for details on the dilation. It follows from Proposition 3.4 that the measures σ_{E_n} converge weak* to σ_{E^*}. $\square$

Proof of Proposition 3.13. This proof is adapted from a similar one given in [44, Theorem 2.6] for the planar case. We will be computing $\sigma_E(E^C)$. We may assume for convenience that $0 \in E$. Then we have

$$\begin{aligned}
\sigma_E(E^C) &= \lim_{R\to\infty} \sigma_E\left(E^C \cap B(0,R)\right) \\
&= \lim_{R\to\infty} \int_{E^C\cap B(0,R)} \frac{1}{(2-N)\omega_N} \Delta d_E^{2-N}(x)\,\mathrm{d}V \\
&= \frac{1}{(2-N)\omega_N} \lim_{R\to\infty} \left(\int_{\partial B(0,R)} \frac{\partial d_E^{2-N}}{\partial \vec{n}}(x)|\,\mathrm{d}x| - \int_{\partial E} \frac{\partial d_E^{2-N}}{\partial \vec{n}}(x)|\,\mathrm{d}x| \right) \\
&= \frac{1}{\omega_N} \lim_{R\to\infty} \int_{\partial B(0,R)} d_E^{1-N}(x) \frac{\partial d_E}{\partial \vec{n}}(x)|\,\mathrm{d}x| - \frac{1}{\omega_N} \int_{\partial E} d_E^{1-N}(x) \frac{\partial d_E}{\partial \vec{n}}(x)|\,\mathrm{d}x|
\end{aligned}$$

where $\vec{n}$ denotes the outward unit normal vector, Proposition 3.1 was used to obtain the second line, and Green's Identity [32, p. 22] was used to obtain the third line.

We consider the integral over the ball first. For $x \in \partial B(0,R)$, we have $R \leq d_E(x) \leq R + \mathrm{diam}(E)$ and hence

$$\big(R + \mathrm{diam}(E)\big)^{1-N} \leq d_E^{1-N}(x) \leq R^{1-N}.$$

Furthermore, we have

$$\delta\left(1 - \frac{\mathrm{diam}(E)}{R}\right) + O(\delta^2) \leq d_E\left((R+\delta)\vec{u}\right) - d_E(R\vec{u}) \leq \delta$$

where the right hand inequality is due to d_E being Lipschitz continuous and the left hand inequality is a consequence of (3.16) and (3.17) from the proof of Proposition 3.10. Thus

$$1 - \frac{\mathrm{diam}(E)}{R} \leq \frac{\partial d_E}{\partial \vec{n}}(x) \leq 1$$

for $x \in \partial B(0,R)$ and we obtain

$$\frac{1}{\omega_N} \lim_{R\to\infty} \int_{\partial B(0,R)} d_E^{1-N}(x) \frac{\partial d_E}{\partial \vec{n}}(x) |\,\mathrm{d}x| = 1.$$

We now consider the integral over ∂E. Recall that E has a support parametrization $\gamma(u)$, $u \in \mathcal{U}^N$, such that the outer normal vector at $\gamma(u)$ is $\vec{u}$. Let $x = \gamma(u)$ be a point in ∂E. Consider the points $\gamma(u)$ and $\gamma(u+\psi)$ where we recall that $\psi \in \mathcal{U}^N$ is given by $(0, \pi, 0, \ldots, 0)$ and the vector in the direction $u+\psi$ is $-\vec{u}$. Then the tangent planes at $\gamma(u)$ and $\gamma(u+\psi)$ are parallel. Since E has constant width it follows that $|\gamma(u) - \gamma(u+\psi)| = \mathrm{diam}(E)$. Therefore, $\gamma(u) = \gamma(u+\psi) + \mathrm{diam}(E)\vec{u}$. Hence, for any point $x \in E$ we have

$$d_E(x) = d_E\big(\gamma(u)\big) = \mathrm{diam}(E)$$

and

$$\frac{\partial d_E}{\partial \vec{n}}(x) = \frac{\partial d_E}{\partial \vec{u}}\big(\gamma(u)\big) = 1$$

where $\vec{n}$ denotes the outward unit normal vector at x. Thus we have

$$\frac{1}{\omega_N} \int_{\partial E} d_E^{1-N}(x) \frac{\partial d_E}{\partial \vec{n}}(x) |\,\mathrm{d}x| = \frac{\big(\mathrm{diam}(E)\big)^{1-N} |\partial E|}{\omega_N}$$

where $|\partial E|$ denotes the $N-1$ dimensional surface area of ∂E.

We complete the computation by applying Kubota's inequality [29, p. 65], which states that

$$|\partial E| \leq \omega_N \big(\mathrm{diam}(E)\big)^{N-1} 2^{1-N},$$

with equality only in the case of the ball. Thus we have

$$\begin{aligned} \sigma_E(E^C) &= 1 - \frac{\big(\mathrm{diam}(E)\big)^{1-N} |\partial E|}{\omega_N} \\ &\geq 1 - 2^{1-N}. \end{aligned}$$

Since σ_E is a unit measure, it follows that $\sigma_E(E) \leq 2^{1-N}$ with equality only in the case of the ball. $\square$

CHAPTER 4

Riesz Potentials and Applications

We previously found that d_E can be expressed via logarithmic or Newtonian potentials. In Section 4.1, we give a representation of d_E via Riesz potentials. We apply it to obtain reverse triangle inequalities for Riesz potentials in Section 4.2. Connections between these results and recent work on polarization inequalities are discussed in Section 4.3. This chapter is based on joint work the author did with Pritsker and Saff [59].

4.1 Representation via Riesz Potentials

For $E \subset \mathbb{R}^N$, $N \geq 3$, the function d_E^{2-N} is a Newtonian potential and for $E \subset \mathbb{R}^2$, the function $\log d_E$ is a logarithmic potential. We can obtain more general results by considering other exponents. Specifically, we will show that for $E \subset \mathbb{R}^N$, $N \geq 2$, the function $d_E^{\alpha-N}$ is a Riesz potential for a certain range of α.

Let μ be a measure in $\mathbb{R}^N$, $N \geq 2$. The α-Riesz potential of μ, with $0 < \alpha < N$, is given by

$$U_\alpha^\mu(x) := \int |x-y|^{\alpha-N}\, \mathrm{d}\mu(y).$$

For $\alpha = 2$ and $N \geq 3$, these are simply the Newtonian potentials we considered in the previous chapters. Note that the logarithmic case $\alpha = N = 2$ is excluded from consideration.

First, we consider the case where $\alpha > 2$. Just as in the Newtonian case, the function $d_E^{\alpha-N}$ is the infimum of Riesz kernels

$$d_E^{\alpha-N}(x) = \inf_{y\in E} d_y^{\alpha-N}(x).$$

We previously showed in Section 2.3 that the Newtonian kernel d_y^{2-N} is superharmonic by calculating its Laplacian. A similar calculation works for all $\alpha > 2$, in which case the Laplacian is given by

$$\Delta d_y^{\alpha-N} = (\alpha - N)(\alpha - 2)d_y^{\alpha-2-N} < 0,$$

and hence by Propositions 2.9 and 2.8, $d_y^{\alpha-N}$ is superharmonic in $\mathbb{R}^N$. It follows, using arguments similar to those in Section 2.3, that for any bounded set $E \subset \mathbb{R}^N$ there exists a unique positive Borel measure μ_α such that

$$d_E^{\alpha-N}(x) = \int_{\mathbb{R}^N} |x-y|^{2-N}\,\mathrm{d}\mu_\alpha(y) = U_2^{\mu_\alpha}(x).$$

Unfortunately, this does not lead to a satisfactory extension of Theorem 2.6 because the resulting measure μ_α is not a unit measure and, in fact, $\mu_\alpha(\mathbb{R}^N) = \infty$.

We turn our attention to the case where $0 < \alpha < 2$. The Riesz kernels $d_y^{\alpha-N}$ are not superharmonic. However, we may use a generalization of superharmonicity which has many similar properties. The following discussion of α-superharmonic functions comes from Landkof [43, Section I.6].

Recall from Proposition 2.8 that a function u is superharmonic if $u(x) \geq \mathcal{M}(u; x, r)$ where $\mathcal{M}(u; x, r)$ is the spherical mean of u. We will choose a new measure and define α-superharmonicity using the mean with respect to this new measure.

For $0 < \alpha < 2$ we define

$$\varepsilon_{\alpha,r}(y) := \begin{cases} 0 & \text{if } |y| < r \\ \frac{2r^\alpha}{\omega_N \Gamma(\alpha/2)\Gamma(\alpha/2-1)}(|y|^2 - r^2)^{-\alpha/2}|y|^{-N} & \text{if } |y| > r. \end{cases}$$

This measure is presented with a slightly different formula in Landkof which can be obtained by noting that $\sin(\pi\alpha/2) = \pi/(\Gamma(\alpha/2)\Gamma(\alpha/2-1))$ and $\omega_N = 2\pi^{N/2}/\Gamma(N/2)$.

This measures $\varepsilon_{\alpha,r}$ converge weak* to the normalized area measure on the sphere of radius r as $\alpha \to 2$.

We denote the average of a function u with respect to this measure by

$$\mathcal{M}_\alpha(u; x, r) := \int u(y)\varepsilon_{\alpha,r}(|y - x|)\, \mathrm{d}y,$$

where u is locally integrable and

$$\int_{|y|>1} \frac{|u(y)|}{|y|^{N+\alpha}}\, \mathrm{d}y < \infty. \tag{4.1}$$

We now define α-superharmonicity using a mean value inequality. The following definition is global, in contrast to the local definition of regular superharmonicity.

Definition 4.1. *Let u be a function in $\mathbb{R}^N$ satisfying (4.1). Then u is α-superharmonic if it is nonnegative, l.s.c. and*

$$u(x) \geq \mathcal{M}_\alpha(u; x, r)$$

for all $x \in \mathbb{R}^N$ and $r > 0$. Further, u is α-harmonic at x_0 if it is continuous at x_0 and for all sufficiently small r

$$u(x) = \mathcal{M}_\alpha(u; x, r).$$

We begin with an example, the Riesz α-kernel $d_y^{\alpha-N}$. Clearly $d_y^{\alpha-N}$ is nonnegative and continuous except at y where it is l.s.c. Further, $d_y^{\alpha-N}(x) \geq \mathcal{M}_\alpha(d_y^{\alpha-N}; x, r)$ for all $x \in \mathbb{R}^N$ and $r > 0$ [43, Appendix] and hence $d_y^{\alpha-N}$ is α-superharmonic.

Many properties of superharmonic functions have analogs for α-superharmonic functions.

Proposition 4.1 (Minimum Principle)**.** *Let u be α-superharmonic. If $u(x_0) = \inf_{x\in\mathbb{R}^N} u(x)$ for some $x_0 \in \mathbb{R}^N$, then $u(x) \equiv u(x_0)$ [43, p. 114].*

Proposition 4.2. *Let $\{u_m\}_{m=0}^\infty$ be a family of α-superharmonic functions. If the functions u_m are increasing, then their supremum, $\sup u_m(x)$, is either $+\infty$ or α-superharmonic [43, p. 114]. Further, their finite infinum $\inf_{m=0,\ldots M} u_m(x)$ for some $M \in N$, is α-superharmonic [43, p. 129].*

Theorem 4.1 (Riesz Decomposition Theorem)**.** *Let u be α-superharmonic. Then there exists a unique positive Borel measure μ on $\mathbb{R}^N$ such that*

$$u(x) = \int |x-y|^{\alpha-N}\,\mathrm{d}\mu(y) + C = U_\alpha^\mu(x) + C$$

where $C \geq 0$ is a constant [43, Theorem 1.30].

We wish to find an analog of Theorem 2.6 for $d_E^{\alpha-N} = \inf_{y\in E} d_y^{\alpha-N}$. Assume E is finite. Applying Proposition 4.2, we find that $d_E^{\alpha-N}$ is α-superharmonic. We use the Riesz Decomposition Theorem to conclude that $d_E^{\alpha-N}$ is a Riesz α-potential. If E is not finite, but is compact, we can reach the same conclusion.

Theorem 4.2. *Let $E \subset \mathbb{R}^N$ be a compact set consisting of at least two points, with $N \geq 2$ and $0 < \alpha \leq 2$. Then there exists a unique positive unit Borel measure σ_α such that*

$$d_E^{\alpha-N}(x) = \int |x-y|^{\alpha-N}\,\mathrm{d}\sigma_\alpha(y) = U_\alpha^{\sigma_\alpha}(x). \tag{4.2}$$

We saw in Section 3.1 that the representing measure σ_E for d_E^{2-N} is, in a suitable sense, the Laplacian of d_E^{2-N}. Similarly, the representing measure σ_α for $d_E^{\alpha-N}$ is the *fractional Laplacian* of $d_E^{\alpha-N}$.

Denote the Riesz α-kernel, $0 < \alpha < N$ by

$$K_\alpha(x) := A(N,\alpha)|x|^{\alpha-N}$$

where $A(N,\alpha)$ is a normalization constant given by

$$A(N,\alpha) := \pi^{\alpha-N/2}\frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)}.$$

Under a variety of suitable conditions on f and α we have the relation

$$f = K_\alpha * (K_{-\alpha} * f)$$

almost everywhere. See [43, p. 45, 48, 74, 118, 120] for examples of sufficient conditions. Thus, under certain conditions, we may set $\mu = K_{-\alpha} * f$ and it will follow that $f = K_\alpha * \mu = U_\alpha^\mu$.

In particular, $d_E^{\alpha-N}$, $N \geq 2$ and $0 < \alpha < 2$, is α-superharmonic and Landkof's proof of Theorem 4.2 shows that the representing measure σ_α for $d_E^{\alpha-N}$ is given by $\sigma_\alpha = K_{-\alpha} * d_E^{\alpha-N}$. This is also written using fractional Laplacians as

$$\sigma_\alpha = (-\Delta)^{\alpha/2}\, d_E^{\alpha-N}.$$

4.2 Reverse Triangle Inequality for Potentials

Let E be a compact set in $\mathbb{C}$. For any set of real-valued functions $f_j, j = 1, \ldots, m$, we have

$$\sum_{j=1}^{m} \sup_E f_j \geq \sup_E \sum_{j=1}^{m} f_j$$

by the triangle inequality. It is not possible to reverse this inequality for arbitrary functions, even by introducing additive constants. However, by restricting the class of functions we can reverse the inequality with sharp additive constants to obtain expressions of the form

$$\sum_{j=1}^{m} \sup_E f_j \leq C + \sup_E \sum_{j=1}^{m} f_j. \tag{4.3}$$

We begin by considering logarithmic potentials $p^\nu(z) = \int \log|z-t|\,\mathrm{d}\nu(t)$. Let ν_j, $j = 1, \ldots, m$, be positive compactly supported Borel measures, normalized so that $\nu := \sum_{j=1}^m \nu_j$ is a unit measure. We want to find a sharp additive constant C depending only on E such that

$$\sum_{j=1}^{m} \sup_E p^{\nu_j} \leq C + \sup_E \sum_{j=1}^{m} p^{\nu_j} = C + \sup_E p^\nu. \tag{4.4}$$

The motivation for such inequalities comes from inequalities for the norms of products of polynomials, which also motivated this study of the farthest distance function. Let $P(z) = \prod_{j=1}^n (z - a_j)$ be a monic polynomial. Then $\log|P(z)| = n \int \log|z-t| d\tau(t)$ where $\tau = \frac{1}{n}\sum_{j=1}^n \delta_{a_j}$ is the normalized counting measure of the zeros of P, with δ_{a_j} being the unit point mass at a_j. Let $||P||_E$ be the uniform (sup) norm on E. Then for polynomials P_j, $j = 1, \ldots, m$, inequality (4.4) can be rewritten as (1.1) which we recall is

$$\prod_{j=1}^{m} ||P_j||_E \leq M^n \left|\left|\prod_{j=1}^{m} P_j\right|\right|_E \tag{4.5}$$

where $M = e^C$ and n is the degree of $\prod_{j=1}^{m} P_j$.

We discussed work on finding sharp constants M for various sets E in Chapter 1. These may be extended to results for C in (4.4) with $C = \log M$, see [58]. Specifically, we can use (1.5) to conclude that

$$C_E = \int \log d_E(z)\, \mathrm{d}\mu_E(z) - \log \operatorname{cap}(E).$$

It follows from [57, 55] that $C_D = \log 2$ is a lower bound for C_E for any compact set E with positive capacity, while $C_{[-1,1]} \approx \log 3.20991$ is an upper bound on C_E for certain classes of sets E. Allowing the constant to be dependent on the number of terms m, Pritsker and Saff [58] found that (4.4) holds for m terms with

$$C_E(m) = \max_{c_k \in \partial E} \int \log \max_{1 \leq k \leq m} |z - c_k|\, \mathrm{d}\mu_E(z) - \log \operatorname{cap}(E).$$

Note that $\lim_{m\to\infty} C_E(m) = C_E$.

These results were generalized to Green potentials by Pritsker [54]. Let p_j, $j = 1, \ldots, m$, be Green potentials [2, p. 96] on a domain $G \subset \overline{\mathbb{C}}$. Then for any compact set $E \subset G$ we have

$$\sum_{j=1}^{m} \inf_E p_j \geq C + M \inf_E \sum_{j=1}^{m} p_j \tag{4.6}$$

where M and C are given in [54] as explicit constants depending only on G and E, and C is sharp.

We will now present a reverse triangle inequality for Newtonian and certain Riesz potentials. We will then consider connections of the reverse triangle inequality with polarization inequalities for Riesz potentials.

Consider a compact set $E \subset \mathbb{R}^N$, $N \geq 2$, and Riesz potentials of the form $U_\alpha^\mu(x) = \int |x-t|^{\alpha-N}\, \mathrm{d}\mu(t)$ for $0 < \alpha \leq 2$. For $\alpha = 2$, these are Newtonian potentials, and they are superharmonic in $\mathbb{R}^N$, $N \geq 3$. If $N = \alpha = 2$ then one may study inequalities for logarithmic potentials as done in [58], but we exclude this case. For $0 < \alpha < 2$, the potentials U_α^μ are not superharmonic, but they are α-superharmonic [43, p. 111]. As discussed in Section 4.1, many of the standard properties of superharmonic functions hold for α-superharmonic functions. Our goal is to find a constant C such that

$$\sum_{j=1}^{m} \inf_E U_\alpha^{\nu_j} \geq C + \inf_E \sum_{j=1}^{m} U_\alpha^{\nu_j}.$$

We begin by stating some known facts. For a compact set $E \subset \mathbb{R}^N$, let $W_\alpha(E) < \infty$ be the minimum α-energy of E and let μ_α be the α-equilibrium measure of E [43, Chapter 2] so that

$$W_\alpha(E) = \int U_\alpha^{\mu_\alpha}\, \mathrm{d}\mu_\alpha.$$

Theorem 4.3 (Frostman's Theorem)**.** *For any compact set $E \subset \mathbb{R}^N$ with $W_\alpha(E) < \infty$, and any $\alpha \in (0, 2]$, we have*

$$U_\alpha^{\mu_\alpha}(x) \leq W_\alpha(E), \ x \in \mathbb{R}^N.$$

Further,

$$U_\alpha^{\mu_\alpha}(x) = W_\alpha(E) \textit{ for quasi-every } x \in E,$$

where quasi-everywhere means except for a set of α-capacity zero [43, p. 137].

Finally, recall from Theorem 4.2 that the function d_E may be expressed via potentials as

$$d_E^{\alpha-N}(x) = U_\alpha^{\sigma_\alpha}(x).$$

We are now prepared to state a reverse triangle inequality.

Theorem 4.4. *Let $E \subset \mathbb{R}^N$ be a compact set with the minimum α-energy $W_\alpha(E) < \infty$, where $0 < \alpha \leq 2$. Suppose that ν_k, $k = 1, \ldots, m$, are positive compactly supported*

Borel measures, normalized so that $\nu := \sum_{k=1}^m \nu_k$ *is a unit measure, with* $m \geq 2$. *Then*

$$\sum_{k=1}^{m} \inf_E U_\alpha^{\nu_k} \geq C_E(\alpha, m) + \inf_E \sum_{k=1}^{m} U_\alpha^{\nu_k}, \tag{4.7}$$

where

$$C_E(\alpha, m) := \min_{c_k \in E} \int \min_{1 \leq k \leq m} |x - c_k|^{\alpha - N} \, \mathrm{d}\mu_\alpha(x) - W_\alpha(E)$$

cannot be replaced by a larger constant for each $m \geq 2$. *Furthermore,* (4.7) *holds with* $C_E(\alpha, m)$ *replaced by*

$$C_E(\alpha) := \int d_E^{\alpha - N}(x) \, \mathrm{d}\mu_\alpha(x) - W_\alpha(E),$$

which does not depend on m.

In the Newtonian case $\alpha = 2$, the minimum principle holds and so the minimum in $C_E(2, m)$ is achieved on the boundary of E. Thus

$$C_E(2, m) = \min_{c_k \in \partial E} \int \min_{1 \leq k \leq m} |x - c_k|^{2 - N} \, \mathrm{d}\mu_2(x) - W_2(E).$$

A closed set $S \subset E$ is called *dominant* if

$$d_E(x) = \max_{t \in S} |x - t| \text{ for all } x \in \operatorname{supp}(\mu_\alpha).$$

When E has at least one finite dominant set, we define a *minimal dominant* set $\mathfrak{D}_E$ as a dominant set with the smallest number of points denoted by $\operatorname{card}(\mathfrak{D}_E)$. Of course, E might not have finite dominant sets at all, in which case we can take any dominant set as the minimal dominant set, e.g., $\mathfrak{D}_E = \partial E$. For example, let E be a polyhedron. The vertices of E are a dominant set, since $d_E(x) = \max_{\text{vertices } t \in E} |x - t|$ everywhere, not just in $\operatorname{supp}(\mu_\alpha)$. However, this need not be the minimal dominant set. For example, let E be a pyramid. If the apex is close to the base, then it will not be in the minimal dominant set. The hemisphere has the equator as the smallest dominant set, however this set is infinite.

Corollary 4.1. *For every $m \geq 2$, we have $C_E(\alpha, m) \geq C_E(\alpha)$. In particular, if $m < \operatorname{card}(\mathfrak{D}_E)$ then $C_E(\alpha, m) > C_E(\alpha)$, while $C_E(\alpha, m) = C_E(\alpha)$ for all $m \geq \operatorname{card}(\mathfrak{D}_E)$. Furthermore, the constants $C_E(\alpha, m)$ are decreasing in m and $\lim_{m\to\infty} C_E(\alpha, m) = C_E(\alpha)$.*

Corollary 4.2. *If $E \subset \mathbb{R}^N$ is a compact set with C^1-smooth boundary and with finitely many connected components, then $C_E(\alpha, m) > C_E(\alpha)$ for all $m \in \mathbb{N}$, $m \geq 2$.*

If $E = L := [-1, 1] \subset \mathbb{R}^2$ and $1 < \alpha < 2$, then $d_L(x) = \max(|x-1|, |x+1|)$, $x \in \mathbb{R}^2$, so that the endpoints form the minimal dominant set with $\operatorname{card}(\mathfrak{D}_L) = 2$. Thus $C_L(\alpha) = C_L(\alpha, 2) = C_L(\alpha, m),\ m \geq 2$.

We finish this section with several explicit examples.

Example 4.1 (Unit circle $\mathbb{T}$ in $\mathbb{C}$)**.** *Let $\mathbb{T} \subset \mathbb{C}$ be the unit circle, and let $1 < \alpha < 2$. We know that $\mathrm{d}\mu_\alpha(e^{i\theta}) = \mathrm{d}\theta/2\pi$ and*

$$W_\alpha(\mathbb{T}) = \frac{2^{\alpha-2}}{\sqrt{\pi}} \frac{\Gamma(\frac{\alpha-1}{2})}{\Gamma(\frac{\alpha}{2})},$$

see [43]. We prove in Section 4.4 that

$$\min_{c_k \in E} \int \min_{1\leq k\leq m} |x - c_k|^{\alpha-2}\, \mathrm{d}\mu_\alpha(x) = 2^{\alpha-2} \frac{2m}{\pi} I\left(\frac{\pi}{2m}\right),$$

where $I(x) = \int_0^x \cos^{\alpha-2}\theta\, d\theta$. It is obvious that $d_{\mathbb{T}}(x) = 2,\ x \in \mathbb{T}$, and that S has no finite dominant set. Therefore,

$$C_{\mathbb{T}}(\alpha, m) = 2^{\alpha-2}\frac{2m}{\pi} I\left(\frac{\pi}{2m}\right) - \frac{2^{\alpha-2}}{\sqrt{\pi}} \frac{\Gamma(\frac{\alpha-1}{2})}{\Gamma(\frac{\alpha}{2})} > C_{\mathbb{T}}(\alpha) = 2^{\alpha-2} - \frac{2^{\alpha-2}}{\sqrt{\pi}} \frac{\Gamma(\frac{\alpha-1}{2})}{\Gamma(\frac{\alpha}{2})}.$$

Example 4.2 (Unit sphere S^{N-1} in $\mathbb{R}^N$)**.** *Let $S^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\},\ N \geq 3$, and let $1 < \alpha \leq 2$. It is known that $\mathrm{d}\mu_\alpha = \mathrm{d}\sigma/\omega_N$ is the normalized surface area on S^{N-1} and*

$$W_\alpha(S^{N-1}) = \frac{2^{\alpha-2}}{\sqrt{\pi}} \frac{\Gamma(\frac{N}{2})\Gamma(\frac{\alpha-1}{2})}{\Gamma(\frac{N+\alpha-2}{2})},$$

see [43]. It is also clear that $d_{S^{N-1}}(x) = 2,\ x \in S^{N-1}$, *and that* S^{N-1} *has no finite dominant set. Hence*

$$C_{S^{N-1}}(\alpha, m) > C_{S^{N-1}}(\alpha) = 2^{\alpha-N} - \frac{2^{\alpha-2}}{\sqrt{\pi}} \frac{\Gamma(\frac{N}{2})\Gamma(\frac{\alpha-1}{2})}{\Gamma(\frac{N+\alpha-2}{2})}.$$

Example 4.3 (Unit ball B^N in $\mathbb{R}^N$)**.** *Let* $B^N := \{x \in \mathbb{R}^N : |x| \le 1\},\ N \ge 2$, *and let* $0 < \alpha \le 2$. *Again,* B^N *has no finite dominant set. The Wiener constant of the ball is*

$$W_\alpha(B^N) = \frac{\Gamma(\frac{N-\alpha+2}{2})\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{N}{2})},$$

see [43].

If $\alpha = 2$ *and* $N \ge 3$ *then the equilibrium measure of the ball* $\mathrm{d}\mu_2 = \mathrm{d}\sigma/\omega_N$ *is the normalized surface area on* $S^{N-1} = \partial B^N$, *so that* $d_{B^N}(x) = 2,\ x \in S^{N-1} = \operatorname{supp}(\mu_2)$. *Hence*

$$C_{B^N}(2, m) > C_{B^N}(2) = 2^{2-N} - 1.$$

If $0 < \alpha < 2$ *then the equilibrium measure of the ball is*

$$\mathrm{d}\mu_\alpha(x) = \frac{\Gamma\left(\frac{N-\alpha+2}{2}\right)}{\pi^{N/2}\Gamma\left(1 - \frac{\alpha}{2}\right)} \frac{R^{\alpha-N}\,\mathrm{d}x}{(R^2 - |x|^2)^{\alpha/2}} \text{ for } |x| < R,$$

see [43, p. 163]. Since $\operatorname{supp}(\mu_\alpha) = B^N$ *in this case, we note that* $d_{B^N}(x) = 1 + |x|,\ x \in B^N$, *so that*

$$C_{B^N}(\alpha, m) > C_{B^N}(\alpha) = \int (1 + |x|)^{\alpha-N}\,\mathrm{d}\mu_\alpha(x) - \frac{\Gamma(\frac{N-\alpha+2}{2})\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{N}{2})},$$

where μ_α *is given above.*

4.3 Connections to Polarization Inequalities

Let E be a compact set in $\mathbb{R}^N$ and let $A_m = \{x_j\}_{j=1}^m$, denote an m-point subset of E. The *Riesz polarization quantities*, introduced by Ohtsuka [51] and recently studied by Erdélyi and Saff [20], are given by

$$M^s(A_m, E) := \inf_{x\in E} \sum_{j=1}^m |x - x_j|^{-s} \text{ and } M_m^s(E) := \sup_{A_m \subset E} M^s(A_m, E), \quad s > 0.$$

Let ν_j denote the normalized point mass δ_{x_j}/m, so that $\sum_{j=1}^m \nu_j$ is a unit measure. The Riesz polarization quantity for $s = N - \alpha$ may be rewritten in terms of potentials as

$$M^{N-\alpha}(A_m, E) = m \inf_E \sum_{j=1}^m U_\alpha^{\nu_j}.$$

As proved by Ohtsuka [51], the normalized limit

$$\mathcal{M}^s(E) := \lim_{m\to\infty} M_m^s(E)/m$$

exists as an extended real number and is called the *Chebyshev constant* of E for the Riesz s-potential. Moreover, he showed that this Chebyshev constant is always greater than or equal to the associated Wiener constant. Combining this fact with Frostman's theorem we deduce the following:

Proposition 4.3. *For $0 < \alpha \le 2$ and any compact set $E \subset \mathbb{R}^N$ there holds*

$$\mathcal{M}^{N-\alpha}(E) = W_\alpha(E). \tag{4.8}$$

Indeed, given a unit Borel measure μ, Frostman's theorem for such α and E gives

$$\inf_E U_\alpha^\mu \le \int U_\alpha^\mu \, \mathrm{d}\mu_\alpha = \int U_\alpha^{\mu_\alpha} \, \mathrm{d}\mu \le W_\alpha(E),$$

so that $\mathcal{M}^{N-\alpha}(E) \le W_\alpha(E)$, which together with Ohtsuka's inequality yields (4.8). Alternatively, one can deduce (4.8) by observing that for the given range of α, a maximum principle holds for the equilibrium potential and appealing to Theorem 11 of Farkas and Nagy [21].

Bounds on the quantity $M_m^{N-\alpha}(E)/m$ and the sets A_m which achieve the maximum in $M_m^{N-\alpha}(E)$ have been the subject of several recent papers [20, 30, 31]. The reverse triangle inequality in Theorem 4.4 is directly connected with $M_m^{N-\alpha}(E)/m$ in the case of atomic measures. Recall that the inequality (4.7) holds for arbitrary positive Borel measures ν_j such that $\sum_{j=1}^m \nu_j$ is a unit measure. We now introduce a similar inequality where each $\nu_j = \delta_{x_j}/m$ is a point mass $1/m$ supported at $x_j \in E$:

$$\frac{1}{m}\sum_{j=1}^{m} \inf_{x\in E} |x-x_j|^{\alpha-N} \geq C_E^{\delta}(\alpha, m) + \frac{1}{m}\inf_{x\in E}\sum_{j=1}^{m}|x-x_j|^{\alpha-N},$$

where $C_E^{\delta}(\alpha, m)$ denotes the largest (best) constant such that the above inequality holds for all $\{x_j\}_{j=1}^m \subset E$. Clearly, we have $C_E^{\delta}(\alpha, m) \geq C_E(\alpha, m)$.

From the definitions of $C_E^{\delta}(\alpha, m)$ and $M_m^{N-\alpha}(E)$ we immediately deduce that for all $\alpha < N$,

$$\max_{A_m\subset E}\frac{1}{m}\sum_{j=1}^{m} d_E^{\alpha-N}(x_j) - \frac{M_m^{N-\alpha}(E)}{m} \geq C_E^{\delta}(\alpha, m).$$

In particular, if E is the unit sphere $S^{N-1} \subset \mathbb{R}^N$, we have

$$2^{\alpha-N} - \frac{M_m^{N-\alpha}(S^{N-1})}{m} = C_{S^{N-1}}^{\delta}(\alpha, m). \tag{4.9}$$

In [30], it is proved that for the unit circle $\mathbb{T} = S^1$ the maximum polarization for any $m \geq 2$ is attained for m distinct equally spaced points. Moreover, this maximum, which occurs at the midpoints of the m subarcs joining adjacent points is known explicitly (in finite terms) when $N - \alpha$ is a positive even integer, and asymptotically for all $-\infty < \alpha < N$. Thereby we obtain the following.

Proposition 4.4. *For the unit circle $\mathbb{T} = S^1$ there holds, for all $-\infty < \alpha < 2$,*

$$C_{\mathbb{T}}^{\delta}(\alpha, m) = 2^{\alpha-2} - \frac{M^{2-\alpha}(A_m^*, \mathbb{T})}{m} = 2^{\alpha-2} - \frac{M_m^{2-\alpha}(\mathbb{T})}{m}, \tag{4.10}$$

where $A_m^* = \{e^{i2\pi k/m} : k = 1, \ldots, m\}$. Moreover the following asymptotic formulas hold as $m \to \infty$:

$$C_{\mathbb{T}}^{\delta}(\alpha, m) \sim \begin{cases} -\dfrac{2\zeta(2-\alpha)}{(2\pi)^{2-\alpha}}\,(2^{2-\alpha}-1)m^{1-\alpha}\,, & 1 > \alpha > -\infty\,, \\ -\dfrac{1}{\pi}\,\log m\,, & \alpha = 1\,, \\ 2^{\alpha-2} - \dfrac{2^{\alpha-2}}{\sqrt{\pi}}\,\dfrac{\Gamma\left(\frac{\alpha-1}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)} = C_{\mathbb{T}}(\alpha)\,, & 1 < \alpha < 2, \end{cases} \tag{4.11}$$

where $\zeta(s)$ denotes the classical Riemann zeta function and $a_m \sim b_m$ means that $\lim_{m\to\infty} a_m/b_m = 1$.

For $1 < \alpha < 2$, we have from Example 2.6 and (4.11) that, for each $m \geq 1$,

$$C_{\mathbb{T}}(\alpha) < C_{\mathbb{T}}(\alpha, m) \leq C_{\mathbb{T}}^{\delta}(\alpha, m),$$

with equality holding throughout in the limit as $m \to \infty$. Consequently, from the formulas in Example 4.1 we have

$$\begin{aligned}
\frac{M_m^{N-\alpha}(\mathbb{T})}{m} &= 2^{\alpha-2} - C_{\mathbb{T}}^{\delta}(\alpha, m) \leq 2^{\alpha-2} - C_{\mathbb{T}}(\alpha, m) \\
&= W_\alpha(\mathbb{T}) + 2^{\alpha-2}\left(1 - \frac{2m}{\pi} I\left(\frac{\pi}{2m}\right)\right) \\
&< W_\alpha(\mathbb{T}).
\end{aligned}$$

We remark that the inequality $M_m^{N-\alpha}(\mathbb{T}) \leq m W_\alpha(\mathbb{T})$ was found by a different method in (3.7) of [20].

Utilizing (4.10) and the polarization formulas in [30], we list the first few explicit formulas for $C_{\mathbb{T}}^{\delta}(\alpha, m)$ that hold whenever α is a nonpositive even integer and $m \geq 1$:

$$\begin{aligned}
C_{\mathbb{T}}^{\delta}(0, m) &= \frac{1}{4} - \frac{m}{4}, \\
C_{\mathbb{T}}^{\delta}(-2, m) &= \frac{1}{16} - \frac{m}{24} - \frac{m^3}{48}, \\
C_{\mathbb{T}}^{\delta}(-4, m) &= \frac{1}{64} - \frac{m}{120} - \frac{m^3}{192} - \frac{m^5}{480}.
\end{aligned}$$

For the unit sphere in higher dimensions, we have the following.

Proposition 4.5. *For the unit sphere $S^{N-1}, N > 2$, in $\mathbb{R}^N$ equation* (4.9) *holds for all $-\infty < \alpha < N$. Moreover, the following asymptotic formulas hold as $m \to \infty$:*

$$C_{S^{N-1}}^{\delta}(\alpha, m) \sim \begin{cases} -\sigma(N-\alpha, N-1)\left(\dfrac{\Gamma(N/2)}{2\pi^{N/2}}\right)^{(N-\alpha)/(N-1)} m^{\frac{1-\alpha}{N-1}}, & 1 > \alpha > -\infty, \\ -\dfrac{\log m}{\sqrt{\pi}} \dfrac{\Gamma(N/2)}{(N-1)\Gamma((N-1)/2)}, & \alpha = 1, \\ 2^{\alpha-N} - W_\alpha(S^{N-1}) = C_{S^{N-1}}(\alpha), & 1 < \alpha < N, \end{cases} \tag{4.12}$$

where $\sigma(N-\alpha, N-1)$ is a positive constant that depends only on α and N (cf. [8]), and where the formulas for $W_\alpha(S^{N-1})$ and $C_{S^{N-1}}(\alpha)$ are given in Example 2.7.

For the unit ball we have the following result.

Proposition 4.6. *For the unit ball B^N in $\mathbb{R}^N$ there holds, for all $-\infty < \alpha < N$,*

$$1 - \frac{M_m^{N-\alpha}(B^N)}{m} \geq C_{B^N}^{\delta}(\alpha, m) \geq 2^{\alpha-N} - \frac{M_m^{N-\alpha}(B^N)}{m}. \tag{4.13}$$

Moreover the following asymptotic formulas hold as $m \to \infty$:

$$C_{B^N}^{\delta}(\alpha, m) \sim \begin{cases} -\sigma(N-\alpha, N) \left(\dfrac{\Gamma(1+N/2)}{\pi^{N/2}} \right)^{(N-\alpha)/N} m^{-\alpha/N}, & 0 > \alpha > -\infty, \\ -\log m, \quad \alpha = 0, \end{cases} \tag{4.14}$$

where $\sigma(N-\alpha, N)$ is a positive constant that depends only on α and N.

We remark that asymptotic formulas similar to those in Proposition 4.6 can be obtained for $C_E^\delta(\alpha, m)$ for a large class of N-dimensional subsets of $\mathbb{R}^N$ by appealing to the results in [9] and [8].

4.4 Proofs

We begin with a lemma that will be used in the proof of Theorem 4.4. Let $F_n = \{x_{k,n}\}_{k=1}^n$ be a set of n points in E. Let τ_n be their normalized counting measure and let $0 < \alpha < N$. We define the *discrete α-energy* of τ_n by

$$E_\alpha[\tau_n] := \frac{2}{n(n-1)} \sum_{1 \leq j < k \leq n} |x_{j,n} - x_{k,n}|^{\alpha - N}.$$

As E is compact, the minimum discrete α-energy is achieved by some set of points. Let $\mathcal{F}_n = \{\xi_{k,n}\}_{k=1}^n$, be a set of n points in E that minimizes the discrete α-energy. For $\alpha = 2$, these are typically called the Fekete points. They provide a way to approximate the α-equilibrium measure.

Lemma 4.1. *Given $0 < \alpha < N$, let $\mathcal{F}_n := \{\xi_{k,n}\}_{k=1}^n$ be the points of E minimizing the discrete α-energy. Let τ_n be the normalized counting measure associated with the set $\mathcal{F}_n$. Then the discrete α-energies of the measures τ_n increase monotonically and converge weak* to the α-equilibrium measure μ_α. Further,*

$$\lim_{n\to\infty} \inf_E U_\alpha^{\tau_n} = \lim_{n\to\infty} \inf_{x\in E} \frac{1}{n} \sum_{k=1}^n |x - \xi_{k,n}|^{\alpha-N} = W_\alpha(E).$$

Proof. The facts that the discrete energies of the measures τ_n increase monotonically and converge weak* to the equilibrium measure are proved in [43, p. 160-162]. Since τ_n is a unit measure, we may apply Tonelli's Theorem followed by Frostman's Theorem 4.3 to find

$$\int U_\alpha^{\tau_n} \,\mathrm{d}\mu_\alpha = \int U_\alpha^{\mu_\alpha} \,\mathrm{d}\tau_n \leq W_\alpha(E).$$

Since $\operatorname{supp}(\mu_\alpha) \subset E$, this implies

$$\inf_E U_\alpha^{\tau_n} \leq W_\alpha(E).$$

On the other hand, for the $(n+1)$-tuple $(x, \xi_{1,n}, \ldots, \xi_{n,n}) \subset E$ we may again apply the extremal property of $\mathcal{F}_n$ to obtain

$$\sum_{1\leq j<k\leq n+1} |\xi_{j,n+1} - \xi_{k,n+1}|^{\alpha-N} \leq \sum_{k=1}^n |x - \xi_{k,n}|^{\alpha-N} + \sum_{1\leq j<k\leq n} |\xi_{j,n} - \xi_{k,n}|^{\alpha-N}.$$

Further, monotonicity of discrete energies gives that

$$\begin{aligned}
\sum_{k=1}^n |x - \xi_{k,n}|^{\alpha-N} &\geq \frac{n(n+1)}{n(n+1)} \sum_{1\leq j<k\leq n+1} |\xi_{j,n+1} - \xi_{k,n+1}|^{\alpha-N} - \sum_{1\leq j<k\leq n} |\xi_{j,n} - \xi_{k,n}|^{\alpha-N} \\
&\geq \frac{n(n+1)}{n(n-1)} \sum_{1\leq j<k\leq n} |\xi_{j,n} - \xi_{k,n}|^{\alpha-N} - \sum_{1\leq j<k\leq n} |\xi_{j,n} - \xi_{k,n}|^{\alpha-N} \\
&= \frac{2}{(n-1)} \sum_{1\leq j<k\leq n} |\xi_{j,n} - \xi_{k,n}|^{\alpha-N},
\end{aligned}$$

which immediately implies that

$$W_\alpha(E) \geq \inf_{x\in E} \frac{1}{n} \sum_{k=1}^n |x - \xi_{k,n}|^{\alpha-N} \geq E_\alpha[\tau_n] \to W_\alpha(E) \text{ as } n \to \infty.$$

$\square$

Proof of Theorem 4.2. We already concluded that $d_E^{\alpha-N}$ is an α-potential if E is finite. If, on the other hand, E is not finite, we consider a sequence of finite subsets $E_m \subset E_{m+1} \subset E$ that are dense in E as $m \to \infty$. Let d_m be the farthest distance function of E_m and let σ_m be the associated measure such that $d_m^{\alpha-N} = U_\alpha^{\sigma_m}$, $m \in \mathbb{N}$. Since $d_m \le d_{m+1}$, it follows that $U_\alpha^{\sigma_m} \ge U_\alpha^{\sigma_{m+1}}$, $m \in \mathbb{N}$. Thus we obtain a decreasing sequence of potentials, and Theorem 3.10 of [43] gives a positive unique Borel measure σ_α such that $\sigma_m \xrightarrow{*} \sigma_\alpha$ and $d_E^{\alpha-N} = U_\alpha^{\sigma_\alpha}$ quasi-everywhere. Since the set of points S where $d_E^{\alpha-N} \neq U_\alpha^{\sigma_\alpha}$ has α-capacity zero, it also has zero volume in $\mathbb{R}^N$, see [43, Theorem 3.13 on p. 196]. Hence $U_\alpha^{\sigma_\alpha} * \varepsilon_{\alpha,r} = d_E^{\alpha-N} * \varepsilon_{\alpha,r}$ for the averaging measure $\varepsilon_{\alpha,r}$ used in the definition of α-superharmonicity in [43, p. 112]. Furthermore, Property (i) [43, p. 114] for α-superharmonic functions gives that

$$U_\alpha^{\sigma_\alpha}(x) = \lim_{r\to 0} U_\alpha^{\sigma_\alpha} * \varepsilon_{\alpha,r}(x) = \lim_{r\to 0} d_E^{\alpha-N} * \varepsilon_{\alpha,r}(x) = d_E^{\alpha-N}(x), \quad x \in \mathbb{R}^N,$$

where we used the fact $\varepsilon_{\alpha,r} \xrightarrow{*} \delta_0$ as $r \to 0$ [43, p. 112] on the last step.

We now consider the mass of the measure σ_α. Assume, without loss of generality, that the origin is a point in E. Consider the ball $B(R)$ of radius $R > \operatorname{diam}(E)$ about the origin. We average

$$d_E^{\alpha-N}(x) = \int_{\mathbb{R}^N} |x-t|^{\alpha-N}\,\mathrm{d}\sigma_\alpha(t)$$

with respect to the α-equilibrium measure τ_R of the ball $B(R)$, to obtain

$$M(R) := \int_{B(R)} d_E^{\alpha-N}(x)\,\mathrm{d}\tau_R(x) = \int_{B(R)}\int_{\mathbb{R}^N} |x-y|^{\alpha-N}\,\mathrm{d}\sigma_\alpha(y)\,\mathrm{d}\tau_R(x). \tag{4.15}$$

The following proof is similar to the proof given in Theorem 2.6 for the $\alpha = 2$ case. In that case we averaged with respect to the usual equilibrium measure on the ball, the normalized surface area. The α-equilibrium measure is given in [43, p. 163] as

$$\mathrm{d}\tau_R(x) = AR^{\alpha-N}(R^2-|x|^2)^{-\alpha/2}\,\mathrm{d}x \text{ for } |x| < R,$$

where A is the constant

$$A = \frac{\Gamma\left(\frac{N-\alpha}{2}+1\right)}{\pi^{N/2}\Gamma\left(1-\frac{\alpha}{2}\right)}.$$

Its potential $U_\alpha^{\tau_R}(x) = \int |x-t|^{\alpha-N}\,\mathrm{d}\tau_R(t)$ is

$$U_\alpha^{\tau_R}(y) = AR^{\alpha-N}\frac{\pi^{N/2+1}}{\Gamma(N/2)\sin(\pi\alpha/2)},$$

for all $|y| \le R$ [43, (A.1)]. Using the fact that

$$\frac{\pi}{\sin(\pi x)} = \Gamma(x)\Gamma(1-x),$$

we calculate for $|y| \le R$ that

$$\begin{aligned} U_\alpha^{\tau_R}(y) =& \frac{\Gamma\left((N-\alpha)/2+1\right)}{\pi^{N/2}\Gamma\left(1-\alpha/2\right)}R^{\alpha-N}\frac{\pi^{N/2}}{\Gamma(N/2)}\Gamma(\alpha/2)\Gamma(1-\alpha/2) \\ =& \frac{\Gamma(\alpha/2)\Gamma\left((N-\alpha)/2+1\right)}{\Gamma(N/2)}\ R^{\alpha-N}. \end{aligned}$$

Introducing the notation

$$c(N,\alpha) = A\frac{\pi^{N/2+1}}{\Gamma(N/2)\sin(\pi\alpha/2)} = \frac{\Gamma(\alpha/2)\Gamma\left((N-\alpha)/2+1\right)}{\Gamma(N/2)},$$

we obtain

$$U_\alpha^{\tau_R}(y) = c(N,\alpha)R^{\alpha-N}, \quad \text{for all } |y| \le R. \tag{4.16}$$

Furthermore, this same value serves as the upper bound of the potential for all $|y| > R$. Notice that $c(N,2) = 1$ and hence (4.16) is a generalization of the fact that $U_2^{\tau_R}(x) = R^{2-N}$ for $|x| \le R$ when $\alpha = 2$.

Consider the left hand side of (4.15). We know $|x| \le d_E(x) \le |x| + \mathrm{diam}(E)$ in $B(R)$. We use the lower bound on d_E to find an upper bound on $M(R)$. Applying the calculations in [43, Appendix] again, we conclude that

$$\begin{aligned} \int_{B(R)} d_E^{\alpha-N}(x)\,\mathrm{d}\tau_R(x) &\le \int_{B(R)} |x|^{\alpha-N}\,\mathrm{d}\tau_R(x) \\ &= AR^{\alpha-N}\int_{B(R)}(R^2-|x|^2)^{-\alpha/2}|x|^{\alpha-N}\,\mathrm{d}x \\ &= AR^{\alpha-N}\frac{\pi^{N/2+1}}{\Gamma(N/2)\sin(\pi\alpha/2)} \\ &= c(N,\alpha)R^{\alpha-N}. \end{aligned}$$

Next we use the upper bound on d_E to obtain a lower bound for $M(R)$. Let $d = \operatorname{diam}(E)$. Then for any $\epsilon > 0$ we have $d \le \epsilon|x|$ for any x not in $B(d/\epsilon)$. Hence

$$\begin{aligned}
\int_{B(R)} d_E^{\alpha-N}(x) d\tau_R(x) &\ge \int_{B(R)} (|x|+d)^{\alpha-N}\, \mathrm{d}\tau_R(x) \\
&> \int_{B(R)\setminus B(d/\epsilon)} (|x|+d)^{\alpha-N}\, \mathrm{d}\tau_R(x) \\
&\ge \int_{B(R)\setminus B(d/\epsilon)} |x|^{\alpha-N}\,(1+\epsilon)^{\alpha-N}\, \mathrm{d}\tau_R(x) \\
&= (1+\epsilon)^{\alpha-N}\, U_\alpha^{\tau_R}(0) - (1+\epsilon)^{\alpha-N} \int_{B(d/\epsilon)} |x|^{\alpha-N}\, \mathrm{d}\tau_R(x) \\
&= (1+\epsilon)^{\alpha-N}\, c(N,\alpha) R^{\alpha-N} - (1+\epsilon)^{\alpha-N} \int_{B(d/\epsilon)} |x|^{\alpha-N}\, \mathrm{d}\tau_R(x)
\end{aligned}$$

Estimating the integral over the ball $B(d/\epsilon)$, we find

$$\begin{aligned}
\int_{B(d/\epsilon)} |x|^{\alpha-N}\, \mathrm{d}\tau_R(x) =& R^{\alpha-N} A\omega_N \int_0^{d/\epsilon} |x|^{\alpha-1}(R^2-|x|^2)^{-\alpha/2}\, \mathrm{d}|x| \\
\le& R^{\alpha-N} \left(R^2 - \frac{d^2}{\epsilon^2}\right)^{-\alpha/2} \frac{A\omega_N d^\alpha}{\alpha\epsilon^\alpha}.
\end{aligned}$$

Since the above integral is also bounded below by zero, it follows that it is $\mathcal{O}(R^{-N})$ and thus

$$(1+\epsilon)^{\alpha-N}\, c(N,\alpha)R^{\alpha-N} - \mathcal{O}(R^{-N}) < M(R) \le c(N,\alpha)R^{\alpha-N}. \tag{4.17}$$

On the other hand, we may apply Tonelli's Theorem on the right hand side of (4.15) to obtain

$$\begin{aligned}
\int_{B(R)} \int_{\mathbb{R}^N} |x-y|^{\alpha-N} d\sigma_\alpha(y)\, \mathrm{d}\tau_R(x) &= \int_{\mathbb{R}^N} \int_{B(R)} |x-y|^{\alpha-N} d\tau_R(x)\, \mathrm{d}\sigma_\alpha(y) \\
&= \int_{\mathbb{R}^N} U_\alpha^{\tau_R}(y)\, \mathrm{d}\sigma_\alpha(y) \\
&= \int_{|y|\le R} U_\alpha^{\tau_R}(y)\, \mathrm{d}\sigma_\alpha(y) + \int_{|y|>R} U_\alpha^{\tau_R}(y)\, \mathrm{d}\sigma_\alpha(y).
\end{aligned}$$

Applying the calculation of the potential in (4.16), we find

$$c(N,\alpha)R^{\alpha-N}\sigma_\alpha(B(R)) \le M(R) < c(N,\alpha)R^{\alpha-N}\sigma_\alpha(\mathbb{R}^N). \tag{4.18}$$

Combining (4.17) and (4.18), dividing by $R^{\alpha-N}$ and then letting $R \to \infty$, we obtain

$$(1+\epsilon)^{\alpha-N} \le \sigma_\alpha(\mathbb{R}^N) \le 1.$$

Finally, we conclude $\sigma_\alpha(\mathbb{R}^N) = 1$ by letting $\epsilon \to 0$. □

Proof of Theorem 4.4. For any positive Borel measure μ, the potential

$$U_\alpha^\mu(t) = \int |t-x|^{\alpha-N}\,\mathrm{d}\mu(x)$$

is lower semicontinuous [43, p. 59], and hence attains its infimum on the compact set E. Thus we may choose $c_k \in E$ such that

$$\inf_E U_\alpha^{\nu_k} = U_\alpha^{\nu_k}(c_k)$$

for each $k = 1, \dots, m$. It follows that

$$\begin{aligned}
\sum_{k=1}^m \inf_E U_\alpha^{\nu_k} &= \sum_{k=1}^m U_\alpha^{\nu_k}(c_k) \\
&= \sum_{k=1}^m \int |c_k - x|^{\alpha-N}\,\mathrm{d}\nu_k(x) \\
&\ge \int \min_{1\le k\le m} |c_k - x|^{\alpha-N}\,\mathrm{d}\nu(x) \\
&= \int \left(\max_{1\le k\le m} |c_k - x|\right)^{\alpha-N} \mathrm{d}\nu(x).
\end{aligned}$$

The function $d_m(x) := \max_{1\le k\le m} |c_k - x|$ is the farthest distance function on the set of points c_k. By Theorem 4.2, there exists a probability measure σ_α such that $U_\alpha^{\sigma_\alpha}(x) = d_m^{\alpha-N}(x)$. Applying Tonelli's Theorem, we have

$$\sum_{k=1}^m \inf_E U_\alpha^{\nu_k} \ge \int U_\alpha^{\sigma_\alpha}(x)\,\mathrm{d}\nu(x) = \int U_\alpha^\nu(t)\,\mathrm{d}\sigma_\alpha(t).$$

We estimate the potential U_α^ν on $\mathbb{R}^N$. Let μ_α be the α-equilibrium measure for E and let $W_\alpha(E)$ be the α-energy for E. Let $g(t) := U_\alpha^{\mu_\alpha}(t) - W_\alpha(E)$. By Frostman's Theorem 4.3, we know $g(t) \le 0$ everywhere. On the other hand, $U_\alpha^\nu(t) - \inf_E U_\alpha^\nu \ge 0$ for $t \in E$. Thus

$$U_\alpha^\nu(t) \ge \inf_E U_\alpha^\nu + U_\alpha^{\mu_\alpha}(t) - W_\alpha(E)$$

on E. It follows by the Principle of Domination [43, Theorem 1.27 on p. 110 for $\alpha = 2$ and Theorem 1.29 on p. 115 for $0 < \alpha < 2$] that this inequality holds in $\mathbb{R}^N$. Thus, noting that σ_α is a unit measure and again applying Tonelli's Theorem, we find

$$\begin{aligned}\sum_{k=1}^{m} \inf_E U_\alpha^{\nu_k} &\geq \int U_\alpha^{\nu}(t)\, \mathrm{d}\sigma_\alpha(t) \\ &\geq \int \left(\inf_E U_\alpha^{\nu} + \int U^{\mu_\alpha}(t) - W_\alpha(E) \right) \mathrm{d}\sigma_\alpha(t) \\ &= \inf_E U_\alpha^{\nu} + \int U_\alpha^{\sigma_\alpha}(x)\, \mathrm{d}\mu_\alpha(x) - W_\alpha(E) \\ &= \inf_E U_\alpha^{\nu} + \int d_m^{\alpha-N}(x)\, \mathrm{d}\mu_\alpha(x) - W_\alpha(E).\end{aligned}$$

By minimizing over all m-tuples c_k, we conclude that

$$\sum_{k=1}^{m} \inf_E U^{\nu_k} \geq C_E(\alpha, m) + \inf_E \sum_{k=1}^{m} U^{\nu_k}$$

where

$$C_E(\alpha, m) := \min_{c_k \in E} \int \min_{1 \leq k \leq m} |x - c_k|^{2-N}\, \mathrm{d}\mu_\alpha(x) - W_\alpha(E).$$

We now show $C_E(\alpha, m)$ is the largest possible constant for a fixed m. We present two proofs of this fact. We begin with the shorter one which requires E to be regular in the sense that $U_\alpha^{\mu_\alpha}(x) = W_\alpha(E)$ for all $x \in E$. Choose a set $c_k^*,\ k = 1, \ldots, m$, such that $\int d_m^{\alpha-N}(x) d\mu_\alpha(x)$ attains its minimum on E^m. Let $d_m^*(x) := \min_{1\leq k\leq m} |x - c_k^*|$ and iteratively define the sets

$$\begin{aligned}S_1 =& \{x \in \mathrm{supp}(\mu_\alpha) : |x - c_1^*| = d_m^*(x)\} \\ S_k =& \{x \in \mathrm{supp}(\mu_\alpha) \setminus \cup_{j=1}^{k-1} S_j : |x - c_k^*| = d_m^*(x)\},\ k = 2, \ldots, m.\end{aligned}$$

It is clear that

$$\mathrm{supp}(\mu_\alpha) = \cup_{k=1}^{m} S_k \text{ and } S_k \cap S_j = \emptyset,\ k \neq j.$$

Hence we can decompose μ_α along the sets S_k such that

$$\nu_k^* := \mu_\alpha|_{S_k} \text{ and } \mu_\alpha = \sum_{k=1}^{m} \nu_k^*.$$

If E is regular, then $\int |x-t|^{\alpha-N}\,\mathrm{d}\mu_\alpha(t) = W_\alpha(E)$ for each $x \in E$ by Frostman's Theorem. Applying this fact, along with Tonelli's Theorem, we obtain

$$\begin{aligned}
\sum_{k=1}^{m} \inf_E U_\alpha^{\nu_k^*} &\le \sum_{k=1}^{m} U_\alpha^{\nu_k^*}(c_k^*) \\
&= \sum_{k=1}^{m} \int |c_k^* - x|^{\alpha-N}\,\mathrm{d}\nu_k^*(x) \\
&= \sum_{k=1}^{m} \int (d_m^*(x))^{\alpha-N}\,\mathrm{d}\nu_k^*(x) \\
&= \int (d_m^*(x))^{\alpha-N}\,\mathrm{d}\mu_\alpha(x) \\
&= \int (d_m^*(x))^{\alpha-N}\,\mathrm{d}\mu_\alpha(x) - W_\alpha(E) + \inf_{t\in E} \int |x-t|^{\alpha-N}\,\mathrm{d}\mu_\alpha(x) \\
&= C_E(\alpha,m) + \sum_{k=1}^{m} \inf_E U_\alpha^{\nu_j^*}.
\end{aligned}$$

Hence $C_E(\alpha, m)$ is sharp. The alternative proof uses points minimizing the discrete α-energy and does not require that E be regular. Let $\mathcal{F}_n = \{\xi_{l,n}\}_{l=1}^n$ be the points of E which minimize the discrete α-energy. We will break the set $\mathcal{F}_n$ up using the points c_k^* just as we broke up $\operatorname{supp}(\mu_\alpha)$ previously. Let $F_{k,n}$ be a subset of $\mathcal{F}_n$ such that $\xi_{l,n} \in \mathcal{F}_{l,n}$ if $d_m^*(\xi_{l,n}) = |\xi_{l,n} - c_k^*|,\ 1 \le l \le n$. If there is overlap between the sets, assign $\xi_{l,n}$ to only one set $\mathcal{F}_{k,n}$. It is clear that for any $n \in \mathbb{N}$,

$$\mathcal{F}_n = \cup_{k=1}^m \mathcal{F}_{k,n} \text{ and } \mathcal{F}_{k,n} \cap \mathcal{F}_{j,n} = \emptyset,\ k \ne j.$$

Define the measures

$$\nu_{k,n}^* = \frac{1}{n} \sum_{\xi_{l,n} \in F_{k,n}} \delta_{\xi_{l,n}},$$

so that for their potentials

$$p_{k,n}^*(x) = \frac{1}{n} \sum_{\xi_{l,n} \in F_{k,n}} |x - \xi_{l,n}|^{\alpha-N},\ k = 1, \ldots, m,$$

we have

$$\inf_E p_{k,n}^*(x) \le \frac{1}{n} \sum_{\xi_{l,n} \in F_{k,n}} |c_k^* - \xi_{l,n}|^{\alpha-N} = \frac{1}{n} \sum_{\xi_{l,n} \in F_{k,n}} (d_m^*(\xi_{l,n}))^{\alpha-N}.$$

It follows from the weak* convergence of $\nu_n := \sum_{k=1}^m \nu_{k,n}^* = \frac{1}{n}\sum_{l=1}^n \delta_{\xi_{l,n}}$ to μ_α, as $n \to \infty$, that

$$\begin{aligned}\limsup_{n\to\infty} \sum_{k=1}^m \inf_E p_{k,n}^*(x) &\leq \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n (d_m^*(\xi_{k,n}))^{\alpha-N} \\ &= \int (d_m^*(x))^{\alpha-N}\,\mathrm{d}\mu_\alpha(x).\end{aligned}$$

Applying Lemma 4.1 to the potential p_n^* of ν_n^* we find that

$$\lim_{n\to\infty} \inf_E p_n^* = W_\alpha(E).$$

It follows that

$$\limsup_{n\to\infty} \sum_{k=1}^m \inf_E p_n^* \leq C_E(\alpha, m) + \lim_{n\to\infty} \inf_E p_n^*.$$

Hence we have asymptotic equality in (4.7) as $n \to \infty$ with $m \geq 2$ being fixed, which shows that $C_E(\alpha, m)$ is the largest possible constant for each m. Since $d_m \leq d_E$ everywhere, we have $C_E(\alpha, m) \geq C_E(\alpha)$. □

Proof of Corollary 4.1. If $m < \operatorname{card}(\mathfrak{D}_E)$, then there is an $x_0 \in \operatorname{supp}(\mu_\alpha)$ such that $d_m^*(x_0) < d_E(x_0)$. As both functions are continuous, the same strict inequality holds in a neighborhood of x_0, so that $\int (d_m^*(x))^{\alpha-N} d\mu_\alpha(x) > \int d_E^{\alpha-N}(x) d\mu_\alpha(x)$ and hence $C_E(\alpha, m) > C_E(\alpha)$. This argument shows that if $\mathfrak{D}_E$ is infinite, then $C_E(\alpha, m) > C_E(\alpha)$ for $m \geq 2$. If $m \geq \operatorname{card}(\mathfrak{D}_E)$ then we may choose the points c_k^* to include $\mathfrak{D}_E$ and hence $d_m^*(x) = d_E(x)$ for $x \in \operatorname{supp}(\mu_\alpha)$. Thus $C_E(\alpha, m) = C_E(\alpha)$.

Let c_k, $k = 1, \ldots, m$, be a set of points in E that minimize the integral in the expression of $C_E(\alpha, m)$. Choose a point $c_{m+1} \in \partial E$. Then

$$\begin{aligned}C_E(\alpha, m) &= \int \min_{1\leq k\leq m} |x - c_k|^{\alpha-N}\,\mathrm{d}\mu_\alpha(x) - W_\alpha(E) \\ &\geq \int \min_{1\leq k\leq m+1} |x - c_k|^{\alpha-N}\,\mathrm{d}\mu_\alpha(x) - W_\alpha(E) \\ &\geq C_E(\alpha, m+1).\end{aligned}$$

Hence the constants $C_E(\alpha, m)$ are decreasing. It remains to show that their limit is $C_E(\alpha)$. Let $\{a_k\}_{k=1}^{\infty}$ be a countable dense subset of E. Then

$$C_E(\alpha) \le C_E(\alpha, m) \le \int \min_{1\le k\le m} |x-a_k|^{\alpha-N}\,\mathrm{d}\mu_\alpha(x) - W_\alpha(E).$$

Further, applying the Dominated Convergence Theorem, we have

$$\begin{aligned}\lim_{m\to\infty}\int \min_{1\le k\le m} |x-a_k|^{\alpha-N}\,\mathrm{d}\mu_\alpha(x) &= \int \lim_{m\to\infty}\min_{1\le k\le m} |x-a_k|^{\alpha-N}\,\mathrm{d}\mu_\alpha(x)\\ &= \int d_E^{\alpha-N}(x)\,\mathrm{d}\mu_\alpha(x).\end{aligned}$$

The result follows. □

Proof of Corollary 4.2. We show the minimal dominant set is infinite and then the result follows from Corollary 4.1. Suppose to the contrary that $\mathfrak{D}_E = \{x_j\}_{j=1}^{s}$ is finite. Let $J \subset \partial E$ be a single connected component of the boundary. Define

$$J_k := \{x \in J : d_E(x) = |x - x_k|\}, k = 1, \ldots, s.$$

For each $x \in J_k$, the segment $[x, x_k]$ is orthogonal to ∂E at x_k, by the smoothness assumption. Hence, each J_k is contained in the normal line to ∂E at x_k, $k = 1, \ldots, s$. We thus obtain that $J = \cup_{k=1}^{s} J_k$ is contained in a union of straight lines which is a contradiction. □

Proof of Example 4.1. To calculate the quantity $\min_{c_k\in\mathbb{T}} \int \min_{1\le k\le m} |x-c_k|^{\alpha-N} d\mu_\alpha(x)$, we follow an idea of Boyd [12]. Let $c_k = -e^{i\psi_k},\ k = 1, \ldots, m$, with $\psi_k < \psi_{k+1}$, and for notational convenience let $\psi_0 = \psi_m$. Then we have $\max_{1\le k\le m} |e^{i\theta} - c_k| = |e^{i\theta} + e^{i\psi_k}| =$

$|e^{i(\theta-\psi_k)}+1|$ for $\frac{\psi_{k-1}+\psi_k}{2} \le \theta \le \frac{\psi_k+\psi_{k+1}}{2}$ and hence

$$\begin{aligned}
\int \min_{1\le k\le m} |x-c_k|^{\alpha-N}\,\mathrm{d}\mu_\alpha(x) =& \frac{1}{2\pi}\sum_{k=1}^m \int_{\frac{\psi_{k-1}+\psi_k}{2}}^{\frac{\psi_k+\psi_{k+1}}{2}} |e^{i(\theta-\psi_k)}+1|^{\alpha-2}\,\mathrm{d}\theta \\
=& \frac{1}{\pi}\sum_{k=1}^m \int_0^{\frac{\psi_k-\psi_{k-1}}{2}} |e^{i\theta}+1|^{\alpha-2}\,\mathrm{d}\theta \\
=& \frac{2^{\alpha-2}}{\pi}\sum_{k=1}^m \int_0^{\frac{\psi_k-\psi_{k-1}}{2}} \cos^{\alpha-2}\left(\frac{\theta}{2}\right)\mathrm{d}\theta \\
=& 2^{\alpha-2}\frac{2}{\pi}\sum_{k=1}^m I(\theta_k),
\end{aligned}$$

where $\theta_k = \frac{\psi_k-\psi_{k-1}}{4}$ and $I(\theta_k) = \int_0^{\theta_k}\cos^{\alpha-2}(\theta)\ d\theta$. Since $I(\theta_k)$ is strictly convex for $0<\theta_k<\frac{\pi}{4}$, and $\sum_{k=1}^m \theta_k = \pi/2$, we have

$$\frac{1}{m}\sum_{k=1}^m I(\theta_k) \ge I\left(\frac{\pi}{2m}\right).$$

Hence

$$\min_{c_k\in\mathbb{T}} \int \min_{1\le k\le m} |x-c_k|^{\alpha-N}\,\mathrm{d}\mu_\alpha(x) = 2^{\alpha-2}\frac{2m}{\pi} I\left(\frac{\pi}{2m}\right),$$

where the outer minimum is clearly attained for the equally spaced points c_k on $\mathbb{T}$. □

Proof of Proposition 4.4. Equation (4.10) is a consequence of (4.9) and the main theorem proved in [30]. The asymptotic formulas in (4.11) follow from (4.10) and the asymptotics for $M_n^s(\mathbb{S}^1)$ given in [30]. □

The proofs of Propositions 4.5 and 4.6 are straightforward consequences of the main theorems on polarization proved in [20], [9] and [8].

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