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TO A MODEL EQUATION FOR

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EXISTENCE AND STABILITY OF SOLUTIONS
TO A MODEL EQUATION FOR
DISPERSION-MANAGED SOLITARY WAVES

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Contents

Abstract

We study the existence and stability of solutions of the initial-value problem for the dispersion-managed nonlinear Schrödinger (DMNLS) equation, a model equation for optical pulses in a dispersion-managed fiber. One interesting feature of the DMNLS equation is that the nonlinear term involves the non-local operator $T(s) = e^{-iD(s)\partial_x^2}$, where the periodic function $D(s)$ governs the dispersive properties of the fiber. Another interesting feature is that even when the average dispersion α is equal to zero, the equation is known to have solitary-wave solutions.

For the Cauchy problem for the DMNLS equation with initial data in H^s with $s \geq 1$, under weak assumptions on the variable dispersion and nonlinear coefficients, we prove local well-posedness for all $\alpha \in \mathbf{R}$, and global well-posedness for $\alpha \neq 0$. We also use a Strichartz estimate on $T(s)$ to establish global well-posedness for initial data in L^2 for all $\alpha \in \mathbf{R}$, and local well-posedness for data in $L^2 \cap L^\infty$ in the case $\alpha = 0$.

We also revisit the proofs of existence and stability of solitary waves due to Zharnitsky et al. in the case $\alpha > 0$ and to Kunze in the case $\alpha = 0$. We show that their arguments, based on a concentration compactness approach to a variational characterization of solitary waves, continue to be valid under weak assumptions on the dispersion and nonlinear coefficients.

Chapter 1

Introduction

1.1 The dispersion-managed nonlinear Schrödinger equation

An important model equation for pulses in fiber optics is the one-dimensional nonlinear Schrödinger equation (NLS),

$$iu_z + u_{\tau\tau} = |u|^2 u, \tag{1.1}$$

which can be derived from Maxwell's equations under the assumption that the pulse is a slowly modulated sinusoidal wave [A]. Here $u = u(z, \tau)$ is the complex-valued envelope of the electromagnetic field. z measures distance along the fiber, and τ is the time. The role of the NLS equation in describing the dynamics of such phenomena makes it central to the understanding and design of long-distance fiber optics communication systems [M]. The NLS equation and its variants also appear in studies of gravity waves, plasma waves, energy transport along molecular chains, and many other applications.

In this thesis we study an averaged version of (1.1) which was derived in [GT] as a model for optical pulses in a dispersion-managed fiber: that is, a fiber which has been treated to alter its dispersive properties to enhance the stability of pulses [A]. For the reader's information we summarize here the assumptions made in the derivation of this equation; for a more detailed description of these assumptions and their regimes of physical validity, a good reference is [M].

The propagation of light in a one-dimensional fiber is described by an elec-

tromagnetic field vector $\mathbf{E} = \mathbf{E}(z, t)$, where z measures distance along the fiber and t is time. Here we assume that $\mathbf{E}$ has small magnitude relative to interatomic electric fields; say $|\mathbf{E}| = O(\epsilon)$ with $\epsilon << 1$. We also assume that $\mathbf{E}$ is, to leading order, polarized in the x-direction, so that if $\mathbf{e}_x$ is the unit vector in the x-direction, then $|\mathbf{E}(z, t) - \epsilon E_1(z, t)\mathbf{e}_x| = O(\epsilon^2)$ for some scalar function $E_1(z, t)$ of order one. Further we assume that E_1 is a nearly monochromatic wave propagating in the z direction, with wave number k and frequency ω, i.e.,

$$E_1(z, t) = u(\epsilon z, \epsilon t)\ e^{i(kz - \omega t)} + \bar{u}(\epsilon z, \epsilon t)\ e^{-i(kz - \omega t)},$$

where the derivatives of u with respect to its arguments $Z = \epsilon z$ and $\tau = \epsilon t$ are order one. In other words, u is a slowly varying function of z and t.

Under the preceding assumptions, a formal argument starting from Maxwell's equations (see [A]) shows that u satisfies approximately the variable-coefficient NLS equation

$$iu_Z + d(Z)u_{\tau\tau} + \gamma(Z)|u|^2 u = 0, \tag{1.2}$$

where $d(Z)$ is determined by the optical properties of the fiber, and affects the dispersion of signals (the spreading of signals due to the fact that components of different wavelengths travel with different velocities), and $\gamma(Z)$ models the amplification and decay of signals due to the presence of amplifiers and/or loss in the fiber.

In the 1990's it was suggested that signals in optical fibers might propagate more stably if one constructed the fiber so that segments with large normal dispersion (in which low frequencies propagate faster than high frequencies) alternate with segments of large anomalous dispersion (in which high frequencies

propagate faster than low frequencies). In the model (1.2), normal dispersion corresponds to regions where $d(Z)$ is negative, and anomalous dispersion to regions where $d(Z)$ is positive. The idea was that the large absolute values of the dispersion would suppress undesirable nonlinear effects, but that if it was arranged that the average value of the dispersion was small (due to cancellation of the negative and positive dispersion with each other), then undesirable spreading of signals would not occur [Tu]. Experimental studies (see [Z2] for references) found that this would indeed be the case. The technique of constructing fibers with such dispersive properties is known as dispersion management.

At around the same time, Gabitov and Turitsyn in [GT] proposed an interesting model equation for optical waves in a dispersion-managed fiber. To derive it, one assumes that the dispersion coefficient function $d(Z)$ in (1.2) is periodic with small period μ, where $\mu << 1$; and that in absolute value, $d(Z)$ is large, of size $O(1/\mu)$, and has mean value α which is $O(1)$. Thus we assume that

$$d(Z) = \alpha + \frac{1}{\mu}\, \Delta\left(\frac{Z}{\mu}\right),$$

where $\Delta(\zeta)$ is a function of period 1 and mean value zero; i.e.,

$$\int_0^1 \Delta(\zeta)\, d\zeta = 0.$$

We also assume that $\gamma(Z)$ is non-negative and periodic with period μ, which, if $\gamma(Z)$ is not constant, corresponds to the assumption that the signal is amplified by amplifiers spaced μ units apart. Let $g(\zeta)$ be defined by $g(\zeta) = \gamma(\mu\zeta)$, so that $\gamma(\zeta)$ is periodic with period 1.

Under the preceding assumptions, one can approximate the solutions of (1.2)

3

by solutions of the averaged equation

$$iu_Z + \alpha u_{\tau\tau} + \int_0^1 g(s)T^{-1}(s)\left[|T(s)u|^2 T(s)u\right] \, ds = 0. \qquad (1.3)$$

(see, e.g., chapter 10 of [A] for a derivation). In fact, in [Z2] it is shown that solutions of (1.3) do approximate those of (1.2) well when μ is small (see Theorem 4.1 of [Z2]).

There does not seem to be a generally accepted name for (1.3) in the literature. We will refer to it below as the dispersion-managed nonlinear Schrödinger (DMNLS) equation.

In equation (1.3), the operator $T(s)$ is defined as follows. Define $D(s)$ for $s \in \mathbf{R}$ by

$$D(s) := \int_0^s \Delta(\zeta)d\zeta. \qquad (1.4)$$

Notice that since $\int_0^1 \Delta(\zeta) \, d\zeta = 0$, then $D(s)$ is periodic with period 1. For $s \in [0,1]$, define $T(s) : L^2 \to L^2$ by

$$T(s) = e^{-iD(s)\partial_x^2}, \qquad (1.5)$$

or in other words

$$T(s)f = \mathcal{F}^{-1} e^{iD(s)\omega^2} \mathcal{F}f, \qquad (1.6)$$

where $\mathcal{F}$ is the Fourier transform. We describe the properties of $T(s)$ in more detail below, but we mention here that clearly $T(s)$ is an invertible (in fact unitary) operator on L^2 with inverse given by $T^{-1}(s) = e^{iD(s)\partial_x^2}$, so the operator $T^{-1}(s)$ in (1.3) is well-defined.

To conform to the usual choice of variables used for NLS and its variants in the literature, from now on we will replace z by t and τ by x, so that (1.3)

becomes

$$iu_t + \alpha u_{xx} + \int_0^1 g(s)T^{-1}(s)\left[|T(s)u|^2 T(s)u\right] \, ds = 0. \qquad (1.7)$$

In (1.7), when $T(s)$ or $T^{-1}(s)$ is applied to a function of two variables x and t, it is understood that t is held fixed and the operators are viewed as acting on functions of the variable x.

The results in this thesis hold under rather general assumptions on the functions $\Delta(s)$ (which determines $T(s)$ through (1.4) and (1.6)) and $g(s)$. We assume, unless otherwise stated, only that $\Delta(s)$ and $g(s)$ are integrable on $[0,1]$.

1.2 Previous results and statement of main results

Two of the central topics in the theory of nonlinear dispersive equations are well-posedness of the initial-value problem, and the existence and stability of important special solutions.

For an initial-value problem to be (globally) well-posed, its solution $u(x,t)$ should exist for all times t, for arbitrary choices of the initial data $u(x,0)$ in some function class, and the solution should be unique and depend continuously on the initial data.

When special solutions $u_s(x,t)$ such as solitary waves or bound-state solutions exist, the question arises of how important these solutions are in the evolution of more general solutions. An important first step in answering this question is to study the stability of the special solutions. To prove stability, one tries to show that if the equation is solved with general initial data $u(x,0)$ that is sufficiently close (in some appropriate function space) to $u_s(x,0)$, the initial data for the special solution, then for all times t the solution $u(x,t)$ arising from $u(x,0)$ stays close to the special solution $u_s(x,t)$. In particular, to prove a stability result one generally also needs to prove a well-posedness result, to

5

guarantee that $u(x,t)$ exists for all time.

For the NLS equation (1.1), global well-posedness of the initial-value problem in H^s for $s \geq 0$ was proved in [T]. Also, (1.1) has important special solutions known as solitary-wave (or bound-state) solutions, which are solutions of the form $u_s(\tau, z) = e^{i\theta z}\phi(\tau)$, where θ is a constant and $\phi(\tau) \to 0$ as $|\tau| \to \infty$. The stability of these solitary waves is a classical result for which it is a little difficult to identify a first author. Certainly any proof of stability owes much to the original proof of stability of solitary waves for the Korteweg-de Vries equation given by Benjamin [Be] and Bona [B]. For the case of the NLS equation (1.1) it is probably fair to give much of the credit to Cazenave and Lions for devising a nice method of proof of stability of solitary-wave solutions (see [C], chapter 8). More recently, in [MP] it has been proved that solitary-wave solutions of (1.1) are stable in L^2.

For the variable-coefficient NLS equation (1.2), global well-posedness in L^2 was proved in [ASS], under the assumptions that $d(Z)$ is periodic and piecewise constant and $g(Z) \equiv 1$. Global well-posedness in H^1 is also known (see Remark 3.3 of [ASS]). Of course, solitary waves as defined above could not exist for the variable-coefficient equation (1.2), but it is an interesting question whether (1.2) has solutions which somehow resemble solitary waves (see, for example, [PZ] for results on this topic).

For the DMNLS equation (1.7), it is noted in [Z2] that it is straightforward to prove global well-posedness in H^1 in the case $\alpha \neq 0$ and local well-posedness in H^1 in the case $\alpha = 0$. Also, [Z2] includes a result on existence of solitary waves and their stability in H^1 in the case $\alpha > 0$. The assumptions used in [Z2] were that $\Delta(s) = D'(s)$ is piecewise constant on $[0,1]$ and that $g(s) = 1$ on $[0,1]$. For the case $\alpha = 0$, Kunze [Z1] used an interesting version of the concentration compactness method to prove the existence of solitary-wave solutions to (1.7)

when $d(s)$ is piecewise constant and $g(s) \equiv 1$. An alternative proof of this result, under more general assumptions on $d(s)$ and $g(s)$, appears in [HL]. We note that the question remains open whether solitary-wave solutions to equation (1.7) exist in the case $\alpha < 0$, although numerical evidence suggests that they do not (see [PZ], p. 749).

The main results of this thesis are as follows. They all apply to the DMNLS equation (1.7).

In Theorem 2.3 below, we include a proof, valid for all $\alpha \in \mathbf{R}$, of global well-posedness of the initial-value problem in H^r for all $r \geq 1$. As indicated in [Z2], the proof is a straightforward contraction-mapping argument, but we wanted to include the details to set the stage for the results which follow. This result only requires that $\Delta(s)$ be integrable and $g(s)$ be bounded and measurable on $[0, 1]$.

In Theorem 2.10, again for all $\alpha \in \mathbf{R}$, we prove global well-posedness of the initial-value problem in L^2. The proof makes use of a Strichartz estimate for the family of operators $T(t)$ (Theorem 2.7). Here we have to assume more about $\Delta(s)$, namely that it is piecewise of one sign, and bounded away from zero (see Assumption D1 below). Our final well-posedness result, Theorem 2.11, is for the case $\alpha = 0$, also under the extra assumption on $\Delta(s)$, and states that the initial-value problem is locally well-posed in $L^2 \cap L^\infty$.

In Chapter 3, we consider the existence and stability of solitary-wave solutions to (1.7). In Section 3.2 we consider the case when $\alpha > 0$, and show in Theorem 3.2 that for every $\lambda > 0$, (1.7) has a non-empty stable set G_λ of solitary-wave solutions $e^{i\theta t}\phi(x)$ satisfying $\int_{-\infty}^{\infty} \phi^2 \, dx = \lambda$. The proof, which proceeds by showing that G_λ is the solution set to a variational problem, is essentially the same as that given in [Z2] for the case when $\Delta(s)$ is piecewise constant and $g(s) = 1$ on $[0, 1]$. We were interested in writing out the details so as to check what are the minimal assumptions on $\Delta(s)$ and $g(s)$ for the proof to

work. It turns out that all that is required is for $\Delta(s)$ and $g(s)$ to be integrable on $[0,1]$ and for $g(s)$ to be non-negative and not identically equal to zero.

In Section 3.3 we consider the case $\alpha = 0$, and state an existence and stability result for solitary waves in Theorem 3.18. Here we have to add additional assumptions on $\Delta(s)$ and $g(s)$; namely that $\Delta(s)$ and $g(s)$ are piecewise absolutely continuous. Since the proof only differs in a couple of places from that given in [K] for the case when $\Delta(s)$ is piecewise constant, we do not give the full proof, but only indicate how the proof in [K] should be modified in the more general case.

Finally we would like to mention a couple of open problems associated to the results in this thesis. It is not yet known whether (1.7) is globally well-posed in H^r for any $r > 1$. In the case $\alpha = 0$, it is not known whether (1.7) is locally well-posed in H^r for any $r > 0$, or whether it is globally well-posed in $L^2 \cap L^\infty$. Also, we would like to know whether a well-posedness result can be proved for equation (1.7) in mixed spaces $L_t^q L_x^p$, along the lines of the result given for equation (1.2) in [ASS].

1.3   Preliminaries

The set of natural numbers $\{1, 2, 3, \cdots\}$ and the set of all integers are written $\mathbf{N}$ and $\mathbf{Z}$, respectively. The set of all real numbers is denoted by $\mathbf{R}$.

For any measurable function f on $\mathbf{R}$ and any $p \in [1, \infty)$, we define

$$\|f\|_{L^p} = \left(\int_{-\infty}^{+\infty} |f(x)|^p \, dx \right)^{\frac{1}{p}},$$

and $L^p = L^p(\mathbf{R})$ denotes the space of all f for which $\|f\|_p$ is finite. The space

L^∞ is defined as the space of all measurable functions f on $\mathbf{R}$ such that

$$\|f\|_{L^\infty} = \text{ess sup}_{x \in \mathbf{R}} |f(x)|$$

is finite.

For any measurable function $f(x,t)$ on $\mathbf{R} \times \mathbf{R}$ and any $p \in [1, \infty)$ and $q \in [1, \infty)$, we define

$$\|f\|_{L_t^q L_x^p} = \left(\int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} |f(x,t)|^p \, dx \right)^{\frac{q}{p}} dt \right)^{\frac{1}{q}},$$

and $L_t^q L_x^p = L_t^q L_x^p(\mathbf{R} \times \mathbf{R})$ denotes the space of all f for which $\|f\|_{L_t^q L_x^p}$ is finite.

If E is a subset of $\mathbf{R}$ then $C_0^\infty(E)$ denotes the space of infinitely differentiable functions with compact support in E. A larger space, the Schwartz space $S(\mathbf{R})$, is defined to be the set of all C^∞ functions on $\mathbf{R}$ such that for every nonnegative integer m and every multi-index α,

$$\sup_{x \in \mathbf{R}} (1 + |x|^2)^{\frac{m}{2}} |D^\alpha u(x)| < \infty. \tag{1.8}$$

If f is any function in $S(\mathbf{R})$, the Fourier transform of f is denoted by $\widehat{f}$ or $\mathcal{F}(f)$ and is defined by

$$\mathcal{F}(f)[\omega] := \widehat{f}(\omega) = \int_{-\infty}^{+\infty} e^{-i\omega x} f(x) \, dx.$$

The inverse of the Fourier transform is given for $f \in S(\mathbf{R})$ by

$$\mathcal{F}^{-1}(f)[x] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\omega x} f(\omega) \, d\omega,$$

and for all f and g in $S(\mathbf{R})$ one has the formula

$$\mathcal{F}(fg) = \mathcal{F}(f) * \mathcal{F}(g), \tag{1.9}$$

where the convolution $p * q$ of two functions p and q is defined by

$$p * q(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} p(\omega - \omega_1)q(\omega_1) \ d\omega_1.$$

The spare $S(\mathbf{R})$ can be given a topology based on the family of seminorms defined in (1.8). The bounded linear functionals on $S(\mathbf{R})$ with respect to this topology are called tempered distributions, and the Fourier transform can be naturally extended to the space of tempered distributions. For any tempered distribution f on $\mathbf{R}$ whose Fourier transform $\hat{f}$ is a function, and any $s \in \mathbf{R}$, we define

$$\|f\|_{H^s}^2 = \left(\int_{-\infty}^{+\infty} (1 + |\omega|^2)^s |\hat{f}(\omega)|^2 \ d\omega \right)$$

and $H^s = H^s(\mathbf{R})$ denotes the Sobolev space of all f for which $\|f\|_{H^s}$ is finite.

If X is any Banach space with norm $\| \cdot \|_X$, and $[a, b] \subset \mathbf{R}$, we define $C([a,b]; X)$ to be the Banach space of all continuous maps $u : [a, b] \to X$ with norm

$$\|u\|_{C([a,b];\mathbf{R})} = \sup_{t \in [a,b]} \|u(t)\|_X.$$

In this thesis we will use C to stand for several constants, whose value can vary from line to line.

We recall the following standard results.

Lemma 1.1. *There exists $C > 0$ such that for all $f \in H^1$, if $g = |f|$ then*

$$\|g\|_{H^1} \leq C\|f\|_{H^1}.$$

10

Proof. See theorem 6.17 in [LL]. □

Theorem 1.2 (Riesz-Thorin Interpolation Theorem). *Suppose X and Y are measurable spaces, $1 \leq p_0 < p_1 \leq \infty$, $1 \leq q_0 < q_1 \leq \infty$, and suppose L is a bounded linear operator from $L^{p_0}(X)$ to $L^{q_0}(Y)$ with norm M_0 and from $L^{p_1}(X)$ to $L^{q_1}(Y)$ with norm M_1. Then for all $\theta \in (0,1)$, L is bounded from $L^{p_\theta}(X)$ to $L^{q_\theta}(Y)$ with norm M_θ such that*

$$M_\theta \leq M_0^{1-\theta} M_1^\theta,$$

where $\dfrac{1}{p_\theta} = \dfrac{1-\theta}{p_0} + \dfrac{\theta}{p_1}$ *and* $\dfrac{1}{q_\theta} = \dfrac{1-\theta}{q_0} + \dfrac{\theta}{q_1}.$

Proof. See [LP]. □

Theorem 1.3 (Hardy-Littlewood Inequality). *Suppose $\beta \in (0,1)$ and $p > 1$. Then for all $f \in L^p$, the integral*

$$I_\beta f(x) = \int_{-\infty}^{+\infty} \frac{f(y)}{|x-y|^{1-\beta}} \, dy$$

is absolutely convergent for almost every $x \in \mathbf{R}$. Moreover, there exists $C > 0$ such that for all $f \in L^p$,

$$\|I_\beta f\|_{L^q} \leq C\|f\|_p,$$

where $\dfrac{1}{q} = \dfrac{1}{p} - \beta.$

Proof. See Theorem 2.6 in [LP]. □

Theorem 1.4 (Banach algebra property). *If $u, v \in H^r(\mathbf{R}^n)$ for $r > \frac{n}{2}$, then $uv \in H^r(\mathbf{R}^n)$ and*

$$\|uv\|_{H^r(\mathbf{R}^n)} \leq C\|u\|_{H^r(\mathbf{R}^n)}\|v\|_{H^r(\mathbf{R}^n)},$$

The constant C depends only on r and n.

Proof. See Theorem 3.4 in [LP]. □

Theorem 1.5 (Minkowski's Integral Inequality)**.** *Let f be a nonnegative function, and let $1 \leq p < \infty$. Then*

$$\left(\int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} f(x,y) \; dy \right)^p dx \right)^{\frac{1}{p}} \leq \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} f(x,y)^p \; dx \right)^{\frac{1}{p}} dy.$$

Proof. See Theorem 2.4 in [LL]. □

Note that Minkowski's integral inequality can be written as

$$\left\| \int_{-\infty}^{+\infty} f(\cdot,y) \; dy \right\|_{L^p} \leq \int_{-\infty}^{+\infty} \| f(\cdot,y) \|_{L^p} \, dy.$$

In fact, this inequality can be generalized by replacing the L^p norm with other Banach function norms, such as the H^1 norm (see for example [S] for a general result). We will use such generalized Minkowski's inequalities freely below.

Theorem 1.6 (Generalized Hölder's Inequality)**.** *Let p_1, $\ldots$, p_m be such that $1 \leq p_j \leq \infty$ for $1 \leq j \leq m$ and $\sum_{j=1}^{m} \dfrac{1}{p_j} = 1$, and suppose $f_j \in L^{p_j}$ for $1 \leq j \leq m$. Then $\prod_{j=1}^{m} f_j = (f_1 f_2 \cdots f_m) \in L^1$, and*

$$\int_{-\infty}^{+\infty} \left| \prod_{j=1}^{m} f_j \right| \; dx \leq \prod_{j=1}^{m} \| f_j \|_{L^{p_j}}.$$

Proof. See Theorem 2.3 in [LL] and the remarks following. □

Theorem 1.7 (Plancherel's Theorem)**.** *Let $f \in L^2$. Then $\hat{f} \in L^2$ and*

$$\| \hat{f} \|_{L^2} = \| f \|_{L^2}.$$

Also, for all $g \in L^2$, $\int_{-\infty}^{+\infty} \hat{f}(x)\overline{\hat{g}(x)} \, dx = \int_{-\infty}^{+\infty} f(x)\overline{g(x)} \, dx$.

Proof. See Theorem 1.3 in [LP]. □

Theorem 1.8 (Gagliardo-Nirenberg Inequality). *Let q, r be any numbers satis-fying $1 \leq q \leq \infty$ and $1 < r \leq \infty$. If u is any function in L^q such that $\frac{du}{dx} \in L^r$, then*

$$\|u\|_{L^p} \leq C \left\|\frac{du}{dx}\right\|_{L^r}^{\theta} \|u\|_{L^q}^{1-\theta},$$

where

$$\frac{1}{p} = \theta \left(\frac{1}{r} - 1\right) + (1 - \theta)\frac{1}{q},$$

for all $\theta \in [0, 1]$.

Proof. See Theorem 9.3, page 29, of [F]. □

Corollary 1.9. *If $u \in H^1$, then $u \in L^\infty$, and*

$$\|u\|_{L^\infty} \leq C\|u\|_{H^1}.$$

Proof. This follows from the Gagliardo-Nirenberg Inequality if we take $p = \infty$, $r = q = 2$, and $\theta = 1/2$. □

We next state a few basic lemmas concerning the operators $T(s)$ defined in (1.6).

Lemma 1.10. *For every $u \in L^2$, $\overline{T(s)u} = T^{-1}(s)\overline{u}$.*

Proof. It is enough to show that the equality holds for $u \in C_0^\infty(\mathbf{R})$.

For such u, we have

$$\overline{T(s)u}(x) = \overline{\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ix\omega} \widehat{T(s)u}(\omega) \ d\omega}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ix\omega} \overline{\widehat{T(s)u}(\omega)} \ d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ix\omega} \overline{e^{i\omega^2 D(s)} \widehat{u}(\omega)} \ d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ix\omega} e^{-i\omega^2 D(s)} \overline{\widehat{u}(\omega)} \ d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ix\omega} e^{-i\omega^2 D(s)} \widehat{\overline{u}}(-\omega) \ d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ix\omega} e^{-i\omega^2 D(s)} \widehat{\overline{u}}(\omega) \ d\omega$$

$$= (T^{-1}(s)\overline{u})(x).$$

$\square$

Lemma 1.11. *For $u, v \in L^2$,*

$$\int_{-\infty}^{+\infty} u(x) \overline{T^{-1}(s)v(x)} \ dx = \int_{-\infty}^{+\infty} T(s)u(x) \overline{v(x)} \ dx.$$

Proof. By Plancherel's Theorem,

$$\int_{-\infty}^{+\infty} u(x) \overline{T^{-1}(s)v(x)} \ dx = \int_{-\infty}^{+\infty} \widehat{u}(\omega) \overline{\widehat{T^{-1}(s)v}(\omega)} \ d\omega$$

$$= \int_{-\infty}^{+\infty} \widehat{u}(\omega) \overline{e^{-i\omega^2 D(s)} \widehat{v}(\omega)} \ d\omega$$

$$= \int_{-\infty}^{+\infty} \widehat{u}(\omega) e^{i\omega^2 D(s)} \overline{\widehat{v}}(\omega) \ d\omega$$

$$= \int_{-\infty}^{+\infty} \widehat{T(s)u}(\omega) \overline{\widehat{v}}(\omega) \ d\omega,$$

$$= \int_{-\infty}^{+\infty} T(s)u(x) \overline{v(x)} \ dx.$$

$\square$

Lemma 1.12. *Let $k \in \mathbf{R}$. Then for all $f \in H^k$,*

$$\|T(s)f\|_{H^k} = \|f\|_{H^k}.$$

Proof. Observe that

$$
\begin{aligned}
\|T(s)f\|_{H^k}^2 &= \int_{-\infty}^{+\infty} (1 + |\omega|^2)^k |\widehat{T(s)f}(\omega)|^2 \, d\omega \\
&= \int_{-\infty}^{+\infty} (1 + |\omega|^2)^k |e^{i\omega^2 D(s)} \widehat{f}(\omega)|^2 \, d\omega \\
&= \int_{-\infty}^{+\infty} (1 + |\omega|^2)^k |\widehat{f}(\omega)|^2 \, d\omega \\
&= \|f\|_{H^k}^2.
\end{aligned}
$$

$\square$

We will also use below the operator $e^{i\partial_x^2 t} : L^2 \to L^2$ defined by $\widehat{e^{i\partial_x^2 t} f}[\omega] = e^{-i\omega^2 t} \widehat{f}(\omega)$. The same proof as given in Lemma 1.12 shows that $\|e^{i\partial_x^2 t} f\|_{H^k} = \|f\|_{H^k}$, for all $f \in H^k$.

Lemma 1.13. *If $u \in H^r$ for $r \geq 1$, and $w(s) := T(s)u$, then $w \in C\left([0,1]; H^r\right)$.*

Proof. Since T is unitary, we have

$$\||T(s)u|^2 T(s)u\|_{H^r} \leq \|T(s)u\|_{H^r}^3 = \|u\|_{H^r}^3.$$

We also have

$$
\begin{aligned}
\|T(s)u - T(s_1)u\|_{H^r}^2 &= \int \left(1 + |\omega|^2\right)^r \left|\left[e^{iD(s)\omega^2} \widehat{u}(\omega) - e^{iD(s_1)\omega^2} \widehat{u}(\omega)\right]\right|^2 \, d\omega \\
&= \int \left(1 + |\omega|^2\right)^r \left|e^{iD(s)\omega^2} - e^{iD(s_1)\omega^2}\right|^2 |\widehat{u}(\omega)|^2 \, d\omega
\end{aligned}
$$

By the assumption on $\Delta(s)$, we have that $D(s)$ is continuous on $[0,1]$, so $\lim_{s \to s_1} e^{iD(s)\omega^2} = e^{iD(s_1)\omega^2}$ uniformly for $s_1 \in [0,1]$. Also, since $u \in H^r$, then

15

$(1 + |\omega|^2)^r |\widehat{u}(\omega)|^2$ is in $L^1(\mathbf{R})$. Therefore, by the Lebesgue Dominated Convergence Theorem, we have

$$\lim_{s \to s_1} \|T(s)u - T(s_1)u\|_{H^r} = 0.$$

This proves $T(s)u \in C\left([0, 1]; H^r\right)$. $\qquad\square$

Chapter 2

Well-posedness of the initial-value problem

Rewrite the DMNLS equation (1.3) as

$$iu_t + \alpha u_{xx} + Q(u) = 0 \qquad (2.1)$$

where $\alpha \in \mathbf{R}$, and we define $Q(u)$ as

$$Q(u) = \int_0^1 g(s)T^{-1}(s)[|T(s)u|^2 T(s)u]ds. \qquad (2.2)$$

Notice that $Q(u)$ will be well-defined if, for example, $|T(s)u|^2 T(s)u \in C\left([0,1]; H^r\right)$ for some $r \in \mathbf{R}$.

Definition 2.1. Suppose $X \subset H^r$ for some $r \in \mathbf{R}$, and suppose $u_0 \in X$ and $M > 0$. We say $u(x,t) \in C([0,M]; X)$ is a strong solution of (2.1) with initial data u_0 if

(a) for every $t \in [0, M]$, if we fix $u = u(t) = u(\cdot, t)$ and define $w(s) = |T(s)u|^2 T(s)u$, then $w(s) \in C([0,1]; H^r)$, so $Q(u)$ is well-defined, and

(b) for all $t \in [0, M]$,

$$u = e^{it\alpha\partial_x^2}u_0 + i \int_0^t e^{i\alpha(t-t')\partial_x^2}Q(u(t'))\, dt'.$$

2.1 Conserved quantities

Define functionals $E : H^1 \to \mathbf{R}$ and $P : H^1 \to \mathbf{R}$ by

$$E(u) = \int_{-\infty}^{+\infty} \int_0^1 \left(\alpha \, |u_x|^2 - \frac{1}{2} \, g(s) \, |T(s)u|^4 \right) \, ds \, dx \qquad (2.3)$$

and

$$P(u) = \int_{-\infty}^{+\infty} |u|^2 \, dx. \qquad (2.4)$$

Theorem 2.2. *If* $u = u(x,t) \in C([0, M]; H^r)$ *is a strong solution of* (2.1) *with r sufficiently large, then $E(u(x,t))$ and $P(u(x,t))$ are independent of $t \in [0, M]$.*

Proof. Suppose $r \in \mathbf{R}$ and $u = u(x,t) \in C([0, M]; H^r)$ is a strong solution of (2.1), so that

$$iu_t + \alpha u_{xx} + \int_0^1 g(s)T^{-1}(s)(|T(s)u|^2 T(s)u)ds = 0. \qquad (2.5)$$

for $t \in [0, M]$. Notice that if $r \geq 1$, then it follows from (2.5), Theorem 1.4, and Lemma 1.13 that $u_t \in C([0, M]; H^{r-2})$. So, for r sufficiently large, the assumption that $u = u(x,t) \in C([0, M]; H^r)$ guarantees that u, u_t and their derivatives with respect to x are smooth enough and tend to zero rapidly enough as $|x| \to \infty$, that the integrations by parts below will be justified. We assume in advance that r has been so chosen.

To prove that $P(u)$ is independent of t, first multiply by $\bar{u}$ and integrate with respect to x over $(-\infty, \infty)$, to get

$$i \int_{-\infty}^{+\infty} \bar{u}u_t \, dx + \alpha \int_{-\infty}^{+\infty} \bar{u}u_{xx} \, dx + \int_{-\infty}^{+\infty} \bar{u} \left\{ \int_0^1 g(s)T^{-1}(s)(|T(s)u|^2 T(s)u) \, ds \right\} \, dx = 0. \qquad (2.6)$$

18

Taking the conjugate gives

$$-i \int_{-\infty}^{+\infty} \overline{u}_t u \, dx + \alpha \int_{-\infty}^{+\infty} \overline{u}_{xx} u \, dx + \int_{-\infty}^{+\infty} u \int_0^1 g(s) \overline{T^{-1}(s)(|T(s)u|^2 T(s)u)} \, ds \, dx = 0,$$

(2.7)

and subtracting equation (2.6) from equation (2.7) gives

$$i \int_{-\infty}^{+\infty} (\overline{u} u_t + \overline{u}_t u) \, dx + \alpha \int_{-\infty}^{+\infty} (\overline{u} u_{xx} - \overline{u}_{xx} u) \, dx + A - B,$$

(2.8)

where

$$A = \int_{-\infty}^{+\infty} \overline{u} \int_0^1 g(s) T^{-1}(s)(|T(s)u|^2 T(s)u) \, ds \, dx$$

and

$$B = \int_{-\infty}^{+\infty} u_{xx} \int_0^1 g(s) \overline{T^{-1}(s)(|T(s)u|^2 T(s)u)} \, ds \, dx.$$

Now

$$A = \int_0^1 g(s) \int_{-\infty}^{+\infty} \overline{u} T^{-1}(s)(|T(s)u|^2 T(s)u) \, dx \, ds$$
$$= \int_0^1 g(s) \int_{-\infty}^{+\infty} \overline{u T^{-1}(s)(|T(s)u|^2 T(s)u)} \, dx \, ds.$$

From Lemma 1.11 it follows that

$$A = \int_0^1 g(s) \int_{-\infty}^{+\infty} \overline{T(s)u |T(s)u|^2 \overline{T(s)}u} \, dx \, ds$$
$$= \int_0^1 g(s) \int_{-\infty}^{+\infty} \overline{|T(s)u|^4} \, dx \, ds$$
$$= \int_0^1 g(s) \int_{-\infty}^{+\infty} |T(s)u|^4 \, dx \, ds.$$

Similarly,

$$B = \int_0^1 g(s) \int_{-\infty}^{+\infty} u \overline{T^{-1}(s)(|T(s)u|^2 T(s)u)} \, dx \, ds,$$

and from Lemma 1.11 it follows that

$$B = \int_0^1 g(s) \int_{-\infty}^{+\infty} T(s)u \overline{(|T(s)u|^2 T(s)u)} \; dx \; ds$$

$$= \int_0^1 g(s) \int_{-\infty}^{+\infty} T(s)u (|T(s)u|^2 \overline{T(s)u}) \; dx \; ds$$

$$= \int_0^1 g(s) \int_{-\infty}^{+\infty} |T(s)u|^4 \; dx \; ds.$$

Therefore, $A - B = 0$. Also, an integration by parts shows that $\int \bar{u}_{xx} u - \bar{u} u_{xx} = 0$. So

$$\frac{d}{dt} \int_{-\infty}^{+\infty} |u|^2 \; dx = \int_{-\infty}^{+\infty} (\bar{u} u_t + \bar{u}_t u) \; dx = 0,$$

and hence $\int_{-\infty}^{+\infty} |u|^2 \; dx$ is independent of t.

To prove E is independent of t, write

$$\frac{dE(u)}{dt} = \frac{d}{dt} \int_{-\infty}^{+\infty} \left\{ \alpha |u_x|^2 - \frac{1}{2} \int_0^1 g(s) \, |T(s)u|^4 \; ds \right\} \; dx$$

$$= \int_{-\infty}^{+\infty} \left\{ \alpha \left(u_x \overline{u_{xt}} + \overline{u_x} u_{xt} \right) - \frac{1}{2} \int_0^1 g(s) \left(2T(s)u T(s)u_t \overline{(T(s)u)}^2 \right. \right.$$

$$\left. \left. + (T(s)u)^2 \, 2\overline{(T(s)u)(T(s)u_t)} \right) \; ds \right\} \; dx,$$

$$= -\alpha \int_{-\infty}^{+\infty} \overline{u_{xx}} u_t \; dx - \int_{-\infty}^{+\infty} \int_0^1 g(s) u_t \overline{T^{-1}(s) \left\{ |T(s)u|^2 \, T(s)u \right\}} \; ds \; dx + c.c.,$$

where in the last step we have used Lemma 1.11, and *c.c.* denotes the complex conjugate of the first two terms on the right-hand side.

Let $V = \alpha u_{xx} + \int_0^1 g(s) T^{-1}(s) \left\{ |T(s)u|^2 \, T(s)u \right\} \; ds$. Then

$$\frac{dE(u)}{dt} = - \left(\int_{-\infty}^{+\infty} u_t \overline{V} \; dx + \int_{-\infty}^{+\infty} \overline{u_t} V \; dx \right).$$

20

But by equation (2.1), $u_t = iV$ and $\overline{u_t} = -i\overline{V}$. So

$$\frac{dE(u)}{dt} = -\left(\int_{-\infty}^{+\infty} iV\overline{V} \ dx + \int_{-\infty}^{+\infty} -i\overline{V}V \ dx \right) = 0.$$

□

Lemma 2.3. *Suppose $\alpha \neq 0$. Then there exists a function $f_\alpha : [0, \infty) \times [0, \infty) \to [0, \infty)$ such that if $a \geq 0$ and $b \geq 0$, and $u \in H^1$ satisfies $P(u) \leq a$ and $|E(u)| \leq b$, then*

$$\|u\|_{H^1} \leq f_\alpha(a, b).$$

Proof. For $u \in H^1$ we have

$$
\begin{aligned}
\|u\|_{H^1}^2 &= \left(\frac{1}{\alpha} \alpha \right) \left(\int_{-\infty}^{+\infty} |u|^2 \ dx + \int_{-\infty}^{+\infty} |u_x|^2 \ dx \right) \\
&= \frac{1}{\alpha} \left(\alpha P(u) + E(u) + \frac{1}{2} \int_{-\infty}^{+\infty} \int_0^1 |T(s)u|^4 \ ds \ dx \right) \\
&\leq \frac{1}{|\alpha|} \left(|\alpha|a + b + C \int_0^1 \|T(s)u_x\|_{L^2} \|T(s)u\|_{L^2}^3 \ ds \right),
\end{aligned}
$$

where in the last step we used the Gagliardo-Nirenberg Inequality, Theorem 1.7, with $p = 4$, $r = q = 2$, and $\theta = 1/4$. Therefore, by Lemma 1.3,

$$
\begin{aligned}
\|u\|_{H^1}^2 &\leq \frac{1}{|\alpha|} \left(|\alpha|a + b + C \int_0^1 \|u_x\|_{L^2} \|u\|_{L^2}^3 \ ds \right) \\
&\leq \frac{1}{|\alpha|} \left(|\alpha|a + b + Ca^3 \|u\|_{H^1} \right).
\end{aligned}
$$

This proves there exists $M_1 > 0$ and $M_2 > 0$, depending only on a, b, and α, such that

$$\|u\|_{H^1}^2 \leq M_1 + M_2 \|u\|_{H^1},$$

and the conclusion of the Lemma follows easily.

□

2.2   Well-posedness in H^1

Here is our well-posedness result for the DMNLS equation (2.1) in Sobolev spaces H^r of order $r \geq 1$.

Theorem 2.4. *Suppose $\alpha \in \mathbf{R}$ and $r \geq 1$.*

(a) *Suppose $\Delta(s)$ and $g(s)$ are integrable on $[0,1]$. For every $u_0 \in H^r$, there exists a number $M > 0$, which can be chosen to depend only on $\|u_0\|_{H^r}$, such that equation (2.1) has a unique strong solution $u \in C([0,M];H^r)$ with initial data u_0. Moreover, the solution depends continuously on the initial data; that is, the map $u_0 \mapsto u$ is continuous from H^r to $C([0,M];H^r)$.*

(b) *With M as in part (a), $E(t)$ and $P(t)$ are independent of t for $t \in [0,M]$.*

(c) *If $\alpha \neq 0$ and $r = 1$, then M in part (a) can be taken arbitrarily large.*

Proof. Suppose $u_0 \in H^r$ is given. If $M > 0$ and $v \in C([0,M];H^r)$, define

$$\phi(v) = \exp\left(it\alpha\partial_x^2\right)u_0 + \int_0^t \exp\left(i\alpha(t-t')\partial_x^2\right)[Q(v)] \; dt',$$

where $Q(v)$ is defined in (2.2). Notice that $Q(v)$ is well-defined by Lemma 1.13 and the comment before Definition 2.1.

Let $||| \cdot |||$ denote the norm in $C([0,M];H^r)$, so $|||v||| = \sup_{t\in[0,M]} \|v(.,t)\|_{H^r}$, and let $E(M,a)$ be the closed ball of radius a in H^r, centered at the origin:

$$E(M,a) = \{v \in C([0,M];H^r) : |||v||| \leq a\}.$$

We want to show that for every $a > 2\|u_0\|_{H^r}$, there exists $M > 0$ such that ϕ defines a contraction map on $E(M,a)$.

First we have to show that if M is chosen small enough, then ϕ maps $E(M, a)$ into itself. Suppose $v \in E(M, a)$ and $a > 2\|u_0\|_{H^r}$. Then for all $t \in [0, M]$, by Minkowski's Integral Inequality,

$$\|\phi(v)\|_{H^r} \leq \|\exp\left(i\alpha t\partial_x^2\right)u_0\|_{H^r} + \int_0^t \|\exp\left(i\alpha(t - t')\partial_x^2\right)Q(v)\|_{H^r}\ dt'.$$

From Lemma 1.12, we then have that

$$\|\phi(v)\|_{H^r} \leq \|u_0\|_{H^r} + \int_0^t \|Q(v)\|_{H^r}\ dt'.$$

Now, again by Minkowski's Integral Inequality, for $v = v(t')$ we have

$$\|Q(v)\|_{H^r} = \left\|\int_0^1 g(s)T^{-1}(s)(|T(s)v|^2 T(s)v)\ ds\right\|_{H^r}$$
$$\leq \beta \int_0^1 \|T^{-1}(s)(|T(s)v|^2 T(s)v)\|_{H^r}\ ds;$$

and hence, by Lemma 1.12,

$$\|Q(v)\|_{H^r} \leq \beta \int_0^1 \||T(s)v|^2 T(s)v\|_{H^r}\ ds.$$

From Lemma 1.4 it follows that

$$\|Q(v)\|_{H^r} \leq \beta C \int_0^1 \||T(s)v\|_{H^r}^2 \|T(s)v\|_{H^r}\ ds,$$

and hence, by Lemma 1.1,

$$\|Q(v)\|_{H^r} \leq \beta C \int_0^1 \|T(s)v\|_{H^r}^3\ ds.$$

23

By Lemma 1.12,

$$\|Q(v)\|_{H^r} \le \beta C \int_0^1 \|v\|_{H^r}^3 \, ds = \beta C \|v(t')\|_{H^r}^3.$$

Since $v \in E(M, a)$, we have

$$\|v(t')\|_{H^r} \le a$$

for all $t' \in [0, M]$. Therefore it follows that

$$\|\phi(v)\|_{H^r} \le \|u_0\|_{H^r} + \int_0^t \|v(t')\|_{H^r}^3 \, dt' \le \|u_0\|_{H^r} + Ca^3 t. \qquad (2.9)$$

Now choose M such that

$$Ca^3 M \le a/2. \qquad (2.10)$$

From (2.9) we obtain

$$\|\phi(v)\|_{H^r} \le a/2 + a/2 = a.$$

This proves that $\phi(v) \in E(M, a)$, whenever (2.10) holds and $a > 2\|u_0\|_{H^r}$.

Also for $u, v \in E(M, a)$, for all $t \in [0, M]$,

$$\|\phi(u) - \phi(v)\|_{H^r} = \left\| \int_0^t \exp\left(i\alpha(t - t')\partial_x^2\right)[Q(u) - Q(v)] \, dt' \right\|_{H^r}.$$

So, by Minkowski's inequality,

$$\|\phi(u) - \phi(v)\|_{H^r} \le \int_0^t \|Q(u) - Q(v)\|_{H^r} \, dt'.$$

Now, for each $t' \in [0, t]$, letting $u = u(t')$ and $v = v(t')$, we have

$$\|Q(u) - Q(v)\|_{H^r} = \left\| \int_0^1 \left(g(s)T^{-1}(s)|T(s)u|^2 T(s)u) - g(s)T^{-1}(s)|T(s)v|^2 T(s)v \right) \, ds \right\|_{H^r}$$

$$\leq \sup_{s \in [0,1]} \left\| T^{-1}(s) \left(|T(s)u|^2 T(s)u - |T(s)v|^2 T(s)v \right) \right\|_{H^r} \int_0^1 |g(s)| \, ds.$$

So, by Lemma 1.12,

$$\|Q(u) - Q(v)\|_{H^r} \leq \gamma \sup_{s \in [0,1]} \left\| |T(s)u|^2 T(s)u - |T(s)v|^2 T(s)v \right\|_{H^r},$$

where $\gamma = \int_0^1 |g(s)| \, ds < \infty$ by assumption.

Now for all $s \in [0, 1]$,

$$\| |T(s)u|^2 T(s)u - |T(s)v|^2 T(s)v \|_{H^r}$$

$$\leq \| |T(s)u|^2 (T(s)u - T(s)v)$$

$$\quad + (T(s)u - T(s)v)\overline{T(s)}uT(s)v + \overline{(T(s)u - T(s)v)}(T(s)v)^2 \|_{H^r}$$

$$\leq \|T(s)u - T(s)v\|_{H^r} \left(\|T(s)v\|_{H^r}^2 + \|T(s)u\|_{H^r}^2 + \|T(s)v\|_{H^r} \|T(s)u\|_{H^r} \right)$$

$$= \|u - v\|_{H^r} \left(\|v\|_{H^r}^2 + \|u\|_{H^r}^2 + \|v\|_{H^r} \|u\|_{H^r} \right)$$

$$\leq 3a^2 \sup_{t \in [0,T]} \|u - v\|_{H^r} = 3a^2 |||u - v|||.$$

So for $u, v \in E(M, a)$

$$|||Q(u) - Q(v)||| = \sup_{0 \leq t \leq M} \|Q(u) - Q(v)\|_{H^r}$$

$$\leq \beta \int_0^1 3a^2 |||u - v||| \, ds$$

$$= 3a^2 \beta |||u - v|||.$$

It follows that

$$|||\phi(u) - \phi(v)||| \leq \sup_{0 \leq t \leq M} \int_0^t 3a^2 \beta |||u - v||| \ dt'$$

$$= \sup_{0 \leq t \leq M} 3a^2 \beta t |||u - v|||$$

$$= 3a^2 \beta C M |||u - v|||.$$

If we now choose M such that $3a^2 \beta C M < \frac{1}{2}$, then it follows that

$$|||\phi(u) - \phi(v)||| < \frac{1}{2}|||u - v|||,$$

for all u, v in $E(M, a)$, so ϕ is a contraction.

Note that we have shown that if M and a are any positive numbers such that

$$\begin{aligned} a &> 2|||u_0|||_{H^r}, \\ M &< \frac{1}{2a^2 C}, \\ M &< \frac{1}{6a^2 \beta C} \end{aligned} \tag{2.11}$$

all hold, then $\phi : E(M, a) \to E(M, a)$ and

$$|||\phi(u) - \phi(v)||| \leq \frac{1}{2}|||u - v|||$$

for all $u, v \in E(M, a)$. In particular, from (2.11) we see that M can be chosen to depend only on $\|u_0\|_{H^r}$, and not otherwise on u_0.

By the Banach Contraction Mapping Theorem, ϕ has a unique fixed point $u \in E(M, a)$, which is therefore a strong solution of (2.1) in $C([0, M]; H^r)$.

To prove the continuity of the solution with respect to the initial data, first

26

fix $u_0 \in H^r$, and define a neighborhood Λ of u_0 in H^r by

$$\Lambda = \left\{ \widetilde{u_0} \in H^r : \|u_0 - \widetilde{u_0}\|_{H^r} < \|u_0\|_{H^r} \right\}. \tag{2.12}$$

Observe that we can extend ϕ to a map depending on the initial data $\widetilde{u_0}$ as a parameter, by defining $\phi : C\left([0, M]; H^r\right) \times \Lambda \to C\left([0, M]; H^r\right)$ as

$$\phi(v, \widetilde{u_0}) := e^{\left(it\alpha\partial_x^2\right)} \widetilde{u_0} + \int_0^t e^{\left(i\alpha(t-t')\partial_x^2\right)} Q(v) \, dt'.$$

Now let

$$a = 10\|u_0\|_{H^r} \tag{2.13}$$

and choose M such that the last two inequalities of (2.11) are satisfied. Then for all $\widetilde{u_0} \in \Lambda$, we have

$$\|\widetilde{u_0}\|_{H^r} \le 2\|u_0\|_{H^r}$$

and so

$$a \ge 2\|\widetilde{u_0}\|_{H^r}. \tag{2.14}$$

Therefore the inequalities (2.11) hold with u_0 replaced by $\widetilde{u_0}$, for all $\widetilde{u_0} \in \Lambda$. As noted above after (2.11), it follows that $\phi : E(M, a) \times \Lambda \to E(M, a)$ and that ϕ is uniformly contractive in the first argument, with constant $1/2$, for all $\widetilde{u_0}$. It then follows from the proof of the Banach Contraction Mapping Theorem (see, for example, Exercise $A.4$ page 18 in [GD]), that for each $\widetilde{u_0} \in \Lambda$, there is a unique fixed point u of $\phi(\cdot, \widetilde{u_0})$ in $E(M, a)$, and the map $\widetilde{u_0} \mapsto u$ is continuous from H^r to $C([0, M]; H^r)$. This then completes the proof of part (a) of the Theorem.

To prove part (b), first approximate u_0 in H^r norm by a sequence u_{0n} of functions in H^{r_0}, for r_0 large enough that Theorem 2.2 holds with $r = r_0$. Then

by part (a), for each n there is a strong solution u_n of (2.1) in $C([0, M]; H^{r_0})$ with initial data u_0. Then from Theorem 2.2, $E(u_n(\cdot, t)) = E(u_{0n})$ and $P(u_n(\cdot, t) = P(u_{0n})$ for $t \in [0, M]$. But as shown in part (a), solutions depend continuously on the initial data in H^r norm, so since $u_{0n} \to u_0$ in H^r, then $u_n \to u$ in $C([0, M]; H^r)$. Since E and P are continuous maps from H^r to $\mathbf{R}$ for $r \geq 1$, it follows that $E(u(\cdot, t)) = E(u_0)$ and $P(u(\cdot, t)) = P(u_0)$ for $t \in [0, M]$, thus proving part (b) of the Theorem.

To prove part (c), suppose $\alpha \neq 0$ and let $u_0 \in H^1$ be given. Let

$$M_s = \sup \left\{ M > 0 : \text{there exists a strong solution } u \text{ of } (2.1) \text{ in } C([0, M]; H^1) \right\}.$$

We claim that $M_s = \infty$. To see this, we suppose to the contrary that $M_s < \infty$ and will get a contradiction.

Let $B = f_\alpha(P(u_0), |E(u_0)|)$, where f_α is the function defined in Lemma 2.3. By part (a), there exists a number $M_1 > 0$ such that if $v_0 \in H^1$ and $\|v_0\|_{H^1} \leq B$, then a strong solution v of (2.1) with initial data v_0 exists in $C([0, M_1]; H^1)$. Let $t_1 = M_s - M_1/2$, and let $v_0(x) = u(x, t_1)$, where $u \in C([0, t_1]; H^1)$ is a strong solution of (2.1) with initial data u_0. By part (b), we have that $P(v_0) = P(u_0)$ and $E(v_0) = E(u_0)$, so by Lemma 2.3, $\|v_0\|_{H^1} \leq B$. Therefore a strong solution v of (2.1) with initial data v_0 exists in $C([0, M_1]; H^1)$.

But it is easy to see that if we define

$$w(x, t) = \begin{cases} u(x, t) & \text{for } t \in [0, t_1] \\ v(x, t - t_1) & \text{for } t \in [t_1, M_1 + t_1], \end{cases}$$

then w is a strong solution of (2.1) in $C([0, M_1 + t_1], H^1)$ with initial data u_0. Since $M_1 + t_1 > M_s$, this contradicts the definition of M_s. So (c) is proved. $\quad\square$

2.3 Well-posedness in L^2 and in $L^2 \cap L^\infty$

Next we would like to prove a well-posedness result for equation (2.1) in L^2.
For this purpose we need to put extra assumptions on $\Delta(s)$; namely that it is
piecewise of one sign and bounded away from zero. So, for the remainder of
Chapter 2, the following assumption will be in force:

Assumption D1. The function $\Delta(s)$ is integrable on $[0,1]$; and there exist $\Delta_0 > 0$
and numbers $s_0, s_1, \ldots, s_n$, with $0 = s_0 < s_1 < s_2 < \cdots < s_{n-1} < s_n = 1$, such
that for all $j \in \{1, \cdots, n\}$, either $\Delta(s) \geq \Delta_0$ for almost every $s \in [s_{j-1}, s_j]$, or
$\Delta(s) \leq -\Delta_0$ for almost every $s \in [s_{j-1}, s_j]$.

Recall that $D(s) = \int_0^s \Delta(\zeta)\, d\zeta$. So it follows from the above assumption
that, for all $j \in \{1, \cdots, n\}$, $D(s)$ is absolutely continuous on $[s_{j-1}, s_j]$ and
either $D'(s) \geq \Delta_0$ almost everywhere on $[s_{j-1}, s_j]$, or $D'(s) \leq -\Delta_0$ almost
everywhere on $[s_{j-1}, s_j]$.

Theorem 2.5 (Strichartz Estimate)**.** *Suppose $2 \leq p \leq \infty$ and*

$$\frac{2}{q} = \frac{1}{2} - \frac{1}{p} \ .$$

There exists $C > 0$ such that for all $f \in L^2$,

$$\|e^{it\partial_x^2} f\|_{L_t^q L_x^p} \leq C\|f\|_2.$$

Proof. See Theorem 4.2 of [LP]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 2.6. *For $\phi \in S(\mathbf{R})$ and $t \in \mathbf{R}$*

$$\exp(-i\partial_x^2 t)\phi(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi it}} \exp\left[-i\left(\frac{(x-y)^2}{4t}\right)\right] \phi(y)\, dy$$

Proof. See [LP], page 17. □

In the next theorem we verify that the Strichartz estimate in Theorem 2.5 remains valid if $e^{it\partial_x^2}$ is replaced by $T(t) = e^{iD(t)\partial_x^2}$. A similar result appears in Lemma 2.5 of [ASS].

Theorem 2.7. *Suppose $D'(s) = \Delta(s)$ satisfies Assumption D1. Assume $q \in [4, \infty]$, and define $r \in [2, \infty]$ by*

$$\frac{2}{q} = \frac{1}{2} - \frac{1}{r}.$$

Then for all $u \in L^2$,

$$\|T(t)u\|_{L_t^q((0,1),L_x^r)} := \left(\int_0^1 \|T(t)u\|_{L^r}^q \, dt \right)^{1/q} \leq C\|u\|_{L^2}, \tag{2.15}$$

with C depending only on q.

Proof. We have

$$
\begin{aligned}
\|T(t)u\|_{L_t^q((0,1),L_x^r)} &= \left(\int_0^1 \|T(t)u\|_{L^r}^q \, dt \right)^{1/q} \\
&= \left(\sum_{j=1}^n \int_{s_{j-1}}^{s_j} \|T(t)u\|_{L^r}^q \, dt \right)^{1/q} \\
&\leq C \sum_{j=1}^n \left(\int_{s_{j-1}}^{s_j} \|T(t)u\|_{L^r}^q \, dt \right)^{1/q} \\
&= C \sum_{j=1}^n \|T(t)u\|_{L_t^q((s_{j-1},s_j),L_x^r)},
\end{aligned}
$$

where $s_0, \ldots, s_n$ are the numbers in Assumption D1. By Assumption D1 it is enough to prove the estimate (2.15) on an arbitrary finite interval $[a, b]$ instead of $[0, 1]$, under the assumption that $D(t)$ is absolutely continuous on $[a, b]$ and either $D'(t) \geq \Delta_0$ almost everywhere on $[a, b]$ or $D'(t) \leq -\Delta_0$ almost everywhere

on $[a, b]$.

From L^p duality it follows that

$$\|T(\cdot)u\|_{L_t^q((a,b),L_x^r)} =$$

$$= \sup \left\{ \left| \int_a^b (T(t)u, \phi(t))_{L^2} \ dt : \phi \in L^{q'}\left((a,b), L^{r'}\right) \text{ and } \|\phi\|_{L^{q'}((a,b),L^{r'})} = 1 \right\},$$

$$(2.16)$$

where $\dfrac{1}{q'} + \dfrac{1}{q} = 1$ and $\dfrac{1}{r'} + \dfrac{1}{r} = 1$. Now

$$\int_a^b (T(t)u, \phi(t))_{L^2} \ dt = \int_a^b (u, T^*(t)\phi(t))_{L^2} \ dt$$

$$= \left(u, \int_a^b T^*(t)\phi(t) \ dt \right)_{L^2} \qquad (2.17)$$

$$\leq \|u\|_{L^2} \left\| \int_a^b T^*(t)\phi(t) \ dt \right\|_{L^2},$$

where $T^*(t)$ is the adjoint of $T(t)$. Notice that since $T(t)$ is unitary by Lemma 1.12, then we have that $T^*(t) = T^{-1}(t)$.

From (2.16) and (2.17) it follows that to prove the theorem, it is enough to show that for all $\phi \in L^{q'}\left((a,b), L^{p'}\right)$,

$$\left\| \int_a^b T^*(t)\phi(t) \ dt \right\|_{L^2} \leq C\|\phi\|_{L^{q'}((a,b),L^{r'})}. \qquad (2.18)$$

We have

$$\left\| \int_a^b T^*(t)\phi(t) \ dt \right\|_{L^2}^2 = \left(\int_a^b T^*(t)\phi(t) \ dt, \int_a^b T^*(s)\phi(s) \ ds \right)_{L^2}$$

$$= \int_a^b \int_a^b (T^*(t)\phi(t), T^*(s)\phi(s))_{L^2} \ dt \ ds \qquad (2.19)$$

$$= \int_a^b \left(\phi(t), \int_a^b T(t)T^*(s)\phi(s) \ ds \right)_{L^2} dt.$$

Let

$$\theta_\phi(t) = \int_a^b T(t)T^*(s)\phi(s) \ ds. \qquad (2.20)$$

Then by using Hölder's Inequality, first in the space variable and then in the time variable, we obtain

$$
\int_a^b \left(\phi(t), \int_a^b T(t)T^*(s)\phi(s)\ ds \right)_{L^2} dt \leq \int_a^b \|\phi(t)\|_{L^{r'}} \|\theta_\phi(t)\|_{L^r}\ dt
$$

$$
\leq \left(\int_a^b \|\phi(t)\|_{L^{r'}}^{q'}\ dt \right)^{1/q'} \left(\int_a^b \|\theta_\phi(t)\|_{L^r}^q\ dt \right)^{1/q} = \|\phi\|_{L^{q'}((a,b),L^{r'})} \|\theta_\phi\|_{L^q((a,b),L^r)}.
$$

$$(2.21)$$

So to prove (2.18), by (2.19) and (2.21), it is enough to show that

$$
\|\theta_\phi\|_{L^q((a,b),L^r)} \leq C\|\phi\|_{L^{q'}((a,b),L^{r'})}. \tag{2.22}
$$

By Minkowski's Integral Inequality, for each $t \in (0,1)$ we have

$$
\|\theta_\phi(t)\|_{L^r} = \left\| \int_a^b T(t)T^*(s)\phi(s)\ ds \right\|_{L^r} \leq \int_a^b \|T(t)T^*(s)\phi(s)\|_{L^r}\ ds.
$$

We have

$$
\mathcal{F}\left(T(t)T^*(s)\phi(s)\right) = \exp\left(i\omega^2(D(t) - D(s))\right) \widehat{\phi}(s),
$$

so

$$
T(t)T^*(s)\phi(s) = \exp\left(-i\partial_x^2(D(t) - D(s))\right)\phi(s),
$$

and hence by Lemma 2.6,

$$
(T(t)T^*(s)\phi(s))[x] = \frac{1}{2\pi} \frac{1}{\sqrt{4\pi i(D(t) - D(s))}} \int_{-\infty}^{+\infty} \exp\left(\frac{-i|x - y|^2}{4(D(t) - D(s))} \right) \phi(s,y)dy.
$$

Taking the supremum over $x \in \mathbf{R}$ gives

$$
\|T(t)T^*(s)\phi(s)\|_{L^\infty} \leq \frac{C\|\phi(s)\|_{L^1}}{|D(t) - D(s)|^{1/2}}. \tag{2.23}
$$

On the other hand, since T is unitary, we have

$$\|T(t)T^*(s)\phi(s)\|_{L^2} = \|\phi(s)\|_{L^2}. \tag{2.24}$$

From (2.23), (2.24) and the Riesz-Thorin Interpolation Theorem (Theorem 1.2), we get

$$\|T(t)T^*(s)\phi(s)\|_{L^r} \leq \left(\frac{C}{\sqrt{|D(t) - D(s)|}} \right)^{1-(2/r)} C^{2/r} \|\phi(s)\|_{L^{r'}}, \tag{2.25}$$

for some constant C independent of ϕ. So

$$\|\theta_\phi(t)\|_{L^r} \leq C \int_a^b \frac{\|\phi(s)\|_{L^{r'}}}{|D(t) - D(s)|^{(1/2)-(1/r)}} \, ds. \tag{2.26}$$

But from our assumption on $D(t)$, it follows that either

$$D(t) - D(s) = \int_s^t \Delta(\zeta) \, d\zeta \geq \Delta_0 |t - s|$$

or

$$D(t) - D(s) = \int_s^t \Delta(\zeta) \, d\zeta \leq -\Delta_0 |t - s|,$$

for all s and t in (a, b), so in any case

$$|D(t) - D(s)| \geq \Delta_0 |t - s|.$$

Therefore

$$\|\theta_\phi(t)\|_{L^r} \leq \frac{C}{\Delta_0^{(1/2)-(1/r)}} \, g(t), \tag{2.27}$$

where

$$g(t) := \int_a^b \frac{\|\phi(s)\|_{L^{r'}}}{|t - s|^{(1/2)-(1/r)}} \, ds.$$

Now we consider separately the cases when $r > 2$ and $r = 2$.

If $r > 2$, we define

$$\beta = \frac{1}{2} + \frac{1}{r},$$

and we have that $\beta \in (0, 1)$. Define $f(s)$ for $s \in \mathbf{R}$ by

$$f(s) := \begin{cases} \|\phi(s)\|_{L^{r'}} & \text{for } s \in [a, b], \\ \\ 0 & \text{for } s \notin [a, b]. \end{cases}$$

Then $g(t)$ is the Hardy-Littlewood fractional integral $g(t) = I_\beta f(t)$, in the notation of Theorem 1.3. From the relation $\dfrac{2}{q} = \dfrac{1}{2} - \dfrac{1}{r}$, we see that $\dfrac{1}{q} = \dfrac{1}{q'} - \beta$. Therefore Theorem 1.3 gives

$$\|I_\beta f\|_{L^q} \le C \|f\|_{L^{q'}},$$

and so

$$\|g\|_{L^q[a,b]} = \|I_\beta f\|_{L^q[a,b]} \le \|I_\beta f\|_{L^q(\mathbf{R})}$$
$$\le C\|f\|_{L^{q'}} = C \left(\int_a^b \|\phi(s)\|_{r'}^{q'} \, ds \right)^{1/q'} = C\|\phi\|_{L^{q'}((a,b),L^{r'})}. \tag{2.28}$$

Together with (2.27), this proves (2.22) when $r > 2$.

In the remaining case, when $r = 2$, we have $q = \infty$ and $q' = 1$, so

$$\|g\|_{L^q[a,b]} = \sup_{t \in [a,b]} \int_a^b \frac{\|\phi(s)\|_{r'}}{|t - s|^0} \, ds = \left(\int_a^b \|\phi(s)\|_{r'}^{q'} \, ds \right)^{1/q'},$$

so (2.28) again holds, and as before (2.22) then follows from (2.27).

As remarked above, from (2.22) we then deduce (2.18), and from there we get (2.15). So the proof of the Theorem is complete. □

Corollary 2.8. *Suppose* $u \in C\left([0, M]; L^2\right)$ *for some* $M > 0$. *Then for every* $t \in [0, M]$, *if* $u = u(\cdot, t)$, *and we define* $v(s) = |T(s)u|^2 T(s)u$, *then* $v \in L^2\left([0, 1]; L^2\right)$.

Proof. Let $t \in [0, M]$ be given. By Theorem 2.7 with $r = 6$ and $q = 6$, we have that for $u = u(., t)$,

$$\int_0^1 \|T(s)u\|_{L^6}^6 \ ds \leq C \|u\|_{L^2}^6 < \infty.$$

Since, for each $s \in [0, 1]$, $\|v(s)\|_{L^2}^2 \leq \|T(s)u\|_{L^6}^6$, it follows that $v \in L^2\left([0, 1]; L^2\right)$.

$\square$

The following Lemma is from [K], where it is proved for the case where $\Delta(t)$ is piecewise constant.

Lemma 2.9. *Suppose* $D'(s) = \Delta(s)$ *satisfies Assumption D1 and* $g(s)$ *is bounded on* $[0, 1]$.

(a) *We have* $Q(u) \in L^\infty$ *for* $u \in L^2$, *and*

$$\|Q(u) - Q(v)\|_{L^\infty} \leq C\left(\|u\|_{L^2} + \|v\|_{L^2}\right)^2 \|u - v\|_{L^2}, \qquad (2.29)$$

for $u, v \in L^2$. *In particular,* $\|Q(u)\|_{L^\infty} \leq C\|u\|_{L^2}^3$ *for* $u \in L^2$.

(b) *We have* $Q(u) \in L^2$ *for* $u \in L^2$, *and*

$$\|Q(u) - Q(v)\|_{L^2} \leq C\left(\|u\|_{L^2} + \|v\|_{L^2}\right)^2 \|u - v\|_{L^2},$$

for $u, v \in L^2$. *In particular,* $\|Q(u)\|_{L^2} \leq C\|u\|_{L^2}^3$ *for* $u \in L^2$.

Proof. We have

$$|Q(u) - Q(v)| \leq C \int_0^1 |T^*(t) \{|T(t)u|^2 T(t)u - |T(t)v|^2 T(t)v\}| \; dt$$

$$= C \sum_{i=1}^n \int_{s_{i-1}}^{s_i} \left|T^*(t) \{|T(t)u|^2 T(t)u - |T(t)v|^2 T(t)v\}\right| \; dt,$$

where $s_0, \ldots, s_n$ are the numbers in Assumption D1. From Lemma 2.6 and the triangle inequality and using that $g(t)$ is bounded on $[0, 1]$, we obtain that

$$|Q(u) - Q(v)| \leq C \sum_{i=1}^n \int_{s_{i-1}}^{s_i} \frac{1}{\sqrt{|D(t)|}} (\||T(t)u|^2 \, [T(t)u - T(t)v] \|_{L^1}$$

$$+ \|T(t)u T(t)v \, \overline{[T(t)u} - \overline{T(t)v]} \|_{L^1} + \||T(t)v|^2 [T(t)u - T(t)v]\|_{L^1}) \; dt,$$

$$\leq C \sum_{i=1}^n \int_{s_{i-1}}^{s_i} \frac{1}{\sqrt{|D(t)|}} (\|T(t)u\|_{L^3}^2 \|T(t)u - T(t)v\|_{L^3}$$

$$+ \|T(t)u\|_{L^3} \|T(t)v\|_{L^3} \|T(t)u - T(t)v\|_{L^3}$$

$$+ \|T(t)v\|_{L^3}^2 \|T(t)u - T(t)v\|_{L^3}) \; dt.$$

Applying the Generalized Hölder's Inequality with exponents $\frac{4}{3}$, 6, and 12, or $\frac{4}{3}$, 12 12, and 12, we get

$$|Q(u) - Q(v)| \leq C \sum_{i=1}^n \left[\int_{s_{i-1}}^{s_i} |D(t)|^{-2/3} \; dt \right]^{3/4}$$

$$\left(\|T(.)u\|_{L_t^{12}(L_x^3)}^2 + \|T(.)u\|_{L_t^{12}(L_x^3)} \|T(.)v\|_{L_t^{12}(L_x^3)} \right) \|T(.)(u-v)\|_{L_t^{12}(L_x^3)},$$

and so, by using Theorem 2.7, we obtain

$$|Q(u) - Q(v)| \leq C \sum_{i=1}^n \left[\int_{s_{i-1}}^{s_i} |D(t)|^{-2/3} \; dt \right]^{3/4} \left(\|u\|_{L^2}^2 + \|v\|_{L^2}^2 \right) \|u - v\|_{L^2}.$$

Now we claim that our assumption D1 on $\Delta(s)$ implies that $D(t) = \int_0^t \Delta(s) \; ds$

36

satisfies $\int_{s_{i-1}}^{s_i} |D(t)|^{-2/3} \, dt < \infty$ for each $i = 1, 2, \cdots, n$. To see this, notice that if $D(t_0) = 0$ for any $t_0 \in [s_{i-1}, s_i]$, then since Assumption D1 implies $|D(t) - D(t_0)| \geq \Delta_0 |t - t_0|$ for all $t \in [s_{i-1}, s_i]$, we have in particular that $|D(t)|^{-2/3} \leq (\Delta_0 |t - t_0|)^{-2/3}$ for t in a neighborhood of t_0, and therefore $|D(t)|^{-2/3}$ is integrable in a neighborhood of t_0, proving the claim.

It follows then that we have

$$|Q(u) - Q(v)| \leq C \left(\|u\|_{L^2}^2 + \|v\|_{L^2}^2 \right) \|u - v\|_{L^2}.$$

This proves part (a) of the Lemma.

To prove part (b), first let us define

$$Q(u, v, w) = \int_0^1 g(s) T^{-1}(s) \left\{ T(s)u \overline{T(s)v} T(s)w \right\} \, ds.$$

Then by applying the Generalized Hölder's Inequality, first in the x-variable with exponents 3, 3, and 3; and then in the s-variable with exponents 6, 6, 6, and 2; we obtain

$$\begin{aligned}
\|Q(u, v, w)\|_{L^2} &\leq C \int_0^1 \|T^{-1}(s) \left(T(s)u \, \overline{T(s)v} \, T(s)w \right)\|_{L^2} \, ds \\
&= C \int_0^1 \| \left(T(s)u \, \overline{T(s)v} \, T(s)w \right) \|_{L^2} \, ds \\
&\leq C \int_0^1 \|T(s)u\|_{L^6} \|T(s)v\|_{L^6} \|T(s)w\|_{L^6} \, ds \\
&\leq C \|T(s)u\|_{L^6((0,1),L_x^6)} \|T(s)v\|_{L^6((0,1),L_x^6)} \|T(s)w\|_{L^6((0,1),L_x^6)} \left(\int_0^1 1^2 \, dx \right)^{1/2} \\
&\leq C \|u\|_{L^2} \|v\|_{L^2} \|w\|_{L^2},
\end{aligned}$$

where in the last inequality we have used Theorem 2.7. Taking $u = v = w$ now gives

$$\|Q(u)\|_{L^2} \leq C \|u\|_{L^2}^3.$$

37

It follows from what we have just proved that

$$\|Q(u) - Q(v)\|_{L^2} = \|Q(u - v, u, u) - Q(v, v - u, u) + Q(v, v, u - v)\|_{L^2}$$

$$\leq \|Q(u - v, u, u)\|_{L^2} + \|Q(v, v - u, u)\|_{L^2} + \|Q(v, v, u - v)\|_{L^2}$$

$$\leq C \left(\|u - v\|_{L^2} \|u\|_{L^2}^2 + \|u - v\|_{L^2} \|u\|_{L^2}^2 \|v\|_{L^2}^2 + \|u - v\|_{L^2} \|v\|_{L^2}^2 \right)$$

$$\leq C \|u - v\|_{L^2} \left(\|u\|_{L^2} + \|v\|_{L^2} \right)^2 .$$

$$(2.30)$$

Thus (b) has been proved. $\qquad\square$

Theorem 2.10. *Suppose $D'(s) = \Delta(s)$ satisfies Assumption D1 and $g(s)$ is bounded on $[0, 1]$.*

(a) *Suppose $\alpha \in \mathbf{R}$. For every $u_0 \in L^2$, there exists $M > 0$ such that the DMNLS equation (2.1) has a unique strong solution $u \in C([0, M]; L^2)$ with initial data u_0. Moreover, the solution depends continuously on the initial data; that is, the map $u_0 \mapsto u$ is continuous from L^2 to $C([0, M]; L^2)$.*

(b) *With M as in part (a), $E(t)$ and $P(t)$ are independent of t for $t \in [0, M]$.*

(c) *The number M in (a) can be taken arbitrarily large.*

Proof. As in the proof of Theorem 2.4, define

$$\phi(v) = \exp(i\alpha t \partial_x^2) u_0 + i \int_0^t \exp(i\alpha(t - t')\partial_x^2)[Q(u)] \, dt',$$

where Q is as defined in (2.2). Notice that since $g(s)$ is bounded, $Q(u)$ is well-defined by Corollary 2.8 and Hölder's inequality.

By Corollary 2.8, if $u \in C([0, M]; L^2)$ then u satisfies part (a) of Definition 2.1 for $r = 0$. Therefore to prove that such a u is a strong solution in L^2, it is enough to show that $\phi(u) = u$, or in other words that ϕ has a fixed point in $C([0, M]; L^2)$ for some $M > 0$.

Let

$$E(M,a) = \{v \in C([0,M]; L^2(\mathbf{R})) : |||v||| = \sup_{t\in[0,M]} \|v(t)\|_{L^2} \le a\}.$$

We want to show that for every $a > 2\|u_0\|_{L^2}$ there exists $M > 0$ such that ϕ defines a contraction map on $E(M,a)$.

First we have to find M such that $\phi : E(M,a) \to E(M,a)$.

By using Lemma 1.12 and Lemma 2.9(b), we get

$$\begin{aligned}
\|\phi(u)\|_{L^2} &\le \|e^{i\alpha\partial_x^2 t}u_0\|_{L^2} + \left\|\int_0^t e^{i\alpha\partial_x^2(t-t')}Q(u)\ dt'\right\|_{L^2} \\
&\le \|u_0\|_{L^2} + \int_0^t \|e^{i\alpha\partial_x^2(t-t')}Q(u)\|_{L^2}dt' \\
&= \|u_0\|_{L^2} + \int_0^t \|Q(u)\|_{L^2}dt' \\
&\le \|u_0\|_{L^2} + CM \sup_{0\le t\le M} \|u\|_{L^2}^3 \\
&\le \frac{a}{2} + CMa^3
\end{aligned}$$

So

$$|||\phi(u)||| \le \frac{a}{2} + CMa^3.$$

Choose M such that

$$Ca^3 M \le a/2, \tag{2.31}$$

then

$$|||\phi(u)|||_M \le a/2 + a/2 = a.$$

This proves $\phi(u) \in E(M,a)$, whenever (2.31) holds and $a > 2\|u_0\|_{L^2}$.

Similarly, by using Lemma 2.9(b), we can show for $u, v \in E(M,a)$, for all

$t \in [0, M]$,

$$|||\phi(u) - \phi(v)||| = \sup_{0 \le t \le M} \|Q(u) - Q(v)\|_{L^2}$$

$$\le \sup_{0 \le t \le M} a^2 Ct|||u - v|||$$

$$= a^2 CM|||u - v|||.$$

Choose M such that $a^2 CM < \frac{1}{2}$, then

$$|||Q(u) - Q(v)||| < \frac{1}{2}|||u - v|||$$

on $E(M, a)$, so ϕ is a contraction map. So by the Banach contraction mapping theorem, ϕ has a fixed point $u \in E(M, a)$. Thus u is a strong solution of (1.3) in $C([0, M]; L^2)$. This proves existence of a solution in $C([0, M]; L^2)$ for some $M > 0$, completing the proof of (a).

Part (b) of the theorem now follows from exactly the same considerations as used to prove part (b) of Theorem 2.4, so we can omit the details here.

The proof of part (c) is even simpler here than in Theorem 2.4, since here we need only the fact that $P(u)$ is independent of time for solutions of (2.1). Let

$$M_s = \sup \left\{ M > 0 : \text{there exists a strong solution } u \text{ of (2.1) in } C([0, M]; L^2) \right\}.$$

We claim that $M_s = \infty$. To see this, we suppose to the contrary that $M_s < \infty$ and will get a contradiction.

Let $B = \|u_0\|_{L^2}$. By part (a), there exists a number $M_1 > 0$ such that if $v_0 \in L^2$ and $\|v_0\|_{H^1} \le B$, then a strong solution v of (2.1) with initial data v_0 exists in $C([0, M_1]; L^2)$. Let $t_1 = M_s - M_1/2$, and let $v_0(x) = u(x, t_1)$, where $u \in C([0, t_1]; L^2)$ is a strong solution of (2.1) with initial data u_0. By part

40

(b), we have that $P(v_0) = P(u_0)$, so $\|v_0\|_{L^2} = \|u_0\|_{L^2} = B$. Therefore a strong

solution v of (2.1) with initial data v_0 exists in $C([0, M_1]; H^1)$. But, as explained

in the proof of Theorem 2.4, it follows that there exists a strong solution w of

(2.1) in $C([0, M_1 + t_1]; L^2)$ with initial data u_0, and since $M_1 + t_1 > M_s$, this

contradicts the definition of M_s. So (c) is proved. $\qquad\square$

Theorem 2.11. *Suppose $\alpha = 0$. Let $u_0 \in L^2 \cap L^\infty$. There exists $M > 0$ such*

that a unique strong solution of (2.1) with initial data u_0 exists in $C([0, M]; L^2 \cap$

$L^\infty)$. Moreover, the solution depends continuously on the initial data; that is,

the map $u_0 \mapsto u$ is continuous from $L^2 \cap L^\infty$ to $C([0, M]; L^2 \cap L^\infty)$.

Proof. According to Definition 2.1 in the case $\alpha = 0$, what we are looking for

is a number $M > 0$ and a function $u \in C([0, M]; L^2 \cap L^\infty)$ such that

$$u(t) = u_0 + i \int_0^t Q(u(t')) \, dt' \qquad (2.32)$$

for all $t \in [0, M]$.

For a given $M > 0$, define $\phi(u)$ for $u \in C([0, M]; L^2 \cap L^\infty)$ by

$$\phi(u) = u_0 + i \int_0^t Q(u(t')) \, dt'. \qquad (2.33)$$

Let

$$E(M, a) = \{u \in C([0, M]; L^2 \cap L^\infty) : |||u||| = \sup_{t \in [0,M]} \|u(.,t)\|_{L^2} + \|u(.,t)\|_{L^\infty} < \infty\}.$$

Suppose $u_0 \in L^2 \cap L^\infty$. We want to show that for every $a > 4(\|u_0\|_{L^2} + \|u_0\|_{L^\infty})$

there exists $M > 0$ such that ϕ defines a contraction map on $E(M, a)$.

First we have to show $\phi : E(M, a) \to E(M, a)$. For all $t \in [0, M]$,

$$\|\phi(u)\|_{L^\infty} \leq \|u_0\|_{L^\infty} + \left\|\int_0^t Q(u) \ dt'\right\|_{L^\infty}$$

$$\leq a/4 + M \sup_{0 \leq t \leq M} \|Q(u)\|_{L^\infty}.$$

By using Lemma 2.9(a), we then get

$$\|\phi(u)\|_{L^\infty} \leq a/4 + M \sup_{0 \leq t \leq M} \|u(t)\|_{L^2}^3$$

$$\leq a/4 + Ma^3$$

Now, for fixed $t \in [0, M]$,

$$\|\phi(u)\|_{L^2} \leq \|u_0\|_{L^2} + \left\|\int_0^t Q(u) \ dt'\right\|_{L^2}$$

$$\leq a/4 + \int_0^t \|Q(u)\|_{L^2} dt'$$

By using Lemma 2.9(b), we get

$$\|\phi(u)\|_{L^2} \leq a/4 + MC\|u\|_{L^2}^3$$

$$\leq a/4 + MCa^3.$$

So

$$|||\phi(u)||| \leq a/2 + 2MCa^3.$$

Choosing M such that

$$Ca^3 M \leq a/4, \tag{2.34}$$

we get

$$|||\phi(u)||| \leq a/2 + a/2 = a.$$

This proves that $\phi(u) \in E(M, a)$, whenever (2.34) holds and

$$a > 4(\|u_0\|_{L^2} + \|u_0\|_{L^\infty})$$

. Similarly, by using Lemma 2.9(a), for $u, v \in E(M, a)$, for all $t \in [0, M]$,

$$
\begin{aligned}
\|\phi(u) - \phi(v)\|_{L^\infty} &= \left\| \int_0^t [Q(u) - Q(v)] \; dt' \right\|_{L^\infty} \\
&\leq \int_0^t \|Q(u) - Q(v)\|_{L^\infty} \; dt' \\
&\leq \beta C \sup_{0 \leq t \leq M} t \left(\|u\|_{L^2} + \|v\|_{L^2} \right)^2 \|u - v\|_{L^2}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \beta C M \sup_{0 \leq t \leq M} \left(\|u\|_{L^2} + \|v\|_{L^2} \right)^2 \|u - v\|_{L^2} \\
&\leq \beta C M a^2 \sup_{0 \leq t \leq M} \|u - v\|_{L^2}.
\end{aligned}
$$

By using Lemma 2.9(b), for $u, v \in E(M, a)$, for all $t \in [0, M]$,

$$
\begin{aligned}
\|\phi(u) - \phi(v)\|_{L^2} &= \left\| \int_0^t [Q(u) - Q(v)] \; dt' \right\|_{L^2} \\
&\leq \int_0^t \|Q(u) - Q(v)\|_{L^2} \; dt'
\end{aligned}
$$

$$
\begin{aligned}
&\leq \beta C \sup_{0 \leq t \leq M} t \left(\|u\|_{L^2} + \|v\|_{L^2} \right)^2 \|u - v\|_{L^2} \\
&\leq \beta C M a^2 \sup_{0 \leq t \leq M} \|u - v\|_{L^2}.
\end{aligned}
$$

Now we have

$$
\begin{aligned}
|||\phi(u) - \phi(v)||| &\leq \sup_{0 \leq t \leq M} \|\phi(u) - \phi(v)\|_{L^2} + \|\phi(u) - \phi(v)\|_{L^\infty} \\
&\leq 2\beta C M a^2 \sup_{0 \leq t \leq M} \|u - v\|_{L^2}.
\end{aligned}
$$

Choose M such that $2a^2\beta CM < \frac{1}{2}$, then

$$|||\phi(u) - \phi(v)||| \leq \frac{1}{2}|||u - v|||$$

on $E(T, a)$, so ϕ is a contraction map. So by the Banach Contraction Mapping Theorem, ϕ has a fixed point $v \in E(M, a)$. Therefore u is a strong solution of (2.1) in $C\left([0, M]; L^2 \cap L^\infty\right)$ for some $M > 0$. $\quad\square$

Chapter 3

Existence and stability of solitary waves

3.1 Variational approach to solitary waves

Definition 3.1. A solution of (1.3) of the form

$$u(t, x) = e^{i\theta t}\phi(x), \tag{3.1}$$

where $\theta \in \mathbf{R}$ and $\phi \in L^2$, is called a solitary-wave solution.

Substituting (3.1) into (1.3), we see that (3.1) defines a solitary-wave solution of (1.3) if and only if $\phi(x)$ satisfies the equation

$$-\theta\phi(x) + \alpha\phi''(x) + Q(\phi(x)) = 0. \tag{3.2}$$

We now observe that (3.2) can be viewed as the Euler-Lagrange equation for the following variational problem.

For $\lambda > 0$, define

$$I_\lambda = \inf \{E(u) : u \in H^1 \quad \text{and} \quad P(u) = \lambda\}.$$

The set of minimizers for the problem of minimizing $E(\psi)$ subject to $P(\psi) = \lambda$ is

$$G_\lambda = \{\phi \in H^1 : E(\phi) = I_\lambda \quad \text{and} \quad P(\phi) = \lambda\}. \tag{3.3}$$

According to the calculus of variations (see, e.g., Theorem 1 on page 243

of [LU]), if ϕ is any element of G_λ, then ϕ must satisfy the Euler-Lagrange equation $E'(\phi) + \tilde{\theta} P'(\phi) = 0$, where $E'(\phi)$ and $P'(\phi)$ are the Frechet derivatives of E and P at ϕ, and $\tilde{\theta}$, the Lagrange multiplier, is a constant.

An elementary computation shows that for $\phi \in H^1$ we have $E'(\phi) = \alpha \phi'' + Q(\phi)$ and $P'(\phi) = 2\phi$. Therefore the elements of G_λ (if any exist) give rise to solitary waves through (3.1), with $\theta = 2\tilde{\theta}$. Such solitary waves are called ground-state solitary waves.

3.2 Existence and stability in H^1 of solitary waves when $\alpha > 0$

In this section we prove the following result on the existence and stability of solitary-wave solutions of the DMNLS equation (2.1) in the case when $\alpha > 0$.

Theorem 3.2. *Suppose $\alpha > 0$. Assume $\Delta(s)$ is an integrable function on $[0,1]$, and $g(s)$ is a non-negative integrable function on $[0,1]$ such that $\int_0^1 g(s)\,ds > 0$.*

Then for every $\lambda > 0$ there exists a non-empty set $G_\lambda \subseteq H^1$ such that for every $\phi \in G_\lambda$, there exists θ such that $\exp(i\theta t)\phi(x)$ is a solitary-wave solution of (2.1) with $\int_{-\infty}^{+\infty} \phi^2\,dx = \lambda$.

The set G_λ is stable in the following sense: for every $\epsilon > 0$, there exists $\delta > 0$ such that if $u_0 \in H^1$ and $\phi \in G_\lambda$ and

$$\|u_0 - \phi\|_{H^1} < \delta, \tag{3.4}$$

then the solution $u(x,t)$ of (2.1) with $u(x,0) = u_0(x)$ satisfies

$$\inf_{\psi \in G_\lambda} \|u(\cdot,t) - \psi(x)\|_{H^1} < \epsilon \tag{3.5}$$

for all $t \geq 0$.

Theorem 3.2 is proved in [Z1] and [Z2] for the case where $g(s) \equiv 1$ and $\Delta(s)$

is piecewise constant. As we show below, it turns out that essentially the same proof works with the more general assumptions on $g(s)$ and $\Delta(s)$ stated above.

We will obtain Theorem 3.2 as a corollary of the following theorem about the behavior of arbitrary minimizing sequences for the variational problem. We define a minimizing sequence for I_λ to be any sequence $\{u_n\}$ of functions in H^1 satisfying

$$P(u_n) = \lambda, \tag{3.6}$$

for all n, and

$$\lim_{n \to \infty} E(u_n) = I_\lambda. \tag{3.7}$$

Theorem 3.3. *The set G_λ is not empty. Moreover, for every minimizing sequence $\{u_n\}$ for I_λ, the following are true:*

1. *there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and a sequence $\{y_k\}$ of real numbers and an element $g \in G_\lambda$ such that*

$$\lim_{k \to \infty} \|u_{n_k}(\cdot + y_k) - g\|_{H^1} = 0.$$

2.

$$\lim_{n \to \infty} \inf_{g \in G_\lambda, y \in \mathbf{R}} \|u_n(.+y) - g\|_{H^1} = 0.$$

3.

$$\lim_{n \to \infty} \inf_{g \in G_\lambda} \|u_n - g\|_{H^1} = 0.$$

The idea behind the proof of Theorem 3.3 is that, for any given minimizing sequence $\{u_n\}$, we can apply the Concentration Compactness Principle to the sequence of non-negative functions ρ_n defined by $\rho_n = |u_n|^2$. This is done as follows. First, we define a sequence of nondecreasing functions $M_n : [0, \infty] \to$

$[0, \lambda]$ by

$$M_n(r) = \sup_{y \in \mathbf{R}} \int_{y-r}^{y+r} \rho_n \; dx. \tag{3.8}$$

From Helley's selection theorem [H], it follows that any uniformly bounded sequence of nondecreasing functions on $[0, \infty)$ must have a subsequence which converges pointwise to a nondecreasing limit function on $[0, \infty)$. Hence $\{M_n\}$ has such a subsequence, which converges to a limit function $M(r)$. Define

$$q = \lim_{r \to \infty} M(r), \tag{3.9}$$

so that $0 \le q \le \lambda$.

The Concentration Compactness Principle, as given in [L], states that we can characterize the behavior of the sequence ρ_n in useful ways based on the value on q.

Theorem 3.4. *Suppose $\lambda > 0$, and let $\{\rho_n\}$ be a sequence of nonnegative functions in L^1 satisfying $\int_{-\infty}^{+\infty} \rho_n(x) \; dx = \lambda$ for all n. Let M_n and M be as defined in (3.8). Then there are three possibilities:*

1. *(Vanishing): If $q = 0$, then there exists a subsequence $\{\rho_{n_k}\}$ of $\{\rho_n\}$ such that for every $R > 0$,*

$$\lim_{k \to \infty} \sup_{y \in \mathbf{R}} \int_{y-R}^{y+R} \rho_{n_k}(x) \; dx = 0.$$

2. *(Dichotomy): If $q \in (0, \lambda)$, then there exists a subsequence $\{\rho_{n_k}\}$ of $\{\rho_n\}$ such that for every $\epsilon > 0$, there exist a number k_0 and nonnegative func-*

48

tions ρ_k^1 and ρ_k^2 in L^1, such that for all $k \geq k_0$,

$$\left\| \rho_{n_k} - \left(\rho_k^1 + \rho_k^2 \right) \right\|_{L^1} \leq \epsilon,$$

$$\left| \int_{-\infty}^{+\infty} \rho_k^1 \, dx - q \right| \leq \epsilon,$$

$$\left| \int_{-\infty}^{+\infty} \rho_k^2 \, dx - (\lambda - q) \right| \leq \epsilon,$$

ρ_k^1 and ρ_k^2 have disjoint support, and

$$dist(supp \; \rho_k^1, supp \; \rho_k^2) \to \infty$$

as $k \to \infty$.

3. (Compactness): If $q = \lambda$, then there is a a subsequence $\{\rho_{n_k}\}$ of $\{\rho_n\}$ and a sequence $\{y_k\}$ of real numbers such that $\rho_{n_k}(\cdot - y_k)$ is tight; i.e, for every $\epsilon > 0$, there is $R > 0$ large enough such that

$$\int_{-R}^{R} \rho_{n_k}(x - y_k) \, dx \geq \lambda - \epsilon$$

for all $k \in \mathbf{N}$.

For a nice exposition of the proof of Theorem 3.4, the reader may consult Lemma 8.3.8 of [C].

The next steps towards a proof of Theorem 3.2 are to show that, for every minimizing sequence $\{u_n\}$ of the variational problem, we must have $q = \lambda$, so that the "compactness" alternative of Theorem 3.4 holds. We will do this by showing in the lemmas which follow that the assumptions that $q = 0$ and $q \in (0, \lambda)$ lead to contradictions.

Lemma 3.5. *For all $\lambda > 0$, one has $I_\lambda > -\infty$.*

Proof. Suppose $u \in H^1$ and $P(u) = \lambda$. Then it follows from Lemma 1.3 and the Gagliardo-Nirenberg Inequality, Theorem 1.7, with $p = 4$, $r = q = 2$, and $\theta = 1/4$, that for every $s \in [0,1]$,

$$\int_{-\infty}^{+\infty} |T(s)u|^4 \; dx = \|T(s)\,u\|_{L^4}^4 \leq C \left\| T(s) \frac{du}{dx} \right\|_{L^2} \|T(s)\,u\|_{L^2}^3$$
$$= C \left\| \frac{du}{dx} \right\|_{L^2} \|u\|_{L^2}^3 = C \left\| \frac{du}{dx} \right\|_{L^2} \lambda^{3/2}.$$

Therefore we have

$$E(u) \geq \alpha \left\| \frac{du}{dx} \right\|_{L^2}^2 - \frac{1}{2} \int_0^1 Cg(s) \left\| \frac{du}{dx} \right\|_{L^2} \lambda^{3/2} \; ds \tag{3.10}$$
$$\geq \alpha \left\| \frac{du}{dx} \right\|_{L^2}^2 - \frac{C\gamma\lambda^{3/2}}{2} \left\| \frac{du}{dx} \right\|_{L^2}$$

where $\gamma = \int_0^1 g(s) \; ds$. Thus

$$E(u) \geq f\left(\left\| \frac{du}{dx} \right\|_{L^2} \right),$$

where $f : \mathbf{R} \to \mathbf{R}$ is the quadratic function defined by $f(x) := \alpha x^2 - \dfrac{C\gamma\lambda^{3/2}}{2} x$. Since the minimum value of $f(x)$ on $\mathbf{R}$ is

$$m := -\frac{\gamma^2 C^2 \lambda^3}{16\alpha},$$

it follows that $E(u) \geq m$ for all $u \in H^1$ such that $P(u) = \lambda$. This proves that

$$I_\lambda \geq m > -\infty.$$

$\square$

Lemma 3.6. *For each $\lambda > 0$, we have $I_\lambda < 0$.*

Proof. It is enough to show that for given $\lambda > 0$ there exists $u \in H^1$ such that $P(u) = \lambda$ and $E(u) < 0$. We follow the proof on page 62 of [Z1], which needs only slight modification here to treat the case where $g(s)$ is nonconstant. Start by defining

$$u(x) = A_0 \exp\left(-x^2/2\sigma_0\right), \tag{3.11}$$

where A_0 and σ_0 are positive real numbers, to be chosen later. Recall that $T(s)[u](x)$ is defined for $s \in [0, 1]$ as

$$T(s)[u](x) = p(s, x), \tag{3.12}$$

where p is the solution of

$$i\frac{\partial p}{\partial s} - D'(s)\frac{\partial^2 p}{\partial x^2} = 0$$
$$p(0, x) = u(x). \tag{3.13}$$

It is straightforward to verify that the solution of (3.13) is given by

$$p(s, x) = A(s) \exp\left(-x^2/2\sigma(s)\right),$$

where

$$\sigma(s) = \sigma_0 - 2iD(s) \tag{3.14}$$

and

$$A(s) = \frac{A_0\sqrt{\sigma_0}}{\sqrt{\sigma(s)}}. \tag{3.15}$$

Here $\sqrt{\sigma(s)}$ is taken as the square root of $\sigma(s)$ which has positive real part. Therefore

$$T(s)[u](x) = A(s) \exp\left(-x^2/2\sigma(s)\right).$$

Since
$$P(u) = A_0{}^2 \int_{-\infty}^{+\infty} \exp\left(-x^2/\sigma_0\right) dx$$
$$= A_0{}^2 \sqrt{\sigma_0 \pi},$$

then to get $P(u) = \lambda$, for a given σ_0 we must choose A_0 so that

$$\lambda = A_0{}^2 \sqrt{\sigma_0 \pi}. \tag{3.16}$$

We also have that

$$\int_{-\infty}^{+\infty} \left|\frac{du}{dx}\right|^2 dx = \frac{A_0{}^2}{\sigma_0^2} \int_{-\infty}^{+\infty} x^2 \exp\left(\frac{-x^2}{\sigma_0}\right) dx$$
$$= \frac{A_0{}^2}{\sqrt{\sigma_0}} \frac{\sqrt{\pi}}{2}$$
$$= \frac{\lambda}{2\sigma_0}.$$

and
$$\int_{-\infty}^{+\infty} \int_0^1 g(s) \left|T\left(s\right)[u]\right|^4 \, ds \, dx$$
$$= A_0{}^4 \sigma_0{}^2 \int_0^1 g(s) \left(\int_{-\infty}^{+\infty} \frac{1}{\left|\sigma\left(s\right)\right|^2} \left|\exp\left(\frac{-x^2}{2\sigma\left(s\right)}\right)\right|^4 dx\right) ds$$
$$= A_0{}^4 \sigma_0{}^2 \int_0^1 \frac{g(s)}{\left|\sigma\left(s\right)\right|^2} \int_{-\infty}^{+\infty} \exp\left(\frac{-2x^2 \Re(\sigma(s))}{\left|\sigma\left(s\right)\right|^2}\right) dx \, ds$$
$$= A_0{}^4 \sigma_0{}^{3/2} \sqrt{\frac{\pi}{2}} \int_0^1 \frac{g(s)}{\left|\sigma\left(s\right)\right|} \, ds$$
$$= \frac{\lambda^2 \sqrt{\sigma_0}}{\sqrt{2\pi}} \int_0^1 \frac{g(s)}{\left|\sigma\left(s\right)\right|} \, ds.$$

Hence
$$E\left(u\right) = \left(\frac{\lambda\alpha}{2\sigma_0}\right)\left(1 - \frac{\lambda \sigma_0{}^{3/2}}{\alpha\sqrt{2\pi}} \int_0^1 \frac{g(s) \, ds}{\sqrt{\sigma_0{}^2 + 4D(s)^2}}\right). \tag{3.17}$$

Now choose σ_0 so large that $\sigma_0 \geq 2D(s)$ for all $s \in [0,1]$, and

$$\sigma_0 \geq \frac{4\pi\alpha^2}{\lambda^2 \gamma^2},$$

where $\gamma = \int_0^1 g(s) \, ds$, and $\gamma > 0$ by assumption. Then

$$\int_0^1 \frac{g(s) \, ds}{\sqrt{\sigma_0{}^2 + 4D^2(s)}} \geq \frac{\gamma}{\sqrt{2} \, \sigma_0},$$

so

$$1 - \frac{\lambda \sigma_0{}^{3/2}}{\alpha \sqrt{2\pi}} \int_0^1 \frac{g(s) \, ds}{\sqrt{\sigma_0{}^2 + 4D(s)^2}} \leq 1 - \frac{\lambda \sigma_0{}^{3/2}}{\alpha \sqrt{2\pi}} \frac{\gamma}{\sqrt{2}\sigma_0} = 1 - \frac{\lambda \sigma_0{}^{1/2} \gamma}{\alpha 2 \sqrt{\pi}} < 0,$$

and hence $E(u) < 0$. Thus, for this choice of σ_0, and with A_0 chosen as in (3.16), we have that u in (3.11) satisfies both $P(u) = \lambda$ and $E(u) < 0$. $\square$

Lemma 3.7. *Suppose $u \in C^\infty(\mathbf{R})$, and for $j \in \mathbf{Z}$, define $Q_j = [j-1, j+1]$.*
Then for all j,

$$\|u\|_{L^\infty(Q_j)} \leq \frac{1}{2} \|u\|_{L^1(Q_j)} + \left\| \frac{du}{dx} \right\|_{L^1(Q_j)}.$$

Proof. Assume $u \in C^\infty(\mathbf{R})$. Then for all $x_0 \in Q_j$ and $y \in Q_j$,

$$u(x_0) = u(y) + \int_y^{x_0} \frac{du}{dx} \, dx, \tag{3.18}$$

so

$$|u(x_0)| \leq |u(y)| + \left\| \frac{du}{dx} \right\|_{L^1(Q_j)}. \tag{3.19}$$

Integrate both sides with respect to y over Q_j to get

$$\int_{j-1}^{j+1} |u(x_0)| \, dy \leq \int_{j-1}^{j+1} |u(y)| \, dy + \int_{j-1}^{j+1} \left\| \frac{du}{dx} \right\|_{L^1(Q_j)} dy \tag{3.20}$$

or

$$2|u(x_0)| \leq \|u\|_{L^1} + 2 \left\| \frac{du}{dx} \right\|_{L^1(Q_j)}. \tag{3.21}$$

Hence

$$\|u\|_{L^\infty(Q_j)} = \sup_{x_0 \in Q_j} |u(x_0)| \le \frac{1}{2} \|u\|_{L^1(Q_j)} + \left\| \frac{du}{dx} \right\|_{L^1(Q_j)} . \qquad (3.22)$$

$\square$

Lemma 3.8. *There exists $C > 0$ such that, for all $u \in H^1(\mathbf{R})$,*

$$\int_{-\infty}^{+\infty} |u|^4 \; dx \le C \left(\sup_{y \in \mathbf{R}} \int_{y-1}^{y+1} |u|^2 \; dx \right) \|u\|_{H^1}^2 \qquad (3.23)$$

Proof. First assume $u \in C_0^\infty(\mathbf{R})$. Applying Lemma 3.7 to u^2 instead of u, we get

$$
\begin{aligned}
\left\| u^2 \right\|_{L^\infty(Q_j)} &\le \frac{1}{2} \left\| u^2 \right\|_{L^1(Q_j)} + \left\| \frac{du^2}{dx} \right\|_{L^1(Q_j)} \\
&\le \frac{1}{2} \|u\|^2_{L^2(Q_j)} + \int_{Q_j} 2\,|u|\,|u_x| \; dx \\
&\le \frac{1}{2} \|u\|^2_{L^2(Q_j)} + 2\|u\|_{L^2(Q_j)} \left\| \frac{du}{dx} \right\|_{L^2(Q_j)} \\
&= \|u\|_{L^2(Q_j)} \left(\frac{1}{2} \|u\|_{L^2(Q_j)} + 2 \left\| \frac{du}{dx} \right\|_{L^2(Q_j)} \right) .
\end{aligned}
$$

Hence

$$\|u\|^2_{L^\infty(Q_j)} \le C \|u\|_{L^2(Q_j)} \left(\|u\|_{L^2(Q_j)} + \left\| \frac{du}{dx} \right\|_{L^2(Q_j)} \right). \qquad (3.24)$$

Since

$$\int_{Q_j} |u|^4 \; dx \; \le \; \|u\|^2_{L^2(Q_j)} \|u\|^2_{L^\infty(Q_j)}, \qquad (3.25)$$

it follows that

$$
\begin{aligned}
\int_{Q_j} |u|^4 \; dx &\le C \|u\|^3_{L^2(Q_j)} \left(\|u\|_{L^2(Q_j)} + \left\| \frac{du}{dx} \right\|_{L^2(Q_j)} \right) \\
&\le C \left(\int_{Q_j} |u|^2 \; dx \right)^{3/2} \left(\int_{Q_j} \left(|u|^2 + \left| \frac{du}{dx} \right|^2 \right) \; dx \right)^{1/2} \\
&\le C \left(\int_{Q_j} |u|^2 \; dx \right) \int_{Q_j} \left(|u|^2 + \left| \frac{du}{dx} \right|^2 \right) \; dx.
\end{aligned}
$$

Now sum over $j \in \mathbf{Z}$ to get

$$2 \int_{-\infty}^{+\infty} |u|^4 \; dx \leq C \left(\sup_{j \in \mathbf{Z}} \int_{Q_j} |u|^2 \; dx \right) \|u\|_{H^1}^2 . \tag{3.26}$$

This proves (3.23) for $u \in C_0^\infty(\mathbf{R})$. The result for general u in H^1 follows by approximating u with a sequence $u_n \in C_0^\infty(\mathbf{R})$ such that $u_n \to u$ in H^1 norm. Then (3.26) holds for each u_n, and we obtain (3.23) by passing to the limit as $n \to \infty$. $\qquad \square$

Lemma 3.9. *Suppose $\{u_n\}$ is a minimizing sequence. Then there exists $B > 0$ such that, for all $n \in \mathbf{N}$, $\|u_n\|_{H^1} \leq B$.*

Proof. Since $\{E\,(u_n)\}$ is a convergent sequence of real numbers, then it is bounded. Moreover, $P(u_n) = \lambda$ for all n. So the conclusion follows immediately from Lemma 2.3. $\qquad \square$

Lemma 3.10. *Suppose $\{u_n\}$ is a minimizing sequence for I_λ. Then there exists $C_0 > 0$ (independent of n) such that for all sufficiently large $n \in \mathbf{N}$, there exists $s_n \in [0,1]$ for which*

$$\sup_{y \in \mathbf{R}} \int_{y-1}^{y+1} |T\,(s_n)\,u_n|^2 \; dx \geq C_0. \tag{3.27}$$

Proof. Since $\{u_n\}$ is a minimizing sequence, we have by Lemma 3.6 that

$$\lim_{n \to \infty} E(u_n) = I_\lambda < 0, \tag{3.28}$$

and hence

$$\liminf_{n \to \infty} \int_0^1 g(s) \left(\int_{-\infty}^\infty |T(s)u_n|^4 \; dx \right) \; ds = C > 0. \tag{3.29}$$

So, by passing to a subsequence if necessary, we can assume that for every

sufficiently large n,

$$\int_0^1 g(s) \int_{-\infty}^{+\infty} |T(s)u_n|^4 \ dx \ ds \geq C,$$

and therefore

$$0 < C \leq \gamma \sup_{s \in [0,1]} \int_{-\infty}^{+\infty} |T(s)u_n|^4 \ dx,$$

where as before we define $\gamma = \int_0^1 g(s) \ ds > 0$. Hence there must exist $s_n \in [0,1]$ such that

$$\int_{-\infty}^{+\infty} |T(s_n)u_n|^4 \ dx \geq \frac{C}{2\gamma} > 0. \tag{3.30}$$

Since $\|u_n\|_{H^1}$ is bounded by Lemma 3.9, then $\|T(s_n)u_n\|_{H^1}$ is bounded by Lemma 1.12. Therefore, from Lemma 3.8 together with (3.30), we obtain that

$$\sup_{y \in \mathbf{R}} \int_{y-1}^{y+1} |T(s_n)u_n|^2 \ dx \geq C_0 \tag{3.31}$$

for some C_0 which does not depend on n. $\qquad\square$

Next we will rule out the possibilities that the sequence $\rho_n = |u_n|^2$ satisfies either of the "vanishing" or "dichotomy" alternatives described in the sense of Theorem 3.4. We follow the argument of [Z2], slightly modified here to take the variable coefficient $\Delta(s)$ into account. For each n and $s \in [0, 1]$, let us define

$$\epsilon_n(s) = \sup_{y \in \mathbf{R}} \int_{y-1}^{y+1} |T(s)u_n(x)|^2 \ dx.$$

Lemma 3.11. *Suppose $\{u_n\}$ is a sequence that is bounded in H^1, satisfies the constraint $\|u_n\|_{L^2} = \lambda$ for all n, and vanishes in the sense of Theorem 3.4, so that $\epsilon_n(0) \to 0$. Then for each $s \in [0, 1]$, the sequence $T(s)u_n$ is also vanishing,*

56

and the following estimate holds:

$$\sup_{s \in [0,1]} \epsilon_n(s) \leq \frac{32}{3} \left(\epsilon_n(0) + \sqrt{\epsilon_n^2(0) + C\epsilon_n(0)} \right). \tag{3.32}$$

Proof. By definition, $T(s)u_n$ satisfies the linear Schrödinger equation

$$i\frac{\partial}{\partial s}(T(s)u_n) = \Delta(s)\frac{\partial^2}{(\partial x)^2}T(s)u_n \tag{3.33}$$

for $s \in \mathbf{R}$ and $x \in \mathbf{R}$. Multiplying equation (3.33) by $\overline{T(s)u_n}$, subtracting the resulting equation from the original, and integrating with respect to x from $-R$ to R, we obtain

$$\frac{d}{ds}\int_{-R}^{+R} |T(s)u_n|^2 \ dx = -2\Delta(s)\left(\Im\left[\overline{T(s)u_n(R)}T(s)u_{nx}(R)\right]\right.$$
$$\left. - \Im\left[\overline{T(s)u_n(-R)}T(s)u_{nx}(-R)\right]\right). \tag{3.34}$$

Integrating equation (3.34) with respect to s from 0 to t, and recalling that $T(0)u_n = u_n$, we obtain

$$\int_{-R}^{+R} |T(t)u_n|^2 \ dx - \int_{-R}^{+R} |u_n|^2 \ dx = -2\int_0^t \Delta(s)\Im\left[\overline{T(s)u_n(R)}T(s)u_{nx}(R)\right] \ ds$$
$$+ 2\int_0^t \Delta(s)\Im\left[\overline{T(s)u_n(-R)}T(s)u_{nx}(-R)\right] \ ds. \tag{3.35}$$

Assume now that $T(t)u_n(x)$ has been translated in x so that

$$\epsilon_n(t) \leq 2\int_{-1}^{1} |T(t)u_n(x)|^2 \ dx.$$

This is always possible since we can shift the initial data using translational

invariance. Using the obvious inequalities

$$\int_{-1}^{1} |T(t)u_n(x)|^2 \ dx \le \int_{-R}^{+R} |T(t)u_n(x)|^2 \ dx$$

and

$$\int_{-R}^{+R} |u_n(x)|^2 \ dx \le (R+2)\,\epsilon_n(0) \le 2R\epsilon_n(0)$$

for $R \ge 2$, we can write

$$\frac{1}{2}\epsilon_n(t) - 2R\epsilon_n(0) \le \left| \int_{-R}^{+R} |T(t)u_n(x)|^2 \ dx - \int_{-R}^{+R} |u_n(x)|^2 \ dx \right|. \qquad (3.36)$$

On the other hand using (3.35), we have

$$\left| \int_{-R}^{+R} |T(t)u_n(x)|^2 \ dx - \int_{-R}^{+R} |u_n(x)|^2 \ dx \right| \le 2 \int_{0}^{1} |\Delta(s)|\, |T(s)u_n(R)|\, |T(s)u_{nx}(R)| \ ds$$

$$+ 2 \int_{0}^{1} |\Delta(s)|\, |T(s)u_n(-R)|\, |T(s)u_{nx}(-R)| \ ds. \qquad (3.37)$$

Assuming that $\epsilon_n(t) > 8\epsilon_n(0)$ (for otherwise we are done), we can find $R_n \ge 2$ such that $\frac{1}{2}\epsilon_n(t) - 4R_n\epsilon_n(0) = 0$.

Combining the inequalities (3.36) and (3.37) gives

$$\frac{1}{2}\epsilon_n(t) - 2R\epsilon_n(0) \le 2 \int_{0}^{1} |\Delta(s)|\, |T(s)u_n(R)|\, |T(s)u_{nx}(R)| \ ds$$

$$+ 2 \int_{0}^{1} |\Delta(s)|\, |T(s)u_n(-R)|\, |T(s)u_{nx}(-R)| \ ds.$$

Now integrating both sides of the last inequality with respect to R, from $R = 1$

to $R = R_n$, we obtain

$$\frac{3\epsilon_n^2(t)}{64\epsilon_n(0)} - \frac{1}{2}\epsilon_n(t) + \epsilon_n(0) \leq 2 \int_0^1 |\Delta(s)| \int_{-\infty}^{+\infty} |T(s)u_n(R)| \, |T(s)u_{nx}(R)| \; dR \, ds$$

$$\leq 2 \int_0^1 |\Delta(s)| \, \|T(s)u_n\|_{L^2} \, \|T(s)u_{nx}\|_{L^2} \; ds$$

$$= 2 \int_0^1 |\Delta(s)| \|u_n\|_{L^2} \, \|u_{nx}\|_{L^2} \; ds.$$

$$(3.38)$$

Since $\|u_n\|_{H^1}$ is bounded and $\int_0^1 |\Delta(s)| \; ds < \infty$, this implies

$$\frac{3\epsilon_n^2(t)}{64\epsilon_n(0)} - \frac{1}{2}\epsilon_n(t) \leq C.$$

Hence

$$\epsilon_n(t) \leq \frac{32}{3} \left(\epsilon_n(0) + \sqrt{\epsilon_n^2(0) + C\epsilon_n(0)} \right),$$

where C is independent of n and t. $\qquad\square$

Now we can rule out the "vanishing" alternative for minimizing sequences.

Lemma 3.12. *If $\{u_n\}$ is a minimizing sequence for I_λ, then $q \neq 0$.*

Proof. If $q = 0$, then for some subsequence of u_n, which we still denote by u_n, we have

$$\lim_{n\to\infty} \epsilon_n(0) = \lim_{n\to\infty} \sup_{y\in\mathbf{R}} \int_{y-1}^{y+1} |u_n|^2 \; dx = 0. \qquad (3.39)$$

But then Lemma 3.11 implies $\lim_{n\to\infty} \epsilon_n(s_n) = 0$, contradicting Lemma 3.10. $\quad\square$

The next lemma is used to describe how minimizing sequences $\{u_n\}$ would behave in the case when $q \in (0, \lambda)$. We follow the proof of inequality (32) in [Z2].

Lemma 3.13. *Suppose $\{u_n\}$ is a minimizing sequence for I_λ and suppose $q \in (0, \lambda)$. Then for some subsequence of $\{u_n\}$, which we continue to denote by*

$\{u_n\}$, *the following is true. For every $\epsilon > 0$, there exist a number $N \in \mathbf{N}$ and sequences $\{v_N, v_{N+1}, \cdots\}$ and $\{w_N, w_{N+1}, \cdots\}$ of H^1 functions such that for every $n \geq N$,*

1. $\left| \int_{-\infty}^{+\infty} |v_n(x)|^2 \, dx - q \right| < \epsilon$

2. $\left| \int_{-\infty}^{+\infty} |w_n(x)|^2 \, dx - (\lambda - q) \right| < \epsilon$

3. $E(u_n) \geq E(v_n) + E(w_n) - \epsilon$

Proof. Suppose $\epsilon > 0$ is given. Then by definition of q, there exists r_0 such that if $r > r_0$, then $q - \epsilon < M(r) \leq q$. Therefore, after passing to a subsequence of $\{M_n\}$ if necessary, we can say that there exist numbers $n_0 \in \mathbf{N}$ and $r_1 > r_0$ and $r_2 > r_0$ such that $r_2 - r_1 \geq 6/\epsilon$ and, for all $n \geq n_0$,

$$q - \epsilon < M_n(r_1) \leq M_n(r_2) < q + \epsilon.$$

It follows that for every $n \geq n_0$, there exists y_n such that

$$q - \epsilon < \int_{y_n - r_1}^{y_n + r_1} |u_n|^2 \, dx \leq \int_{y_n - r_2}^{y_n + r_2} |u_n|^2 \, dx < q + \epsilon. \qquad (3.40)$$

Now introduce smooth cut-off functions ρ and θ, defined on $\mathbf{R}$) with values in $[0, 1]$, such that $|\rho_x| < \epsilon$ and $|\theta_x| < \epsilon$ for all $x \in \mathbf{R}$; $\rho(x) = 1$ for $|x| \leq r_1$; $\rho(x) = 0$ for $|x| \geq r_1 + 2/\epsilon$; $\theta(x) = 1$ for $|x| \geq r_2$; and $\theta(x) = 0$ for $|x| \leq r_2 - 2/\epsilon$. Define $v_n(x) := \rho(y_n - x)u_n(x)$ and $w_n(x) := \theta(y_n - x)u_n(x)$. Then it follows easily from (3.40) that statements 1 and 2 of the Lemma hold, and it remains only to prove statement 3.

Now we can write

$$u_n = v_n + w_n + h_n,$$

where $\|h_n\|_{L^2}^2 < \epsilon$. Then we have

$$E(u_n) = \int_{-\infty}^{+\infty} \int_0^1 \left(\alpha |v_{nx} + w_{nx} + h_{nx}|^2 - \frac{1}{2} |T(s)(v_n + w_n + h_n)|^4 \right) \, ds \, dx,$$

(3.41)

which can be rewritten as

$$\begin{aligned}
E(u_n) = {}& E(v_n) + E(w_n) + 2\alpha \Re \int_{-\infty}^{+\infty} (\overline{v_{nx}} \, w_{nx} + \overline{v_{nx}} \, h_{nx} + \overline{w_{nx}} \, h_{nx}) \, dx \\
& + \alpha \int_{-\infty}^{+\infty} |h_{nx}|^2 \, dx - \Re \int_{-\infty}^{+\infty} \int_0^1 \Big(2|T(v_n + w_n)|^2 |Th_n|^2 + \frac{1}{2} |Th_n|^4 \\
& + 2|T(v_n + w_n)|^2 T(v_n + w_n) Th_n + (T(v_n + w_n))^2 (\overline{Th_n})^2 \\
& + 2T(v_n + w_n) |Th_n|^2 \overline{Th_n} + 2|Tv_n|^2 |Tw_n|^2 + 2|Tv_n|^2 Tv_n \overline{Tw_n} \\
& + (Tv_n)^2 (\overline{Tw_n})^2 + 2|Tw_n|^2 Tv_n \overline{Tw_n} \, \Big) \, ds \, dx,
\end{aligned}$$

(3.42)

where we have used T as an abbreviation for $T(s)$.

To prove statement 3, it is enough to show that each of the integrals on the right-hand side of (3.42) can be bounded below by terms which go to zero with ϵ, uniformly in n.

First, notice that

$$\int_{-\infty}^{+\infty} \overline{v_{nx}} \, w_{nx} \, dx = 0,$$

since v_n and w_n have disjoint supports.

To estimate the second term in the integral on the first line of (3.42), write

$$\begin{aligned}
\overline{v_{nx}} h_{nx} &= (\rho \overline{u_n})_x (u_n - \rho u_n - \theta u_n)_x \\
&= (\rho \overline{u_n})_x (u_n - \rho u_n)_x = (\rho_x \overline{u_n} + \rho \overline{u_{nx}})(-\rho_x u_n + (1 - \rho) u_{nx}),
\end{aligned}$$

(3.43)

where we have used ρ as an abbreviation for $\rho(y_n - x)$. Then we have

$$
\begin{aligned}
2\alpha\Re \int_{-\infty}^{+\infty} \overline{v_{nx}} h_{nx} \, dx &= -2\alpha \int_{-\infty}^{+\infty} \rho_x^2 |u_n|^2 \, dx + 2\alpha \int_{-\infty}^{+\infty} \rho(1-\rho)|u_{nx}|^2 \, dx \\
&\quad - 2\alpha\Re \int_{-\infty}^{+\infty} \rho\rho_x \overline{u_{nx}} u_n \, dx + 2\alpha\Re \int_{-\infty}^{+\infty} (1-\rho)\rho_x \overline{u_n} u_{nx} \, dx \\
&\geq -2\alpha \left(\|\rho_x\|_{L^\infty}^2 \|u_n\|_{L^2}^2 + \|\rho_x\|_{L^\infty} \|u_n\|_{L^2} \|u_{nx}\|_{L^2} \right) \\
&\geq -C\epsilon,
\end{aligned}
$$

$$(3.44)$$

since $\|\rho_x\|_{L^\infty} \leq \epsilon$ and u_n is bounded in H^1 by Lemma 3.9.

A similar argument proves the desired estimate for the third term in the integral on the first line of (3.42).

The first term in the second line of (3.42) is non-negative, and so trivially satisfies the desired estimate.

The second term in the second line of (3.42) can be estimated by writing

$$
\begin{aligned}
\int_{-\infty}^{+\infty} \int_0^1 &|T(v_n + w_n)|^2 |Th_n|^2 \, ds \, dx \\
&\leq \int_0^1 \left(\int_{-\infty}^{+\infty} |T(v_n + w_n)|^4 \, dx \int_{-\infty}^{+\infty} |Th_n|^4 \, dx \right)^{1/2} ds \\
&\leq C \int_0^1 \left(\|T(v_n + w_n)\|_{L^2} \|T(v_n + w_n)_x\|_{L^2}^3 \|Th_n\|_{L^2} \|T(h_n)_x\|_{L^2}^3 \right)^{1/2} ds \\
&= C \int_0^1 \|v_n + w_n\|_{L^2}^{1/2} \|(v_n + w_n)_x\|_{L^2}^{3/2} \|h_n\|_{L^2}^{1/2} \|h_{nx}\|_{L^2}^{3/2} \, ds \\
&\leq C\epsilon^{1/4}.
\end{aligned}
$$

$$(3.45)$$

Here we have used that $\|h_n\|_{L^2}^2 \leq \epsilon$, and that $\{w_n\}$, $\{v_n\}$, and $\{h_n\}$ are bounded sequences in H^1, due to the fact that $\{u_n\}$ is bounded in H^1.

The remaining terms in the second and third lines of (3.42), along with the first term in the fourth line of (3.42), are estimated similarly. We have already

shown in the preceding paragraph that

$$\int_{-\infty}^{+\infty} \int_0^1 |Th_n|^4 \ ds \ dx \leq C\sqrt{\epsilon},$$

and we have also that

$$\left| \int_0^1 \int_{-\infty}^{+\infty} |T(v_n + w_n)|^2 T(v_n + w_n) Th_n \ dx \ ds \right|$$

$$\leq C \int_0^1 \|T(v_n + w_n)\|_{L^4}^3 \|Th_n\|_{L^4} \ ds$$

$$\leq C\|v_n + w_n\|_{L^2}^{3/4} \|(v_n + w_n)_x\|_{L^2}^{9/4} \|h_n\|_{L^2}^{1/4} \|h_{nx}\|_{L^2}^{3/4}$$

$$\leq C\epsilon^{1/8},$$

$$\left| \int_0^1 \int_{-\infty}^{+\infty} (T(v_n + w_n))^2 \overline{Th_n} \ dx \ ds \right| \leq C\epsilon^{1/4},$$

and

$$\left| \int_0^1 \int_{-\infty}^{+\infty} (T(v_n + w_n)) |Th_n|^2 \overline{Th_n} \ dx \ dt \right| \leq C\epsilon^{3/8}.$$

To estimate the last four terms on the right-hand side of (3.42), which do not involve h_n, we first need to establish the following facts. Let $r_c = \dfrac{1}{2}(r_1 + r_2)$. We claim that

$$\int_{y_n - r_c}^{y_n + r_c} |T(s)w_n|^2 \ dx \leq C\epsilon \tag{3.46}$$

and

$$\int_{|x - y_n| \geq r_c} |T(s)v_n|^2 \ dx \leq C\epsilon \tag{3.47}$$

for all $n \geq n_0$.

To prove these estimates, we use the argument of Lemma 3.11. As in equation (3.35) we obtain, for all r such that $r_c \leq r \leq r_c + 1/\epsilon$, the relation

$$\int_{y_n - r}^{y_n + r} |T(t)w_n|^2 \ dx = 2 \int_0^t \Delta(s) \Im \left[\overline{T(s)w_n} \ T(s)w_{nx} \right]_{y_n - r}^{y_n + r} \ ds.$$

Integrating this relation with respect to r over the interval $[r_c, r_c + 1/\epsilon]$, and estimating the left-hand side from below and the right-hand side from above, we obtain

$$
\begin{aligned}
\frac{1}{\epsilon} \int_{y_n - r_c}^{y_n + r_c} |T(t)w_n|^2 \ dx &\leq \int_{r_c}^{r_c + 1/\epsilon} \int_{y_n + r}^{y_n - r} |T(t)w_n|^2 \ dx \ dr \\
&= 2 \left| \int_{r_c}^{r_c + 1/\epsilon} \int_0^t \Delta(s) \Im \left[\overline{Tw_n} \ Tw_{nx} \right]_{y_n - r_c}^{y_n + r_c} \ ds \ dr \right| \quad (3.48) \\
&\leq C \int_0^t \Delta(s) \|T(s)w_n\|_{L^2} \|T(s)w_{nx}\|_{L^2} \ ds \leq C.
\end{aligned}
$$

This proves (3.46). The estimate (3.47) for v_n is obtained similarly.

Now using (3.46) and (3.47), we can write

$$
\begin{aligned}
\int_0^1 \int_{-\infty}^{+\infty} |Tv_n|^2 \ |Tw_n|^2 \ dx \ ds \\
= \int_0^1 \int_{|x - y_n| \leq r_c} |Tv_n|^2 \ |Tw_n|^2 \ dx \ ds + \int_0^1 \int_{|x - y_n| \geq r_c} |Tv_n|^2 \ |Tw_n|^2 \ dx \ ds \\
\leq C \int_0^1 \|Tv_n\|_{L^\infty}^2 \int_{|x - y_n| \leq r_c} |Tw_n|^2 \ dx \ ds \\
+ C \int_0^1 \|Tw_n\|_{L^\infty}^2 \int_{|x - y_n| \leq r_c} |Tv_n|^2 \ dx \ ds \\
\leq C\epsilon (\|Tv_n\|_{H^1}^2 + \|Tw_n\|_{H^1}^2) = C\epsilon (\|v_n\|_{H^1}^2 + \|w_n\|_{H^1}^2) \leq C\epsilon,
\end{aligned}
$$

$$(3.49)$$

where we have used Corollary 1.9, applied to Tv_n and Tw_n.

Similar estimates apply to the remaining terms in (3.42). $\qquad \square$

Corollary 3.14. *Suppose $\{u_n\}$ is a minimizing sequence for I_λ and $0 < q < \lambda$. Then*

$$I_\lambda \geq I_q + I_{\lambda - q}. \qquad (3.50)$$

Proof. First observe that if v is a function such that $|P(v) - q| < \epsilon$, then $P(\beta v) = q$, where $\beta = \sqrt{q/P(v)}$ satisfies $|\beta - 1| < A_1 \epsilon$ with A_1 independent of

g and ϵ. Hence

$$I_\alpha \leq E(\beta v) \leq E(v) + A_2\epsilon, \tag{3.51}$$

where A_2 depends only on A_1 and $\|v\|_1$. A similar result holds for functions w such that

$$|P(w) - (\lambda - q)| < \epsilon. \tag{3.52}$$

From these observations and Lemma 3.13 it follows easily that there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and corresponding functions v_{n_k} and w_{n_k} such that for all k,

$$E\left(v_{n_k}\right) \geq I_q - \frac{1}{k}, \tag{3.53}$$

$$E\left(w_{n_k}\right) \geq I_{\lambda-\alpha} - \frac{1}{k}, \tag{3.54}$$

and

$$E\left(u_{n_k}\right) \geq E\left(v_{n_k}\right) + E\left(w_{n_k}\right) - \frac{1}{k}, \tag{3.55}$$

so

$$E\left(v_{n_k}\right) \geq I_\alpha + I_{\lambda-q} - \frac{1}{k}. \tag{3.56}$$

The desired result is now obtained by taking the limit as $k \to \infty$ of both sides of the inequality (3.56). $\qquad\square$

Lemma 3.15. *For all $\lambda_1 > 0$ and $\lambda_2 > 0$, one has*

$$I_{(\lambda_1+\lambda_2)} < I_{\lambda_1} + I_{\lambda_2}. \tag{3.57}$$

Proof. First we claim that for all $\theta > 1$, and $\lambda > 0$

$$I_{\theta\lambda} < \theta I_\lambda \tag{3.58}$$

To see this, let ϕ_n be such that $\int_{-\infty}^{+\infty} \phi_n{}^2 = \lambda$ and $E(\phi_n) \to I_\lambda$, and let η be such that $\int_{-\infty}^{+\infty} (\eta \phi_n)^2 = \theta \lambda$, or $\eta = \sqrt{\theta}$. Then

$$I_{\theta \lambda} = \inf \{ E(u) : u \in H^1 \text{ and } \int |u|^2 = \theta \lambda \}$$

$$\leq E(\eta \phi_n)$$

$$= \eta^2 \left(\alpha \int |\phi_{nx}|^2 \right) - \frac{\eta^4}{2} \int_{-\infty}^{+\infty} \int_0^1 g(s) |T(s)\phi_n|^4 \ ds \ dx$$

$$= \eta^2 \alpha \left(\int |\phi_{nx}|^2 \right) - \frac{\eta^2}{2} \int_{-\infty}^{+\infty} \int_0^1 g(s) |T(s)\phi_n|^4 \ ds \ dx$$

$$+ \frac{1}{2} \left(\eta^2 - \eta^4 \right) \int_{-\infty}^{+\infty} \int_0^1 g(s) |T(s)\phi_n|^4 \ ds \ dx$$

$$= \theta E(\phi_n) + \frac{1}{2} \left(\eta^2 - \eta^4 \right) \int_{-\infty}^{+\infty} \int_0^1 g(s) |T(s)\phi_n|^4 \ ds \ dx.$$

So, taking $n \to \infty$, we get, since $\eta > 1$ and therefore $\eta^2 - \eta^4 < 0$, that

$$I_{\theta \lambda} \leq \theta I_\lambda + \frac{1}{2} \left(\eta^2 - \eta^4 \right) \liminf_{n \to \infty} \int_{-\infty}^{+\infty} \int_0^1 g(s) |T(s)\phi_n|^4 \ ds \ dx.$$

Since $I_\lambda < 0$ by Lemma 3.6, then we have

$$\beta = \liminf_{n \to \infty} \int_{-\infty}^{\infty} \int_0^1 g(s) |T(s)\phi_n|^4 \ ds \ dx > 0, \tag{3.59}$$

and so

$$I_{\theta \lambda} \leq \theta I_\lambda + \frac{1}{2} \left(\eta^2 - \eta^4 \right) \beta < \theta I_\lambda. \tag{3.60}$$

Now suppose, without loss of generality, that $\lambda_1 \geq \lambda_2$. Then from the claim just proved, it follows that

$$I_{(\lambda_1 + \lambda_2)} = I_{\lambda_1 (1 + \lambda_2/\lambda_1)} < (1 + \lambda_2/\lambda_1) I_{\lambda_1}$$

$$\leq I_{\lambda_1} + (\lambda_2/\lambda_1)(\lambda_1/\lambda_2) I_{\lambda_2}$$

$$= I_{\lambda_1} + I_{\lambda_2},$$

so

$$I_{(\lambda_1+\lambda_2)} < I_{\lambda_1} + I_{\lambda_2}$$

as desired. □

Now we can rule out the "dichotomy" alternative of Theorem 3.4 for minimizing sequences for I_λ.

Corollary 3.16. *Suppose $\{u_n\}$ is a minimizing sequence for I_λ. Then $q \notin (0, \lambda)$.*

Proof. From Corollary 3.14, if $q \in (0, \lambda)$, then by taking $\lambda_1 = q$ and $\lambda_2 = \lambda - q$ we get

$$I_{(\lambda_1+\lambda_2)} \leq I_{\lambda_1} + I_{\lambda_2} \tag{3.61}$$

which contradicts Lemma 3.15. □

Finally we examine what happens in the only remaining alternative from Theorem 3.4, the case of "compactness".

Lemma 3.17. *Suppose $q = \lambda$. Then there exists a sequence of real numbers $\{y_1, y_2, y_3, \cdots\}$ such that*

1. *for every $z < \lambda$ there exists $r = r(z) > 0$ and an integer $N(z)$ such that for all $n > N(z)$,*

$$\int_{y_n-r}^{y_n+r} |u_n|^2 \ dx > z. \tag{3.62}$$

2. *the sequence $\{\widetilde{u_n}\}$ defined by $\widetilde{u_n}(x) = u_n(x - y_n)$ for $x \in \mathbf{R}$ has a subsequence which converges in H^1 norm to a function $\phi \in G_\lambda$.*

In particular, G_λ is nonempty.

Proof. Since $q = \lambda$, then there exists r_0 such that for all sufficiently large values of n we have

$$M_n(r_0) = \sup_{y \in \mathbf{R}} \int_{y-r_0}^{y+r_0} |u_n|^2 \ dx > \lambda/2. \tag{3.63}$$

Hence for each sufficiently large n we can find y_n such that

$$\int_{y_n-r_0}^{y_n+r_0} |u_n|^2 \ dx > \lambda/2. \tag{3.64}$$

Now let $z < \lambda$ be given; clearly we may assume $z > \lambda/2$. Again, since $q = \lambda$ then we can find $r_0(z)$ and $N(z)$ such that if $n > N(z)$ then

$$\int_{y_n(z)-r_0(z)}^{y_n(z)+r_0(z)} |u_n|^2 \ dx > z \tag{3.65}$$

for some $y_n(z) \in \mathbf{R}$. Since $\int_{-\infty}^{\infty} |u_n|^2 \, dx = \lambda$, it follows that for large n the intervals $[y_n-r_0, y_n+r_0]$ and $[y_n(z)-r_0(z), y_n(z)+r_0(z)]$ must overlap. Therefore, defining $r = r(z) = 2r_0(z) + r_0$, we have that $[y_n - r, y_n + r]$ contains $[y_n(z) - r_0(z), y_n(z) + r_0(z)]$, and so (3.62) follows from (3.65), for all $n > N(z)$. This proves statement 1.

Now statement 1 implies that for every $k \in \mathbf{N}$, there exists $r_k > 0$ such that for all sufficiently large n,

$$\int_{-r_k}^{r_k} |\widetilde{u_n}|^2 \ dx > \lambda - \frac{1}{k}. \tag{3.66}$$

By Lemma 3.9, the sequence $\{\widetilde{u_n}\}$ is uniformly bounded in H^1, and therefore also in $H^1(-r_k, r_k)$. Therefore, from Rellich's Lemma ([E], page 272), it follows that some subsequence of $\{\widetilde{u_n}\}$ converges in $L^2(-r_k, r_k)$ norm to a limit function

$\phi \in L^2(-r_k, r_k)$ satisfying

$$\int_{-r_k}^{r_k} |\phi|^2 \, dx \geq \lambda - \frac{1}{k}. \tag{3.67}$$

A Cantor diagonalization argument, together with the fact that $\int_{-\infty}^{\infty} |\widetilde{u_n}|^2 \, dx = \lambda$ for all n, then shows that some subsequence of $\{\widetilde{u_n}\}$ converges in $L^2(\mathbf{R})$ norm to a function $\phi \in L^2(\mathbf{R})$ satisfying $\int_{-\infty}^{\infty} |\phi|^2 \, dx = \lambda$. For ease of notation we continue to denote this subsequence by $\widetilde{u_n}$. Again using Lemma 3.9, together with the Gagliardo-Nirenberg Inequality, we have

$$\|T(s)\,(\widetilde{u_n} - \phi)\,\|_{L^4} \leq C\|T(s)\,(\widetilde{u_n} - \phi)\,\|_{H^1}^{1/4}\|T(s)\,(\widetilde{u_n} - \phi)\,\|_{L^2}^{3/4} \leq C\|\widetilde{u_n} - \phi\|_{L^2}^{3/4}. \tag{3.68}$$

Now (3.68), together with the Lebesgue Dominated Convergence Theorem, implies that

$$\lim_{n\to\infty} \int_0^1 \int_{-\infty}^{+\infty} g(s)\,|T(s)\widetilde{u_n}|^4 \, dx \, ds = \int_0^1 \int_{-\infty}^{+\infty} g(s)\,|T(s)\phi|^4 \, dx \, ds. \tag{3.69}$$

Furthermore, by the weak compactness of the unit sphere and the weak lower continuity of the norm in Hilbert space, we can assume, by passing to a subsequence again if necessary, that $\widetilde{u_n}$ converges weakly to ϕ in H^1, and that

$$\|\phi\|_{H^1} \leq \liminf_{n\to\infty} \|\widetilde{u_n}\|_{H^1}. \tag{3.70}$$

It follows then that

$$E(\phi) \leq \lim_{n\to\infty} E(\widetilde{u_n}) = I_\lambda, \tag{3.71}$$

69

and since $\widetilde{u_n}$ converges in L^2 to ϕ, we also have that $P(\phi) = \lim_{n\to\infty} P(\widetilde{u_n}) = \lambda$. From the definition of I_λ we conclude that we must have $E(\phi) = I_\lambda$ and $\phi \in G_\lambda$. Finally, $E(\phi) = \lim_{n\to\infty} E(\widetilde{u_n})$, (3.69), and $\|\phi\|_{L^2} = \lim_{n\to\infty} \|\widetilde{u_n}\|_{L^2}$ together imply that $\|\phi\|_{H^1} = \lim_{n\to\infty} \|\widetilde{u_n}\|_{H^1}$, and from an elementary exercise in Hilbert space theory it then follows that $\widetilde{u_n}$ converges to ϕ in H^1 norm. $\qquad\square$

We can now prove Theorem 3.3.

Proof of Theorem 3.3. Let q be as defined in (3.9). By Corollary 3.16, q cannot be in $(0, \lambda)$, and by Lemma 3.12 we cannot have $q = 0$. So it follows that $q = \lambda$. Hence by Lemma 3.17, the set G_λ is nonempty and statement 1 of Theorem 3.3 holds. Now suppose statement 2 does not hold; then there exist a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and a number $\epsilon_0 > 0$ such that

$$\inf_{\phi\in G_\lambda, y\in \mathbf{R}} \|u_{n_k}(. + y) - \phi\|_{H^1} \geq \epsilon_0 \tag{3.72}$$

for all $k \in \mathbf{N}$. But since $\{u_{n_k}\}$ is itself a minimizing sequence for I_λ, from statement 1 it follows that there exist a sequence $\{y_k\}$ and $\phi_0 \in G_\lambda$ such that

$$\lim_{k\to\infty} \|u_{n_k}(. + y_k) - \phi_0\|_{H^1} = 0. \tag{3.73}$$

This contradiction proves statement 2.

Since the functionals E and P are invariant under translations, then G_λ clearly contains any translate of ϕ_0 if it contains ϕ_0, and hence statement 3 follows immediately from statement 2. $\qquad\square$

Finally we can complete the proof of Theorem 3.2.

Proof of Theorem 3.2. We have already seen in Theorem 3.3 that the set G_λ defined in (3.3) is non-empty. Also, as explained in Section 3.1, for every $\phi \in G_\lambda$

70

there exists $\theta \in \mathbf{R}$ such that $e^{i\theta t}\phi(x)$ is a solitary-wave solution of (1.7). Now suppose G_λ is not stable. Then there exist a number $\epsilon_0 > 0$, a sequence $\{\psi_n\}$ of functions in H^1, and a sequence of times $\{t_n\}$ such that

$$\inf_{\phi \in G_\lambda} \|\psi_n - \phi\|_{H^1} < \frac{1}{n} \tag{3.74}$$

and

$$\inf_{\phi \in G_\lambda} \|u_n(\cdot, t_n) - \phi\|_{H^1} \geq \epsilon_0$$

for all n, where $u_n(x,t)$ solves (2.1) with $u_n(x,0) = \psi_n$. Then since $\psi_n \to G_\lambda$ in H^1, and $E(\phi) = I_\lambda$ and $P(\phi) = \lambda$ for $\phi \in G_\lambda$, we have $E(\psi_n) \to I_\lambda$ and $P(\psi_n) \to \lambda$. Choose $\{\alpha_n\}$ such that $P(\alpha_n \psi_n) = \lambda$ for all n; thus $\alpha_n \to 1$. Hence the sequence $v_n = \alpha_n u_n(., t_n)$ satisfies $P(v_n) = \lambda$ and

$$\lim_{n \to \infty} E(v_n) = \lim_{n \to \infty} E(u_n(\cdot, t_n)) = \lim_{n \to \infty} E(\psi_n) = I_\lambda, \tag{3.75}$$

and is therefore a minimizing sequence for I_λ. From Theorem 3.3 it follows that for all n sufficiently large there exists $\phi_n \in G_\lambda$ such that $\|v_n - \phi_n\|_{H^1} < \epsilon_0/2$. But then, for large n,

$$\epsilon_0 \leq \|u_n(., t_n) - \phi_n\|_{H^1} \leq \|u_n(., t_n) - v_n\|_{H^1} + \|v_n - \phi_n\|_{H^1}$$

$$\leq |1 - \alpha_n| . \|u_n(., t_n)\|_{H^1} + \frac{\epsilon_0}{2},$$

and taking $n \to \infty$ gives $\epsilon_0 \leq \epsilon_0/2$, a contradiction. $\qquad \square$

3.3   Existence and stability in L^2 of solitary waves when $\alpha = 0$

The proof of existence and stability of solitary-wave solutions to the DMNLS equation (2.1) given in Section 3.2 relied crucially in several places on the as-

sumption that $\alpha > 0$. In particular, without the assumption that $\alpha > 0$ one cannot control the H^1 norm of minimizing sequences, and so cannot use the Rellich Lemma as in Lemma 3.17 to prove the convergence of minimizing sequences. Remarkably, however, Kunze has shown ([K], see also [KMZ]) that a variational proof of existence and stability of solitary waves can be given in the case $\alpha = 0$ by using a novel version of the method of concentration compactness.

In [K, KMZ] it is assumed that $D(s)$ is piecewise constant and that $g(s) \equiv 1$. In this section we verify that Kunze's arguments carry through to the case when $D(s)$ satisfies the more general assumption D1 given above in Section 2, together with the further assumption:

Assumption D2. The numbers $s_0, \ldots, s_n$ in Assumption D1 can be chosen so that $\Delta(s)$ is absolutely continuous on the interval (s_{j-1}, s_j) for each $j = 1, 2, \ldots, n$.

In addition, for the case of variable $g(s)$ considered here, we need an extra regularity assumption on $g(s)$; namely that $g(s)$ is piecewise absolutely continuous on $[0, 1]$.

Then the existence and stability result for solitary waves is as follows.

Theorem 3.18. *Suppose $\alpha = 0$. Assume that $D'(s) = \Delta(s)$ satisfies assumptions D1 and D2, and that $g(s)$ is a non-negative measurable function on $[0, 1]$ with $\int_0^1 g(s)\, ds > 0$ which is piecewise absolutely continuous on $[0, 1]$.*

Then for every $\lambda > 0$, there exists a non-empty set $\widetilde{G_\lambda} \subseteq L^2 \cap L^\infty$ such that for every $\phi \in \widetilde{G_\lambda}$, there exists ω such that $\exp(i\omega t)\phi(x)$ is a solitary-wave solution of (2.1) with $\int_{-\infty}^{+\infty} \phi^2\, dx = \lambda$.

The set $\widetilde{G_\lambda}$ is stable in the following sense: for every $\epsilon > 0$, there exists

$\delta > 0$ *such that if* $u_0 \in L^2$ *and*

$$\|u_0 - \phi\|_{L^2} < \delta \tag{3.76}$$

for some $\phi \in \widetilde{G_\lambda}$, *then the solution* $u\,(x,t)$ *of* (2.1) *with* $u\,(x,0) = u_0\,(x)$ *satisfies*

$$\inf_{\Psi \in \widetilde{G_\lambda}} \|u\,(x,t) - \Psi\,(x)\|_{L^2(dx)} < \epsilon \tag{3.77}$$

for all $t \geq 0$.

In the case when $\alpha = 0$, the energy functional $E(u)$ is given by

$$E(u) = -\frac{1}{2} \int_{-\infty}^{+\infty} \int_0^1 |T(s)u|^4 \ ds \ dx. \tag{3.78}$$

With $P(u)$ defined as before, we now consider the variational problem of minimizing $E(u)$ over the set of all $u \in L^2$ satisfying $P(u) = \lambda$. That is, we define

$$\widetilde{I_\lambda} = \inf \{E(u) : u \in L^2 \quad \text{and} \quad P\,(u) = \lambda\},$$

and

$$\widetilde{G_\lambda} = \{\phi \in L^2 : E\,(\phi) = I_\lambda \quad \text{and} \quad P\,(\phi) = \lambda\}.$$

The key to the proof of Theorem 3.18 is Kunze's solution of this variational problem, in which he showed that arbitrary minimizing sequences for $\widetilde{I_\lambda}$ have subsequences which converge in L^2 to the solution set $\widetilde{G_\lambda}$.

Another way to view Kunze's result is to consider the estimate

$$\int_{-\infty}^{+\infty} \int_0^1 |T(s)u|^4 \ ds \ dx \leq C \|u\|_{L^2}^4,$$

which, as can be seen from the proof of Corollary 2.8 above, is valid for all

$u \in L^2$, for some $C > 0$ which is independent of u. Kunze's result shows that the best possible constant C in this estimate is attained on a non-empty set of functions in L^2.

Lemma 3.19. *For every $\lambda > 0$, we have*

$$0 > \widetilde{I_\lambda} > -\infty.$$

Proof. To see that $\widetilde{I_\lambda} > -\infty$, observe that it follows from Theorem 2.7 with $r = 4$ and $q = 8$ that for each $u \in L^2$, we have

$$-E(u) = \int_0^1 \|T(s)u\|_{L^4}^4 \, ds$$

$$\leq \left(\int_0^1 \|T(s)u\|_{L^4}^8 \, ds \right)^{1/2} \left(\int_0^1 1 \, ds \right)^{1/2}$$

$$= \|T(\cdot)u\|_{L_t^8((0,1), L_x^4)}^4 \leq C\|u\|_{L^4}^4.$$

It follows that $E(u) \geq -C\lambda^2$ for all u such that $P(u) = \lambda$, and therefore $\widetilde{I_\lambda} \geq -C\lambda^2 > -\infty$.

To prove that $\widetilde{I_\lambda} < 0$, observe that the proof of Lemma 3.6 shows that, for u as given in (3.11), we have in place of (3.17) that

$$E(u) = -\frac{\lambda^2 \sqrt{\sigma_0}}{2\sqrt{2\pi}} \int_0^1 \frac{g(s) \, ds}{\sqrt{\sigma_0{}^2 + 4D(s)^2}} \; .$$

Since $E(u) < 0$ when $\sigma_0 > 0$, it follows that $\widetilde{I_\lambda} < 0$. $\qquad\square$

Here is a simple version of the classical Van der Corput lemma for oscillatory integrals (see, e.g., Chapter 1 of [P] for more general versions).

Lemma 3.20. *Suppose $D'(s) = \Delta(s)$ satisfies assumptions D1 and D2, and $g(s)$ satisfies the assumptions of Theorem 3.18. Then there exists a constant*

$C > 0$ *such that for all* $w \in \mathbf{R}$,

$$\left| \int_0^1 g(s) e^{iwD(s)} \, ds \right| \leq \frac{C}{1 + |w|}. \tag{3.79}$$

Proof. Since $g(s)$ is piecewise absolutely continuous on $[0, 1]$, by taking the intervals $[s_{j-i}, s_j]$ in Assumptions D_1 and D_2 smaller if necessary, we can assume $g(s)$ is absolutely continuous on $[s_{j-i}, s_j]$ for each $j = 1, \cdots, n$. We have

$$\left| \int_0^1 g(s) e^{iwD(s)} \, ds \right| = \left| \sum_{j=1}^n \int_{s_{j-1}}^{s_j} g(s) e^{iwD(s)} \, ds \right| \leq \sum_{j=1}^n \left| \int_{s_{j-1}}^{s_j} g(s) e^{iwD(s)} \, ds \right|,$$

so it is enough to prove the result with the integral over $[0, 1]$ replaced by the integral over an arbitrary interval $[a, b]$, under the assumption that $D(s)$ is absolutely continuous on $[a, b]$ with $|D'(s)| = |\Delta(s)| \geq \Delta_0$ on $[a, b]$. Also, since the left-hand side of (3.79) is clearly less than 1 for all $w \in \mathbf{R}$, we only need to prove that

$$\left| \int_a^b g(s) e^{iwD(s)} \, ds \right| \leq \frac{C}{|w|} \tag{3.80}$$

for all w such that $|w| \geq 1$.

Using integration by parts, we can write

$$\left| \int_a^b g(s) \exp(iwD(s)) \, ds \right| =$$

$$= \left| \frac{g(b) \exp(iwD(b))}{iwD'(b)} - \frac{g(a) \exp(iwD(a))}{iwD'(a)} - \frac{1}{iw} \int_a^b \exp(iwD(s)) \frac{d}{ds} \left(\frac{g(s)}{D'(s)} \right) \, ds \right|$$

$$\leq \frac{1}{|w|} \left(\frac{1}{|D'(b)|} + \frac{1}{|D'(a)|} + \int_a^b \left| \frac{d}{ds} \left(\frac{g(s)}{D'(s)} \right) \right| \, ds \right).$$

The assumptions on $\Delta(s) = D'(s)$ and $g(s)$ guarantee that

$$I = \int_a^b \left| \frac{d}{ds}\left(\frac{g(s)}{D'(s)} \right) \right| \, ds = \int_a^b \left| \frac{d}{ds}\left(\frac{g(s)}{\Delta(s)} \right) \right| \, ds$$
$$\leq \int_a^b \frac{|\Delta(s)||g'(s)| + |g(s)||\Delta'(s)|}{\Delta_0^2} \, ds < \infty.$$

Also

$$\frac{1}{|D'(b)|} + \frac{1}{|D'(a)|} \leq \frac{2}{\Delta_0}.$$

So it follows that (3.80) holds with $C = (2/\Delta_0) + I$. $\qquad\qquad \square$

As in (2.2), let us define

$$Q(u) := \int_0^1 g(s)T^{-1}(s)[|T(s)u|^2 T(s)u]ds.$$

Recall that if $u \in L^2$, then $Q(u) \in L^2$ also, by Lemma 2.9(b).

Lemma 3.21. *Suppose $u \in L^2$. Then the Fourier transform of $Q(u) \in L^2$ is given by*

$$\widehat{Q}(u)(\omega) = \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} Z(\omega, \omega_1, \omega_2)\widehat{u}(\omega - \omega_1 - \omega_2)\overline{\widehat{u}}(\omega_1)\widehat{u}(\omega_2) \, d\omega_1 \, d\omega_2,$$

(3.81)

where

$$Z(\omega, \omega_1, \omega_2) = \int_0^1 g(s)e^{i\alpha(\omega,\omega_1,\omega_2)D(s)} \, ds,$$

with

$$\alpha(\omega, \omega_1, \omega_2) = \omega_2 + \omega_1^2 - \omega_2^2 - (\omega - \omega_1 - \omega_2)^2 = 2(\omega - \omega_2)(\omega_1 + \omega_2).$$

For all $\omega \in \mathbf{R}$,

$$\left|\widehat{Q}(u)(\omega)\right| \leq C \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\left|\widehat{u}\left(\omega - \omega_1 - \omega_2\right)\widehat{\overline{u}}(\omega_1)\widehat{u}(\omega_2)\right|}{1 + |\alpha(\omega, \omega_1, \omega_2)|} \, d\omega_1 \omega_2. \qquad (3.82)$$

Proof. Using (1.6) and (1.9), we have

$$\begin{aligned}
\widehat{Q}(u)(\omega) &= \int_0^1 g(s)\mathcal{F}\left\{T^{-1}(s)[|T(s)|^2 T(s)u]\right\}(\omega) \, ds \\
&= \int_0^1 g(s)e^{-iD(s)\omega^2}\mathcal{F}\left\{T(s)u \cdot \overline{T(s)u} \cdot T(s)u\right\}(\omega) \, ds \\
&= \int_0^1 g(s)e^{-i\omega^2 D(s)}\mathcal{F}\left\{T(s)u\right\} * \mathcal{F}\left\{\overline{T(s)u}\right\} * \mathcal{F}\left\{T(s)u\right\} \, ds \\
&= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left[\int_0^1 g(s)e^{i\alpha(\omega,\omega_1,\omega_2)D(s)} \, ds\right] \widehat{u}(\omega - \omega_1 - \omega_2)\widehat{\overline{u}}(\omega_1)\widehat{u}(\omega_2) \, d\omega_1 \, d\omega_2,
\end{aligned}$$
$$(3.83)$$

which proves (3.81).

The estimate (3.82) then follows immediately from (3.81) and Lemma 3.20.

$\square$

Proof of Theorem 3.18. The same argument as used at the end of Section 3.2 to prove Theorems 3.3 and 3.2 shows that the statement of Theorem 3.18 will follow if we can show that for every minimizing sequence $\{u_n\}$ for $\widetilde{(I_\lambda)}$ in L^2, there exists a subsequence $\{u_{n_k}\}$ and a sequence of numbers y_k such that $u_{n_k}(x - y_k)$ converges strongly in L^2.

The proof of the latter statement is given in Kunze's paper [K] for the case when $D(s)$ is piecewise constant and $g(s) \equiv 1$. It is enough to check, then, that Kunze's proof still works under the assumptions on $D(s)$ and $g(s)$ given in Theorem 3.18.

Only a few modifications are needed to Kunze's proof, which is rather long. Therefore rather then repeating all the details of Kunze's proof here, we only list the necessary modifications. We use the notation Lemma $x.y$K to refer to

Lemma $x.y$ in [K].

We already proved above in Theorem 2.7 and Lemma 2.9 that Lemma 2.1K and Lemma 2.2K still hold in our case, except that the formula for $\widehat{Q(u)}$ given in Lemma 2.2K has to be replaced here by the formula in Lemma 3.21.

It is easy to see that Lemmas 2.4K and 2.5K hold as well in our case.

Lemma 2.6K is replaced here by Lemma 3.19, and Lemmas 2.7K and 2.8K are replaced here by Lemma 3.11 and the estimates obtained in the proof of Lemma 3.13.

Lemma 2.9K and its proof need no modification. The proof of Lemmas 2.10K and 2.11K are the same in our case, except that in our case to prove the estimates (2.23) and (2.31) in [K], one uses our Lemma 3.21.

Finally Lemma 2.12K and its proof remain unchanged.

Once the Lemmas in Section 2 of [K] have been proved, the proof given in Sections 3 and 4 of [K] applies almost without change to prove the desired convergence result for minimizing sequences. The only modification needed is that in the proof of Lemma 4.1K, the variable coefficient $\Delta(t)$ should be inserted in the formula for $\dot{I}(t)$, but the estimate for $\dot{I}(t)$ still holds because our assumptions imply that $\Delta(t)$ is bounded on $[0, 1]$. $\qquad\square$

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