SYSTOLIC FREEDOM OF 3-MANIFOLDS

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Abstract:

In this thesis, we study the $\mathbb{Z}_2$-coefficient homology (1, 2)-systolic freedom of 3-manifolds. In 1994, Bérard-Bergery and Katz proved the $\mathbb{Z}$-coefficient homology (1, 2)-systolic freedom of $S^2 \times S^1$. More generally, compact and orientable 3-manifolds are of $\mathbb{Z}$-coefficient homology (1, 2)-systolic freedom due to the work of Babenko and Katz. Later in 1999, Freedman showed that $S^2 \times S^1$ is of $\mathbb{Z}_2$-coefficient homology (1, 2)-systolic freedom, which is a counterexample to Gromov's conjecture. In the thesis, we show that the 3-manifold $\mathbb{RP}^3 \# \mathbb{RP}^3$ is of $\mathbb{Z}_2$-coefficient homology (1, 2)-systolic freedom. The proof is based on the semibundle structure property of $\mathbb{RP}^3 \# \mathbb{RP}^3$ and the application of Freedman's technique on $S^2 \times S^1$. We show the details of how Dehn surgery changes metric on mapping torus in Freedman's example. Then with respect to the sequence of metrics constructed, we calculate the lower bound estimates of $\mathbb{Z}_2$-coefficient homology 1-systole and $\mathbb{Z}_2$-coefficient homology 2-systole, as well as the upper bound estimates of the volume of $S^2 \times S^1$ in details. The 3-manifold $\mathbb{RP}^3 \# \mathbb{RP}^3$ has a sphere semibundle structure. We employ Freedman's technique to construct a sequence of Riemannian metrics on $\mathbb{RP}^3 \# \mathbb{RP}^3$. By an investigation of 3-manifolds with semibundle structure, we prove a lower bound estimate of $\mathbb{Z}_2$-coefficient homology 1-systole of $\mathbb{RP}^3 \# \mathbb{RP}^3$. A lower bound estimate for $\mathbb{Z}_2$-coefficient homology 2-systole of $\mathbb{RP}^3 \# \mathbb{RP}^3$ is obtained in terms of the semibundle structure and Freedman's result on $S^2 \times S^1$. Based on these estimations, we prove the $\mathbb{Z}_2$-coefficient homology (1, 2)-systolic freedom of $\mathbb{RP}^3 \# \mathbb{RP}^3$.

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CHAPTER 1

Introduction

Let M be a compact non-simply connected Riemannian manifold with a Riemannian metric $\mathcal{G}$, denoted by $(M, \mathcal{G})$. The systolic inquality of $(M, \mathcal{G})$ studies lower bounds on the Riemannian volume in terms of infimum volume of representatives of nonzero homotopy classes or homology classes.

Roughly speaking, the homotopy 1-systole of $(M,\, \mathcal{G})$ is the infimum length of all noncontractible loops of M. We denote it by $\operatorname{Sys}\pi_1(M,\, \mathcal{G})$. The first systolic inequality in literature is proved by C. Loewner, see [33] or [42]. For a Riemannian torus $(\mathbb{T}^2,\, \mathcal{G})$, we use $\mathrm{Area}_{\mathcal{G}}(\mathbb{T}^2)$ to denote the area of $\mathbb{T}^2$ under the metric $\mathcal{G}$. C. Loewner showed that for every Riemannian metric $\mathcal{G}$ on a torus $\mathbb{T}^2$,

$$\operatorname{Sys}\pi_1(M,\, \mathcal{G})^2 \leqslant \frac{2}{\sqrt{3}}\,\mathrm{Area}_{\mathcal{G}}(\mathbb{T}^2), \tag{1.1}$$

where the equality holds if $(\mathbb{T}^2,\, \mathcal{G})$ is a flat hexagonal torus, i.e., the metric $\mathcal{G}$ on $\mathbb{T}^2$ is flat and the torus $\mathbb{T}^2$ is defined by $\mathbb{R}^2/\Lambda$, where Λ is the lattice in $\mathbb{R}^2$ generated by vectors $(1,\, 0)$ and $(1/2,\, \sqrt{3}/2)$. Moreover, we define the systolic ratio $\mathrm{SR}(\mathcal{G})$ of $(\mathbb{T}^2,\, \mathcal{G})$ as

$$\frac{\mathrm{Area}_{\mathcal{G}}(\mathbb{T}^2)}{\operatorname{Sys}\pi_1(\mathbb{T}^2,\, \mathcal{G})^2},$$

and define the optimal systolic ratio $\mathrm{SR}(\mathbb{T}^2)$ as

$$\inf_{\mathcal{G}} \mathrm{SR}(\mathcal{G}),$$

where the infimum runs over all Riemannian metrics $\mathcal{G}$ on $\mathbb{T}^2$. Loewner's inequality implies that for every Riemannian metric $\mathcal{G}$ on $\mathbb{T}^2$ we have

$$\mathrm{SR}(\mathcal{G}) \geqslant \frac{\sqrt{3}}{2},$$

and the optimal systolic ratio of $\mathbb{T}^2$ is equal to $\sqrt{3}/2$. After C. Loewner, P. Pu proved another systolic inequality in the nonorientable surface case in 1952, see [33] or [42]. He showed that for every Riemannian metric $\mathcal{G}$ on a real projective plane $\mathbb{RP}^2$, the inequality

$$\mathrm{Sys}\,\pi_1(\mathbb{RP}^2, \mathcal{G})^2 \leqslant \frac{\pi}{2}\,\mathrm{Area}_{\mathcal{G}}(\mathbb{RP}^2) \tag{1.2}$$

holds, and equality holds for a metric $\mathcal{G}$ with constant Gaussian curvature. Hence for every Riemannian metric $\mathcal{G}$ on $\mathbb{RP}^2$, we have

$$\mathrm{SR}(\mathcal{G}) \geqslant \frac{2}{\pi},$$

and the optimal systolic ratio $\mathrm{SR}(\mathbb{RP}^2)$ is equal to $2/\pi$.

Due to a result of M. Gromov, the Pu inequality can be generalized to a closed and connected surface Σ which is not homeomorphic to the 2-sphere S^2. Here we use the convention that a surface is closed if it is compact and without boundary.

Theorem 1.0.1 (Croke and Katz [13]) *If Σ is a closed surface other than S^2, then for every Riemannian metric $\mathcal{G}$ on Σ,*

$$Sys\,\pi_1(\Sigma, \mathcal{G})^2 \leqslant \frac{\pi}{2}\,Area_{\mathcal{G}}(\Sigma), \tag{1.3}$$

where the equality holds if Σ is a real projective plane with the metric $\mathcal{G}$ of constant Gaussian curvature.

For closed and non-simply connected surfaces, currently we only know the optimal systolic ratio of $\mathbb{T}^2$, $\mathbb{RP}^2$ and the Klein bottle $\mathbb{RP}^2\#\mathbb{RP}^2$. The optimal systolic ratio $\mathrm{SR}(\mathbb{RP}^2\#\mathbb{RP}^2)$ is equal to $2\sqrt{2}/\pi$, which is achieved by a singular metric, see [3] or [33].

In 1983, M. Gromov proved a systolic inequality for an n-dimensional($n \geqslant 3$) essential Riemannian manifold M in [21]. A topological space K is aspherical if all higher homotopy groups $\pi_i(K)$ vanish for $i \geqslant 2$. A connected and orientable closed manifold M of dimension n is essential if there exists an aspherical space K and a map $f : M \to K$ such that the image $f_*([M])$ of integral fundamental class $[M] \in H_n(M; \mathbb{Z})$ is nonzero in $H_n(K; \mathbb{Z})$. When the manifold M is nonorientable, we use the homology group $H_n(M; \mathbb{Z}_2)$ to define essential manifolds.

Theorem 1.0.2 (Gromov [21]) *For every Riemannian metric $\mathcal{G}$ on a closed essential Riemannian manifold M of dimension n,*

$$Sys\, \pi_1(M, \mathcal{G})^n \leqslant C(n)\; Vol_{\mathcal{G}}(M), \tag{1.4}$$

where the constant $C(n)$ is equal to $\left(6(n+1)(n^n)\sqrt{(n+1)!}\right)^n$.

The Loewner inequality, Pu inequality and inequality (1.4) are involved with homotopy 1-systole, which is measured in terms of lengths of noncontractible loops on a given Riemannian manifold. There are also interesting geometric properties for higher systoles, which are defined in terms of infimum area or infimum volume of cycles representing nonzero homology classes. Let $(M, \mathcal{G})$ be a compact Riemannian manifold of dimension n. Let k be an integer satisfying $1 \leqslant k \leqslant n$. We use $\mathbb{Z}$ to denote the integer coefficient ring for homology group, and use $\mathbb{Z}_2$ to denote the coefficient ring $\mathbb{Z}/2\mathbb{Z}$ for homology group. We define the infimum volume of all cycles representing nonzero classes in $H_k(M; \mathbb{Z})$ as $\mathbb{Z}$-coefficient homology k-systole, denoted by $\mathrm{Sys}\, H_k(M, \mathcal{G}; \mathbb{Z})$. And we define the infimum volume of all cycles representing nonzero classes in $H_k(M; \mathbb{Z}_2)$ as $\mathbb{Z}_2$-coefficient homology k-systole, denoted by $\mathrm{Sys}\, H_k(M, \mathcal{G}; \mathbb{Z}_2)$. For $\mathbb{Z}$-coefficient homology k-systoles, there are examples of violations of systolic inequalities, see [1, 2, 6, 31, 32, 36, 41]. We define such a phenomenon as systolic freedom. In [6], L. Bergery and M. Katz proved that the 3-manifold $S^2 \times S^1$ has $\mathbb{Z}$-coefficient homology (1, 2)-systolic freedom, i.e., we have

$$\inf_{\mathcal{G}} \frac{\mathrm{Vol}_{\mathcal{G}}(S^2 \times S^1)}{\mathrm{Sys}\, H_1(S^2 \times S^1, \mathcal{G}; \mathbb{Z}) \cdot \mathrm{Sys}\, H_2(S^2 \times S^1, \mathcal{G}; \mathbb{Z})} = 0,$$

where the infimum runs over all Riemannian metrics $\mathcal{G}$ on $S^2 \times S^1$. And in [1], a result of $\mathbb{Z}$-coefficient homology $(1, n-1)$-systolic freedom of n-manifold is proved.

Theorem 1.0.3 (Babenko and Katz [1]) *Every compact and orientable n-manifold M is of $\mathbb{Z}$-coefficient homology $(1,\, n-1)$-systolic freedom, i.e.,*

$$\inf_{\mathcal{G}} \frac{Vol_{\mathcal{G}}(M)}{Sys\, H_1(M,\, \mathcal{G};\, \mathbb{Z}) \cdot Sys\, H_{n-1}(M,\, \mathcal{G};\, \mathbb{Z})} = 0,$$

where the infimum is over all Riemannian metrics $\mathcal{G}$ on M.

Moreover, M. Freedman [18] proved the 3-manifold $S^2 \times S^1$ is of $\mathbb{Z}_2$-coefficient homology $(1,\, 2)$-systolic freedom, i.e.,

$$\inf_{\mathcal{G}} \frac{\mathrm{Vol}_{\mathcal{G}}(S^2 \times S^1)}{\mathrm{Sys}\, H_1(S^2 \times S^1,\, \mathcal{G};\, \mathbb{Z}_2) \cdot Sys\, H_2(S^2 \times S^1,\, \mathcal{G};\, \mathbb{Z}_2)} = 0,$$

where the infimum is taken over all Riemannian metrics $\mathcal{G}$ on $S^2 \times S^1$. This result is a counterexample to M. Gromov's conjecture, see [18]. And it is the first example of $\mathbb{Z}_2$-coefficient homology $(1,\, 2)$-systolic freedom for 3-manifolds.

In M. Freedman's work (see [18] or [19]), a sequence of 3-manifolds with mapping torus structure is constructed, where the fiber surfaces of these mapping tori are arithmetic hyperbolic and the monodromy maps are periodic. Then according to Thurston's classification theorem, the Riemannian metrics on surface bundles are locally isometric to the product metric on $\mathbb{H}^2 \times \mathbb{R}$. The inverse of the periodic monodromy map can be decomposed into the product of Dehn twists on arithmetic hyperbolic surfaces by Lickorish Twist Theorem. We perform Dehn surgeries on the mapping torus to finish these Dehn twists. After performing additional Dehn surgeries to kill a collection of loops representing the homology basis of the arithmetic hyperbolic fiber surface, we obtain the 3-manifold $S^2 \times S^1$. The Riemannian metrics on $S^2 \times S^1$ are derived after the metric change made by Dehn surgeries.

M. Freedman described the geometric properties of all Dehn surgeries. However, an explicit expression of metrics on $S^2 \times S^1$ is not provided. In the thesis, we show details of how metrics are changed in the Dehn surgery performed on mapping tori, see Chapter 5. We use a cutoff function technique in the gluing procedure of Dehn surgeries to obtain smooth Riemannian metrics on $S^2 \times S^1$. Then following M. Freedman's outline, in terms of the metric property of Dehn surgery, we obtain lower bound estimations for the $\mathbb{Z}_2$-coefficient homology 1-systole and the $\mathbb{Z}_2$-coefficient homology 2-systole, as well as the upper bound estimation of the volume of $S^2 \times S^1$ in terms of the genus of arithmetic hyperbolic surfaces. These techniques will be further employed in Chapter 7 on $\mathbb{RP}^3 \# \mathbb{RP}^3$.

Let $\mathbb{RP}^3 \# \mathbb{RP}^3$ be the 3-manifold of the connected sum of two copies of real projective 3-space, which has a semibundle structure. In the semibundle structure of $\mathbb{RP}^3 \# \mathbb{RP}^3$, the regular fiber surface is the sphere S^2. Moreover, the sphere semibundle $\mathbb{RP}^3 \# \mathbb{RP}^3$ can be doubly covered by the sphere surface bundle $S^2 \times S^1$. We use a way similar to M. Freedman's method to construct a sequence of Riemannian metrics on $\mathbb{RP}^3 \# \mathbb{RP}^3$. Let Σ_g be the arithmetic hyperbolic surface of genus g in M. Freedman's example. If we perform Dehn surgeries on a semibundle with the regular fiber surface Σ_g, then we have the sphere semibundle $\mathbb{RP}^3 \# \mathbb{RP}^3$. On $\mathbb{RP}^3 \# \mathbb{RP}^3$, we employ M. Freedman's technique to estimate the $\mathbb{Z}_2$-coefficient homology 1-systole and $\mathbb{Z}_2$-coefficient homology 2-systole. We get the semibundle back by doing reverse Dehn surgeries on $\mathbb{RP}^3 \# \mathbb{RP}^3$. By an investigation of properties between the semibundle and its covering surface bundle, we show that on a semibundle, the $\mathbb{Z}_2$-coefficient homology 1-systole of the semibundle can be bounded below by the homotopy 1-systole of the covering surface bundle. M. Freedman's method on surface bundles is applied to get the lower bound estimations of the $\mathbb{Z}_2$-coefficient homology 1-systole of $\mathbb{RP}^3 \# \mathbb{RP}^3$. Every nonseparating surface of $\mathbb{RP}^3 \# \mathbb{RP}^3$ is one sided, which is doubly covered by a nonseparating surface in $S^2 \times S^1$. Therefore, the area minimizing nonseparating surface in $\mathbb{RP}^3 \# \mathbb{RP}^3$ is doubly covered by a nonseparating surface in $S^2 \times S^1$. Moreover, in $S^2 \times S^1$ we only have one homology class in $H_2(S^2 \times S^1; \mathbb{Z}_2)$. Then we employ M. Freedman's technique on $S^2 \times S^1$ to get the lower bound estimation of the $\mathbb{Z}_2$-coefficient homology 2-systole of $\mathbb{RP}^3 \# \mathbb{RP}^3$. Based on the above estimations, we establish the $\mathbb{Z}_2$-coefficient homology (1, 2)-systolic freedom for $\mathbb{RP}^3\#\mathbb{RP}^3$, see Chapter 6 and Chapter 7.

The thesis is organized as follows. In Chapter 2, we introduce the definition of homotopy 1-systole, homology k-systoles and stable systole. In Chapter 3, preliminary knowledge of geometric manifolds is given. In Chapter 4, we have a review of systolic inequalities on surfaces. In Chapter 5, we have a further investigation of M. Freedman's theorem of $\mathbb{Z}_2$-coefficient homology (1, 2)-systolic freedom of $S^2 \times S^1$. The metric change made by Dehn surgeries is provided, which is not given in M. Freedman's paper. In Chapter 6, we introduce properties of 3-manifolds with semibundle structure. Two systolic propositions of semibundles are proved in this chapter. In Chapter 7, we prove the $\mathbb{Z}_2$-coefficient homology (1, 2)-systolic freedom of $\mathbb{RP}^3\#\mathbb{RP}^3$.

CHAPTER 2

Definition of systoles

In this chapter, we give definitions of various systolic variants according to homotopy and homology. These definitions will be used later in this thesis. Moreover, in order to be complete, we also introduce the definition of stable systole of manifolds.

2.1 Homotopy 1-systole

Let M be a Riemannian manifold of dimension n with a Riemannian metric $\mathcal{G}$, denoted by $(M, \mathcal{G})$. We further assume that M is compact and non simply connected. Roughly speaking, the homotopy 1-systole of $(M, \mathcal{G})$ is the shortest length of all non-contractible loops in M. For the purpose of being consistent with the definition of homology systoles, we give the following definition of homotopy 1-systole.

We use $\mathrm{length}_{\mathcal{G}}(\gamma)$ to denote the length of a smooth loop γ in $(M, \mathcal{G})$. Given a homotopy class $\alpha \in \pi_1(M)$, we define its length as

$$\mathrm{length}_{\mathcal{G}}(\alpha) = \inf_{\gamma} \mathrm{length}_{\mathcal{G}}(\gamma),$$

where the infimum is over all smooth loops γ representing α.

We define the homotopy 1-systole in terms of the length of nonzero homotopy classes in $\pi_1(M)$.

Definition 2.1.1 *The homotopy* 1*-systole* $Sys\pi_1(M, \mathcal{G})$ *of* $(M, \mathcal{G})$ *is defined as*

$$\inf_{\alpha \in \pi_1(M) \setminus \{0\}} length_{\mathcal{G}}(\alpha)$$

where the infimum is over all nonzero homotopy classes α *in* $\pi_1(M)$.

2.2 Homology systoles

2.2.1 $\mathbb{Z}$-coefficient homology 1-systole

We define the $\mathbb{Z}$-coefficient homology 1-systole in terms of smooth loops representing nonzero $\mathbb{Z}$-coefficient homology classes.

For a homology class $\beta \in H_1(M; \mathbb{Z})$, we define its length as

$$\text{length}_{\mathcal{G}}(\beta) = \inf_{\ell} \text{length}_{\mathcal{G}}(\ell),$$

where the infimum is over all smooth loops representing β.

Definition 2.2.1 *We define $\mathbb{Z}$-coefficient homology $1-$systole $Sys\, H_1(M, \mathcal{G}; \mathbb{Z})$ as*

$$\inf_{\beta \in H_1(M; \mathbb{Z}) \setminus \{0\}} \textit{length}_{\mathcal{G}}(\beta),$$

where the infimum is over all nonzero homology classes β in $H_1(M; \mathbb{Z})$.

2.2.2 $\mathbb{Z}_2$-coefficient homology 1-systole

We define $\mathbb{Z}_2$-coefficient homology 1-systole in terms of smooth loops representing nonzero $\mathbb{Z}_2$-coefficient homology classes.

For a nonzero homology class $\delta \in H_1(M; \mathbb{Z}_2)$, we define its length as

$$\text{length}_{\mathcal{G}}(\delta) = \inf_{\ell} \text{length}_{\mathcal{G}}(\ell),$$

where the infimum is over all smooth loops ℓ representing δ.

Definition 2.2.2 *We define $\mathbb{Z}_2$-coefficient homology 1-systole $Sys\, H_1(M, \mathcal{G}; \mathbb{Z}_2)$ as*

$$\inf_{\delta \in H_1(M, \mathcal{G}; \mathbb{Z}_2) \setminus \{0\}} \textit{length}_{\mathcal{G}}(\delta), \tag{2.1}$$

where the infimum is over all nonzero homology classes δ in $H_1(M; \mathbb{Z}_2)$.

Remark 2.2.1 *On an oriented surface $(\Sigma, \mathcal{G})$, we have*

$$Sys\, H_1(\Sigma, \mathcal{G}; \mathbb{Z}) = Sys\, H_1(\Sigma, \mathcal{G}; \mathbb{Z}_2).$$

However, when Σ is nonoriented, there are examples that homology 1-systoles with different coefficients are not the same, see 2.A. of [22].

In general, for different 1-systoles of a Riemannian manifold $(M, \mathcal{G})$, we have

$$\mathrm{Sys}\, H_1(M, \mathcal{G}; \mathbb{Z}_2) \geqslant \mathrm{Sys}\, H_1(M, \mathcal{G}; \mathbb{Z}) \geqslant \mathrm{Sys}\, \pi_1(M, \mathcal{G}).$$

2.2.3 Higher homology k-systoles

For $1 \leqslant k \leqslant n$, we define the homology k-systole of a Riemannian manifold with dimension n in terms of k-cycles representing nonzero homology classes.

Definition 2.2.3 *A map $f : \mathbb{R}^m \to \mathbb{R}^n$ is called a Lipschitz map if there exists a positive constant C such that*

$$|f(x) - f(y)|_{\mathbb{R}^m} \leqslant C|x - y|_{\mathbb{R}^n},$$

where $|\cdot|_{\mathbb{R}^m}$ and $|\cdot|_{\mathbb{R}^n}$ are standard Euclidean norms.

We have the following theorem of Lipschitz functions.

Theorem 2.2.1 (Rademacher's Theorem, 3.1.6 of [17] or 3.2 of [40]) *A Lipschitz function $f : \mathbb{R}^m \to \mathbb{R}^n$ is differentiable almost everywhere.*

We define a Lipschitz singular k-simplex σ_k in M as the map

$$\sigma_k : \Delta^k \to M,$$

where σ_k is a Lipschitz map if it is composed with a chart map in an atlas of a differentiable structure on M, with Δ^k the standard k-simplex in $\mathbb{R}^k$. In terms of Rademacher's Theorem, we define the volume of σ_k as

$$\mathrm{Vol}_{\mathcal{G}}(\sigma_k) = \int_{\Delta^k} \mathrm{dV}_{\sigma_k^*(\mathcal{G})},$$

where $\mathrm{dV}_{\sigma_k^*(\mathcal{G})}$ is the volume form of the pullback metric $\sigma_k^*(\mathcal{G})$ on Δ^k.

We use R to denote the coefficient ring $\mathbb{Z}$ or $\mathbb{Z}_2$ in homology group. For a singular chain $\sigma = \sum_{i=1}^{k} r_i \sigma_i$, where r_i are elements in the coefficient ring R, we define its volume as

$$\mathrm{Vol}_{\mathcal{G}}(\sigma) = \sum_{i=1}^{k} |r_i| \, \mathrm{Vol}_{\mathcal{G}}(\sigma_k).$$

Definition 2.2.4 *We define the R-coefficient homology k-systole $Sys\,H_k(M, \mathcal{G}; R)$ of a Riemannian manifold $(M, \mathcal{G})$ as*

$$\inf_c Vol_{\mathcal{G}}(c),$$

where the infimum is over all cycles c representing nonzero homology classes in $H_k(M; R)$.

2.3 Stable systoles

For a homology class $\alpha \in H_k(M; \mathbb{Z})$, we define its norm as

$$\|\alpha\| = \inf_c \mathrm{Vol}_{\mathcal{G}}(c),$$

where the infimum runs over all integral cycles representing α. We also define the norm for homology classes in $H_k(M; \mathbb{Z}_2)$ and $H_k(M; \mathbb{R})$ by the same approach.

The integer coefficient homology group $H_k(M; \mathbb{Z})$ can be embedded into the real coefficient homology group $H_k(M; \mathbb{R})$. For $\alpha \in H_k(M; \mathbb{Z})$, we denote its image in $H_k(M; \mathbb{R})$ by $\alpha_{\mathbb{R}}$. The stable norm $\|\alpha\|_s$ of α is defined to be the norm $\|\alpha_{\mathbb{R}}\|$ of $\alpha_{\mathbb{R}}$ in $H_k(M; \mathbb{R})$. We have the following proposition of stable norms.

Proposition 2.3.1 (Gromov, 4.18 of [23]) *The stable norm of $\alpha \in H_k(M; \mathbb{Z})$ satisfies identity*

$$\|\alpha\|_s = \lim_{i \to \infty} \frac{\|i\alpha\|}{i},$$

where i stands for positive integers.

Definition 2.3.1 *We define the stable k-systole $Stsys_k(M)$ of $(M, \mathcal{G})$ by*

$$\inf_{\alpha \in H_k(M;\mathbb{Z}) \setminus \{0\}} \|\alpha\|_s,$$

where the infimum is over all nonzero integral homology classes in $H_k(M; \mathbb{Z})$.

The stable systoles will not be studied in this thesis. For a reference of stable systoles, we can see [8, 23, 33].

CHAPTER 3

Geometric manifolds

3.1 Geometric structures on manifolds

3.1.1 Definition

Let X be a simply connected differentiable manifold. Let G be the group of self diffeomorphisms of X.

Definition 3.1.1 *A differentiable manifold M has an $(X, G)-$geometric structure, if there exists an open covering $\{U_i\}$ on M satisfying:*

1. *For each U_i, there exists an open differmorphism $\varphi_i : U_i \to X$;*
2. *When $i \neq j$ and $U_i \cap U_j \neq \emptyset$, the restriction of $\varphi_i \circ \varphi_j^{-1}$ to $\varphi_j(U_i \cap U_j)$ is an element of G.*

A group Γ acts on manifold M freely if there are no fixed points for each $\gamma \in \Gamma$, i.e., $\{x \in M | \gamma x = x\} = \emptyset$. The action is properly and discontinuously, if for two different compact subsets K, H of M, $\{\gamma \in \Gamma | \gamma H \cap K \neq \emptyset\}$ is finite. When Γ acts on M freely, properly and discontinuously, the quotient space M/Γ is also a manifold.

When a connected complete manifold M has (X, G)-geometric structure, we have the following theorem.

Theorem 3.1.1 (Benedetti and Petronio, [5, Theorem B.1.7]) *The fundamental group of M can be identified with a subgroup of G acting feely and properly discontinuously on M.*

Therefore we know that an (X, G)-manifold M is homeomorphic to X/H, where H is a subgroup of G which can be identified with $\pi_1(M)$. On the other hand, if H is a subgroup of G acting freely and properly discontinuously on M, the quotient manifold X/H has the (X, G)-geometric structure.

3.1.2 Examples of geometric structures

Let $\mathbb{R}^n$ be the Euclidean n-space, S^n be the unit n-sphere, and H^n be the hyperbolic n-space (see Chapter 3 of [43]). There are three standard geometric structures: the Euclidean structure $(\mathbb{R}^n, O(n) \ltimes \mathbb{R}^n)$, the spherical structure $(S^n, O(n+1))$, and the hyperbolic structure $(\mathbb{H}^n, PO(n,1))$, where $O(n)$ is the orthogonal group of $n \times n$ real matrices, $O(n) \ltimes \mathbb{R}^n$ is the semidirect product, and $PO(n,1)$ is the positive Lorentz group (see Chapter 3 of [43]). Properties related to these geometric structures can be found in [5, 43, 48]. On $\mathbb{R}^n$, we have the defined standard Riemannian metric $< x, y >= \Sigma_{i=1}^n x_i y_i$. Under this metric, $O(n) \ltimes \mathbb{R}^n$ is the isometry group. The standard Riemannian metric on S^n is the metric induced from the standard metric of $\mathbb{R}^{n+1}$, and $O(n+1)$ is the isometry group. If on $\mathbb{R}^{n+1}$, we define metric

$$< x, y >_{(n,1)} = -x_1 y_1 + \cdots + x_{n+1} y_{n+1},$$

then $\{x \in \mathbb{R}^{n+1} | < x, x >_{(n,1)} = -1, x_1 > 0\}$ is the hyperboloid model of H^n, with the positive Lorentz group $PO(n,1)$ as the isometry group.

Equipped with the standard Riemannian metrics, $\mathbb{R}^n$ is a Riemannian manifold of sectional curvature 0, S^n is a Riemannian manifold of sectional curvature $+1$, and $\mathbb{H}^n$ has sectional curvature -1. If a manifold M has $(\mathbb{R}^n, O(n) \ltimes \mathbb{R}^n)$-structure, we call M a flat manifold; when M has $(S^n, O(n+1))$-structure, it is called a spherical manifold or elliptic manifold; and if M has $(\mathbb{H}^n, PO(n,1))$-structure, it is called a hyperbolic manifold.

In dimension 2, the above three geometric structures are the only ones. For 3-manifolds, according to Thurston's geometrization theory [46, 47, 48], there are eight geometric structures.

For references of geometric structures of manifolds, we can see [5], [43], [46] and [48].

3.2 Geometric structures of surfaces

A surface is a 2-dimensional manifold. We use the convention that a surface is closed if it is compact and without boundary. We assume that all surfaces are connected in the following.

3.2.1 Topology of surfaces

A closed surface is homeomorphic either to the 2-sphere S^2, or to the connected sum of $n (n \geqslant 1)$ tori, or to the connected sum of $n (n \geqslant 1)$ real projective planes, see [39]. The surface homeomorphic to the connected sum of n tori is orientable and with genus n. The surface homeomorphic to the connected sum of n real projective planes is nonorientable and with genus n.

Assume that S_g is a closed surface of genus g. The Euler characteristic of S_g is given by the formula

$$\chi(S_g) = \sum_{k=0}^{2} \operatorname{rank}(H_k(S_g)).$$

When S_g is orientable, we have $\chi(S_g) = 2-2g$; and $\chi(S_g)$ is $2-g$ if S_g is nonorientable.

3.2.2 Geometric surfaces

There are three geometric structures on a closed orientable surface.

Theorem 3.2.1 (Benedetti and Petronio, [5, Theorem B.3.5.]) *Assume that Σ_g is an orientable surface of genus g.*

1. *If $g = 0$, i.e., Σ_g is homeomorphic to the 2-sphere S^2, Σ_g has spherical geometric structure S^2;*

2. *If $g = 1$, i.e., Σ_g is homeomorphic to the torus $\mathbb{T}^2$, Σ_g has flat geometric structure $\mathbb{R}^2$;*

3. *If $g \geqslant 2$, Σ_g has hyperbolic geometric structure $\mathbb{H}^2$.*

Every non-orientable surface has a two-sheeted orientable covering. The above theorem provides a geometric classification of all closed surfaces. For example, the real projective plane $\mathbb{R}P^2$ is doubly covered by S^2, hence it is spherical; and the Klein bottle K is doubly covered by T^2, thus it is flat. When $g \geqslant 3$, the non-orientable surface S_g is hyperbolic.

On a closed surface, the geometric structure is unique. The uniqueness can be deduced from the Gauss-Bonnet formula, see [5, Proposition B.3.1.].

Theorem 3.2.2 (Gauss-Bonnet formula, [5, Theorem B.3.2.]) *If S_g is a closed surface with Euler characteristic $\chi(S_g)$, then we have*

$$\int_{S_g} \kappa(x)\, dV = -2\pi\chi(S_g), \tag{3.1}$$

where $\kappa(x)$ is the Gaussian curvature at x of S_g.

3.2.3 Hyperbolic surfaces and Fuchsian groups

A hyperbolic surface is a closed surface on which a Riemannian metric with Gaussian curvature -1 is defined. On hyperbolic plane $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 \,|y > 0\}$, we have the Riemannian metric

$$\mathcal{G}_{\mathbb{H}^2} = \frac{1}{y^2}(dx^2 + dy^2)$$

with Gaussian curvature -1, which is called the hyperbolic metric on $\mathbb{H}^2$. The isometry group of $(\mathbb{H}^2, \mathcal{G}_{\mathbb{H}^2})$ is $\mathrm{PSL}(2, \mathbb{R})$, see [30].

A hyperbolic surface Σ has geometric structure $(\mathbb{H}^2, \mathrm{PSL}(2, \mathbb{R}))$, so that $\Sigma = \mathbb{H}^2/\Gamma$, where $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ is called a Fuchsian group.

Definition 3.1 (Fuchsian Group) *A Fuchsian group is a discrete subgroup of $PSL(2, \mathbb{R})$.*

More properties of Fuchsian group can be found in [30].

3.3 Mapping class group

3.3.1 Nielsen-Thurston classification

Let Σ_g be a closed orientable surface with genus g. We use Diff(Σ) to denote the group of orientation preserving self diffeomorphisms on Σ_g, and use $\mathrm{Diff}_0(\Sigma_g)$ to denote the group of self diffeomorphisms which are isotopic to the identity.

Definition 3.3.1 *The mapping class group $\mathcal{M}(\Sigma_g)$ of Σ_g is defined as*

$$\mathsf{Diff}(\Sigma_g)/\,\mathsf{Diff}_0(\Sigma_g).$$

The elements of the mapping class group $\mathcal{M}(\Sigma_g)$ are classified by the Nielsen-Thurston Theorem.

Theorem 3.3.1 (Nielsen-Thurston classification, [16, Theorem 13.2]) *Each $f \in \mathcal{M}(\Sigma_g)$ is in one of the following three types:*

1. *periodic, or finite order;*
2. *reducible;*
3. *pseudo-Anosov.*

Let f be an element in $\mathcal{M}(\Sigma_g)$, and the homeomorphism $\phi : \Sigma_g \to \Sigma_g$ is a representative of f. The 3-manifold with surface bundle over unit circle structure is homeomorphic to the mapping torus

$$M_g = \Sigma_g \times [0,1]/(x,0) \sim (\phi(x),1),$$

or equivalently

$$M_g = \Sigma_g \times \mathbb{R}/(x,t) \sim (\phi(x),t+1).$$

We call f as the monodromy of the surface bundle M_g. Thurston's theorem determines the geometric structure of surface bundle M_g.

Theorem 3.3.2 (Thurston, [49, Proposition 2.6]) *Let $f \in \mathcal{M}(\Sigma_g)$, the mapping torus M_g has monodromy f.*

1. *M_g has geometric structure $\mathbb{H}^2 \times \mathbb{R}$ if and only if f is periodic.*
2. *M_g contains an incompressible torus if and only if f is reducible.*
3. *M_g is hyperbolic if and only if f is pseudo-Anosov.*

3.3.2 Lickorish twist theorem

Let Σ_g be a closed orientable surface of genus g.

Definition 3.3.2 (Dehn Twist) *Suppose γ is a simple loop in Σ_g. The Dehn twist $D_\gamma : \Sigma_g \to \Sigma_g$ is a homeomorphism, which is the operation of cutting along γ, and doing a π rotation of one end, then gluing back. The operation is done in a tubular neighborhood of γ.*

Let S^1 be the unit circle in the complex plane. On annulus $A = S^1 \times [0, 1]$ we define the Dehn twist T along loop $\ell = \left\{ \left(e^{i\theta}, \frac{1}{2}\right) \middle| \, \theta \in [0, 2\pi] \right\}$ as

$$(e^{i\theta}, t) \mapsto (e^{i(\theta+2\pi t)}, t),$$

where $(e^{i\theta}, t) \in S^1$.

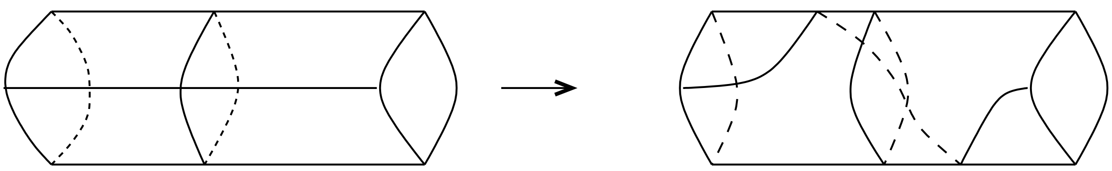

Figure 3.1: Dehn Twist at Annulus

For a simple loop $\gamma \in \Sigma_g$, let $C(\gamma)$ be a regular neighborhood of γ, which is homeomorphic to the annulus A through the homeomorphism $h : C(\gamma) \to A$. The Dehn twist D_γ is isotopic to the homeomorphism $h^{-1} \circ T \circ h$.

The mapping class group can be generated from a finite number of isotopy classes of Dehn twists.

Theorem 3.3.3 (Lickorish, see [16, Theorem 4.13]) *The mapping class group $\mathcal{M}(\Sigma_g)$ of a closed orientable surface Σ_g is generated by the isotopy classes of finitely many Dehn twists along nonseparating simple loops of Σ_g.*

Remark 3.3.1 *The elements of $\mathcal{M}(\Sigma_g)$ can be generated from the isotopy classes of Dehn twists along $3g-1$ nonseparating simple loops.*

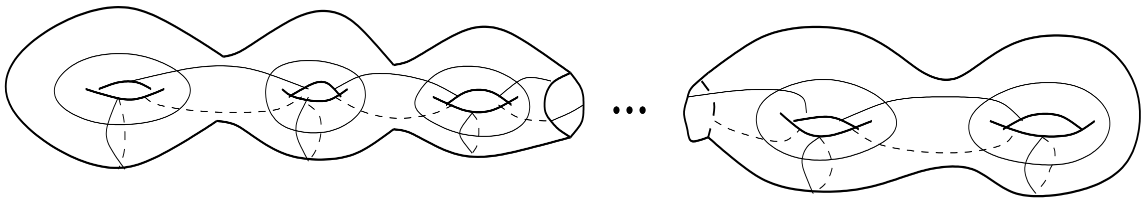

Figure 3.2: Lickorish Twist Theorem

CHAPTER 4

Systolic inequality of surfaces

4.1 Loewner inequality and Pu inequality

4.1.1 Flat torus

A lattice Λ in $\mathbb{R}^n$ is the set of points with the form $\lambda_1 a_1 + \lambda_2 a_2 + \cdots \lambda_n a_n$, where $\{a_1, a_2, \cdots, a_n\}$ is a basis of $\mathbb{R}^n$, and $\lambda_i \in \mathbb{Z}$. Flat torus can be constructed through lattices in $\mathbb{R}^2$.

A flat 2-dimensional torus $\mathbb{T}_0$ has Riemannian metric with zero curvature, and its metric is induced from the Euclidean metric on universal covering space $\mathbb{R}^2$ of $\mathbb{T}_0$. From geometric structure theory of surfaces (see Chapter 3), a flat torus $\mathbb{T}^2$ is homeomorphic to $\mathbb{R}^2/\Gamma$, where Γ is a lattice in $\mathbb{R}^2$. The classification of flat tori is given in terms of the classification of lattices in $\mathbb{R}^2$. The following theorem is from Chapter 2 of [20].

Theorem 4.1 ([20, Chapter 2]) *Two Riemannian metrics $\mathcal{G}_1$ and $\mathcal{G}_2$ on flat tori $\mathbb{R}^2/\Gamma_1$ and $\mathbb{R}^2/\Gamma_2$ respectively are isometric to each other, if there exists an isometry $f : \mathbb{R}^2 \to \mathbb{R}^2$ sending Γ_1 to Γ_2.*

Up to scaling of a positive factor, we have the classification of all lattices in $\mathbb{R}^2$. Assume that the shortest vector in a lattice Γ of $\mathbb{R}^2$ is $e_1 = (1, 0)$. Let z be the shortest vector not proportional to e_1. Each vector w located in

$$\left\{ (x, y) \middle| x \in \left[0, \frac{1}{2}\right], x^2 + y^2 \geqslant 1 \right\}$$

represents a class of lattices. When $w = \left(1/2, \sqrt{3}/2\right)$, the lattice with basis $\{e_1, z\}$

is called a hexagonal lattice, with the corresponding flat torus called flat hexagonal torus.

Theorem 4.2 *If λ is the shortest closed geodesic in a flat torus $\mathbb{T}_0$, then we have*

$$\lambda^2 \leqslant \frac{2}{\sqrt{3}} Area(\mathbb{T}_0),$$

where equality holds if $\mathbb{T}_0$ is a hexagonal torus.

4.1.2 Uniformization theorem of compact Riemann surfaces

Definition 4.1.1 *A Riemann surface Σ is a 1-dimensional complex manifold.*

Let $\mathbb{C}$ be the complex plane. On a Riemann surface Σ, a chart $z : U \to \mathbb{C}$ is a homeomorphism from Σ to $\mathbb{C}$, where U is an open set of Σ. Two charts (U_α, z_α) and (U_β, z_β) are compatible with each other if either $U_\alpha \cap U_\beta = \emptyset$, or $U_\alpha \cap U_\beta \neq \emptyset$ with the transition map $z_\alpha \circ z_\beta^{-1} : z_\beta(U_\alpha \cap U_\beta) \to z_\alpha(U_\alpha \cap U_\beta)$ being holomorphic. The set of collection of all compatible charts is an atlas of Σ. We call an atlas a conformal structure on Σ. A Riemann surface is a two dimensional manifold equipped with a conformal structure.

Theorem 4.3 (Uniformization Theorem, [29, Chapter 4]) *If Σ_g is a compact Riemann surface with genus g, then there exists a conformal diffeomorphism $f : \Sigma \to S$, with S*

1. *a Riemann sphere S^2 if $g = 0$;*
2. *a torus T^2 if $g = 1$;*
3. *a closed surface constructed as $\mathbb{H}^2/\Gamma$ by a discrete subgroup of $PSL(2, \mathbb{R})$, if $g \geqslant 2$.*

4.1.3 Loewner inequality

The investigation of systolic inequality was initiated by Loewner.

Theorem 4.4 (Loewner, see [42]) *For every Riemannian metric $\mathcal{G}$ on a torus $\mathbb{T}^2$,*

$$Sys\pi_1\,(\mathbb{T}^2, \mathcal{G})^2 \leqslant \frac{2}{\sqrt{3}}\, Area_{\mathcal{G}}(\mathbb{T}^2), \tag{4.1}$$

where equality holds on a flat hexagonal torus $\mathbb{T}^2$.

The Loewner inequality can be deduced from the following lemma.

Lemma 4.1 (Croke and Katz [13, (2.4)]) *On a Riemannian torus $(\mathbb{T}^2, \mathcal{G})$, there exists two distinct noncontractible geodesic loops σ_1 and σ_2 such that*

$$length_{\mathcal{G}}(\sigma_1) \cdot length_{\mathcal{G}}(\sigma_2) \leqslant \frac{2}{\sqrt{3}}\, Area_{\mathcal{G}}(\mathbb{T}^2), \tag{4.2}$$

with the homotopy classes of σ_1 and σ_2 forming a generating set for $\pi_1(\mathbb{T}^2)$.

4.1.4 Pu inequality

After Loewner, Pu proved another systolic inequality on a real projective plane $\mathbb{RP}^2$.

Theorem 4.5 (Pu, [42]) *For every Riemannian metric $\mathcal{G}$ on a real projective plane $\mathbb{RP}^2$,*

$$Sys\,\pi_1(\mathbb{RP}^2, \mathcal{G}^2) \leqslant \frac{\pi}{2} Area_{\mathcal{G}}(\mathbb{RP}^2, \mathcal{G}), \tag{4.3}$$

where equality holds if the metric $\mathcal{G}$ has constant Gaussian curvature.

A proof of the Pu inequality is provided in Chapter 6 of [33]. We sketch the main steps here.

Proof. Assume that $\widetilde{\mathbb{RP}^2}$ is the double covering of $\mathbb{RP}^2$, which is homeomorphic to S^2. Let SO(3) be the 3-dimensional special orthogonal group, which is diffeomorphic to the unit tangent space TS^2 of S^2. We have the fibration $q : \mathrm{SO}(3) \to \widetilde{\mathbb{RP}^2}$, with fiber the collection of unit vectors tangent to a great circle on S^2. Each fiber ν is projected to a great circle on the sphere. Hence we consider $\widetilde{\mathbb{RP}^2}$ as the configuration space of oriented great circles on the sphere. We have the following geometric identity

$$\mathrm{Area}_{\mathcal{G}}(S^2) = \frac{1}{2\pi} \int_{\widetilde{\mathbb{RP}^2}} E_{\mathcal{G}}(\nu) d\nu, \tag{4.4}$$

where $E_{\mathcal{G}}(\nu)$ is the energy integration.

Remark 4.1.1 *We have used the duality of Radon transform in identity (4.4), see Proposition 2.2 of [27]. There is another fibration $p : SO(3) \to S^2$, where $SO(2)$ is the fiber, see [27] or [33]. We view $\widetilde{\mathbb{RP}^2}$ and S^2 as homogeneous spaces in duality. The proof of (4.4) is based on the duality between $\mathbb{RP}^2$ and S^2.*

By the uniformization theorem, the Riemannian metric $\mathcal{G}$ on S^2 is conformally equivallent to the standard round metric $\mathcal{G}_0$ with curvature $+1$. So there exists a positive function f, such that $\mathcal{G} = f^2\mathcal{G}_0$. Under this conformal expression, the energy integration becomes

$$E_{\mathcal{G}}(\nu) = \int_{\nu} f^2 dt.$$

Therefore, by the geometric identity (4.4), we can find a great circle ν_0 such that

$$\left(\int_{\nu_0} f dt\right)^2 \leqslant \pi \int_{S^2} f^2 d\sigma. \tag{4.5}$$

If we use L to denote $\mathrm{length}_{\mathcal{G}}(\nu_0)$, (4.5) implies that $L^2 \leqslant \pi \,\mathrm{Area}_{\mathcal{G}}(S^2)$.

For a Riemannian metric $\mathcal{G}$ defined on $\mathbb{RP}^2$, it is lifted by the covering map to a centrally symmetric metric on S^2. Hence we obtain the Pu inequality by (4.5). In (4.5), the identity holds if and only if f is a constant function. And the metric $\mathcal{G}$ has constant curvature if and only if f is a constant function. Therefore, the equality in Pu inequality (4.3) holds if and only if the metric has constant curvature. ■

The systolic inequality on Klein bottle is established by Bavard.

Theorem 4.6 (Bavard [3]) *For every Riemannian metric $\mathcal{G}$ on a Klein bottle $\mathbb{RP}^2\#\mathbb{RP}^2$,*

$$Sys\pi_1(\mathbb{RP}^2\#\mathbb{RP}^2,\mathcal{G})^2 \leqslant \frac{2\sqrt{2}}{\pi} Area_{\mathcal{G}}(\mathbb{RP}^2\#\mathbb{RP}^2), \tag{4.6}$$

where the equality is reached by a metric with sigularity.

4.2 Systolic inequality of hyperbolic surfaces

Gromov [21] proved a systolic inequality for all surfaces with infinite fundamental group, i.e., surfaces which are not homeomorphic to S^2 and $\mathbb{RP}^2$.

Theorem 4.7 (Gromov [21]) *If S is a closed surface with infinite fundamental group, then for every Riemannian metric $\mathcal{G}$ on S,*

$$Sys\pi_1(S,\mathcal{G})^2 \leqslant \frac{4}{3} Area_{\mathcal{G}}(S). \tag{4.7}$$

The proof of this theorem is based on Gromov's estimation of the area of balls with radii controlled by the systole of the surface S. Gromov [21] showed that at every point on S, the ball with radius $\mathrm{Sys}\,\pi_1(S,\mathcal{G})/2$ has volume bounded below by $\frac{3}{4}\left(\mathrm{Sys}\,\pi_1(S,\mathcal{G})\right)^2$.

Combined with the Pu inequality (4.3), for every Riemannian metric $\mathcal{G}$ on a closed surface Σ other than S^2, we have

$$\mathrm{Sys}\,\pi_1(\Sigma,\mathcal{G})^2 \leqslant \frac{\pi}{2}\mathrm{Area}_{\mathcal{G}}(\Sigma), \tag{4.8}$$

where the equality holds for metrics $\mathcal{G}$ with constant Gaussian curvature.

4.3 Systolic inequality of surfaces with large genus

For closed surfaces with large genus, Gromov [21] proved a systolic inequality with better constants.

Theorem 4.3.1 (Gromov [21]) *Let Σ_g be a closed genus g surface other than S^2. For every Riemannian metric $\mathcal{G}$ on Σ_g,*

$$Sys\,\pi_1(\Sigma_g, \mathcal{G})^2 \leqslant \frac{64}{4\sqrt{g}+27}\,Area_{\mathcal{G}}(\Sigma_g). \tag{4.9}$$

The inequality (4.9) implies that when the genus g increases to infinity, the ratio

$$\frac{\mathrm{Area}_{\mathcal{G}}(\Sigma_g)}{\mathrm{Sys}\,\pi_1(\Sigma_g, \mathcal{G})^2}$$

is going to infinity.

The inequality (4.9) is improved further in [22].

Theorem 4.8 (Gromov [22]) *Let Σ_g be a closed surface with genus $g \geqslant 2$. For every Riemannian metric $\mathcal{G}$ on Σ_g,*

$$Sys\,\pi_1(\Sigma_g, \mathcal{G})^2 \leqslant C\,\frac{(\log g)^2}{g}\,Area_{\mathcal{G}}(\Sigma_g), \tag{4.10}$$

where C is a positive constant independent of g.

Remark 4.3.1 *The inequality (4.10) also holds for $\mathbb{Z}_2$-coefficient homology 1-systole, i.e., for every Riemannian metric $\mathcal{G}$ on Σ_g,*

$$Sys\,H_1(\Sigma_g, \mathcal{G}; \mathbb{Z}_2)^2 \leqslant C\,\frac{(\log g)^2}{g}\,Area_{\mathcal{G}}(\Sigma_g),$$

where C is a positive constant independent of g.

4.4 Optimal systolic ratio

For a closed Riemannian surface $(\Sigma, \mathcal{G})$, we define the systolic ratio as

$$\frac{\mathrm{Area}_{\mathcal{G}}(\Sigma)}{\mathrm{Sys}\,\pi_1(\Sigma, \mathcal{G})^2},$$

denoted by $\mathrm{SR}(S, \mathcal{G})$.

Definition 4.4.1 *The optimal systolic ratio of a closed surface* Σ *other than* S^2 *is defined to be*

$$\inf_{\mathcal{G}} SR(M, \mathcal{G}),$$

denoted by $SR(\Sigma)$*, where the infimum is taken over all Riemannian metrics* $\mathcal{G}$ *on* Σ*.*

By Loewner inequality (4.1), for every Riemannian metric $\mathcal{G}$ on a torus $\mathbb{T}^2$,

$$\mathrm{SR}(\mathbb{T}^2, \mathcal{G}) \geqslant \frac{\sqrt{3}}{2}.$$

Hence we have $\mathrm{SR}(\mathbb{T}^2) = \sqrt{3}/2$, with the realized metric flat hexagonal. The Pu inequality (4.3) implies that for every Riemannian metric $\mathcal{G}$ on $\mathbb{RP}^2$,

$$\mathrm{SR}(\mathbb{R}P^2, \mathcal{G}) \geqslant \frac{\pi}{2}.$$

Hence we have $\mathrm{SR}(\mathbb{R}P^3) = 2/\pi$, with the realized metrics of constant Gaussian curvature. Bavard's inequality (4.6) yields that the optimal systolic ratio $\mathrm{SR}(\mathbb{RP}^2\#\mathbb{RP}^2)$ of a Klein bottle $\mathbb{RP}^2\#\mathbb{RP}^2$ is equal to $\pi/2\sqrt{2}$, which is realized by a singular metric. For closed surfaces with nontrivial fundamental group, currently we only know the optimal systolic ratio for $\mathbb{T}^2, \mathbb{RP}^2, \mathbb{RP}^2\#\mathbb{RP}^2$. The inequality (4.8) implies that for a closed surface Σ other than S^2,

$$\mathrm{SR}(\Sigma) \geqslant \frac{\pi}{2}.$$

4.4.1 Optimal systolic ratio of hyperbolic surfaces

Let Σ_g be a closed surface with genus g. Gromov's inequality (4.10) yields that

$$\mathrm{SR}(\Sigma_g) \geqslant C\frac{g}{(\log g)^2},$$

where C is a positive constant independent of g.

Katz and Sabourau [34] showed an improvement to the optimal systolic ratio of closed orientable surfaces with genus g. We introduce their work as follows.

For a Riemannian manifold $(M, \mathcal{G})$, let $\tilde{M}$ be the univeral covering manifold with the induced covering metric $\tilde{\mathcal{G}}$.

Definition 4.4.2 *The volume entropy of a Riemannian manifold* $(M, \mathcal{G})$ *is defined as*

$$h(\mathcal{G}) = \lim_{R \to \infty} \frac{\log\left(Vol_{\tilde{\mathcal{G}}}(B(\tilde{x}, R))\right)}{R},$$

where $B(\tilde{x}, R)$ *is the ball of radius* R *on the Riemannian universal covering manifold* $\tilde{M}$.

Gromov [21] showed that each nonsimply orientable surface Σ_g admits a generalized extremal Riemannian metric $\mathcal{G}_{ex}$, such that $\mathrm{SR}(\Sigma_g, \mathcal{G}_{ex}) = \mathrm{SR}(\Sigma_g)$. Based on M. Gromov's result, Katz and Sabourau [34] proved an upper bound for the volume entropy under the extremal metric.

Theorem 4.9 (Katz and Sabourau [34]) *Let* $(\Sigma_g, \mathcal{G}_{ex})$ *be a closed orientable surface of genus* g, *with the extremal metric* $\mathcal{G}_{ex}$. *Assume that* α *and* β *are two positive numbers such that* $4\alpha + \beta > 0$. *Then we have the following upper bound of the volume entropy with respect to the extremal metric* $\mathcal{G}_{ex}$,

$$h(\mathcal{G}_{ex}) \leqslant -\frac{\log\left(2\alpha^2\, SR(\Sigma_g)^{-1}\right)}{\beta\, SR(\Sigma_g)^{-1}}. \tag{4.11}$$

Katok proved the following lower bound of the volume entropy on a closed Riemannian surface $(\Sigma_g, \mathcal{G})$ with negative Euler characteristic,

$$h(\mathcal{G}) \geqslant \frac{2\pi\chi(\Sigma_g)}{\mathrm{Area}_{\mathcal{G}}(\Sigma_g)},$$

see [34]. If we combine Katok's lower bound with the upper bound estimate (4.11), an estimate of the optimal systolic ratio is obtained.

Theorem 4.10 (Katz and Sabourau [34]) *Suppose* α *and* β *are two positive numbers satisfying* $4\alpha + \beta < \frac{1}{2}$. *On a closed orientable Riemannian surface* $(\Sigma_g, \mathcal{G})$, *we have*

$$\frac{\log^2\left(2\alpha^2\, SR(\Sigma_g)^{-1}\right)}{SR(\Sigma_g)^{-1}} \geqslant 4\pi\beta^2(g-1). \tag{4.12}$$

The inequality (4.12) improves the optimal systolic ratio of closed orientable surfaces.

Theorem 4.11 (Katz and Sabourau [34]) *Let Σ_g be a closed orientable surface with genus g. For every positive constant $\lambda \in (0, \pi)$, we have*

$$SR(\Sigma_g) \geqslant \lambda \frac{g}{(\log g)^2}$$

if g is large enough.

4.4.2 Loewner surface

Definition 4.1 *Let Σ_g be a closed genus g surface other than S^2. A Riemannian metric $\mathcal{G}$ on Σ_g is Loewner if the inequality*

$$Sys\pi_1(\Sigma_g, \mathcal{G})^2 \leqslant \frac{2}{\sqrt{3}} Area_{\mathcal{G}}(\Sigma_g)$$

holds.

The torus is Loewner by the inequality (4.1). By (4.9), Σ_g is Loewner when $g \geqslant 50$.

Katz and Sabourau [35] proved that hyperelliptic surfaces are Loewner.

Definition 4.2 *A hyperelliptic involution J of a Riemann surface Σ_g is a holomorphic involution with $2g+2$ fixed points.*

A Riemann surface with a hyperelliptic involution J is called a hyperelliptic Riemann surface.

Theorem 4.12 (Katz and Sabourau [35]) *Let Σ be an orientable surface. Then for every metric $\mathcal{G}$ in a hyperelliptic conformal class, $(\Sigma, \mathcal{G})$ is Loewner.*

Every genus 2 Riemann surface is hyperelliptic. Hence an orientable surface with genus two is Loewner. Moreover, Katz and Sabourau [34] proved the following theorem.

Theorem 4.13 (Katz and Sabourau [34]) *Let Σ_g be an orientable surface with genus g. Every Riemannian metric $\mathcal{G}$ is Loewner if $g \geqslant 20$.*

The above results imply that for an orientable surface Σ_g with genus g, every Riemannian metric $\mathcal{G}$ on Σ_g is Loewner if $g = 2$ or $g \geqslant 20$; the problem is open if $3 \leqslant g \leqslant 19$.

4.5 A lower bound for systoles of arithmetic hyperbolic surfaces

4.5.1 Arithmetic Fuchsian group

Let $\mathbb{Q}$ be the field of rational numbers. A number field is a field extension over $\mathbb{Q}$. Suppose that $\mathcal{K}$ is a number field, and a quaternion algebra Q over $\mathcal{K}$ is an 4 dimensional vector space over $\mathcal{K}$. We use $1, i, j, k$ to denote the basis of a quaternion algebra Q. By Hilbert symbol, the quaternion algebra Q is expressed as

$$\left(\frac{a, b}{\mathcal{K}}\right),$$

where $a = i^2$, $b = j^2$, and $k = ij = -ji$.

Definition 4.3 (Arithmetic Fuchsian Group) *Assume that $\mathcal{K}$ is a totally real number field. Let A be a quaternion algebra over $\mathcal{K}$ which is ramified at all places except one. Let ρ be an embedding which embeds A into $M_2(\mathbb{R})$. Let $\mathcal{O}$ be an order in A. Let $\mathcal{O}^1$ be the set of elements in $\mathcal{O}$ with reduced norm 1. A subgroup F of $SL(2, \mathbb{R})$ is called an arithmetic Fuchsian group if it is commensurable with $\rho(\mathcal{O}^1)$.*

For a reference of arithmetic Fuchsian group and its properties, see Maclachlan and Reid [38].

4.5.2 Sarnak's result

Let a and b be two integers. Let A be the following quaternion algebra:

$$\left(\frac{a, b}{\mathbb{Q}}\right).$$

An element $X \in A$ is expressed as $X_0 + X_1 i + X_2 j + X_3 k$, where $\{1, i, j, k\}$ is a basis of the quaternion algebra A satisfying $i^2 = a, j^2 = b, ij = k$, and X_0, X_1, X_2, X_3 are elements of $\mathbb{Q}$. We define a reduced norm on A. For $X \in A$, its reduced norm is $N(X) = X_0^2 - X_1^2 a - X_2^2 b + X_3^2 ab$.

Let $\mathcal{O}^1$ be the group of unit norm elements of A. We define an embedding $\rho : \mathcal{O}^1 \to \mathrm{SL}_2(\mathbb{R})$ as

$$X = X_0 + X_1 i + X_2 j + X_3 k \longrightarrow \begin{pmatrix} X_0 + X_1\sqrt{a} & X_2 + X_3\sqrt{a} \\ b(X_2 - X_3\sqrt{a}) & X_0 - X_1\sqrt{a} \end{pmatrix}.$$

We use Γ to denote the embedded group $\rho(\mathcal{O}^1)$. The group Γ is an arithmetic Fuchsian group.

Let p be an odd prime number. We define the arithmetic congruence group $\Gamma(p)$ of Γ as follows

$$\Gamma(p) = \left\{ \rho(X) | \, X = X_0 + X_1 i + X_2 j + X_3 k \in \mathcal{O}^1, X \equiv 1 (\mathrm{mod}\, p) \right\}.$$

If we choose a and b such that $N(X) = 0$ if and only if $X = 0$, then A is a division algebra. Hence $\Sigma = \mathbb{H}^2/\Gamma(p)$ is a closed hyperbolic Riemann surface. Buser and Sarnak [11] proved the following proposition on Σ.

Proposition 4.5.1 (Buser and Sarnak [11])

1. *The genus g_p of $\Sigma_{g_p} = \mathbb{H}^2/\Gamma(p)$ is equal to $p(p-1)(p+1)+1$;*

2. *There exists a positive number c such that $\lambda_1(\Sigma) \geqslant c$, where λ_1 is the first eigenvalue of the Laplacian on Σ.*

A lower bound of the homotopy 1-systole of Σ_g is proved in [11].

Theorem 4.14 (Buser and Sarnak [11]) *If Σ_g is the arithmetic hyperbolic surface constructed in terms of the arithmetic congruence group $\Gamma(p)$, then the homotopy 1-systole satisfies the inequality*

$$Sys\pi_1(\Sigma_g, \mathcal{G}_{\mathbb{H}^2}) \geqslant \frac{4}{3}\log g + C, \tag{4.13}$$

where C is a constant which only depends on a and b, and $\mathcal{G}_{\mathbb{H}^2}$ is a hyperbolic metric on Σ_g.

CHAPTER 5

$\mathbb{Z}_2$-coefficient homology $(1, 2)$-systolic freedom of $S^2 \times S^1$

5.1 Systolic freedom of higher homology k-systoles

Let $(M, \mathcal{G})$ be a Riemannian manifold of dimension n. Let $1 \leqslant k \leqslant n-1$. The homology k-systole of $(M, \mathcal{G})$ is defined as the infimum volume of representatives of nonzero homology classes. With different coefficient groups $\mathbb{Z}$ and $\mathbb{Z}_2$, we have $\mathbb{Z}$-coefficient homology k-systoles and $\mathbb{Z}_2$-coefficient homology k-systoles, see Chapter 2. We use A to denote the coefficient group $\mathbb{Z}$ or $\mathbb{Z}_2$. There are two types of systolic inequalities for higher homology k-systoles. One type is for the homology systole of middle dimension, i.e., if n is even, whether we have

$$\inf_{\mathcal{G}} \frac{\mathrm{Vol}_{\mathcal{G}}(M)}{\mathrm{Sys}\, H_{n/2}(M, \mathcal{G};\, A)^2} > 0,$$

where the infimum is over all Riemannian metrics $\mathcal{G}$ on M. The second type is for homology systoles in a pair of distinct complementary dimensions, i.e., if $n = d_1 + d_2$ and $d_1 \neq d_2$, whether we have

$$\inf_{\mathcal{G}} \frac{\mathrm{Vol}_{\mathcal{G}}(M)}{\mathrm{Sys}\, H_{d_1}(M, \mathcal{G};\, A) \cdot \mathrm{Sys}\, H_{d_2}(M, \mathcal{G};\, A)} > 0,$$

where the infimum is over all Riemannian metrics $\mathcal{G}$ on M.

5.1.1 Systolic freedom of $\mathbb{Z}$-coefficient homology k-systoles

Let $2 \leqslant k \leqslant n-1$. Different from the homotopy or homology 1-systole, the violation of systolic inquality generally exists for $\mathbb{Z}$-coefficient homology k-systoles. We call such a phenomenon systolic freedom. In particular, we have the following definition in a pair of complementary dimensions.

Definition 5.1 *Let M be a manifold of dimension n. Let $1 \leqslant k \leqslant n/2$. The manifold M has $\mathbb{Z}$-coefficient $(k,\, n-k)$-systolic freedom if we have*

$$\inf_{\mathcal{G}} \frac{Vol_{\mathcal{G}}(M)}{Sys\,H_k(M,\,\mathcal{G};\,\mathbb{Z}) \cdot Sys\,H_{n-k}(M,\,\mathcal{G};\,\mathbb{Z})} = 0,$$

where the infimum is over all Riemannian metrics $\mathcal{G}$ on M.

For $\mathbb{Z}_2$-coefficient homology k-systoles, we define the $(k,\, n-k)$-systolic freedom in the same way.

We list some results of systolic freedom for $\mathbb{Z}$-coefficient homology systoles in the following. For a reference of more details and many other results, we can see [1], [2], [6], [18], [31], [32], [36], [41], and section 7.2.3 of [7], Appendix D of [23].

In [6], L. Bergery and M. Katz proved the $\mathbb{Z}$-coefficient homology $(1,\, 2)$-systolic freedom of the 3-manifold $S^2 \times S^1$.

Theorem 5.1.1 (Bergery and Katz, [6]) *The 3-manifold $S^2 \times S^1$ has $\mathbb{Z}$-coefficient $(1,\, 2)$-systolic freedom, i.e, we have*

$$\inf_{\mathcal{G}} \frac{Vol_{\mathcal{G}}(S^2 \times S^1)}{Sys\,H_1(S^2 \times S^1,\,\mathcal{G};\,\mathbb{Z}) \cdot Sys\,H_2(S^2 \times S^1,\,\mathcal{G};\,\mathbb{Z})} = 0,$$

where the infimum is over all Riemannian metrics $\mathcal{G}$ on $S^2 \times S^1$.

Moreover, I. Babenko and M. Katz showed the $\mathbb{Z}$-coefficient $(1,\, n-1)$-systolic freedom on an n-dimensional compact orientable manifold M.

Theorem 5.1.2 (Babenko and Katz, [1]) *Let M be a compact and orientable manifold of dimension n, with $n \geqslant 3$. Then M has $\mathbb{Z}$-coefficient homology $(1,\, n-1)$-systolic freedom, i.e., we have*

$$\inf_{\mathcal{G}} \frac{Vol_{\mathcal{G}}(M)}{Sys\,H_1(M,\,\mathcal{G};\,\mathbb{Z}) \cdot Sys\,H_{n-1}(M,\,\mathcal{G};\,\mathbb{Z})} = 0. \tag{5.1}$$

For systolic freedom of $\mathbb{Z}$-coefficient homology k-systoles involving middle dimension, the following result is proved in [2].

Theorem 5.1.3 (Babenko and Katz and Suciu, [2]) *Let M be a closed orientable manifold of dimension $2m$, where $m \geqslant 3$. If $H_m(M; \mathbb{Z})$ is torsion free, we have*

$$\inf_{\mathcal{G}} \frac{Vol_{\mathcal{G}}(M)}{Sys\, H_m(M, \mathcal{G}; \mathbb{Z})^2} = 0,$$

where the infimum is over all Riemannian metrics $\mathcal{G}$ on M.

5.1.2 Systolic freedom of $\mathbb{Z}_2$-coefficient homology k-systoles

It is mentioned in [18] that M. Gromov conjectured the existence of systolic inequality for $\mathbb{Z}_2$-coefficient homology k-systoles. However, M. Freedman found a counterexample on 3-manifold $S^2 \times S^1$ in 1999. He proved that the 3-manifold $S^2 \times S^1$ has $\mathbb{Z}_2$-coefficient homology (1, 2)-systolic freedom.

Theorem 5.1.4 (Freedman, [18] and [19]) *The 3-manifold $S^2 \times S^1$ exhibits $\mathbb{Z}_2$-coefficient (1, 2)-systolic freedom, i.e., we have*

$$\inf_{\mathcal{G}} \frac{Vol_{\mathcal{G}}(S^2 \times S^1)}{Sys\, H_1(S^2 \times S^1, \mathcal{G}; \mathbb{Z}_2) \cdot Sys\, H_2(S^2 \times S^1, \mathcal{G}; \mathbb{Z}_2)} = 0,$$

where the infimum is over all Riemannian metrics $\mathcal{G}$ on $S^2 \times S^1$.

For the proof of Theorem (5.1.4), a sequence of 3-dimensional Riemannian mapping tori with geometric structure $\mathbb{H}^2 \times \mathbb{R}$ is constructed. And a sequence of 3-manifolds $S^2 \times S^1$ is obtained by performing Dehn surgeries on mapping tori. In the next section, based on M. Freedman's outline of proof, we show the details of metric change in Dehn surgeries. Then we precisely calculate the growth estimation of $\mathbb{Z}_2$-coefficient 1-systoles and $\mathbb{Z}_2$-coefficient 2-systoles as well as the volume in terms of the sequence of smooth Riemannian metrics constructed on $S^2 \times S^1$. Then we get the $\mathbb{Z}_2$-coefficient (1, 2)-systolic freedom of $S^2 \times S^1$.

5.2 $\mathbb{Z}_2$-coefficient homology (1, 2)-systolic freedom of $S^2 \times S^1$

5.2.1 Arithmetic hyperbolic surfaces

The arithmetic Fuchsian group is defined in Section 4.5. Let Γ be a Fuchsian group derived from the quaternion algebra

$$\left(\frac{p, -1}{\mathbb{Q}}\right),$$

where p is a prime number such that $p \equiv 3 \pmod 4$. After being embedded into $\mathrm{PSL}(2, \mathbb{R})$, the Fuchsian group Γ can be expressed as

$$\Gamma = \left\{ \left(\begin{array}{cc} a + b\sqrt{p} & -c + d\sqrt{p} \\ c + d\sqrt{p} & a - b\sqrt{p} \end{array} \right) \middle| \, a, b, c, d \in \mathbb{Z}, \det = 1 \right\} \Big/ \pm \mathrm{I},$$

where I is the 2×2 unit matrix, see [45]. We use $\Gamma(N)$ to denote the N-th congruence subgroup of Γ, which is defined as

$$\Gamma(N) = \left\{ \left(\begin{array}{cc} 1 + N(a + b\sqrt{p}) & N(-c + d\sqrt{p}) \\ N(c + d\sqrt{p}) & 1 + N(a - b\sqrt{p}) \end{array} \right) \middle| \, a, b, c, d \in \mathbb{Z}, \det = 1 \right\} \Big/ \pm \mathrm{I},$$

where $N \geqslant 2$ is an integer. In the following, we use $\mathbb{H}^2$ to denote the hyperbolic plane. We use the arithmetic Fuchsian group $\Gamma(N)$ to construct an arithmetic hyperbolic surface $\mathbb{H}^2/\Gamma(N)$. Some properties of the hyperbolic surface $\mathbb{H}^2/\Gamma(N)$ are listed as follows. The verification of these properties can be found in [18] and [45].

Proposition 5.2.1 ([45]) *1. $\mathbb{H}^2/\Gamma(N)$ is a closed hyperbolic surface.*

2. *For every element $g \in \Gamma(N)$, we have*

$$|tr(g)| \geqslant N^2 - 2,$$

where $tr(g)$ denotes the trace of g.

3. *Let g_N be the genus of $\mathbb{H}^2/\Gamma(N)$. We have the following estimation*

$$A_p N^2 \leqslant g_N \leqslant B_p N^3,$$

where A_p and B_p are fixed constant numbers which only depend on p.

Let g be a hyperbolic isometry. The translation length of g is defined as

$$T = \inf_{z \in \mathbb{H}^2} \rho(z, g(z)),$$

where ρ is the distance induced by the hyperbolic metric. The relation between translation length and the trace is provided in the following Lemma.

Lemma 5.2.1 ([4]) *Let g be a hyperbolic isometry with the translation length T. If tr is the trace of g, we have*

$$\frac{1}{2}\,|tr| = \cosh \frac{T}{2}.$$

Hence if Σ_g is the arithmetic hyperbolic surface defined by the quotient $\mathbb{H}^2/\Gamma(N)$, we have

$$\log g \geqslant C \log N,$$

where C is a constant independent of N. By the previous lemma, we can show the following proposition.

Proposition 5.2.2 *Assume that Σ_g is the genus g arithmetic hyperbolic surface defined as $\mathbb{H}^2/\Gamma(N)$.*

1. *The homotopy 1-systole of Σ_g satisfies*

   $$Sys\,\pi_1(\Sigma_g, \mathcal{G}_{\mathbb{H}^2}) \geqslant c_1 \log g,$$

   where c_1 is a positive constant which does not depend on the genus g, and $\mathcal{G}_{\mathbb{H}^2}$ is the hyperbolic metric on Σ_g.

2. *The first eigenvalue λ_1 of the Laplacian on Σ_g satisfies*

   $$\lambda_1(\Sigma_g) \geqslant c_2,$$

   where c_2 is a positive constant which does not depend on g.

Remark 5.2.1 *In fact, we can show that* $Sys\,\pi_1(\Sigma_g, \mathcal{G}_{\mathbb{H}^2}) = \mathcal{O}(\log g)$, *i.e., there exists positive constants* c_1, c_1' *which do not depend on the genus* g, *such that*

$$c_1 \, \log g \leqslant Sys\,\pi_1(\Sigma_g, \, \mathcal{G}_{\mathbb{H}^2}) \leqslant c_1' \, \log g.$$

Let $N \to \infty$, we get a sequence of arithmetic hyperbolic surfaces

$$\{\Sigma_{g_k}\}_{k=1}^{\infty},$$

such that $\{g_k\}_{k=1}^{\infty}$ is a strictly increasing sequence without upper bound.

5.2.2 Riemannian mapping torus

Let Σ_g be a surface in the sequence $\{\Sigma_{g_k}\}_{k=1}^{\infty}$, and Σ_h is another one in the sequence such that

$$\log g \geqslant (\log h)^2 .$$

Assume that γ is the shortest closed geodesic in Σ_h. We use $[\gamma]$ to denote the homotopy class in $\pi_1(\Sigma_h)$ represented by the loop γ. When h is chosen large enough, we have

$$\mathrm{Sys}\,\pi_1(\Sigma_h, \mathcal{G}_{\mathbb{H}^2}) \leqslant c_1' \log h \leqslant c_1' \, (\log g)^{1/2} < c_1 \log g.$$

Hence the class $[\gamma] \in \Gamma(N_h)$ is not an element of $\Gamma(N_g)$, where we assume that $\Sigma_h = \mathbb{H}^2/\Gamma(N_h)$ and $\Sigma_g = \mathbb{H}^2/\Gamma(N_g)$. Let $\tau : \Sigma_g \to \Sigma_g$ be the self isometry map on Σ_g defined by the covering translation of $[\gamma]$. We have the estimation for the order of τ as follows.

Proposition 5.2.3 *There exists a constant* c_3 *independent of* g *such that*

$$Order\,(\tau) \geqslant c_3 \, (\log g)^{1/2} .$$

Moreover, we have an estimation for the homotopy 1-systole of the quotient surface $S_g = \Sigma_g / < \tau >$. Let $\mathcal{G}_{\mathbb{H}^2}$ be the covering metric. We have a hyperbolic metric $\mathcal{G}'_{\mathbb{H}^2}$ on S_g.

Proposition 5.2.4 *There exists a positive constant* c_4 *which is independent of* g *such that*

$$Sys\, \pi_1(S_g,\, \mathcal{G}'_{\mathbb{H}^2}) \geqslant c_4\, (\log g)^{1/2}\,. \tag{5.2}$$

Letting the genus increase, we can find a subsequence of $\{\Sigma_{g_k}\}$, so that on each surface in the subsequence we construct a finite order isometry τ_k satisfying Proposition 5.2.3 and Proposition 5.2.4. Without any confusion, we still denote this subsequence by

$$\{(\Sigma_{g_k}, \tau_k)\}_{k=1}^{\infty},$$

where $\tau_k : \Sigma_{g_k} \to \Sigma_{g_k}$ is the finite order isometry map constructed as above. For each hyperbolic surface Σ_{g_k}, we construct a Riemannian mapping torus M_{g_k} with the monodromy represented by τ_k as follows

$$M_{g_k} = \Sigma_{g_k} \times [0,\, 1]/(x,\, 0) \sim (\tau_k(x),\, 1).$$

As τ_k has finite order, according to Theorem 3.3.2 the mapping torus M_{g_k} has the geometric structure $\mathbb{H}^2 \times \mathbb{R}$. Then we have a Riemannian metric $\mathcal{G}_k$ defined on M_{g_k}, which is locally isometric to the standard product metric on $\mathbb{H}^2 \times \mathbb{R}$.

Hence we have a sequence of Riemannian 3-manifolds $\{(M_{g_k},\, \mathcal{G}_k)\}_{k=1}^{\infty}$, where $2 \leqslant g_1 < g_2 < \cdots < g_k < \cdots$, and $\lim\limits_{k\to\infty} g_k = \infty$.

5.2.3 Dehn surgery

Let S be a compact subset of a manifold. In the following, we use S° to denote the interior of S.

Definition 5.2.1 *Let* M *be a compact* 3*-manifold. Let* K *be a knot in* M. *Let* $\mathcal{N}(K)$ *be a tubular neighborhood of* K, *with the boundary torus* $\partial\mathcal{N}$. *Let* γ *be a loop on the boundary torus* $\partial\mathcal{N}$. *The Dehn surgery around* K *is an operation containing the following two procedures:*

1. *Remove the tubular neighborhood $\mathcal{N}(K)$ from M.*

2. *Glue in a solid torus T to $M \setminus (\mathcal{N}(K))^{\circ}$, such that the boundary torus ∂T is glued with $\partial\mathcal{N}$ by a homeomorphism $\varphi : \partial T \to \partial\mathcal{N}$, and the meridian loop μ of ∂T is glued with γ. This step is also called Dehn filling.*

After Dehn surgery, we will have a new 3-manifold

$$M' = (M - \mathcal{N}(K)^{\circ}) \cup_{\varphi} T,$$

where $\varphi : \partial T \to \partial\mathcal{N}$ is a homeomorphism, and $\varphi(\mu) = \gamma$.

Remark 5.2.2 *More generally, the Dehn surgery can be defined along a link. A link is a disjoint union of knots in a 3-manifold. In the link case, we need to glue in a union of solid tori to the link complement, see [44].*

We perform a series of Dehn surgeries on the mapping torus M_{g_k} to get a 3-manifold homeomorphic to $S^2 \times S^1$. By the Lickorish twist theorem (see Theorem 3.3.3), $\tau_k^{-1} = \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_{n_k}$, where $\sigma_1, \sigma_2, \cdots, \sigma_{n_k}$ are Dehn twists along $3g_k - 1$ nonseparating simple loops of Σ_{g_k}. For each σ_i, we perform a Dehn surgery to finish the twist. After all of these n_k Dehn surgeries, the monodromy of the obtained mapping torus M'_{g_k} is represented by $\tau_k^{-1} \circ \tau_k$, which is the identity. Then we have

$$M'_{g_k} = \Sigma_{g_k} \times [0, 1]/(x, 0) \sim (\tau_k^{-1} \circ \tau_k (x), 1),$$

which is homeomorphic to $\Sigma_{g_k} \times S^1$.

Assume that $\{\lambda_1, \mu_1, \lambda_2, \mu_2 \cdots, \lambda_{g_k}, \mu_{g_k}\}$ is a system of loops which represent a homology basis of $H_1(\Sigma_{g_k}; \mathbb{Z})$. We perform Dehn surgeries around these loops. In each Dehn surgery, the meridian curve of the glued in solid torus is glued with the loop λ_i or μ_i. After these additional $2g_k$ Dehn surgeries, λ_i and μ_i become contractible loops in the 3-manifold M''_{g_k} obtained. By Stalling's fibration theorem, the 3-manifold M''_{g_k} is a surface bundle over circle with fiber surface the sphere S^2, i.e., M''_{g_k} is homeomorphic to $S^2 \times S^1$. We use $S^2 \times S^1_{g_k}$ to denote M''_{g_k} in the following.

For convenience to get growth estimations of $\mathbb{Z}_2$-coefficient homology systoles, we let all Dehn surgeries be performed at different surface levels in the mapping torus M_{g_k}. Then all glued in solid tori of Dehn surgeries are pairwise disjoint. We will restrict these $n_k + 2g_k$ Dehn surgeries at surface levels in (1/2, 1), see [18].

Hence, after $n_k + 2g_k$ Dehn surgeries, we have a sequence of 3-manifolds

$$\{S^2 \times S^1_{g_k}\}_{k=1}^{\infty},$$

where the index sequence $\{g_k\}$ is strictly increasing and $\lim\limits_{k\to\infty} g_k = \infty$. We will define a smooth Riemannian metric $\hat{\mathcal{G}}_k$ on $S^2 \times S^1_{g_k}$ in terms of the metric change happened in Dehn surgeries.

5.2.4 Metric change in Dehn surgeries

Recall that on the mapping torus M_{g_k} we have a Riemannian metric $\mathcal{G}_k$. A smooth Riemannian metric $\hat{\mathcal{G}}_k$ on $S^2 \times S^1_{g_k}$ is defined after the change of $\mathcal{G}_k$ in Dehn surgeries. We show how the metric $\mathcal{G}_k$ changes during Dehn surgeries on M_{g_k} in the following. First we express the metric $\mathcal{G}_k$ in terms of Fermi coordinates. Then we use cutoff function technique to obtain the smooth Riemannian metric $\hat{\mathcal{G}}_k$ after all Dehn surgeries. In each Dehn surgery, we remove a solid torus in the drilling procedure. Then another solid torus is filled in during the filling procedure. We will show metric changes in terms of these two steps.

Fermi coordinates Let γ be a closed geodesic in the hyperbolic surface Σ_{g_k}. A point x in the collar neighborhood of γ can be expressed by the Fermi coordinate $(t,\, \rho)$, where the variable ρ stands for the distance from x to γ which is equal to the length of perpendicular geodesic arc from x to γ. If we have the unit speed parametrization on γ, denoted by $\gamma(s)$, then the coordinate t is equal to the value of s, such that the perpendicular geodesic arc from x to γ intersects with γ at the point $\gamma(s)$. Under the Fermi coordinate, the hyperbolic metric on $\mathbb{H}^2$ is expressed as $ds^2 = \cosh^2\rho\, dt^2 + d\rho^2$.

We give an expression of the metric $\mathcal{G}_k$ on M_{g_k} in terms of the above Fermi coordinate. The metric $\mathcal{G}_k$ is locally isometric to the product metric on $\mathbb{H}^2 \times \mathbb{R}$. If we use h to denote the $\mathbb{R}$ direction coordinate, the metric $\mathcal{G}_k$ on M_{g_k} has the following expression:

$$ds^2 = \cosh^2\rho\, dt^2 + d\rho^2 + dh^2.$$

Drilling As mentioned above, by the Lickorish twist theorem (see Theorem 3.3.3), we have $\tau_k^{-1} = \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_{n_k}$, where each σ_i is a Dehn twist along a nonseparating simple loop γ_i of Σ_{g_k}. We further assume that γ_i is a geodesic loop.

We use $\mathscr{D}_i$ to denote the Dehn surgery corresponding to the geodesic loop γ_i. In the drilling step of $\mathscr{D}_i$, we remove a solid torus T_{i,ε_k} of radius ε_k from M_{g_k}. Moreover, for the purpose to control geometric properties after Dehn surgeries, we assume that the radius ε_k is equal to $\frac{1}{g_k^2}$, which is small enough when k is sufficiently large. The solid torus T_{i,ε_k} is a tubular neighborhood of a geodesic loop in M_{g_k}, which can be expressed as

$$T_{i,\varepsilon_k} = \{(r,\theta,t)\,|0 \leqslant r \leqslant \varepsilon_k,\ 0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant t \leqslant L_{i,k}\}\,/\sim,$$

where $L_{i,k}$ stands for the length of the geodesic loop γ_i. Moreover, we assume that the longitude loop of T_{i,ε_k} is γ_i. In terms of Fermi coordinate, the metric $\mathcal{G}_k$ restricted to T_{i,ε_k} is expressed by

$$ds^2 = dr^2 + r^2\, d\theta^2 + \cosh^2(r\cos\theta)\, dt^2.$$

After the drilling step to remove out a solid torus in M_{g_k}, we have a 3-manifold $M'_{g_k} = M_{g_k} - (T_{i,\varepsilon_k})^\circ$ with torus boundary $\partial T_{i,\varepsilon_k}$.

Dehn filling Let $\delta_k = \varepsilon_k/4$. We fill in a solid torus $\tilde{T}_{i,\varepsilon_k+\delta_k}$ with radius $(L_{i,k}/2\pi)+\delta_k$ to M'_{g_k}. We define the solid torus $\tilde{T}_{i,\varepsilon_k+\delta_k}$ as follows.

First we define the solid torus $\bar{T}_{i,\,\varepsilon_k+\delta_k}$ with radius $(L_{i,\,\varepsilon_k}/2\pi)+\delta_k$ as

$$\left\{(r,\,\theta,\,t)\,\middle|\,0\leqslant r\leqslant\frac{L_{i,\,k}}{2\pi}+\delta_k,\,0\leqslant\theta\leqslant 2\pi,\,0\leqslant t\leqslant 2\pi\varepsilon_k\right\}\Big/\sim,$$

where $\sim$ is the identification $(r,\,\theta,\,0)\equiv(r,\,\theta,\,2\pi\varepsilon_k)$. The Riemannian metric $\bar{\mathcal{G}}_{i,\,k}$ on $\bar{T}_{i,\,\varepsilon_k+\delta_k}$ is defined to be the Euclidean one $ds^2=dr^2+r^2d\theta^2+dt^2$, which is rotationally symmetric. Moreover, in the following we use $\bar{T}_{i,\,\varepsilon_k}$ to denote the solid torus with the radius $L_{i,\,\varepsilon_k}/2\pi$, which is a subset of $\bar{T}_{i,\,\varepsilon_k}$, expressed in coordinates as

$$\left\{(r,\,\theta,\,t)\,\middle|\,0\leqslant r\leqslant\frac{L_{i,\,k}}{2\pi},\,0\leqslant\theta\leqslant 2\pi,\,0\leqslant t\leqslant 2\pi\varepsilon_k\right\}\Big/\sim,$$

where $\sim$ is the identification $(r,\,\theta,\,0)\equiv(r,\,\theta,\,2\pi\varepsilon_k)$. We will use $\partial\bar{T}_{i,\,\varepsilon_k}$ to denote the boundary torus of $\bar{T}_{i,\,\varepsilon_k}$.

In Dehn surgery $\mathscr{D}_i$, we twist the solid torus $\bar{T}_{i,\,\varepsilon_k+\delta_k}$ before gluing it to M'_{g_k}. We define the twisting map $\beta_{i,\,k}:\bar{T}_{i,\,\varepsilon_k+\delta_k}\to\bar{T}_{i,\,\varepsilon_k+\delta_k}$ as

$$\beta_{i,\,k}(r,\theta,t)=\begin{cases}\left(r,\theta+\frac{\pi}{\varepsilon_k}\left(t-\pi\varepsilon_k+\varepsilon_k\right),t\right), & \text{if } \pi\varepsilon_k-\varepsilon_k\leqslant t\leqslant\pi\varepsilon_k+\varepsilon_k,\\ (r,\theta,t), & \text{otherwise.}\end{cases}$$

The twisting map $\beta_{i,\,k}$ is continuous but not differentiable at points

$$(r,\theta,t)\in\bar{T}_{i,\,\varepsilon_k+\delta_k}$$

with $t=(\pi-1)\varepsilon_k$ or $t=(\pi+1)\varepsilon_k$. Hence in order to obtain a smooth Riemannian metric after twisting by the pullback, we use the standard mollifier to smooth $\beta_{i,\,k}$. The standard mollifier φ is defined as

$$\varphi(x)=\begin{cases}C\cdot\exp\left(\frac{1}{x^2-1}\right) & \text{if } |x|<1,\\ 0 & \text{if } |x|>1,\end{cases}$$

where $x\in\mathbb{R}^3$ and C is a positive constant such that

$$\int_{\mathbb{R}^3}\varphi(x)dx=1.$$

Moreover, we define

$$\varphi_a(x) = \frac{1}{a}\varphi\left(\frac{x}{a}\right),$$

which has compact support in the closure of the ball

$$B(0,\, a) = \{x \in \mathbb{R}^3 |\ \|x\| \leqslant a\}$$

with a a small positive constant satisfying $0 < a < 1$.

Then let

$$\beta^{(2)}_{i,\,k}(r,\theta,t) = \begin{cases} \theta + \frac{\pi}{\varepsilon_k}\left(t - \pi\varepsilon_k + \varepsilon_k\right) & \text{if } \pi\varepsilon_k - \varepsilon_k \leqslant t \leqslant \pi\varepsilon_k + \varepsilon_k, \\ \theta & \text{otherwise.} \end{cases}$$

We define

$$\widetilde{\beta}^{(2)}_{i,\,k}(x) = \int_{\mathbb{R}^3} \varphi_a(x-y)\beta^{(2)}_{i,\,k}(y)dy,$$

where $x = (r,\, \theta,\, t) \in \bar{T}_{i,\,\varepsilon_k+\delta_k}$. We further define a smooth cutoff function ψ on $\bar{T}_{i,\,\varepsilon_k+\delta_k}$ as

$$\psi(r,\,\theta,\,t) = \begin{cases} 1, & \text{if } t \in \left[\pi\varepsilon_k - \frac{3\varepsilon_k}{2}, \pi\varepsilon_k - \frac{\varepsilon_k}{2}\right] \cup \left[\pi\varepsilon_k + \frac{\varepsilon_k}{2}, \pi\varepsilon_k + \frac{3\varepsilon_k}{2}\right]; \\ \in (0,1), & \text{if } t \in \left(\pi\varepsilon_k - \frac{7\varepsilon_k}{4}, \pi\varepsilon_k - \frac{3\varepsilon_k}{2}\right) \cup \left(\pi\varepsilon_k - \frac{\varepsilon_k}{2}, \pi\varepsilon_k - \frac{\varepsilon_k}{4}\right) \\ & \cup \left(\pi\varepsilon_k + \frac{\varepsilon_k}{4}, \pi\varepsilon_k + \frac{\varepsilon_k}{2}\right) \cup \left(\pi\varepsilon_k + \frac{3\varepsilon_k}{2}, \pi\varepsilon_k + \frac{7\varepsilon_k}{4}\right); \\ 0 & \text{otherwise.} \end{cases}$$

Then we define the smooth twisting map $\tilde{\beta}_{i,\,k}$ on $\bar{T}_{i,\,\varepsilon_k+\delta_k}$ as

$$\tilde{\beta}_{i,\,k}(r,\,\theta,\,t) = \left(r,\ (1-\psi)\,\widetilde{\beta}^{(2)}_{i,\,k}(r,\,\theta,\,t) + \psi\,\widetilde{\beta}^{(2)}_{i,\,k}(r,\,\theta,\,t),\ t\right),$$

which is differentiable. The map $\tilde{\beta}_{i,\,k}$ is equal to $\beta_{i,\,k}$ except at small neighborhoods of the set of points with $t = \pi\varepsilon_k - \varepsilon_k$ or the set of points with $t = \pi\varepsilon_k + \varepsilon_k$.

Now we let the smooth twisting map $\tilde{\beta}_{i,\,k}$ act on the solid torus $\bar{T}_{i,\,\varepsilon_k+\delta_k}$. We define

$$\tilde{T}_{i,\,\varepsilon_k+\delta_k} = \tilde{\beta}_{i,\,k}\left(\bar{T}_{i,\,\varepsilon_k+\delta_k}\right).$$

The Riemannian metric $\tilde{\mathcal{G}}_{i,\,k}$ on $\tilde{T}_{i,\,\varepsilon_k+\delta_k}$ is then defined to be the pullback $\left(\tilde{\beta}_{i,\,k}^{-1}\right)^{*}\bar{\mathcal{G}}_{i,\,k}$. Let

$$\bar{m}_{i,\,k} = \left\{ \left(\frac{L_{i,\,k}}{2\pi},\, \theta,\, \pi\varepsilon_k \right) \middle| \, 0 \leqslant \theta \leqslant 2\pi \right\}$$

be the meridian loop of the boundary torus $\partial\bar{T}_{i,\,\varepsilon_k}$. We use $\tilde{m}_{i,\,k}$ to denote the meridian loop $\tilde{\beta}_{i,\,k}(\bar{m}_{i,\,k})$ of the boundary torus $\partial\tilde{T}_{i,\,\varepsilon_k}$. From the expression of the twisting map $\tilde{\beta}_{i,\,k}$, we can see that the meridian loop $\tilde{m}_{i,\,k}$ is obtained from the π-rotation of $\bar{m}_{i,\,k}$. Both of $\bar{m}_{i,\,k}$ and $\tilde{m}_{i,\,k}$ have the same length, because the metric $\bar{\mathcal{G}}_{i,\,k}$ is rotationally symmetric, and the twisting map $\tilde{\beta}_{i,\,k}$ is an isometry.

In the second step of Dehn surgeries, that is, for those Dehn surgeries performed around $2g_k$ geodesic loops which are representatives of a homology basis of $H_1(\Sigma_{g_k};\,\mathbb{Z})$, we don't need to twist the filled in solid tori. Hence we have $\tilde{T}_{i,\,\varepsilon_k+\delta_k} = \bar{T}_{i,\,\varepsilon_k+\delta_k}$ and $\tilde{\mathcal{G}}_{i,\,k} = \bar{\mathcal{G}}_{i,\,k}$. Also the meridian loop $\tilde{m}_{i,\,k}$ of $\tilde{T}_{i,\,\varepsilon_k}$ is identical with the meridian loop $\bar{m}_{i,\,k}$ of $\bar{T}_{i,\,\varepsilon_k}$. The following gluing procedures are the same for both types of Dehn surgeries.

Next we glue the replacement solid torus $\tilde{T}_{i,\,\varepsilon_k+\delta_k}$ to M'_{g_k}. The gluing map is $f_{i,\,k} : \tilde{Y}_{i,\,k} \to Y_{i,k}$, where $\tilde{Y}_{i,\,k} = \tilde{\beta}_{i,\,k}(\bar{Y}_{i,\,k})$, with

$$\bar{Y}_{i,\,k} = \left\{ (r,\,\theta,\,t) \,\middle|\, \frac{L_{i,k}}{2\pi} \leqslant r \leqslant \frac{L_{i,k}}{2\pi} + \delta_k,\, 0 \leqslant \theta \leqslant 2\pi,\, 0 \leqslant t \leqslant 2\pi\varepsilon_k \right\} \Big/ \sim,$$

and

$$Y_{i,\,k} = \{\, (r,\,\theta,\,t) |\, \varepsilon_k \leqslant r \leqslant \varepsilon_k + \delta_k, 0 \leqslant \theta \leqslant 2\pi, 0 \leqslant t \leqslant L_{i,k} \} \,/ \sim .$$

In the above, $\bar{Y}_{i,\,k}$ is an annulus product in $\bar{T}_{i,\,\varepsilon_k+\delta_k}$, while $Y_{i,\,k}$ is an annulus product in M'_{g_k} which is homeomorphic to $\partial T_{i,\,\varepsilon_k} \times [0,\,\delta_k]$. The gluing map $f_{i,\,k}$ is defined as

$$(r,\,\theta,\,t) \mapsto \left(\frac{L_{i,\,k}}{2\pi}\theta_k,\, \frac{1}{\varepsilon_k}t,\, r + \varepsilon_k - \frac{L_{i,k}}{2\pi} \right).$$

After gluing, the meridian loop $\tilde{m}_{i,\,k}$ is glued to the longitude loop

$$\gamma_i = \{\, (\varepsilon_k,\,\pi,\,t) |\, 0 \leqslant t \leqslant L_{i,k} \}$$

on the boundary torus $\partial T_{i,\varepsilon_k}$, i.e. $f_{i,k}(\tilde{m}_{i,k}) = \gamma_i$. In the second step of Dehn surgeries, the longitude loop γ_i is from the set of loops λ_i or μ_i which represent the basis of $H_1(\Sigma_{g_k};\mathbb{Z})$.

We use M''_{g_k} to denote the manifold after gluing, which is the 3-manifold

$$(M_{g_k} - (T_{i,\varepsilon_k})^\circ) \cup_{f_{i,k}} \tilde{T}_{i,\varepsilon_k+\delta_k}.$$

In order to have a smooth metric after gluing, we use a smooth cutoff function to attach two metrics together. We define the smooth cutoff function $\alpha_{i,k}$ on M''_{g_k} as follows:

$$\alpha_{i,k}(x) = \begin{cases} 0, & \text{if } x \in M'_{g_k}, \\ \in (0,1), & \text{if } x \in \left(\hat{Y}_{i,k}\right)^\circ, \\ 1, & \text{if } x \in \tilde{T}_{i,\varepsilon_k}, \end{cases}$$

where $\hat{Y}_{i,k} = Y_{i,k} \cup_{f_{i,k}} \tilde{Y}_{i,k}$. Under the cutoff function $\alpha_{i,k}$, we define the Riemannian metric on M''_{g_k} as

$$\mathcal{G}''_k = \begin{cases} \mathcal{G}_k & \text{if restricted to } M'_{g_k}, \\ (1-\alpha_{i,k})\mathcal{G}_k + \alpha_{i,k}\left(f_{i,k}^{-1}\right)^* \tilde{\mathcal{G}}_{i,k} & \text{if restricted to } \hat{Y}_{i,k}, \\ \tilde{\mathcal{G}}_{i,k} & \text{if restricted to } \tilde{T}_{i,\varepsilon_k}. \end{cases}$$

After finishing all Dehn surgeries around $\gamma_1, \gamma_2, \cdots, \gamma_{g_k}$, we have a new 3-manifold which is homeomorphic to $\Sigma_{g_k} \times S^1$. And when we finish each Dehn surgery $\mathscr{D}_i$, a cutoff function $\alpha_{i,k}$ is used to define a smooth Riemannian metric on the obtained 3-manifold.

In the Dehn surgery around loops λ_i and μ_i which represent a homology basis of $H_1(\Sigma_{g_k};\mathbb{Z})$, the procedure is similar to the previous ones. First we remove out a solid torus T_{i,ε_k} around λ_i or μ_i, then glue in a solid torus $\tilde{T}_{i,\varepsilon_k+\delta_k}$. Here we don't use twisting to obtain the filled in solid torus $\tilde{T}_{i,\varepsilon_k+\delta_k}$. We use a cutoff function to obtain the smooth Riemannian metric in the gluing of Dehn surgeries. As mentioned above, after these additional $2g_k$ number of Dehn surgeries, we have a 3-manifold homeomorphic to $S^2 \times S^1$, denoted by $S^2 \times S^1_{g_k}$. The Riemannian metric $\hat{\mathcal{G}}_k$ on $S^2 \times S^1_{g_k}$ is obtained after the use of cutoff function technique. Hence after a total of $n_k + 2g_k$ Dehn surgeries on M_{g_k}, we obtain the 3-manifold $S^2 \times S^1_{g_k}$. Now we have a sequence of smooth Riemannian 3-manifolds

$$\left\{ \left(S^2 \times S^1_{g_k}, \hat{\mathcal{G}}_k \right) \right\}_{k=1}^{\infty}.$$

5.2.5 $\mathbb{Z}_2$-coefficient homology 1-systole

We show the estimation of lower bounds of $\mathbb{Z}_2$-coefficient homology 1-systole of the Riemannian 3-manifold $(S^2 \times S^1_{g_k}, \mathcal{G}_k)$ in this section. In M. Freedman's paper [18], this lower bound estimation is provided in Proposition 2.3. The following proof is based on M. Freedman's work and the Riemannian metric $\hat{\mathcal{G}}_k$ described above.

Theorem 5.1 *The $\mathbb{Z}_2$-coefficient homology 1-systole $Sys\, H_1(S^2 \times S^1_{g_k}, \hat{\mathcal{G}}_k; \mathbb{Z}_2)$ satisfies the following lower bound*

$$Sys\, H_1(S^2 \times S^1_{g_k}, \hat{\mathcal{G}}_k; \mathbb{Z}_2) \geqslant c_5 \, (\log g_k)^{\frac{1}{2}} \tag{5.3}$$

when k is large enough, where c_5 is a positive constant which does not depend on g_k.

To prove this theorem, we first show that an estimation similar to (5.3) holds on the mapping torus M_{g_k}. Some properties of the degree of smooth maps will be used in the proof.

Let M and N be two n-dimensional differentiable manifolds. Let f be a smooth map from M to N. If $y_0 \in N$ is a regular value of f, the degree of f, denoted by $\deg(f)$, is defined as follows:

$$\deg(f) = \sum_{\substack{x_i \in M, \\ f(x_i) = y_0}} \operatorname{Sgn} \left(\frac{\partial y_0^\beta}{\partial x_i^\alpha} \right),$$

where y_0^β is a coordinate chart of y_0, and x_i^α is the coordinate chart at x_i. We have the following theorem for the relation between the degree and the integration. Let Ω be an n-dimensional differential form defined on the manifold N.

Theorem 5.2 ([14]) *If $f^*\Omega$ is the pullback of the differential form Ω, we have*

$$\int_M f^*\Omega = deg(f) \int_N \Omega.$$

We have the following proposition on mapping torus M_{g_k}.

Proposition 5.2.5 *On mapping torus M_{g_k}, there exists a positive constant c such that*

$$Sys\,\pi_1(M_{g_k}, \mathcal{G}_k) \geqslant c\,(\log g_k)^{1/2},$$

where the constant c is independent of g_k.

Proof. Assume that no such c exists, we can find a subsequence $\{g_m\}$ of $\{g_k\}$ such that

$$\mathrm{Sys}\,\pi_1(M_{g_m}, \mathcal{G}_m) < \frac{1}{m}\,(\log g_m)^{1/2}.$$

Hence in M_{g_m}, there exists a noncontractible geodesic loop γ_m such that

$$\mathrm{length}_{\mathcal{G}_m}(\gamma_m) < \frac{2}{m}\,(\log g_m)^{1/2}.$$

If we use the map $\gamma_m : S^1 \to M_{g_m}$ to denote the loop γ_m, and the surface bundle map is assumed to be $f : M_{g_m} \to S^1 = [0,1]/(0) \sim (1)$. By Theorem 5.2, we know that the degree of the composition map $f \circ \gamma_m$ satisfies

$$\deg(f \circ \gamma_m) \cdot \int_{S^1} ds = \int_{S^1} (f \circ \gamma_m)^*\, ds,$$

where s is the arc length parametrization. Then we have

$$\begin{aligned}
\deg(f \circ \gamma_m) &= \int_{S^1} \gamma_m^* f^* ds \\
&\leqslant \int_{S^1} \gamma_m^* \mathrm{dV}_{\mathcal{G}_m} \\
&= \mathrm{length}_{\mathcal{G}_m}(\gamma_m) \\
&< \frac{2}{m}\,(\log g_m)^{1/2}.
\end{aligned}$$

In the following we use h to denote $\deg(f \circ \gamma_m)$. Let $x \in M_{g_k}$ be a point on γ_m, and assume that $(\tilde{x}, t) \in \Sigma_{g_m} \times \mathbb{R}$ is a lifting of x. There exists a lifting $\tilde{\gamma}_m$ of γ_m with the initial point $(\tilde{x}, t)$. The endpoint of $\tilde{\gamma}_m$ is $(\tau_m^h(\tilde{x}), t+h)$, where τ_m is the isometry representing the monodromy of M_{g_m}. Let m be large enough, we have

$$\begin{aligned} \mathrm{Order}(\tau_m) &\geqslant c_3 (\log g_m)^{1/2} \\ &> h. \end{aligned}$$

The geodesic arc connecting $\tilde{x}$ and $\tau_m^h(\tilde{x})$ is the lifting a geodesic loop γ_m' in $\Sigma_{g_m}/ < \tau_m >$. By Proposition 5.2.4, we have

$$\begin{aligned} \mathrm{length}_{\mathcal{G}_m}(\gamma_m) &= \mathrm{length}(\tilde{\gamma}_m) \\ &\geqslant \mathrm{length}(\gamma_m') \\ &\geqslant c_4 (\log g_m)^{1/2}, \end{aligned}$$

which contradicts the previous assumption if m is large enough.

■

Theorem 5.1 is yielded by the above proposition and the metric change in Dehn surgeries.

Proof. Let γ_k be a noncontractible geodesic loop contained in $S^2 \times S^1_{g_k}$. If the intersection of γ_k and one of the filled in solid tori is not empty, i.e., $\gamma_k \cap \tilde{T}_{i,\,\varepsilon_k+\delta_k} \neq \emptyset$, we use a piecewise geodesic arc on the boundary torus of $\tilde{T}_{i,\,\varepsilon_k+\delta_k}$ to replace the intersection arc. As there is a collar neighborhood along the longitude geodesic, we can find an arc segment on the boundary torus with shorter length, or the length increase is no more than the meridian loop of $\tilde{T}_{i,\,\varepsilon_k+\delta_k}$ which is equal to $2\pi(\varepsilon_k + \delta_k)$. We denote the new loop obtained after this modification by γ_k'. Then we have

$$\begin{aligned} \left|\mathrm{length}_{\hat{\mathcal{G}}_k}(\gamma_k) - \mathrm{length}_{\mathcal{G}_k}(\gamma_k')\right| &\leqslant 2\pi(\varepsilon_k + \delta_k) \cdot (3g_k - 1 + 2g_k) \\ &\leqslant \frac{5\pi}{2}(5g_k - 1)\varepsilon_k, \end{aligned}$$

where the fact $\delta_k = \varepsilon_k/4$ is used in the last inequality. Therefore, we have

$$\begin{aligned}\text{length}_{\hat{\mathcal{G}}_k}(\gamma_k) &\geqslant \text{length}_{\mathcal{G}_k}(\gamma'_k) - \frac{5\pi}{2}(5g_k - 1)\varepsilon_k \\ &\geqslant \text{length}_{\mathcal{G}_k}(\gamma'_k) - \frac{25\pi}{2g_k}\end{aligned}$$

if we let $\varepsilon_k = \left(\dfrac{1}{g_k}\right)^2$.

Then we perform reverse Dehn surgeries on $S^2 \times S^1_{g_k}$. We have the mapping torus M_{g_k} back and γ'_k is a non-contractible loop contained in it. By Proposition 5.2.5, the following estimation holds,

$$\text{length}_{\hat{\mathcal{G}}_k}(\gamma_k) \geqslant c\,(\log g_k)^{1/2} - \frac{25\pi}{2g_k}.$$

If g_k is large enough, we can find a positive constant c_5 which is independent of g_k, such that

$$\text{length}_{\hat{\mathcal{G}}_k}(\gamma_k) \geqslant c_5\,(\log g_k)^{1/2}.$$

If we take the infimum over all noncontractible loops γ_k in $S^2 \times S^1_{g_k}$, we will have

$$\text{Sys}\,\pi_1(S^2 \times S^1_{g_k},\, \hat{\mathcal{G}}_k) \geqslant c_5\ (\log g_k)^{1/2}.$$

Then the estimation

$$\begin{aligned}\text{Sys}\,H_1(S^2 \times S^1_{g_k},\, \hat{\mathcal{G}}_k; \mathbb{Z}_2) &\geqslant Sys\,\pi_1(S^2 \times S^1_{g_k},\, \hat{\mathcal{G}}_k) \\ &\geqslant c_5\ (\log g_k)^{1/2}\end{aligned}$$

holds when k is large enough. ■

5.2.6 $\mathbb{Z}_2$-coefficient homology 2-systole

For $\mathbb{Z}_2$-coefficient homology 2-systole of $(S^2 \times S^1_{g_k},\, \hat{\mathcal{G}}_k)$, we have the following estimation. The proof of the theorem is based on M. Freedman's work in Proposition 2.2 of [18].

Theorem 5.2.1 *When k is large enough, we have*

$$Sys\, H_2(S^2 \times S^1_{g_k}, \hat{\mathcal{G}}_k; \mathbb{Z}_2) \geqslant c_6\, g_k, \tag{5.4}$$

where c_6 is a positive constant which does not depend on k.

In the proof of this theorem, we need to use P. Buser's isoperimetric inequality and the coarea formula.

Definition 5.2.2 (Cheeger's Constant) *Let M be a compact n-dimensional Riemannian manifold. We define Cheeger's constant as*

$$\inf_A \frac{Vol(\partial A)}{Vol(A)},$$

denoted by $h(M)$, where A is over all open subsets with volume not more than half of the total volume.

In [9], P. Buser proved the following theorem.

Theorem 5.2.2 (P. Buser) *If the Ricci curvature of a compact closed Riemannian manifold M is bounded below by $-(n-1)\delta^2$ $(\delta \geqslant 0)$, we have*

$$\lambda_1(M) \leqslant c\,(\delta h + h^2),$$

where c is a positive constant which depends only on the dimension n.

Next we introduce the coarea formula. Let $(X, \mathcal{G}_X)$ and $(Y, \mathcal{G}_Y)$ be two Riemannian manifolds. Let $F : X \to Y$ be a C^1 map, such that for any $p \in X$ the differential map

$$D_pF : T_pX \mapsto T_{F(p)}Y$$

is surjective. We use $J_{F(p)}$ to denote the Jacobian of this map.

Theorem 5.2.3 (Coarea Formula) *For any nonnegative fucntion $\varphi : X \to \mathbb{R}$ which is measurable with respect to the measure defined by the volume form dV_X,* *we have*

$$\int_X J_F(p)\varphi(p)\,dV_X(p) = \int_Y \left(\int_{F^{-1}(q)} \varphi(p)\,dV_{F^{-1}(q)}(p)\right) dV_Y(q),$$

where $dV_{F^{-1}(q)}$ *is the volume form induced by the restriction of the metric* $\mathcal{G}_X$ *to* $F^{-1}(q)$.

We prove Theorem 5.2.1 in the following.

Proof. Suppose that we cannot find a constant c_6 independent of g_k, such that the inequality (5.4) holds. Then there exists a subsequence $\{g_m\}$ of $\{g_k\}$ such that

$$\operatorname{Sys} H_2(S^2 \times S^1_{g_m}, \hat{\mathcal{G}}_m; \mathbb{Z}_2) < \frac{1}{m}\, g_m.$$

Let X_m be a cycle which is area minimizing among all cycles representing the nonzero homology class of $H_2(S^2 \times S^1_{g_m}; \mathbb{Z}_2)$. By the regularity theorem of geometric measure theory, we know that X_m is a smooth embedded surface in $S^2 \times S^1_{g_m}$, see [17] or [40]. Then we have

$$\begin{aligned} \operatorname{Area}_{\hat{\mathcal{G}}_m}(X_m) &= \operatorname{Sys} H_2(S^2 \times S^1_{g_m}, \hat{\mathcal{G}}_m; \mathbb{Z}_2) \\ &< \frac{g_m}{m}. \end{aligned}$$

Moreover, in $H_2(S^2 \times S^1_{g_m}; \mathbb{Z}_2)$ we have only one homology class. When we do Dehn surgeries on the mapping torus M_{g_m}, all of them are performed at fiber surface levels in $(1/2,\, 1)$. Therefore, for $t \in (0,\, 1/2)$, the fiber surface $\Sigma_{g_m} \times \{t\}$ is in $S^2 \times S^1_{g_m}$, which is homologous to X_m, see [18].

By Sard's theorem, the intersection $X_m \cap (\Sigma_{g_m} \times \{t\})$ is transversal almost everywhere. Let $W_t = X_m \cap (\Sigma_{g_m} \times \{t\})$ be the intersection curve, then it is an embedded smooth curve for $t \in (0,\, 1/2)$ almost everywhere. Assume that

$$F:\, S^2 \times S^1_{g_m} \to S^1 = [0,\, 1]/(0) \sim (1)$$

is the bundle projection map. Let $\widetilde{F} = F|_{X_m}$. By the coarea formula, we have

$$\begin{aligned}\int_{X_m} J_{\widetilde{F}}\, \mathrm{dV}_{X_m} &= \int_0^1 \int_{\widetilde{F}^{-1}(t)} \mathrm{dV}_{\widetilde{F}^{-1}(t)} dt \\ &= \int_0^1 \mathrm{length}_{\hat{\mathcal{G}}_m} \left(X_m \cap \Sigma_{g_m} \times \{t\}\right) dt \\ &\geqslant \int_0^{1/2} \mathrm{length}_{\hat{\mathcal{G}}_m} (W_t) dt,\end{aligned}$$

where the left side of the above inequality is bounded above by $c' \,\mathrm{Area}_{\hat{\mathcal{G}}_m}(X_m)$, with c' is an upper bound of the Jacobian on X_m. Then we have the inequality

$$\begin{aligned}\int_0^{1/2} \mathrm{length}_{\hat{\mathcal{G}}_m}(W_t) dt &\leqslant c' \,\mathrm{Area}_{\hat{\mathcal{G}}_m}(X_m) \\ &< c' \,\frac{g_m}{m}.\end{aligned}$$

Hence there must exist a $t_0 \in (0,\, 1/2)$, such that

$$\mathrm{length}_{\hat{\mathcal{G}}_m}(W_{t_0}) < 2c' \,\frac{g_m}{m}.$$

As mentioned above, the surface $\Sigma_{g_m} \times \{t_0\}$ and X_m are in the same nonzero homology class of $H_2(S^2 \times S^1_{g_m}; \mathbb{Z}_2)$. Therefore, there exists a subset $B \subset S^2 \times S^1_{g_m}$ such that $\partial B = X_m \cup (\Sigma_{g_m} \times \{t_0\})$, see [18].

The surface X_m may intersect with the glued in solid torus $\tilde{T}_{i,\,\varepsilon_m+\delta_m}$ of Dehn surgeries. If this happens, we use surface pieces on the boundary torus $\partial\tilde{T}_{i,\,\varepsilon_m+\delta_m}$ cut by X_m to replace surface pieces of X_m intersecting with $\tilde{T}_{i,\,\varepsilon_m+\delta_m}$. After all such replacements on X_m, we have a new 2-cycle X'_m, which has no intersection with the solid tori $T_{i,\,\varepsilon_m+\delta_m}$ in $S^2 \times S^1_{g_m}$. The 2-cycle X'_m represents a nontrivial homology class in $H_2(S^2 \times S^1; \mathbb{Z}_2)$. Moreover, we let $B' = B \setminus (\cup\tilde{T}_{i,\,\varepsilon_m+\delta_m})^\circ$. The 2-cycle X'_m is homologous to $\Sigma_{g_m} \times \{t_0\}$ via B', i.e., we have $\partial B' = X'_m \cup \Sigma_{g_m} \times \{t_0\}$.

Then we perform reverse Dehn surgeries on $S^2 \times S^1_{g_m}$ to get the mapping torus M_{g_m} back. The 2-cycle X'_m has no intersections with all solid tori of Dehn surgeries. Therefore, after reverse Dehn surgeries X'_m persists as a 2-cycle in M_{g_m}. The area difference between X_m and X'_m is estimated as follows,

$$\begin{aligned}\left|\mathrm{Area}_{\hat{\mathcal{G}}_m}(X_m) - \mathrm{Area}_{\mathcal{G}_m}(X'_m)\right| &\leqslant \mathrm{Area}_{\hat{\mathcal{G}}_m}(\partial T_{i,\,\varepsilon_m+\delta_m}) \cdot (5g_m - 1) \\ &\leqslant 3\pi C\, \varepsilon_m(5g_m - 1)\log g_m\end{aligned}$$

where C is the constant such that $L_{i,\,m} \leqslant C\log g_m$; here we use $L_{i,\,m}$ to denote the length of the core geodesic loop of solid torus $T_{i,\,\varepsilon_m}$ in Dehn surgeries. Therefore, we have

$$\mathrm{Area}_{\mathcal{G}_m}(X'_m) \leqslant \mathrm{Area}_{\hat{\mathcal{G}}_m}(X_m) + 3\pi C\, \varepsilon_m(5g_m - 1)\log g_m.$$

We have assumed that $\varepsilon_m = 1/g_m^2$, so that

$$\mathrm{Area}_{\mathcal{G}_m}(X'_m) < \frac{g_m}{m} + c'_6, \tag{5.5}$$

where c'_6 is a positive constant which can be taken small enough when m is sufficiently large.

If Y is the surface piece with smaller area bounded by W_{t_0} on $\Sigma_{g_m} \times \{t_0\}$, the isoperimetric inequality of Buser implies that

$$\begin{aligned}\mathrm{Area}_{\hat{\mathcal{G}}_m}(Y) &\leqslant c''\,\mathrm{length}_{\hat{\mathcal{G}}_m}(W_{t_0}) \\ &\leqslant 2c''c'\,\frac{g_m}{m}.\end{aligned}$$

We cut along W_{t_0} on $\Sigma_{g_m} \times \{t_0\}$, and glue two copies of Y along the boundary loop W_{t_0} to have a new 2-cycle X''_m in M_{g_m}. Then we modify the subset B' to have a subset B'', such that $\partial B'' = X''_m \cup (\Sigma_{g_m} \times \{t_0\})$. We lift X''_m to the covering space $\Sigma_{g_m} \times \mathbb{R}$. Denote one lifting of X''_m by $\widetilde{X''_m}$. We have

$$\begin{aligned}\mathrm{Area}(\widetilde{X''_m}) &= \mathrm{Area}_{\mathcal{G}_m}(X'_m) + 2\mathrm{Area}_{\hat{\mathcal{G}}_m}(Y) \\ &\leqslant (1 + 4c''c')\left(\frac{g_m}{m} + c'_6\right),\end{aligned}$$

where $\mathrm{Area}(\widetilde{X''_m})$ is evaluated under covering metric on $\Sigma_{g_m} \times \mathbb{R}$. Let $\tilde{\Sigma}_{g_m} \times \{t_0\}$ be a lifting of $\Sigma_{g_m} \times \{t_0\}$, let $\widetilde{B''}$ be the lifting of B'' such that $\partial\widetilde{B''} = \widetilde{X''_m} \cup (\tilde{\Sigma}_{g_m} \times \{t_0\})$.

Let φ_t be a divergenceless flow on $\widetilde{B''}$, i.e., we have $\mathrm{div}(\varphi_t) = 0$. We further assume that the direction of the flow is orthogonal to $\Sigma_{g_m} \times \{t_0\}$. By the Divergence Theorem we have

$$\int_{\widetilde{B''}} \mathrm{div}(\varphi_t) \mathrm{dV}_{\widetilde{B''}} = \int_{\partial\widetilde{B''}} < \varphi_t,\, n > \mathrm{dV}_{\partial\widetilde{B''}},$$

where n is the outward unit normal vector field. Therefore, we have

$$\begin{aligned} \mathrm{Area}(\tilde{\Sigma}_{g_m} \times \{t_0\}) &= -\int_{\widetilde{X''_m}} < \varphi_t,\, n > \mathrm{dV} \\ &\leqslant \mathrm{Area}(\widetilde{X''_m}). \end{aligned}$$

Combined with the area estimation of $\widetilde{X''_m}$, we have

$$4\pi(g_m - 1) \leqslant (1 + 4c''c')\left(\frac{g_m}{m} + c'_6\right),$$

which is a contradiction if m is going to infinity. ■

5.2.7 Volume estimation

An upper bound of the volume is given in the following theorem. In M. Freedman's paper [18], the estimation is given in Proposition 2.1. The following proof of the estimation is based on both M. Freedman's work and the metric change described above.

Theorem 5.2.4 *When k is large enough, the volume $Vol_{\hat{\mathcal{G}}_k}(S^2 \times S^1_{g_k})$ satisfies the following upper bound,*

$$Vol_{\hat{\mathcal{G}}_k}(S^2 \times S^1_{g_k}) \leqslant c_7\, g_k, \tag{5.6}$$

where c_7 is a postive constant independent of g_k.

Proof. First we have

$$\begin{aligned} \mathrm{Vol}_{\hat{\mathcal{G}}_k}(S^2 \times S^1_{g_k}) =& \mathrm{Vol}_{\mathcal{G}_k}(M_{g_k}) - \sum_i \mathrm{Vol}_{\mathcal{G}_k}(T_{i,\,\varepsilon_k+\delta_k}) + \sum_i \Big(\mathrm{Vol}_{\tilde{\mathcal{G}}_k}(\tilde{T}_{i,\,\varepsilon_k}) \\ &+ \mathrm{Vol}_{\hat{\mathcal{G}}_k}(\hat{Y}_{i,\,k})\Big), \end{aligned}$$

where $T_{i,\,\varepsilon_k}$ and $\tilde{T}_{i,\,\varepsilon_k+\delta_k}$ are the solid tori in Dehn surgeries. Then we have

$$\begin{aligned}\left|\mathrm{Vol}_{\hat{\mathcal{G}}_k}(S^2\times S^1_{g_k})-\mathrm{Vol}_{\mathcal{G}_k}(M_{g_k})\right| \leqslant \sum_i \Big(&\mathrm{Vol}_{\mathcal{G}_k}(T_{i,\,\varepsilon_k+\delta_k})+\mathrm{Vol}_{\tilde{\mathcal{G}}_k}(\tilde{T}_{i,\,\varepsilon_k})\\ &+\mathrm{Vol}_{\hat{\mathcal{G}}_k}(\hat{Y}_{i,\,k})\Big)\,.\end{aligned}$$

We estimate the volume of $T_{i,\,\varepsilon_k+\delta_k}$ as follows,

$$\begin{aligned}\mathrm{Vol}_{\mathcal{G}_k}(T_{i,\,\varepsilon_k+\delta_k}) &= \int_0^{L_{i,\,k}}\int_0^{\varepsilon_k+\delta_k}\int_0^{2\pi} r\cosh\left(r\cos\theta\right)d\theta dr dt\\ &\leqslant \frac{25\pi}{8}C_1\,\varepsilon_k^2\log g_k,\end{aligned}$$

where C_1 is the constant in the estimation of $L_{i,\,k}=\mathrm{length}(\gamma_i)\leqslant C_1\log g_k$, and the last inequality holds as $\delta_k=\varepsilon_k/4$. The volume of $\tilde{T}_{i,\,\varepsilon_k}$ can be estimated similarly,

$$\begin{aligned}\mathrm{Vol}_{\tilde{\mathcal{G}}_k}(\tilde{T}_{i,\,\varepsilon_k}) &= \mathrm{Vol}_{\bar{\mathcal{G}}_k}(\bar{T}_{i,\,\varepsilon_k})\\ &= \int_0^{2\pi\varepsilon_k}\int_0^{\frac{L_{i,\,k}}{2\pi}}\int_0^{2\pi} r d\theta dr dt\\ &\leqslant \frac{C_1^2}{2}\,\varepsilon_k(\log g_k)^2.\end{aligned}$$

When the metic $\hat{\mathcal{G}}_k$ is restricted to $\hat{Y}_{i,\,k}=Y_{i,\,k}\cup_{f_{i,\,k}}\tilde{Y}_{i,\,k}$, the volume form can be estimated accordingly,

$$\begin{aligned}\mathrm{dV}_{\hat{\mathcal{G}}_k} &= \sqrt{\det\left[(g_{ij})_{\hat{\mathcal{G}}_k}\right]}\;dt\,dr\,d\theta\\ &\leqslant C\,\varepsilon_k\;dt\,dr\,d\theta,\end{aligned}$$

where C is a positive constant independent of g_k. Hence we have

$$\begin{aligned}\mathrm{Vol}_{\hat{\mathcal{G}}_k}(\hat{Y}_{i,\,k}) &= \int_0^{L_{i,\,k}}\int_{\varepsilon_k}^{\varepsilon_k+\delta_k}\int_0^{2\pi}\mathrm{dV}_{\hat{\mathcal{G}}_k}\\ &\leqslant C_2\,\varepsilon_k^2\log g_k,\end{aligned}$$

where C_2 is a positive constant independent of g_k.

Finally we have

$$\begin{aligned}\left|\mathrm{Vol}_{\hat{\mathcal{G}}_k}(S^2 \times S^1_{g_k}) - \mathrm{Vol}_{\mathcal{G}_k}(M_{g_k})\right| &\leqslant \left(\frac{25\pi}{8} C_1\, \varepsilon_k^2 \log g_k + \frac{C_1^2}{2}\, \varepsilon_k (\log g_k)^2 + C_2\, \varepsilon_k^2 \log g_k\right) \cdot \\ &\quad (5g_k - 1) \\ &\leqslant A\, \frac{\log g_k}{g_k^3} + B\, \frac{(\log g_k)^2}{g_k}\,,\end{aligned}$$

where the constant A is equal to $\dfrac{125\pi C_1}{8} + 5C_2$ and $B = \dfrac{5C_1^2}{2}$. If we let k be large enough, the above upper bound can be less than 1. So for convenience, in the following we assume that k is large enough and the above upper bound is equal to 1.

On the other hand,

$$\begin{aligned}\mathrm{Vol}_{\mathcal{G}_k}(M_{g_k}) &= \mathrm{Area}(\Sigma_{g_k}) \cdot \mathrm{length}(S^1) \\ &= 4\pi(g_k - 1),\end{aligned}$$

where the last equality holds as $\mathrm{Area}(\Sigma_{g_k}) = 4\pi(g_k - 1)$ by Gauss-Bonnet formula. Therefore we have

$$\begin{aligned}\mathrm{Vol}_{\hat{\mathcal{G}}_k}(S^2 \times S^1_{g_k}) &\leqslant \mathrm{Vol}_{\mathcal{G}_k}(M_{g_k}) + A\, \frac{\log g_k}{g_k^3} + B\, \frac{(\log g_k)^2}{g_k} \\ &\leqslant 4\pi(g_k - 1) + 1 \\ &\leqslant c_7\, g_k,\end{aligned}$$

where c_7 is a positive constant which can be taken as $4\pi + 1$. ■

5.2.8 $\mathbb{Z}_2$-coefficient homology $(1,\,2)$-systolic freedom of $S^2 \times S^1$

Based on above three estimations, we show that the 3-manifold $S^2 \times S^1$ exhibits $\mathbb{Z}_2$-coefficient $(1,\,2)$-systolic freedom. In M. Freedman's paper [18], the following theorem is corresponding to Theorem 2.4.

Theorem 5.2.5 *The 3-manifold $S^2 \times S^1$ exhibits $\mathbb{Z}_2$-coefficient homology $(1,\,2)$-systolic freedom, i.e., we have*

$$\inf_{\mathcal{G}} \frac{Vol(S^2 \times S^1,\, \mathcal{G};\, \mathbb{Z}_2)}{Sys\, H_2(S^2 \times S^1,\, \mathcal{G};\, \mathbb{Z}_2) \cdot SysH_1(S^2 \times S^1,\, \mathcal{G};\, \mathbb{Z}_2)} = 0,$$

where $\mathcal{G}$ runs over all Riemannian metrics on $S^2 \times S^1$.

Proof. As described above, we use the sequence of arithmetic hyperbolic surfaces $\{\Sigma_{g_k}\}_{k=1}^{\infty}$ and Dehn surgeries to construct the sequence of Riemannian 3-manifolds $\{(S^2 \times S^1_{g_k},\, \hat{\mathcal{G}}_k)\}_{k=1}^{\infty}$, where g_k is the index of the sequence, and $\lim_{k\to\infty} g_k = \infty$. From (5.1), (5.4) and (5.6), we conclude that

$$\frac{\mathrm{Vol}_{\hat{\mathcal{G}}_k}(S^2 \times S^1)}{\mathrm{Sys}\, H_2(S^2 \times S^1_{g_k},\, \hat{\mathcal{G}}_k;\, \mathbb{Z}_2) \cdot \mathrm{Sys}\, H_1(S^2 \times S^1_{g_k},\, \hat{\mathcal{G}}_k;\, \mathbb{Z}_2)} \leqslant \frac{c_7\, g_k}{c_5\, (\log g_k) \cdot c_6\, g_k}$$

$$\to 0 \qquad (\text{when}\, k \to 0).$$

Hence by definition, the 3-manifold $S^2 \times S^1$ exhibits $\mathbb{Z}_2$-coefficient homology (1, 2)-systolic freedom. ■

CHAPTER 6

3-manifolds with semibundle structure

6.1 Semibundle structure

Roughly speaking, a semibundle decomposition of a closed orientable 3-manifold M is the decomposition $M_1 \cup M_2$, with both M_1 and M_2 are twisted I-bundles over nonorientable surfaces. The twisted I-bundles M_1 and M_2 have the same boundary $M_1 \cap M_2$, which is a closed orientable surface. In the following we introduce a definition from [50].

Let M be a closed, connected and orientable 3-manifold. If H is an index 2 subgroup of $\pi_1(M)$, we call H a halving of M. For each halving H of M, there exists a two-sheeted covering $Q_H : M_H \to M$, where Q_H denotes the covering map. We consider the case where the 2-sheeted covering M_H is a 3-manifold with surface bundle structure, i.e., M_H has the fibration $F : M_H \to S^1$ with fiber a closed orientable surface. Assume that $\alpha_H : M_H \to M_H$ is the covering translation, which is a free involution and thus $M = M_H/\alpha_H$. If we view S^1 as the unit circle on the complex plane $\mathbb{C}$, then we define $\tau : S^1 \to S^1$ as the complex conjugation and $q : S^1 \to D^1$ as the projection to the real line, where $D^1 = [-1, 1]$. Hence for $e^{i\theta} \in S^1$, we have $\tau(e^{i\theta}) = e^{-i\theta}$ and $q(e^{i\theta}) = \mathrm{Re}(e^{i\theta})$. Employing the same notation in [50], we use a map $f : M \to D^1$ to express the fiber relation in semibundle, which is doubly covered by the surface bundle map $F : M_H \to S^1$. Then we have the following definition of semibundle.

Definition 6.1.1 *Let M be a closed orientable 3-manifold with halving H. The 3-*

manifold M has semibundle structure $f : M \to D^1$, if we have $F \circ \alpha_h = \tau \circ F$ and $f \circ Q_H = q \circ F$.

Remark 6.1.1 *1. Suppose that the fiber surface of M_H is a closed orientable surface Σ_g. Then when $t \in (-1, 1)$, $f^{-1}(t)$ is homeomorphic to Σ_g and it is covered by two copies of Σ_g in M_H. When $t = -1$ or $t = 1$, $f^{-1}(t)$ is doubly covered by $\Sigma_g = F^{-1}(q^{-1}(t))$.*

2. Let $J_1 = [-1, 0]$, $J_2 = [0, 1]$. We have that $f^{-1}(J_1)$ is homeomorphic to $f^{-1}(J_2)$. The common boundary of them is the surface Σ_g. Both of them are twisted I-bundles with the regular fiber surface Σ_g. Hence the semibundle M is a union of two twisted I-bundles, which are glued together along their common boundary surface. This closed orientable surface Σ_g is called as the regular fiber surface of the semibundle M.

6.2 Twisted cohomology and fibration theorem

6.2.1 Twisted cohomology group

Let Π be a group. Let A be another abelian group with the automorphism group $\mathrm{Aut}(A)$. Assume that $\theta : \Pi \to \mathrm{Aut}(A)$ is a homomorphism. For $\gamma \in \Pi$ and $a \in A$, we define a group action of Π on A as $\gamma \cdot a = [\theta(\gamma)](a)$. A crossed homomorphism $f : \Pi \to A$ is defined through the identity: $f(\gamma \cdot \eta) = f(\gamma) + \gamma \cdot f(\eta)$, where $\gamma, \eta \in \Pi$. For each $a \in A$, we define a principal crossed homomorphism p_a as $p_a(\gamma) = a - \gamma \cdot a$. We use $Q(\Pi, A)$ to denote the group of crossed homomorphisms from Π to A, and we use $P(\Pi, A)$ to denote the group of principal crossed hommorphisms. We define the quotient group $H^1(\Pi; A) = Q(\Pi, A)/P(\Pi, A)$ as the first cohomology group with coeffiecients in A.

Let H be an index two subgroup of Π. We say that (Π, H) acts semitrivially on A, if for $a \in A$, we have $\gamma \cdot a = a$ when $\gamma \in H$, and we have $\gamma \cdot a = -a$

when $\gamma \notin H$. Assume that (Π, H) acts on A semitrivially. The group $H^1(\Pi; A)$ is called the first cohomology group with twisted coefficients in A. If $f : M \to D^1$ is an H-semibundle with the covering surface bundle $F : M_H \to S^1$, and we further assume that $(\pi_1(M), H)$ acts semitrivially on $\mathbb{Z}$, there exists a crossed homomorphism $F_* : \pi_1(M) \to \mathbb{Z}$ defined as follows. Let γ be a noncontractible loop in M. We use $[\gamma]$ to denote the class in $\pi_1(M)$ represented by γ. Suppose that $\tilde{\gamma}$ is the lifting of γ in M_H. Then $F_*([\gamma])$ is defined as the image of $[\tilde{\gamma}]$ under the homomorphism induced by F on $\pi_1(M_H)$. Hence F_* defines an element $[F_*]$ in $H^1(\pi_1(M); \mathbb{Z})$.

6.2.2 Fibration theorem for semibundles

Proposition 6.2.1 (Proposition 5.1, [50]) *If $f : M \to D^1$ is a semibundle covered by the surface bundle $F : M_H \to S^1$, the induced crossed homomorphism $F_* : \pi_1(M) \to \mathbb{Z}$ satisfies:*

1. *$F_*|_H$ is surjective;*
2. *$Ker(F_*|_H)$ is finitely generated.*

In [50], there is a theorem of semibundles which is similar to Stalling's fibration theorem of surface bundles.

Theorem 6.2.1 (Theorem 5.2, [50]) *Let M be a connected, irreducible, closed and orientable 3-manifold. Let H be a halving of M. Assume that $[\theta] \in H^1(\pi_1(M); \mathbb{Z})$. If the crossed homomorphism $\theta : \pi_1(M) \to \mathbb{Z}$ satisfies*

(1) $\theta|_H$ is surjective,

(2) $Ker(\theta|_H)$ is finitely generated,

then M is a semibundle with the halving H. Moreover, if Σ is the regular fiber surface of M, we have $\pi_1(\Sigma) = Ker(\theta|_H)$.

6.3 Systolic properties of semibundles

Let $(M, \mathcal{G})$ be a Riemannian manifold with semibundle structure. Let $\mathcal{G}$ be the Riemannian metric on M. Assume that the halving of M is H. Let $(M_H, \tilde{\mathcal{G}})$ be the 2-sheeted covering surface bundle of $(M, \mathcal{G})$, with $\tilde{\mathcal{G}}$ the Riemannian covering metric induced from $\mathcal{G}$. We use $Q_H : M_H \to M$ to denote the covering map. We have the following systolic propositions on Riemannian semibundle $(M, \mathcal{G})$.

Proposition 6.3.1 *The homotopy* 1*-systole of* $(M, \mathcal{G})$ *satisfies*

$$Sys\,\pi_1(M, \mathcal{G}) \geqslant \frac{1}{2}\, Sys\,\pi_1(M_H, \tilde{\mathcal{G}}). \tag{6.1}$$

Proof. If $\gamma \subset M$ is a noncontractible loop, we use $[\gamma]$ to denote the homotopy class in $\pi_1(M)$ represented by γ. There are two possibilities here, either $[\gamma] \in H$, or $[\gamma] \in \pi_1(M) \setminus H$.

For the first case, we have $[\gamma] \in H$. If we lift γ to the covering space M_H, there are two liftings $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$. Both of them are noncontractible loops, as the induced homomorphism $(Q_H)_*$ acts on $\pi_1(M_H)$ injectively. Therefore, we have

$$\text{length}_{\tilde{\mathcal{G}}}(\tilde{\gamma}_1) = \text{length}_{\tilde{\mathcal{G}}}(\tilde{\gamma}_2) = \text{length}_{\mathcal{G}}(\gamma).$$

For the second case, we have $[\gamma] \notin H$. As $[\pi_1(M) : H] = 2$, we should have $[\gamma]^2 \in H$. Moreover, the two liftings $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are arcs with same endpoints. The loop $\tilde{\gamma} = \tilde{\gamma}_1 \cup \tilde{\gamma}_2$ is noncontractible in M_H, and we have

$$\text{length}_{\widetilde{\mathcal{G}}}(\widetilde{\gamma}) = 2\,\text{length}_{\mathcal{G}}(\gamma).$$

On the other hand, any noncontractible loop $\tilde{\gamma}$ in M_H descends to a noncontractible loop $\gamma = Q_H(\tilde{\gamma})$ in M. Therefore, we have

$$\mathrm{Sys}\,\pi_1(M, \mathcal{G}) \geqslant \frac{1}{2} \mathrm{Sys}\,\pi_1(M_H, \tilde{\mathcal{G}}).$$

■

Moreover, we have another property of $\mathbb{Z}_2$-coefficient homology 2-systole of the Riemannian semibundle $(M, \mathcal{G})$.

Proposition 6.3.2 *For $\mathbb{Z}_2$-coefficient homology 2-systole $Sys\,H_2(M, \mathcal{G}; \mathbb{Z}_2)$, we have the following estimation:*

$$Sys\,H_2(M, \mathcal{G}; \mathbb{Z}_2) \geqslant \frac{1}{2}\,Sys\,H_2(M_H, \tilde{\mathcal{G}}; \mathbb{Z}_2). \tag{6.2}$$

Proof. Let Σ be a nonseparating surface embedded into M, either one-sided or two-sided. If Σ can be lifted to a surface $\tilde{\Sigma}$ in M_H, the lifting $\tilde{\Sigma}$ is also nonseparating. And we have

$$\text{Area}_{\mathcal{G}}(\Sigma) = \text{Area}_{\tilde{\mathcal{G}}}(\tilde{\Sigma}).$$

By the lifting criterion, the sufficient and necessary condition of the existence of a lifting is $\pi_1(\Sigma) \subset H$. We assume that $\pi_1(M) = H \cup aH$, where $a \in \pi_1(M) \setminus H$ and $a^2 \in H$. The surface Σ cannot be lifted to M_H when $a \in \pi_1(\Sigma)$. However, in this case the double covering surface $\tilde{\Sigma}$ of Σ is an embedded surface in M_H, which is nonseparating if Σ is nonseparating in M. An example for this case is the nonseparating fiber surface $f^{-1}(-1)$ or $f^{-1}(1)$ cannot be lifted to M_H. However, the double covering of $f^{-1}(1)$ or $f^{-1}(-1)$ is the fiber surface Σ_g in M_H, which is nonseparating. Therefore, we have

$$\text{Area}_{\mathcal{G}}(\Sigma) = \frac{1}{2}\,\text{Area}_{\tilde{\mathcal{G}}}(\tilde{\Sigma}).$$

Combine the above two cases together, we have

$$\text{Sys}\,H_2(M, \mathcal{G}; \mathbb{Z}_2) \geqslant \frac{1}{2}\,\text{Sys}\,H_2(M_H, \tilde{\mathcal{G}}; \mathbb{Z}_2).$$

∎

CHAPTER 7

$\mathbb{Z}_2$-coefficient homology $(1, 2)$-systolic freedom of $\mathbb{RP}^3\#\mathbb{RP}^3$

Let $\mathbb{RP}^3\#\mathbb{RP}^3$ be the 3-manifold of the connected sum of two copies of real projective 3-space $\mathbb{RP}^3$. In this chapter, we establish the $\mathbb{Z}_2$-coefficient homology $(1, 2)$-systolic freedom on $\mathbb{RP}^3\#\mathbb{RP}^3$.

7.1 Main theorem

The semibundle with regular fiber surface S^2 is a union of two twisted I-bundles over $\mathbb{RP}^2$, which is homeomorphic to $\mathbb{RP}^3\#\mathbb{RP}^3$.

Theorem 7.1.1 *The 3-manifold $\mathbb{RP}^3\#\mathbb{RP}^3$ is of $\mathbb{Z}_2$-coefficient homology $(1, 2)$-systolic freedom.*

7.2 Construction of metrics

We construct a sequence of Riemannian metrics $\hat{\mathscr{G}}_k$ on $\mathbb{RP}^3\#\mathbb{RP}^3$ exhibiting systolic freedom. The construction is separated into three steps.

Step 1:

Let Σ_g be the genus g arithmetic hyperbolic surface constructed in M. Freedman's example, see Chapter 5. The surface Σ_g satisfies the following properties:

(1) For homotopy 1-systole of Σ_g, we have

$$\operatorname{Sys}\pi_1(\Sigma_g, \mathcal{G}_{\mathbb{H}^2}) \geqslant c_1 \log g,$$

where $\mathcal{G}_{\mathbb{H}^2}$ is a hyperbolic metric on Σ_g, with c_1 a positive constant independent of g.

(2) There exists an isometry map $\tau : \Sigma_g \to \Sigma_g$ with the property

$$\mathrm{Order}(\tau) \geqslant c_2 (\log g)^{1/2}, \tag{7.1}$$

where c_2 is a positive constant independent of g.

Let the genus g increase, we have a sequence of arithmetic hyperbolic surfaces $\{\Sigma_{g_k}\}_{k=1}^{\infty}$, with $\lim\limits_{k\to\infty} g_k = \infty$. On each Σ_{g_k}, there is an isometry map $\tau_k : \Sigma_{g_k} \to \Sigma_{g_k}$ with order satisfying (7.1).

Let $I_1 = [-1, 0]$, $I_2 = [0, 1]$. Let $\Sigma_{g_k} \widetilde{\times} I_1$ and $\Sigma_{g_k} \widetilde{\times} I_2$ be two twisted I-bundles. With respect to each arithmetic hyperbolic surface (Σ_{g_k}, τ_k), we construct a semibundle

$$N_{g_k} = (\Sigma_{g_k} \widetilde{\times} I_1) \cup_{\tau_k} (\Sigma_{g_k} \widetilde{\times} I_2),$$

where two twisted I-bundles $\Sigma_{g_k} \widetilde{\times} I_1$ and $\Sigma_{g_k} \widetilde{\times} I_2$ are glued together along their common boundary surface Σ_{g_k}. Let $D^1 = [-1, 1]$. We use $f_k : N_{g_k} \to D^1$ to denote the semibundle N_{g_k}. The semibundle N_{g_k} is doubly covered by the surface bundle $M_{g_k} = \Sigma_{g_k} \times [0, 1]/(x, 0) \sim (\tau_k(x), 1)$, which is the surface bundle constructed in Freedman's example, see Chapter 5. On N_{g_k}, there is a Riemannian metric $\mathscr{G}_k$ locally isometric to the product metric on $\mathbb{H}^2 \times \mathbb{R}$. Moreover, the metric $\mathcal{G}_k$ on surface bundle M_{g_k} is the Riemannian covering metric induced by the covering map $Q_k : M_{g_k} \to N_{g_k}$. Let $H = \pi_1(M_{g_k})$. The semibundle N_{g_k} has halving H, so that in the following we call N_{g_k} as an H-semibundle. In conclusion, we have constructed a sequence of Riemannian semibundles:

$$\{(N_{g_k}, \mathscr{G}_k)\}_{k=1}^{\infty},$$

with $2 \leqslant g_1 < g_2 < \cdots$ and $\lim\limits_{k\to\infty} g_k = \infty$.

Step 2:

By Lickorish twist theorem (Theorem 3.3.3), every element in the mapping class group of a closed orientable surface can be decomposed into a product of isotopy classes of Dehn twists. Hence we have $\tau_k^{-1} = \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_{n_k}$, where each σ_i is a Dehn twist along a nonseparating simple geodesic loop γ_i of Σ_{g_k}. Moreover, the nonseparating simple geodesic loop γ_i is from a set of $3g_k - 1$ nonseparating geodesic loops of Σ_{g_k}, see Chapter 5. We perform a Dehn surgery along each γ_i in the semibundle N_{g_k}. For $1 \leqslant i \leqslant n_k$, we do Dehn surgeries at the following distinct surface levels

$$\gamma_1 \times \left(\frac{1}{2} + \frac{1}{3n_k+1}\right), \quad \gamma_2 \times \left(\frac{1}{2} + \frac{2}{3n_k+1}\right), \quad \cdots \quad , \gamma_{n_k} \times \left(\frac{1}{2} + \frac{n_k}{3n_k+1}\right).$$

If we let the radius ε_k of the replacement solid tori in Dehn filling be small enough, all the solid tori in Dehn surgeries will not intersect with each other.

After all of the above n_k Dehn surgeries, we have a semibundle

$$N'_{g_k} = (\Sigma_{g_k} \widetilde{\times} I_1) \cup_{\tau_k^{-1} \circ \tau_k} (\Sigma_{g_k} \widetilde{\times} I_2),$$

which is doubly covered by the surface bundle $\Sigma_{g_k} \times S^1$.

Step 3:

Assume that the set of simple geodesic loops $\{\lambda_1, \mu_1, \lambda_2, \mu_2, \cdots \lambda_{g_k}, \mu_{g_k}\}$ represents a homology basis of $H_1(\Sigma_{g_k}; \mathbb{Z})$. We do a Dehn surgery around each geodesic loop λ_i or μ_i to kill it, i.e., in the Dehn surgery we glue the meridian loop of each replacement solid torus to the loop λ_i or μ_i so that λ_i or μ_i becomes contractible, see Chapter 5 for details. Similarly we do Dehn surgeries at different fiber surface levels to separate them. For example, these additional $2g_k$ Dehn surgeries can be performed at the following fiber surface levels

$$\begin{aligned} &\lambda_1 \times \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{6(2g_k+1)}\right), \qquad \mu_1 \times \left(\frac{1}{2} + \frac{1}{3} + \frac{2}{6(2g_k+1)}\right), \qquad \cdots \\ &, \lambda_{g_k} \times \left(\frac{1}{2} + \frac{1}{3} + \frac{2g_k - 1}{6(2g_k+1)}\right), \quad \mu_{g_k} \times \left(\frac{1}{2} + \frac{1}{3} + \frac{2g_k}{6(2g_k+1)}\right). \end{aligned}$$

After these additional $2g_k$ Dehn surgeries, we have a 3-manifold which is homeomorphic to $\mathbb{RP}^3 \# \mathbb{RP}^3$.

Proposition 7.2.1 *The 3-manifold obtained after above $n_k + 2g_k$ Dehn surgeries is the semibundle with S^2 as the corresponding 0-sphere bundle. Hence it is homeomorphic to $\mathbb{RP}^3\#\mathbb{RP}^3$.*

Proof. Assume that the semibundle $f'_k : N'_{g_k} \to D^1$ is doubly covered by the surface bundle $F_k : \Sigma_{g_k} \times S^1 \to S^1$. Then $(F_k)_* : \pi_1(N'_{g_k}) \to \mathbb{Z}$ is a crossed homomorphism determined by f'_k. For definition of crossed homomorphism and cohomology group with twisted coefficients, see Chapter 6. We denote $\pi_1(\Sigma_{g_k} \times S^1)$ by H. By Proposition 6.2.1, we have

(1) $(F_k)_*|_H$ is surjective;

(2) $\mathrm{Ker}((F_k)_*|_H)$ is finitely generated.

Assume that N''_{g_k} is the 3-manifold obtained after a pair of Dehn surgeries to kill $\{\lambda_1, \mu_1\}$, then $H' = \pi_1(\Sigma_{g_k-1} \times S^1)$ is a halving of N''_{g_k}. Let $(F'_k)_*$ be the restriction of $(F_k)_*$ to $\pi_1(N''_{g_k})$, then $(F'_k)_* : \pi_1(N''_{g_k}) \to \mathbb{Z}$ is a crossed homomorphism. Based on above two facts of the crossed homomorphism F_*, we have

(1) $(F'_k)_*|_{H'}$ is surjective;

(2) $\mathrm{Ker}\,((F'_k)_*|_{H'})$ is finitely generated.

Theorem 6.2.1 yields that N''_{g_k} is a semibundle with the regular fiber surface Σ_{g_k-1}, as $\mathrm{Ker}((F'_k)_*|_{H'}) = \pi_1(\Sigma_{g_k-1})$. We repeat the above procedure. After g_k pairs of Dehn surgeries, we will kill all $2g_k$ geodesic loops representing the homology basis $H_1(\Sigma_{g_k-1}; \mathbb{Z})$. The 3-manifold finally obtained is a semibundle with the corresponding 0-sphere bundle S^2. Hence it is homeomorphic to $\mathbb{RP}^3\#\mathbb{RP}^3$, denoted by $\mathbb{RP}^3\#\mathbb{RP}^3_{g_k}$. ■

Now we have a sequence of Riemannian 3-manifolds $\{(\mathbb{RP}^3\#\mathbb{RP}^3_{g_k}, \hat{\mathscr{G}_k})\}_{k=1}^{\infty}$, where $\lim\limits_{k\to\infty} g_k = \infty$, with the Riemannian metric $\hat{\mathscr{G}_k}$ obtained through cutoff function technique in Dehn surgeries, see Chapter 5 for details. The Riemannian semibundle

$$(\mathbb{RP}^3\#\mathbb{RP}^3_{g_k},\, \hat{\mathscr{G}}_k)$$

has the two-sheeted Riemannian covering surface bundle $(\widetilde{\mathbb{RP}^3\#\mathbb{RP}^3}_{g_k},\, \tilde{\mathscr{G}}_k)$, where $\tilde{\mathscr{G}}_k$ is the Riemannian metric induced by the covering map. The properties about $\mathbb{Z}_2$-coefficient homology 1- and 2-systoles of $(\widetilde{\mathbb{RP}^3\#\mathbb{RP}^3}_{g_k},\, \tilde{\mathscr{G}}_k)$ is similar to $(S^2\times S^1,\, \hat{\mathcal{G}}_k)$, see Chapter 5.

7.3 Proof of main theorem 7.1.1

We will show the following fact:

$$\lim_{k\to\infty} \frac{\mathrm{Vol}_{\hat{\mathscr{G}}_k}(\mathbb{RP}^3\#\mathbb{RP}^3_{g_k})}{\mathrm{Sys}\, H_1(\mathbb{RP}^3\#\mathbb{RP}^3_{g_k},\, \hat{\mathscr{G}}_k;\, \mathbb{Z}_2)\cdot \mathrm{Sys}\, H_2(\mathbb{RP}^3\#\mathbb{RP}^3_{g_k},\, \hat{\mathscr{G}}_k;\, \mathbb{Z}_2)} = 0, \tag{7.2}$$

which implies the property of $\mathbb{Z}_2$-coefficient homology (1, 2)-systolic freedom on the 3-manifold $\mathbb{RP}^3\#\mathbb{RP}^3$.

In the estimation of $\mathbb{Z}_2$-coefficient homology 1- and 2-systoles, as well as the volume of $(\mathbb{RP}^3\#\mathbb{RP}^3,\, \hat{\mathscr{G}}_k)$, similar to Section 5.2, we assume that the radius of solid tori in Dehn surgeries on the semibundle N_{g_k} is equal to $\frac{1}{g_k^2}$.

7.3.1 The estimate of $\mathbb{Z}_2$-coefficient homology 1-systole

For $\mathbb{Z}_2$-coefficient homology 1-systole of $(\mathbb{RP}^3\#\mathbb{RP}^3,\, \hat{\mathscr{G}}_k)$, we have the following estimation.

Proposition 7.3.1 *There exists a positive constant s_1 independent of g_k such that*

$$\mathit{Sys}\, H_1(\mathbb{RP}^3\#\mathbb{RP}^3_{g_k},\, \hat{\mathscr{G}}_k;\, \mathbb{Z}_2) \geqslant s_1\, (\log g_k)^{1/2}.$$

Proof. Let γ be a nonseparating geodesic loop of $\mathbb{RP}^3\#\mathbb{RP}^3_{g_k}$ representing a homology class in $H_1(\mathbb{RP}^3\#\mathbb{RP}^3_{g_k};\, \mathbb{Z}_2)$. We perform reverse Dehn surgeries to have a noncontractible loop γ' in the semibundle N_{g_k}. The construction of γ' is similar to the construction in $S^2 \times S^1_{g_k}$, see section 5.2.5 . By a calculation of the length difference made by Dehn surgeries, we have

$$\begin{aligned}|\mathrm{length}_{\hat{\mathscr{G}}_k}(\gamma) - \mathrm{length}_{\mathscr{G}_k}(\gamma')| &\leqslant \frac{25\pi}{2g_k} \\ &\leqslant s_1'\end{aligned}$$

where s_1' is a small positive constant, which can be taken sufficiently small when k is large enough. By Proposition (6.3.1), we have

$$\begin{aligned}\mathrm{length}_{\hat{\mathscr{G}}_k}(\gamma) &\geqslant \mathrm{length}_{\mathscr{G}_k}(\gamma') - s_1' \\ &\geqslant \mathrm{Sys}\,\pi_1(N_{g_k}, \mathscr{G}_k) - s_1' \\ &\geqslant \frac{1}{2}\,\mathrm{Sys}\,\pi_1(M_{g_k}, \mathcal{G}_k) - s_1' \\ &\geqslant \frac{1}{2}c\,(\log g_k)^{1/2} - s_1',\end{aligned}$$

where c is the positive constant from Proposition (5.2.5). If k is large enough, there exists positive constant s_1 independent of g_k such that

$$\mathrm{length}_{\hat{\mathscr{G}}_k}(\gamma) \geqslant s_1\,(\log g_k)^{1/2}.$$

Hence we have

$$\mathrm{Sys}\,H_1(\mathbb{RP}^3\#\mathbb{RP}^3, \hat{\mathscr{G}}_k; \mathbb{Z}_2) \geqslant s_1\,(\log g_k)^{1/2}. \tag{7.3}$$

■

7.3.2 The estimate of $\mathbb{Z}_2$-coefficient homology 2-systole

For $\mathbb{Z}_2$-coefficient homology 2-systole of $(\mathbb{RP}^3\#\mathbb{RP}^3_{g_k}, \hat{\mathscr{G}}_k)$, we have the following estimation.

Proposition 7.3.2 *There exists a positive constant s_2 independent of g_k such that*

$$\mathit{Sys}\,H_2(\mathbb{RP}^3\#\mathbb{RP}^3_{g_k}, \hat{\mathscr{G}}_k; \mathbb{Z}_2) \geqslant s_2\,g_k.$$

Proof. Assume that X_k is a nonseparating and area minimizing 2-cycle in $\mathbb{RP}^3 \# \mathbb{RP}^3_{g_k}$. According to the regularity theory of area minimizing currents in geometric measure theory, the 2-cycle X_k is a smooth embedded surface in $\mathbb{RP}^3 \# \mathbb{RP}^3_{g_k}$, see [17] or [40]. Moreover, X_k is an one sided surface in $\mathbb{RP}^3 \# \mathbb{RP}^3_{g_k}$. The nonseparating one sided surface X_k can be doubly covered by a nonseparating surface $\tilde{X}_k$ in the 2-sheeted Riemannian covering surface bundle $(\widetilde{\mathbb{RP}^3 \# \mathbb{RP}^3_{g_k}}, \tilde{\mathscr{G}}_k)$. Furthermore, there is only one homology class in $H_2(\widetilde{\mathbb{RP}^3 \# \mathbb{RP}^3_{g_k}}, \tilde{\mathscr{G}}_k)$. If we repeat the way to estimate $\mathrm{Sys}H_2(S^2 \times S^1_{g_k}, \hat{\mathcal{G}}_k; \mathbb{Z}_2)$ in section 5.2.6, there will be a positive constant s_2 independent of g_k such that

$$\mathrm{Sys}H_2(\widetilde{\mathbb{RP}^3 \# \mathbb{RP}^3_{g_k}}, \tilde{\mathscr{G}}_k; \mathbb{Z}_2) \geqslant s_2 g_k.$$

Hence we have

$$\begin{aligned}
\mathrm{Area}_{\hat{\mathscr{G}}_k}(X_k) &= \frac{1}{2}\,\mathrm{Area}_{\tilde{\mathscr{G}}_k}(\tilde{X}_k) \\
&\geqslant \mathrm{Sys}\, H_2(\widetilde{\mathbb{RP}^3 \# \mathbb{RP}^3_{g_k}}, \tilde{\mathscr{G}}_k; \mathbb{Z}_2) \\
&\geqslant s_2\, g_k.
\end{aligned}$$

The above inequality implies that

$$\mathrm{Sys}\, H_2(\mathbb{R}P^3 \# \mathbb{R}P^3_{g_k}, \hat{\mathscr{G}}_k; \mathbb{Z}_2) \geqslant s_2\, g_k. \tag{7.4}$$

■

7.3.3 The estimate of volume

We have the following estimate of the volume of $(\mathbb{RP}^3 \# \mathbb{RP}^3, \hat{\mathscr{G}}_k)$.

Proposition 7.3.3 *There exists a positive constant s_3 independent of g_k such that*

$$Vol_{\hat{\mathscr{G}}_k}(\mathbb{RP}^3 \# \mathbb{RP}^3_{g_k}) \leqslant s_3\, g_k.$$

Proof. When k is large enough, the radius ε_k of solid tori in Dehn surgeries on the semibundle N_{g_k} would be small enough. Similar to the estimation method in Theorem

5.2.4, we have

$$\left|\mathrm{Vol}_{\hat{\mathscr{G}}_k}(\mathbb{RP}^3\#\mathbb{RP}^3_{g_k}) - \mathrm{Vol}_{\mathscr{G}_k}(N_{g_k})\right| \leqslant A\,\frac{\log g_k}{g_k^3} + B\,\frac{(\log g_k)^2}{g_k}$$
$$\leqslant 1,$$

if k is sufficiently large. Then we have the following estimation on volume of $\mathbb{RP}^3\#\mathbb{RP}^3_{g_k}$,

$$\begin{aligned}\mathrm{Vol}_{\hat{\mathscr{G}}_k}(\mathbb{RP}^3\#\mathbb{RP}^3_{g_k}) &\leqslant \mathrm{Vol}_{\mathscr{G}_k}(N_{g_k}) + 1\\ &= \frac{1}{2}\,\mathrm{Vol}_{\mathcal{G}_k}(M_{g_k}) + 1.\end{aligned}$$

By the Gauss-Bonett formula (3.1) of hyperbolic surfaces, we have

$$\begin{aligned}\mathrm{Vol}_{\mathcal{G}_k}(M_{g_k}) &= \mathrm{Area}_{\mathcal{G}_{\mathbb{H}^2}}(\Sigma_{g_k})\cdot 1\\ &= 4\pi(g_k - 1),\end{aligned}$$

where $\mathcal{G}_{\mathbb{H}^2}$ is the hyperbolic metric on Σ_{g_k}. Hence when k is large enough, there exists a positive constant s_3 independent of g_k such that

$$\mathrm{Vol}_{\hat{\mathscr{G}}_k}(\mathbb{RP}^3\#\mathbb{RP}^3_{g_k}) \leqslant s_3\, g_k. \tag{7.5}$$

■

7.3.4 $\mathbb{Z}_2$-coefficient homology (1, 2)-systolic freedom

In terms of above estimations (7.3), (7.4) and (7.5), we have

$$\frac{\mathrm{Vol}_{\hat{\mathscr{G}}_k}(\mathbb{RP}^3\#\mathbb{RP}^3_{g_k})}{\mathrm{Sys}\,H_1(\mathbb{RP}^3\#\mathbb{RP}^3_{g_k},\,\hat{\mathscr{G}}_k;\,\mathbb{Z}_2)\cdot \mathrm{Sys}\,H_2(\mathbb{RP}^3\#\mathbb{RP}^3_{g_k},\,\hat{\mathscr{G}}_k;\,\mathbb{Z}_2)} \leqslant \frac{s_3 g_k}{s_1(\log g_k)^{1/2}\cdot s_2 g_k}.$$

The right side of the above inequality is zero if we let $k \to \infty$. Hence the formula (7.2) holds.

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