UNIVERSITY OF OKLAHOMA

GRADUATE COLLEGE

DEHN FUNCTIONS OF THE STALLINGS-BIERI GROUPS AND CONSTRUCTIONS OF NON-UNIQUE PRODUCT GROUPS

A DISSERTATION

SUBMITTED TO THE GRADUATE FACULTY

in partial fulfillment of the requirements for the

Degree of

DOCTOR OF PHILOSOPHY

By

WILLIAM PAUL CARTER
Norman, Oklahoma
2015

DEHN FUNCTIONS OF THE STALLINGS-BIERI GROUPS AND CONSTRUCTIONS OF NON-UNIQUE PRODUCT GROUPS

A DISSERTATION APPROVED FOR THE DEPARTMENT OF MATHEMATICS

BY

Dr. Max Forester, Chair

Dr. Noel Brady

Dr. Jonathan Kujawa

Dr. Andrew Miller

Dr. Piers Hale

Acknowledgments

I would like to thank my past and present advisors, Dr. Peter Linnell and Dr. Max Forester, for their guidance, and encouragement during my graduate studies. Both of the projects that formed the basis for this thesis greatly benefited from their insightful advice and comments. I am extremely fortunate to have such fantastic mentors. I would also like to give my sincere thanks to Dr. Noel Brady, in addition to Max Forester and Peter Linnell, for years of stimulating and inspirational conversations.

In addition, I would also like to thank the other members of my committee, Dr. Piers Hale, Dr. Jonathan Kujawa, and Dr. Andy Miller, for their time and effort. I would also like to extend my thanks to the anonymous referee, for a careful reading and insightful comments regarding the second half of this thesis.

On personal note, I am very thankful to Brian Archer, Sang Rae Lee, Funda Gultepe, and Elizabeth Pacheco for their friendship. Finally, none of this would have been possible without the unconditional love and support of my wife Kellie Woodson.

Contents

Abstract

This thesis will consist of two separate halves in which we will present results concerning two different families of finitely generated torsion-free groups. The themes of each half are quite different and are related to certain geometric and non-geometric properties of groups. Considerations for both types of properties allow us to better understand the algebraic structure of such groups.

In the first half of the thesis, we examine the Dehn functions of a family of subgroups of right-angled Artin groups. If G is a finitely generated group, then the Dehn function gives the optimal isoperimetric function of a simply connected 2-complex that is quasi-isometric to the Cayley graph of G. In the first half of this thesis, we will show that if a graph Γ can be decomposed as a non-trivial join of three smaller subgraphs, then the Bestvina-Brady group, B_Γ, has a quadratic Dehn function. This result proves that the Stallings-Bieri groups SB_n have quadratic Dehn functions, for $n \geq 3$, establishing a claim made by Bridson in [9].

The second half of this thesis is motivated by the study of two well known conjectures in the theory of group rings over torsion-free groups, namely Kaplansky's Zero Divisor Conjecture and Non-trivial Units Conjecture. Together, they represent basic information we would like to know about any given group ring. It is known that if a group satisfies the unique product property, then any group ring over this group also satisfies both conjectures, and so torsion-free groups that do not satisfy this property are likely candidates for counter examples. Unfortunately, very little is known about torsion-free groups that do not have

this property. Specifically, there are only two known examples of such groups, a single group with an explicit presentation [26], and a family of groups produced via a complex procedure [27]. From these two examples, one can trivially produce infinitely many examples via products and embeddings; however, it is currently unclear how to produce genuinely new examples of such groups. Currently, to demonstrate that such a group does not satisfy this property requires producing two finite sets whose product has no uniquely represented element. We will refer to such sets as a pair of non-unique product sets. In the second part of this thesis, we construct a new family of examples with explicit presentations and show that these groups contain arbitrarily large non-unique product sets.

Chapter 1

Introduction

In this thesis, we examine the properties of two different families of torsion-free, finitely presented groups. Our results in each half of the thesis are very different; however, there is a connecting theme where we show that groups are capable of satisfying specific properties by constructing explicit examples of such groups. In the first half, we demonstrate that there are groups with a quadratic Dehn function that satisfy stated finiteness properties. In the second half, we construct new examples of torsion-free groups that do not satisfy the unique product property.

As stated above, this thesis is divided into two distinct halves and each topic is presented separately. Considered separately, these can be thought of as representing certain geometric and non-geometric properties of groups respectively. Chapters 2-5 will present material on the first topic, and chapters 6-9 will present material on the second topic. Overviews for each of the two topics are contained in Chapters 2 and 6 respectively. An outline of the Chapters is given as follows.

1.1 Outline of Thesis

- In Chapter 2, we give an overview of isoperimetric functions and the Dehn function from the point of view of constructing quasi-isometric invariants for a group.

- In Chapter 3, we describe several key definitions and constructions that we need to accurately describe our results in this half of the thesis.
- In Chapter 4, we describe the Bestvina-Brady construction, which gives a family of subgroups of right-angled Artin groups. We will also describe a space on which these groups act geometrically. At the end of this chapter, we will describe an embedding lemma for the spaces associated to certain Bestvina-Brady groups.
- In Chapter 5, we present the main result for this half of the thesis. Using properties of the embedded subspaces defined in Chapter 4, we will show that a family of Bestvina-Brady groups has a quadratic Dehn Function.
- In Chapter 6, we give an overview of groups that do not satisfy the unique product property from the point of view of the Kaplansky conjectures for group rings.
- In Chapter 7, we describe the algebraic properties of the groups P_k. Of particular interest, we will establish that these groups are not already trivial examples of non-unique product groups. We will do this by showing that if $k > 1$, then P_k does not contain Promislow's example.
- In Chapter 8, we present a pair of non-unique product sets for each group P_k. We then proceed to verify by inspection that every element in the product set is not uniquely represented.
- In Chapter 9, we extend the results in Chapter 8 to show that these groups contain arbitrarily large pairs of non-unique product sets.

Chapter 2

Overview and Statement of Main Results: Part 1

The study of groups from a geometric viewpoint, or geometric group theory, can be traced to three fundamental decision theoretic questions asked by Max Dehn in 1912 [15]. These questions represent information we would like to know about a group G, given a presentation for the group. In our discussion, we will be primarily concerned with questions 1 and 3, but we will include all three questions for completeness. These questions are stated as follows.

1. The *Word Problem* asks if there is a procedure to determine whether or not two words w_1 and w_2 in G in the generators S represent the same element of G.

2. The *Conjugacy Problem* asks if there is a procedure to determine whether or not two words, w_1 and w_2, are conjugate in G.

3. The *Isomorphism Problem* asks if there is a procedure to determine whether or not two groups are isomorphic.

In question 3, it is generally assumed that these two groups are given by a presentation; for our purposes, we will also assume that these are finite presentations. It is known that even for finitely presented groups, the solution to this problem is unsolvable in general. However, there are large classes of groups for which this problem is solvable, and one way to determine the solution to this problem is to understand the geometric invariants of the spaces on which these groups act. In fact, a major theme in geometric group theory is the search for nice spaces and nice group actions.

2.1 Quasi-Isometries

The study of such invariants begins with the simple, but powerful observation that if G is a finitely generated group and S is a finite generating set, then G can be given the structure of a metric space under the *word metric*. As a metric space, G is isometric to its Cayley graph endowed with the graph metric, so blurring the distinction between G and its Cayley graph, we can study how certain algebraic properties are reflected in this geometry. It is important to point out that under different generating sets, the resulting spaces are not isometric, but rather are quasi-isometric in the following sense.

Definition 2.1. A (not necessarily continuous) map $f : X \to Y$ between metric spaces is a (K, C)-quasi-isometry if there exists constants K and C so that for all $x_1,\ x_2 \in X$ and $y \in Y$

$$\frac{1}{K}d(x_1,\ x_2) - C \leq d(f(x_1),\ f(x_2)) \leq Kd(x_1,\ x_2) + C \text{ and } d(y, f(X)) \leq C.$$

Up to quasi-isometry, the geometric structure of G is invariant under different choices of finite generating sets. More generally, we can consider groups up to quasi-isometry, and this opens up the study of group theory to geometric methods; such methods are motivated by the following question.

Question 2.2. If G and H are quasi-isometric groups, to what extent do G and H share the same algebraic properties?

We refer to the algebraic properties of a group that are preserved under quasi-isometries as *geometric properties*. In the case of the Isomorphism Problem above, an understanding of the geometric properties of a group gives us some insight into its isomorphism class. Specifically, if H and G are finitely presented isomorphic groups, then they must also be quasi-isometric. So by the contrapositive, if H and G are finitely presented groups that are not quasi-isometric, then they are also not isomorphic.

In the context of Question 2.2, we would like to consider more sophisticated spaces than the Cayley graph. For $n \in \mathbb{N}$, a group G is said to be of type F_{n+1} if it acts freely, faithfully, cellularly, properly, and cocompactly on an n-connected cell complex X, or equivalently, the group has an Eilenberg-MacLane complex, $K(G, 1)$, with a finite $(n + 1)$-skeleton. The Milnor-Švarc Theorem allows us to connect the quasi-isometry classes groups to the quasi-isometry classes of more interesting spaces that they act nicely on [10].

Definition 2.3. Suppose that a group G acts on a topological space X.

- We say that this action is *cocompact* if the quotient space X/G is compact.
- We will say that this action is *properly discontinuous* if each $x \in X$ has an open neighbourhood U such that $gU \cap U = \varnothing$ for all but finitely many $g \in G$.

Theorem 2.4 (Milnor-Švarc)**.** *Let (X, d_X) be a metric space. If G is group of isometries that acts properly discontinuously and cocompactly on X, then G is finitely generated, and the map $g \mapsto gx$ induces a quasi-isometry $G \to X$ for any $x \in X$.*

2.2 Isoperimetric Inequalities and Dehn Functions

The quasi-isometry invariants that we will study in this thesis are closely related to the word problem stated above. Arguably, the most direct way to approach the Word Problem is to replace the relators that occur as subwords in a word w with the identity element until we can see that this word is the identity element. This approach is practical if there is a bound on the number of replacements that would need to be made in order to deduce that $w = 1$. For a finite presentation, the Dehn function gives us such a bound as a function of the length of the word w.

Definition 2.5. Given a finite presentation $P = \langle X \mid R \rangle$ of a group G, we say that a word w in X is *null-homotopic* if w is equivalent in G to the identity $1 \in G$ (denoted $w =_G 1$). We define the algebraic area of such a word to be

$$\mathrm{Area}_P(w) = \min\{N \mid w \overset{free}{=} \prod_{i=1}^{N} x_i^{-1} r_i x_i \text{ with } x_i \in F(X), r_i \in R\}.$$

The Dehn function corresponding to P is the function $\delta_P : \mathbb{N} \to \mathbb{N}$ defined by

$$\delta_P(n) = \max\{\mathrm{Area}(w) \mid w =_G 1, \text{ and } |w| \leq n\}.$$

More generally, we can also define the Dehn function and its higher dimensional analogues in terms of the geometry of the spaces that a groups acts on. If G is of type F_{n+1}, then it acts as described above on an n-connected space X. Since X is n-connected, every map $f : S^n \to X$ is homotopic to a constant map. The *n-dimensional Dehn functions* give bounds on the number of $(n+1)$-cells necessary to realize this homotopy.

In particular, if a group G is of type F_2, or finitely presented, then G acts on a simply connected 2-complex X that is quasi-isometric to its Cayley graph. Every nullhomotopic word in G corresponds to a closed edge circuit in X. Since X is simply connected, this closed edge circuit bounds a disk in X. The 1-dimensional Dehn function (generally also referred to as the Dehn function) gives the optimal isoperimetric function of X in the following sense.

Definition 2.6. A non-negative, monotone function $f : \mathbb{N} \to \mathbb{N}$ is an *isoperimetric function* for a simply connected combinatorial 2-complex X if $\mathrm{Area}_X(w) \leq f(\mathrm{length}(w))$ for every edge circuit w, where $\mathrm{length}(w)$ is the combinatorial length of the edge circuit w and $\mathrm{Area}_X(w)$ is the minimum number of 2-cells enclosed by w.

Given two such functions, $f, g : \mathbb{N} \to \mathbb{N}$, we say that $f \preceq g$ if there exists a constant $c \geq 0$ so that

$$f(n) \leq cg(cn) + cn + c \text{ for all } n \in \mathbb{N},$$

and two functions are $\simeq$-equivalent if $f \preceq g$ and $g \preceq f$.

In particular, the geometric *Dehn function*, $\delta_X : \mathbb{N} \to \mathbb{N}$, given by

$$\delta_X(n) = max\{\mathrm{Area}_X(w) \mid \mathrm{length}(w) \leq n\}$$

is an isoperimetric function for X and if f is any other isoperimetric function for X and $n \in \mathbb{N}$, then $\delta_X(n) \preceq f(n)$. So, we can view the Dehn function as a minimal isoperimetric function for X.

It is clear that both of these definitions of the Dehn function coincide for a finite presentation of a $P = \langle X \mid R \rangle$ with presentation 2-complex K. More generally, the following theorem shows that the Dehn function is a quasi-isometric invariant for a finitely presented group G.

Proposition 2.7. *[1] If X and Y are two simply connected combinatorial* 2-*complexes with quasi-isometric* 1- *skeleta, then $\delta_X \simeq \delta_Y$.*

If $P = \langle X \mid R \rangle$ and $P' = \langle Y \mid S \rangle$ are finite presentations for quasi-isometric groups, then Proposition 2.7 says that these groups have the same Dehn function. So, up to $\simeq$-equivalence, we can define the Dehn function of a group G, denoted δ_G, in an obvious way. Moreover, if G acts properly discontinuously and cocompactly on a space X, then Propositions 2.4 and 2.7 allow us to suitably estimate the Dehn function of G in terms of isoperimetric functions on X.

The higher dimensional Dehn function does not have a strict group theoretic interpretation as in the 1-dimensional case, but rather is given by the cellular structure of the space on which G acts. Nevertheless, these higher dimensional isoperimetric inequalities give us quasi-isometric invariants for a group having appropriate finiteness properties. There are two types of questions that motivate the search and study of these invariants.

Question 2.8. for a given n, which functions can be the n-dimensional Dehn function of a finitely presented group?

The most comprehensive information for this question in the 1-dimensional case is given by the work of Birget, Rips, and Sapir [28]. Additionally, Brady, Bridson, Forester, and Shankar [6] and Brady and Forester [7] provide detailed information about n-dimensional Dehn functions of the form x^α. If there is a finitely presented group with Dehn function of the form x^α, then the exponent α is defined as an *isoperimetric exponent* and the collection of all isoperimetric exponents is called the *isoperimetric spectrum*.

Question 2.9. Given a group G (or family of groups), what are the n-dimensional Dehn functions of G (or what Dehn functions are possible for this family of groups)?

The most comprehensive information regarding this question is known for hyperbolic and CAT(0) groups. The following facts are known about hyperbolic and CAT(0) groups:

- In [20], Gromov showed that a group is hyperbolic if and only if it has a subquadratic (asymptotically grows slower than x^2) Dehn function. Alternate proofs of this result have been given in [23] [24] [4].

- If G is hyperbolic, then its higher dimensional Dehn functions are linear in all dimensions. In the case of CAT(0) groups, the work of Gromov [21] and Wenger [31] shows that the n-dimensional Dehn function is bounded by $x^{\frac{n+1}{n}}$. In particular, all CAT(0) groups satisfy a quadratic isoperimetric inequality.

- The family of CAT(0) groups includes free groups, free abelian groups, right-angled Artin groups, small cancellation groups, and Coxeter groups.

- It is worth noting that, unlike in the hyperbolic case, a quadratic Dehn function is not sufficient to deduce that a group is CAT(0).

In this thesis, we will explicitly give a family of subgroups of CAT(0) groups that are not CAT(0) themselves, but have quadratic Dehn function. Specifically, we will prove the following theorem.

Theorem A. *If a graph Γ can be decomposed as the join of three nonempty graphs, then the Bestvina-Brady Group, $B_\Gamma \leq A_\Gamma$, has quadratic Dehn function.*

As a corollary to this, we will show that the Stallings-Bieri groups, SB_n, also have a quadratic Dehn function for $n \geq 3$.

Corollary A. *If $n \geq 3$, then the Stallings-Bieri group, SB_n, have quadratic Dehn function.*

This corollary answers two open questions. The first is due to Gersten, who asked what the exact Dehn functions for this family of groups was in [19]. The second question is due to Bridson [9][17], and is related to the finiteness properties of groups with quadratic Dehn functions. Specifically, it asks if there exists a group which is of Type F_{n-1} but not of Type F_n and yet has quadratic Dehn function, for each $n \geq 4$? The work of Dison, Elder, Riley, and Young establishes this result for the case $n = 4$ [17], and in general, the Stallings-Bieri groups are considered as possible candidates for groups that satisfy this phenomena. We will establish this result in Chapter 5. Bridson's question is of interest, because it is well-known that groups with linear Dehn functions are of Type F_n for all $n \in \mathbb{N}$.

Corollary B. *For each $n \geq 4$, there exists a group with quadratic Dehn function that is of Type F_{n-1} but not Type F_n.*

Chapter 3

Preliminaries

In this chapter, we will outline the standard definitions and notations that we will need in order to accurately describe our results about Bestvina-Brady groups.

3.1 Graphs and Flag Complexes

For our purposes, the only graphs that we will consider are finite simplicial graphs, i.e. graphs with no loops or multiple edges. Associated to each graph Γ will be a set of vertices $V(\Gamma)$ and a set of edges $E(\Gamma)$. We will say that two vertices $v, w \in V(\Gamma)$ are *adjacent* if there exists an edge $vw \in E(\Gamma)$ (or $wv \in E(\Gamma)$) connecting v to w in Γ. Associated to these graphs, we make the following definitions.

Definition 3.1. If Γ is a graph, then the graph Γ' is an *induced subgraph* if Γ' is isomorphic to the graph whose vertex set is $V' \subset V(\Gamma)$ and whose edge set E' consists of all the edges in $E(\Gamma)$ that connect the vertices in V'

Definition 3.2. Given two graphs Γ_1 and Γ_2, we define the *join*, denoted $\Gamma_1 * \Gamma_2$, to be the graph with vertex set $V(\Gamma) = V(\Gamma_1) \sqcup V(\Gamma_2)$ and edge set $E(\Gamma) = E(\Gamma_1) \sqcup E(\Gamma_2) \sqcup E$, where E is the set of all edges connecting the vertices $V(\Gamma_1)$ and $V(\Gamma_2)$. Of particular interest, we will also define the cone of a graph Γ_1, denoted $C\Gamma_1$, to be the join of Γ_1 with the graph consisting of a single vertex.

Definition 3.3. We say that a simplicial complex L is a *flag complex* if the following condition holds:

For any $\{v_0, \ldots, v_n\}$ of L, if v_i and v_j are adjacent in L for every $i, j \in \{0, \ldots, n\}$ with $i \neq j$, then the vertex set $\{v_0, \ldots, v_n\}$ spans an n-simplex in L.

From the preceding definition, it is worth noting that if Γ is a simplicial graph, then Γ determines a flag complex L whose 1-skeleton $L^{(1)}$ coincides with Γ. Specifically, every complete subgraph on n vertices in Γ corresponds to an n-simplex in L.

3.2 Right-Angled Artin Groups

The significance of the graphs described in the preceding section is that we can associate to each graph Γ a torsion-free group A_Γ.

Definition 3.4. If Γ is a finite simplicial graph with vertex set $V(\Gamma) = \{v_1, \ldots, v_n\}$, we define the *right-angled Artin group*, A_Γ, to be the group given by the presentation

$$\langle a_1, \ldots, a_n \mid [a_i, a_j] = 1 \text{ if the vertices } v_i \text{ and } v_j \text{ are adjacent in } \Gamma \rangle.$$

Several algebraic properties of the group can be inferred from the structure of the graph. Indeed, from the definition above, we can see that the family of right-angled Artin groups contains every free group (graphs consisting of just vertices) as well as every free abelian group (complete graphs) as members. Since every graph exists somewhere between these two extremes, we can think of a generic right-angled Artin group as interpolating between these two extremes. It is clear that the induced subgraphs of Γ are subgroups of A_Γ, and we can observe that certain product structures of the group can be inferred from properties of the graph as well. The following proposition is immediate from the defining presentations.

Proposition 3.5. *If Γ_1 and Γ_2 are nontrivial induced subgraphs of a graph Γ, then:*

1. *$A_\Gamma = A_{\Gamma_1} \times A_{\Gamma_2}$ if and only if $\Gamma = \Gamma_1 * \Gamma_2$;*

2. *$A_\Gamma = A_{\Gamma_1} * A_{\Gamma_2}$ if and only if Γ is the disjoint union of Γ_1 and Γ_2.*

From this proposition, we can characterize when a right-angled Artin group splits as certain free or direct products over its "induced" subgroups. For example, the pentagonal graph C_5 corresponds to a right-angled Artin group that does not split as a free or direct product over any of its induced subgroups. While these subgroups allow us to understand the structure of some right-angled Artin groups, it is worth pointing out that these are certainly not the only subgroups of A_Γ, and that the study of subgroups of these groups is currently an active and fruitful topic of research. In Chapter 4, we will study a specific family of subgroups of right-angled Artin groups.

3.3 Spaces

The properties of the graph are not our only means of understanding the algebraic structure of A_Γ. In this section, we will associate to every right-angled right-angled Artin group, A_Γ, a CAT(0) cubical complex $\tilde{S}_\Gamma$ on which A_Γ acts properly discontinuously and cocompactly by isometries.

Definition 3.6. A geodesic metric space is CAT(0) if the distance between any two points on a geodesic triangle is less than or equal to the distance between the corresponding points of a comparison triangle in Euclidean space.

Definition 3.7. A group G is called a CAT(0) group if acts properly discontinuously and cocompactly by isometries on a CAT(0) space.

Given the presentation of A_Γ, as in Definition 3.4, we construct a finite CW-complex called the Salvetti Complex, denoted S_Γ as follows. Beginning with a wedge of n circles labeled by the generators meeting at a point x_0, we attach an m-torus for each set of m mutually commuting generators with faces labeled by the commutator relations. From the 2-skeleton, we observe $\pi_1(S_\Gamma) = A_\Gamma$. Taking the universal cover $\tilde{S}_\Gamma$ of S_Γ gives the desired cubical complex.

For example, suppose that Γ is the graph with vertex set $V(\Gamma) = \{a, b, c\}$ and edge set $E(\Gamma) = \{ab, bc\}$. In this case, the Salvetti complex consists of two tori glued together along the curve labeled by the generator corresponding to the vertex b. The universal cover of this space is the direct product of a four valent tree labeled with the generators corresponding to a and c and a line labeled by the generator corresponding to b. Alternatively, if we view this graph as the cone of a 0-sphere, we can infer that A_Γ is the direct product of a free group with $\mathbb{Z}$. The Cayley graph of A_Γ is clearly quasi-isometric to this space.

It is not difficult to see that $\tilde{S}_\Gamma$ is a CAT(0) cubical complex via Gromov's Link Condition [10]. The link condition reduces the problem of showing that a simply connected metric space is CAT(0) to a local condition on the link of each vertex. Stated precisely, the *link of a vertex* v in a cubical complex is the simplicial complex with one k-simplex for each $(k+1)$-cube containing v. Intuitively, we can view the link as the boundary of a ball in the metric induced by the l^1 norm centered at v.

Theorem 3.8 (Gromov)**.** *A finite dimensional, simply connected, cubical complex is CAT(0) if and only if all its vertex links are flag complexes.*

As the universal cover of S_Γ, the link of each vertex in $\tilde{S}_\Gamma$ is isometric to the link of the unique vertex in S_Γ. If we imagine gluing together the corners of a k-torus, we can observe that each k-torus in the Salvetti complex contributes a

$(k-1)$-sphere to the link. In particular, each vertex $a \in \Gamma$ contributes a 0-sphere to the link that has two vertices a^+ and a^-. The vertex set of the link consists of signed copies of the vertices in Γ, we will denote by $V(\Gamma)^{\pm}$. From the definition of the link, any two vertices in $V(\Gamma)^{\pm}$ are adjacent in the link if and only if their unsigned counterparts are adjacent in Γ. We can deduce that the vertices in the link span a simplex if and only if their unsigned counterparts span a simplex in Γ. Since Γ is flag by assumption, the result follows.

For example, consider the graph in Figure 3.1. The right-angled Artin group A_T associated to the triangle T on the left is $\mathbb{Z}^3$. By the definition of the link, this corresponds to a 3-sphere in the l^1 metric. The figure on the right is this octahedron with appropriate vertex labels as described above. In the link, the signed vertices of subtriangles in the octahedron correspond to the vertices of the triangle.

Remark 3.9. As a cubical complex, we can endow this $\tilde{S}_\Gamma$ with a canonical metric which makes each cube isometric to the regular Euclidean cube of side length 1 in Euclidean space. Under this metric, $\tilde{S}_\Gamma$ is a CAT(0) cubical complex. In this setting, each n-cube is also contained in a canonical cubical complex isomorphic to $\mathbb{R}^n$ with its usual cubing. This $\mathbb{R}^n$ has coordinates $x_1, \ldots, x_n$ with each cube given by

$$[k_1, k_1 + 1] \times \cdots \times [k_n, k_n + 1],$$

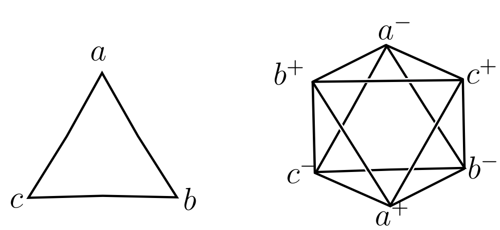

Figure 3.1: A Graph and its Associated Link

with $k_1, \ldots, k_n \in \mathbb{Z}$ (where $k_i \leq x_i \leq k_i + 1$ for all $i \in \{1, \ldots, n\}$).

Definition 3.10. Let (X, d_1) and (Y, d_2) be metric spaces and let $1 \leq p < \infty$. Define the p-product metric d_p on $X \times Y$ by

$$d_p((x_1, y_1), (x_2, y_2)) := (d_1(x_1, x_2)^p + d_2(y_1, y_2)^p)^{1/p}$$

There is a natural metric on the product of CAT(0) spaces coming from the following result from [10].

Proposition 3.11. *If (X, d_2) and (Y, d_2) are CAT(0) spaces, then the product $X \times Y$ with the 2-product metric*

$$d((x_1, y_1), (x_2, y_2))^2 = d(x_1, x_2)^2 + d(y_1, y_2)^2$$

is also CAT(0) space.

If each factor space is endowed with the cubical metric, then in the 2-product metric, each cube is isometric to a Euclidean cube. Since both of these path metrics agree on each cube, we can deduce that they define the same geometry on the product space.

Chapter 4

Bestvina-Brady Kernels

As mentioned above, we have a very clear understanding of the Dehn functions of hyperbolic groups and CAT(0) groups. In both cases, we can construct geometric arguments that verify that hyperbolic groups have linear Dehn functions, and CAT(0) groups can have either linear or quadratic Dehn functions. We can consider subgroups of these groups in hopes that our understanding of these geometric arguments might carry over. Specifically, we can ask questions similar to Questions 2.8 and 2.9 related to various subgroups of these groups.

We will restrict our attention to a very specific family of subgroups of right-angled Artin groups, the Bestvina-Brady groups, denoted B_Γ. These groups were initially introduced in [2] to solve a long standing open problem about whether or not there exists a group that is of type FP_2, but not of type F_2. More generally, the Bestvina-Brady construction can be applied to produce examples of groups that satisfy interesting finiteness properties. These groups arise as kernels of the natural epimorphism $\phi : A_\Gamma \to \mathbb{Z}$ that sends each generator of the right-angled Artin group A_Γ to the generator of $\mathbb{Z}$. To understand the structure of these groups, we can construct a space that B_Γ acts on properly discontinuously and co-compactly.

For each right-angled Artin group A_Γ, there exists a height function $\tilde{h} : \tilde{S}_\Gamma \to \mathbb{R}$ constructed as follows. From [2] (Theorem 5.12), we observe that the homomorphism

$$\phi : A_\Gamma \to \mathbb{Z}$$

that sends each generator of A_Γ to the generator 1 in $\mathbb{Z}$ is induced by a continuous map $h : S_\Gamma \to S^1$ that sends each 1-cell in S_Γ homeomorphically onto S^1. Lifting this map to the universal covers gives a continuous map

$$\tilde{h} : \tilde{S}_\Gamma \to \mathbb{R}$$

that is linear on each m-cube of $I^m \subset \tilde{S}_\Gamma$.

This construction allows us to relate topological properties of a finite flag complex to finiteness properties of the kernel in the following way.

Proposition 4.1 (Bestvina-Brady). *Let L be a finite non-empty flag complex, and let A be the corresponding right-angled Artin group; let $\phi : A \to \mathbb{Z}$ be the epimorphism taking all generators a_i to $1 \in \mathbb{Z}$,and let $B = ker(\phi)$. Then,*

1. *B has type F_n if and only if L is $(n-1)$-connected;*

2. *B has type FP_n if and only if L is $(n-1)$-acyclic.*

In our discussion of these groups, we will not make use of the second statement in the theorem. In lieu of a discussion of these properties, we will direct the interested reader to the following references [8], [11].

Regarding Question 2.9, a general isoperimetric function for any Bestvina-Brady group was given by Dison in [16].

Theorem 4.2 (Dison). *If L is a simply connected flag complex with $L^{(1)} = \Gamma$, then the Dehn function δ of the Bestvina-Brady group associated to Γ satisfies $\delta(n) \preceq n^4$.*

Brady has given examples that show this upper bound is sharp [8]. Crisp has given an example with cubic Dehn function [13], and it is easy to construct an example with quadratic Dehn function (the triangle graph above furnishes an example with quadratic Dehn function). It is currently unknown whether or not there exist Bestvina-Brady groups with a Dehn function having an isoperimetric exponent that is not an integer. Moreover, it is unknown what other types of functions can be the Dehn function of a Bestvina-Brady group.

The Stallings-Bieri groups are a particularly well studied family of groups that can be realized as Bestvina-Brady groups. Stallings group was initially constructed as an example of a group that is of type F_2 but not F_3 [30], and in [3], Bieri recognized this group to be given by the kernel of the map

$$\Phi : F(x_1, y_1) \times F(x_2, y_2) \times F(x_3, y_3) \to \mathbb{Z}$$

that sends the generator of each free group in the product to the generator $1 \in \mathbb{Z}$. As a generalization of Stallings group, the Stallings-Bieri groups, denoted SB_n, are given by the kernel of the homomorphism

$$\Phi : F(x_1, y_1) \times \cdots \times F(x_n, y_n) \to \mathbb{Z}$$

that sends the generator of each free group in the product to the generator $1 \in \mathbb{Z}$. Such groups display a range of finiteness properties as it is known that SB_k is of type F_{k-1}, but not type F_k.

Proposition 4.3. *If $n \geq 3$, then SB_n has a finite presentation given by*

$$\mathcal{P} = \langle u_1, \ldots, u_{n-1}, v_1, \ldots, v_{n-1}, t \mid R_1 \cup R_2 \cup R_3 \cup T \rangle$$

with defining relations

$$R_1 = \{[u_i, v_j] \mid \textit{ for } i, j \in \{1, \dots, n-1\} \textit{ with } i \neq j\},$$

$$R_2 = \{[u_i, u_j] \mid \textit{ for all } i, j \in \{1, \dots, n-1\}\},$$

$$R_3 = \{[v_i, v_j] \mid \textit{ for all } i, j \in \{1, \dots, n-1\}\}, \textit{ and}$$

$$T = \{t^{s_1} = t^{s_2} \mid s_1, s_2 \in \{v_1, \dots, v_{n-1}, u_1, \dots u_{n-1}\}\}.$$

Proof. From the definition above, SB_n is given as the kernel of the map

$$\Phi : F(x_1, y_1) \times F(x_2, y_2) \times \cdots \times F(x_n, y_n) \to \mathbb{Z}.$$

Since

$$Z = \{z_1 z_2^{-1} \mid z_1, z_2 \in \{x_1, \dots, x_n, y_1, \dots, y_n\} \text{ and } z_1 \neq z_2\}$$

is a generating set for SB_n, it can also be generated by the $2n-1$ element subset consisting of elements of the form $x_n y_i^{-1}$, $x_n x_i^{-1}$, and $y_n x_n^{-1}$ with $i \in \{1, \dots, n-1\}$. The set map that sends $u_i \mapsto x_n y_i^{-1}$, $v_i \mapsto x_n x_i^{-1}$, and $t \mapsto y_n x_n^{-1}$ for all $i \in \{1, \dots, n-1\}$, induces a surjective homomorphism $f : G(\mathcal{P}) \to SB_n$. Since the generators in SB_n satisfy the relations in $G(\mathcal{P})$, we need only show that this homomorphism is injective. From the defining relations above, we have the relations

$$z u_1^i t u_1^{-i} = u_1^{i+1} t u_1^{-(i+1)} z$$

for each $z \in \{u_1, \dots, u_{n-1}, v_1, \dots, v_{n-1}\}$. So if $w \in ker(f)$, we can write w as

$$w = w_1(u_1, v_1) w_2(u_2, v_2) \dots w_{n-1}(u_{n-1}, v_{n-1}) w_n(Q)$$

with $Q = \{u_1^i t u_1^{-i} \mid i \in \mathbb{Z}\}$. Mapping this normal form into SB_n, it is easy to see that if w maps to 1, then each $w_k(u_k, v_k) = 1$ for $1 \leq k \leq n-1$ and $w_n = 1$. Therefore $w = 1$ in $G(\mathcal{P})$ and so f is an isomorphism. $\square$

Isoperimetric inequalities for these groups have been studied in [19], [9], and [17], but the Dehn function for SB_n is still not known for $n > 3$. Using the presentation given above, Gersten has shown that these groups satisfy a polynomial isoperimetric inequality by producing a fifth degree polynomial isoperimetric function. In the case of SB_3, it was an open question for some time as to whether or not it had a polynomial Dehn function. This was settled by Dison, Elder, Riley, and Young by showing that that Stallings group has a quadratic Dehn function [17]. The four authors achieved their result via an algebraic argument applied to Gersten's presentation,

$$\langle a, b, c, d, t \mid [a, c], [b, c], [b, d], t^a = t^b = t^c = t^d \rangle,$$

with $a = x_3 x_1^{-1}$, $b = x_3 y_1^{-1}$, $c = x_3 x_2^{-1}$, $d = x_3 y_2^{-1}$, $t = y_3 x_3^{-1}$.

4.1 Geometric Structure

In this section, we will introduce a new cellular structure on the cube complex $\tilde{S}_\Gamma$. For each m, we subdivide each m-cube $I^m \subset \tilde{S}_\Gamma$ so that for all $n \in \mathbb{Z}$, $\tilde{h}^{-1}(n) \cap I^m$ is either empty or a subcomplex of I^m. We will write X_Γ to indicate $\tilde{S}_\Gamma$ with this sliced cubical structure. Subdividing $\tilde{S}_\Gamma$ in this way, each level set $\tilde{h}^{-1}(n)$ is a subcomplex of X_Γ. In particular, the zero level set, $(\tilde{S}_\Gamma)_0 = \tilde{h}_{\tilde{S}_\Gamma}^{-1}(0)$, is a subcomplex of X_Γ. It is worth noting that in the sliced cubical complex, the cells no longer correspond to cubes, but rather distinct pieces of cubes.

- The 0-cells are the same as the 0-cells of $\tilde{S}_\Gamma$.

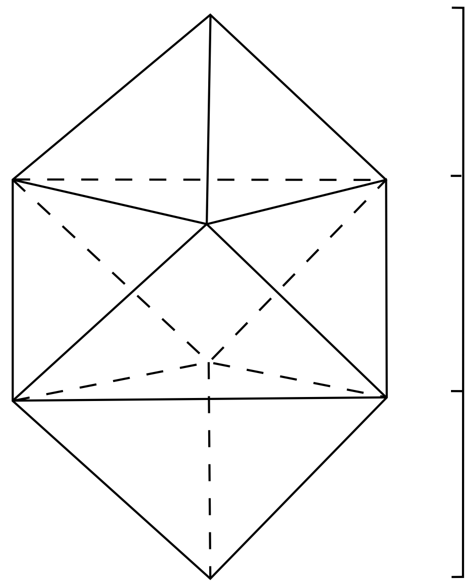

Figure 4.1: Cellular Structure of a Sliced 3-cube

- Each 1-cell is either a 1-cell in $\tilde{S}_\Gamma$ or a 1-cell in a level set.
- Each 2-cell is either an isosceles right triangle coming from a split square or an equilateral triangle in the level sets.
- Each 3-cell is either a tetrahedron or an octahedron.
- The general n-cell is described in Remark 4.4 below.

This sliced cellular structure is illustrated in Figure 4.1 for a standard 3-cube, and shows the 1, 2, and 3-cells in the sliced cubical complex.

Remark 4.4. In order to build some intuition regarding the cellular structure of this sliced cubical complex, recall that each n-cube in $\tilde{S}_\Gamma$ is contained in a canonical copy of $\mathbb{R}^n$ with appropriate coordinates as outlined in Remark 3.9. The subdivision coming from X_Γ slices this cube into n cells, each of which is characterized by one of the inequalities:

$$K \leq \sum_{i=1}^{n} x_i \leq K+1,\ K+1 \leq \sum_{i=1}^{n} x_i \leq K+2,\ \ldots,\ K+n-1 \leq \sum_{i=1}^{n} x_i \leq K+n$$

where $K = \sum_{i=1}^{n} k_i \in \mathbb{Z}$.

It is clear that B_Γ acts on $(\tilde{S}_\Gamma)_0$ properly discontinuously and cocompactly by isometries. Indeed, as a subgroup, there is an natural action B_Γ on X_Γ by height preserving isometries; restricting this action to $\tilde{h}_{S_\Gamma}^{-1}(0)$ gives the desired action.

In the case that Γ is the join of Γ_1 and Γ_2, the height function on $\tilde{S}_\Gamma = \tilde{S}_{\Gamma_1} \times \tilde{S}_{\Gamma_2}$ is given by $\tilde{h} : \tilde{S}_{\Gamma_1} \times \tilde{S}_{\Gamma_2} \to \mathbb{R}$ with

$$\tilde{h}(x_1, x_2) = \tilde{h}_1(x_1) + \tilde{h}_2(x_2),$$

where $\tilde{h}_i$ is the height function on $\tilde{S}_{\Gamma_i}$. In the case of the cone, we establish the following Lemma.

Lemma 4.5. *Let Φ be a simplicial graph and let $C\Phi$ denote the cone of Φ. There exists a cellular isomorphism*

$$F : X_\Phi \to (\tilde{S}_{C\Phi})_0.$$

Proof. Since the cone $C\Phi$ is formed by taking the join of Φ with a vertex v, we observe that $\tilde{S}_{C\Phi}$ decomposes as the product space $\tilde{S}_\Phi \times \mathbb{R}$. Further, we will set $L : \mathbb{R} \to \mathbb{R}$ to be the line in the product space whose edges are labeled by the single generator of A_v and parameterized by the height function on the factor $\mathbb{R}$.

By definition and the properties of the height function,

$$(\tilde{S}_{C\Phi})_0 = \{(x_1, L(t)) \mid \tilde{h}(x_1) + L(t) = 0\}.$$

So we can define a continuous map

$$F : X_\Phi \to (\tilde{S}_{C\Phi})_0$$

by $F(x) = (x, L(-\tilde{h}(x)))$. By construction, F is clearly both injective and surjective. We will show that the restriction of F to each coordinate flat is a cellular isomorphism onto its image. Suppose that Y is a coordinate n-flat in X_ϕ. Via F, each point $y = (y_1, \ldots, y_n)$ corresponds uniquely to some point $z \in (\tilde{S}_{C\Phi})_0 = (\tilde{S}_\Phi \times \mathbb{R})_0$. If we identify this point z with

$$z = ((y_1, \ldots, y_n), -\sum_{i=1}^{n} y_i),$$

then the restriction of F to this n-flat is given by

$$(y_1, \ldots y_n) \mapsto ((y_1, \ldots, y_n), -\sum_{i=1}^{n} y_i).$$

Letting each y_i range through all of its values in $\mathbb{R}$, it is clear that the image of Y describes an n-flat $(\tilde{S}_{C\Phi})_0$.

We will next show that F sends a sliced n-cell to a sliced n-cell in the image. We start by observing that each n-cell in $(\tilde{S}_{C\Phi})_0$ is contained inside of an $(n+1)$-cube in $\tilde{S}_{C\Phi}$. So by Remark 3.9, this n-cell is also contained inside of a canonical copy of $\mathbb{R}^{n+1}$ with coordinates $(y_1, \ldots, y_n, y_{n+1})$ and satisfies the inequalities

$$m_1 \le y_1 \le m_1 + 1,\ \ldots,\ m_n \le y_n \le m_n + 1,\ m_{n+1} \le y_{n+1} \le m_{n+1} + 1$$

for some integers $m_1, \ldots, m_{n+1}$. This cell is also in the zero level set, and so it also satisfies the requirement that $\sum_{i=1}^{n+1} y_i = 0$. Let $M = \sum_{i=1}^{n} m_i$. Since m_{n+1} is an integer bounded between $-M-n$ and $-M-1$, and since $y_{n+1} = -\sum_{i=1}^{n} y_i$, the inequality $m_{n+1} \le y_{n+1} \le m_{n+1} + 1$ allows us to deduce that this n-cell in $(\tilde{S}_{C\Phi})_0$ is a sliced n-cell as described in Remark 4.4.

Given an n-cell in X_Φ contained in a canonical flat as described Remark 4.4, say with $K + j \le \sum_{i=1}^{n} y_i \le K + j + 1$ for some fixed $j \in \{0, \ldots, n-1\}$. The image of this cell under F is an n-cell in the corresponding flat that satisfies the inequalities:

$$k_1 \le y_1 \le k_1 + 1,\ \ldots,\ k_n \le y_n \le k_n + 1,\ k_{n+1} \le y_{n+1} \le k_{n+1} + 1$$

with $y_{n+1} = -\sum_{i=1}^{n} y_i$ and $k_{n+1} = -(K + j + 1)$. Hence, restriction of F to a flat is a cellular isomorphism onto its image. Since F is bijective, it follows that F is also a cellular isomorphism. $\square$

Since $\tilde{S}_\Gamma$ is a CAT(0) space, it satisfies either a sharp linear or quadratic isoperimetric inequality. Via the cellular isomorphism in Lemma 4.5, it is clear that the space $(\tilde{S}_{C\Gamma})_0$ also satisfies the same type of isoperimetric inequality. This gives other examples of Bestvina-Brady groups with linear and quadratic Dehn functions. We state this result as a corollary.

Corollary 4.6. *If Γ is a simplicial graph, then the Bestvina-Brady group $B_{C\Gamma}$ has the same Dehn function as A_Γ.*

More generally, suppose that Γ is a join of two nontrivial graphs Γ_1 and Γ_2. For each vertex $v \in V(\Gamma_i)$, there is a graph embedding of $C\Gamma_j \hookrightarrow \Gamma$ (with $i \neq j$) as a induced subgraph. Given $x = (x_1, x_2) \in (\tilde{S}_\Gamma)_0$, we let L_i be the bi-infinite geodesic line in $\tilde{S}_{\Gamma_i}$ passing through x_i with edges labeled by a, the generator corresponding to v, and parameterized so that $\tilde{h}_i(L_i(t)) = t$. We will refer to such lines as *monotone lines*. The graph embedding induces an embedding $\tilde{S}_{C\Gamma_j} \to \tilde{S}_\Gamma$ as the subcomplex $\tilde{S}_{\Gamma_j} \times L_i$. Further, in terms of the zero level sets, $(\tilde{S}_{\Gamma_j} \times L_i)_0$ is a subcomplex of $(\tilde{S}_\Gamma)_0$ containing x that is isomorphic to $(\tilde{S}_{C\Gamma_j})_0$. Composing this isomorphism with the map in Lemma 4.5, we have also the following corollary.

Corollary 4.7. *Suppose that $\Gamma = \Gamma_1 * \Gamma_2$ with each Γ_i non-trivial and $x = (x_1, x_2) \in (\tilde{S}_\Gamma)_0$. If L_i is a monotone line containing x_i and $i \neq j$, then there exists a combinatorial embedding*

$$F_{L_i} : X_{\Gamma_j} \to (S_\Gamma)_0$$

where $F_{L_i}(X_{\Gamma_j})$ contains x.

Observe that there may be several monotone lines passing through a single point x, and this embedding depends on this monotone line. For a specific monotone line L_i, we will write $Y(L_i) = F_{L_i}(X_{\Gamma_j})$ to denote the embedded subspace.

Chapter 5

Isoperimetric Inequalities for Certain Bestvina-Brady Groups

In this chapter, we will prove that if the simplicial graph Γ can be decomposed as the join of three nontrivial graphs, then the Dehn function of the Bestvina-Brady group B_Γ is quadratic. Let $\Gamma = \Gamma_1 * \Gamma_2 * \Gamma_3$ for fixed simplicial graphs Γ_i.

Remark 5.1. Our argument relies on special combinatorially embedded subspaces that are constructed by Corollary 4.7. Suppose we are given three specific monotone lines L_1, L_2, and L_3 with $L_i \subset \tilde{S}_{\Gamma_i}$. We will write $Y(L_1) = (L_1 \times \tilde{S}_{\Gamma_2} \times \tilde{S}_{\Gamma_3})_0$, $Y(L_2) = (\tilde{S}_{\Gamma_1} \times L_2 \times \tilde{S}_{\Gamma_3})_0$, and $Y(L_3) = (\tilde{S}_{\Gamma_1} \times \tilde{S}_{\Gamma_2} \times L_3)_0$. We will write $Y(L_i,\ L_j)$ to denote the intersection $Y(L_i) \cap Y(L_j)$. An argument similar to Corollary 4.7 shows that this subspace can be identified with $X_{\Gamma_k} \times \mathbb{R}$. In the same way, we will write the intersection $Y(L_1) \cap Y(L_2) \cap Y(L_3)$ as $Y(L_1, L_2, L_3) = (L_1 \times L_2 \times L_3)_0$; this subspace is identified with $\mathbb{R}^2$ having the sliced cell structure.

Lemma 5.2. *Suppose that $L_i \subset \tilde{S}_{\Gamma_i}$ is a monotone line for $i \in \{1, 2, 3\}$. There exists a constant k_1 so that if Y is one of the embedded subspaces $Y(L_i)$, $Y(L_i,\ L_j)$, or $Y(L_1,\ L_2,\ L_3)$ and ρ is a loop of length n contained in Y, then*

$$\mathrm{Area}_{(\tilde{S}_\Gamma)_0}(\rho) \leq \mathrm{Area}_Y(\rho) \leq k_1 n^2.$$

Proof. Choose k_1 sufficiently large so that $k_1 n^2$ is an upper bound for the area in any of the CAT(0) spaces described in Remark 5.1 (there are only finitely many of these spaces to consider up to isometry). Let Y be one of the subspaces listed above, and let ρ be a loop of length n contained in Y. From the identifications stated in Remark 5.1, there exists a loop ρ' in the CAT(0) space identified with Y. Each cell enclosed by ρ' can be identified with a cell enclosed by ρ in Y. Hence, $\text{Area}_Y(\rho) \leq k_1 n^2$. For the first inequality it is clear that any 2-cell in Y is also a 2-cell in $(\tilde{S}_\Gamma)_0$. So by definition, $\text{Area}_{(\tilde{S}_\Gamma)_0}(\rho) \leq \text{Area}_Y(\rho)$. □

In addition to the special subspaces described in remark 5.1, we will also make use of special paths in $(\tilde{S}_\Gamma)_0$ that pass through these subspaces joined along a point in a common subspaces.

Lemma 5.3. *Suppose that $x,\ y, \in (\tilde{S}_\Gamma)_0$ and $i, j \in \{1,\ 2,\ 3\}$ are distinct, with $x \in Y(L_i)$, $y \in Y(L_j)$. There exists a constant k_2 and a path $\gamma^1(L_i, L_j) \subset (\tilde{S}_\Gamma)_0$, joining x to y and passing through a point in $Y(L_i, L_j)$ such that*

$$\text{length}(\gamma^1(L_i,\ L_j)) \leq k_2 dist_{(\tilde{S}_\Gamma)_0}(x,\ y).$$

Proof. To avoid excessive notation, we will suppose, without loss of generality, that $i = 1$ and $j = 2$; the other cases are handled in an identical way. Let $x = (x_1, x_2, x_3) \in Y(L_1)$ and $y = (y_1, y_2, y_3) \in Y(L_2)$ for monotone lines L_1 and L_2 that pass through x_1 and y_2 respectively. Let $a_1 = dist_{X_{\Gamma_1}}(L_1(-\tilde{h}_2(y_2) - \tilde{h}_3(y_3),\ y_1)$, $a_2 = dist_{X_{\Gamma_2}}(x_2,\ y_2)$, $a_3 = dist_{X_{\Gamma_3}}(x_3,\ y_3)$, and $a = a_1 + a_2 + a_3$. Parameterize shortest edge paths

$$\rho_1 : [a_2 + a_3,\ a] \to X_{\Gamma_1}$$

$$\rho_2 : [0,\ a_2] \to X_{\Gamma_2}$$

$$\rho_3 : [a_2,\ a_2 + a_3] \to X_{\Gamma_3}$$

connecting the appropriate points in the factor spaces, and extend these maps to the entire interval $[0,\ a]$ by having them be constant outside the subintervals given in the definitions of the ρ_k. Using these edge paths, we define a path in $(\tilde{S}_\Gamma)_0$,

$$\gamma^1(L_1,\ L_2) : [0,\ a] \to (\tilde{S}_\Gamma)_0$$

by

$$\gamma^1(t) = \begin{cases} (L_1(-\tilde{h}_2(\rho_2(t)) - \tilde{h}_3(\rho_3(t))), \rho_2(t), \rho_3(t)), & 0 \leq t \leq a_2 + a_3 \\ (\rho_1(t), L_1(-\tilde{h}_2(\rho_2(t)) - \tilde{h}_3(\rho_3(t))), \rho_3(t)), & a_2 + a_3 \leq t \leq a \end{cases}$$

that connects x to y.

By construction, $p(L_1,\ L_2) = \gamma^1(L_1,\ L_2)(a_2 + a_3) \in Y(L_1,\ L_2)$. Also, the number of 1-cells in $(\tilde{S}_\Gamma)_0$ traversed by $\gamma^1(L_i,\ L_j)$ is bounded by $b_1 + 2(a_2 + a_3)$, where $b_1 = dist_{X_{\Gamma_1}}(x_1, y_1)$. We obtain our final estimate comparing the 1-product metric on X_Γ with the ambient 2-product metric:

$$\text{length}(\gamma^1(L_1,\ L_2)) \leq 4dist_1(x,\ y) \leq 4\sqrt{3}dist_2(x,\ y) \leq 4\sqrt{3}dist_{(\tilde{S}_\Gamma)_0}(x,\ y).$$

This construction can be generalized for other values of i and j. □

Remark 5.4. This path described in the preceding proof consists of two steps described as follows.

- The first part of the path travels along ρ_2 in the second factor followed by travelling along ρ_3 in the third factor while simultaneously moving appropriately along L_1 in the first factor so that the path stays in $(\tilde{S}_\Gamma)_0$.

- The second part of the path travels along ρ_1 in the first factor while simultaneously moving along L_2 in the second factor so that the path stays in $(\tilde{S}_\Gamma)_0$.

From the description above, it is clear that the first part of the path lies in $Y(L_1)$, the second part lies in $Y(L_2)$, and that the common point $p(L_1,\ L_2) = \gamma^1(L_1,\ L_2)(a_2+a_3)$ lies in $Y(L_1,\ L_2)$. It is worth noting that in the construction, $\gamma^1(L_i,\ L_j) \neq \gamma^1(L_j,\ L_i)$, so we must necessarily specify a direction.

We will refer to such a path defined in Lemma 5.3 as a *two-step path of type 1*. A similar construction in the following lemma will furnish the second type of paths that we will consider.

Lemma 5.5. *Suppose that $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$, and $z = (z_1, z_2, z_3)$ are points in $(S_\Gamma)_0$. There exists a constant k_3 so that if there are two-step paths of type* 1 *connecting x to y and y to z, then there exists a path, $\gamma^2((L_i,\ L_k), L_j) \subset (\tilde{S}_\Gamma)_0$, connecting $p(L_i,\ L_j)$ to $p(L_j,\ L_k)$ and passing through a point in $Y(L_1, L_2, L_3)$ with*

$$\text{length}(\gamma^2((L_i,\ L_k), L_j)) \leq k_3 dist_{(\tilde{S}_\Gamma)_0}(p(L_i,\ L_j), p(L_j,\ L_k)).$$

Proof. Let x, y, and z be as in the statement of the Lemma 5.3. As above, we will proceed without loss of generality by presenting this construction using an example, $i = 1$, $j = 2$, and $k = 3$. Given monotone lines L_1, L_2, and L_3 with $x_1 \in L_1 \subset \tilde{S}_{\Gamma_1}$, $y_2 \in L_2 \subset \tilde{S}_{\Gamma_2}$, and $z_3 \in L_3 \subset \tilde{S}_{\Gamma_3}$, from Lemma 5.3, our two step paths of type 1 pass through the common points

$$p(L_1,\ L_2) = (L_1(-\tilde{h}_2(y_2) - \tilde{h}_3(y_3)),\ y_2,\ y_3) \text{ and}$$

$$p(L_2,\ L_3) = (z_1,\ L_2(-\tilde{h}_1(z_1) - \tilde{h}_3(z_3)),\ z_3).$$

Parameterize shortest edge paths

$$\rho_1' : [b_3, b_3 + b_1] \to X_{\Gamma_1} \text{ and } \rho_3' : [0, b_3] \to X_{\Gamma_3}$$

joining $L_1(-\tilde{h}_2(y_2) - \tilde{h}_3(y_3))$ to z_1 and y_3 to z_3, respectively, with

$$b_1 = dist(z_1,\ L_1(-\tilde{h}_2(y_2) - \tilde{h}_3(y_3))) \text{ and } b_3 = (y_3, z_3).$$

Let $b = b_1 + b_3$. We define the path $\gamma^2 : [0, b] \to (\tilde{S}_\Gamma)_0$ by

$$\gamma^2(t) = \begin{cases} (L_1(-\tilde{h}_2(y_2) - \tilde{h}_3(y_3)), L_2(-\tilde{h}_1(\rho_1'(t)) - \tilde{h}_3(z_3)), \rho_3'(t)), & 0 \le t \le b_3 \\ (\rho_1'(t), L_1(-\tilde{h}_1(y_1) - \tilde{h}_3(\rho_3'(t))), z_3), & b_3 \le t \le b \end{cases}.$$

By construction, $\gamma^2((L_i,\ L_k), L_j)(b_3) \in Y(L_1,\ L_2,\ L_3)$. Also, the number of 1-cells in $(\tilde{S}_\Gamma)_0$ traversed by $\gamma^2((L_i,\ L_k), L_j)$ is bounded by $b_1 + b_3$. A length calculation similar to the one used in the proof of Lemma 5.3 establishes the desired inequality. □

Note that the first part of the path lies in $Y(L_1,\ L_2)$, the second part lies in $Y(L_2,\ L_3)$, and, as stated above, $\gamma^2((L_i,\ L_k), L_j)(b_3) \in Y(L_1,\ L_2,\ L_3)$. We refer to paths as described in Lemma 5.5 as *two-step paths of type 2.*

Lemma 5.6. *Let $x,\ y,\ z \in (\tilde{S}_\Gamma)_0$ be distinct points, and let $T(L_1, L_2, L_3)$ be a directed triangle formed by two-step paths of type 1 connecting these points, for some choice of monotone lines $L_i \subset \tilde{S}_{\Gamma_i}$. There exists a constant k_4 such that if the perimeter of T is l, then* $\mathrm{Area}_{(\tilde{S}_\Gamma)_0}(T) \le k_4 l^2$.

Proof. Let $x = (x_1,\ x_2,\ x_3),\ y = (y_1,\ y_2,\ y_3)$, and $z = (z_1,\ z_2,\ z_3)$ be points

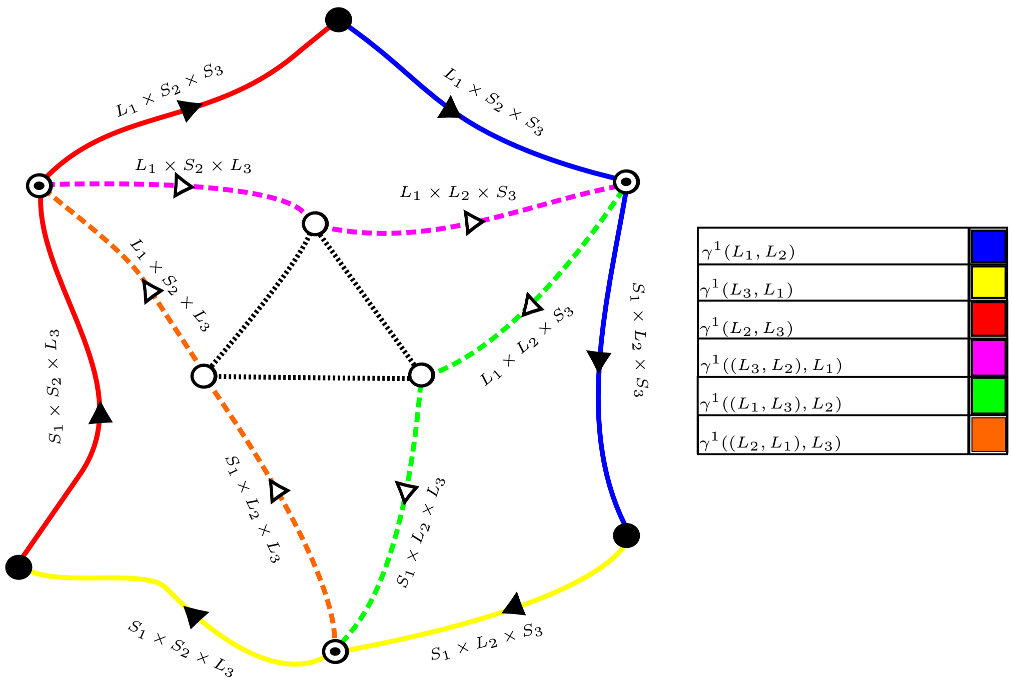

Figure 5.1: The triangle $T(L_1, L_2, L_3)$ formed by two-step paths

in $(\tilde{S}_\Gamma)_0$, and let $c = \max\{k_2, k_3\}$. Without loss of generality, choose monotone lines $L_k \subset X_{\Gamma_k}$ such that $x_1 \in L_1$, $y_2 \in L_2$ and $z_3 \in L_3$. Let $T(L_1,\ L_2,\ L_3)$ denote the directed triangle connecting x, y, and z by two-step paths of type 1 as described in Lemma 5.3 forming a clockwise loop

$$\gamma^1(L_1,\ L_2) \cup \gamma^1(L_2,\ L_3) \cup \gamma^1(L_3,\ L_1);$$

relabeling x, y, and z if necessary. This path is illustrated in Figure 5.1 by solid directed paths connecting the black vertices. Next, we subdivide T by connecting $p(L_1,\ L_2)$, $p(L_2,\ L_3)$, and $p(L_3,\ L_1)$ by two-step paths of type 2 as described in Lemma 5.5 to form the clockwise directed loop

$$\gamma^2((L_1\ , L_3), L_2) \cup \gamma^2((L_2,\ L_1), L_3) \cup \gamma^2((L_3,\ L_2), L_1).$$

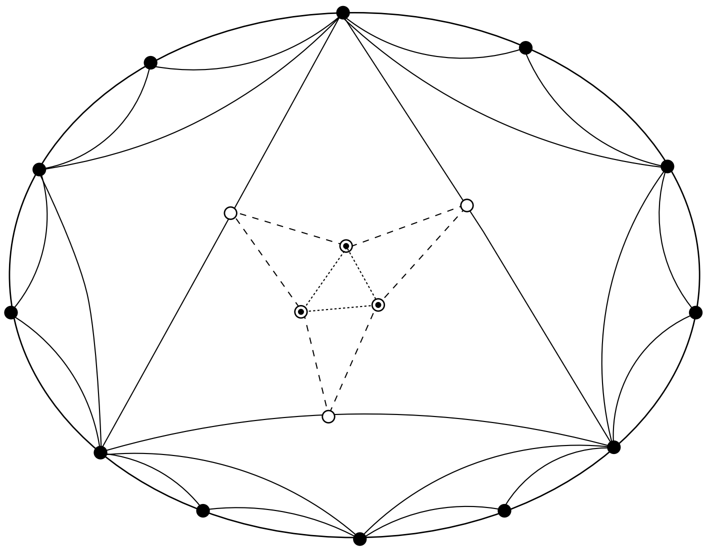

Figure 5.2: Filling Pattern of a Loop ρ by Two-step Triangles with the Central Triangle Subdivided as in Lemma 5.6

This is illustrated in Figure 5.1 by the directed dashed lines connecting the white vertices. Finally, we connect $\gamma^2((L_2\ ,L_1),L_3)(b_1)$, $\gamma^2((L_3,\ L_2),L_1)(b_2)$, and $\gamma^2((L_1,\ L_3),L_2)(b_3) \in Y(L_1,\ L_2,\ L_3)$, by shortest edge paths in $Y(L_1,L_2,L_3)$, where the b_k are given by the shortest edge path ρ'_k in the construction of $\gamma^2((L_i,\ L_k),L_j)$ in Lemma 5.5.

Connecting the relevant vertices in this way subdivides T into 7 subtriangles. By construction, each subtriangle is a loop with edges contained one in of the subspaces $Y(L_i)$, $Y(L_i,\ L_j)$ or $Y(L_1,\ L_2,\ L_3)$ described in Remark 5.1. The perimeter of each triangle is bounded above by $12cl$. Estimating the area of each triangle in its respective subspace via Lemma 5.2 gives the stated upper bound on its area in $(\tilde{S}_\Gamma)_0$. $\square$

Theorem 5.7. *If a graph Γ can be decomposed as the join of three nonempty graphs, then the Bestvina-Brady Group B_Γ has quadratic Dehn function.*

Proof. Parameterize ρ with respect to arc length, and let $\rho(t_1) = (x_1,\ x_2,\ x_3) =$

x, $\rho(t_2) = (y_1,\ y_2,\ y_3) = y$, and $\rho(t_3) = (z_1,\ z_2,\ z_3) = z$ be three equally spaced points on ρ (approximated to the closest vertex along ρ), and let $c = \max\{k_2, k_3\}$. Choose monotone lines L_i with $x_1 \in L_1$, $y_2 \in L_2$ and $z_3 \in L_3$ to construct a directed triangle $T_0 = T(L_1,\ L_2,\ L_3)$ as described in Lemma 5.6. This central triangle can be filled by a disc in $(\tilde{S}_\Gamma)_0$ with area at most $k_4 c^2 (l+2)^2$.

As illustrated in figure 5.2, there are three loops to consider that are formed by $\rho([t_i,\ t_j]) \cup \gamma^1(L_i, L_j)$, for appropriate values of i and j. Consider the loop formed by $\rho([t_1,\ t_2]) \cup \gamma^1(L_1, L_2)$. For the approximate midpoint, $w = (w_1,\ w_2,\ w_3)$ (the closest vertex on ρ to $\rho(\frac{t_1+t_2}{2})$), choose a monotone line $L_3' \subset X_{\Gamma_3}$ containing w_3, and connect w to x and y by the two-step paths $\gamma^1(L_1, L_3')$ and $\gamma^1(L_3', L_2)$ to form the triangle $T_1 = T(L_1,\ L_2,\ L_3')$. The side lengths of $\gamma^1(L_1, L_3')$ and $\gamma^1(L_3', L_2)$ are bounded by

$$\text{length}(\gamma^1(-)) \leq c\frac{l+2}{6} + 1 \leq c\frac{l+2}{3},$$

and so the perimeter of T_1 is bounded by $c(l+2)$. Construct similar triangles in the loops $\rho([t_2,\ t_3]) \cup \gamma^1(L_2, L_3)$ and $C([t_3,\ t_1]) \cup \gamma^1(L_3, L_1)$.

- Stage 1 creates three triangles, each with perimeter at most $c(l+2)$, area bounded by $c^2 k_4 (l+2)^2$, and six new loops to consider.

- In the second stage, the length of the attached two-step paths is bounded by
$$\text{length}(\gamma^1(-)) \leq c(\frac{l+2}{2^2 3} + 1) \leq c\frac{l+2}{(2)(3)}.$$
So, we have 6 triangles, each with perimeter bounded by $c\frac{2}{3}(l+2)$, area bounded by $\frac{4}{9}c^2 k_4 (l+2)^2$, and 12 new loops to consider.

- Proceeding inductively, at the *nth* stage, the length of the attached two-step paths is bounded by

$$\text{length}(\gamma^1(-)) \leq c(\frac{l+2}{2^n 3} + 1) \leq c\frac{l+2}{2^{n-1}3},$$

since we will only be considering the values of n with $\frac{l+2}{2^n 3} \geq 1$. So, we have $2^{n-1}3$ triangles, each with perimeter bounded by $\frac{1}{2^{n-3}3}c(l+2)$, area bounded by $\frac{1}{2^{2n-6}9}c^2 k_4 (l+2)^2$, and $2^n 3$ new loops to consider.

The largest value of n with $\frac{l+2}{2^n 3} \geq 1$ is an upper bound on the number of stages required to fill ρ. Summing up the contributions from each of the subtriangles we have

$$\text{Area}(\rho) \leq 4k_4 c^2 (l+2)^2 + \sum_{j=2}^{\infty} \frac{1}{2^{j-5}3} k_4 c^2 (l+2)^2 = \frac{28}{3} k_4 c^2 (l+2)^2.$$

The result follows by choosing a sufficiently large k. $\square$

Corollary 5.8. *If Γ can be decomposed as the join of 3 non-trivial graphs then the Bestvina-Brady Group B_Γ has quadratic Dehn Function.*

Proof. Theorem 5.7 established a quadratic upper bound on the Dehn function. In light of the gap in the isoperimetric spectrum between 1 and 2, it is sufficient to show that B_Γ is not hyperbolic. Since Γ is the join of three non-trivial graphs, B_Γ necessarily contains $\mathbb{Z}^2$ subgroups, and thus cannot be hyperbolic. The result follows. $\square$

As mentioned earlier, we have the following corollaries that establish claims made by Bridson in [9]. Specifically, the Stallings-Bieri groups have quadratic Dehn function, since each SB_n can be realized as a the Bestvina-Brady group B_{Γ_n} where Γ_n is a join of n 0-spheres.

Corollary 5.9. *If $n \geq 3$, then the Stallings-Bieri groups, SB_n, have quadratic Dehn function.*

Corollary 5.10. *For each $n \geq 4$, there exists a group with quadratic Dehn function that is of Type F_{n-1} but not Type F_n.*

Chapter 6

Overview and Statement of Main Results: Part 2

If $k[G]$ is a group ring over a torsion-free group, two natural questions that can be asked are what are the zero divisors, and what are the units? Both questions are very well known and considered to be two of the least tractable questions in the theory of group rings. A detailed discussion of the history of these problems (and other interesting open questions) can be found in [25].

Conjecture 1. Zero Divisor Conjecture (Kaplansky) *If G is a torsion-free group and K is an integral domain, then the group ring $K[G]$ has no zero divisors.*

Similarly, the second conjecture, which implies Conjecture 1, can be stated as.

Conjecture 2. Nontrivial Units Conjecture (Kaplansky) *If G is a torsion-free group and K is a field, then the only units in $K[G]$ are the trivial ones, i.e. those of the form kg where $k \in K$ and $g \in G$.*

The unique product property was initially conceived as an attempt to solve these conjectures. A group is G is said to satisfy the *unique product property* if given any two non-empty finite sets X, $Y \subset G$ then at least one element, say z in the product set $XY = \{xy \mid x \in X \text{ and } y \in Y\}$ can be written uniquely as a product, $z = xy$ where $x \in X$ and $y \in Y$. Many familiar groups satisfy this property, for example, orderable groups [25], diffuse groups [5] and locally indicable groups [12]. In particular, it is well known that every right orderable group satisfies this property. The converse, however, is still open.

Any group with torsion does not satisfy the unique product property, so the only interesting examples of groups without this property would necessarily be torsion-free. There are only two known examples of torsion-free groups that do not satisfy the unique product property (excluding, of course, torsion-free groups that contain either of these two examples as a proper subgroup).

The first example was given by E. Rips and and Y. Segev. The authors showed that there exists a family of torsion-free groups that do not satisfy this property [27]. In their examples, given predetermined sets, relations for a group were carefully constructed that in such a way that the resulting group is torsion-free and contains the two sets as a pair of non-unique product sets. Many seemingly natural questions regarding these groups are still open. In particular, nothing is known about these groups in relation to Conjectures 1 or 2.

The second known example of a group that does not satisfy the unique product property and the only known explicit example of such a group was given by D. Promislow in [26]. By means of a random search algorithm, he found a 14 element set S in the group

$$P = \langle x, y \mid xy^2x^{-1}y^2,\ yx^2y^{-1}x^2 \rangle$$

with the property that SS has no uniquely represented element. We will call such a set S a *non-unique product set*. Given the nature of the search, very little is known about other non-unique product sets in P or about how to extend this result to other groups.

A result due to Lewin, [22], shows that P satisfies Conjecture 1.

Theorem 6.1. (Lewin) *If $G = G_1 *_{G_N} G_2$ a free product with amalgamation, where*

1. G_N *is normal in both* G_1 *and* G_2*;*

2. $F[G_1]$ *and* $F[G_2]$ *have no zero divisors;*

3. $F[G_N]$ *satisfies the Ore condition.*

Then $F[G]$ *has no zero divisors.*

To see this, note that $P \cong K *_{\mathbb{Z}^2} K$, where K is a Klein bottle group and we identify index 2 subgroups that are isomorphic to $\mathbb{Z}^2$ in each copy of K. The second condition holds since group rings over locally indicable groups satisfy Conjecture 1. For the last condition, it is well known that a group ring over an abelian group satisfies the Ore condition. It is still unknown whether P satisfies Conjecture 2.

The purpose of this section is to generate new simple examples of groups that do not satisfy the unique product property and to produce non-unique product sets whose existence can be inferred from the relations in the group. Currently, it is not all together clear where to look for such groups or even sets within these groups. Certainly these groups must be non-left orderable. In fact, this is precisely why P was initially seen as a likely candidate [18]; however, this does not tell us how to find such sets or even if they exist (clearly, any finite pair of subsets will not work). The hope is that generating more examples will lead to a better understanding of the structure of such groups. In Chapters 7 and 8, we do so by generalizing P in the following way.

Theorem 6.2. *For each* $k > 0$*, the torsion-free group*

$$P_k = \langle a, b, \mid ab^{2^k}a^{-1}b^{2^k},\ ba^2b^{-1}a^2 \rangle$$

does not satisfy the unique product property, and for $k > 1$*, does not contain* P*.*

Note that the group P_1 is the same as Promisow's example P. The relations of P_1 and P_k are similar, but the groups are quite different. For example, it is well known that P is a finite extension of $\mathbb{Z}^3$ and as such is supersolvable. In contrast, the groups P_k for $k > 1$ are much larger. One can show P_k contains a finite index subgroup isomorphic to $\mathbb{Z}^2 \times F$, where F is a finitely generated free group. In particular, these groups are also not amenable and hence are not solvable. An argument, identical to the one above, shows that each P_k satisfies the hypotheses of Theorem 6.1 and thus every group P_k satisfies Conjecture 1.

These groups are generalizations of P in the sense that each P_k is is an amalgamation of Klein bottle groups over $\mathbb{Z}^2$. However, we wish to emphasize that the non-unique product sets we construct in Chapter 8 are not generalizations of Promislow's set S found [26], but rather arise from a careful study of the geometry of the Cayley graph given by the presentation above. Roughly, the idea is to construct specific paths in the Cayley graph taken sufficiently long so that the Klein bottle relations force certain paths from the product set to overlap nicely. In Chapter 9, this idea is extended to longer paths in the Cayley graph to prove the following result.

Theorem 6.3. *Each group P_k contains arbitrarily large non-unique product sets.*

Chapter 7

Properties of the Groups P_k

7.1 Bass Serre Theory

In this section we will recall several facts from Bass-Serre theory, which studies the algebraic structure of groups acting on simplicial trees. If a group G acts by automorphisms on a simplicial tree T, without inversion, then T is called a G-tree. The action is said to be *trivial* if G fixes a point and *minimal* if there is no invariant G-subtree.

In this setting, an automorphism is said to be *elliptic* if it fixes a point and *hyperbolic* otherwise. If g is elliptic, we define $Fix(g)$ to be the set of all points fixed by g. Following [29], we can characterize these automorphisms in the following way.

Proposition 7.1. *Let G be group that acts on a simplicial tree T by automorphisms.*

1. *If $g \in G$, then either g acts on a unique simplicial line in T by translations or $Fix(g) \neq \emptyset$.*

2. *If $g_1,\ g_2 \in G$ and $Fix(g_1),\ Fix(g_2)$ are nonempty and disjoint, then $Fix(g_1 g_2) = \emptyset$.*

3. *If G is generated by a finite set of elements $s_1,\ s_2,\ \ldots,\ s_m$ such that s_j and $s_i s_j$ fix points in T for all $i,\ j$, then G the action of G is trivial.*

The unique simplicial line in (1) is called the *axis* of g and denoted A_g. Further, following [14], we can describe minimal subtrees in the following way.

Proposition 7.2. *If G is finitely generated and T is a non-trivial G-tree then T contains a unique minimal G-invariant subtree, which is the union of the axes of all the hyperbolic elements in G.*

A natural setting for groups acting on G-trees is when G splits as a free product with amalgamation, an HNN extension, or more generally as the fundamental group of a graph of groups. From [29] there exists a tree T, referred to as the Bass-Serre tree, on which G acts simplicially. For our purposes, we need only consider the case in which $G \cong A *_C B$. In this case, such a tree is described as follows. The vertices of the tree T are given by $G/A \cup G/B$. The edges are given by G/C, with initial vertices $v_i(gC) = gA$ and the terminal vertices $v_t(gC) = gB$. The stabilizers of the vertices are the conjugates of A and B, and the edge stabilizers are the conjugates of C.

7.2 Group Structure

Note that just as in P, each group P_k is a free product with amalgamation. To see this, fix $k > 0$, and take two Klein bottle groups

$$K_1 = \langle a,\ x \mid axa^{-1}x \rangle \text{ and } K_2 = \langle y,\ b \mid byb^{-1}y \rangle$$

with subgroups

$$A_1 = \langle a^2,\ x \rangle \cong \mathbb{Z}^2 \text{ and } A_2 = \langle b^{2^k},\ y \rangle \cong \mathbb{Z}^2.$$

respectively. If we define the isomorphism

$$\phi : A_1 \to A_2 \text{ by } x \mapsto b^{2^k} \text{ and } a^2 \mapsto y,$$

then the free product of K_1 and K_2 with amalgamation of A_1 and A_2, by ϕ has the presentation

$$K_1 *_{A_1} K_2 \cong \langle a,\ b,\ x,\ y \mid axa^{-1}x,\ byb^{-1}y,\ x = b^{2^k} y = a^2 \rangle \cong P_k.$$

For concreteness, we will choose transversal

$$T_{K_1} = \{1,\ a\} \text{ and } T_{K_2} = \{1,\ b,\ \ldots,\ b^{2^k-1}\}.$$

So, as an amalgamated product with transversal T_{K_1} we have the following results.

Proposition 7.3. (Normal Forms)

Every element $w \in P_k$ can be written uniquely in the form:

$$w = a^{2u} b^{2^k v} a^{\alpha} b^{\beta_1} a b^{\beta_2} a \ldots b^{\beta_l} a b^{\beta}$$

where $u,\ v \in \mathbb{Z}$, $\alpha \in \{0,\ 1\}$, $\beta_i \in \{1,\ b,\ \ldots,\ b^{2^k-1}\}$, and
$\beta \in \{0,\ 1,\ b,\ \ldots,\ b^{2^k-1}\}$.

As an amalgamated product of torsion-free groups, from [29] we have

Proposition 7.4. *Every group P_k is torsion-free.*

Ultimately, we want to show that every group P_k does not satisfy the unique product property and hence gives an infinite family of simple concrete examples.

One issue that needs to be addressed is that some of the groups P_k $(k > 1)$ could contain P and hence not be truly new examples. We will show that every group does not contain P. This will be done by showing the following:

- If $A,\ B \in P_k$ where $\langle A,\ B\rangle$ fixes a line L in P_k, and $\langle A,\ B\rangle$ acts on L with no global fixed point, then the relations

$$AB^2A^{-1}B^2 = 1 \text{ and } BA^2B^{-1}A^2 = 1$$

  can not simultaneously hold in P_k.

- If $P \leq P_k$, then the induced action of P on P_k fixes a line L_k in T_k.

Lemma 7.5. *Suppose $\langle A,\ B\rangle$ fixes a line L in T_k. If A and B are hyperbolic, then neither of the relations*

$$AB^2A^{-1}B^2 = 1 \textit{ and } BA^2B^{-1}A^2 = 1$$

can hold in P_k.

Proof. Suppose A and B are hyperbolic elements that stabilize the same line U. Then there $m,\ n \in \mathbb{Z}$ so that A^nB^{-m} fixes L pointwise. So $A^nB^{-m} \in \langle a^2,\ b^{2^k}\rangle$ or rather $A^n = a^{2s}b^{2^kt}B^m$, for some $s,\ t \in \mathbb{Z}$. So if the relation $BA^2B^{-1}A^2 = 1$ holds, then so does $1 = BA^{2n}B^{-1}A^{2n}$. It follows then that $B^{4m} \in \langle a^2,\ b^{2^k}\rangle$, contradicting the fact that B is hyperbolic. A similar result holds if we assume that $AB^2A^{-1}B^2 = 1$ holds. $\square$

Lemma 7.6. *If A is hyperbolic and B is elliptic, then the following relations*

$$AB^2A^{-1}B^2 = 1 \textit{ and } BA^2B^{-1}A^2 = 1$$

can not simultaneously hold in P_k.

Proof. Suppose otherwise. Conjugating if necessary, we may assume that either

$$B = a^{2s}b^{2^k t}a \text{ or } B = a^{2s}b^{2^k t}b^{2^{k-1}}$$

and from Proposition 7.3 we may write

$$A = a^{2u}b^{2^k v}a^{\alpha}b^{\beta_1}ab^{\beta_2}a \dots b^{\beta_1}ab^{\beta}$$

as a word in reduced normal form. In either case of B, the idea of the proof is to analyze the possible values of $\alpha,\ \beta_1,\ \beta_2,\ \dots,\ \beta_1,$ and β, and show that no such word A exists.

Consider the case $B = a^{2s}b^{2^k}a$. The first relation says that

$$1 = AB^2A^{-1}B^2 = a^{4s+2+\sigma_b(A)(4s+2)}$$

which is true if and only if $\sigma_b(A) = -1$, where

$$\sigma_b(A) = \begin{cases} 1 & \text{if the sum of all the powers of } b \text{ in } A \text{ is even} \\ -1 & \text{if the sum of all the powers of } b \text{ in } A \text{ odd} \end{cases}.$$

Suppose the relation

$$1 = BA^2B^{-1}A^2 = a^{2q}b^{2^k r}a(a^{\alpha}b^{\beta_1}a \dots b^{\beta_l}ab^{\beta})^2a^{-1}(a^{\alpha}b^{\beta_1}a \dots b^{\beta_l}ab^{\beta})^2. \qquad (7.1)$$

holds. By assumption, A is a hyperbolic element, and so $A^2 \notin \langle a^2,\ b^{2^k}\rangle$. We claim that cancellation must occur in the subword $ab^{\beta}a^{-1}a^{\alpha}b^{\beta_1}$. Otherwise, say in the case where $\alpha = 0$ and $\beta \neq 0$, then the right hand side of (7.1) above can be written as a non-trivial word in normal form contradicting Proposition 7.3. Similarly, in the case where $\alpha = 1$ and $\beta = 0$, the right hand side of (7.1) reduces to a non-trivial word in normal form, which also contradicts Proposition 7.3. Hence, the only cases that need to be considered are when $\alpha = 0$ and $\beta = 0$ or when $\alpha = 1$ and $\beta \neq 0$. We will handle both cases at the same time, so for concreteness, relabel $\beta = \beta_{l+1}$. After reduction of the pair aa^{-1}, right hand side of (7.1) contains a subword of the form $b^{\beta_i+\beta_j}$. If $\beta_i + \beta_j = 2^k$, move $b^{\beta_i+\beta_j}$ and the resulting a^2 to the far left in (7.1) as described by Proposition 7.3. Repeat this process for the next resulting subword $b^{\beta_{i-1}+\beta_{j+1}}$. If at any stage of the reduction, we have $b^{\beta_s+\beta_t} \neq 2^k$, then the reduced word in (7.1) is a non-trivial word in normal form, leading to a contradiction of Proposition 7.3. Pairing off the powers of b in this way, we have either:

1. $\alpha = 0$, $\beta = 0$, $\beta_l + \beta_1 = 2^k$, $\beta_{l-1} + \beta_2 = 2^k$, $\ldots$, $\beta_{\frac{l}{2}+1} + \beta_{\frac{l}{2}} = 2^k$ (if l is even),

2. $\alpha = 0$, $\beta = 0$, $\beta_l + \beta_1 = 2^k$, $\beta_{l-1} + \beta_2 = 2^k$, $\ldots$, $\beta_{\frac{l+1}{2}} + \beta_{\frac{l+1}{2}} = 2^k$ (if l is odd),

3. $\alpha = 1$, $\beta \neq 0$, $\beta + \beta_1 = 2^k$, $\beta_l + \beta_2 = 2^k$, $\ldots$, $\beta_{\frac{l+2}{2}} + \beta_{\frac{l+2}{2}} = 2^k$ (if l is even), or

4. $\alpha = 1$, $\beta \neq 0$, $\beta + \beta_1 = 2^k$, $\beta_l + \beta_2 = 2^k$, $\ldots$, $\beta_{\frac{l+1}{2}+1} + \beta_{\frac{l+1}{2}} = 2^k$ (if l is odd),

In any event, this forces $\sigma_b(A) = 1$ giving a contradiction.

Consider the other case, where $B = a^{2s}b^{2^k t}b^{2^{k-1}}$. Using the same normal form for A as above, the relation

$$1 = AB^2A^{-1}B^2 = a^{4s+\sigma_b(A)4s}b^{2^{k+1}t+2^k+\sigma_a(A)(2^{k+1}t+2^k)}$$

holds provided $\sigma_a(A) = -1$ and either $\sigma_b(A) = -1$ or $s = 0$, where

$$\sigma_a(w) = \begin{cases} 1 & \text{if the sum of all the powers of } a \text{ in } w \text{ is even} \\ -1 & \text{if the sum of all the powers of } a \text{ in } w \text{ odd} \end{cases}$$

and $\sigma_b(A)$ is as above.

An argument similar to the one above applied to the relation

$$1 = BA^2B^{-1}A^2$$

shows

1. $\alpha = 0$, $\beta \neq 0$, $\beta + \beta_1 = 2^{k-1}$, $\beta_l + \beta_2 = 2^k$, $\ldots$, $\beta_{\frac{l+1}{2}+1} + \beta_{\frac{l+1}{2}} = 2^k$,

2. $\alpha = 0$, $\beta = 0$, $\beta_1 = 2^{k-1}$, $\beta_l + \beta_2 = 2^k$, $\ldots$, $\beta_{\frac{l+1}{2}} + \beta_{\frac{l+1}{2}+1} = 2^k$, or

3. $\alpha = 1$, $\beta = 2^{k-1}$, $\beta_l + \beta_1 = 2^k$, $\beta_{l-1} + \beta_2 = 2^k$, $\ldots$, $\beta_{\frac{l}{2}} + \beta_{\frac{l}{2}+1} = 2^k$.

and so in every case, $\sigma_b(A) = 1$.

So we must have that $s = 0$. If we simply count the number of exponents in a of $BA^2B^{-1}A^2$, one checks that after all possible cancellations, this is $8u + 4(2j + 1)$ for some integer j, i.e. this is true by our description of A and B if no cancellations occur and any cancellation reduces the total number of exponents in a by 8. Since $8u + 4(2j + 1) = 0$ has no integer solution, this relation holding would contradict Proposition 7.4. □

Lemma 7.7. *If $\langle A,\ B\rangle \subset P_k$ fixes some line L in the Bass-Serre Tree T_k where A and B are elliptic elements with disjoint fixed point sets, then the following relations*

$$AB^2A^{-1}B^2 = 1 \text{ and } BA^2B^{-1}A^2 = 1$$

can not simultaneously hold in P_k.

Proof. If A and B are elliptic elements with disjoint fixed point sets, then AB acts as a translation on U. Moreover that $\langle AB,\ B\rangle = \langle AB,\ B\rangle$ and if A and B satisfy the relations above, then so do AB and B. So $\langle AB,\ B\rangle$ satisfied the hypotheses of the preceding lemma and both relations which contradicts the preceding lemma. □

Theorem 7.8. *For $k > 1$, P_k does not contain P.*

Proof. Fix $k > 1$ and suppose that $\langle A,\ B\rangle \cong P$ is a subgroup of P_k. Since P_k acts on the Bass-Serre tree T_k, there is an induced action of P on T_k by isometries without edge inversion. It follows that the action of P on T_k has no global fixed point; otherwise, $P \leq K_1^g$ or $P \leq K_2^g$ for some $g \in P_k$ and in particular, this implies that the surface groups K_1^g or K_2^g contain a free Abelian group of rank 3. Since P is finitely generated and T_k is non-trivial, by Proposition 7.2, T_k contains a unique minimal P-invariant subtree which we will denote by L. By Proposition 7.1, L contains at least one axis. On the other hand, since P is a finite extension of $\mathbb{Z}^3$, the largest tree P can act on is a line. So, if P is a subgroup of P_k, we can deduce that P acts simplicially on a line $L \subset T_k$. Applying Lemmas 7.5, 7.6, and 7.7 gives us the desired contradiction. □

Chapter 8

Non-unique Product Sets

Let k be a fixed positive integer that we will use for the remainder of the paper. In this section, we will show that P_k does not satisfy the unique product property. Recall, that given a torsion-free group G, a subset of the form $\{xr^i \mid l \leq i \leq m\}$ for some $x, r \in G$ and $l, m \in \mathbb{Z}$ is said to be a *left progression of ratio* r, or simply a *left r-progression*. In P_k, consider the following b-progressions

$$X_0 = \{a^{-1},\ a^{-1}b\},$$

$$X_i = \{b^i a^{-1} b^j \mid\ 0 \leq j \leq 2^k + 1\},$$

$$Y_l = \{b^l a b^j \mid\ 1 \leq j \leq 2^k + 1\},$$

$$Z_0 = \{b^j \mid\ -2^k \leq j \leq 2^k\}$$

where $1 \leq i \leq 2^k - 1$ and $0 \leq l \leq 2^k - 1$. Set

$$T = \bigcup_{i=0}^{2^k-1} X_i \cup \bigcup_{j=0}^{2^k-1} Y_j \cup Z_0$$

and for convenience, set $X = \bigcup_{i=0}^{2^k-1} X_i$ and $Y = \bigcup_{j=0}^{2^k-1} Y_j$. Proposition 7.3 shows every element in T is distinct we will show that every element in TT has no unique representation as follows. First, decompose TT into smaller product sets of the form

$$X_iX_j,\ Y_iX_j,\ X_iY_j,\ Y_iY_j,\ Z_0X_i,\ X_iZ_0,\ Z_0Y_i,\ Y_iZ_0,\ \text{ and } Z_0Z_0.$$

From there, we decompose these product sets further into progressions that are obtained as the product of single element in T with one of the sets X_i, Y_j, or Z_0, which we will refer to as *slices*.

Showing TT is a non-unique product set requires careful bookkeeping to make keeping track of the specific slices easier, we will adopt the following conventions. Write $x_{(n,m)} = b^n a^{-1} b^m$, $y_{(n,m)} = b^n a b^m$, and $z_{(0,n)} = b^n$ and if $u_{(m,i)} \in T$ and $W_n = \{w_{(n,j)} \mid l_n \leq j \leq m_n\}$ is one of our b-progressions listed above, we will denote the slices by

$$u_{(m,i)}W_n = \{u_{(m,i)}w_{(n,j)} \mid l_n \leq j \leq m_n\}.$$

Clearly, any product in TT that belongs to two of these slices has two different representations in TT. Using the our choice of the b-progressions, we can efficiently show most of these slices are contained in at least one other slice. This reduces the number of elements we need to check to a much smaller set. For the remaining slices, the Klein bottle relations are used to show the remaining slices are contained in at least two of the subproduct sets listed above and hence have two distinct representations.

The following equalities and containments hold for subproduct sets in TT as a result of the structure of the progressions. These are perhaps easiest to see visually, as in figures 8.1, 8.2, and 8.3, by writing the respective products U_iY, U_iX, and U_iZ_0 in table form, where U_i is an arbitrary progression in T. In Tables 8.2 and 8.1, the rows are labeled by individual words in a progression (written in order from the starting value $u_{i,s}$ to the ending value $u_{i,e}$) and the columns are labeled by the progressions in X and Y respectively. In Table 8.3, both row and column are labeled by words in the respective progressions (also written in the order of the progression). In each the figures, the circled slices are those that are not paired up by the structure of the progressions mentioned above.

$$
\begin{array}{c|cccccc}
 & Y_0 & Y_1 & Y_2 & \dots & Y_{2^k-2} & Y_{2^k-1} \\
\hline
u_{(i,s)} & * & * & * & \dots & * & * \\
u_{(i,s+1)} & * & * & * & \dots & * & * \\
u_{(i,s+2)} & * & * & * & \dots & * & * \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
u_{(i,e-1)} & * & * & * & \dots & * & * \\
u_{(i,e)} & * & * & * & \dots & * & *
\end{array}
$$

Figure 8.1: Matching Patterns for Products of the Form U_iY

Case 1: Consider products of the form U_iY. As illustrated in Figure 8.1, the slices along the diagonal lines are equal since we always have

$$u_{(i,v+1)}Y_u = \{b^i a^\epsilon b^{v+1} b^u ab^j \mid 1 \le j \le 2^k + 1\} = u_{(i,v)}Y_{u+1},$$

where $\epsilon \in \{-1,\ 0,\ 1\}$ and u and v are taken in the appropriate range. So the only slices we need consider separately, are those of the form

$$u_{(i,s)}Y_0 \text{ and } u_{(i,e)}Y_{2^k-1}$$

for appropriate starting values s and ending values e of each progression.

Case 2 Consider products of the form U_iX. Just as in Case 1, we have similar

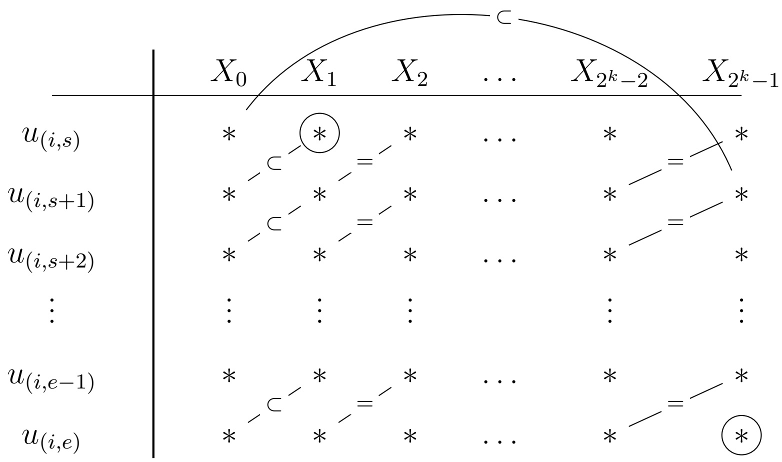

Figure 8.2: Matching Patterns for Products of the Form U_iX

identifications along the diagonal lines for all the slices with the same cardinality, as illustrated in Figure 8.2. However, we also have proper containments, since the slices $u_{(i,j)}X_0$ only have cardinality 2. There are two containments of particular interest, namely $u_{(i,s)}X_0 \subset u_{(i,s+1)}X_{2^k-1}$ and $u_{(i,s+1)}X_0 \subset u_{(i,s)}X_1$. The former always occurs, since

$$u_{(i,s)}X_0 \subset \{b^i a^\epsilon b^s a^{-1} b^j \mid -2^k \leq j \leq 1\} = u_{(i,s+1)}X_{2^k-1}.$$

Containment in the latter case is clear, but it is worth mentioning this containment plays a very important role, later. Therefore, the only slices we need to consider separately are those of the form

$$u_{(i,e)}X_{2^k-1} \text{ and the shortened } u_{(i,s)}X_1 \text{ written as } \{u_{(i,s)}ba^{-1}b^j \mid 2 \leq j \leq 2^k+1\},$$

where once again s and e are the appropriate starting and ending values of the progression U_i.

Case 3: Consider products of the form U_iZ_0. As illustrated in Figure

$$
\begin{array}{c|ccccc}
 & z_{(0,-2^k)} & z_{(0,-2^k+2)} & \cdots & z_{(0,2^k-1)} & z_{(0,2^k))} \\
\hline
u_{(i,s)} & * & * & \cdots & * & * \\
u_{(i,s+1)} & * & * & \cdots & * & * \\
u_{(i,s+2)} & * & * & \cdots & * & * \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
u_{(i,e-1)} & * & * & \cdots & * & * \\
u_{(i,e)} & * & * & \cdots & * & *
\end{array}
$$

Figure 8.3: Matching Patterns for Products of the Form U_iZ_0

8.3, each product has exactly two elements $\{u_{(i,s)}b^{-2^k},\ u_{(i,e)}b^{2^k}\}$ that are not identified within the table. If $U_i \neq Z_0$, then it is clear that

$$u_{(i,s)}b^{-2^k} = b^{2^k}u_{(i,s)} \subset Z_0U_i \text{ and } u_{(i,s)}b^{2^k} = b^{-2^k}u_{(i,s)} \subset Z_0U_i$$

and Z_0U_i is contained in either Z_0X or Z_0Y. Hence, these elements have no unique representation in TT. If $U_i = Z_0$, the elements not identified within the table are $\{b^{-2^{k+1}},\ b^{2^{k+1}}\}$. Since we have

$$b^{-2^{k+1}} = ab^{2^k+1}b^{2^k-1}a^{-1} \in y_{(0,2^k+1)}X_{2^k-1}$$

$$b^{2^{k+1}} = b^{2^k-1}a^{-1}ab^{2^k+1} \in x_{(2^k-1,0)}Y_0.$$

these elements also have no unique representation in TT.

We can extend this idea further to account for the remaining slices in Z_0Y and Z_0X. As illustrated in Figure 8.1, the slices we have yet to account for in the subproduct set Z_0Y are

$$z_{(0,-2^k)}Y_0 = \{ab^j \mid 2^k+1 \leq j \leq 2^{k+1}+1\} \subset Y_0Z_0$$

and

$$z_{(0,2^k)}Y_{2^k-1} = \{b^{2^k-1}ab^j \mid 1-2^k \leq j \leq 1\} \subset Y_{2^k-1}Z_0.$$

Similarly, as illustrated in Figure 8.2, the slices we have yet to account for in the subproduct set Z_0X are subsets of the slices

$$z_{(0,-2^k)}X_1 = \{ba^{-1}b^j \mid 2^k \leq 2^{k+1}+1\} \subset X_1Z_0$$

and

$$z_{(0,2^k)}X_{2^k-1} = \{b^{2^k-1}a^{-1}b^j \mid -2^k \leq j \leq 1\} \subset X_{2^k-1}Z_0.$$

This accounts for all the subproduct sets of the form Z_0U_i and U_iZ_0.

Remaining Elements in TT		
Slice	Rewritten Elements	Remaining Values for j
$x_{(0,0)}X_1$	$abab^j \subset Y_0Y_0$	$2 \leq j \leq 2^k+1$
$x_{(0,1)}X_{2^k-1}$	$a^{-2}b^j \subset Y_{2^k-1}Y_{2^k-1}$	$-2^k \leq j \leq 1$
$x_{(l,0)}X_1$	$b^labab^j \subset Y_lY_0$	$2 \leq j \leq 2^k+1$
$x_{(l,2^k+1)}X_{2^k-1}$	$a^2b^j \subset Y_{l-1}Y_{2^k-1}$	$l-2^{k+1} \leq j \leq l+1-2^k$
$x_{(m,0)}X_1$	$b^mabab^j \subset Y_mY_0$	$2 \leq j \leq 2^k+1$
$x_{(m,2^k+1)}X_{2^k-1}$	$a^{-2}b^j \subset Y_{m-1}Y_{2^k-1}$	$m-2^{k+1} \leq j \leq m+1-2^k$
$x_{(2^k-1,0)}X_1$	$b^{2^k-1}abab^j \subset Y_{2^k-1}Y_0$	$2 \leq j \leq 2^k+1$
$x_{(2^k-1,2^k+1)}X_{2^k-1}$	$a^2b^j \subset Y_{2^k-2}Y_{2^k-1}$	$-1-2^k \leq j \leq 0$
$y_{(n,1)}X_1$	$b^na^{-1}b^2ab^j \subset X_nY_1$	$2 \leq j \leq 2^k+1$
$y_{(n,2^k+1)}X_{2^k-1}$	$b^j \subset Z_0Z_0$	$n-2^{k+1} \leq j \leq n+1-2^k$
$y_{(0,1)}Y_0$	$abab^j \subset X_0X_1$	$1 \leq j \leq 2^k+1$
$y_{(0,2^k+1)}Y_{2^k-1}$	$a^2b^j \subset X_1X_{2^k-1}$	$1-2^{k+1} \leq j \leq 1-2^k$
$y_{(l,1)}Y_0$	$b^labab^j \subset X_lX_1$	$1 \leq j \leq 2^k+1$
$y_{(l,2^k+1)}Y_{2^k-1}$	$a^{-2}b^j \subset X_{l+1}X_{2^k-1}$	$l+1-2^{k+1} \leq j \leq l+1-2^k$
$y_{(m,1)}Y_0$	$b^mabab^j \subset X_mX_1$	$1 \leq j \leq 2^k+1$
$y_{(m,2^k+1)}Y_{2^k-1}$	$a^2b^j \subset X_{m+1}X_{2^k-1}$	$m+1-2^{k+1} \leq j \leq m+1-2^k$
$y_{(2^k-1,1)}Y_0$	$b^{2^k-1}abab^j \subset X_{2^k-1}X_1$	$1 \leq j \leq 2^k+1$
$y_{(2^k-1,2^k+1)}Y_{2^k-1}$	$a^{-2}b^j \subset X_0X_{2^k-1}$	$-2^k \leq j \leq 0$
$x_{(n,0)}Y_0$	$b^j \subset Z_0Z_0$	$n+1 \leq j \leq n+1+2^k$
$x_{(0,1)}Y_{2^k-1}$	$b^j \subset Z_0Z_0$	$1-2^k \leq j \leq 1$
$x_{(n,2^k+1)}Y_{2^k-1}$	$b^j \subset Z_0Z_0$	$n+1-2^{k+1} \leq j \leq n+1-2^k$

As mentioned above, we will show that these sets are contained in two of the smaller product sets. Clearly, $u_{(m,i)}W_n \subset U_m W_n$, so for each remaining slice, we need only find some other product set that contains it. In the chart given above, we list all of the remaining slices as well as a reduced form for each of the words obtained by applying the relations

$$aba = a^{-1}ba^{-1},\ ba^2b^{-1}a^2 = 1,\ \text{and}\ ab^{2^k}a^{-1}b^{-2^k} = 1,$$

where

$$l \in \{1,\ 3,\ \ldots,\ 2^k - 3\},\ m \in \{2,\ 4,\ \ldots,\ 2^k - 2\},\ \text{and}\ n \in \{0,\ 1,\ \ldots,\ 2^k - 1\}.$$

In case 2 above, we used a smaller progression X_0 to shorten the length of the remaining words in the slices $u_{(i,s)}X_1$ so that they will fit inside the product sets Y_iY_0. In the chart above, we will list only those elements that have not been accounted for by the structure of the progressions. In each case, containment is verified by considering the reduced words and length of the remaining values in j. Inspection shows that every product in TT is not uniquely represented, and so T is a non-unique product set. Since k is arbitrary, this shows that each P_k does not have the unique product property.

Chapter 9

Cardinalities of Non-unique Product Sets

From the standpoint of Conjectures 1 and 2, it seems natural to consider the cardinality of the possible non-unique product sets in G. Indeed if the cardinality of such sets were bounded, then one need only consider products in $k[G]$ of bounded support size. In this section, we will show that this is not possible in general, by showing that each P_k contains arbitrarily large square non-unique product sets.

The construction in the preceding section shows that P_k contains T a set with cardinality $2^{2k+1} + 2^{k+2} + 1$ with the property that TT has no uniquely represented elements. We will construct larger sets as follows. Let p be any fixed positive odd integer and choose an odd integer q so that $q-1$ is a multiple of 2^k. For these odd integers p and q, consider the following b-progressions in P_k.

$$X_0(p,q) = \{a^{-p}b^j \ \mid \ -q+1 \le j \le (2^k+1)q - 2^k\},$$

$$X_i(p,q) = \{b^i a^{-p} b^j \ \mid \ -q+1 \le j \le (2^k+1)q\},$$

$$Y_l(p,q) = \{b^l a^p b^j \ \mid \ -q+2 \le j \le (2^k+1)q\},$$

$$Z_0(p,q) = \{b^j \ \mid \ -2^k(\frac{q+1}{2}) - (q-1) \le j \le 2^k(\frac{q+1}{2}) + (q-1)\}$$

where $0 \leq i \leq 2^k - 1$ and $0 \leq j \leq 2^k - 1$. We want to show that

$$T(p,q) = \bigcup_{i=0}^{2^k-1} X_i(p,q) \cup \bigcup_{j=0}^{2^k-1} Y_j(p,q) \cup Z_0(p,q) \subset P_k$$

has the property that the product set $T(p,q)T(p,q)$ has no uniquely represented element.

Once again, our normal forms ensure that every element in $T(p,q)$ is distinct. The method of showing this set has no uniquely represented element is analogous to the case where p and q are 1, as given in Chapter 8. In fact, the matchings in Figures 8.1, 8.2, and 8.3 are identical here as well. Given this similarity, we will only list those elements that are not matched via the progressions in the table below.

Remaining Elements in $T(p,q)T(p,q)$	
Slice	Rewritten Elements
Remaining Values for j	
$x_{(0,-q+1)}X_1$	$a^p b a^p b^j \subset Y_0 Y_0$
$2^k q + 2q - 2^k \leq j \leq 2^k q + 2q - 1$	
$x_{(0,(2^k+1)q-2^k)}X_{2^k-1}$	$a^{-2p} b^j \subset Y_{2^k-1} Y_{2^k-1})$
$2 - 2^k q - 2q \leq j \leq 1$	
$x_{(l,-q+1)}X_1$	$b^l a^p b a^p b^j \subset Y_l Y_0$
$2^k q + 2q - 2^k \leq j \leq 2^k q + 2q - 1$	
$x_{(l,(2^k+1)q)}X_{2^k-1}$	$a^{2p} b^j \subset Y_{l-1} Y_{2^k-1}$
$l + 2 - 2^k q - 2q - 2^k \leq j \leq l + 1 - 2^k$	
$x_{(m,-q+1)}X_1$	$b^m a^{-p} b a^{-p} b^j \subset Y_m Y_0$
$2^k q + 2q - 2^k \leq j \leq 2^k + 2q - 1$	
Continued on next page-	

-Continued from previous page	
Slice	Remaining Elements
Remaining Values for j	
$x_{(m,(2^k+1)q)}X_{2^k-1}$	$a^{-2p}b^j \subset Y_{m-1}Y_{2^k-1}$
$m+2-2^kq-2q-2^k \leq j \leq m+1-2^k$	
$x_{(2^k-1,-q+1)}X_1$	$b^{2^k-1}a^pba^pb^j \subset Y_{2^k-1}Y_0$
$2^kq+2q-2^k \leq j \leq 2^kq+2q-1$	
$x_{(2^k-1,(2^k+1)q)}X_{2^k-1}$	$a^{2p}b^j \subset Y_{2^k-2}Y_{2^k-1}$
$1-2^kq-2q \leq j \leq 0$	
$y_{(n,-q+2)}X_1$	$b^na^pb^2a^{-p}b^j \subset X_nY_1$
$2^kq+2q-2^k \leq j \leq 2^kq+2q-1$	
$y_{(n,(2^k+1)q)}X_{2^k-1}$	$b^j \subset Z_0Z_0$
$n+2-2^kq-2q-2^k \leq j \leq n+1-2^k$	
$y_{(0,-q+2)}Y_0$	$a^pba^pb^j \subset X_0X_1$
$1 \leq j \leq 2^kq+2q-1$	
$y_{(0,(2^k+1)q)}Y_{2^k-1}$	$a^{2p}b^j \subset X_1X_{2^k-1}$
$3-2^kq-2q-2^k \leq j \leq 1-2^k$	
$y_{(l,-q+2)}Y_0$	$b^la^pba^pb^j subset X_lX_1$
$1 \leq j \leq 2^kq+2q-1$	
$y_{(l,(2^k+1)q)}Y_{2^k-1}$	$a^{-2p}b^j \subset X_{l+1}X_{2^k-1}$
$l+2-2^kq-2q-2^k \leq j \leq l+1-2^k$	
$y_{(m,-q+2)}Y_0$	$b^ma^pba^pb^j \subset X_mX_1$
$1 \leq j \leq 2^kq+2q-1$	
$y_{(m,(2^k+1)q)}Y_{2^k-1}$	$a^{2p}b^j \subset X_{m+1}X_{2^k-1}$
$m+3-2^kq-2q-2^k \leq j \leq m+1-2^k$	
$y_{(2^k-1,-q+2)}Y_0$	$b^{2^k-1}a^pba^pb^j \subset X_{2^k-1}X_1$
Continued on next page-	

-Continued from previous page	
Slice	Remaining Elements
Remaining Values for j	
$1 \leq j \leq 2^q + 2q - 1$	
$y_{(2^k-1,(2^k+1)q)}Y_{2^k-1}$	$a^{-2p}b^j \subset X_0 X_{2^k-1}$
$2 - 2^k q - 2q \leq j \leq 0$	
$x_{(n,-q+1)}Y_0$	$b^j \subset Z_0 Z_0$
$n + 1 \leq j \leq n + 2^k q + 2q - 1$	
$x_{(0,(2^k+1)q-2^k)}Y_{2^k-1}$	$b^j \subset Z_0 Z_0$
$3 - 2^k q - 2q \leq j \leq 1$	
$x_{(n,(2^k+1)q)}Y_{2^k-1}$	$b^j \subset Z_0 Z_0$
$n + 3 - 2^k q - 2q - 2^k \leq j \leq n + 1 - 2^k$	

For the remaining slices, we summarize the results in the table above (suppressing (p, q)), the argument is similar to results in Chapter 8 and the words the slices are rewritten using the relations

$$a^p b a^p = a^{-p} b a^{-p},\ b a^{2p} b^{-1} a^{2p} = 1, \text{ and } a^p b^{2^k} a^{-p} b^{2^k} = 1,$$

in P_k. As a result of the containments, $T(p, q)$ is also a non-unique product set. Further, note that our construction does not depend on a specific choice of p and q. Since each set $T(p, q) \subset P_k$ has cardinality $(2^{2k+1} + 5 \times 2^k + 2)q - (2^k + 1)$ this establishes Theorem 6.3. In our construction, we only needed that p was an odd positive integer, if we consider

$$\{T(2n - 1,\ q) \mid\ n \geq 1 \text{ and } q - 1 \text{ is a fixed multiple of } 2^k\},$$

this also shows there are infinitely many distinct square non-unique product sets for any fixed cardinality.

Bibliography

[1] Juan M. Alonso. Inégalités isopérimétriques et quasi-isométries. *C. R. Acad. Sci. Paris Sér. I Math.*, 311(12):761–764, 1990.

[2] Mladen Bestvina and Noel Brady. Morse theory and finiteness properties of groups. *Invent. Math.*, 129(3):445–470, 1997.

[3] Robert Bieri. *Homological dimension of discrete groups*. Mathematics Department, Queen Mary College, London, 1976. Queen Mary College Mathematics Notes.

[4] B. H. Bowditch. A short proof that a subquadratic isoperimetric inequality implies a linear one. *Michigan Math. J.*, 42(1):103–107, 1995.

[5] B. H. Bowditch. A variation on the unique product property. *J. London Math. Soc. (2)*, 62(3):813–826, 2000.

[6] Noel Brady, Martin R. Bridson, Max Forester, and Krishnan Shankar. Snowflake groups, Perron-Frobenius eigenvalues and isoperimetric spectra. *Geom. Topol.*, 13(1):141–187, 2009.

[7] Noel Brady and Max Forester. Density of isoperimetric spectra. *Geom. Topol.*, 14(1):435–472, 2010.

[8] Noel Brady, Tim Riley, and Hamish Short. *The geometry of the word problem for finitely generated groups*. Advanced Courses in Mathematics. CRM Barcelona. Birkhäuser Verlag, Basel, 2007. Papers from the Advanced Course held in Barcelona, July 5–15, 2005.

[9] Martin R. Bridson. Doubles, finiteness properties of groups, and quadratic isoperimetric inequalities. *J. Algebra*, 214(2):652–667, 1999.

[10] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.

[11] Kenneth S. Brown. Lectures on the cohomology of groups. In *Cohomology of groups and algebraic K-theory*, volume 12 of *Adv. Lect. Math. (ALM)*, pages 131–166. Int. Press, Somerville, MA, 2010.

[12] R. G. Burns and V. W. D. Hale. A note on group rings of certain torsion-free groups. *Canad. Math. Bull.*, 15:441–445, 1972.

[13] John Crisp. On the CAT(0) dimension of 2-dimensional Bestvina-Brady groups. *Algebr. Geom. Topol.*, 2:921–936 (electronic), 2002.

[14] Marc Culler and John W. Morgan. Group actions on **R**-trees. *Proc. London Math. Soc. (3)*, 55(3):571–604, 1987.

[15] Max Dehn. *Papers on group theory and topology.* Springer-Verlag, New York, 1987. Translated from the German and with introductions and an appendix by John Stillwell, With an appendix by Otto Schreier.

[16] Will Dison. An isoperimetric function for Bestvina-Brady groups. *Bull. Lond. Math. Soc.*, 40(3):384–394, 2008.

[17] Will Dison, Murray Elder, Timothy R. Riley, and Robert Young. The Dehn function of Stallings' group. *Geom. Funct. Anal.*, 19(2):406–422, 2009.

[18] Colin D. Fox. Can a Fibonacci group be a unique products group? *Bull. Austral. Math. Soc.*, 19(3):475–477, 1978.

[19] S. M. Gersten. Finiteness properties of asynchronously automatic groups. In *Geometric group theory (Columbus, OH, 1992)*, volume 3 of *Ohio State Univ. Math. Res. Inst. Publ.*, pages 121–133. de Gruyter, Berlin, 1995.

[20] M. Gromov. Hyperbolic groups. In *Essays in group theory*, volume 8 of *Math. Sci. Res. Inst. Publ.*, pages 75–263. Springer, New York, 1987.

[21] Mikhael Gromov. Filling Riemannian manifolds. *J. Differential Geom.*, 18(1):1–147, 1983.

[22] Jacques Lewin. A note on zero divisors in group-rings. *Proc. Amer. Math. Soc.*, 31:357–359, 1972.

[23] A. Yu. Ol′shanskiĭ. Hyperbolicity of groups with subquadratic isoperimetric inequality. *Internat. J. Algebra Comput.*, 1(3):281–289, 1991.

[24] Panagiotis Papasoglu. On the sub-quadratic isoperimetric inequality. In *Geometric group theory (Columbus, OH, 1992)*, volume 3 of *Ohio State Univ. Math. Res. Inst. Publ.*, pages 149–157. de Gruyter, Berlin, 1995.

[25] Donald S. Passman. *The algebraic structure of group rings.* Robert E. Krieger Publishing Co. Inc., Melbourne, FL, 1985. Reprint of the 1977 original.

[26] S. David Promislow. A simple example of a torsion-free, nonunique product group. *Bull. London Math. Soc.*, 20(4):302–304, 1988.

[27] Eliyahu Rips and Yoav Segev. Torsion-free group without unique product property. *J. Algebra*, 108(1):116–126, 1987.

[28] Mark V. Sapir, Jean-Camille Birget, and Eliyahu Rips. Isoperimetric and isodiametric functions of groups. *Ann. of Math. (2)*, 156(2):345–466, 2002.

[29] Jean-Pierre Serre. *Trees*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. Translated from the French original by John Stillwell, Corrected 2nd printing of the 1980 English translation.

[30] John Stallings. A finitely presented group whose 3-dimensional integral homology is not finitely generated. *Amer. J. Math.*, 85:541–543, 1963.

[31] Stefan Wenger. A short proof of Gromov's filling inequality. *Proc. Amer. Math. Soc.*, 136(8):2937–2941, 2008.