PREDICTION OF THE RESPONSE AND OPTIMAL CONTROL OF STOCHASTIC PARAMETRICALLY AND EXTERNALLY EXCITED NONLINEAR SYSTEMS

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CHAPTER I

INTRODUCTION

General Definition and Historical Background

The random behavior of physical systems which are described by using differential equations have attracted several researchers in the areas of random vibration and stochastic control recently. These random systems, according to their physical origins and mathematical formulations, can be classified into two categories (Arnold and Lefever, 1983). The first class of random differential equations is deterministic systems with random nonhomogeneous terms or under additive noise excitations. Since Langevin derived the first stochastic differential equation of this type in 1908 (Gardiner, 1983), intensive studies have been done on the areas of random vibration and stochastic control (Crandall and Zhu, 1983; Maybeck, 1982). Indeed, differential equations of this type represent any deterministic dynamic systems operated under a noisy environment. The second class of random systems is characterized by the differential equations with random coefficients or under parametric noise excitations. The developments and applications of differential equations of this type have been progressing only recently (Arnold and Lefever, 1983; Soong, 1973). An earlier study in this area can be traced back to the investigation of the propagation of sound in a medium with stochastic refraction index in 1946 (Soong, 1973). In the areas of random vibration and stochastic control, there are some examples which include the continuous stirred tank reactor with random fluctuations in the volume and flow rate (King, 1968), the control of momentum exchange for regulating the angular precession of rotating spacecraft (McLane, 1971), the prediction of ship motion in random sea waves (Ibrahim, 1985), and the deformation of a noise excited cylindrical shell (Scheurkogel and Elishakoff, 1985).

The random dynamic systems which are described by using differential equations also can be classified into linear and nonlinear ones (Arnold and Lefever, 1983). A linear dynamic system is one in which the output response is in linear proportion to the input excitation. Real systems such as economic, biological, and physical systems are usually nonlinear. Actually, real dynamic systems which are linearizable can be modeled by linear differential equations only when the systems are operated in a small range about the nominal states.

The stochastic process of parametric or external excitation for random dynamic systems is usually modeled as a zero-mean Gaussian white noise process (Maybeck, 1982). This process is defined as a stationary process with constant spectral density. In the time domain, the values of the signal at different times are completely uncorrelated, and further the process has infinite variance. The pathological nature of such a process causes some analytical difficulties in conventional mathematics. Some of the difficulties are avoided through the concepts of using the Wiener process or Brownian-motion process (Kailath and Frost, 1968). Even though the violent and irregular Brownian motion is not a differentiable signal, a convenient mathematical technique is to treat the "formal" derivative of the Wiener process as a Gaussian white noise.

Linear differential equations with Gaussian white noise nonhomogeneous terms are the most analytically tractable forms used in the study of random vibration and stochastic control. Since most dynamic systems subjected to stochastic external excitation are nonlinear, certain linearization techniques are usually applied to analyze the nonlinear phenomena of real physical systems. One of the most useful linearization approaches for dynamic analysis and controller design of nonlinear stochastic systems is the statistical linearization approach (Beaman, 1984; Gelb and Vander Velde, 1968). This approach is a kind of equivalent linearization technique. Historically, the concepts of using equivalent linearization methods were first introduced by Krylov and Bogoliubov (1943) for deterministic nonlinear oscillators. When nonlinear oscillators are subjected to Gaussian white noise excitation, the approach was independently extended by Booton (1954) and Caughey (1959) to the statistical linearization method. Since the response of the states of linear systems subjected to Gaussian excitation is still jointly Gaussian, in the application of statistical linearization techniques, the simple and useful jointly Gaussian distribution is usually applied to the evaluation of expectations of certain nonlinear functions of states to derive the equivalent linear functions.

Nonlinear differential equations with Gaussian white noise coefficients and nonhomogeneous terms are the most general and difficult models used in the study of random vibration and stochastic control (Ibrahim, 1985; Maybeck, 1982). Since the differential equations of this type include state-noise multiplicative nonlinear terms, the convenient Ito stochastic integrals (Gardiner, 1983) are usually applied to derive the solution process. For an nth-order differential equation, an n-dimensional vector Ito differential equation can be formulated through the concepts of state equation and the Ito integral. The solution process thus forms a certain class of Markov processes and the complete description of the solution process is provided by the transition joint probability density function of the states. As a consequence of the definition of the Ito integral, the Ito differential rule is derived. Since the formal rules of calculus are invalid for random differential equations of this type, applications of the Ito stochastic differential equation play an important role in the study of random behavior of stochastic parametrically and externally excited nonlinear systems.

Statement of the Problem

There are two central problems in the fields of random vibration and stochastic control selected for the present study. The first problem of interest is the prediction of the statistical moments of nonlinear dynamic systems subjected to both stochastic parametric and external excitations. The second problem selected is the synthesis of optimal controllers for those stochastic systems given in the first problem. Actually, the first problem in the field of random vibration is just the problem of dynamic response of closed-loop systems in the area of stochastic control. These two problems are closely related as shown in Fig. 1.

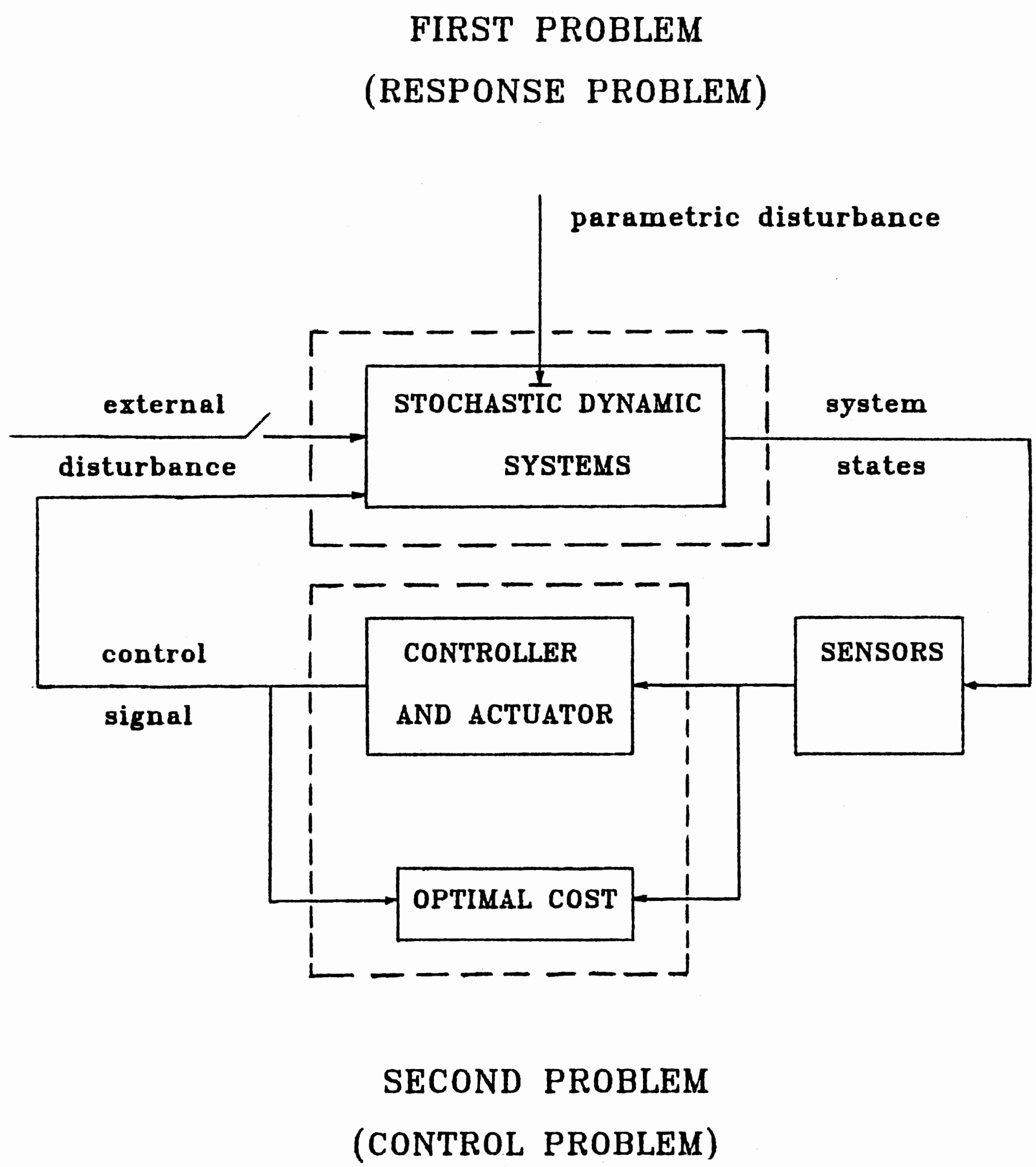

Figure 1. Description of the Problem

Assumptions and Limitations

The assumptions and limitations of the present research for dynamic response and optimal control of stochastic parametrically and externally excited nonlinear systems are listed as follows.

1. The nonlinear systems are restricted to a type of systems with non-memory type nonlinearities. A hard or soft spring or a saturating element are typical examples for nonlinearities of this type.

2. The parametrically or externally excited noise is described as Gaussian white noise which can be modeled as a Wiener process with independent increments.

3. The nonlinear stochastic dynamic systems are interpreted in the Ito sense. This is the most popular model used in the study of the dynamic response and control of stochastic systems.

4. The dynamic response and control of the states of stochastic systems are restricted to statistically stationary states of the second moment.

5. The nonlinear control systems are described as systems without state-control multiplicative terms.

6. The nonlinear control systems are assumed to provide complete state information. This means that all states of the control systems are measurable directly.

Scope of the Study

In this research, the investigations are given to the developments of new analytical approaches for the accurate prediction of output response and effective controller design of stochastic parametrically and externally excited nonlinear systems. The analytical approaches which include nonlinearization and linearization techniques are developed. Through a new physical concept of equivalent external excitation, a nonlinearization technique is developed to predict the accurate stationary output variances and a nonlinear controller is synthesized to reduce the output variances of nonlinear stochastic systems. For the prediction of output variances and controller design by utilizing linearization techniques, a useful statistical linearization approach is first modified and extended from external noise excited systems to include parametric noise excited terms. Then, through the concept of equivalent external excitation, the techniques of non-Gaussian linearization and a Gaussian criterion which is used to determine when Gaussian linearization is appropriate are derived for the accurate prediction of stationary output variances. The above techniques are illustrated by using a typical Duffing-type nonlinear stochastic system under strong nonlinearities and noise intensities and the validity of the present research is supported by employing Monte Carlo-based numerical simulations.

This thesis includes seven chapters. Chapter II gives a brief review of literature. Chapter III provides a necessary mathematical background for the present studies of dynamic response and controller design of parametrically and externally excited nonlinear systems. The dynamic response of nonlinear stochastic systems predicted by the nonlinearization techniques is presented in Chapter IV. In this chapter, a key physical concept called equivalent external excitation is first introduced. By following this concept and utilizing matching conditions, the accurate prediction of stationary output variances of stochastic parametrically and externally excited nonlinear systems is obtained. The concepts of equivalent external excitation are further extended and applied to the problems of Gaussian and non-Gaussian linearization of stochastic parametrically and externally excited nonlinear systems. These problems are considered in Chapter V. After the response problems have been investigated by developing both the nonlinearization and linearization approaches, the stochastic optimal control problems are presented in Chapter VI. The linearized and nonlinearized designs of controllers are given to compare their stationary output performances. Finally, the present research will be concluded and future research is recommended in Chapter VII.

CHAPTER II

REVIEW OF LITERATURE

Dynamic Response of Nonlinear Stochastic Systems

The prediction of the response of nonlinear systems subjected to stochastic parametric and external excitations is a central problem in the field of random vibration. In contrast to the problem of the stationary response of nonlinear oscillators subjected to only external stochastic excitation which has been widely studied (Nigam, 1983), the stationary response of nonlinear oscillators under both stochastic parametric and external excitations has been studied only for few nonlinear oscillators (Dimentberg, 1982; Wu and Lin, 1984). Actually, in the prediction of the response of these oscillators (very few nonlinear oscillators can be solved exactly by using the Fokker-Planck-Kolmogorov equation), even the response of a simple nonlinear oscillator such as a Duffing system can be obtained only by certain approximate methods under restricted assumptions. Several approximate approaches which have been applied to the problems of nonlinear oscillators subjected to external stochastic excitation such as the Gaussian statistical linearization method, perturbation method (Nigam, 1983), Gram-Charlier expansion method (Assaf and Zirkle, 1976), and the approximate method to solve the Fokker-Planck-Kolmogorov equation (Nigam, 1983) are based on the implicit or explicit assumption that the stationary probability densities of the states of the nonlinear oscillators can be approximated as jointly Gaussian. Unfortunately, this assumption implies that the nonlinear oscillators are excited by weak noise intensity and/or the oscillators can be classified as weak nonlinear oscillators. For the stochastic parametrically and externally excited nonlinear oscillators, the Stratonovich method (Stratonovich, 1963) can be applied to solve for the distributions of amplitude and phase processes, although it is restricted to lightly damped oscillators. Recently, Wu and Lin (1984) applied the cumulant-neglect closure method (Soong, 1973) to solve for the response of stochastic parametrically excited oscillators. This method is very general; however, it will become rather difficult when the higher order cumulants need to be retained. For example, the retaining of the cumulants up to the fourth order used to solve a second-order nonlinear oscillator usually requires the solution of ten simultaneous nonlinear algebraic equations if one cannot eliminate certain variables by trivial substitution. Furthermore, it is formidable to apply the cumulant-neglect method when the nonlinearity is not of a polynomial form (Liu, 1969).

Optimal Control of Nonlinear Stochastic Systems

The study of optimal control of nonlinear stochastic systems has been given considerable attention in recent years. The stochastic systems which are described by using the Ito stochastic differential equations with the Gaussian parametric and/or external noise excitations are the most popular models used for this study (Maybeck, 1982). Previous work on the optimal control of nonlinear stochastic systems has concentrated on the Ito systems subjected to external Gaussian white noise excitations (Beaman, 1984; Gelb and Vander Velde, 1968). The performance measure used is the quadratic function of states and control inputs. By the combined use of Gaussian statistical linearization and linear quadratic Gaussian (LQG) theory, a sub-optimal linear state feedback controller is usually synthesized (Beaman, 1984; Hess, 1970; Kwakernaak and Sivan, 1972; Rajarao and Mahalanabis, 1970). The LQG problem was extended to a nonquadratic Gaussian problem recently (Jacobson, 1977; Yoshida, 1984). To improve the performance over the LQG design, Yoshida applied the Gaussian statistical linearization approach and interpreted the linearization in reverse sense to obtain the corresponding nonlinear controller. This approach is similar to the inverse describing function synthesis method (Taylor, 1983) for the deterministic systems. When nonlinear control systems are subjected to both stochastic parametric and external excitations, the controller design, even by using the statistical linearization approach, has not been investigated. Most results which have been derived for the stochastic parametrically and externally excited control systems are restricted to the linear cases (McLane, 1971; Mohler and Kolodziej, 1980; Phillis, 1985; Wonham 1967).

Gaussian and Non-Gaussian Linearization of Nonlinear Stochastic Systems

A practical method for analyzing nonlinear random vibration and stochastic control is the statistical linearization approach. In the applications of the statistical linearization approach, the simple and effective Gaussian linearization techniques are usually applied (Beaman, 1984). Although Gelb (1974) concluded that there was little difference in the linearizing gain when either the non-Gaussian distributed signal including uniform and triangular distributed signal or the Gaussian distributed signal was passed through a saturating element, the non-negligible effects of large nonlinearities on the Gaussian linearization of external noise excited nonlinear systems were reported by Crandall in 1980. Recently, Beaman (1980) applied a fourth-order Gram-Charlier expansion method to include the non-Gaussian effect in the linearization techniques for the dynamic analysis and controller design of nonlinear stochastic externally excited systems. When the dynamic systems are subjected to both stochastic parametric and external excitations, in the fields of random vibration and stochastic control, the applications of the Gaussian or non-Gaussian statistical linearization approach have not been investigated.

In this chapter, previous work on the dynamic response and controller design of nonlinear stochastic dynamic systems has been briefly reviewed. Since the present research is concerned with the nonlinear systems with state-noise multiplicative terms, Ito's stochastic differential equations need to be employed for the present studies. Thus, before new analytical approaches are developed for the investigations of the dynamic response and controller design of stochastic parametrically and externally excited nonlinear systems, the mathematical background of Ito's stochastic differential equations and their related topics will be briefly introduced in Chapter III.

CHAPTER III

MATHEMATICAL BACKGROUND

The purpose of this chapter is to provide necessary mathematical background of probability theory and the theory of stochastic processes for the studies of the dynamic response and controller design of nonlinear stochastic systems. The following material has been collected from Gardiner (1983), Ibrahim (1985), Maybeck (1982), and Papoulis (1984).

Random Variables and Stochastic Processes

Probability Space

In probability theory, the probability space is defined as a triplet (Ω, $\mathcal{L}$, P). Ω is called the sure event which is a space of points, ω. $\mathcal{L}$ is called the set of events and is a sigma field of subsets of Ω. P is called a probability measure which is a function mapping $P : \mathcal{L} \rightarrow R$ and satisfies the following axioms:

1. $P(A) \geqq 0$ (3.1a)

where A is an event and P(A) is called the probability of the event A.

2. $P(\Omega) = 1$ (3.1b)

3. $P(A \cup B) = P(A) + P(B)$ (3.1c)

provided that $A, B \in \mathcal{L}$ and $A \cap B = \phi$ (ϕ is called the impossible event).

Random Variables and Related Properties

In a probability space $(\Omega, \mathcal{L}, P)$, a function mapping $X : \Omega \to R$ is a random variable if and only if X is measurable with respect to the field $\mathcal{L}$. The function $F(x) = P\{\omega |\ X(\omega) \leq x\}$ is defined as the distribution function of $X(\omega)$. The distribution function is a non-decreasing function with the properties, $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x) = 1$. If F(x) is continuous, then

$$p(x) = \frac{dF(x)}{dx} \tag{3.2a}$$

p(x) is called the probability density function of $X(\omega)$ or we can express F(x) by using

$$F(x) = \int_{-\infty}^{x} p(y)\ dy \tag{3.2b}$$

For an n-dimensional vector random variable, the distribution function is defined as

$$F(x_1, x_2, \ldots x_n) = P\{\omega |\ X_i(\omega) \leq x_i,\ i = 1, \ldots n\} \tag{3.3}$$

The random variables $X_1(\omega)$, $X_2(\omega)$, ... $X_n(\omega)$ are said to be independent if

$$p(x_1, x_2, \ldots x_n) = p(x_1)\ p(x_2) \ldots p(x_n) \tag{3.4}$$

The probability density function of $X_2(\omega)$ conditioned on $X_1(\omega)$ is denoted by $p(x_2 |\ x_1)$. A useful relationship (Bayes Theorem) for the conditional probability density function is given as

$$\begin{aligned} p(x_1, x_2) &= p(x_1|x_2)\ p(x_2) \\ &= p(x_2|x_1)\ p(x_1) \end{aligned} \tag{3.5}$$

For simplicity, the dependence of the random variable X on ω is usually omitted. Then, the moments of X which are defined as the expected values of X^n are expressed as

$$E[x^n] = \int_{-\infty}^{\infty} x^n \, p(x) \, dx \tag{3.6}$$

From (3.6), the mean of X, μ_x is given by setting n equal to unity. The variance of X, which describes the mean-square deviation from the mean value, is defined by

$$Var\{x\} = E[\, (x - \mu_x)^2 \,] \tag{3.7}$$

For an n-dimensional vector random variable, the covariance of $\underline{X}$ is defined as

$$P_x = E[\, (\underline{X} - \underline{\mu}_x)\cdot(\underline{X} - \underline{\mu}_x)^T \,] \tag{3.8}$$

A further important statistical quantity which is related to the moments of $\underline{X}$ is called the cumulant. The higher-order cumulant of the random variable $\underline{X}$ contains less statistical 'information' than the lower-order one. The cumulants of the random variable $\underline{X}$ are denoted by $\langle\langle\, x_1^m \, x_2^m \, \ldots \, x_n^m \,\rangle\rangle$ and related to the moments as given by

$$E[x_j] = \langle\langle\, x_j \rangle\rangle \tag{3.9a}$$

$$E[x_j x_k] = \langle\langle x_j x_k \rangle\rangle + \langle\langle x_j \rangle\rangle \, \langle\langle x_k \rangle\rangle \tag{3.9b}$$

$$E[x_j x_k x_l] = \langle\langle x_j x_k x_l \rangle\rangle + 3\{\langle\langle x_j \rangle\rangle \, \langle\langle x_k x_l \rangle\rangle\}_s$$

$$+ \langle\langle x_j \rangle\rangle \, \langle\langle x_k \rangle\rangle \, \langle\langle x_l \rangle\rangle \tag{3.9c}$$

$$E[x_j x_k x_l x_m] = \langle\langle x_j x_k x_l x_m \rangle\rangle + 3\{\langle\langle x_j x_k \rangle\rangle \, \langle\langle x_l x_m \rangle\rangle\}_s$$

$$+ 4\{\langle\langle x_j \rangle\rangle \, \langle\langle x_k x_l x_m \rangle\rangle\}_s$$

$$+ 6\{\langle\langle x_j\rangle\rangle \langle\langle x_k\rangle\rangle \langle\langle x_l x_m\rangle\rangle\}_s$$

$$+ \langle\langle x_j\rangle\rangle \langle\langle x_k\rangle\rangle \langle\langle x_l\rangle\rangle \langle\langle x_m\rangle\rangle \qquad (3.9d)$$

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where $\{\cdot\}_s$ represents a symmetrizing operation with respect to all its arguments. This is done by taking the arithmetic mean of different permuted terms similar to the one within the braces (Soong, 1973; Wu and Lin, 1984).

Stochastic Processes

A stochastic process $X(t,\omega)$ is a family of random variables defined on the probability space $(\Omega, \mathcal{L}, P)$ and indexed by time t. For fixed time, we obtain a random variable which is measurable with respect to $\mathcal{L}$. For each $\omega \in \Omega$, we obtain a function mapping $X : T \rightarrow R$ called a sample function. The relationship between a random variable and a random process is illustrated in Fig. 2 by selecting several sample functions corresponding to different outcomes ω_i.

Stochastic Convergence

For a stochastic process $X(t)$ and a sequence $\{X(t_i)\}$ defined on the probability space $(\Omega, \mathcal{L}, P)$, there are several different definitions of stochastic convergence given as follows.

Definition 1. Convergence in distribution:

For any continuous bounded function $f(x)$ (e.g. probability distribution function), $X(t_n)$ converges in distribution if

$$\lim_{n\to\infty} E[\ f(x(t_n))\ ] = E[\ f(x)\ ] \qquad (3.10)$$

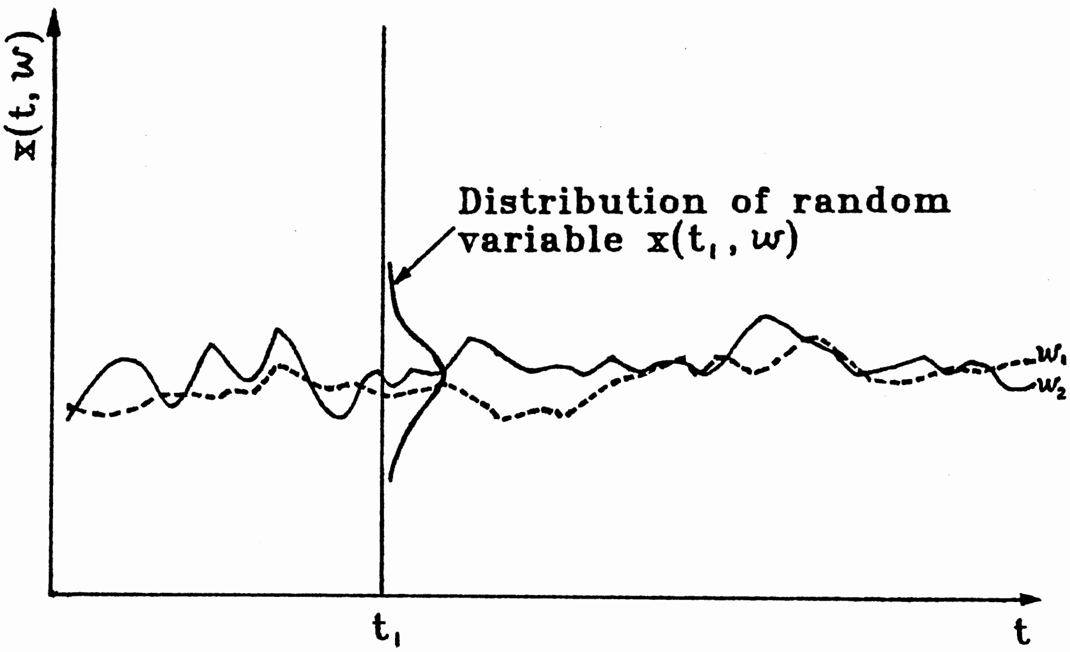

Figure 2. Relationship of Random Variables and Stochastic Processes

Definition 2. Convergence in probability:

$X(t_n)$ converges in probability to $X(t)$ if, for each $\varepsilon > 0$

$$\lim_{n\to\infty} P\{|X(t_n) - X(t)| \geq \varepsilon\} = 0 \tag{3.11}$$

Definition 3. Almost Sure (a.s.) Convergence:

$X(t_n)$ converges to $X(t)$ almost surely if

$$P\{\lim_{n\to\infty} X(t_n) = X(t)\} = 1 \tag{3.12a}$$

or written as

$$\lim_{n\to\infty} X(t_n) = X(t) \text{ a.s.} \tag{3.12b}$$

Definition 4. Convergence in the mth mean:

$X(t_n)$ converges in the mth mean to $X(t)$ if, for some $m > 0$,

$$\lim_{n\to\infty} E[\,|X(t_n) - X(t)|^m\,] = 0 \tag{3.13}$$

If $m = 2$, the mth-mean convergence gives the mean-square convergence,

$$\lim_{n\to\infty} E[\,|X(t_n) - X(t)|^2\,] = 0 \tag{3.14a}$$

or written in the form

$$\text{l.i.m. } X(t_n) = X(t) \tag{3.14b}$$

where l.i.m. reads limit-in-the-mean.

The relationship of the above definitions is shown in Fig. 3.

Classification of Stochastic Processes

Time Dependencies of Distributions

Stationary Processes. A stochastic process $X(t)$ is strictly stationary if the joint probability density of the sequence $\{X(t_1), X(t_2), \ldots X(t_n)\}$ is invariant under a time shift, τ. Consequently,

$$p(x_1, t_1; x_2, t_2; \ldots x_n, t_n) = p(x_1, t_1 + \tau; x_2, t_2 + \tau; \ldots x_n, t_n + \tau) \tag{3.15}$$

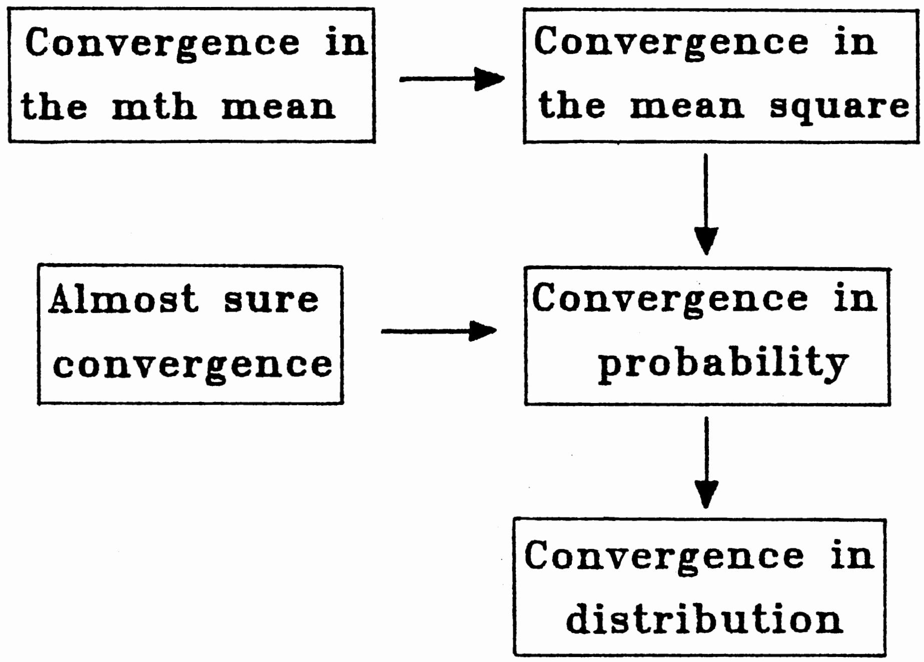

Figure 3. Relationship of Stochastic Convergence

In many applications, the definition of strictly stationary is too strict to be applied and hence a weakly stationary process is usually more practical. A weakly stationary process is defined to satisfy

1. $E[\ X^2\ (t)\ ] < \infty$ (3.16a)

2. $\mu_X(t) = E[\ X(t)\ ]$ is a constant (3.16b)

3. $E[\ (X(t) - X(s))\ (X(t) - X(s))\ ]$ depends only on the time difference t-s. (3.16c)

Ergodic Processes. A stochastic process is said to be ergodic if the time average of sample functions is equal to their ensemble averages. Thus, all statistical properties of an ergodic process can be determined from a single sample function. In mathematics, for any measurable function f such that $E[\ |f(x(t_0))|\ ] < \infty$, then the ergodic process X(t) gives

$$\lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\ dt = E[\ f(x(t_0))\ ] \tag{3.17}$$

Some Simple Stochastic Processes

Gaussian Process. A random process X(t) is a Gaussian process if for any finite collection of n parametric values at t_1, t_2, ... t_n the corresponding n random variables $X(t_1)$, $X(t_2)$, ... $X(t_n)$ are jointly Gaussian. i.e. The joint probability density function of the random variables X_1, X_2, ... X_n can be given by

$$p(x_1,\ t_1;\ x_2,\ t_2;\ \dots\ x_n,\ t_n) = \frac{1}{(2\pi)^{n/2}|\Gamma|^{1/2}} \cdot \exp\left(-\frac{1}{2}\,[\underline{X} - \underline{\mu}_X]^T \cdot \Gamma^{-1} \cdot [\underline{X} - \underline{\mu}_X]\right) \tag{3.18}$$

where

$$\underline{\mu}_x = E[\ \underline{X}\ ]$$

$$\Gamma = E[\ \underline{X}\cdot\underline{X}^T] - \underline{\mu}_x \cdot \underline{\mu}_x^T$$

$$\underline{X} = [x_1,\ x_2,\ \dots\ x_n]^T$$

Markov Process. A stochastic process X(t) defined on a probability space (Ω, £, P) is Markovian if for $t_1 < t_2 \dots < t_n \dots < t_{n+m}$

$$p(x_{n+1},\ t_{n+1};\ \dots;\ x_{n+m}, t_{n+m} | x_1,\ t_1;\ \dots\ ;\ x_n,\ t_n) = p(x_{n+1},\ t_{n+1};\ \dots\ x_{n+m},\ t_{n+m} |\ x_n,\ t_n) \qquad (3.19)$$

From the Markovian properties of X(t) and by using the relationship in (3.5) then

$$p(x_1,\ t_1;\ x_2,\ t_2;\ \dots\ x_n,\ t_n) = p(x_n,\ t_n | x_{n-1},\ t_{n-1})\ \dots\ p(x_2,\ t_2 | x_1,\ t_1)\ p(x_1,\ t_1) \qquad (3.20)$$

For the Markov process, the conditional probability density function $p(x_i,\ t_i | x_{i-1},\ t_{i-1})$ is called the transition probability density function. The transition probability density function thus gives the complete description of the statistical characteristics of a Markov process.

Wiener Process. The stochastic process W(t) defined on (Ω, £, P) is a Wiener process with intensity 2q if

1. W(0) = 0 (a.s.) (3.21)

2. W(t) is a stationary process with independent increments, i.e., for $t_1 < t_2 < \dots < t_n$, the increments of the process $(W(t_2) - W(t_1))$, $(W(t_3) - W(t_2))$, ... $(W(t_n) - W(t_{n-1}))$ are mutually independent.

3. For every t, s < t, the increments W(t) - W(s) are Gaussian

distributed with

(1) $E[\ W(t) - W(s)\ ] = 0$ (3.22a)

(2) $E[\ (W(t) - W(s))^2\ ] = 2q|t-s|$ (3.22b)

Formally, the properties of 2 and 3 can be expressed in terms of differentials:

1. $dW(t_i)$, $i = 1, n$ are mutually independent. (3.23a)
2. $E[\ dW(t)\ ] = 0$ (3.23b)
3. $E[\ (dW(t))^2\ ] = 2q \cdot dt$ (3.23c)

The Wiener process also has the Levy oscillation property, i.e., if $[t_0, t_1, \ldots t_n]$ is a partition of the interval $[t_0, t_n]$ and $\Delta = \max_i |t_i - t_{i-1}|$, then

$$\underset{\Delta \to 0}{\text{l.i.m.}} \sum_{i=1}^{n} (W(t_i) - W(t_{i-1}))^2 = 2q \cdot (t_n - t_0) \tag{3.24}$$

<u>Gaussian White Noise Process</u>. A Gaussian process V(t) defined on $(\Omega, \pounds, P)$ is a white noise process if its mean and covariance functions are given by

1. $E[\ V(t)\ ] = 0$ (3.25a)
2. $E[\ V(t) \cdot V(s)\ ] = 2q \cdot \delta(t-s)$ (3.25b)

where $\delta(t-s)$ is the Dirac Delta function. From the definition of the Wiener process W(t) and the Gaussian white noise process V(t), these two processes are related by

$$W(t) = \int_0^t V(\tau)\, d\tau, \quad t \geq 0 \tag{3.26}$$

or the Gaussian white noise process can be written as the "formal" derivative of the Wiener process to yield

$$V(t) = dW(t) / dt \tag{3.27}$$

Stochastic Integral and Differential

Stochastic Integral

The stochastic integral is given by

$$I = \int_{t_0}^{t_n} G(s)\, dW(s) \tag{3.28}$$

where G(t) is the arbitrary function of W(t), a Wiener process. The integral is defined as a kind of Riemann integral by using the following procedure.

1. Partition the interval $[t_0, t_n]$ into n subintervals and give $t_0 \leq t_1 \leq t_2 \ldots \leq t_{n-1} \leq t_n$ and $\Delta = \max_i |t_i - t_{i-1}|$.
2. Define intermediate point τ_i to satisfy $t_{i-1} \leq \tau_i \leq t_i$.
3. The stochastic integral is given as the limit-in-the-mean as

$$\int_{t_0}^{t_n} G(s)\, dW(s) = \underset{\Delta \to 0}{\text{l.i.m.}} \sum_{i=1}^{n} G(\tau_i)\, [W(t_i) - W(t_{i-1})] \tag{3.29}$$

If $\tau_i = t_{i-1}$, the integral defines an Ito integral; however, if $\tau_i = \frac{1}{2}(t_i + t_{i-1})$, the integral defines a Stratonovich integral.

Example:

Find the following stochastic integral in the Ito sense,

$$I = \int_{t_0}^{t_n} W(s)\, dW \tag{3.30}$$

where W is a Wiener process with unit intensity.

By using the definition of stochastic integral, one obtains

$$I = \underset{\Delta \to 0}{\text{l.i.m.}} \left\{ \sum_{i=1}^{n} W_{i-1} (W_i - W_{i-1}) \right\} \tag{3.31}$$

Define $\Delta W_i = W_i - W_{i-1}$, then

$$I = \underset{\Delta \to 0}{\text{l.i.m.}} \left\{ \frac{1}{2} \sum_{i=1}^{n} \left[(W_{i-1} + \Delta W_i)^2 - (W_{i-1})^2 - (\Delta W_i)^2 \right] \right\}$$

$$= \underset{\Delta \to 0}{\text{l.i.m.}} \left\{ \frac{1}{2} \left[W(t_n)^2 - W(t_0)^2 \right] - \frac{1}{2} \sum_{i=1}^{n} (\Delta W_i)^2 \right\}$$

$$= \underset{\Delta \to 0}{\text{l.i.m.}} \left\{ \frac{1}{2} \left[W(t_n)^2 - W(t_0)^2 \right] \right\} - \underset{\Delta \to 0}{\text{l.i.m.}} \left\{ \frac{1}{2} \sum_{i=1}^{n} (\Delta W_i)^2 \right\} \tag{3.32}$$

Note that (3.24) gives

$$\underset{\Delta \to 0}{\text{l.i.m.}} \sum_{i=1}^{n} \left\{ (\Delta W_i)^2 \right\} = t_n - t_o \tag{3.33}$$

thus,

$$\int_{t_o}^{t_n} W(s)dW = \underset{\Delta \to 0}{\text{l.i.m.}} \left\{ \frac{1}{2} \left[W(t_n)^2 - W(t_0)^2 - (t_n - t_o) \right] \right\} \tag{3.34}$$

<u>Ito's Stochastic Differential Rule</u>

For a continuous, real-valued function $h(W(t))$, where $W(t)$ is a Wiener process with intensity $2q \cdot dt$, the value of $h(W(t + \Delta t))$ can be expanded in the Taylor series to yield

$$h(W(t + \Delta t)) = h(W(t) + \Delta W(t))$$

$$= h(W(t)) + \frac{dh(W(t))}{dW(t)} \Delta W(t)$$

$$+ \frac{1}{2} \frac{d^2h(W(t))}{dW(t)^2} (\Delta W(t))^2 + \ldots \tag{3.35}$$

where the time derivative of h(W(t)) is defined by

$$\frac{dW(t)}{dt} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \{h(W(t + \Delta t)) - h(W(t))\} \tag{3.36}$$

Substituting (3.35) into (3.36) and taking the expected value of both sides of (3.36), one obtains

$$E[\frac{dh(W(t))}{dt}] = \lim_{\Delta t \to 0} \frac{1}{\Delta t} E[\frac{dh(W(t))}{dW(t)} \Delta W(t)]$$

$$+ \lim_{\Delta t \to 0} E[\frac{1}{2} \frac{d^2h(W(t))}{dW(t)^2}(\Delta W(t))^2] + \ldots \tag{3.37}$$

Since the mean-square of $\Delta W(t)$ is of order Δt not Δt^2, (3.37) gives

$$E[\frac{dh(W(t))}{dt}] = q \cdot E[\frac{d^2h(W(t))}{dW(t)^2}] \tag{3.38}$$

Note that the ordinary calculus will yield a zero value for the right hand side of (3.38) instead of a non-zero value. Thus, the Ito's differential rule is applied when one deals with a random function of the Wiener process. For an n-dimensional vector Wiener process $\underline{X}$, the Ito's differential rule can be expressed as

$$h(\underline{x}(t + \Delta t),t) = h(\underline{x},t) + \frac{\partial h}{\partial t}dt + \frac{\partial h}{\partial \underline{x}}d\underline{x}(t) + \frac{1}{2}d\underline{x}^T(t)\frac{\partial^2 h}{\partial \underline{x}^2}d\underline{x}(t)+\ldots \tag{3.39}$$

Ito's Stochastic Differential Equations

Ito's Stochastic Differential Equations

The random behavior of nonlinear physical systems which are described by using Ito's differential equation is defined as

$$d\underline{x} = \underline{f}[\underline{x}(t), t]\, dt + G[\underline{x}(t), t]\, d\underline{\alpha}(t) \tag{3.40}$$

where $\underline{f}[\underline{x}(t), t]$ or $G[\underline{x}(t), t]$ is a nonlinear function of states $\underline{x}$ and $\underline{\alpha}$ is the zero-mean Wiener process with intensities $E[d\underline{\alpha}(t)\cdot d\underline{\alpha}(t)^T]$ = $Q\cdot dt$. Equation (3.40) could be interpreted heuristically as

$$\dot{\underline{x}} = \underline{f}[\underline{x}(t), t] + G[\underline{x}(t), t]\, \underline{W}(t) \tag{3.41}$$

where $\underline{W}(t)$ is the Gaussian white noise process and is treated as the "formal" derivative of the Wiener process $\underline{\alpha}(t)$. Also, (3.40) is to be understood in the sense that

$$\underline{x}(t) = \underline{x}(t_o) + \int_{t_0}^{t} \underline{f}[\underline{x}(s), s]\, ds + \int_{t_o}^{t} G[\underline{x}(s), s]\, d\underline{\alpha}(s) \tag{3.42}$$

the last term on the right hand side of (3.42) constructs the Ito integral in (3.28). The existence and uniqueness conditions of solution in the mean-square sense to (3.40) were established by Ito, namely:

1. Lipschitz conditions

There exists a $K < \infty$ such that

$$\| \underline{f}[\underline{x} + \Delta\underline{x}, t] - \underline{f}[\underline{x}, t] \| \leq K \| \Delta\underline{x} \| \tag{3.43a}$$

and

$$\| G[\underline{x} + \Delta\underline{x}, t] - G[\underline{x}, t] \| \leq K \| \Delta\underline{x} \| \tag{3.43b}$$

for all $\underline{x}$ and $\Delta\underline{x}$, and t in the range $[t_0, t_n]$, where the norm function is defined as

$$\| A \| = [\ \mathrm{trace}(A\cdot A^T)\ ] \tag{3.44}$$

where A is an arbitrary vector or matrix.

2. Growth condition

$f[\underline{x}, t]$ and $G[\underline{x}, t]$ are continuous functions of t which satisfy

$$\| \underline{f}[\underline{x}, t] \|^2 \leq K(1 + \| \underline{x} \|) \tag{3.45a}$$

and

$$\| G[\underline{x}, t] \|^2 \leq K(1 + \| \underline{x} \|) \qquad (3.45b)$$

The Ito's stochastic differential equation as (3.40) can be interpreted in the Stratonovich sense. The relationship of the Ito and Stratonovich solution of (3.40) is related by the Wong-Zaikai transformation and is given simply by

$$f_i^s = f_i - \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} G_{kj} \frac{\partial G_{ij}}{\partial x_k}, \text{ if } Q(t) = I \text{ (Unit matrix)} \qquad (3.46a)$$

and

$$G_{ij}^s = G_{ij} \qquad (3.46b)$$

The Ito solution of (3.40) is the same as the Stratonovich solution of the Ito equation with a diffusional correction term which is the second term in the right hand side of (3.46a). The Stratonovich sense of (3.40) is usually employed to model a 'physical' white noise (Stratonovich, 1963).

Differential Generator

From the Ito's stochastic differential rule, one rewrites (3.39) as

$$dh(\underline{x}, t) = \frac{\partial h}{\partial t}dt + \frac{\partial h}{\partial \underline{x}}d\underline{x}(t) + \frac{1}{2}d\underline{x}^T(t) \frac{\partial^2 h}{\partial \underline{x}^2}d\underline{x}(t) \qquad (3.47)$$

By substituting (3.40) into the last term of (3.47) and applying the Levy oscillation property (3.24), one retains only terms up to the first order in dt to get

$$dh(\underline{x}, t) = \frac{\partial h}{\partial t}dt + \frac{\partial h}{\partial \underline{x}}d\underline{x}(t) + \frac{1}{2} \underline{\alpha}^T G^T \frac{\partial^2 h}{\partial \underline{x}^2} G \, d\underline{\alpha}(t)$$

$$= \frac{\partial h}{\partial t}dt + \frac{\partial h}{\partial \underline{x}}d\underline{x}(t) + \frac{1}{2}\text{trace}\{GQ(t)G^T \frac{\partial^2 h}{\partial \underline{x}^2}\}dt \qquad (3.48)$$

Then, (3.40) is substituted for the dx in (3.48) to get

$$dh(\underline{x}, t) = \frac{\partial h}{\partial t}dt + \frac{\partial h}{\partial \underline{x}}G(\underline{x}, t)d\underline{\alpha}(t) + D_x\{h(\underline{x}, t)\}dt \tag{3.49}$$

where

$$D_x\{h(\underline{x}, t)\} = \frac{\partial h}{\partial \underline{x}}\underline{f}(\underline{x}, t) + \frac{1}{2}\text{trace}\{G(\underline{x}, t)Q(t)G^T(\underline{x}, t)\frac{\partial^2 h}{\partial \underline{x}^2}\} \tag{3.50}$$

(3.50) is called the differential generator.

In applications, (3.49) can be used to derive the moment equations if one takes the expected value of (3.49) and divides it by dt to yield

$$\frac{dE[h(t)]}{dt} = E[\frac{\partial h}{\partial t}] + E[\frac{\partial h}{\partial \underline{x}}\underline{f}(\underline{x},t)] + \frac{1}{2}E[\text{trace}\{G(\underline{x},t)Q(t)G^T(\underline{x},t)\frac{\partial^2 h}{\partial \underline{x}^2}\}] \tag{3.51}$$

and sets

$$h(\underline{x},t) = x_1^{k_1} x_2^{k_2} x_3^{k_3} \cdots x_n^{k_n} \tag{3.52}$$

Note that (3.51) also can be expressed as

$$\frac{dE[h(t)]}{dt} = E[\frac{\partial h}{\partial t}] + \sum_{j=1}^{n} E\,[f_j \frac{\partial h}{\partial x_j}] + \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{2}E[(GQG^T)_{ij}\frac{\partial^2 h}{\partial x_i \partial x_j}] \tag{3.53}$$

<u>Example:</u>

For a simple first-order dynamic system subjected to parametric noise excitation as given by

$$\dot{x} = x(t)\alpha(t),\ x(0) = 1 \tag{3.54a}$$

where $\alpha(t)$ is a Gaussian white noise process with intensity

$$E[\alpha(t)\cdot\alpha(s)] = 2q\cdot\delta(t - s) \tag{3.54b}$$

The stochastic differential equation of (3.54) becomes

$$dx = x(t)\ dW(t) \tag{3.55a}$$

where W(t) is a Wiener process with intensity

$$E[\ dW(t) \cdot dW(s)\ ] = 2q \cdot dt \tag{3.55b}$$

The Stratonovich solution of (3.54) is

$$x(t) = \exp(\ W(t)\ ) \tag{3.56}$$

From the Wong and Zaikai transformation as (3.46), the Stratonovich solution is the same as the Ito solution of

$$dx(t) = x(t)\ dW(t) + q \cdot x(t)\ dt \tag{3.57}$$

Then, the Ito solution of (3.55) is derived by using the Ito's differential rule (3.49) with $h(x(t),t) = \ln(x(t))$ and gives

$$x(t) = \exp(\ W(t) - q \cdot t\ ) \tag{3.58}$$

Fokker-Planck-Kolmogorov Equation

Forward and Backward Form

For the Ito's stochastic differential equation given as (3.40), the forward Fokker-Planck-Kolmogorov equation which describes the propagation of the transition probability density function $p(\underline{x},\ t|\underline{x}_0,\ t_0)$ is given as

$$\frac{\partial p(\underline{x},t|\underline{x}_0,t_0)}{\partial t} = - \sum_{j=1}^{n} \frac{\partial}{\partial x_j} [f_j(\underline{x},t)p(\underline{x},t|\underline{x}_0,t_0)] + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{2} \frac{\partial^2}{\partial x_i \partial x_j}[(GQG^T)_{ij}p(\underline{x},t|\underline{x}_0,t_0)] \tag{3.59}$$

The forward Fokker-Planck-Kolmogorov equation is subjected to the initial condition

$$\lim_{t \to t_0} p(\underline{x}|\underline{x}_0) = \delta(\underline{x} - \underline{x}_0) \tag{3.60}$$

and the boundary condition

$$p(\infty,\ t) = p(-\infty,\ t) = 0 \tag{3.61}$$

(3.59) is called the forward form because the derivative of $p(\underline{x}, t | \underline{x}_0, t)$ is taken with respect to the future time. The forward form is usually called the Fokker-Planck-Kolmogorov (FPK) equation for convenience. It can be used to derive the moment equations which are given by Maybeck (1983).

1. Propagation of mean

$$\dot{\underline{\mu}}_x = E[\underline{f}(\underline{x},t)] \tag{3.62a}$$

2. Propagation of covariance

$$\dot{P}_x = E[\underline{f}(\underline{x},t)\cdot\underline{x}^T] - E[\underline{f}(\underline{x},t)\cdot\underline{\mu}_x^T] + E[\underline{x}\cdot\underline{f}^T(\underline{x},t)]$$

$$- \underline{\mu}_x\cdot E[\underline{f}^T(\underline{x},t)] + E[G(\underline{x},t)\ Q(t)G^T(\underline{x},t)] \tag{3.62b}$$

When the derivative of $p(\underline{x},t|\underline{x}_0,t_0)$ is taken with respect to the earlier time, then the propagation of the transition probability density function satisfies the backward FPK equation. The backward FPK equation is of little interest in this thesis and will not be discussed here.

Some Exact Solutions of the FPK Equation

The exact solution of the stationary FPK equation has been derived for a large class of nonlinear systems subjected to external Gaussian white noise excitation (Fuller, 1969). Some exact solutions of the stationary FPK equations for the second-order nonlinear random systems are given in this section.

Consider a second-order nonlinear random system given by

$$\ddot{x} + h(x_1, x_2) = W'(t) \tag{3.63a}$$

where

$$x_1 = x,\ x_2 = \dot{x} \tag{3.63b}$$

with

$$E[W'(t)W'(s)] = 2q\cdot\delta(t - s) \tag{3.63c}$$

The exact solutions of the stationary FPK equations for (3.63) are given as follows.

Case 1.

$$h(x_1, x_2) = cx_2 + f(x_1) \tag{3.64}$$

where

$$F(x_1) = \int_0^{x_1} f(x)\, dx \tag{3.65}$$

and $F(x_1) \to \infty$ as $|x_1| \to \infty$

The stationary FPK equation is given by

$$- x_2 \frac{\partial p}{\partial x_1} + cp + (cx_2 + f(x_1)) \frac{\partial p}{\partial x_2} + q \frac{\partial^2 p}{\partial x_2^2} = 0 \tag{3.66}$$

The solution of (3.66) is derived as (Lin, 1972)

$$p(x_1, x_2) = N\cdot\exp[- \frac{c}{q} [\frac{1}{2} x_2^2 + \int_0^{x_1} f(x)dx]] \tag{3.67}$$

where N is the normalized constant and determined by

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x_1, x_2)\, dx_1 dx_2 = 1 \tag{3.68}$$

Case 2.

$$h(x_1,x_2) = g(H)x_2 + f(x_1) \tag{3.69}$$

where

$$H = \frac{1}{2}x_2^2 + \int_0^{x_1} f(x)dx \tag{3.70}$$

and $G(H) = \int_0^{H} g(r)dr$ (3.71)

and $G(H) \to \infty$ as $H \to \infty$

The stationary FPK equation is given by

$$-x_2 \frac{\partial p}{\partial x_1} + f(x_1) \frac{\partial p}{\partial x_2} + \frac{\partial}{\partial x_2} [g(H)x_2 p + q \frac{\partial p}{\partial x_2}] = 0 \tag{3.72}$$

The solution of (3.72) is derived as

$$p(x_1, x_2) = N \cdot \exp[- \frac{1}{q} \int_0^H g(r)dr] \tag{3.73}$$

where N is a normalized constant.

<u>Example</u>:

The Duffing oscillator subjected to the stochastic external excitation is given by

$$\ddot{x} + \xi_0 \dot{x} + x + \mu_0 x^3 = W'(t) \tag{3.74a}$$

with

$$E[W'(t)\ W'(s)] = 2q \cdot \delta(t - s) \tag{3.74b}$$

Since

$$h(x_1, x_2) = \xi_0 x_2 + x_1 + \mu_0 x_1^3$$

$$= \xi_0 x_2 + f(x_1) \tag{3.75}$$

From (3.67), the stationary joint probability density function of states x_1 and x_2 is obtained as

$$p(x_1, x_2) = N \cdot \exp[- \frac{\xi_0}{q} (\frac{1}{2} x_2^2 + \frac{1}{2} x_1^2 + \frac{\mu_0 x_1^4}{4})] \tag{3.76}$$

or

$$p(x_1) = \frac{\exp[-\frac{\xi_0}{q}(\frac{x_1^2}{2} + \frac{\mu_0 x_1^4}{4})]}{\int_{-\infty}^{\infty} \exp[-\frac{\xi_0}{q}(\frac{x_1^2}{2} + \frac{\mu_0 x_1^4}{4})]dx_1} \tag{3.77}$$

$$p(x_2) = \frac{\exp[-\frac{\xi_0}{q}(\frac{1}{2}x_2^2)]}{\int_{-\infty}^{\infty} \exp[-\frac{\xi_0}{q}(\frac{1}{2}x_2^2)]dx_2} \tag{3.78}$$

where $p(x_1)$ and $p(x_2)$ are the stationary probability density functions of x_1 and x_2, respectively.

Stochastic Stability

There are several definitions of stochastic stability in terms of convergence in probability, convergence in the mth mean and almost sure convergence. These definitions are the extensions of Lyapunov stability for deterministic systems. The deterministic Lyapunov stability is defined as follows.

The equilibrium solution is stable if, given $\varepsilon > 0$, there exists a $\delta > 0$ such that $\| \underline{x}_0 \| < \delta$ implies $\sup_{t \geq t_0} \| \underline{x}(t) \| < \varepsilon$ where

$$\| \underline{x} \| = \sum_{i=1}^{n} |x_i| \tag{3.79}$$

The definitions of stochastic stability are then formed through the concepts of convergence (3.10) to (3.14) used with the above definition.

<u>Definition 1. Lyapunov stability in probability</u>:

The equilibrium solution is said to be stable in probability if, given $\varepsilon > 0$, $\varepsilon' > 0$, there exists a δ such that $\| \underline{x}_0 \| < \delta$ implies

$$P\{ \sup_{t \geq t_0} \| \underline{x} \| > \varepsilon' \} < \varepsilon \tag{3.80}$$

Definition 2. Lyapunov stability in the mth mean:

The equilibrium solution is stable in the mth mean if the mth moments exist and if, given $\varepsilon > 0$, there exists a δ such that $\| \underline{x}_0 \| < \delta$ implies

$$E[\sup_{t \geq t_0} \| \underline{x}(t) \|^m] < \varepsilon \tag{3.81}$$

where

$$\| \underline{x} \|^m = \sum_{i=1}^{n} |x_i|^m \tag{3.82}$$

Definition 3. Almost sure Lyapunov stability:

The equilibrium solution is said to be almost surely stable if

$$P\{ \lim_{\|\underline{x}_0\| \to 0,} \sup_{t \geq t_0} \| \underline{x}(t) \| = 0 \} = 1 \tag{3.83}$$

In addition to the above definitions which depend on the sample behavior on the infinite half line (t_0, ∞), the following stability concepts are also considered.

Definition 4. Lyapunov stability of the probability:

The equilibrium solution possesses stability of probability if, given ε, $\varepsilon' > 0$, there exists a $\delta > 0$ such that $\| \underline{x}_0 \| < \delta$ implies

$$P\{\| \underline{x}(t) \| > \varepsilon'\} < \varepsilon \tag{3.84}$$

Definition 5. Lyapunov stability of the mth mean:

The equilibrium solution is said to possess stability of the mth mean if, given $\varepsilon > 0$, there exists a δ such that $\| \underline{x}_0 \| < \delta$ implies

$$E[\| \underline{x}(t) \|^m] < \varepsilon \tag{3.85}$$

The last two definitions which depend on the sample behavior at one time only are not as strong as Definitions 1 to 3. Furthermore, the above definitions of stability can be extended to asymptotic stability by following the example given below.

Definition 6. Asymptotic stability in the mth mean:

The equilibrium solution is said to be asymptotically stable in the mth mean if the equilibrium solution is stable in the mth mean and if there exists a $\delta' > 0$ such that $\| \underline{x}_0 \| < \delta'$ implies

$$\lim_{\delta \to \infty} E \left[\sup_{t \geq \delta} \| \underline{x} \|^m \right] = 0 \tag{3.86}$$

Summary

This chapter has laid the mathematical groundwork for the further investigations of the dynamic response and controller design of a nonlinear system subjected to both stochastic parametric and external excitations. The most pertinent material concerning stochastic processes, FPK equation, and Ito's stochastic differential equations has been presented. We will now turn our attention to the central problems in the areas of random vibration and control. The problems of dynamic response of nonlinear stochastic systems will be first explored in the next chapter.

CHAPTER IV

DYNAMIC RESPONSE OF NONLINEAR STOCHASTIC SYSTEMS PREDICTED BY NONLINEARIZATION TECHNIQUES

In this chapter, a new physical concept of equivalent external excitation is introduced and a nonlinearization technique is developed for the accurate prediction of stationary output variances of the states of nonlinear systems subjected to both stochastic parametric and external excitations. By following the concept of equivalent external excitation, a stochastic parametrically and externally excited nonlinear oscillator is interpreted as one which is excited solely by an external zero-mean stochastic process. Then, the FPK equation is applied to solve for the density functions and the stationary variances of the states are derived by using matching conditions. Four examples which include polynomial, nonpolynomial, and Duffing-type nonlinear oscillators are used to illustrate this approach. The validity of the present approach is compared with some exact solutions and with Monte Carlo simulations.

Derivation of the Equivalent External Excitation Method

The derivation of the equivalent external excitation method is illustrated by using the following oscillator. Consider a second-order linear oscillator with stochastic parametric and external excitations

$$\ddot{x} + (\xi_0 + \xi')\dot{x} + (\mu_0 + \mu')x = w' \tag{4.1}$$

where ξ_0 and μ_0 are some constants. ξ', μ', and w' are independent zero-mean 'physical' Gaussian white noise processes with covariances $E[\xi'(t)\ \xi'(s)] = 2q_{22}\delta(t-s)$, $E[\mu'(t)\mu'(s)] = 2q_{11}\delta(t-s)$, and $E[w'(t)\ w'(s)]=2q_{33}\delta(t-s)$, respectively. On introducing $x_1=x$ and $x_2=\dot{x}$, the state equation with diffusional correction term from (3.46a) is

$$\begin{bmatrix} dx_1 \\ \\ dx_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \\ -\mu_0 x_1 - (\xi_0 - q_{22})x_2 \end{bmatrix} dt + \begin{bmatrix} 0 & 0 & 0 \\ & & \\ -x_1 & -x_2 & 1 \end{bmatrix} \begin{bmatrix} dw_1 \\ dw_2 \\ dw_3 \end{bmatrix} \tag{4.2}$$

where $dw_1 = \mu' dt$, $dw_2 = \xi' dt$, and $dw_3 = w' dt$ and w_1, w_2, and w_3 are Wiener processes with independent increments. The propagation of moments can be derived by using Ito's differential rule (3.53) to yield

$$\begin{aligned} \dot{m}_{10} &= m_{01} \\ \dot{m}_{01} &= -\mu_0 m_{10} - (\xi_0 - q_{22})m_{01} \\ \dot{m}_{20} &= 2m_{11} \\ \dot{m}_{11} &= m_{02} - \mu_0 m_{20} + (q_{22} - \xi_0)m_{11} \\ \dot{m}_{02} &= 2[-\mu_0 m_{11} - (\xi_0 - q_{22})m_{02} + q_{11}m_{20} + q_{22}m_{02} + q_{33}] \end{aligned} \tag{4.3}$$

where m_{ij} is defined as the expected value of $x_1^i x_2^j$ with $i,j = 0,1,2$. If one rewrites (4.1) as

$$\ddot{x} + \xi_0\dot{x} + \mu_0 x = -\mu' x - \xi'\dot{x} + w' \tag{4.4}$$

and interprets the noise excitation term on the right hand side as an equivalent external excitation with intensity $2q_{11}m_{20}+2q_{22}m_{02}+2q_{33}$, then the equivalent state equation with diffusional correction term can be expressed in the form

$$\begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\mu_0 x_1 - (\xi_0 - q_{22})x_2 \end{bmatrix} dt + \begin{bmatrix} 0 \\ 1 \end{bmatrix} dw \qquad (4.5)$$

where $\dot{w}$ is a Gaussian white noise with variance $E[\dot{w}(t)\dot{w}(s)] = (2q_{11}m_{20} + 2q_{22}m_{02} + 2q_{33})\delta(t-s)$. By inspection, (4.5) maintains the same equations of the propagation of moments up to the second order as (4.3). However, the oscillator with stochastic parametric and external excitations now becomes an equivalent oscillator subjected to external stochastic excitation. Accordingly, the well-known FPK equation can be applied to solve for the joint probability density function of x_1 and x_2 without any restrictions on the intensity of the equivalent external noise. The stationary FPK equation to (4.5) is given by using (3.59)

$$-x_2 \frac{\partial P}{\partial x_1} + \mu_0 x_1 \frac{\partial P}{\partial x_2} + \frac{\partial(\xi_0 - q_{22})x_2 P}{\partial x_2} + \tfrac{1}{2}Q \frac{\partial^2 P}{\partial x_2^2} = 0 \qquad (4.6)$$

where $Q=2q_{11}m_{20}+2q_{22}m_{02}+2q_{33}$ and P is the stationary joint probability density of x_1 and x_2. The solution of (4.6) is obtained from (3.67) as

$$P(x_1, x_2) = N \exp[-(\xi_0 - q_{22})(\mu_0 x_1^2 + x_2^2)/Q] \qquad (4.7)$$

with $\xi_0 - q_{22} > 0$, for convergent solution, and N is a normalized constant. It follows that the stationary probability densities of x_1 and x_2 are expressed as

$$P(x_1) = \sqrt{\mu_0(\xi_0 - q_{22})/\pi Q}\ \exp[-(\xi_0 - q_{22})\ \mu_0 x_1^2/Q]$$

$$P(x_2) = \sqrt{(\xi_0 - q_{22})/\pi Q}\ \exp[-(\xi_0 - q_{22})x_2^2/Q] \tag{4.8}$$

The next step is to solve (4.8) by matching the variances of x_1 and x_2 as follows:

$$m_{20} = \int_{-\infty}^{\infty} P(x_1)x_1^2\, dx_1 = Q/2(\xi_0 - q_{22})\mu_0$$

$$m_{02} = \mu_0 m_{20} \tag{4.9}$$

Finally, the solutions of m_{20} and m_{02} are given by substituting $Q=2q_{11}m_{20}+2q_{22}m_{02}+2q_{33}$ and solving the algebraic equation (4.9) to yield

$$m_{20} = m_{02}/\mu_0 = q_{33}/(\xi_0\mu_0 - q_{11} - 2q_{22}\mu_0) \tag{4.10}$$

Specifically, when $q_{11}= \mu_0 q_{22}$, then $m_{20}=q_{33}/\mu_0(\xi_0-3q_{22})$. If one defines $\alpha=q_{33}/q_{11}$, $\beta=((\xi_0/q_{22})+1)/2$,. it follows that the stationary variance of x_1 is $\alpha/2(\beta-2)$.

Dimentberg (1982) has solved the exact stationary probability densities of x_1 and x_2 to (4.1) using the FPK equation

$$P(x_1) = \frac{\alpha^{\beta-1}\ \Gamma(\beta - ½)}{\sqrt{\pi}\ \ \Gamma(\beta-1)}\ (\alpha + x_1^2)^{-(\beta - ½)} \tag{4.11}$$

where $\Gamma(\cdot)$ is the gamma function. From (4.11), the variance of x_1 can be integrated to obtain $\alpha/2(\beta-2)$. The variance of x_1 is exactly the same as the results obtained by using the equivalent external excitation method. Also, for the linear oscillator (4.1), the stationary variance of x_1 can be derived from (4.3) by setting all derivative terms equal to zero. Again, one derives $m_{20} = m_{02}/\mu_0 = \alpha/2(\beta-2)$. In summary, the present method can be applied by the following procedure.

1. Replace the stochastic parametrically and externally excited non-linear oscillator by an equivalent one with diffusional correction term which is excited solely by external excitation. The variance of the equivalent external excitation equals the variance of the original external noise plus an equivalent variance which is attributed to the parametrically excited terms. The equivalent variance is obtained by taking the summation of the expected values of the square of the parametrically excited terms multiplied by the corresponding variances of the excited coefficients.
2. Obtain the stationary probability density functions of the states of the equivalent nonlinear oscillator from Step 1 by using the Fokker-Planck-Kolmogorov equation.
3. Match the expected value of the square of each of the parametrically excited terms in the oscillator by taking the expected value of those terms through the probability density functions of the states obtained from Step 2.
4. Solve the simultaneous algebraic or integral equations by iterative methods. This step is exactly the same as the procedure when one applies the statistical linearization method (Hedrick and Arslan, 1980).

5. Find the stationary variances of the states of the oscillator by substituting the probability density functions which have been derived from Steps 1 to 4.

Prediction of the Output Stationary Response

The following examples follow the same noise notations and definitions as given above. The first example selected is a nonpolynomial type nonlinear oscillator which can not be solved by using the cumulant-neglect method.

Example 1:

A second-order nonlinear oscillator with nonlinear spring $F(x) = C \tan(\pi x/2d)$ for $-d<x<d$ is given by (Liu, 1969)

$$\ddot{x} + \xi_0 \dot{x} + F(x) = w' \qquad (4.12)$$

The stationary probability density of x is obtained through the FPK equation as

$$P(x) = N \exp[(8d\xi_0 C/\pi 2q_{33})\ln(\cos(\pi x/2d))] \qquad (4.13)$$

If the nonlinear oscillator with parametric and external noise excitations is in the form

$$\ddot{x} + \xi_0 \dot{x} + (1 + \mu') F(x) = w' \qquad (4.14)$$

one can derive the stationary variance of x by the following procedure.

Step 1. and 2. Substitute $2q_{33}$ with $Q=2q_{33}+2q_{11}E[F(x)^2]$ then,

$$P(x) = N \exp[(8d\xi_0 C/\pi Q)\ln(\cos(\pi x/2d))] \qquad (4.15)$$

Step 3. Match the expected value of $F(x)^2$ by writing

$$E[F(x)^2] = \frac{Q - 2q_{33}}{2q_{11}}$$

$$= NC^2 \int_{-d}^{d} \tan^2(\frac{\pi x}{2d})\exp[(\frac{8d\xi_0 C}{\pi Q})\ln(\cos\frac{\pi x}{2d})]dx \qquad (4.16)$$

Step 4. Since (4.16) is an integral equation with unknown Q, the value of Q can be obtained by using numerical iteration.

Step 5. Substituting Q into (4.15), one obtains the probability density function of x. Thus, the stationary variance of x is derived by using P(x) and integrating

$$m_{20} = \int_{-d}^{d} x^2 P(x)dx \qquad (4.17)$$

This example illustrates that the existing solution of the probability density function of oscillators subjected to external noise excitation can be extended to predict the response of the nonlinear oscillators subjected to both parametric and external noise excitations by using the equivalent external excitation approach. The second example chosen for the present investigation is a nonlinear oscillator considered by Dimentberg (1982).

Example 2:

Consider the nonlinear oscillator given by

$$\ddot{x} + (\xi_0 + \xi')\dot{x} + \rho\dot{x}(x^2 + \dot{x}^2/\mu_0) + (\mu_0 + \mu')x = w' \qquad (4.18)$$

Step 1. The equivalent oscillator with diffusional correction term is given by

$$\ddot{x} + (\xi_0 - q_{22})\dot{x} + \rho\dot{x}(x^2 + \dot{x}^2/\mu_0) + \mu_0 x = w$$

$$\text{with } Q = 2q_{11}m_{20} + 2q_{22}m_{02} + 2q_{33} \tag{4.19}$$

Step 2. From (3.59), the stationary FPK equation is expressed as

$$-x_2 \frac{\partial P}{\partial x_1} + \mu_0 x_1 \frac{\partial P}{\partial x_2} + \frac{\partial}{\partial x_2}\{[(\xi_0 - q_{22}) + \rho(x_1^2 + \frac{x_2^2}{\mu_0})]x_2 P + \tfrac{1}{2}Q\frac{\partial P}{\partial x_2}\} = 0 \tag{4.20}$$

From (4.20), the solution of the stationary probability density function of P is (Caughey, 1971)

$$P(x_1, x_2) = N \exp[- \frac{2(\xi_0 - q_{22})}{Q} H - \frac{2\rho}{Q\mu_0} H^2]$$

$$\text{with } H = \frac{x_2^2}{2} + \frac{\mu_0}{2} x_1^2 \tag{4.21}$$

Step 3. The matching conditions can be applied directly at this step; however, it is easier to apply if one considers the propagation of H. The probability density function of H is in the form

$$P(H) = N' \exp[- \frac{2(\xi_0 - q_{22})}{Q} H - \frac{2\rho}{Q\mu_0} H^2] \tag{4.22}$$

Introducing $a = \rho/Q\mu_0$, $b = \mu_0(\xi_0 - q_{22})/2\rho$, one obtains

$$N' = 1/\int_0^\infty \exp[- \frac{2(\xi_0 - q_{22})}{Q} H - \frac{2\rho}{Q\mu_0} H^2]\, dH$$

$$= 1/[(\sqrt{\pi}/2\sqrt{2a})\text{erfc}(b\sqrt{2a})], \quad a > 0 \tag{4.23}$$

where $\mathrm{erfc}(b\sqrt{2a})=1-\mathrm{erf}(b\sqrt{2a})$ is the complementary error function. From (4.21), it is seen that the probability density functions of x_2 and $\sqrt{\mu_0}x_1$ are equal. Taking the expected value of H and using $m_{02} = \mu_0 m_{20}$, one derives

$$m_{02} = E[H] = \int_0^{\infty} H\ P(H)dH = \frac{e^{-2ab^2}}{\sqrt{2a\pi}\ \mathrm{erfc}(b\sqrt{2a})} - b \tag{4.24}$$

Step 4. m_{02} is readily given by the numerical solution of (4.24). Since the exact solution of m_{20} can be derived through the FPK equation provided that the noise intensities q_{11} and q_{22} satisfy $q_{11} = \mu_0 q_{22}$, one chooses the appropriate values of noise intensities and parameters to compare several approaches. For $2q_{11} = 2q_{22} = 0.1$ and $\rho = \mu_0 = \xi_0 = 1.0$, (4.24) becomes

$$m_{20} = \frac{e^{-0.451/Q}}{\sqrt{2\pi/Q}\ \mathrm{erfc}(0.672/\sqrt{Q})} - 0.475$$

$$\text{with } Q = 0.2m_{20} + 2q_{33} \tag{4.25}$$

The cumulant-neglect method up to the fourth order is derived by solving nine simultaneous nonlinear algebraic equations and gives (Wu and Lin, 1984)

$$192m_{20}^3 + 61.2m_{20}^2 + (2.55 - 28(2q_{33}))m_{20} - 1.5(2q_{33}) = 0 \tag{4.26}$$

By applying the Gaussian closure method, one derives

$$m_{20} = \frac{18 - \sqrt{324 + 3160(2q_{33})}}{-158} \tag{4.27}$$

The exact solution of m_{20} is given by (Dimentberg, 1982)

$$m_{20} = \frac{1}{20} \left[-9.5 + \frac{(100(2q_{33}))^{100(2q_{33})-9.5}\, e^{-100(2q_{33})}}{\Gamma(100(2q_{33})-9.5,\ 100(2q_{33}))}\right] \tag{4.28}$$

where $\Gamma(u,x)$ is the incomplete gamma function. The comparisons of (4.25), (4.26), (4.27), and (4.28) with varying $s_3=2q_{33}$ are obtained by numerical solution and given in Fig. 4. It is observed that the present approach is in close agreement with the exact solution even when the intensity of the external noise is of the same order as that of parametric noise. The next example selected is governed by a cubic nonlinear spring oscillator and can be easily extended to a higher order nonlinear spring oscillator. By using the cumulant-neglect method such an increase will increase the difficulty in derivation because more cumulant terms are required to express higher order moments. However, this is not a concern when using the equivalent external excitation method.

Example 3:

A nonlinear oscillator with cubic nonlinearity is described by

$$\ddot{x} + (\xi_0 + \xi')\dot{x} + (\mu_0 + \mu')x^3 = w' \tag{4.29}$$

Step 1. The equivalent oscillator with diffusional correction term is written as

$$\ddot{x} + (\xi_0 - q_{22})\dot{x} + \mu_0 x^3 = w$$

$$\text{with } Q = 2q_{11}m_{60} + 2q_{22}m_{02} + 2q_{33} \tag{4.30}$$

Step 2. The probability density functions of states x_1 and x_2 can be derived from (3.67) to yield

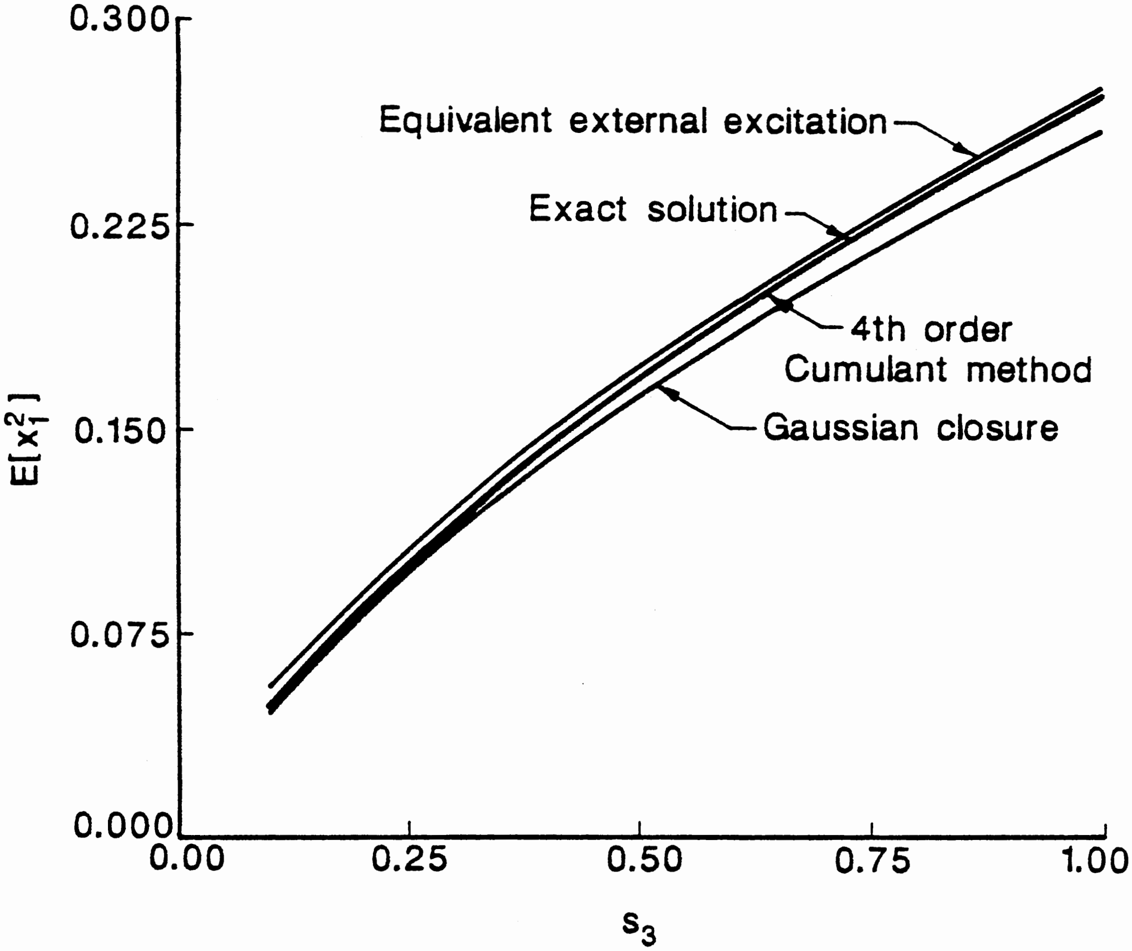

Figure 4. The Prediction of the Stationary Variance of Displacement With Varied External Excitation Intensity S_3 by Equation (4.25) and Several Other Approaches

$$P(x_1) = \frac{2K_1^{1/4}}{\Gamma(1/4)} \exp(-k_1 x_1^4), \; k_1 = \frac{(\xi_0 - q_{22})\mu_0}{2Q} > 0 \tag{4.31a}$$

$$P(x_2) = \sqrt{\frac{k_2}{\pi}} \exp(-k_2 \, x_2^2), \; k_2 = \frac{2k_1}{\mu_0} \tag{4.31b}$$

Step 3. The matching conditions for m_{60} and m_{02} are expressed as

$$m_{60} = \int_{-\infty}^{\infty} P(x_1)x_1^6 dx_1 = 0.254(2Q/(\xi_0 - q_{22})\mu_0)^{3/2} \tag{4.32}$$

$$m_{02} = \int_{-\infty}^{\infty} P(x_2)x_2^2 dx_2 = Q/2(\xi_0 - q_{22}) \tag{4.33}$$

Step 4. From (4.32) and (4.33), substituting Q and eliminating m_{60}, one derives

$$(4.113q_{11}^2/\mu_0^3)m_{02}^3 - (\xi_0 - 2q_{22})^2 m_{02}^2 + 2q_{33}(\xi_0 - 2q_{22})m_{02} - q_{33}^2 = 0 \tag{4.34}$$

Step 5. m_{20} is derived by substituting $P(x_1)$ and integrating as

$$m_{20} = \int_{-\infty}^{\infty} P(x_1)x_1^2 dx_1 = 0.676(m_{02}/\mu_0)^{1/2} \tag{4.35}$$

The comparisons of (4.35) with a 500 run Monte Carlo simulation of (4.29) by choosing $\xi_0 = 1.0$, $\mu_0 = 5.0$ and with varying q_{33}/q_{11} and q_{33}/q_{22} are illustrated in Fig. 5, 6, and 7. It is seen that these figures show good agreement in the stationary response of m_{20} between the present method and Monte Carlo simulations. Since the stationary response of an oscillator will be dominated by the parametric noise excitation when the intensity of the external noise is much less than that of the parametric noise, the concept of using equivalent external excitation to approximate the parametric noise excitation as an equivalent external

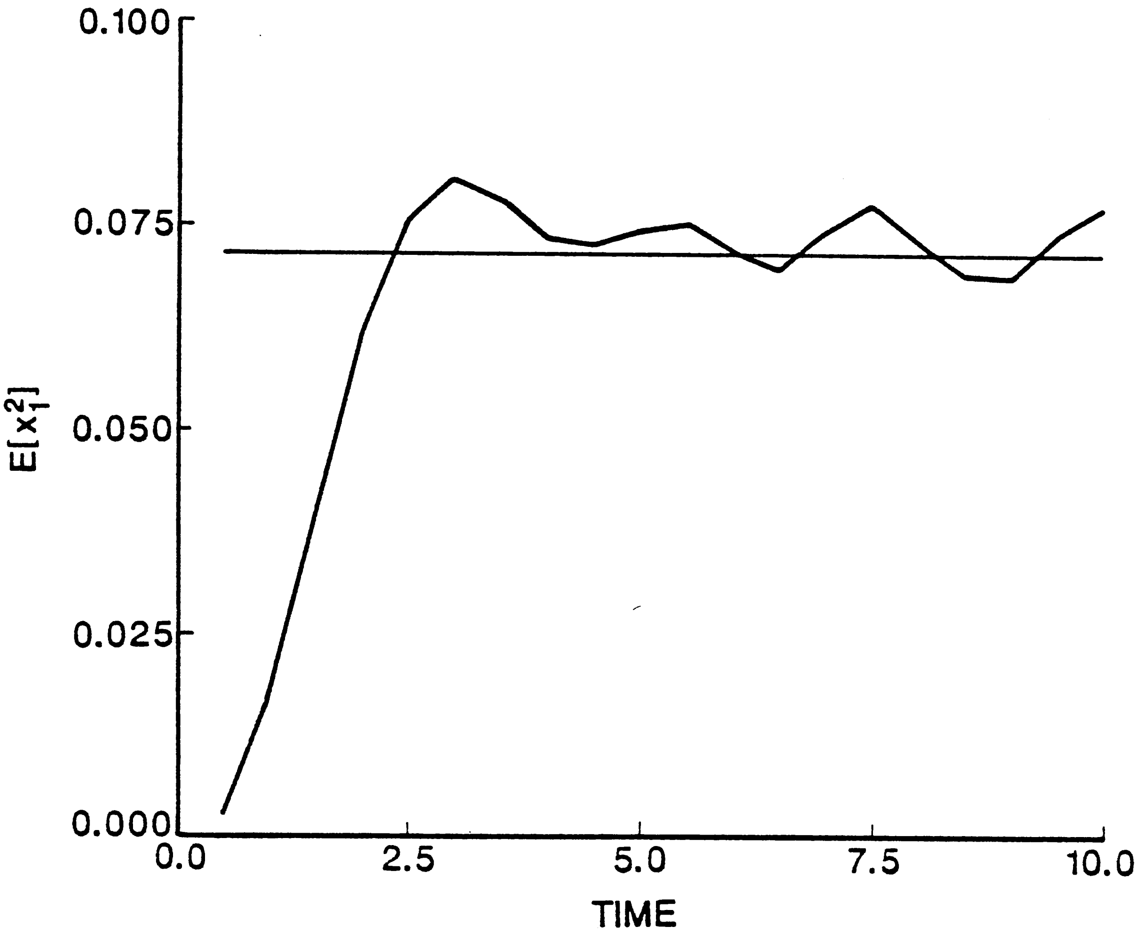

Figure 5. The Prediction of the Stationary Variance of Displacement by Equation (4.35) and Monte Carlo Simulation Using $2q_{11} = 0.0$, $2q_{22} = 0.1$, $2q_{33} = 0.1$. i.e. $q_{33}/q_{22} = 1.0$

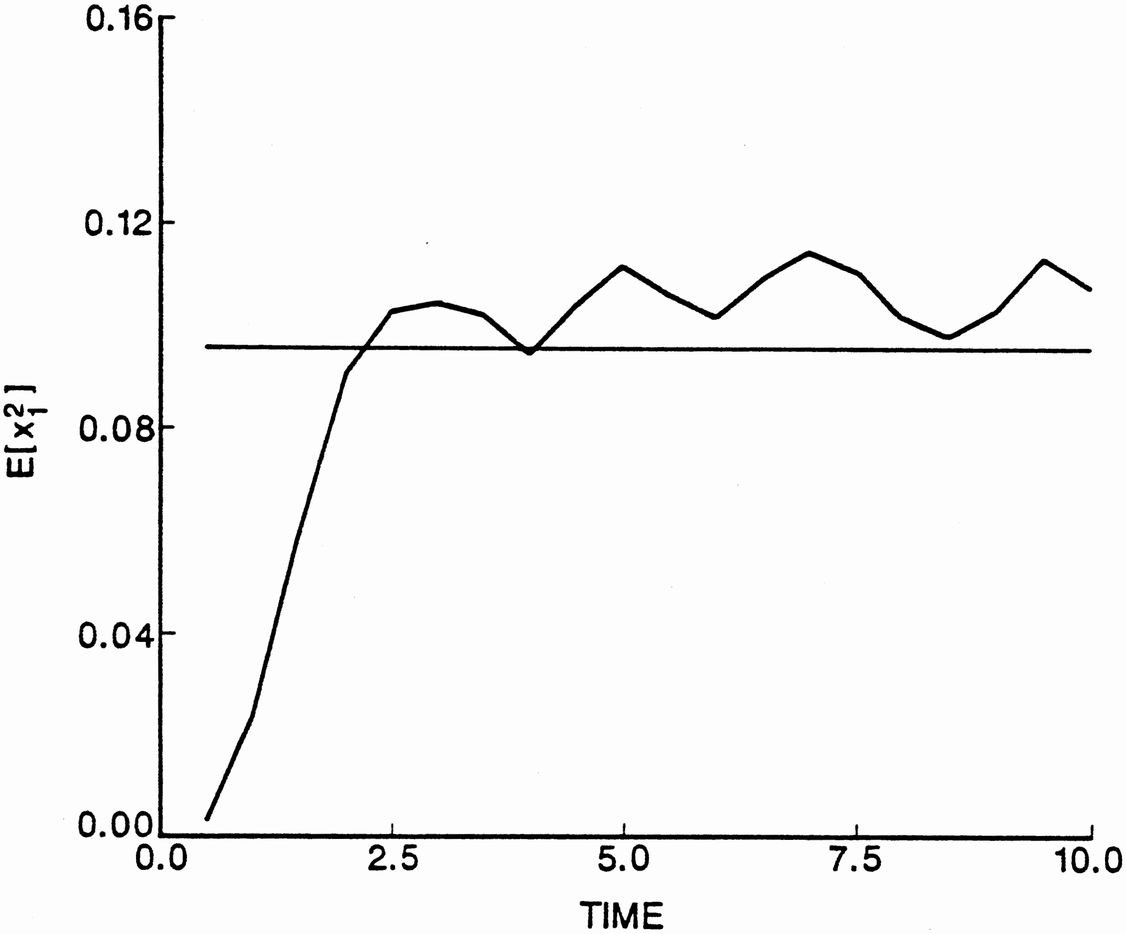

Figure 6. The Prediction of the Stationary Variance of Displacement by Equation (4.35) and Monte Carlo Simulation Using $2q_{11} = 0.0$, $2q_{22} = 0.5$, $2q_{33} = 0.1$. i.e. $q_{33}/q_{22} = 0.2$

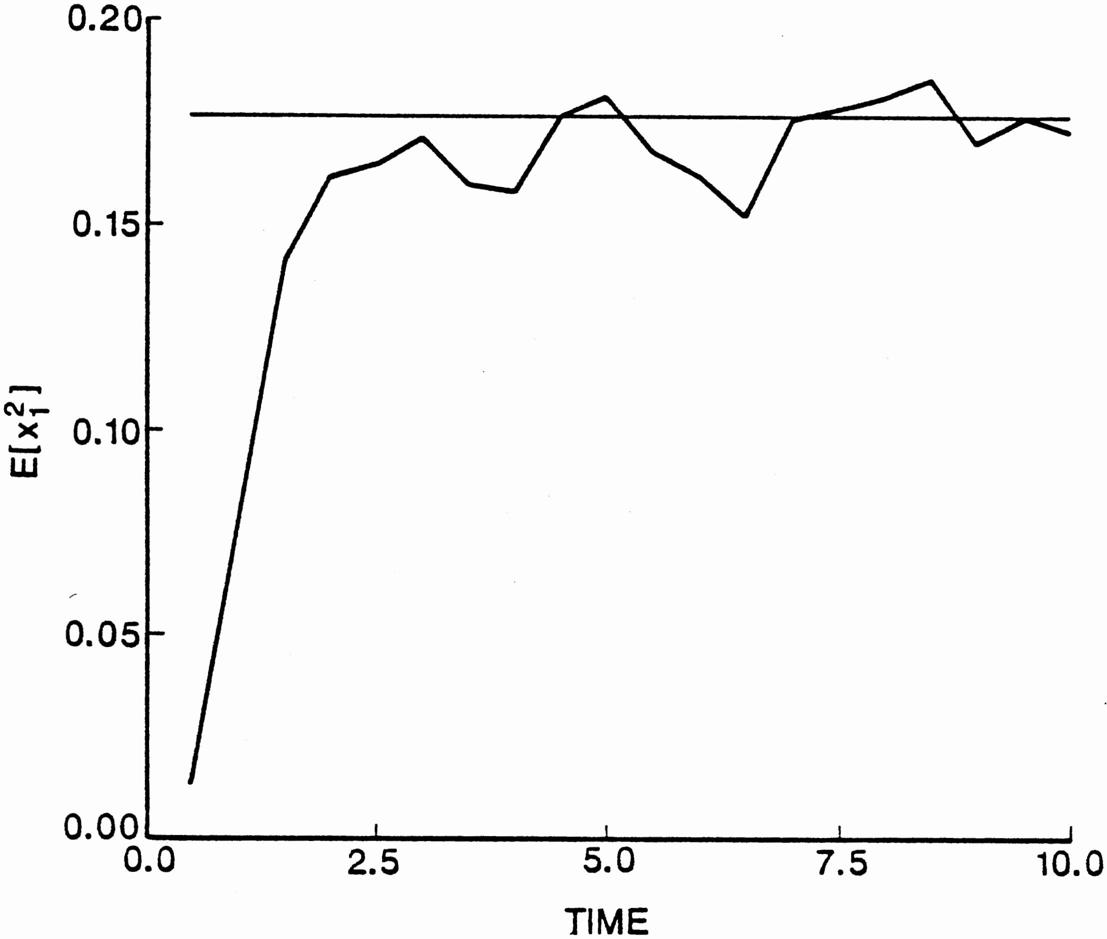

Figure 7. The Prediction of the Stationary Variance of Displacement by Equation (4.35) and Monte Carlo Simulation Using $2q_{11} = 5.0$, $2q_{22} = 0.0$, $2q_{33} = 0.5$. i.e. $q_{33}/q_{11} = 0.1$

noise excitation should induce more error in the prediction of the stationary response. Thus, the inaccuracy in the prediction of the stationary m_{20} by using the present approach is expected when the intensity of the damping noise is higher than that of the external noise as shown in Fig. 6. However, the present approach still predicts very accurate stationary response of m_{20} even when the intensity of the spring noise is ten times that of the external noise as shown in Fig. 7. The last example chosen for discussion is the Duffing oscillator.

Example 4:

A Duffing oscillator with parametric and external noise excitations is expressed as

$$\ddot{x} + (\xi_0 + \xi')\dot{x} + x + (\mu_0 + \mu')x^3 = w' \tag{4.36}$$

Step 1. The equivalent oscillator with diffusional correction term is

$$\ddot{x} + (\xi_0 - q_{22})\dot{x} + x + \mu_0 x^3 = w$$

$$\text{with } Q = 2q_{11}m_{60} + 2q_{22}m_{02} + 2q_{33} \tag{4.37}$$

Step 2. The probability density functions of states x_1 and x_2 are given by using (3.67)

$$P(x_1) = \frac{\exp[-k(x_1^2 + \frac{\mu_0 x_1^4}{2})]}{\int_{-\infty}^{\infty} \exp[-k(x_1^2 + \frac{\mu_0 x_1^4}{2})]dx_1}$$

$$P(x_2) = \sqrt{\frac{k}{\pi}}\,\exp(-kx_2^2), \quad k = \frac{\xi_0 - q_{22}}{Q} > 0 \tag{4.38}$$

Steps 3 and 4. The matching condition of m_{02} is used to yield

$$m_{02} = \int_{-\infty}^{\infty} P(x_2)x_2^2\, dx_2 = Q/2(\xi_0 - q_{22}) \tag{4.39}$$

By using the matching condition of m_{60} and substituting K and Q, one derives

$$m_{02} = [(\frac{q_{11} \int_0^{\infty} \exp[-(\frac{1}{2m_{02}})(x_1^2 + \frac{\mu_0 x_1^4}{2})]x_1^6 dx_1}{\int_0^{\infty} \exp[-(\frac{1}{2m_{02}})(x_1^2 + \frac{\mu_0 x_1^4}{2})]\, dx_1}) + q_{33}]/(\xi_0 - 2q_{22}) \tag{4.40}$$

Step 5. The stationary m_{20} is obtained by

$$m_{20} = \frac{\int_0^{\infty} \exp[-(\frac{1}{2m_{02}})(x_1^2 + \frac{\mu_0 x_1^4}{2})]x_1^2 dx_1}{\int_0^{\infty} \exp[-(\frac{1}{2m_{02}})(x_1^2 + \frac{\mu_0 x_1^4}{2})]dx_1} \tag{4.41}$$

The comparison of (4.41) with a 500 run Monte Carlo simulation of (4.36) by choosing ξ_0 =1.0, μ_0 =5.0 and $2q_{11}$ =5.0, q_{22} =0.0, $2q_{33}$ =1.0 is shown in Fig. 8. It is seen that the accurate prediction of the stationary variance of m_{20} is obtained by using the present approach. When the intensity of the external noise is at least the same order as that of the spring noise, say q_{33}/q_{11} = 1.0, better results are obtained as can be expected.

Summary

A new approach by using equivalent external excitation to predict the stationary variances of the states of nonlinear oscillators subjected to both stochastic parametric and external excitations has

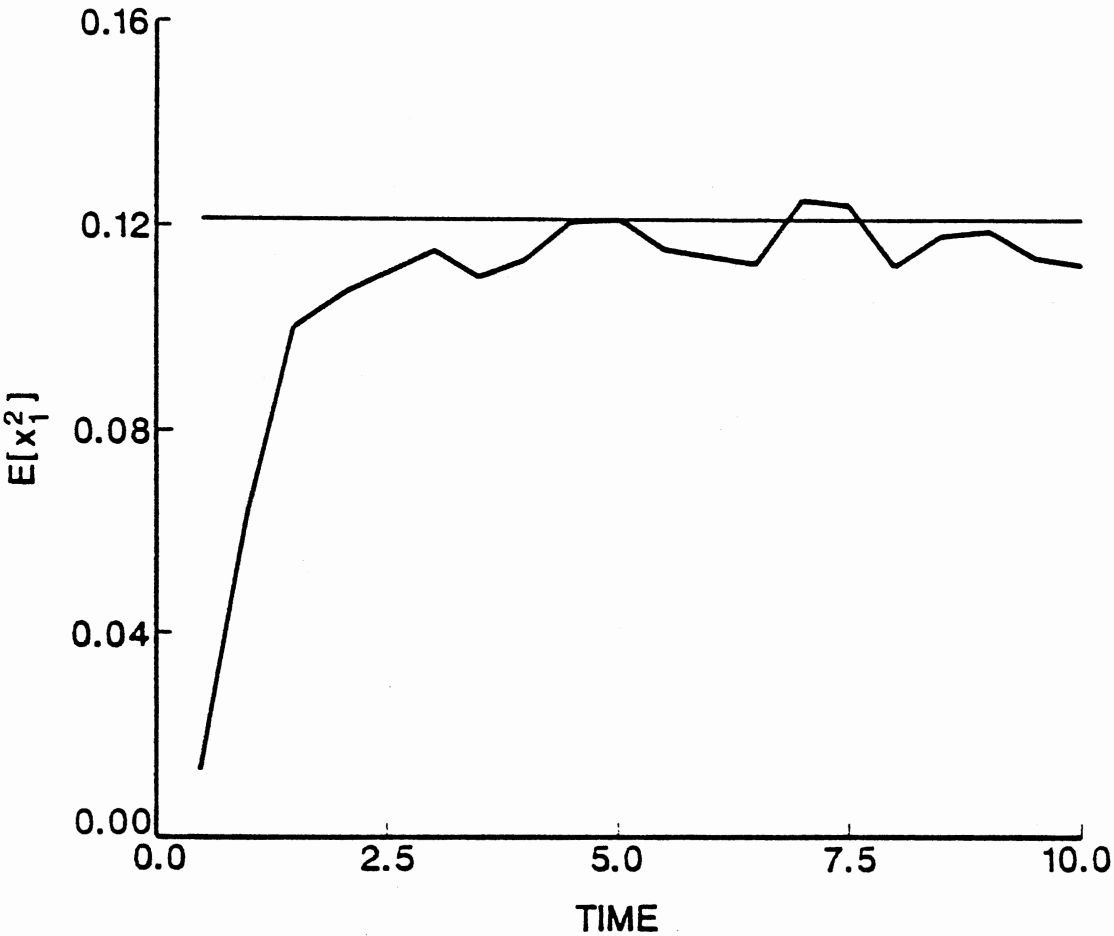

Figure 8. The Prediction of the Stationary Variance of Displacement by Equation (4.41) and Monte Carlo Simulation Using $2q_{11} = 5.0$, $2q_{22} = 0.0$, $2q_{33} = 1.0$. i.e. $q_{33}/q_{11} = 0.2$

been introduced in this chapter. By using the concept of equivalent external excitation, one can easily extend the existing solution of the response of nonlinear oscillators subjected to external excitation to the problem of parametric and external excitations. Since the present approach is to reformulate the stochastic parametric excitation to an equivalent external excitation and apply the FPK equation, this approach can be applied to nonpolynomial nonlinearities and strong noise excitations in nonlinear oscillators. The validity of the prediction of stationary variance by using the present approach has been compared with some exact solutions and Monte Carlo simulations. The results show that better prediction is obtained when the intensity of the external excitation is at least the same order as that of the parametric excitation. When the parametric excitation enters through the spring term, it was shown through an example that the present approach is valid even when the intensity of the spring noise is ten times that of the external excitation. By using the present approach, the computational effort is almost the same as the statistical linearization method and the restricted assumption that the probability densities of the stationary states are jointly Gaussian need not be invoked. Here, the only assumption of the equivalent external excitation approach is based on the validity of using the equivalent external excitation to maintain the same propagation of moments up to the second order.

For the accurate prediction of the stationary output second moments of the states of nonlinear systems, the above nonlinearization technique has been developed for the nonlinear systems subjected to strong nonlinearities and/or noise intensities. When the system nonlinearities and/or excited noise intensities are not too strong, a useful statistical linearization approach can be applied. For the effective use of statistical linearization approach, the concepts and techniques of both Gaussian and non-Gaussian linearizations for the stochastic parametrically and externally excited nonlinear systems will be presented in the following chapter.

CHAPTER V

GAUSSIAN AND NON-GAUSSIAN LINEARIZATION OF STOCHASTIC PARAMETRICALLY AND EXTERNALLY EXCITED NONLINEAR SYSTEMS

This chapter is concerned with the techniques and criterion of Gaussian and non-Gaussian linearization derived for the accurate prediction of the stationary output variances of the states of nonlinear oscillators subjected to both stochastic parametric and external excitations. The applications of both Gaussian and non-Gaussian linearizations of stochastic parametrically and externally excited nonlinear systems are first derived. A parametric and external noise excited oscillator with cubic nonlinear spring is then selected to illustrate these applications. The non-Gaussian linearization is implemented through a non-Gaussian density which is derived by using the concepts of equivalent external excitation developed in Chapter IV. By following these concepts and their extensions, a Gaussian criterion is further proposed. A stochastic parametrically and externally excited Duffing oscillator is selected to illustrate this application. The validity of using the Gaussian criterion for the investigations of non-Gaussian effects of system nonlinearities and noise intensities is also supported by performing the Chi-square Gaussian goodness-of-fit test.

Statistical Linearization of Stochastic Parametrically and Externally Excited Nonlinear Systems

Consider a second-order stochastic parametrically and externally excited nonlinear system described by

$$\ddot{x} + \sum_{i=1}^{n} (a_i + \alpha_i)\, H_i(x, \dot{x}) = W' \tag{5.1}$$

where the a_i are constants and the α_i and W' are mutually independent zero-mean Gaussian white noises with intensities $E[\alpha_i(t)\alpha_i(s)] = 2q_{ii}\cdot\delta(t - s)$ and $E[W'(t)W'(s)] = 2q_{n+1\ n+1}\cdot\delta(t - s)$, respectively. The equivalent linearization system of (5.1) is expressed as

$$\ddot{x} + \sum_{i=1}^{n} (a_i + \alpha_i)(g_i\dot{x} + f_i x) = W' \tag{5.2}$$

where $f_i = f_i(m_{20}, m_{02})$, $g_i = g_i(m_{20}, m_{02})$ and m_{ij} is defined as the expected value of $x_1^i x_2^j$ with $x_1 = x$ and $x_2 = \dot{x}$. The equation difference between the left hand side of (5.1) and (5.2) is given by

$$\begin{aligned} e' &= \sum_{i=1}^{n} (a_i + \alpha_i)\, [H_i - (g_i\dot{x} + f_i x)] \\ &= \sum_{i=1}^{n} (a_i + \alpha_i)\, e_i \end{aligned} \tag{5.3}$$

where e_i is defined as an error and given by

$$e_i = H_i - g_i\dot{x} - f_i x \tag{5.4}$$

Thus, the techniques of statistical linearization which were originally derived for nonlinear systems subjected to stochastic external excitation (Booton, 1954; Caughey, 1959) are now extended to include the parametric noise excited terms if the error of equation difference between (5.1) and (5.2) is defined as e_i rather than e', which is an error usually defined for nonlinear systems subjected to stochastic external excitation. This concept is illustrated in Fig. 9 by using a simple cubic nonlinearity. By following the usual concepts of statistical linearization, f_i and g_i are selected such that the mean-square error of e_i will be minimized. Utilizing the following equations,

$$\frac{\partial E[e_i^2]}{\partial f_i} = 0 \tag{5.5a}$$

$$\frac{\partial E[e_i^2]}{\partial g_i} = 0 \tag{5.5b}$$

and substituting e_i from (5.4), f_i and g_i are then derived as

$$f_i = \frac{m_{02}\ E[x \cdot H_i(x,\ \dot{x})] - m_{11}\ E[\dot{x} \cdot H_i(x,\ \dot{x})]}{m_{20}m_{02} - m_{11}^2} \tag{5.6a}$$

$$g_i = \frac{m_{20}E[\dot{x} \cdot H_i(x,\ \dot{x})] - m_{11}E[x \cdot H_i(x,\ \dot{x})]}{m_{20}m_{02} - m_{11}^2} \tag{5.6b}$$

By applying Ito's differential rule (3.53) to (5.1), $\dot{m}_{20}$ is derived as

$$\dot{m}_{20} = 2m_{11} \tag{5.7}$$

If stationary, (5.6a) and (5.6b) are further simplified by substituting

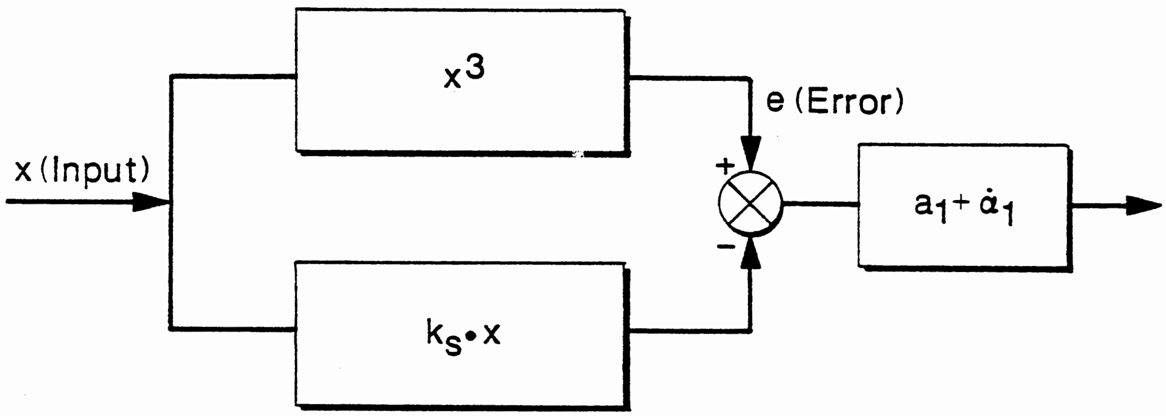

Figure 9. Statistical Linearization of Stochastic Parametrically and Externally Excited Nonlinear Systems

$m_{11} = 0$ to yield

$$f_i = \frac{E[x \cdot H_i(x, \dot{x})]}{m_{20}} \tag{5.8a}$$

$$g_i = \frac{E[\dot{x} \cdot H_i(x, \dot{x})]}{m_{02}} \tag{5.8b}$$

The equivalent linearization system (5.2) is then rewritten as

$$\ddot{x} + (\zeta_0 + \zeta')\,\dot{x} + (\mu_0 + u')x = W' \tag{5.9}$$

where

$$\zeta_0 = \sum_{i=1}^{n} a_i g_i$$

$$\mu_0 = \sum_{i=1}^{n} a_i f_i$$

$$E[\zeta'(t)\zeta'(s)] = 2S\delta(t - s)$$

$$= (\sum_{i=1}^{n} 2q_{ii} g_i^2)\ \delta(t - s) \tag{5.10}$$

$$E[\mu'(t)\mu'(s)] = 2R\ \delta(t - s)$$

$$= (\sum_{i=1}^{n} 2q_{ii} f_i^2)\ \delta(t - s)$$

The stationary output variances of x and $\dot{x}$ of (5.9) can be derived from the moment equations with diffusional correction term (4.10) as

$$m_{20} = m_{02}\mu_0 \tag{5.11a}$$

$$= q_{n+1\ n+1}/\zeta_0\mu_0 - R - 2S\mu_0 \tag{5.11b}$$

By substituting (5.10) into (5.11a) and (5.11b), two simultaneous algebraic equations with unknown m_{20} and m_{02} are then derived to yield

$$m_{02} - (\sum_{i=1}^{n} a_i f_i)\, m_{20} = 0 \tag{5.12a}$$

$$(\sum_{i=1}^{n} a_i g_i)(\sum_{i=1}^{n} a_i f_i) m_{20} - (\sum_{i=1}^{n} q_{ii} f_i^2) m_{20}$$

$$- 2 (\sum_{i=1}^{n} q_{ii} g_i^2)(\sum_{i=1}^{n} a_i f_i) m_{20} - q_{n+1\ n+1} = 0 \tag{5.12b}$$

Thus, m_{20} and m_{02} are readily obtained by solving (5.12a) and (5.12b) with f_i and g_i derived by using the Gaussian or non-Gaussian density function in (5.8a) and (5.8b), respectively. When the Gaussian (non-Gaussian) density function is applied to evaluate f_i and g_i from (5.8a) and (5.8b), the statistical linearization approach is called the Gaussian (non-Gaussian) linearization approach.

The non-Gaussian densities of the states of (5.9) under certain conditions can be derived through the FPK equation. For $R = \mu_0 S$, Dimentberg (1982) has solved the stationary joint probability density of the states to (5.9) as

$$p(x_1, x_2) = \frac{(\beta - 1)\alpha^{\beta-1}}{\pi\sqrt{\mu_0}} \frac{1}{(\alpha + x_2^2/\mu_0 + x_1^2)^{\beta}} \tag{5.13}$$

where $\alpha = q_{n+1\ n+1}/R$ and $\beta = (\zeta_0/S + 1)/2$, $\beta > 1$. From (5.13), it is seen that the stationary states of x_1 and x_2 are not independent in spite of $m_{11} = 0$ from (5.7). Thus, the highly non-Gaussian effects in the linearization of stochastic parametrically and externally excited nonlinear systems need to be considered. One approach of the non-Gaussian linearizations of stochastic parametrically and externally excited nonlinear systems as proposed is implemented by first deriving the non-Gaussian probability density from the concepts of equivalent external excitation given in Chapter IV. Then, the equivalent linear gains f_i and g_i are readily obtained by using the density function in (5.8a) and (5.8b), respectively. The applications of the non-Gaussian linearization are illustrated by using the following Duffing-type nonlinear oscillator.

<u>Example 1</u>:

A stochastic parametrically and externally excited nonlinear system is described by

$$\ddot{x} + (\zeta_0 + \zeta')\dot{x} + (\mu_0 + \mu')x^3 = W' \tag{5.14}$$

where $E[\mu'(t)\mu'(s)] = 2q_{11}\ \delta(t - s)$, $E[\zeta'(t)\zeta'(s)] = 2q_{22}\delta(t - s)$, and $E[W'(t)W'(s)] = 2q_{33}\ \delta(t - s)$. From (5.2) and (5.6a), the equivalent linearization system of (5.14) is written as

$$\ddot{x} + (\zeta_0 + \zeta')\ \dot{x} + (\mu_0 + \mu')f_1 x = W' \tag{5.15}$$

where

$$f_1 = m_{40}/m_{20} \tag{5.16}$$

The applications of non-Gaussian linearization are first illustrated. By following the concepts of equivalent external excitation and using the FPK equation, $p(x_1)$ and $p(x_2)$ are derived from (4.31) to yield

$$p(x_1) = \frac{2k_1^{1/4}}{\Gamma(1/4)} \exp(-k_1 x_1^4), \; k_1 = \frac{(\zeta_0 - q_{22})\mu_0}{2Q} > 0 \qquad (5.17a)$$

$$p(x_2) = \sqrt{k_2/\pi} \exp(-k_2 x_2^4), \; k_2 = \frac{2k_1}{\mu_0} \qquad (5.17b)$$

where $\Gamma(\cdot)$ is the Gamma function and $Q = 2q_{11}m_{60} + 2q_{22}m_{02} + 2q_{33}$. By employing (5.17a) to evaluate m_{20} and m_{40}, respectively to derive

$$m_{20} = \frac{\Gamma(3/4)}{\Gamma(1/4)} \cdot \frac{1}{k_1^{1/2}} \qquad (5.18a)$$

$$m_{40} = \frac{1}{4k_1} \qquad (5.18b)$$

then (5.16) becomes

$$f_1 = \frac{\Gamma(1/4)}{4\Gamma(3/4)} \cdot \frac{1}{k_1^{1/2}} \qquad (5.19)$$

From (5.18a) and by substituting $k_1^{1/2}$ into (5.19), then (5.19) is expressed as

$$f_1 = 2.188 \, m_{20} \qquad (5.20)$$

By using (5.12a) and (5.12b) with f_1 given by (5.20), one derives

$$4.787 \, q_{11} m_{20}^3 + (4.376 \, q_{22} \, \mu_0 - 2.188 \, \mu_0 \zeta_0) \, m_{20}^2 + q_{33} = 0 \qquad (5.21a)$$

$$m_{02} = 2.188 \, \mu_0 \, m_{20}^2 \qquad (5.21b)$$

Hence, the stationary output variances of the states x_1 and x_2 are derived by solving (5.21a) and substituting m_{20} into (5.21b) to obtain

m_{02}. Next, the Gaussian linearization approach is applied to derive f_1. From (5.16), one obtains

$$f_1 = 3.0m_{20} \tag{5.22}$$

Thus, following the derivations of (5.21a) and (5.21b), the stationary output variances of x_1 and x_2 can be derived from

$$9q_{11}m_{20}^3 + (6q_{22}\mu_0 - 3\zeta_0\mu_0)m_{20}^2 + q_{33} = 0 \tag{5.23a}$$

$$m_{02} = 3\mu_0 m_{20}^2 \tag{5.23b}$$

The comparisons of (5.21a) and (5.23a) by choosing $\mu_0 = 5.0$, $\zeta_0 = 1.0$, $2q_{11} = 5.0$, and $2q_{22} = 0.0$, with varying q_{33} are shown in Fig. 10. When $2q_{33}$ equals 0.5, a 1000-run Monte Carlo simulation to (5.14) is given to compare the results derived by using (5.21a) and (5.23a) as shown in Fig. 11. From Fig. 10 and 11, it is seen that the Gaussian linearization approach predicts lower variance of x_1 than that by the non-Gaussian linearization approach. Also, better prediction of the stationary output variance of x_1 is obtained by utilizing the present non-Gaussian linearization approach. Although the present non-Gaussian linearization approach can be developed through the non-Gaussian density derived by using the concepts of equivalent external excitation, this approach is applied only for some practical nonlinear stochastic systems. Actually, if the Gaussian assumption is deemed valid by using a quantitative measure, the simple and effective Gaussian linearization approach can be applied with high confidence in the resulting solution. As a result, a quantitative measure called the Gaussian criterion will be derived to help in determining when the application of

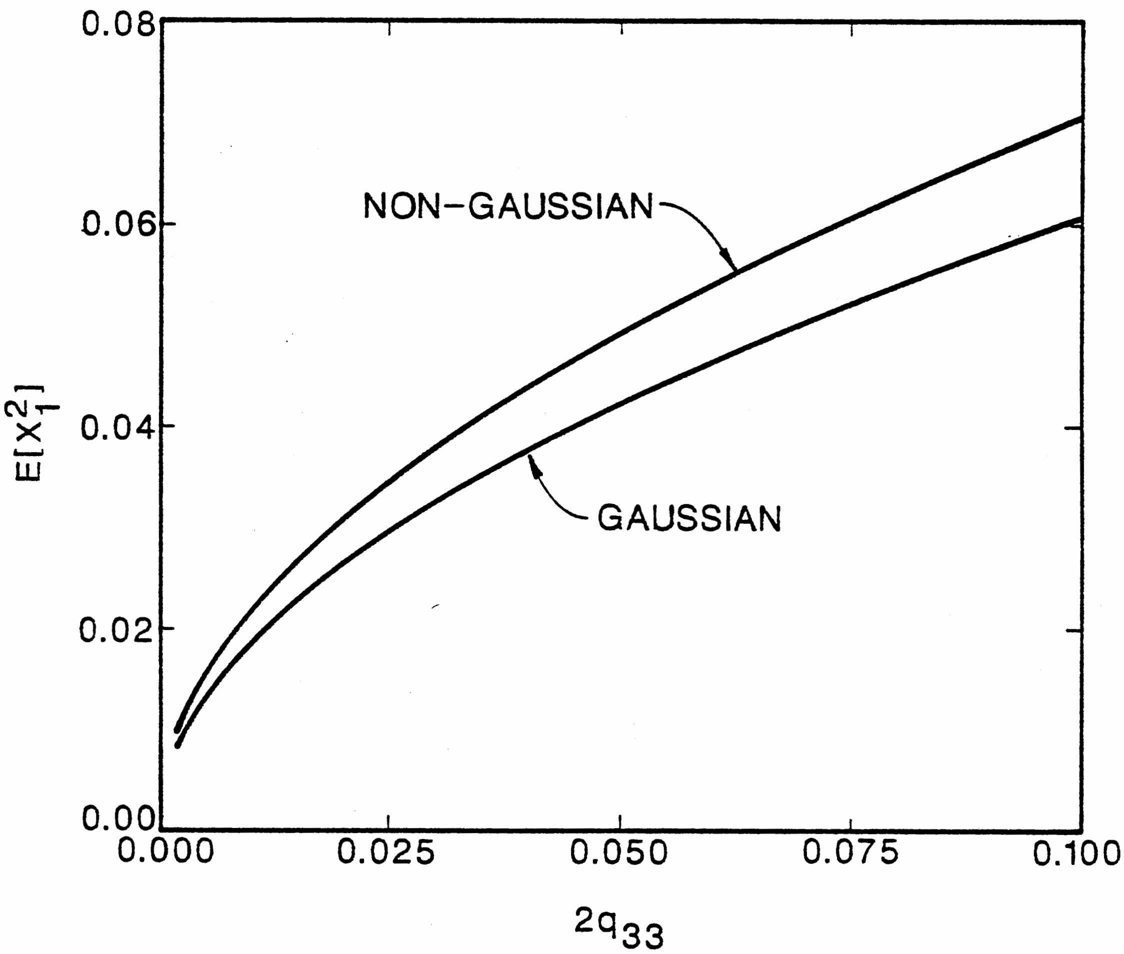

Figure 10. The Comparisons of Output Mean-square Displacement by (5.21a) and (5.23a) using $\mu_0 = 5.0$, $\xi_0 = 1.0$, $2q_{11} = 5.0$, $2q_{22} = 0.0$, and With Varying $2q_{33}$

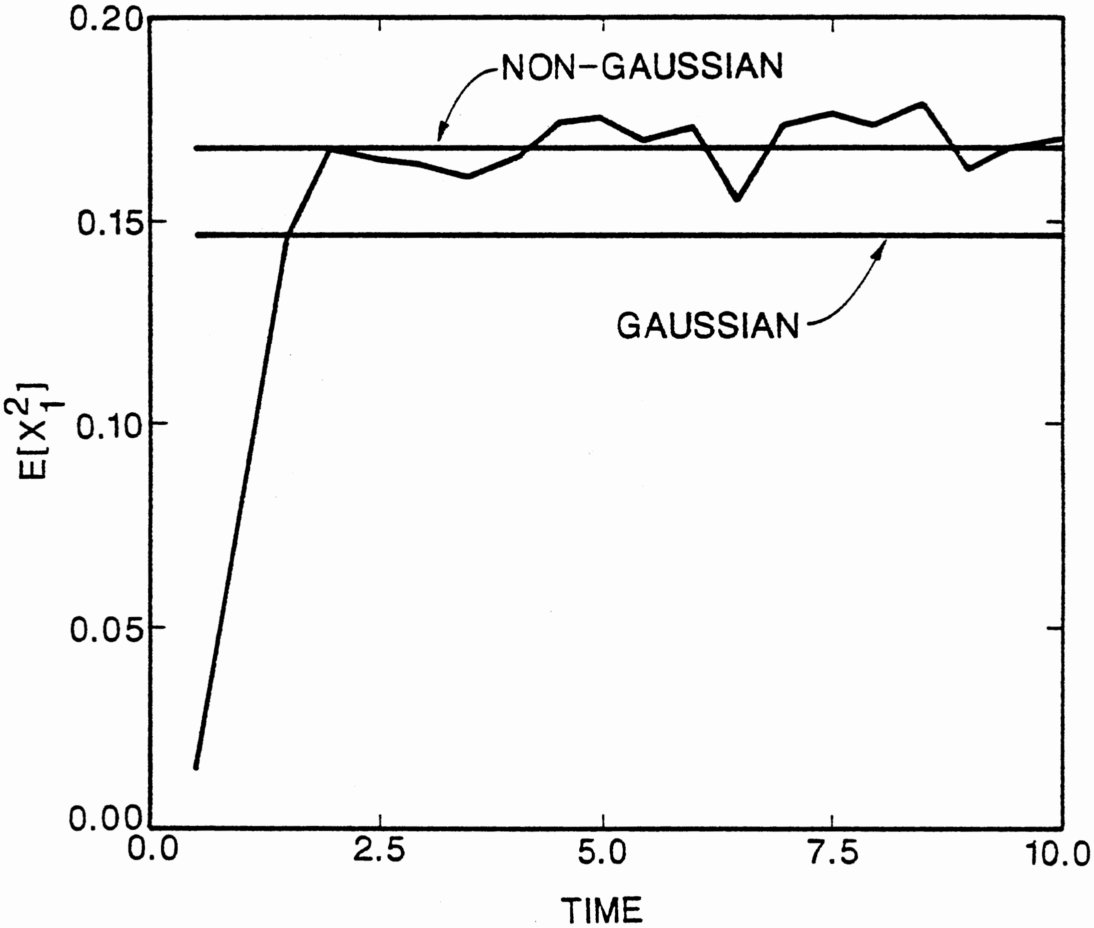

Figure 11. The Comparisons of Output Mean-square Displacement by (5.21a), (5.23a), and 1000-run Monte Carlo Simulation Using μ_0 = 5.0, ξ_0 = 1.0, $2q_{11}$ = 5.0, $2q_{22}$ = 0.0, and $2q_{33}$ = 0.5

Gaussian linearization is appropriate.

Gaussian Criterion (GC)

The determination of using Gaussian linearization for the accurate prediction of stationary output variance of nonlinear stochastic systems has not been investigated by establishing a concrete Gaussian measure. The successful applications of Gaussian linearization for stochastic externally excited 'weak' nonlinear oscillators are only supported by comparing the linearization solution with the exact solution of some solvable examples, e.g., Duffing oscillator. When nonlinear systems are subjected to both stochastic parametric and external excitations, the non-Gaussian effects of parametric noise have not been investigated although one may give some vague physical interpretations such as "When the noise intensity is increased, the output densities become 'less' Gaussian." Thus, a concrete mathematical criterion needs to be established for the investigation of the non-Gaussian effects of system nonlinearities and noise intensities. For a stochastic parametric and external noise excited nonlinear system described by (5.1), the GC is established by the following arguments.

Let (5.1) be rewritten as

$$\ddot{x} + \sum_{i=1}^{n} a_i H_i(x, \dot{x}) = W' - \sum_{i=1}^{n} \alpha_i H_i(x, \dot{x}) \tag{5.24}$$

By following the concepts of equivalent external excitation given in Chapter IV, the stochastic parametrically excited term $\alpha_i H_i(x, \dot{x})$ in (5.24) can be interpreted as an equivalent external noise excited term with intensity $2q_{ii}\ E[H_i^2(x, \dot{x})]$. Thus, the stationary output response of (5.24) will be dominated by the external noise term W' if

$$\sum_{i=1}^{n} 2q_{ii} \; E[H_i^2 \, (x, \dot{x})] << 2q_{n+1 \; n+1} \tag{5.25}$$

The inequality (5.25) provides the relationship between system parameters and noise intensities such that (5.24) becomes a nonlinear system excited solely by an external noise. Since it is well known that the output variance of stochastic externally excited nonlinear systems predicted by the Gaussian linearization method are accurate within ten percent error in some cases in spite of rather strong system nonlinearities (Hedrick, 1980) the non-Gaussian effects of parametric noise can be investigated by using (5.25) without considering the effects of system nonlinearities. If the effects of system nonlinearities are considered, the strength of system nonlinearities needs to be quantified by certain mathematical expressions. One may interpret the strength of system nonlinearities in (5.24) as $\sum_{i=1}^{n} a_i^2$ $E[H_i^2(x, \dot{x})]$, which may give an overestimate (underestimate) of the strength of system nonlinearities when a_i is large (small), then (5.25) is modified to

$$\sum_{i=1}^{n} (2q_{ii} + a_i^2) \; E[H_i^2(x, \dot{x})] << 2q_{n+1 \; n+1} \tag{5.26}$$

For the convenient use of (5.26), the Gaussian criterion is defined as

$$GC = 2q_{n+1 \; n+1} / \sum_{i=1}^{n} (2q_{ii} + a_i^2) \; E[H_i^2(x, \dot{x})] \tag{5.27}$$

Since the higher-order moment terms are required to express $E[H_i^2(x, \dot{x})]$ in (5.27), certain approximate estimates need to be applied. In considering the derivation of (5.27), it is based on physical interpretation that the output response will be dominated by the linear part of a system equation if the intensity of both system nonlinearities and parametric noise excitations can be suppressed by that of external noise. Thus, the intensity of the nonlinear term $H_i(x, \dot{x})$ is replaced by that of equivalent linear term $g_i \dot{x} + f_i x$ as given by

$$E[H_i^2(x, \dot{x})] = f_i^2 m_{20} + g_i^2 m_{02} \tag{5.28}$$

By substituting (5.28) into (5.27), the GC is finally obtained as

$$GC = 2q_{n+1\ n+1} / \sum_{i=1}^{n} (2q_{ii} + a_i^2)(f_i^2 m_{20} + g_i^2 m_{02}) \tag{5.29}$$

In (5.29), the unknown m_{20} and m_{02} are derived by solving (5.12a) and (5.12b) and f_i and g_i are obtained by using (5.8a) and (5.8b), respectively. Hence, from (5.26) and (5.27), the Gaussian assumption is valid if

$$GC \gg 1.0 \tag{5.30}$$

Here, 1.0 is given as the critical value of the GC (CVGC). The applications of the GC are illustrated by using the following stochastic Duffing oscillator.

Example 2:

For a stochastic parametrically and externally excited Duffing oscillator described as

$$\ddot{x} + (\zeta_0 + \zeta')\dot{x} + x + (\mu_0 + \mu')x^3 = W' \tag{5.31}$$

the GC is established from (5.29) and given as

$$GC = 2q_{33}/((2q_{11} + \mu_0^2)(f_1^2 m_{20}) + 2q_{22} m_{02}) \tag{5.32}$$

where f_1 is given by (5.22). By substituting (5.22) into (5.32), one obtains

$$GC = 2q_{33}/((2q_{11} + \mu_0^2)(9m_{20}^3) + 2q_{22}m_{02}) \tag{5.33}$$

The unknown m_{20} and m_{02} in the above equation are derived by using the Gaussian linearization approach. By using (5.12a) and (5.12b), one obtains

$$m_{02} - (3\mu_0 m_{20} + 1)m_{20} = 0 \tag{5.34a}$$

$$\zeta_0(3\mu_0 m_{20} + 1)m_{20} - (q_{11}9m_{20}^2)m_{20} \tag{5.34b}$$

$$-2(q_{33})(3\mu_0 m_{20} + 1)m_{20} - q_{33} = 0$$

Thus, the GC (5.33) is evaluated by substituting m_{02} and m_{20} which are derived from solving (5.34a) and (5.34b).

The applications of the GC are illustrated by first plotting (5.33) as a function of system parameters or noise intensities. Then, the GC test is performed by using the criterion (5.30). The non-Gaussian effects of μ_0 and ζ_0 for the stochastic externally excited Duffing oscillator are shown in Fig. 12. It is seen that the non-Gaussian effects due to the decreasing of damping coefficient are much stronger than the effects due to the increasing of nonlinear-spring coefficient. In addition, the Gaussian linearization approach can be applied to a nonlinear system subjected to a rather strong external-noise excitation if the nonlinear system is not lightly damped. The non-Gaussian effects of $2q_{11}$ and $2q_{22}$ for the stochastic parametrically and externally excited Duffing oscillator are shown in Fig. 13. It is seen that the non-Gaussian effects of noise intensity in damping term

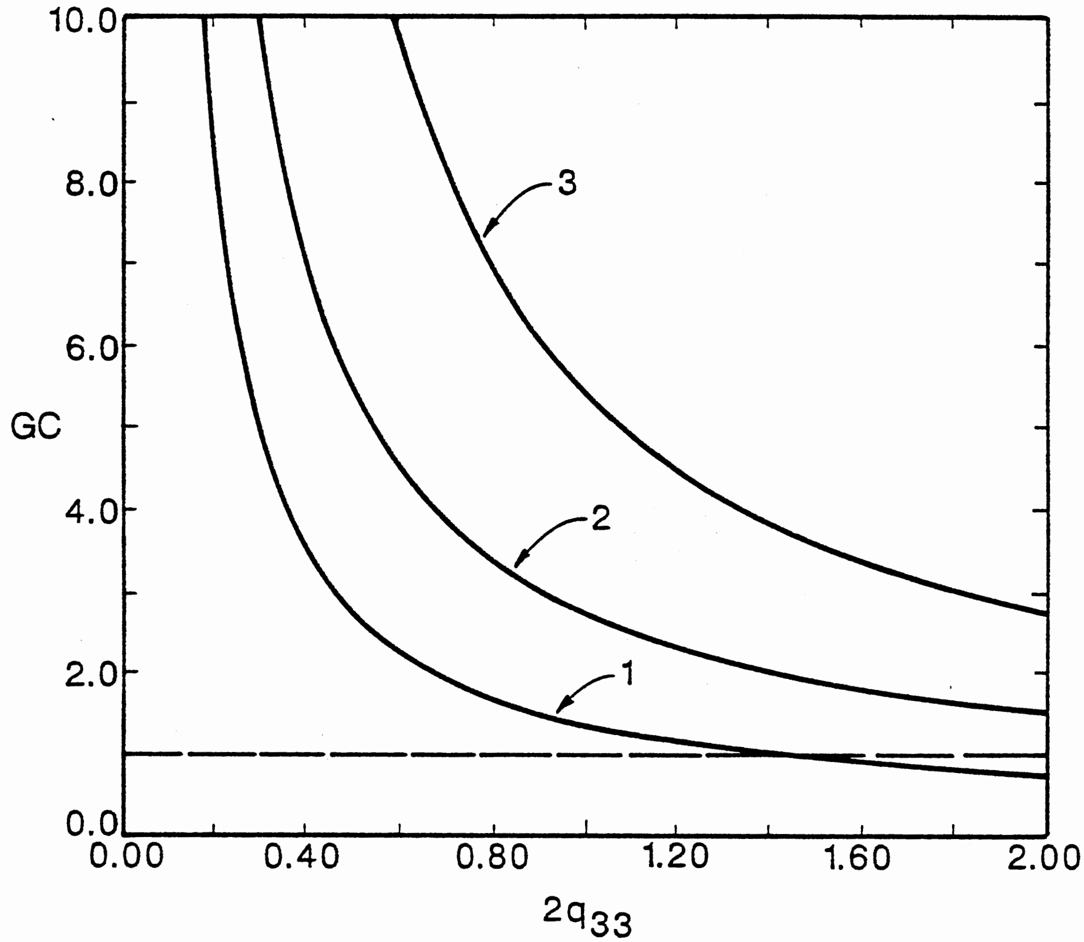

Figure 12. The Non-Gaussian Effects of μ_0 and ξ_0 for the Stochastic Externally Excited Duffing Oscillator, 1- ξ_0 = 0.5, μ_0 = 1.0, 2- ξ_0 = 1.0, μ_0 = 2.0, 3- ξ_0 = 1.0, μ_0 = 1.0

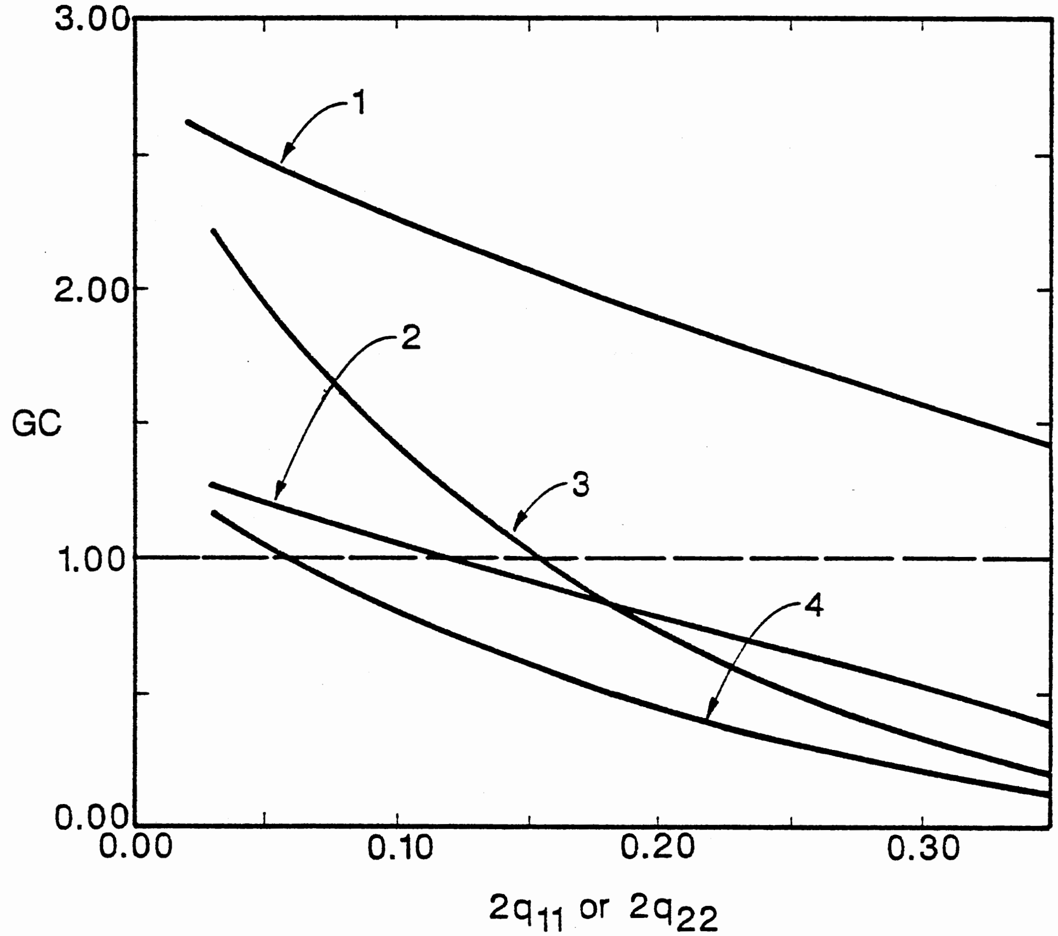

Figure 13. The Non-Gaussian Effects of Parametric Noise $2q_{11}$ and $2q_{22}$ for the Stochastic Parametrically and Externally Excited Duffing Oscillator With $\mu_0 = 1.0$ and $\xi_0 = 0.5$, 1- X-axis Represents $2q_{11}$, $2q_{22} = 0.0$, $2q_{33} = 0.5$, 2- X-axis Represents $2q_{11}$, $2q_{22} = 0.0$, $2q_{33} = 1.0$, 3- $2q_{11} = 0.0$, X-axis Represents $2q_{22}$, $2q_{33} = 0.5$, 4- $2q_{11} = 0.0$, X-axis Represents $2q_{22}$, $2q_{33} = 1.0$

are much stronger than than of noise intensity in spring term. Also, when a nonlinear system is subjected to both stochastic parametric and external excitations, the increasing of external-noise intensity can easily drive the system response into a non-Gaussian region. Thus, the applications of the GC provide a good insight into the non-Gaussian effects of system parameters and/or noise intensities. The CVGC which is used to classify the output response into a Gaussian or non-Gaussian region is further investigated by using the Chi-square Gaussian goodness-of-fit test.

Chi-square Gaussian Goodness-of-Fit (CSGGF) Test

The applications of the GC and its critical value are further supported by performing the CSGGF test. Here, the CSGGF test is used as a measure of the discrepancy between the output non-Gaussian process of a nonlinear stochastic dynamic system and the Gaussian density function. The CSGGF test is first performed by collecting 5000 weakly stationary random data and arranging them into 25 groups of 200 data each. The test of normality then follows an equal frequency procedure with 16 class intervals under 5% level of significance (Bendat and Piersol, 1971) Since the random data are collected from the stationary response of a dynamic system excited by a pseudo-random number generator, the 25 groups' Chi-square value calculated by using 200 random data each will fluctuate. Thus, it is proposed that the hypothesis of Gaussian distribution is accepted if the average Chi-square value of the 25 groups is less than 22.36 (Bendat and Piersol, 1971) and if at least one-half of the number of groups have Chi-square values less than 22.36 (i.e. 13 in this case). Before the CSGGF test is performed on the stochastic Duffing oscillator, the output of a pseudo-random number generator which simulates the Gaussian white noise and the output of the states of a second-order linear system excited externally by the pseudo-random number generator are first tested as shown in Fig. 14. It is seen that the output of both pseudo-random number generator and state $\dot{x}$ satisfies the Gaussian hypothesis at 5% level of significance but the output of state x does not satisfy the above hypothesis test. This phenomenon contradicts the fact that the Gaussian distribution after being transformed by a linear transformation including the differential operator should still be Gaussian. This phenomenon has not been reported to the authors knowledge and can be interpreted as the effects of numerical integration of inexact Gaussian noise which amplify the discrepancy between the output process x and the Gaussian density function. Since the number of numerical integrations used to obtain the output process x from the pseudo-random number generator is one more than that used to derive the output process $\dot{x}$, the output distribution of the process x will be 'less' Gaussian than that of the process $\dot{x}$. This interpretation is further supported by using a third-order linear system excited externally by the pseudo-random number generator. For the third-order system, by following the above interpretation, the selection of state $\ddot{x}$ is better than $\dot{x}$ and $\dot{x}$ is better than x in performing the CSGGF test. The above interpretations are verified by the simulated results as given in Table I which includes the maximum and average values of 25 groups' Chi-square value, number of Chi-square values which is greater than 22.36, and the Gaussian (G) or non-Gaussian (N-G) distribution which is classified by using the above simulated Chi-square results and following the present Gaussian

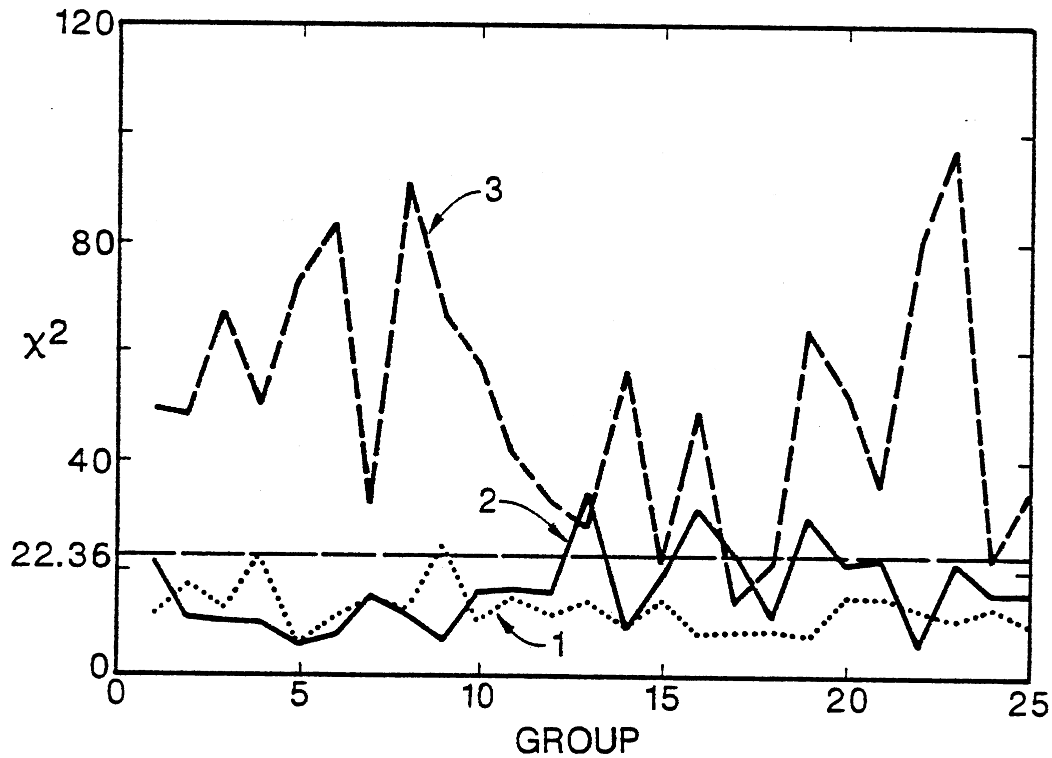

Figure 14. The Chi-square Gaussian Goodness-of-fit Test for a Second- order Linear System Subjected to Externally Gaussian White-noise Excitation With μ_0 = 1.0 and ξ_0 = 1.0, 1- Output of a Pseudo-random Number Generator, 2- Output Process of Velocity, 3- Output Process of Displacement

TABLE I

THE CSGGF TEST FOR THE SECOND-ORDER AND THIRD-ORDER LINEAR SYSTEMS SUBJECTED TO EXTERNALLY GAUSSIAN WHITE NOISE EXCITATION

Systems	Distribution of χ^2	Maximum value of χ^2	Average value of χ^2	No. of χ^2 values > 22.36	Gaussian or Non-Gaussian
Second-order linear system	x	97.66	50.65	21	N-G
	$\dot{x}$	34.81	17.14	5	G
Third-order linear system	x	138.74	67.65	24	N-G
	$\dot{x}$	107.01	51.98	25	N-G
	$\ddot{x}$	38.71	20.03	9	G

hypothesis test. Hence, the following CSGGF test for the second-order stochastic Duffing oscillator will be performed by using state $\dot{x}$. The non-Gaussian effects and the CVGC for the external noise excited Duffing oscillator are first investigated by using both the CSGGF and GC tests. The simulated results are given in Table II which is divided into three groups with each group arranged in the increasing order of GC. The first group is simulated with $\zeta_0 = 0.5$, $2q_{33} = 1.0$, and with varying μ_0. It is seen that the output response of the moderately damped Duffing oscillator can be classified as a Gaussian distribution from the CSGGF test even if the strength of nonlinearity varies over a wide range. The decreasing of μ_0 from 100 to 0.1 results in a nonlinear system with 'weak' strength of system nonlinearity. This fact is reflected in the increasing of GC form 0.06 to 21.27 but not in the decreasing of the Chi-square value. Thus, it implies that the CSGGF test cannot be used for sensitivity studies of the non-Gaussian effects of system parameters; however, it can be studied by using the GC test as shown in Fig. 12. For the selection of CVGC, the value of 1.0 gives a conservative Gaussian boundary in classification because even though GC is equal to 0.06, the CSGGF test still confirms the acceptance of Gaussian hypothesis. This phenomenon is due to the fact that $9\mu_0^2 m_{20}^3$ gives an overestimate of the strength of system nonlinearity. However, in considering the applications of the Gaussian linearization approach, when the GC is greater than 1.0, it implies that the strength of system nonlinearity is weak. Hence, the accurate prediction of output variance by employing the linearization approach is guaranteed if the GC is greater than 1.0. The non-Gaussian effects of $2q_{33}$ are simulated and given in the second group. It is interesting to note that the

TABLE II

THE CSGGF AND GC TESTS FOR THE STOCHASTIC EXTERNALLY EXCITED DUFFING OSCILLATOR

Group	Case	μ_0	ζ_0	$2q_{33}$	GC	Max. value of χ^2	Avg. value of χ^2	No. of χ^2 values > 22.36	Gaussian or Non-Gaussian
	1	100	0.5	1.0	0.06	48.32	18.60	5	G
	2	50	0.5	1.0	0.09	61.48	22.22	9	G
1	3	10	0.5	1.0	0.24	43.00	19.74	8	G
	4	1	0.5	1.0	1.36	52.51	21.27	12	G
	5	0.1	0.5	1.0	21.27	43.58	20.75	8	G
	1	1.0	0.5	100	0.06	48.32	18.60	5	G
	2	1.0	0.5	50	0.09	61.48	22.22	9	G
2	3	1.0	0.5	10	0.24	43.00	19.74	8	G
	4	1.0	0.5	1.0	1.36	52.50	21.27	12	G
	5	1.0	0.5	0.1	21.27	43.58	20.75	8	G
	1	0.5	0.1	10.0	0.03	148	40.50	16	N-G
	2	0.5	0.1	2.0	0.08	117	41.78	19	N-G
3	3	1.0	0.2	2.0	0.15	89.5	27.54	12	N-G
	4	0.2	0.2	1.0	1.08	59.8	30.94	15	N-G
	5	0.2	0.2	0.2	8.51	40.4	27.49	18	N-G

increasing of $2q_{33}$ plays the same role as that of μ_0 to drive the output response of the nonlinear stochastic system into a 'less' Gaussian region. The simulated results show that the Chi-square characteristics are exactly the same if one interchanges the values of μ_0 and ζ_0 between the first and second group for each Chi-square test. The simulated results of an underdamped stochastic Duffing oscillator are given in the third group. From cases 1 to 3 in this group, it is seen that if the GC is less than one, the hypothesis of Gaussian distribution is rejected. From cases 4 and 5, it is seen that although the GC is greater than one, the hypothesis of Gaussian distribution is still rejected. This phenomenon is owing to the fact that $9\mu_0^2 m_{20}^3$ now gives an underestimate of the strength of system nonlinearity. Thus, by applying the GC test to determine whether the Gaussian linearization can be used to predict accurate output variance, it is applicable to a nonlinear system when its coefficients of system nonlinearities are greater than one. When the coefficients of system nonlinearities are less than one, the accurate output variance can be predicted by using the Gaussian linearization approach if the system is not lightly damped as seen from the Gaussian and non-Gaussian response of the moderately damped oscillator in Group 1, case 5 and the lightly damped oscillator in Group 3, case 4, respectively. Next, the non-Gaussian effects and the CVGC for the stochastic parametrically and externally excited Duffing oscillator are investigated. The simulated results are given in Table III with cases 4 and 5 also shown in Fig. 15. It is seen that the combined effects of both parametric and external noise excitations will easily drive the system response into a non-Gaussian region. Also, the non-Gaussian effects of damping noise is stronger than that of spring

TABLE III

THE CSGGF AND GC TESTS FOR THE STOCHASTIC PARAMETRICALLY AND EXTERNALLY EXCITED DUFFING OSCILLATOR

Case	μ_0	ζ_0	$2q_{11}$	$2q_{22}$	$2q_{33}$	GC	Max. value of χ^2	Avg. value of χ^2	No. of χ^2 values > 22.36	Gaussian or Non-Gaussian
1	1.	.5	0.00	0.35	1.0	0.14	59.4	25.0	13	N-G
2	1.	.5	0.00	0.24	1.0	0.34	81.0	25.3	12	N-G
3	1.	.5	0.00	0.25	0.5	0.51	108.3	24.9	10	N-G
4	1.	.5	0.00	0.15	1.0	0.60	62.0	24.2	14	N-G
5	1.	.5	0.20	0.00	1.0	0.78	42.4	22.2	11	G
6	1.	.5	0.00	0.05	1.0	1.05	39.7	21.6	8	G
7	1.	.5	0.00	0.00	1.0	1.36	52.5	21.3	12	G
8	1.	.5	0.34	0.00	0.5	1.46	38.5	20.7	10	G
9	1.	.5	0.00	0.09	0.5	1.50	56.5	21.9	8	G
10	1.	.5	0.00	0.00	0.5	2.70	45.7	21.8	12	G

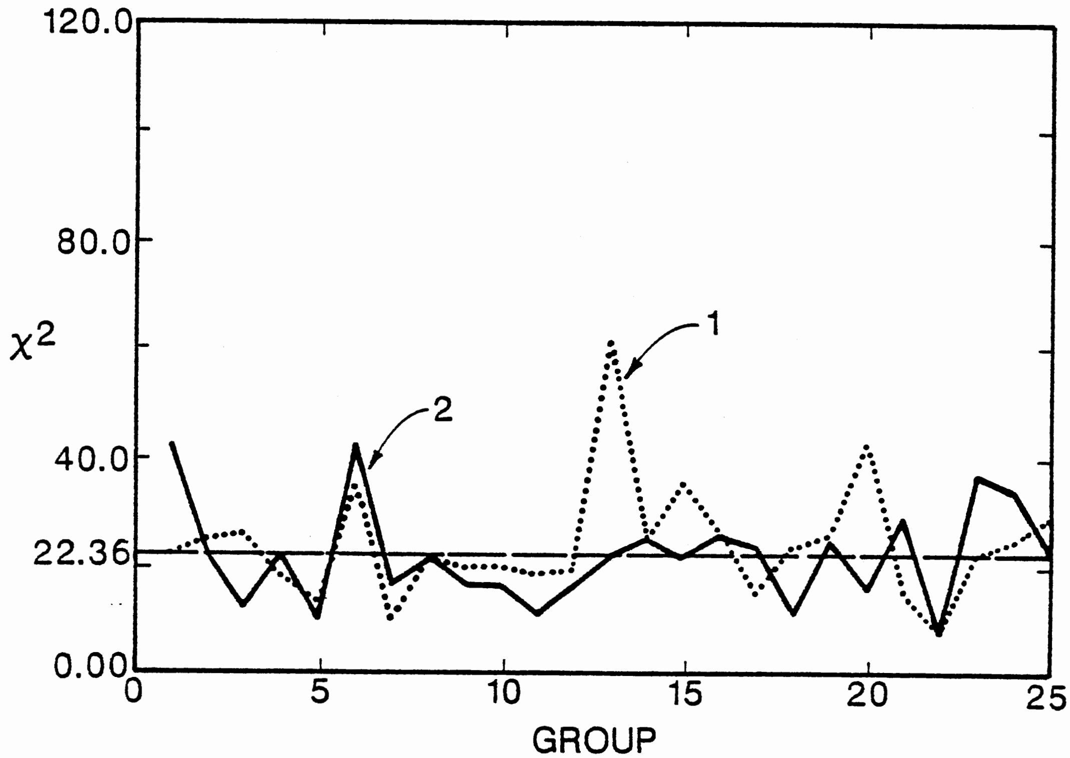

Figure 15. The Chi-square Gaussian Goodness-of-fit Test for Stochastic Parametrically and Externally Excited Duffing Oscillator With μ_0 = 1.0 and ε_0 = 0.5, 1- $2q_{11}$ = 0.0, $2q_{22}$ = 0.15, $2q_{33}$ = 1.0 (GC = 0.60), 2- $2q_{11}$ = 0.2, $2q_{22}$ = 0.0, $2q_{33}$ = 1.0 (GC = 0.78)

noise as shown in Fig. 13. Here, in applying the GC test, the CVGC is defined as 1.0 which is also acceptable to help in determining whether Gaussian linearization can predict accurate output variance.

Summary

The techniques and criterion of Gaussian and non-Gaussian linearization of nonlinear stochastic parametrically and externally excited nonlinear oscillators have been developed in this chapter. The developments of the present non-Gaussian linearization techniques are illustrated by using a nonlinear oscillator with cubic spring through the concepts of equivalent external excitation. The simulated results show that the applications of Gaussian linearization for the nonlinear oscillator provide an underestimate of output variance. By using the present non-Gaussian linearization approach, the accurate prediction of stationary output variance is obtained; however, it is applicable to certain classes of practical nonlinear stochastic systems. Thus, by the extension of the concepts of equivalent external excitation, the Gaussian criterion is established to determine when the accurate prediction of stationary output variance can be obtained by employing the Gaussian linearization techniques. The applications of the present GC test are illustrated by using a stochastic parametrically and externally excited Duffing oscillator. First, it provides the understanding of the strong non-Gaussian effects of the system damping coefficient and of combined system damping and external noise. Then, in the determination of using Gaussian linearization for the accurate prediction of output variance, the present GC test is simply performed by testing whether the GC is greater than the CVGC. The simulated results by using the GC test and CSGGF test, which is performed at 5% level of significance for the state $\dot{x}$, show that the CVGC can be defined as 1.0 when the coefficients of system nonlinearities are greater than one. Thus, with the applications of the concepts of equivalent external excitation, the techniques and criterion of non-Gaussian linearization of nonlinear stochastic systems can be readily developed.

Thus far, the dynamic response of nonlinear stochastic systems has been investigated by utilizing both nonlinearization and linearization techniques. Since the response problem of this type is the problem of the dynamic response of a closed-loop system in the area of control, the concepts and techniques which have been developed are readily applicable to stochastic control problems. Thus, a representative optimal control problem is selected for investigations in the next chapter.

CHAPTER VI

OPTIMAL CONTROL OF STOCHASTIC PARAMETRICALLY AND EXTERNALLY EXCITED NONLINEAR SYSTEMS

The accurate prediction of stationary output variances of the states of nonlinear systems subjected to both stochastic parametric and external excitations by utilizing the equivalent external excitation approach or statistical linearization approach has been presented in Chapters IV and V, respectively. The present chapter will focus on the stochastic optimal control problems. Here, the optimal stochastic control problems are of nonlinear quadratic type with complete state information. We begin with the problem formulation of optimal control of certain nonlinear stochastic systems. The nonlinear controller is synthesized to compensate the system nonlinearities and to incorporate linear state feedback. Then, the linear feedback gains which satisfy the modified Riccati equation and covariance equation are derived. For illustration, a first-order nonlinear stochastic system is first used to demonstrate the performance of the present design. Then, a second-order stochastic parametrically and externally excited Duffing type system is selected to illustrate the applications of statistical linearization techniques to the optimal control of nonlinear stochastic systems and to compare the performance with the present design.

Problem Formulation

Consider an nth-order nonlinear stochastic control system which is interpreted in the Ito sense and satisfies the uniform Lipschitz condition (3.43) and uniform growth conditions (3.45) and is given by

$$
\begin{aligned}
dx_1 &= x_2 dt \\
dx_2 &= x_3 dt \\
&\vdots \\
dx_n &= -\sum_{i=1}^{n} (a_i x_i dt + x_i d\alpha_i) - h(x_1, x_2, \ldots x_n)dt + dW' + udt
\end{aligned}
\qquad (6.1)
$$

where x_1, x_2, $\ldots x_n$ forms the n-dimensional state vector $\underline{x}$, a_i are constants, u is the scalar control input, α_i and W' are independent zero-mean Wiener processes with intensities

$$
E[d\alpha_i \cdot d\alpha_j] = \begin{cases} 0 & , i \neq j \\ 2q_{ii}dt, & i = j = 1, n \end{cases}
\qquad (6.2)
$$

$$
E[dW' \cdot dW'] = 2q_{n+1,n+1}dt
$$

and h is the nonlinear function of states. The control problem is to find a feedback control $u=u(\underline{x})$ which regulates the system (6.1) and minimizes, in steady state, the quadratic cost function

$$
J = \frac{1}{2} E[\underline{x}^T Q \underline{x} + u^2 r] \qquad (6.3)
$$

where Q is a symmetric and positive definite matrix and r is a positive scalar.

The existence of optimal control to this problem is usually assumed. Even if the dynamic programming technique can be applied to derive the Bellman equation (Bryson and Ho, 1969; Jacobson, 1977), it still can be solved for only very special cases by a numerical approach. When the nonlinear systems are only subjected to stochastic external excitation, sub-optimal controllers synthesized by using the Gaussian statistical linearization technique are usually applied. The Gaussian assumption, which is good for systems with weak nonlinearities and/or under weak external noise excitation, will become questionable especially when the nonlinear systems are subjected to both stochastic parametric and external excitations. Actually, even the distributions of states of a second-order linear system of this type are not jointly Gaussian (Dimentberg, 1982). Thus, a nonlinear controller which can be used to compensate the system nonlinearities should overcome the drawback of using the Gaussian linearization techniques.

External Linearization

Although the design of nonlinear controllers by using the external linearization techniques has been developed for deterministic robotic systems (Gibbert and Ha, 1984; Kokotovic, 1985), the applications to the stochastic systems have not been investigated. For a nonlinear control system under Gaussian noise excitations, the stationary covariances of states can be evaluated by using the Gaussian properties if the nonlinear controller is used to compensate the system nonlinearities and the noise intensities follow the concepts of equivalent external excitation (Young and Chang, 1986a). By following these concepts, stochastic parametrically excited systems are reformulated to equivalent externally excited ones through the replacing of the variance of parametric and external noise by the equivalent external variance. As a result, the stochastic parametrically excited nonlinear systems which have been compensated become linear systems with equivalent external excitations and the Gaussian properties are readily applied in the design of nonlinear controllers. Also, noise intensities are not constrained to weak excitations, which are the implicit assumptions of the noise intensities invoked by applying the techniques of Gaussian statistical linearization. The techniques of using the external linearization are given as follows.

Consider the nonlinear function h which can be separated, in general, into noise-free term h_{nf} and noise-dependent term h_{nd} as given by

$$h = h_{nf} + h_{nd} \tag{6.4}$$

By using the nonlinear compensator and incorporating the state feedback

$$u = h_{nf} - \sum_{j=1}^{n} k_j x_j \tag{6.5}$$

the closed-loop system becomes

$$\begin{aligned} dx_1 &= x_2 dt \\ dx_2 &= x_3 dt \\ &\vdots \\ dx_n &= -\sum_{i=1}^{n} ((a_i + k_i) x_i dt + x_i d\alpha_i) - h_{nd} dt + dW' \end{aligned} \tag{6.6}$$

The noise-dependent nonlinearity h_{nd} can be expressed as

$$h_{nd} = \sum_{\ell=1}^{m} d\beta_\ell h_\ell(\underline{x})/dt \tag{6.7}$$

where β_ℓ is a zero-mean Wiener process with intensity

$$E[d\beta_\ell \cdot d\beta_\ell] = 2\, v_{\ell\ell} dt \tag{6.8}$$

The linear state-multiplicative noise excited system with the equivalent external noise excitation of (6.6) is reformulated as

$$\begin{aligned} dx_1 &= x_2 dt \\ dx_2 &= x_3 dt \\ &\vdots \\ dx_n &= -\sum_{i=1}^{n} ((a_i + k_i)x_i dt + x_i d\alpha_i) + dW'' \end{aligned} \tag{6.9}$$

where W" is a Wiener process with the equivalent noise intensity

$$E[dW'' \cdot dW''] = \{2q_{n+1,n+1} + \sum_{\ell=1}^{m} 2v_{\ell\ell}\, E[h_\ell^2]\}\, dt \tag{6.10}$$

Since the stationary density functions by (6.9) can be approximated as the Gaussian ones by following the concepts of equivalent external excitation (Young and Chang, 1986a), the expected value of h_ℓ^2 is then expressed as a function of moments up to the second order as

$$E[h_\ell^2] = f_\ell(P) \tag{6.11}$$

where P is a covariance matrix. From (6.9), the nth-order stochastic system with the initial conditions can be written as a matrix form

$$d\underline{x}(t) = A\underline{x}(t) + \sum_{i=1}^{n} x_i(t) D_i(t) d\underline{\alpha} + \underline{e} d\zeta \tag{6.12}$$

$$\underline{x}(t_0) = \underline{x}_0$$

where $\underline{x}$ is the n-dimensional state vector, and $\underline{\alpha}$ and ζ are zero-mean Wiener processes with the following intensities:

$$E[d\underline{\alpha}\cdot d\underline{\alpha}^T] = \begin{bmatrix} 2q_{11} & & & 0 \\ & 2q_{22} & & \\ & & \ddots & \\ 0 & & & 2q_{nn} \end{bmatrix} dt = Sdt \qquad (6.13)$$

$$E[d\zeta \cdot d\zeta] = [2q_{n+1,n+1} + \sum_{\ell=1}^{m} 2v_{\ell\ell} f_\ell(P)]dt = v(P)dt \qquad (6.14)$$

A, D, $\underline{e}$, and $d\underline{\alpha}$ are given as,

$$A = A' + \underline{b}\,\underline{k}$$

$$A' = \begin{bmatrix} 0 & 1 & & & 0 \\ & \ddots & \ddots & & \vdots \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & 1 \\ -a_1 & -a_2 & \cdots & \cdots & -a_n \end{bmatrix} \quad \underline{b} = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 1 \end{bmatrix} \quad \underline{k}^T = \begin{bmatrix} -k_1 \\ -k_2 \\ \vdots \\ -k_n \end{bmatrix}$$

$$D_1 = \begin{bmatrix} 0 & \cdot & \cdot & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ -1 & & & 0 \end{bmatrix} \quad D_2 = \begin{bmatrix} 0 & \cdot & & \cdot & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \vdots & \vdots \\ 0 & -1 & \cdot & \cdot & 0 \end{bmatrix} \quad \cdots \qquad (6.15)$$

$$D_n = \begin{bmatrix} 0 & \cdot & \cdot & 0 \\ \vdots & \vdots & & \vdots \\ \vdots & & \vdots & \vdots \\ 0 & \cdot & \cdot & -1 \end{bmatrix}$$

$$\underline{e} = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix} \qquad d\underline{\alpha} = \begin{bmatrix} d\alpha_1 \\ d\alpha_2 \\ \vdots \\ d\alpha_n \end{bmatrix}$$

From (6.3) and (6.5) and applying the Gaussian properties for y(P) and $\underline{z}$(P), then one derives

$$J = \frac{1}{2}tr[(Q + \underline{k}^T r\underline{k})P] + r\underline{k}\ \underline{z}(P) + \frac{r}{2}y(P) \tag{6.16}$$

where

$$\underline{z}(p) = \begin{bmatrix} E[x_1 \cdot h_{nf}] \\ E[x_2 \cdot h_{nf}] \\ \vdots \\ E[x_n \cdot h_{nf}] \end{bmatrix} \tag{6.17}$$

$$y(P) = E[h_{nf}^2]$$

The nonlinear control system (6.1) with the quadratic cost function (6.3) becomes an almost linear system (6.12) with non-quadratic performance index (6.16) by using a nonlinear controller (6.5). The problem formulation thus becomes the optimal control with non-quadratic measure as it is given by Yoshida (1984). The covariance equation is then derived by using the Ito's equation as (Maybeck, 1982; Wonham, 1967)

$$\dot{P} = AP + PA + G(P) + \underline{e}v(P)\ \underline{e}^T \tag{6.18}$$

where

$$G(P) = \sum_{i=1}^{n} \sum_{j=1}^{n} (P)_{ij} D_i S D_j^T \qquad (6.19)$$

The optimal control problem is now stated as choosing the optimal feedback gain $\underline{k}$ to minimize J subjected to the constraint (6.18). By using the Lagrange multiplier L to form the Hamiltonian

$$H = tr(\dot{P}L) + \frac{1}{2}tr[(Q+\underline{k}^T r\underline{k})P] + r\underline{k}\ \underline{z}(P) + \frac{r}{2}y(P) \qquad (6.20)$$

the necessary conditions for the minimization of H with respect to $\underline{k}$, L, and P, respectively are expressed as the gradient matrices

$$\frac{\partial H}{\partial \underline{k}} = 0 \qquad (6.21)$$

$$\frac{\partial H}{\partial L} = 0 \qquad (6.22)$$

$$\frac{\partial H}{\partial P} = 0 \qquad (6.23)$$

From (6.21) to (6.23), equations (6.24), (6.25), and (6.26) are derived, respectively.

$$\underline{k} = -2r^{-1}(\underline{b}^T L) - \underline{z}^T P^{-1} \qquad (6.24)$$

$$AP + PA^T + G + \underline{e}v\underline{e}^T = 0 \qquad (6.25)$$

$$\frac{1}{2}(Q+\underline{k}^T r \underline{k}) + LA + A^T L + \Gamma + \Delta + \phi + \psi = 0 \qquad (6.26)$$

where

$$(\Gamma)_{ij} = tr[D_i^T L \, D_j S]$$

$$\Delta = [tr[\underline{e}^T L \underline{e} \, \frac{\partial v(P)}{\partial (P)_{ij}}]]$$

$$\phi = r\underline{k} \, \frac{\partial \underline{z}(P)}{\partial P} \qquad (6.27)$$

$$\psi = \frac{r}{2} \, \frac{\partial y(P)}{\partial P}$$

Here, (6.26) is known as a modified Riccati equation. Finally, the optimal stochastic control problem is given by the solution of the simultaneous nonlinear algebraic equations (6.24) to (6.26). Further, the sufficient and necessary conditions for the mean-square stability are assumed by the appropriate choice of system parameters (Willems, 1976).

Analytical Examples and Discussion

The following first-order control system selected for comparisons is given by Yoshida. The performances of output variance and cost expense are illustrated by using Yoshida's, Beaman's, and the present approach.

Example 1:

A first-order nonlinear control system subjected to stochastic external excitation is given by

$$dx = (-ax^3 + bu)dt + dW' \tag{6.28}$$

with $E[dW' \cdot dW'] = W \cdot dt$

$$J = \frac{1}{2} E[qx^2 + ru^2], \; q, r, > 0 \tag{6.29}$$

a. Yoshida's approach

By applying the statistical linearization technique, Yoshida derives the linear state feedback as

$$u = -\{ -3a \bar{p}_{11}/b + [(q/r) + (9a^2 \bar{p}_{11}^2/b^2)]^{1/2}\}x \tag{6.30}$$

Here, (6.30) is a linear controller; however, he stated that better performance can be achieved by using the following nonlinear controller:

When $q/r >> 9a^2 \bar{p}_{11}^2/b^2$,

$$u \simeq - \{(q/r)^{1/2} - 3a \bar{p}_{11}/b\}x$$

$$\simeq - (q/r)^{1/2}x + (a/b)x^3 \tag{6.31}$$

Note that the above nonlinear controller is derived by interpreting $3\bar{p}_{11}$ x as x^3 in the reverse sense of statistical linearization approach. The closed-loop system is thus given by substituting (6.31) in (6.28)

$$dx = -b(q/r)^{1/2}xdt + dW' \tag{6.32}$$

From (6.32), the stationary output variance of x is

$$p_{11} = W/2(q/r)^{1/2}b \tag{6.33}$$

b. Beaman's approach

Following Beaman's approach, by using the statistical linearization techniques, one derives the linear state feedback as

$$u = -\left(-3a\,\bar{p}_{11}/b + W/2b\,\bar{p}_{11}\right)x \tag{6.34}$$

where $\bar{p}_{11}$ satisfies the following equation

$$(27a^2)\,\bar{p}_{11}^{\,4} + (b^2q/r - 3Wa)\,\bar{p}_{11}^{\,2} - 0.25\,W^2 = 0 \tag{6.35}$$

The closed-loop system now becomes

$$dx = (-ax^3 - bcx)dt + dW' \tag{6.36}$$

where

$$c = -3a\,\bar{p}_{11}/b + W/2b\,\bar{p}_{11} \tag{6.37}$$

The stationary output variance is derived through the FPK equation (3.59) and yields

$$p_{11} = \frac{\int_0^\infty x^2 \exp\{(\frac{2}{W})(-\frac{a}{4}x^4 - \frac{bc}{2}x^2)\}dx}{\int_0^\infty \exp\{(\frac{2}{W})(-\frac{a}{4}x^4 - \frac{bc}{2}x^2)\}dx} \tag{6.38}$$

c. Present approach

By applying the nonlinear compensator and incorporating linear state feedback as

$$u = (-cx + ax^3)/b, \quad c > 0 \tag{6.39}$$

the closed-loop system becomes a linear system subjected to stochastic external excitation as

$$dx = - cxdt + dW' \tag{6.40}$$

Substituting (6.39) into (6.29) and applying the Gaussian properties, one derives

$$J = \frac{1}{2} (q + c^2r/b^2)p_{11} - (3cra/b^2)p_{11}^2 + (15ra^2/2b^2)p_{11}^3 \tag{6.41}$$

From (6.40), the stationary output variance yields

$$p_{11} = W/2c \tag{6.42}$$

The necessary condition of the minimization of J is derived by substituting (6.42) in (6.41) and setting the derivative of J with respect to c equal to zero to yield

$$c^4 + (3aW - qb^2/r)c^2 - 11.25a^2 W^2 = 0 \tag{6.43}$$

Thus, p_{11} is obtained by solving for the positive root of c in (6.43) and substituting this value of c into (6.42).

The comparisons of the performances of the above three approaches are given in Fig. 16 and 17. By choosing a=1.0, q/r=1.0, b=1.0, and with varying W in (6.33), (6.38), and (6.42) to obtain the output variance of p_{11}, then the corresponding cost is evaluated by substituting each p_{11} and the system parameters into (6.29). It is seen that the output variance is greatly reduced with moderate increase in the cost by applying the present nonlinear controller. Actually, the present design will approach Yoshida's nonlinear controller if the intensity of external noise and/or the effect of nonlinearity is negligible. i.e. If we set $W \cdot a=0$ in (6.43) to get $c=b(q/r)$, then the controller of (6.39) becomes (6.31). The second example which is chosen for the present investigation is a nonlinear Duffing type system under both stochastic parametric and external excitations.

Example 2:

A second-order stochastic Duffing system with control input is given as

$$dx_1 = x_2 dt$$

$$dx_2 = -(a_2 x_2 + a_1(d\ x_1 + \varepsilon x_1^3) - u)dt$$

$$-\left(x_2 d\alpha_2 + (d\ x_1 + \varepsilon x_1^3)d\alpha_1\right) \pm dW' \qquad (6.44)$$

where α_1, α_2, and W' are independent zero-mean Wiener processes with intensities $E[d_{\alpha_1} d_{\alpha_1}]=2q_{11}dt$, $E[d_{\alpha_2} d_{\alpha_2}]=2q_{22}dt$, and $E[dW'dW']=2q_{33}dt$, respectively. The cost function is given by (6.16). The nonlinear controller is designed as

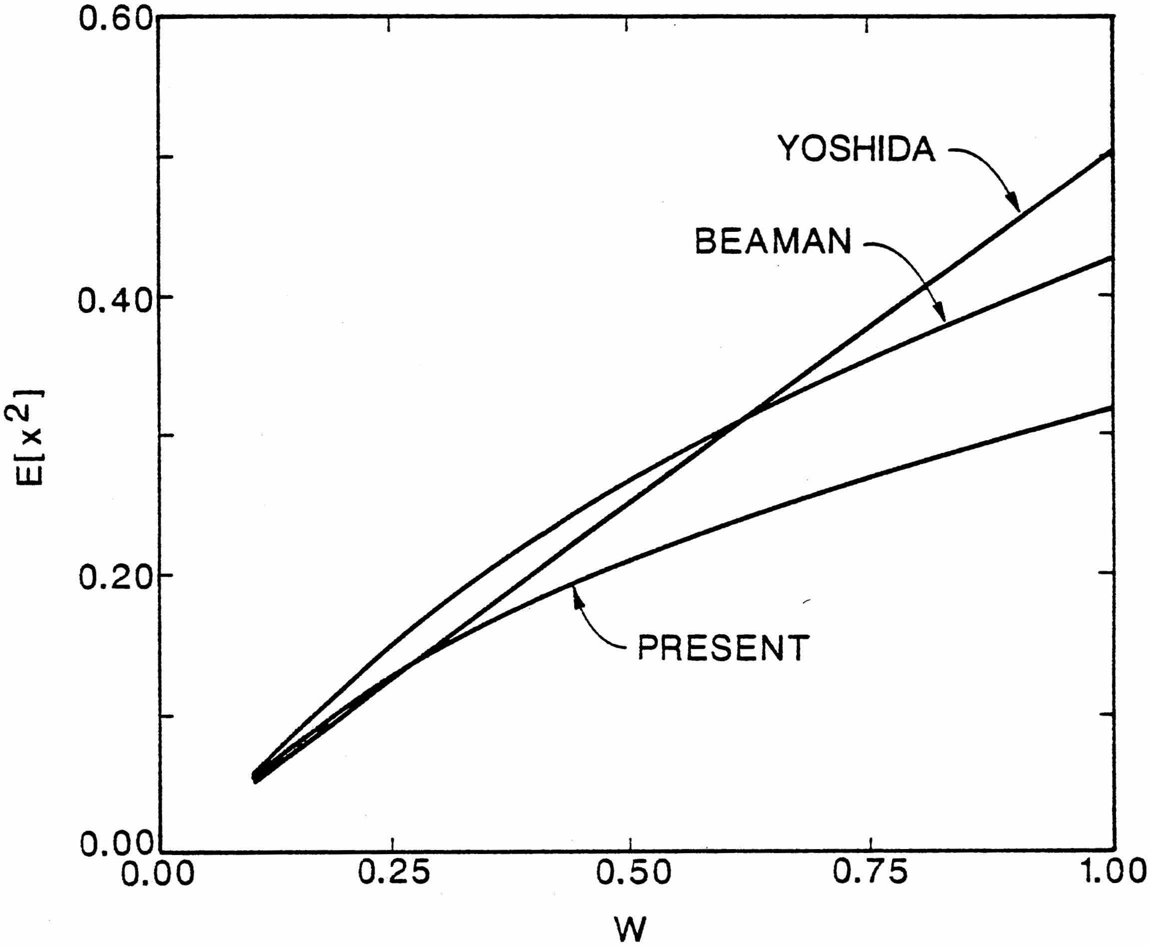

Figure 16. Stationary Output Variance of Control System (6.28) With Varied External Excitation Intensity W by Several Designs

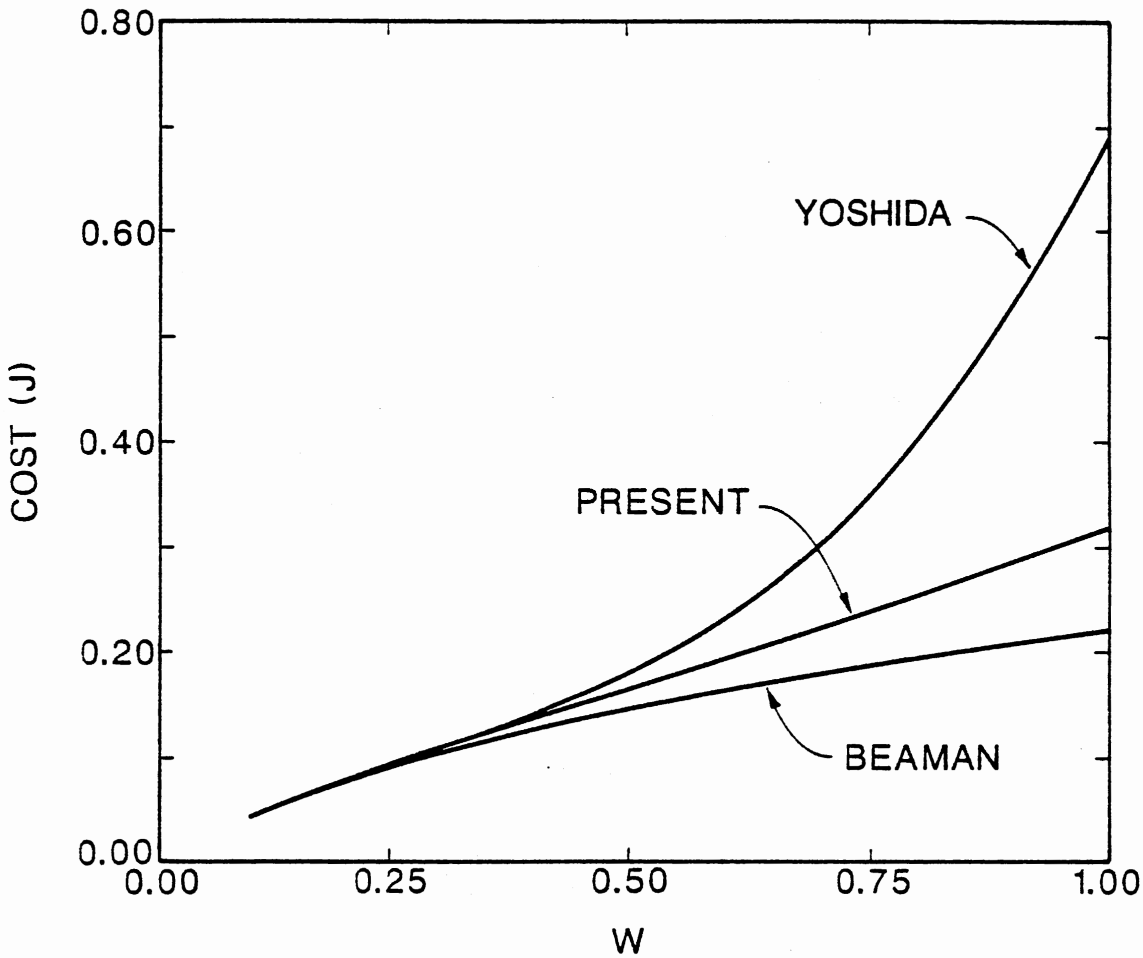

Figure 17. Stationary Cost of Control System (6.28) With Varied External Excitation Intensity W by Several Designs

$$u = a_1 \varepsilon x_1^3 - k_1 x_1 - k_2 x_2 \quad , \quad k_1, k_2 > 0 \tag{6.45}$$

Substituting (6.45) in (6.44) and expressing the closed-loop system in matrix form, one has

$$d\underline{x} = A\underline{x}dt + \sum_{i=1}^{2} x_i D_i d\underline{\alpha} + \underline{e}d\zeta \tag{6.46}$$

where

$$A = A' + \underline{b}\,\underline{k} = \begin{bmatrix} 0 & 1 \\ -a_1 d & -a_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot [-k_1 \;\; -k_2]$$

$$D_1 = \begin{bmatrix} 0 & 0 \\ -d & 0 \end{bmatrix} , \quad D_2 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \tag{6.47}$$

$$v(P)dt = E[d\zeta \cdot d\zeta] = (2q_{33} + 30q_{11}\,\varepsilon^2 p_{11}^3)dt$$

$$Sdt = E\left\{ \begin{bmatrix} d\alpha_1 \\ d\alpha_2 \end{bmatrix} \cdot [d\alpha_1 \;\; d\alpha_2] \right\} = \begin{bmatrix} 2q_{11} & 0 \\ 0 & 2q_{22} \end{bmatrix} dt$$

By substituting (6.47) into (6.24), (6.25), and (6.26), equations (6.48), (6.49), and (6.50) are derived, respectively

$$k_1 = 2\ell_{12}/r + 3\varepsilon a_1 p_{11} \tag{6.48a}$$

$$k_2 = 2\ell_{22}/r \tag{6.48b}$$

$$p_{22} - a_1 d p_{11} - k_1 p_{11} = 0 \tag{6.49a}$$

$$-2a_2p_{22} - 2k_2p_{22} + 2q_{11}d^2p_{11} + 2q_{22}p_{22} + 2q_{33} + 30q_{11}\varepsilon^2p_{11}^3 = 0 \qquad (6.49b)$$

$$(Q_{11} + rk_1^2)/2 - 2a_1d\ell_{12} - 2k_1\ell_{12} + 2d^2q_{11}\ell_{22}$$

$$+ 90\ell_{22}q_{11}\varepsilon^2p_{11}^2 - 6ra_1\varepsilon k_1p_{11} + 22.5ra_1^2\,\varepsilon^2p_{11}^2 = 0 \qquad (6.50a)$$

$$(Q_{22} + rk_2^2)/2 + 2\ell_{12} - 2a_2\ell_{22} - 2k_2\ell_{22} + 2q_{22}\ell_{22} = 0 \qquad (6.50b)$$

The feedback gains of k_1 and k_2 are derived by solving the simultaneous nonlinear algebraic equations (6.48) to (6.50). Specifically, for the nonlinear stochastic system with a_1=a_2=d=ε=1.0, and $2q_{11}$=0, $2q_{22}$=0.4, and $2q_{33}$=2.0, one can choose the appropriate values of Q_{11}, Q_{22}, and r such that the solution of positive feedback gains gives the mean-square stable response. By choosing Q_{11}=Q_{22}=r=1.0 and the system parameters given above, the solution of (6.48) to (6.50) gives p_{11}=0.281, p_{22}=0.635, k_1=1.261, k_2=0.774, l_{12}=0.209, and l_{22}=0.387. Thus, the nonlinear controller is given by

$$u = x_1^3 - 1.261x_1 - 0.774x_2 \qquad (6.51)$$

Gaussian Linearization Approach

The present design is also compared with the linear state feedback control by applying the Gaussian linearization approach. By using the feedback law $u=\underline{k}\cdot\underline{x}$ and k_s=$E[x^3\cdot x]/E[x^2]=3p_{11}$ from (5.22), the closed-loop system becomes

$$dx_1 = x_2dt$$

$$dx_2 = -\big((a_2 + k_2)x_2 + (a_1 d + 3a_1\varepsilon p_{11} + k_1)x_1\big)dt$$

$$-x_2 d\alpha_2 - (d + 3\varepsilon p_{11})x_1 d\alpha_1 + dW' \tag{6.52}$$

The stationary covariance equation is derived by using Ito's equation (3.53) and simplifying to yield

$$p_{22} = 3a_1\varepsilon p_{11}^2 + (a_1 d + k_1)p_{11} \tag{6.53}$$

$$-9q_{11}\varepsilon^2 p_{11}^3 + (3a_1\varepsilon a_2 + 3a_1\varepsilon k_2 - 3a_1\varepsilon q_{22} - 6d\varepsilon q_{11})p_{11}^2$$

$$+ (a_1 a_2 d + a_1 dk_2 - a_1 dq_{22} + a_2 k_1 + k_1 k_2 - k_1 q_{22} - q_{11} d^2)p_{11} - q_{33}$$

$$= 0 \tag{6.54}$$

The cost function is derived by substituting $u=\underline{k}\cdot\underline{x}$ into (6.3) and using (6.53) to derive

$$J = (1.5a_1\varepsilon Q_{22} + 1.5a_1\ \varepsilon r k_2^2)p_{11}^2 + (0.5\ Q_{11} + 0.5rk_1^2 + 0.5Q_{22}a_1 d$$

$$+ 0.5Q_{22}k_1 + 0.5\ ra_1 dk_2^2 + 0.5rk_1 k_2^2)p_{11} \tag{6.55}$$

Forming the Hamiltonian, H, by (6.55) and (6.54) through the Lagrange multiplier λ, one derives (6.56) to (6.59) by using $\partial H/\partial p_{11}=0$, $\partial H/\partial k_1=0, \partial H/\partial k_2=0$, and $\partial H/\partial \lambda=0$, respectively.

$$(3a_1\varepsilon Q_{22} + 3a_1\varepsilon r k_2^2)p_{11} + (0.5Q_{11} + 0.5rk_1^2 + 0.5Q_{22}a_1 d$$

$$+ 0.5Q_{22}k_1 + 0.5ra_1 dk_2^2 + 0.5\ rk_1 k_2^2) \tag{6.56}$$

$$+ \lambda\{(-27\varepsilon^2 q_{11})p_{11}^2 + (6a_1\varepsilon a_2 + 6a_1\varepsilon k_2 - 6a_1\varepsilon q_{22} - 12d\varepsilon q_{11})p_{11}$$

$$+ (a_1 a_2 d + a_1 dk_2 - a_1 dq_{22} + a_2 k_1 + k_1 k_2 - k_1 q_{22} - q_{11} d^2)\} = 0$$

$$rk_1 + 0.5Q_{22} + 0.5rk_2^2 + \lambda(a_2 + k_2 - q_{22}) = 0 \qquad (6.57)$$

$$3a_1\varepsilon rk_2p_{11} + ra_1dk_2 + rk_1k_2 + \lambda(3a_1\varepsilon p_{11} + a_1d + k_1) = 0 \qquad (6.58)$$

$$(-9q_{11}\varepsilon^2)p_{11}^3 + (3a_1\varepsilon a_2 + 3a_1\varepsilon k_2 - 3a_1\varepsilon q_{22} - 6d\varepsilon q_{11})p_{11}^2 \qquad (6.59)$$

$$+ (a_1a_2d + a_1dk_2 - a_1dq_{22} + a_2k_1 + k_1k_2 - k_1q_{22} - q_{11}d^2)p_{11}$$

$$- q_{33} = 0$$

By substituting the parameters given in this example, the simultaneous solution of (6.56) to (6.59) gives p_{11}=0.331, λ=-0.602, k_1=0.163, and k_2=0.602. From (6.53), P_{22} equals 0.711. The linear controller becomes

$$u = -0.163x_1 - 0.602x_2 \qquad (6.60)$$

The comparisons of the output responses by using the present and the linearized design are shown in Fig. 18, 19, and 20. In Fig. 18 and 19, the improvement of the output variances of states by using (6.51) and (6.60) in (6.44), respectively is illustrated through a 2000-run Monte Carlo simulation. Since the parametric noise excitation enters through the damping term only, the cubic spring nonlinearity can be completely compensated and the closed-loop system becomes a linear stochastic parametrically excited system. Thus, the propagation of variance can be directly simulated by using the covariance equation without using Monte Carlo techniques. However, the covariance equation needs to be interpreted in the Stratonovich sense by incorporating the diffusional correction term in considering the fourth-order Runge-Kutta integration algorithm used in the Monte Carlo simulations (Wright, 1974). The improvement of the output variance is also shown in Fig. 20

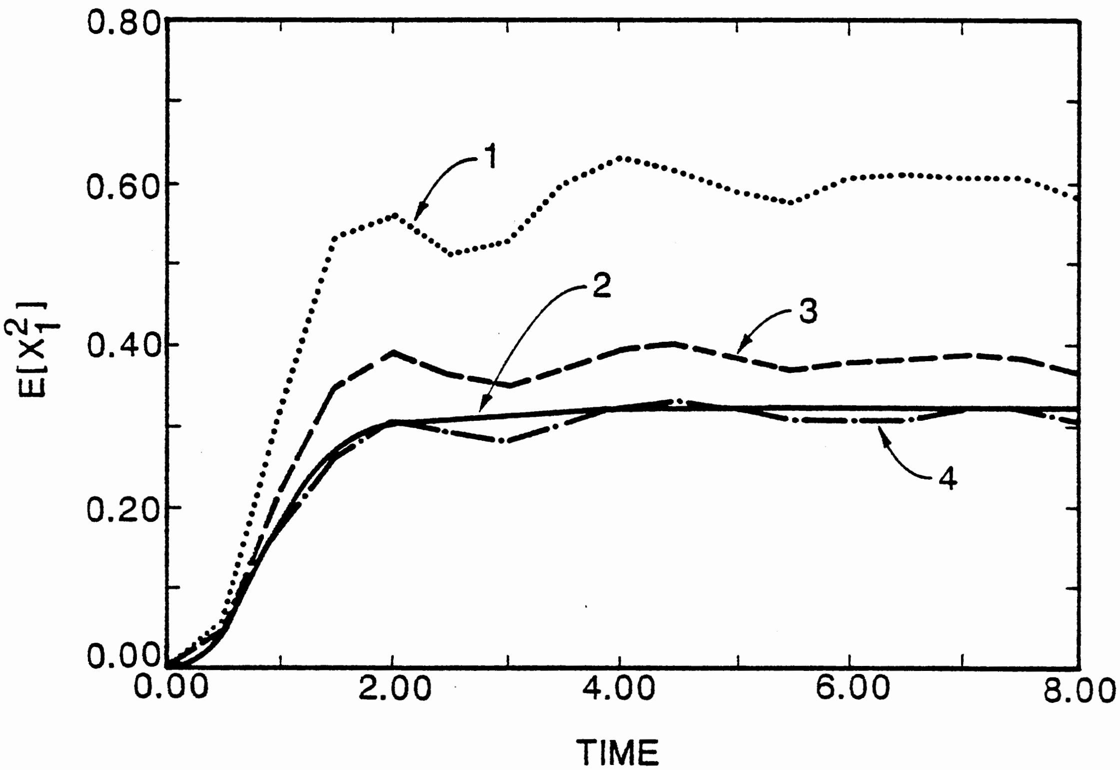

Figure 18. Mean-square Displacement of Duffing-type Control System Under Damping and External Noise Excitations- $2q_{11} = 0.0$, $2q_{22} = 0.4$, $2q_{33} = 2.0$. 1- Without Control, 2- Present Design (Direct Integration), 3- Linearized Design, 4- Present Design

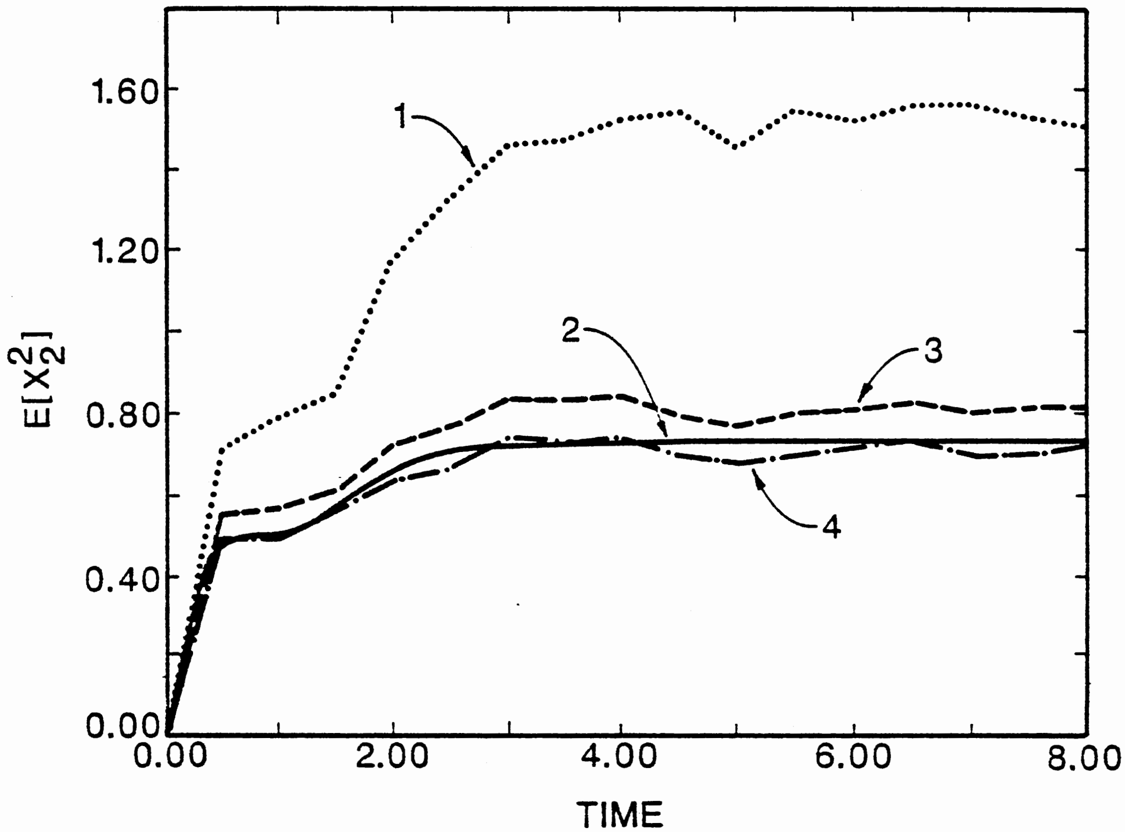

Figure 19. Mean-square Velocity of Duffing-type Control System Under Damping and External Noise Excitations- $2q_{11} = 0.0$, $2q_{22} = 0.4$, $2q_{33} = 2.0$, 1- Without Control, 2- Present Design (Direct Integration), 3- Linearized Design, 4- Present Design

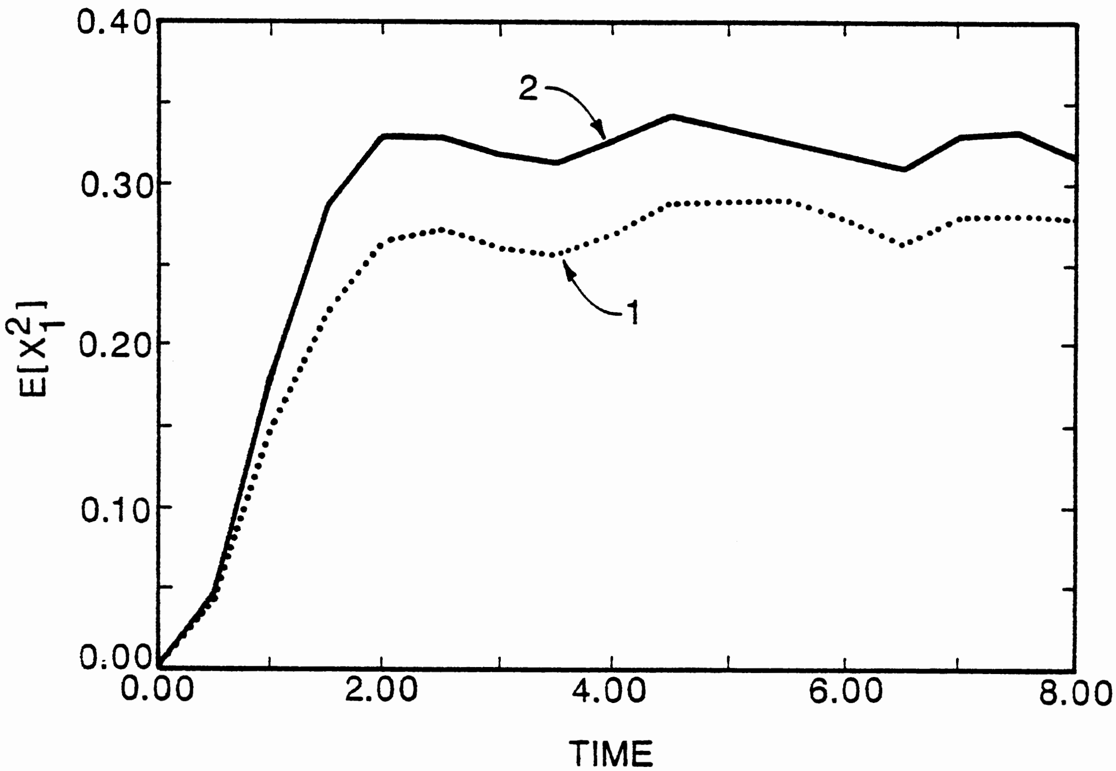

Figure 20. Mean-square Displacement of Duffing-type Control System Under Spring and External Noise Excitations- $2q_{11} = 0.1$, $2q_{22} = 0.0$, $2q_{33} = 2.0$, 1- Present Design, 2- Linearized Design

by the present nonlinear controller through a 1000-run Monte Carlo simulation. It is seen from these figures, the decrease of the output variances over the linearized design is greater than ten percent by using the present approach.

Summary

Nonlinear controllers which are designed by using the external linearization techniques are presented for the optimal control of nonlinear stochastic parametrically and externally excited nonlinear systems. For the externally noise excited nonlinear systems, since the nonlinear controllers are designed to compensate the system nonlinearities, there are no limitations of the strength of system nonlinearities and/or excited noise intensities in applying the Gaussian properties by the present approach. However, it is a concern when linear controllers are designed by using the Gaussian statistical linearization approach. The simulated results of the first-order external noise excited nonlinear system show that the output variance can be reduced with moderate increase in the cost expense by applying the present nonlinear controller design as it is compared with Yoshida's and Beaman's approaches, which follow the concepts of the Gaussian statistical linearization approach. For stochastic parametrically and externally excited nonlinear systems, the concepts of equivalent external excitation are incorporated with the nonlinear compensator to apply the Gaussian properties and derive the matrix feedback gain, covariance, and modified Riccati equation. A second-order Duffing type stochastic parametrically and externally excited system was selected to illustrate the applications of these matrix equations. Also, the concepts of statistical linearization extended from the externally excited nonlinear systems to those which are excited both parametrically and externally are presented and applied to design the sub-optimal linear controller for the Duffing type control system. The simulated results show that the improvement of the output variances of states by applying the present design over the statistical linearization approach is greater than ten percent.

CHAPTER VII

CONCLUSIONS AND RECOMMENDATIONS

Conclusions

In this thesis, the nonlinearization and statistical linearization techniques along with the Gaussian criterion have been developed for accurate prediction of stationary output variance and effective controller design of nonlinear systems excited by both stochastic parametric and external excitations. The development of a new physical concept called equivalent external excitation introduced in Chapter IV have prompted later studies of dynamic response and controller design. The developments of a nonlinearization approach for the accurate prediction of stationary output variances of the states of stochastic parametrically and externally excited nonlinear systems are also presented in this chapter. The techniques are implemented through the concepts of equivalent external excitation by employing matching conditions to adjust the effects of noise intensity. By utilizing the nonlinearization approach, very good agreement exists between results obtained using the present approach and the exact solution or the Monte Carlo simulation over a wide range of parametric and external excitation intensities and with strong system nonlinearities. For the linearization approach, the concepts and techniques of the Gaussian and non-Gaussian linearization for stochastic parametrically and externally excited nonlinear systems are developed in Chapter V. The simulated results show that more accurate stationary output variances are obtained by the present non-Gaussian linearization approach than that by the Gaussian linearization approach. Also, in order to utilize the statistical linearization techniques in the prediction of accurate output variances and design of effective optimal controllers for nonlinear stochastic systems, a deterministic Gaussian criterion is established in this chapter through the concepts and extensions of equivalent external excitation. The simulated results of the Duffing-type oscillator show that the non-Gaussian effects of the damping coefficient and the combined effects of damping and external noise will easily drive the system response into a non-Gaussian region. Furthermore, the critical value of the Gaussian criterion which is used to determine whether the applications of the Gaussian linearization can predict accurate stationary output variances can be defined as 1.0 if the coefficients of the system nonlinearities are greater than one. The validity of utilizing the Gaussian criterion is supported by performing the Chi-square Gaussian goodness-of-fit test. Chapter VI is concerned with the developments of a nonlinearization approach for the optimal control of nonlinear systems subjected to both stochastic parametric and external excitations. Through the concepts of equivalent external excitation and by employing a nonlinear compensator to compensate the system nonlinearities, a sub-optimal nonlinear controller is synthesized. The simulated results show that the improvements of the stationary output variances by the present nonlinear controller over the linearized design by using the Gaussian linearization approach are more than ten percent for the systems under study.

Recommendations

The dynamic response and controller design of stochastic mechanical systems which are described by utilizing lumped-parameter models have been investigated in this thesis. In the area of structural engineering, although the dynamics of structural systems need to be described by employing distributed-parameter models, the distributed-parameter models usually can be 'lumped' into lumped-parameter ones by several approaches and the techniques developed for the lumped-parameter systems can be readily applied. Therefore, in extending existing results to nonlinear stochastic parametrically and externally excited structural systems, there are many avenues which could be taken. What follows are the descriptions of recommended research in three areas which are of immediate concern.

1. Investigate the effects of parametric noise excitation on reliability problems which include level crossing, peak distribution, and first passage time problems. The key step for the level crossing and peak distribution problems is to solve for the probability densities of states of parametric noise excited systems. If stationary, the density functions can be derived by using the equivalent external excitation approach. Thus, the classical work of these problems given by Rice (1944) and its extension (Nigam, 1983) are readily applied. For the first passage time problem, it is very difficult to derive the solution when a stochastic system is nonlinear. One approach is to solve for the survival probability by using a backward Kolmogorov equation through the Galerkin technique (Spanos, 1982). Although Spanos applied this technique to stochastic externally excited nonlinear

systems, the technique can be extended to stochastic parametrically and externally excited systems if the concepts of equivalent external excitation are appropriately incorporated.

2. Modify the method of equivalent external excitation to develop a method to predict the response of stochastic parametrically and externally excited structural elements which include the beam, plate, and shell. This technique will provide the basis for the analysis and design of randomly excited structural systems because a structural system is most commonly modeled as an assembly of structural elements. In implementing this technique, the displacement of a structural element is first expressed as a combination of normal modes multiplied by their corresponding generalized coordinates through the concepts of eigenfunction expansion (Bolotin, 1984). Then, the generalized coordinates are formulated to satisfy the Ito type stochastic differential equations by using the Galerkin approach. Since the Ito type differential equations include noise coefficient terms, the random processes of generalized coordinates are expected to be derived through the technique of equivalent external excitation. Thus, the displacement of a parametric noise excited structural element such as the deformation of a cylindrical shell (Scheurkogel and Elishakoff, 1985) can be readily derived. Completion of this step of the research will enable a more accurate description of structural systems including large space structures than those obtained from methods which utilize more restrictive assumptions.
3. Utilize the concepts of equivalent external excitation and the present nonlinear controller design for the active control of

random vibration of stochastic parametrically and externally excited nonlinear structural systems. This problem is an extension of active control of stochastic external noise excited structures (Aleksander, 1986). Since the techniques for prediction of dynamic response of stochastic parametrically and externally excited structures are proposed in the above, the control problem is proposed after the above response problem has been solved. By using the techniques of solving moment propagation equations and nonlinear controller design described in Chapter VI, the techniques for active control of parametrically excited structural systems will be developed. Due to the close relationship between the prediction of system response and feedback control, once an effective method has been developed for response prediction, controller algorithms can then be developed and verified. This will lead to improved system response of structural systems.

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APPENDIX

LISTING OF COMPUTER PROGRAMS

```
c     +++++++++++++++++++++++++++++++++++++++++++++++++++++
c     +                                                   +
c     +    COMPUTER ALGORITHMS FOR MONTE CARLO SIMULATION   +
c     +   NOTE: THIS PROGRAM IS A MODIFICATION OF MONTE     +
c     +         CARLO PROGRAM GIVEN IN THE COURSE ECEN 5783. +
c     +                                                   +
c     +++++++++++++++++++++++++++++++++++++++++++++++++++++
c2345678
c+++++++++++++++++ MAIN PROGRAM ++++++++++++++++++++++++++++
c IMPORTANT: THE USER MUST FURNISH A SUBROUTINE NAMED SYSEQN +
c              FOR THE SIMULATION OF RANDOM RESPONSE.        +
c++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
      implicit real*8 (a-h,o-z)
      dimension x(2),dx(2),xavg(2,20),xvar(2,20)
      common /blk1/xmean,sig,ix,uprev,xnorm
      common /blk2/mtot,xnum,xavg,xvar
      common /blk3/kutta,dt,nx,x,dx
      common /blk4/ynorm,ymean,siy
      common /blk5/qwc,qwy
      common /blk6/x20,x02
      common/blk7/pmuc,dr
c +++++++++++++++++SYSTEM PARAMETERS ++++++++++++++++++++++
c +   pmuc: spring consatant. dr: damping coefficient.      +
c +++++++++++++++++++++++++++++++++++++++++++++++++++++++++
      write(6,11)
   11 format(1x,'read pmuc and damping ratio')
      read(5,*)pmuc,dr
c +++++++ SET PARAMETERS FOR MONTE CARLO LOOPS ++++++++++++++
c +   nx: no. of states. lt*dt*mtot: simulation time. lt*dt +
c +   is the time step for print. dt: time step. num: Monte +
c +   Carlo run. ix,uprev: initial values for random        +
c +   generator.                                            +
c +++++++++++++++++++++++++++++++++++++++++++++++++++++++++
      nx=2
      lt=10
      mtot=20
      dt=0.05
```

```
      num=500
      ix=31571
      uprev=0.1
c +++++++++++++++ DEFINE GAUSSIAN WHITE NOISE +++++++++++++++
c +   xmean: mean value of external noise. ymean: mean      +
c +   value of parametric noise. qwc: variance of external  +
c +   noise. qwy: variance of parametric noise.             +
c +++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
      xmean=0.0
      ymean=0.0
      write(6,66)
   66 format(1x,'read external and spring noise intensity')
      read(5,*)qwc,qwy
c ++++++++++++++++ CLEAR "xavg and xvar" +++++++++++++++++++
      do 10 i=1,nx
      do 20 j=1,mtot
      xavg(i,j)=0.0
      xvar(i,j)=0.0
  20  continue
  10  continue
      xnum=num
c + CONVERT CONTINUOUS GAUSSIAN WHITE NOISE TO DISCRETE ONE +
      sig=sqrt(qwc/dt)
      siy=sqrt(qwy/dt)
c +++++++++++++++++++ MONTE CARLO LOOPS ++++++++++++++++++++
      do 30 i=1,num
      x(1)=0.0
      x(2)=0.0
c PERFORM INTEGRATIONS AND ACCUMULATE DATA FOR MTOT INTERVALS
      do 40 j=1,mtot
c ++ INTEGRATIONS WITHIN SUBINTERVALS BETWEEN ACCUMULATIONS ++
      do 50 l=1,lt
      call randg
      call rungk
  50  continue
c +++++++++ ACCUMULATE SUMMED AND SUM-SQUARED VALUES +++++++++
      do 70 ni=1,nx
      xavg(ni,j)=xavg(ni,j)+x(ni)
      xvar(ni,j)=xvar(ni,j)+x(ni)*x(ni)
```

```
   70  continue
   40  continue
   30  continue
c PERFORM STATISTICAL COMPUTATIONS FOR ESTIMATES AND PRINT OUT
       call statcp
       open(unit=9,file='duffmc')
       write(9,65)
   65  format(1h1,//////////)
       write(9,75) pmuc,dr,qwc,qwy
   75  format(10x,'pmu=',f8.2,'dr=',f8.2,'exn=',f8.2,'inn=',f8.2)
       write(9,55)
   55  format(2x,'t',11x,'xavg(1)',7x,'xavg(2)',7x,'xvar(1)'
      1,7x,'xvar(2)',/)
       do 80 i=1,mtot
       write(9,85) i*dt*lt,xavg(1,i),xavg(2,i),xvar(1,i),xvar(2,i)
   85  format(2x,f5.2,4(2x,f12.6))
   80  continue
       stop
       end
c
c
c
c +++++++++++++++++ SUBROUTINE SYSEQN ++++++++++++++++++++++++
c +   DYNAMICAL SYSTEM EQUATIONS WITH RANDOM EXCITATIONS       +
c +   NOTE: THE GIVEN EXAMPLE IS A DUFFING OSCILLATOR.         +
c +   x(1),x(2): states. xnorm,ynorm: noise terms.             +
c ++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
       subroutine syseqn
       implicit real*8 (a-h,o-z)
       dimension x(2),dx(2)
       common /blk1/xmean,sig,ix,uprev,xnorm
       common /blk3/kutta,dt,nx,x,dx
       common /blk4/ynorm,ymean,siy
       common/blk7/pmuc,dr
       pmu=pmuc+ynorm
       dx(1)=x(2)
       dx(2)=-dr*x(2)-pmu*x(1)*x(1)*x(1)-x(1)+xnorm
       return
       end
```

```
c
c
c
c +++++++++++++++++++ SUBROUTINE RANDG ++++++++++++++++++++++
c +      MULTIPLICATIVE PSEUDO-RANDOM NUMBER GENERATOR        +
c +   XNORM AND YNORM ARE GAUSSIANLY DISTRIBUTED.             +
c +   U IS UNIFORMLY DISTRIBUTED.                             +
c +   THE BOX-MULLER TRANSFORMATION IS USED TO CONVERT FROM   +
c +   UNIFORM TO GAUSSIAN DISTRIBUTION.                       +
c +++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
      subroutine randg
      implicit real*8 (a-h,o-z)
      common /blk1/xmean,sig,ix,uprev,xnorm
      common /blk4/ynorm,ymean,siy
      iy=1366853*ix
      iyp=iy/2147483647
      ix=iy-iyp*2147483647
      ax=ix
      u=ax/2147483647.
      if(u) 5,5,6
  5   u=-u
  6   continue
      ix=iy
      aaa=-2.0*dlog(uprev)
      y=sqrt(aaa)*sig
      z=sqrt(aaa)*siy
      xnorm=y*cos(6.28318*u)+xmean
      ynorm=z*sin(6.28318*u)+ymean
      uprev=u
      return
      end
c
c
c
c ++++++++++++++++ SUBROUTINE STATCP +++++++++++++++++++
c +     UNBIASED ESTIMATES OF THE MEAN AND VARIANCE      +
c +++++++++++++++++++++++++++++++++++++++++++++++++++++++
      subroutine statcp
      implicit real*8 (a-h,o-z)
```

```
      dimension xavg(2,20),xvar(2,20),x(2),dx(2)
      common /blk2/mtot,xnum,xavg,xvar
      common /blk3/kutta,dt,nx,x,dx
      b1=1.0/xnum
      b2=1.0/(xnum-1.0)
      do 10 i=1,mtot
      do 20 j=1,nx
      xavg(j,i)=xavg(j,i)*b1
      xvar(j,i)=b2*(xvar(j,i)-xnum*xavg(j,i)*xavg(j,i))
 20   continue
 10   continue
      return
      end
c
c
c
c +++++++++++++++++ SUBROUTINE RUNGK +++++++++++++++++++++
c +         FOURTH-ORDER RUNGE-KUTTA INTEGRATION           +
c ++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
      subroutine rungk
      implicit real*8 (a-h,o-z)
      dimension x(2),dx(2),xa(2),dxa(2)
      common/blk1/xmean,sig,ix,uprev,xnorm
      common /blk3/kutta,dt,nx,x,dx
      call syseqn
 10   hdt=0.5*dt
      do 20 i=1,nx
      xa(i)=x(i)
      dxa(i)=dx(i)
      x(i)=x(i)+hdt*dx(i)
 20   continue
      call syseqn
 30   do 40 i=1,nx
      dxa(i)=dxa(i)+dx(i)+dx(i)
      x(i)=xa(i)+hdt*dx(i)
 40   continue
      call syseqn
 50   do 60 i=1,nx
      dxa(i)=dxa(i)+dx(i)+dx(i)
```

```
      x(i)=xa(i)+dt*dx(i)
60    continue
      call syseqn
70    vdt=dt*0.1666667
      do 80 i=1,nx
      x(i)=xa(i)+vdt*(dxa(i)+dx(i))
80    continue
100   continue
      return
      end
```

```
c      ++++++++++++++++++++++++++++++++++++++++++++++++++
c      +                                                +
c      +      COMPUTER ALGORITHMS FOR CHI-SQUARE TEST     +
c      +    NOTE: THIS PROGRAM IS A MODIFICATION OF CHI-  +
c      +          SQUARE PROGRAM GIVEN IN THE COURSE      +
c      +          ECEN 5783.                              +
c      +                                                +
c      ++++++++++++++++++++++++++++++++++++++++++++++++++
c2345678
c +++++++++++++++++ MAIN PROGRAM ++++++++++++++++++++++
c  IMPORTANT: THE USER MUST FURNISH A SUBROUTINE NAMED   +
c             SYSEQN FOR THE CHI-SQUARE TEST OF         +
c             STATIONARY RESPONSE OF SYSTEM STATES.     +
c ++++++++++++++++++++++++++++++++++++++++++++++++++++++
      implicit real*8 (a-h,o-z)
      dimension xavg(2,20),xvar(2,20)
      dimension x(2),dx(2)
      dimension xd(6001)
      dimension a(16)
      dimension xnob(200)
      real*4 t(20),y(20),z(20),w(20),v(20)
      common /blk1/xmean,sig,ix,uprev,xnorm
      common /blk3/kutta,dt,nx,x,dx
      common /blk4/ynorm,ymean,siy
      common /blk5/qwc,qwy
      common /blk6/x20,x02
      common/blk7/pmuc,dr
c +++++++++++++++++ SYSTEM PARAMETERS ++++++++++++++++++++
c +   pmuc: spring constant. dr: damping coefficient.     +
c +++++++++++++++++++++++++++++++++++++++++++++++++++++++
      write(6,11)
  11  format(1x,'Read spring const. and damping coeff.')
      read(5,*) pmuc,dr
c +++++++++++++++ INPUT NOISE INTENSITIES +++++++++++++++
c +   qwc: variance of external noise. qwy: variance of   +
c +   parametric noise.                                   +
c +++++++++++++++++++++++++++++++++++++++++++++++++++++++
```

```
      write(6,66)
  66  format(1x,'Read external and spring noise intensity')
      read(5,*) qwc,qwy
c +++++++++++ SET PARAMETERS FOR CHI-SQUARE TEST ++++++++++
c +   ngroup: no. of groups(times) for chi-square test.   +
c +   nsamp: no. of samples in one group. nfi: no. of     +
c +   frequency intervals (degrees of freedom = nfi - 1)  +
c +   lt: total samples (stationary samples + transient   +
c +   samples)                                            +
c ++++++++++++++++++++++++++++++++++++++++++++++++++++++++
      nsamp=200
      ngroup=25
      igroup=0
      nfi=16
      lt=nsamp*ngroup+200
      lt1=lt+1
c ++++++++ SET PARAMETERS FOR MONTE CARLO SIMULATION +++++
      dt=0.1
      nx=2
      xmean=0.0
      ymean=0.0
      ix=31571
      uprev=0.1
      sig=sqrt(qwc/dt)
      siy=sqrt(qwy/dt)
      x(1)=0.0
      x(2)=0.0
      lc=0
      do 40 l=1,lt
      call randg
      call rungk
c ++++++++++ NEGLECT THE TRANSIENT RESPONSE +++++++++++++
      if(l.le.200) go to 40
      xd(l-200)=x(2)
      lc=lc+1
   40 continue
      open(unit=9,file='chiduf')
      open(unit=10,file='dataduf')
c ++++++++++++++ PERFORM CHI-SQUARE TEST +++++++++++++++
```

```
      chisum=0.0
      do 800 ij=1,ngroup
      igroup=igroup+1
      write(9,903) igroup
 903  format(1h0,5x,'igroup=',i3)
      do 77 ni=1,nsamp
      nig=nsamp*(ij-1)
      xnob(ni)=xd(ni+nig)
  77  continue
c +++++++++++++++ FIND THE MEAN AND VARIANCE ++++++++++++
      call ameva(nsamp,xnob,xb,va)
      write(9,901) xb,va
 901  format(10x,'xbar=',f10.6,5x,'var=',f10.6)
c +++++++++++++++ FREQUENCY COUNTER ++++++++++++++++++++
      call afc(xnob,nsamp,nfi,a,xb,va)
c ++++++++++++ CALCULATE THE CHI-SQUARE VALUE +++++++++++
      call chites(nsamp,a,nfi,chi)
      chisum=chi+chisum
      write(9,905) chi
 905  format(1h0,5x,'chi=',e9.3)
      write(10,*) igroup,chi
 800  continue
      write(9,65)
  65  format(1h1,//////////)
      write(9,75) pmuc,dr,qwc,qwy
  75  format(10x,'pmu=',f8.2,'dr=',f8.2,'exn=',f8.2,'inn=',f8.2)
      chiav=chisum/ngroup
      write(9,80) chiav
  80  format(1x,'chiave=',f8.2)
      stop
      end
c
c
c +++++++++++++++ SUBROUTINE SYSEQN ++++++++++++++++++++++
c +   DYNAMICAL SYSTEM EQUATIONS WITH RANDOM EXCITATIONS  +
c ++++++++++++++++++++++++++++++++++++++++++++++++++++++++
      subroutine syseqn
      implicit real*8 (a-h,o-z)
      dimension x(2),dx(2)
```

```
      common /blk1/xmean,sig,ix,uprev,xnorm
      common /blk3/kutta,dt,nx,x,dx
      common /blk4/ynorm,ymean,siy
      common/blk7/pmuc,dr
      pmu=pmuc+ynorm
      dx(1)=x(2)
      dx(2)=-dr*x(2)-pmu*x(1)*x(1)*x(1)-x(1)+xnorm
      return
      end
c
c
c ++++++++++++++++ SUBROUTINE RANDG ++++++++++++++++++++
c +     MULTIPLICATIVE PSEUDO-RANDOM NUMBER GENERATOR   +
c +++++++++++++++++++++++++++++++++++++++++++++++++++++++
      subroutine randg
      implicit real*8 (a-h,o-z)
      common /blk1/xmean,sig,ix,uprev,xnorm
      common /blk4/ynorm,ymean,siy
      iy=1366853*ix
      iyp=iy/2147483647
      ix=iy-iyp*2147483647
      ax=ix
      u=ax/2147483647.
      if(u) 5,5,6
  5   u=-u
  6   continue
      ix=iy
      aaa=-2.0*dlog(uprev)
      y=sqrt(aaa)*sig
      z=sqrt(aaa)*siy
      xnorm=y*cos(6.28318*u)+xmean
      ynorm=z*sin(6.28318*u)+ymean
      uprev=u
      return
      end
c
c
c ++++++++++++++ SUBROUTINE RUNGK +++++++++++++++++++
c +     FOURTH-ORDER RUNGE-KUTTA INTEGRATION         +
```

```
c ++++++++++++++++++++++++++++++++++++++++++++++++++++
      subroutine rungk
      implicit real*8 (a-h,o-z)
      dimension x(2),dx(2),xa(2),dxa(2)
      common/blk1/xmean,sig,ix,uprev,xnorm
      common /blk3/kutta,dt,nx,x,dx
      call syseqn
  10  hdt=0.5*dt
      do 20 i=1,nx
      xa(i)=x(i)
      dxa(i)=dx(i)
      x(i)=x(i)+hdt*dx(i)
  20  continue
      call syseqn
  30  do 40 i=1,nx
      dxa(i)=dxa(i)+dx(i)+dx(i)
      x(i)=xa(i)+hdt*dx(i)
  40  continue
      call syseqn
  50  do 60 i=1,nx
      dxa(i)=dxa(i)+dx(i)+dx(i)
      x(i)=xa(i)+dt*dx(i)
  60  continue
      call syseqn
  70  vdt=dt*0.1666667
      do 80 i=1,nx
      x(i)=xa(i)+vdt*(dxa(i)+dx(i))
  80  continue
  100 continue
      return
      end
c
c
c ++++++++++++++++ SUBROUTINE AMEVA +++++++++++++++++++++
c +   UNBIASED ESTIMATES OF THE MEAN AND VARIANCE       +
c +++++++++++++++++++++++++++++++++++++++++++++++++++++++
      subroutine ameva(nsamp,xnob,xbar,var)
      implicit real*8 (a-h,o-z)
      dimension xnob(nsamp)
```

```
      xbsum=0.
      do 100 j=1,nsamp
      xbsum=xbsum+xnob(j)
 100  continue
      xbar=xbsum/nsamp
      sumvar=0.0
      do 200 k=1,nsamp
      sumsg=(xnob(k)-xbar)**2
      sumvar=sumvar+sumsg
 200  continue
      var=sumvar/(nsamp-1)
      return
      end
c
c
c
c ++++++++++++++++ SUBROUTINE AFC ++++++++++++++++++++
c +             AUTOMATIC FREQUENCY COUNTER             +
c ++++++++++++++++++++++++++++++++++++++++++++++++++++++
      subroutine afc(xnob,nsamp,nfi,a,xb,va)
      implicit real*8 (a-h,o-z)
      dimension achi(16)
      dimension a(nfi)
      dimension xnob(nsamp)
      achi(1)=-1.530
      achi(2)=-1.150
      achi(3)=-0.89
      achi(4)=-0.68
      achi(5)=-0.49
      achi(6)=-0.34
      achi(7)=-0.16
      achi(8)=0.0
      achi(9)=0.16
      achi(10)=0.34
      achi(11)=0.49
      achi(12)=0.68
      achi(13)=0.89
      achi(14)=1.15
      achi(15)=1.53
```

```
      achi(16)=5.0
      do 30 i=1,nfi
      a(i)=0.
 30   continue
      do 300 mmm=1,nsamp
      xnor=(xnob(mmm)-xb)/sqrt(va)
      if(xnor.lt.achi(1)) go to 301
      if(xnor.lt.achi(2)) go to 302
      if(xnor.lt.achi(3)) go to 303
      if(xnor.lt.achi(4)) go to 304
      if(xnor.lt.achi(5)) go to 305
      if(xnor.lt.achi(6)) go to 306
      if(xnor.lt.achi(7)) go to 307
      if(xnor.lt.achi(8)) go to 308
      if(xnor.lt.achi(9)) go to 309
      if(xnor.lt.achi(10)) go to 310
      if(xnor.lt.achi(11)) go to 311
      if(xnor.lt.achi(12)) go to 312
      if(xnor.lt.achi(13)) go to 313
      if(xnor.lt.achi(14)) go to 314
      if(xnor.lt.achi(15)) go to 315
      if(xnor.lt.achi(16)) go to 316
301   a(1)=a(1)+1
      go to 300
302   a(2)=a(2)+1
      go to 300
303   a(3)=a(3)+1
      go to 300
304   a(4)=a(4)+1
      go to 300
305   a(5)=a(5)+1
      go to 300
306   a(6)=a(6)+1
      go to 300
307   a(7)=a(7)+1
      go to 300
308   a(8)=a(8)+1
      go to 300
309   a(9)=a(9)+1
```

```
      go to 300
 310  a(10)=a(10)+1
      go to 300
 311  a(11)=a(11)+1
      go to 300
 312  a(12)=a(12)+1
      go to 300
 313  a(13)=a(13)+1
      go to 300
 314  a(14)=a(14)+1
      go to 300
 315  a(15)=a(15)+1
      go to 300
 316  a(16)=a(16)+1
      go to 300
 300  continue
      return
      end
c
c
c ++++++++++++ SUBROUTINE CHITES ++++++++++++++++++++++
c +         CALCULATE THE CHI-SQUARE VALUES            +
c +++++++++++++++++++++++++++++++++++++++++++++++++++++
      subroutine chites(nsamp,a,nfi,chi)
      implicit real*8 (a-h,o-z)
      dimension aa(16),a(nfi)
      aa(1)=0.063
      aa(2)=0.0621
      aa(3)=0.0616
      aa(4)=0.0616
      aa(5)=0.0638
      aa(6)=0.0548
      aa(7)=0.0695
      aa(8)=0.0636
      aa(9)=0.0636
      aa(10)=0.0695
      aa(11)=0.0548
      aa(12)=0.0638
      aa(13)=0.0616
```

```
      aa(14)=0.0616
      aa(15)=0.0621
      aa(16)=0.0630
      chi=0.0
      do 400 i=1,nfi
      chi=chi+((aa(i)*nsamp)-a(i))**2/(aa(i)*nsamp)
400   continue
      return
      end
```

VITA

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Doctor of Philosophy

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