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A COUPLED CHEMO-THERMO-POROELASTIC SYSTEM:
WELL-POSEDNESS AND APPROXIMATION
FOR CONTROL APPLICATIONS

A DISSERTATION APPROVED FOR THE
DEPARTMENT OF MATHEMATICS

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Table of Contents

List of Tables

List of Figures

Abstract

This dissertation addresses well-posedness and approximation problems for coupled parabolic-elliptic systems with applications to geomechanics. This work is motivated by problems of borehole stability in porous formations that involve the modeling of fully coupled thermal, chemical, hydraulic, and mechanical processes. The underlying thermo-chemo-poroelastic (TCPu) model is a system of time dependent parabolic equations modeling thermal, solute, and fluid diffusions coupled with Navier-type elliptic equations that attempt to capture the elastic behavior of rock around the borehole. The obtained results are fundamental to posing optimal control problems using borehole parameters as controls to achieve desired stress distributions in the neighborhood of the borehole.

Sufficient conditions for well-posedness (in the sense of Hadamard) of the coupled parabolic-elliptic initial-boundary value problems are obtained for the two- and three-dimensional TCPu models. These results provide a mathematical basis for the general theory of fully coupled chemical thermo-poroelasticity in homogeneous isotropic porous media saturated by a fluid.

Numerical methods for solving fully coupled parabolic-elliptic initial-boundary value problems for two- and three-dimensional TCPu processes, including radially non-symmetric processes in an inclined borehole, are developed. These methods are based on a hybrid Fourier-finite-element approximation technique proposed in this research. The technique also involves a boundary penalization strategy and captures borehole geometry and mechanics. The numerical methods allow numerical analysis of a control problem in which the temperature and pressure on the borehole boundary are considered as control parameters used to achieve desirable stresses in the neighborhood of the borehole.

Chapter 1

Introduction

Natural systems are usually complex and composed of numerous interacting parts. Such interactions require coupled models that, in turn, create a need for advanced mathematical analysis and control techniques. In this dissertation, we consider initial-boundary value problems for coupled parabolic-elliptic systems with applications to geomechanics. This work is motivated by the problems of borehole stability in porous formations saturated by a fluid that involve the modeling of fully coupled thermal, chemical, hydraulic, and mechanical (elastic) processes.

The poroelasticity theory describing the coupled processes of elastic deformation and pore fluid diffusion in fluid-saturated isothermal porous media can be traced back to the pioneering works of von Terzaghi [41, 42] (1925, 1943) and Biot [5] (1941). In 1956, Biot [6] pointed out a complete mathematical analogy between poroelasticity and thermoelasticity with the temperature playing the same role as the fluid pressure and heat conduction corresponding to fluid flow. A complete mathematical analysis of the well-posedness of a general initial-boundary value problem for a system of coupled partial differential equations that describes the Biot consolidation model [5] in poroelasticity, as well as a coupled quasi-static problem in thermoelasticity, has been carried out by Showalter [37, 38]. The corresponding Biot consolidation model has the form

$$-(\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}(x,t)) - \mu\Delta\mathbf{u}(x,t) + \alpha\nabla p(x,t) = \mathbf{f}(x,t) \tag{1.0.1}$$

$$\frac{\partial}{\partial t}(c_0 p(x,t) + \alpha\nabla\cdot\mathbf{u}(x,t)) - \nabla\cdot k\nabla p(x,t) = h(x,t) \tag{1.0.2}$$

consisting of the equilibrium equation for momentum (1.0.1) and the diffusion equation for Darcy flow (1.0.2). Here $\mathbf{u}(x,t)$ is the displacement of the stucture; $p(x,t)$

is the fluid pore pressure; the positive Lamé constants λ and μ are the dilation and shear moduli of elasticity, respectively; $\alpha > 0$ is the Biot-Willis constant that accounts for the pressure-deformation coupling; the constant $c_0 \geq 0$ combines the porosity of the medium and the compressibility of the fluid; $k > 0$ involves the permeability of the medium and the viscosity of the fluid; f and g are given data, typically the volume-distributed external forces and the volume-distributed source density, respectively. In the context of the coupled thermoelasticity system, $p(x,t)$ denotes the temperature; $c_0 > 0$ is the specific heat of the medium; and $k > 0$ is the conductivity.

Based on the theory of linear degenerate evolution equations in Hilbert spaces, the existence and uniqueness of strong and weak solutions, as well as regularity theory for the system (1.0.1)-(1.0.2) supplemented by initial and boundary conditions, were developed. The earlier works on the mathematical issues of analysis and numerical approximation for the Biot consolidation model include [2, 7, 8, 29, 44, 45, 47]. The a priori analysis of Euler-Galerkin approximations for the Biot consolidation problem is presented in the works of Murad, Loula, and Thomée [30, 31, 32], including the semi-discrete and fully discrete cases and long-time behavior; and the a posteriori error analysis was developed by Ern and Meunier [17].

In 1971, Schiffman [35] first extended the Biot theory for non-isothermal systems to account for the effects of thermal expansion of both the pore fluid and the elastic matrix. Since then, a substantial literature on thermo-poroelasticity theory and the modeling of the coupled hydro-thermo-mechanical behavior of a fluid-saturated porous media has been developed for geomechanics, including petroleum and geothermal borehole stability [1, 4, 10, 14, 23, 24, 26, 28, 33, 39, 43]. Considerable attention has been recently placed on the impact of chemical processes in porous media on drilling and borehole stability. Ghassemi and Diek [20] developed a linear chemo-poroelastic model that couples solute transfer in the mud/shale system to stress and pore pressure within the framework of a Biot-like isotropic poroelastic theory. Thereafter, Zhou and Ghassemi [46] and Rafieepour et al. [34] presented linear and nonlinear chemo-thermo-poroelastic models, respectively. However, all those models are not rigorously developed from the first and second laws of thermodynamics. In 2011, Diek et al. [12, 13, 11] first presented fully coupled thermo-poroelastic and chemo-thermo-poroelastic models for drilling in geological formations that satisfy the first and second laws of thermodynamics.

Due to the complexity of cross-coupling mechanisms involved in thermo-poroelastic and chemo-thermo-poroelastic models, the majority of corresponding initial-boundary value problems are solved by various numerical and computational techniques [18, 19, 20, 25, 34]. Very few analytical solutions of the fully coupled thermo-poroelastic and chemo-poroelastic equations are currently available. Typically, the solutions are derived under the assumptions that some of the couplings can be neglected [10, 23, 24, 28]. For instance, in the works of Kodashima and Kurashige [23, 24], the displacement field is decoupled from the temperature and pore fluid pressure fields; and in Coussy [10], an analytical solution is presented under the assumption that the temperature equation can be decoupled and approximated by a purely diffusive equation. Belotserkovets and Prevost [4], using the Laplace transformation and the residue theorem, developed an analytical method and presented an exact unique solution for a specific case of the fully coupled thermo-hydro-mechanical response of a fluid saturated porous sphere under mechanical pulse load.

In this dissertation, we study the model that is based on equations derived by Diek [11] and constitutes the general theory of fully coupled chemo-thermo-poroelasticity in homogeneous isotropic porous media saturated by a fluid. The theory satisfies the first and second laws of thermodynamics and is based on concepts of irreversible thermodynamics, a novel rock constitutive relation, and Onsager's transport phenomenology [11]. Since the model is derived from physical laws, it requires a mathematical basis for its well-posedness. On the other hand, the geomechanical nature of the problem demands analytical and numerical methods that capture both the geometry and the physical background of the problem. With this in mind, the aims of the dissertation are to establish the mathematical theory of well-posedness for coupled linear parabolic-elliptic systems modeling fully coupled mechanical, hydraulic, chemical, and thermal processes in homogeneous isotropic media and to develop and experimentally validate numerical methods for approximation and boundary control problems in geomechanical applications.

The dissertation is organized in chapters as follows. Chapter 2 describes the underlying fully coupled thermo-chemo-poroelastic (TCPu) model for fluid-saturated porous rock formations and formulates the corresponding coupled parabolic-elliptic initial-boundary value problems for two- and three-dimensional regions. Chapter 3 addresses the well-posedness theory for the initial-boundary value problems introduced in Chapter 2. Chapter 4 develops numerical methods for approximation and boundary control problems for the two- and three-dimensional TCPu models. Chapter 5 presents experimental validation of the numerical methods developed in Chapter 4. Chapter 6 contains the conclusions and potential directions for further research.

Chapter 2

Coupled TCPu System: Underlying Equations

The underlying TCPu model [11] is a system of time-dependent parabolic partial differential equations (PDEs) coupled with Navier-type elliptic PDEs with time, $t \in (0, t_f)$, as a parameter. The parabolic equations represent heat, solute, and fluid diffusions, and the Navier-type elliptic equations attempt to capture the elastic behavior of rock, while incorporating thermal, chemical, and porous media effects. The equations are developed in terms of the following independent variables: the absolute temperature T, the solute mass fraction C, the pore pressure p, and the vector of solid displacements $\vec{u}$. The coupled partial differential equations supplemented by the appropriate initial and boundary conditions constitute an initial-boundary value problem defined in a region exterior to the borehole. Although this region is infinite, it is specified with a far-field radius such that, without loss of generality, we may assume that, on the far-field boundary, the absolute temperature, solute mass fraction, pore pressure, and displacements are time-independent, and the displacement is negligibly small.

2.1 Two-dimensional Model

The initial-boundary value problem that constitutes our model is defined in a region Ω in $\mathbb{R}^2$ exterior to the borehole. We assume that Ω is a bounded open domain with a sufficiently smooth boundary Γ. The boundary Γ is expressed as the union of disjoint sets Γ_B and Γ_F having nonempty interiors relative to the set Γ. Specifically, $\Omega \subset \mathbb{R}^2$ is an annular region with an inner radius R_B and an outer radius R_F.

Let (X, Y) be a borehole Cartesian coordinate system such that

x -axis is in the direction of maximum horizontal principal stress σ_H and

y -axis is in the direction of minimum horizontal principal stress σ_h.

A point $\bar{x}$ in $\mathbb{R}^2$ is associated with position vector $\bar{x} = x\vec{i} + y\vec{j}$, where $\vec{i}$ and $\vec{j}$ are the basis vectors of the Cartesian coordinate system (X, Y). Displacements of points within Ω are described in terms of a vector-valued displacement function of the position $\bar{x}$ and time t as $\vec{u}(\bar{x}, t) = u(\bar{x}, t)\vec{i} + v(\bar{x}, t)\vec{j}$.

The boundary Γ of Ω is $\Gamma = \Gamma_B \cup \Gamma_F$, where

$$\Gamma_B = \left\{ \bar{x} = (x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} = R_B \right\}$$
$$\Gamma_F = \left\{ \bar{x} = (x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} = R_F \right\}$$

Figure 2.1 shows the two-dimensional region Ω in $\mathbb{R}^2$ exterior to the borehole.

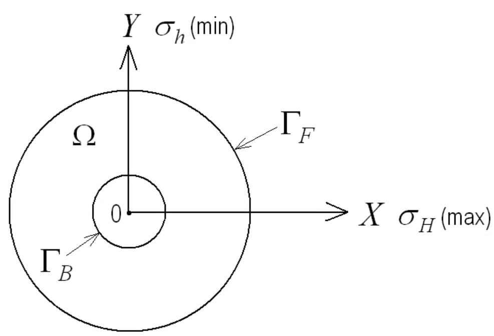

Figure 2.1: Two-dimensional region exterior to the borehole

The initial-boundary value problem that constitutes the two-dimensional radially nonsymmetric TCPu model is the following.

Thermal diffusion:

$$\Lambda\dot{T} + \Sigma\dot{C} + \Phi\dot{p} - \frac{k^T}{T_F}\nabla^2 T - \frac{\rho_f R T_F D^T}{M^s(1 - C_F)}\nabla^2 C + K^T \nabla^2 p = -\zeta(\nabla \cdot \dot{\vec{u}}) \qquad \text{in } \Omega \times (0, t_f) \tag{2.1.1}$$

with boundary conditions

$$T(\bar{x},t) = T_B(t) \qquad \text{on } \Gamma_B \times [0,t_f) \tag{2.1.2}$$

$$T(\bar{x},t) = T_F(\bar{x}) \qquad \text{on } \Gamma_F \times [0,t_f) \tag{2.1.3}$$

and initial conditions

$$T(\bar{x},0) = \tilde{T}_I(\bar{x}) \qquad \text{in } \Omega \tag{2.1.4}$$

Solute diffusion:

$$\phi\dot{C} - C_F D^T \nabla^2 T - D\nabla^2 C + \frac{k\mathfrak{R}}{\eta}\nabla^2 p = 0 \qquad \text{in } \Omega \times (0,t_f) \tag{2.1.5}$$

with boundary conditions

$$C(\bar{x},t) = C_B(t) \qquad \text{on } \Gamma_B \times [0,t_f) \tag{2.1.6}$$

$$C(\bar{x},t) = C_F(\bar{x}) \qquad \text{on } \Gamma_F \times [0,t_f) \tag{2.1.7}$$

and initial conditions

$$C(\bar{x},0) = \tilde{C}_I(\bar{x}) \qquad \text{in } \Omega \tag{2.1.8}$$

Fluid diffusion:

$$\Gamma\dot{T} + \chi\dot{C} + \Psi\dot{p} + K^T\nabla^2 T + \frac{\rho_f \mathfrak{R}\Omega k}{\eta}\nabla^2 C - \frac{k}{\eta}\nabla^2 p = -\alpha(\nabla\cdot\dot{\vec{u}}) \qquad \text{in } \Omega\times(0,t_f) \tag{2.1.9}$$

with boundary conditions

$$p(\bar{x},t) = p_B(t) \qquad \text{on } \Gamma_B \times [0,t_f) \tag{2.1.10}$$

$$p(\bar{x},t) = p_F(\bar{x}) \qquad \text{on } \Gamma_F \times [0,t_f) \tag{2.1.11}$$

and initial conditions

$$p(\bar{x},0) = \tilde{p}_I(\bar{x}) \qquad \text{in } \Omega \tag{2.1.12}$$

Navier-type elastic equations:

$$\left(K + \frac{G}{3}\right)\nabla(\nabla \cdot \dot{\vec{u}}) + G\nabla^2\dot{\vec{u}} - \tilde{\zeta}\nabla\dot{T} + \xi\nabla\dot{C} - \tilde{\alpha}\nabla\dot{p} = 0 \qquad \text{in } \Omega \times (0, t_f) \quad (2.1.13)$$

with boundary conditions

$$\vec{u}(\bar{x}, t) \approx 0 \qquad \text{on } \Gamma_F \times (0, t_f) \quad (2.1.14)$$

$$\dot{\vec{u}}(\bar{x}, t) \approx 0 \qquad \text{on } \Gamma_F \times (0, t_f) \quad (2.1.15)$$

and initial conditions

$$\vec{u}(\bar{x}, 0) = 0 \qquad \text{in } \bar{\Omega} \quad (2.1.16)$$

The description and values of physical constants are presented in Appendix A.

It is convenient to consider the thermal diffusion, solute diffusion, and fluid diffusion as a system (TCP system). Hence, we introduce the following matrices and vectors:

$$M = \begin{bmatrix} \Lambda & \Sigma & \Phi \\ 0 & \phi & 0 \\ \Gamma & \chi & \Psi \end{bmatrix}, \quad A = \begin{bmatrix} \dfrac{k^T}{T_F} & \dfrac{\rho_f R T_F D^T}{M^s(1 - C_F)} & -K^T \\ C_F D^T & D & -\dfrac{k\mathfrak{R}}{\eta} \\ -K^T & -\dfrac{\rho_f \mathfrak{R}\Omega k}{\eta} & \dfrac{k}{\eta} \end{bmatrix}, \quad \vec{b}_0 = \begin{bmatrix} \zeta \\ 0 \\ \alpha \end{bmatrix} \quad (2.1.17)$$

$$\bar{V}(\bar{x}, t) = \begin{bmatrix} T \\ C \\ p \end{bmatrix}(\bar{x}, t), \qquad V_B = \begin{bmatrix} T_B \\ C_B \\ p_B \end{bmatrix}, \qquad V_F = \begin{bmatrix} T_F \\ C_F \\ p_F \end{bmatrix} \quad (2.1.18)$$

With the notations (2.1.17) and (2.1.18), the initial-boundary value problem (2.1.1)-(2.1.12) has the form

$$M\dot{\bar{V}} - A\nabla^2\bar{V} = -\vec{b}_0(\nabla \cdot \dot{\vec{u}}) \qquad \text{in } \Omega \times (0, t_f) \quad (2.1.19)$$

with boundary conditions

$$\bar{V}(\bar{x},t) = \begin{cases} V_B(t) & \text{on } \Gamma_B \times [0,t_f) \\ V_F(\bar{x}) & \text{on } \Gamma_F \times [0,t_f) \end{cases} \tag{2.1.20}$$

and initial conditions

$$\bar{V}(\bar{x},0) = \tilde{V}_I(\bar{x}) \qquad \text{in } \Omega \tag{2.1.21}$$

Note that $\vec{b}_0$ is the factor that couples the displacement velocity to the absolute temperature, the pore pressure, the solute mass fraction (TCP), and their velocities.

By the principle of minimum total potential energy, the region Ω shall displace to a position that minimizes the total potential energy; it is the stable configuration for equilibrium.

The total potential energy of our system is

$$\mathcal{V}(\vec{u}) = \mathcal{V}_S(\vec{u}) - W_b(\vec{u}) - W_S(\vec{u}) \tag{2.1.22}$$

where $\mathcal{V}_S(\vec{u})$ is the elastic energy of the system; $W_b(\vec{u})$ is work done by body forces due to the absolute temperature, the solute mass fraction, and the pore pressure; and $W_S(\vec{u})$ is work done by applied boundary stress.

Now we specify $\mathcal{V}_S(\vec{u})$, $W_b(\vec{u})$, and $W_S(\vec{u})$. Here and in the following we will suppress the time dependence of the displacement vector for the sake of brevity.

Given the displacement

$$\vec{u} = u\vec{i} + v\vec{j}$$

the linearized strain is a second order symmetric tensor

$$\varepsilon = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{bmatrix} = \frac{1}{2}\left(\nabla\vec{u} + \nabla\vec{u}^T\right)$$

where

$$\varepsilon_{11} = u_x, \qquad \varepsilon_{12} = \varepsilon_{21} = \frac{1}{2}\left(u_y + v_x\right), \qquad \varepsilon_{22} = v_y$$

In the linear elasticity theory, a relation between the stress tensor τ and the linearized strain tensor $\varepsilon(\vec{u})$ is given by

$$\tau_{ij}(\vec{u}) = a_{ijkh}\varepsilon_{kh}(\vec{u}) \tag{2.1.23}$$

where a_{ijkh} are the coefficients of elasticity, independent of the strain tensor.

The coefficients of elasticity have properties of symmetry

$$a_{ijkh} = a_{jihk} = a_{khij} \tag{2.1.24}$$

and of ellipticity: There exists a constant $\alpha_1 > 0$ such that

$$a_{ijkh}\varepsilon_{ij}\varepsilon_{kh} \geq \alpha_1\varepsilon_{ij}\varepsilon_{ij}, \qquad \forall\ \varepsilon_{ij}. \tag{2.1.25}$$

In equations (2.1.23)-(2.1.25), we made use of the Einstein summation convention concerning repeated indices.

For a homogeneous elastic isotropic medium, the stress-strain relation (2.1.23) in terms of the bulk modulus K and the shear modulus G has the form

$$\tau = \begin{bmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{bmatrix} = 2G\varepsilon + \big(K - \frac{2}{3}G\big)(\mathrm{tr}\varepsilon)I \tag{2.1.26}$$

That is,

$$\begin{aligned} \tau_{11} &= 2Gu_x + \big(K - \frac{2}{3}G\big)\big(u_x + v_y\big) \\ \tau_{12} &= \tau_{21} = G\big(u_y + v_x\big) \\ \tau_{22} &= 2Gv_y + \big(K - \frac{2}{3}G\big)\big(u_x + v_y\big) \end{aligned} \tag{2.1.27}$$

The elastic strain energy of the system is

$$\mathcal{V}_S(\vec{u}) = \frac{1}{2}\int_\Omega \tau_{11}(\vec{u})\varepsilon_{11}(\vec{u}) + 2\tau_{12}(\vec{u})\varepsilon_{12}(\vec{u}) + \tau_{22}(\vec{u})\varepsilon_{22}(\vec{u})\,\mathrm{d}\Omega \tag{2.1.28}$$

The work done by body forces due to the absolute temperature, the solute mass fraction, and the pore pressure is

$$W_b(\vec{u}) = \int_\Omega f\big(\mathrm{tr}\varepsilon(\vec{u})\big)\,\mathrm{d}\Omega \tag{2.1.29}$$

where

$$f = \vec{b}_1^T \bar{V}, \qquad \vec{b}_1 = [\tilde{\zeta} \quad -\xi \quad \tilde{\alpha}]^T, \qquad \bar{V} = [T \quad C \quad p]^T \tag{2.1.30}$$

Note that $\vec{b}_1$ is the factor that couples the absolute temperature, the pore pressure, and the solute mass fraction (TCP) to the displacement.

The work done by applied boundary stress is

$$W_S(\vec{u}) = \int_\Gamma (\hat{\sigma}\vec{n}) \cdot \vec{u}\,\mathrm{d}\Gamma \tag{2.1.31}$$

where $\hat{\sigma}$ is applied boundary stress and $\vec{n}$ is the outward unit normal vector on the boundary.

Applying (2.1.28), (2.1.29), and (2.1.31) to (2.1.22), we have

$$\begin{aligned}\mathcal{V}(\vec{u}) = \frac{1}{2}\int_\Omega & \tau_{11}(\vec{u})\varepsilon_{11}(\vec{u}) + 2\tau_{12}(\vec{u})\varepsilon_{12}(\vec{u}) + \tau_{22}(\vec{u})\varepsilon_{22}(\vec{u})\,\mathrm{d}\Omega \\ & - \int_\Omega f\big(\mathrm{tr}\varepsilon(\vec{u})\big)\,\mathrm{d}\Omega - \int_\Gamma (\hat{\sigma}\vec{n}) \cdot \vec{u}\,\mathrm{d}\Gamma \end{aligned} \tag{2.1.32}$$

To aid in giving a mathematical formulation, let

$$\begin{aligned} H &= L^2\big(\Omega;\mathbb{R}^2\big) = \big\{L^2\big(\Omega;\mathbb{R}\big)\big\}^2 \\ V &= H^1\big(\Omega;\mathbb{R}^2\big) = \big\{H^1\big(\Omega;\mathbb{R}\big)\big\}^2 \\ V_0 &= \big\{\varphi \in V : \varphi\big|_{\Gamma_F} = 0\big\} \\ \vec{\Phi}(\bar{x}) &= \phi(\bar{x})\vec{i} + \psi(\bar{x})\vec{j} \end{aligned}$$

At this point, we wish to express the elastic strain energy $\mathcal{V}_S(\vec{u})$ as a functional on V. To that end, define the bilinear form on V by

$$a_E(\vec{u},\vec{\Phi}) = \int_\Omega \tau_{11}(\vec{u})\varepsilon_{11}(\vec{\Phi}) + 2\tau_{12}(\vec{u})\varepsilon_{12}(\vec{\Phi}) + \tau_{22}(\vec{u})\varepsilon_{22}(\vec{\Phi})\,\mathrm{d}\Omega, \qquad \forall\ \vec{u},\vec{\Phi} \in V \tag{2.1.33}$$

Then from (2.1.28) and (2.1.33),

$$\mathcal{V}_S(\vec{u}) = \frac{1}{2}a_E(\vec{u},\vec{u}) \tag{2.1.34}$$

and, from (2.1.32) and (2.1.34), the total potential energy has the form

$$\mathcal{V}(\vec{u}) = \frac{1}{2}a_E(\vec{u},\vec{u}) - \int_\Omega f\big(\mathrm{tr}\varepsilon(\vec{u})\big)\,\mathrm{d}\Omega - \int_\Gamma (\hat{\sigma}\vec{n})\cdot\vec{u}\,\mathrm{d}\Gamma \tag{2.1.35}$$

Define the following vectors:

$$\vec{\tau}_1 = [\tau_{11} \quad \tau_{12}]^T, \qquad \vec{\tau}_2 = [\tau_{21} \quad \tau_{22}]^T$$

Then

$$\begin{aligned}
a_E(\vec{u},\vec{\Phi}) &= \int_\Omega \tau_{11}\phi_x + \tau_{12}\phi_y + \tau_{21}\psi_x + \tau_{22}\psi_y\,\mathrm{d}\Omega \\
&= \int_\Omega \vec{\tau}_1\cdot\nabla\phi + \vec{\tau}_2\cdot\nabla\psi\,\mathrm{d}\Omega \\
&= \int_\Omega \nabla\cdot(\phi\vec{\tau}_1) - \phi\nabla\cdot\vec{\tau}_1 + \nabla\cdot(\psi\vec{\tau}_2) - \psi\nabla\cdot\vec{\tau}_2\,\mathrm{d}\Omega \\
&= \int_\Gamma \phi(\vec{\tau}_1\cdot\vec{n}) + \psi(\vec{\tau}_2\cdot\vec{n})\,\mathrm{d}\Gamma - \int_\Omega \phi\nabla\cdot\vec{\tau}_1 + \psi\nabla\cdot\vec{\tau}_2\,\mathrm{d}\Omega \\
&= \int_\Gamma (\tau\vec{n})\cdot\vec{\Phi}\,\mathrm{d}\Gamma - \int_\Omega [\nabla\cdot\vec{\tau}_1 \quad \nabla\cdot\vec{\tau}_2]^T\cdot\vec{\Phi}\,\mathrm{d}\Omega, \qquad \forall\ \vec{\Phi}\in V
\end{aligned} \tag{2.1.36}$$

Thus, we obtain the following Green's formula:

$$a_E(\vec{u},\vec{\Phi}) = \int_\Gamma (\tau\vec{n})\cdot\vec{\Phi}\,\mathrm{d}\Gamma - \int_\Omega [\nabla\cdot\vec{\tau}_1 \quad \nabla\cdot\vec{\tau}_2]^T\cdot\vec{\Phi}\,\mathrm{d}\Omega, \qquad \forall\ \vec{\Phi}\in V \tag{2.1.37}$$

Referring to (2.1.14) and (2.1.15), we assume that $\vec{u}$ and $\dot{\vec{u}}$ are negligible on Γ_F and therefore, the set of admissible displacements is

$$\mathcal{U}_{ad} = \big\{\vec{u}\in V_0 : \dot{\vec{u}} \in V_0\big\}$$

Using the principle of minimum total potential energy, the displacement $\vec{u}$ that the region Ω undergoes is given by

$$D\mathcal{V}(\vec{u})\vec{\Phi} = 0, \qquad \forall\, \vec{\Phi} \in V_0 \tag{2.1.38}$$

where

$$D\mathcal{V}(\vec{u})\vec{\Phi} = \frac{d}{d\delta}\mathcal{V}(\vec{u} + \delta\vec{\Phi})\Big|_{\delta=0}$$

is the Gâteaux differential of $\mathcal{V}$ with increment $\vec{\Phi}$.

From (2.1.35), (2.1.38)

$$a_E(\vec{u}, \vec{\Phi}) - \int_\Omega f\big(\mathrm{tr}\varepsilon(\vec{\Phi})\big)\,\mathrm{d}\Omega - \int_{\Gamma_B} (\hat{\sigma}\vec{n}) \cdot \vec{\Phi}\,\mathrm{d}\Gamma = 0, \qquad \forall\, \vec{\Phi} \in V_0 \tag{2.1.39}$$

From Green's formula (2.1.37) we have

$$a_E(\vec{u}, \vec{\Phi}) = \int_{\Gamma_B} (\tau\vec{n}) \cdot \vec{\Phi}\,\mathrm{d}\Gamma - \int_\Omega [\nabla \cdot \vec{\tau}_1 \quad \nabla \cdot \vec{\tau}_2]^T \cdot \vec{\Phi}\,\mathrm{d}\Omega, \qquad \forall\, \vec{\Phi} \in V_0 \tag{2.1.40}$$

Now

$$\begin{aligned}
\int_\Omega f\big(\mathrm{tr}\varepsilon(\vec{\Phi})\big)\,\mathrm{d}\Omega &= \int_\Omega f(\nabla \cdot \vec{\Phi})\,\mathrm{d}\Omega \\
&= \int_\Omega \nabla \cdot (f\vec{\Phi}) - \nabla f \cdot \vec{\Phi}\,\mathrm{d}\Omega \\
&= \int_{\Gamma_B} f\vec{\Phi} \cdot \vec{n}\,\mathrm{d}\Gamma - \int_\Omega \nabla f \cdot \vec{\Phi}\,\mathrm{d}\Omega \\
&= \int_{\Gamma_B} (fI\vec{n}) \cdot \vec{\Phi}\,\mathrm{d}\Gamma - \int_\Omega \nabla f \cdot \vec{\Phi}\,\mathrm{d}\Omega, \qquad \forall\, \vec{\Phi} \in V_0
\end{aligned} \tag{2.1.41}$$

Applying (2.1.40) and (2.1.41) to (2.1.39), we have

$$\int_{\Gamma_B} (\tau\vec{n}) \cdot \vec{\Phi}\,\mathrm{d}\Gamma - \int_\Omega [\nabla \cdot \vec{\tau}_1 \quad \nabla \cdot \vec{\tau}_2]^T \cdot \vec{\Phi}\,\mathrm{d}\Omega -$$

$$- \int_{\Gamma_B} (fI\vec{n}) \cdot \vec{\Phi}\,\mathrm{d}\Gamma + \int_\Omega \nabla f \cdot \vec{\Phi}\,\mathrm{d}\Omega - \int_{\Gamma_B} (\hat{\sigma}\vec{n}) \cdot \vec{\Phi}\,\mathrm{d}\Gamma = 0, \qquad \Rightarrow$$

$$\int_{\Gamma_B} (\tau - fI - \hat{\sigma})\vec{n} \cdot \vec{\Phi}\, \mathrm{d}\Gamma + \int_{\Omega} \big(-[\nabla \cdot \vec{\tau}_1 \quad \nabla \cdot \vec{\tau}_2]^T + \nabla f \big) \cdot \vec{\Phi}\, \mathrm{d}\Omega = 0, \qquad \forall\, \vec{\Phi} \in V_0$$

It follows that

$$[\nabla \cdot \vec{\tau}_1 \quad \nabla \cdot \vec{\tau}_2]^T = \nabla f \qquad \text{in } \Omega \tag{2.1.42}$$

$$\tau \vec{n} = (fI + \hat{\sigma})\vec{n} \qquad \text{on } \Gamma_B \tag{2.1.43}$$

From (2.1.27),

$$\begin{aligned}
\nabla \cdot \vec{\tau}_1 = \tau_{11_x} + \tau_{12_y} &= \big(K + \frac{G}{3}\big)u_{xx} + \big(K - \frac{2G}{3}\big)v_{yx} + Gu_{xx} + Gu_{yy} + Gv_{xy} \\
&= \big(K + \frac{G}{3}\big)u_{xx} + \big(K + \frac{G}{3}\big)v_{yx} + Gu_{xx} + Gu_{yy} \\
\nabla \cdot \vec{\tau}_2 = \tau_{21_x} + \tau_{22_y} &= Gu_{yx} + Gv_{xx} + \big(K + \frac{G}{3}\big)v_{yy} + \big(K - \frac{2G}{3}\big)u_{xy} + Gv_{yy} \\
&= \big(K + \frac{G}{3}\big)u_{xy} + \big(K + \frac{G}{3}\big)v_{yy} + Gv_{xx} + Gv_{yy} \qquad \Rightarrow
\end{aligned}$$

$$\begin{bmatrix} \nabla \cdot \vec{\tau}_1 \\ \nabla \cdot \vec{\tau}_2 \end{bmatrix} = \big(K + \frac{G}{3}\big) \begin{bmatrix} (u_x + v_y)_x \\ (u_x + v_y)_y \end{bmatrix} + G \begin{bmatrix} (u_{xx} + u_{yy}) \\ (v_{xx} + v_{yy}) \end{bmatrix} \qquad \Rightarrow$$

$$[\nabla \cdot \vec{\tau}_1 \quad \nabla \cdot \vec{\tau}_2]^T = \big(K + \frac{G}{3}\big)\nabla(\nabla \cdot \vec{u}) + G\nabla^2 \vec{u} \tag{2.1.44}$$

From (2.1.42)-(2.1.44) and (2.1.15),

$$\big(K + \frac{G}{3}\big)\nabla(\nabla \cdot \vec{u}) + G\nabla^2 \vec{u} = \nabla f \qquad \text{in } \Omega \times (0, t_f) \tag{2.1.45}$$

$$\tau \vec{n} = (fI + \hat{\sigma})\vec{n} \qquad \text{on } \Gamma_B \times (0, t_f) \tag{2.1.43}$$

$$\vec{u} \approx 0 \qquad \text{on } \Gamma_F \times (0, t_f) \tag{2.1.14}$$

Differentiating (2.1.45) with respect to time and applying (2.1.30), we obtain the system (2.1.13). On the other hand, since $\dot{\vec{u}} \in V_0$, from (2.1.39) we have

$$a_E(\dot{\vec{u}}, \vec{\Phi}) - \int_\Omega f\big(\mathrm{tr}\varepsilon(\vec{\Phi})\big)\, \mathrm{d}\Omega - \int_{\Gamma_B} (\hat{\sigma}\vec{n}) \cdot \vec{\Phi}\, \mathrm{d}\Gamma = 0, \qquad \forall\, \vec{\Phi} \in V_0 \tag{2.1.46}$$

Using the same argument as above and applying (2.1.30), we obtain the equivalence of systems (2.1.45), (2.1.43), (2.1.14) and (2.1.13)-(2.1.15).

Remark 2.1 ([3]). The condition for mechanical equilibrium is

$$\nabla \cdot \sigma = 0$$

where $\sigma = \tau - fI$ is the poroelastic stress tensor.

From this relation, (2.1.27), and (2.1.30), we obtain the system (2.1.45) that constitutes the equations of equilibrium.

2.2 Three-dimensional Model

The initial-boundary value problem that constitutes our model is defined in a bounded open domain Ω in $\mathbb{R}^3$ exterior to the borehole, with a sufficiently smooth boundary Γ. Specifically, $\Omega \subset \mathbb{R}^3$ is an inclined cylindrical region with an inner radius R_B and an outer radius R_F.

Let (X, Y, Z) be a Cartesian coordinate system such that

x -axis is in the direction of maximum horizontal principal stress $\sigma_H(z)$,

y -axis is in the direction of minimum horizontal principal stress $\sigma_h(z)$, and

z -axis is in the direction of overburden stress σ_V and

(X_b, Y_b, Z_b) be a borehole Cartesian coordinate system.

A point $\bar{x}$ in $\mathbb{R}^3$ is associated with position vector $\bar{x} = x\vec{i} + y\vec{j} + z\vec{k}$, where $\vec{i}$, $\vec{j}$, and $\vec{k}$ are the basis vectors of the borehole Cartesian coordinate system (X_b, Y_b, Z_b). Displacements of points within Ω are described in terms of a vector-valued displacement function of the position $\bar{x}$ and time t as $\vec{u}(\bar{x}, t) = u(\bar{x}, t)\vec{i} + v(\bar{x}, t)\vec{j} + w(\bar{x}, t)\vec{k}$.

The boundary Γ of Ω is $\Gamma = \Gamma_B \cup \Gamma_F \cup \tilde{\Gamma}_T \cup \tilde{\Gamma}_B$, where

$$\Gamma_B = \left\{(x, y, z) \in \mathbb{R}^3 : \sqrt{x^2 + y^2} = R_B,\ z_T \leq z \leq z_B\right\}$$

$$\Gamma_F = \left\{(x, y, z) \in \mathbb{R}^3 : \sqrt{x^2 + y^2} = R_F,\ z_T \leq z \leq z_B\right\}$$

$$\tilde{\Gamma}_T = \left\{(x, y, z_T) \in \mathbb{R}^3 : R_B \leq \sqrt{x^2 + y^2} \leq R_F\right\}$$

$$\tilde{\Gamma}_B = \left\{(x, y, z_B) \in \mathbb{R}^3 : R_B \leq \sqrt{x^2 + y^2} \leq R_F\right\}$$

Figure 2.2 shows the two-dimensional region Ω in $\mathbb{R}^3$ exterior to the borehole.

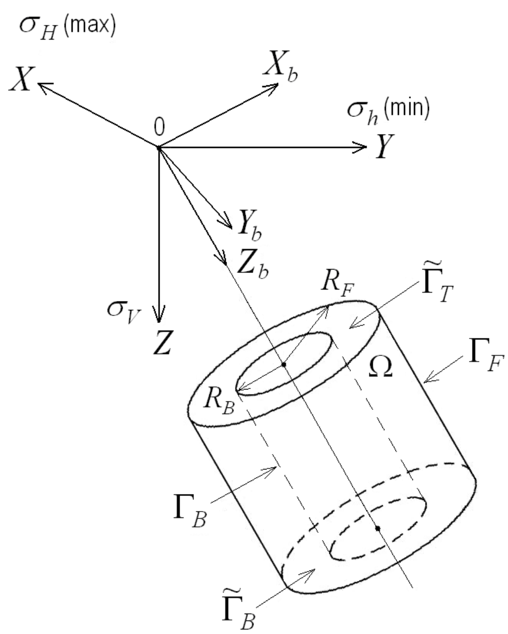

Figure 2.2: Three-dimensional region exterior to the borehole

Now we formulate the initial-boundary value problem that constitutes the three-dimensional radially nonsymmetric TCPu model. The thermal diffusion, solute diffusion, and fluid diffusion equations, as well as the Navier-type elastic equations for the three-dimensional problem are the same as in the two-dimensional case and thus, the problem is the following.

Thermal diffusion:

$$\Lambda\dot{T}+\Sigma\dot{C}+\Phi\dot{p}-\frac{k^T}{T_F}\nabla^2 T-\frac{\rho_f R T_F D^T}{M^s(1-C_F)}\nabla^2 C+K^T\nabla^2 p=-\zeta(\nabla\cdot\dot{\vec{u}}) \qquad \text{in } \Omega\times(0,t_f) \tag{2.2.1}$$

with boundary conditions

$$T(\bar{x},t)=T_B(z,t) \qquad \text{on } \Gamma_B\times[0,t_f) \tag{2.2.2a}$$

$$T(\bar{x},t)=T_F(\bar{x}) \qquad \text{on } \Gamma_F\times[0,t_f) \tag{2.2.2b}$$

$$T(\bar{x},t)=\tilde{T}_T(x,y,t) \qquad \text{on } \tilde{\Gamma}_T\times[0,t_f) \tag{2.2.2c}$$

$$T(\bar{x},t)=\tilde{T}_B(x,y,t) \qquad \text{on } \tilde{\Gamma}_B\times[0,t_f) \tag{2.2.2d}$$

compatibility conditions

$$\tilde{T}_T(t)\big|_{\Gamma_B}=T_B(t)\big|_{\tilde{\Gamma}_T} \tag{2.2.3a}$$

$$\tilde{T}_T(t)\big|_{\Gamma_F}=T_F\big|_{\tilde{\Gamma}_T} \tag{2.2.3b}$$

$$\tilde{T}_B(t)\big|_{\Gamma_B}=T_B(t)\big|_{\tilde{\Gamma}_B} \tag{2.2.3c}$$

$$\tilde{T}_B(t)\big|_{\Gamma_F}=T_F\big|_{\tilde{\Gamma}_T} \tag{2.2.3d}$$

and initial conditions

$$T(\bar{x},0)=\tilde{T}_I(\bar{x}) \qquad \text{in } \Omega \tag{2.2.4}$$

Solute diffusion:

$$\phi\dot{C}-C_F D^T\nabla^2 T-D\nabla^2 C+\frac{k\mathfrak{R}}{\eta}\nabla^2 p=0 \qquad \text{in } \Omega\times(0,t_f) \tag{2.2.5}$$

with boundary conditions

$$C(\bar{x},t)=C_B(z,t) \qquad \text{on } \Gamma_B\times[0,t_f) \tag{2.2.6a}$$

$$C(\bar{x},t)=C_F(\bar{x}) \qquad \text{on } \Gamma_F\times[0,t_f) \tag{2.2.6b}$$

$$C(\bar{x},t)=\tilde{C}_T(x,y,t) \qquad \text{on } \tilde{\Gamma}_T\times[0,t_f) \tag{2.2.6c}$$

$$C(\bar{x},t)=\tilde{C}_B(x,y,t) \qquad \text{on } \tilde{\Gamma}_B\times[0,t_f) \tag{2.2.6d}$$

compatibility conditions

$$\tilde{C}_T(t)\big|_{\Gamma_B} = C_B(t)\big|_{\tilde{\Gamma}_T} \tag{2.2.7a}$$

$$\tilde{C}_T(t)\big|_{\Gamma_F} = C_F\big|_{\tilde{\Gamma}_T} \tag{2.2.7b}$$

$$\tilde{C}_B(t)\big|_{\Gamma_B} = C_B(t)\big|_{\tilde{\Gamma}_B} \tag{2.2.7c}$$

$$\tilde{C}_B(t)\big|_{\Gamma_F} = C_F\big|_{\tilde{\Gamma}_T} \tag{2.2.7d}$$

and initial conditions

$$C(\bar{x},0) = \tilde{C}_I(\bar{x}) \qquad \text{in } \Omega \tag{2.2.8}$$

Fluid diffusion:

$$\Gamma\dot{T} + \chi\dot{C} + \Psi\dot{p} + K^T\nabla^2 T + \frac{\rho_f \Re\Omega k}{\eta}\nabla^2 C - \frac{k}{\eta}\nabla^2 p = -\alpha(\nabla\cdot\dot{\vec{u}}) \quad \text{in } \Omega\times(0,t_f) \tag{2.2.9}$$

with boundary conditions

$$p(\bar{x},t) = p_B(z,t) \qquad \text{on } \Gamma_B \times [0,t_f) \tag{2.2.10a}$$

$$p(\bar{x},t) = p_F(\bar{x}) \qquad \text{on } \Gamma_F \times [0,t_f) \tag{2.2.10b}$$

$$p(\bar{x},t) = \tilde{p}_T(x,y,t) \qquad \text{on } \tilde{\Gamma}_T \times [0,t_f) \tag{2.2.10c}$$

$$p(\bar{x},t) = \tilde{p}_B(x,y,t) \qquad \text{on } \tilde{\Gamma}_B \times [0,t_f) \tag{2.2.10d}$$

compatibility conditions

$$\tilde{p}_T(t)\big|_{\Gamma_B} = p_B(t)\big|_{\tilde{\Gamma}_T} \tag{2.2.11a}$$

$$\tilde{p}_T(t)\big|_{\Gamma_F} = p_F\big|_{\tilde{\Gamma}_T} \tag{2.2.11b}$$

$$\tilde{p}_B(t)\big|_{\Gamma_B} = p_B(t)\big|_{\tilde{\Gamma}_B} \tag{2.2.11c}$$

$$\tilde{p}_B(t)\big|_{\Gamma_F} = p_F\big|_{\tilde{\Gamma}_T} \tag{2.2.11d}$$

and initial conditions

$$p(\bar{x},0) = \tilde{p}_I(\bar{x}) \qquad \text{in } \Omega \tag{2.2.12}$$

Navier-type elastic equations:

$$\Big(K + \frac{G}{3}\Big)\nabla(\nabla\cdot\dot{\vec{u}}) + G\nabla^2\dot{\vec{u}} - \tilde{\zeta}\nabla\dot{T} + \xi\nabla\dot{C} - \tilde{\alpha}\nabla\dot{p} = 0 \qquad \text{in } \Omega\times(0,t_f) \quad (2.2.13)$$

with boundary conditions

$$\vec{u}(\bar{x},t) \approx 0 \qquad \text{on } \Gamma_F\times(0,t_f) \quad (2.2.14)$$

$$\dot{\vec{u}}(\bar{x},t) \approx 0 \qquad \text{on } \Gamma_F\times(0,t_f) \quad (2.2.15)$$

and initial conditions

$$\vec{u}(\bar{x},0) = 0 \qquad \text{in } \bar{\Omega} = \Omega\cup\Gamma \quad (2.2.16)$$

The description and values of physical constants are presented in Appendix A.

Using the matrices and vectors (2.1.17) and (2.1.18) defined for the two-dimensional case, we have the following TCP initial-boundary value problem:

$$M\dot{\bar{V}} - A\nabla^2\bar{V} = -\vec{b}_0(\nabla\cdot\dot{\vec{u}}) \qquad \text{in } \Omega\times(0,t_f) \quad (2.2.17)$$

with boundary conditions

$$\bar{V}(\bar{x},t) = \begin{cases} V_B(z,t) & \text{on } \Gamma_B\times[0,t_f) \\ V_F(\bar{x}) & \text{on } \Gamma_F\times[0,t_f) \\ \tilde{V}_T(x,y,t) & \text{on } \tilde{\Gamma}_T\times[0,t_f) \\ \tilde{V}_B(x,y,t) & \text{on } \tilde{\Gamma}_B\times[0,t_f) \end{cases} \quad (2.2.18)$$

compatibility conditions

$$\tilde{V}_T(t)\big|_{\Gamma_B} = V_B(t)\big|_{\tilde{\Gamma}_T} \quad (2.2.19a)$$

$$\tilde{V}_T(t)\big|_{\Gamma_F} = V_F\big|_{\tilde{\Gamma}_T} \quad (2.2.19b)$$

$$\tilde{V}_B(t)\big|_{\Gamma_B} = V_B(t)\big|_{\tilde{\Gamma}_B} \quad (2.2.19c)$$

$$\tilde{V}_B(t)\big|_{\Gamma_F} = V_F\big|_{\tilde{\Gamma}_T} \quad (2.2.19d)$$

and initial conditions

$$\bar{V}(\bar{x},0) = \tilde{V}_I(\bar{x}) \qquad \text{in } \Omega \tag{2.2.20}$$

As before, $\vec{b}_0$ is the factor that couples the displacement velocity to the absolute temperature, the pore pressure, the solute mass fraction (TCP), and their velocities.

Next, using the principle of minimum total potential energy, we will show that the three-dimensional Navier-type elastic system (2.2.13)-(2.2.15) is equivalent to the system

$$\big(K + \frac{G}{3}\big)\nabla(\nabla\cdot\vec{u}) + G\nabla^2\vec{u} = \nabla f \qquad \text{in } \Omega\times(0,t_f) \tag{2.2.21}$$

$$\tau\vec{n} = (fI + \hat{\sigma})\vec{n} \qquad \text{on } \Gamma\setminus\Gamma_F\times(0,t_f) \tag{2.2.22}$$

$$\vec{u} \approx 0 \qquad \text{on } \Gamma_F\times(0,t_f) \tag{2.2.14}$$

where f is given by (2.1.30).

Here and in the following we will suppress the time dependence of the displacement vector for the sake of brevity. Given the displacement

$$\vec{u} = u\vec{i} + v\vec{j} + w\vec{k}$$

the linearized strain is a second order symmetric tensor

$$\varepsilon = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} = \frac{1}{2}\big(\nabla\vec{u} + \nabla\vec{u}^T\big)$$

where

$$\varepsilon_{11} = u_x, \qquad \varepsilon_{12} = \varepsilon_{21} = \frac{1}{2}\big(u_y + v_x\big), \qquad \varepsilon_{22} = v_y$$
$$\varepsilon_{13} = \varepsilon_{31} = \frac{1}{2}\big(u_z + w_x\big), \qquad \varepsilon_{23} = \varepsilon_{32} = \frac{1}{2}\big(v_z + w_y\big), \qquad \varepsilon_{33} = w_z$$

For a homogeneous elastic isotropic medium, the stress-strain relation (2.1.23) in terms of the bulk modulus K and the shear modulus G has the form

$$\tau = \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{13} & \tau_{23} & \tau_{33} \end{bmatrix} = 2G\varepsilon + \big(K - \frac{2}{3}G\big)(\mathrm{tr}\varepsilon)I \tag{2.2.23}$$

That is,

$$\begin{aligned}
\tau_{11} &= 2Gu_x + \big(K - \frac{2}{3}G\big)\big(u_x + v_y + w_z\big) \\
\tau_{12} &= \tau_{21} = G\big(u_y + v_x\big) \\
\tau_{22} &= 2Gv_y + \big(K - \frac{2}{3}G\big)\big(u_x + v_y + w_z\big) \\
\tau_{13} &= \tau_{31} = G\big(u_z + w_x\big) \\
\tau_{23} &= \tau_{32} = G\big(v_z + w_y\big) \\
\tau_{33} &= 2Gw_z + \big(K - \frac{2}{3}G\big)\big(u_x + v_y + w_z\big)
\end{aligned} \tag{2.2.24}$$

The total potential energy of our system is

$$\mathcal{V}(\vec{u}) = \mathcal{V}_S(\vec{u}) - W_b(\vec{u}) - W_S(\vec{u}) \tag{2.2.25}$$

where

$$\begin{aligned}
\mathcal{V}_S(\vec{u}) = \frac{1}{2}\int_\Omega & \tau_{11}(\vec{u})\varepsilon_{11}(\vec{u}) + 2\tau_{12}(\vec{u})\varepsilon_{12}(\vec{u}) + \tau_{22}(\vec{u})\varepsilon_{22}(\vec{u}) + \\
& + 2\tau_{13}(\vec{u})\varepsilon_{13}(\vec{u}) + 2\tau_{23}(\vec{u})\varepsilon_{23}(\vec{u}) + \tau_{33}(\vec{u})\varepsilon_{33}(\vec{u})\,\mathrm{d}\Omega
\end{aligned} \tag{2.2.26}$$

is the elastic strain energy of the system;

$$W_b(\vec{u}) = \int_\Omega f\big(\mathrm{tr}\varepsilon(\vec{u})\big)\,\mathrm{d}\Omega \tag{2.2.27}$$

with f given by (2.1.30), is work done by body forces due to the absolute temperature, the solute mass fraction, and the pore pressure ($\vec{b}_1$ is the factor that couples the absolute temperature, the pore pressure, and the solute mass fraction (TCP) to the displacement); and

$$W_S(\vec{u}) = \int_\Gamma (\hat{\sigma}\vec{n}) \cdot \vec{u}\,\mathrm{d}\Gamma \tag{2.2.28}$$

is the work done by applied boundary stress $\hat{\sigma}$. Here $\vec{n}$ is the outward unit normal vector on the boundary.

Applying (2.2.26), (2.2.27), and (2.2.28) to (2.2.25), we have

$$\begin{aligned}\mathcal{V}(\vec{u}) = \frac{1}{2}\int_\Omega & \tau_{11}(\vec{u})\varepsilon_{11}(\vec{u}) + 2\tau_{12}(\vec{u})\varepsilon_{12}(\vec{u}) + \tau_{22}(\vec{u})\varepsilon_{22}(\vec{u}) + \\ & + 2\tau_{13}(\vec{u})\varepsilon_{13}(\vec{u}) + 2\tau_{23}(\vec{u})\varepsilon_{23}(\vec{u}) + \tau_{33}(\vec{u})\varepsilon_{33}(\vec{u})\,\mathrm{d}\Omega \\ & - \int_\Omega f\big(\mathrm{tr}\varepsilon(\vec{u})\big)\,\mathrm{d}\Omega - \int_\Gamma (\hat{\sigma}\vec{n}) \cdot \vec{u}\,\mathrm{d}\Gamma \end{aligned} \tag{2.2.29}$$

To aid in giving a mathematical formulation, let

$$\begin{aligned} H &= L^2\big(\Omega;\mathbb{R}^3\big) = \big\{L^2\big(\Omega;\mathbb{R}\big)\big\}^3 \\ V &= H^1\big(\Omega;\mathbb{R}^3\big) = \big\{H^1\big(\Omega;\mathbb{R}\big)\big\}^3 \\ V_0 &= \big\{\varphi \in V : \varphi\big|_{\Gamma_F} = 0\big\} \\ \vec{\Phi}(\bar{x}) &= \phi(\bar{x})\vec{i} + \psi(\bar{x})\vec{j} + \omega(\bar{x})\vec{k} \end{aligned}$$

Define the bilinear form on V by

$$\begin{aligned} a_E(\vec{u},\vec{\Phi}) = \int_\Omega & \tau_{11}(\vec{u})\varepsilon_{11}(\vec{\Phi}) + 2\tau_{12}(\vec{u})\varepsilon_{12}(\vec{\Phi}) + \tau_{22}(\vec{u})\varepsilon_{22}(\vec{\Phi}) + 2\tau_{13}(\vec{u})\varepsilon_{13}(\vec{\Phi}) \\ & + 2\tau_{23}(\vec{u})\varepsilon_{23}(\vec{\Phi}) + \tau_{33}(\vec{u})\varepsilon_{33}(\vec{\Phi})\,\mathrm{d}\Omega, \qquad \forall\, \vec{u},\vec{\Phi} \in V \end{aligned} \tag{2.2.30}$$

Then from (2.2.26) and (2.2.30)

$$\mathcal{V}_S(\vec{u}) = \frac{1}{2}a_E(\vec{u},\vec{u}) \tag{2.2.31}$$

and, from (2.2.29) and (2.2.31), the total potential energy has the form

$$\mathcal{V}(\vec{u}) = \frac{1}{2}a_E(\vec{u},\vec{u}) - \int_\Omega f\big(\mathrm{tr}\varepsilon(\vec{u})\big)\,\mathrm{d}\Omega - \int_\Gamma (\hat{\sigma}\vec{n}) \cdot \vec{u}\,\mathrm{d}\Gamma \tag{2.2.32}$$

Define the following vectors:

$$\vec{\tau}_1 = [\tau_{11} \quad \tau_{12} \quad \tau_{13}]^T, \quad \vec{\tau}_2 = [\tau_{21} \quad \tau_{22} \quad \tau_{23}]^T, \quad \vec{\tau}_3 = [\tau_{31} \quad \tau_{32} \quad \tau_{33}]^T$$

Then

$$\begin{aligned}
a_E(\vec{u}, \vec{\Phi}) &= \int_\Omega \tau_{11}\phi_x + \tau_{12}\phi_y + \tau_{13}\phi_z + \tau_{21}\psi_x + \tau_{22}\psi_y + \tau_{23}\psi_z + \\
&\qquad + \tau_{31}\omega_x + \tau_{32}\omega_y + \tau_{33}\omega_z \,\mathrm{d}\Omega \\
&= \int_\Omega \vec{\tau}_1 \cdot \nabla\phi + \vec{\tau}_2 \cdot \nabla\psi + \vec{\tau}_3 \cdot \nabla\omega \,\mathrm{d}\Omega \\
&= \int_\Omega \nabla \cdot (\phi\vec{\tau}_1) - \phi\nabla \cdot \vec{\tau}_1 + \nabla \cdot (\psi\vec{\tau}_2) - \psi\nabla \cdot \vec{\tau}_2 + \nabla \cdot (\omega\vec{\tau}_3) - \omega\nabla \cdot \vec{\tau}_3 \,\mathrm{d}\Omega \\
&= \int_\Gamma \phi(\vec{\tau}_1 \cdot \vec{n}) + \psi(\vec{\tau}_2 \cdot \vec{n}) + \omega(\vec{\tau}_3 \cdot \vec{n}) \,\mathrm{d}\Gamma - \\
&\qquad - \int_\Omega \phi\nabla \cdot \vec{\tau}_1 + \psi\nabla \cdot \vec{\tau}_2 + \omega\nabla \cdot \vec{\tau}_3 \,\mathrm{d}\Omega \\
&= \int_\Gamma (\tau\vec{n}) \cdot \vec{\Phi} \,\mathrm{d}\Gamma - \int_\Omega [\nabla \cdot \vec{\tau}_1 \quad \nabla \cdot \vec{\tau}_2 \quad \nabla \cdot \vec{\tau}_3]^T \cdot \vec{\Phi} \,\mathrm{d}\Omega, \qquad \forall\, \vec{u},\ \vec{\Phi} \in V
\end{aligned} \tag{2.2.33}$$

Thus, we obtain the following Green's formula:

$$a_E(\vec{u}, \vec{\Phi}) = \int_\Gamma (\tau\vec{n}) \cdot \vec{\Phi} \,\mathrm{d}\Gamma - \int_\Omega [\nabla \cdot \vec{\tau}_1 \quad \nabla \cdot \vec{\tau}_2 \quad \nabla \cdot \vec{\tau}_3]^T \cdot \vec{\Phi} \,\mathrm{d}\Omega, \qquad \forall\, \vec{u},\ \vec{\Phi} \in V \tag{2.2.34}$$

Referring to (2.2.14) and (2.2.15), we assume that $\vec{u}$ and $\dot{\vec{u}}$ are negligible on Γ_F and therefore, the set of admissible displacements is

$$\mathcal{U}_{ad} = \left\{ \vec{u} \in V_0 : \dot{\vec{u}} \in V_0 \right\}$$

Using the principle of minimum total potential energy, the displacement $\vec{u}$ that the region Ω undergoes is given by

$$D\mathcal{V}(\vec{u})\vec{\Phi} = 0, \qquad \forall\, \vec{\Phi} \in V_0 \tag{2.2.35}$$

where

$$D\mathcal{V}(\vec{u})\vec{\Phi} = \frac{d}{d\delta}\mathcal{V}(\vec{u} + \delta\vec{\Phi})\Big|_{\delta=0}$$

is the Gâteaux differential of $\mathcal{V}$ with increment $\vec{\Phi}$.

From (2.2.32), (2.2.35)

$$a_E(\vec{u}, \vec{\Phi}) - \int_{\Omega} f\big(\mathrm{tr}\varepsilon(\vec{\Phi})\big)\,\mathrm{d}\Omega - \int_{\Gamma\setminus\Gamma_F} (\hat{\sigma}\vec{n}) \cdot \vec{\Phi}\,\mathrm{d}\Gamma = 0, \qquad \forall\,\vec{\Phi} \in V_0 \tag{2.2.36}$$

From Green's formula (2.2.34) we have

$$a_E(\vec{u}, \vec{\Phi}) = \int_{\Gamma\setminus\Gamma_F} (\tau\vec{n}) \cdot \vec{\Phi}\,\mathrm{d}\Gamma - \int_{\Omega} [\nabla \cdot \vec{\tau}_1 \quad \nabla \cdot \vec{\tau}_2 \quad \nabla \cdot \vec{\tau}_3]^T \cdot \vec{\Phi}\,\mathrm{d}\Omega, \quad \forall\,\vec{\Phi} \in V_0 \tag{2.2.37}$$

Now

$$\begin{aligned}
\int_{\Omega} f\big(\mathrm{tr}\varepsilon(\vec{\Phi})\big)\,\mathrm{d}\Omega &= \int_{\Omega} f(\nabla \cdot \vec{\Phi})\,\mathrm{d}\Omega \\
&= \int_{\Omega} \nabla \cdot (f\vec{\Phi}) - \nabla f \cdot \vec{\Phi}\,\mathrm{d}\Omega \\
&= \int_{\Gamma\setminus\Gamma_F} f\vec{\Phi} \cdot \vec{n}\,\mathrm{d}\Gamma - \int_{\Omega} \nabla f \cdot \vec{\Phi}\,\mathrm{d}\Omega \\
&= \int_{\Gamma\setminus\Gamma_F} (fI\vec{n}) \cdot \vec{\Phi}\,\mathrm{d}\Gamma - \int_{\Omega} \nabla f \cdot \vec{\Phi}\,\mathrm{d}\Omega, \qquad \forall\,\vec{\Phi} \in V_0
\end{aligned} \tag{2.2.38}$$

Applying (2.2.37) and (2.2.38) to (2.2.36), we have

$$\begin{aligned}
&\int_{\Gamma\setminus\Gamma_F} (\tau\vec{n}) \cdot \vec{\Phi}\,\mathrm{d}\Gamma - \int_{\Omega} [\nabla \cdot \vec{\tau}_1 \quad \nabla \cdot \vec{\tau}_2 \quad \nabla \cdot \vec{\tau}_3]^T \cdot \vec{\Phi}\,\mathrm{d}\Omega \\
&- \int_{\Gamma\setminus\Gamma_F} (fI\vec{n}) \cdot \vec{\Phi}\,\mathrm{d}\Gamma + \int_{\Omega} \nabla f \cdot \vec{\Phi}\,\mathrm{d}\Omega - \int_{\Gamma\setminus\Gamma_F} (\hat{\sigma}\vec{n}) \cdot \vec{\Phi}\,\mathrm{d}\Gamma = 0
\end{aligned}$$

and so,

$$\int_{\Gamma\setminus\Gamma_F} (\tau - fI - \hat{\sigma})\vec{n} \cdot \vec{\Phi}\,\mathrm{d}\Gamma + \int_{\Omega} \big(-[\nabla \cdot \vec{\tau}_1 \quad \nabla \cdot \vec{\tau}_2 \quad \nabla \cdot \vec{\tau}_3]^T + \nabla f\big) \cdot \vec{\Phi}\,\mathrm{d}\Omega = 0, \quad \forall\,\vec{\Phi} \in V_0$$

It follows that

$$[\nabla \cdot \vec{\tau}_1 \quad \nabla \cdot \vec{\tau}_2 \quad \nabla \cdot \vec{\tau}_3]^T = \nabla f \qquad \text{in } \Omega \tag{2.2.39}$$

$$\tau \vec{n} = (fI + \hat{\sigma})\vec{n} \qquad \text{on } \Gamma \setminus \Gamma_F \tag{2.2.40}$$

From (2.2.24),

$$\begin{aligned}
\nabla \cdot \vec{\tau}_1 &= \tau_{11_x} + \tau_{12_y} + \tau_{13_z} \\
&= \big(K + \frac{G}{3}\big)u_{xx} + \big(K - \frac{2G}{3}\big)(v_{yx} + w_{zx}) + \\
&\quad + Gu_{xx} + Gu_{yy} + Gv_{xy} + Gu_{zz} + Gw_{xz} \\
&= \big(K + \frac{G}{3}\big)u_{xx} + \big(K + \frac{G}{3}\big)(v_{yx} + w_{zx}) + G(u_{xx} + u_{yy} + u_{zz}) \\
&= \big(K + \frac{G}{3}\big)(\nabla \cdot \vec{u})_x + G\nabla^2 u
\end{aligned}$$

$$\begin{aligned}
\nabla \cdot \vec{\tau}_2 &= \tau_{21_x} + \tau_{22_y} + \tau_{23_z} \\
&= G(u_{yx} + v_{xx}) + \big(K + \frac{G}{3}\big)v_{yy} + \big(K - \frac{2G}{3}\big)(u_{xy} + w_{zy}) + \\
&\quad + Gv_{yy} + G(v_{zz} + w_{yz}) \\
&= \big(K + \frac{G}{3}\big)(u_{xy} + v_{yy} + w_{zy}) + G(v_{xx} + v_{yy} + v_{zz}) \\
&= \big(K + \frac{G}{3}\big)(\nabla \cdot \vec{u})_y + G\nabla^2 v
\end{aligned}$$

$$\begin{aligned}
\nabla \cdot \vec{\tau}_3 &= \tau_{31_x} + \tau_{32_y} + \tau_{33_z} \\
&= G(u_{zx} + w_{xx}) + G(v_{zy} + w_{yy}) + \big(K + \frac{G}{3}\big)w_{zz} + \\
&\quad + \big(K - \frac{2G}{3}\big)(u_{xz} + v_{yz}) + Gw_{zz} \\
&= \big(K + \frac{G}{3}\big)(u_{xz} + v_{yz} + w_{zz}) + G(w_{xx} + w_{yy} + w_{zz}) \\
&= \big(K + \frac{G}{3}\big)(\nabla \cdot \vec{u})_z + G\nabla^2 w
\end{aligned}$$

It follows that

$$\begin{bmatrix}\nabla\cdot\vec{\tau}_1\\ \nabla\cdot\vec{\tau}_2\\ \nabla\cdot\vec{\tau}_3\end{bmatrix} = \big(K+\frac{G}{3}\big)\begin{bmatrix}(\nabla\cdot\vec{u})_x\\ (\nabla\cdot\vec{u})_y\\ (\nabla\cdot\vec{u})_z\end{bmatrix} + G\begin{bmatrix}\nabla^2 u\\ \nabla^2 v\\ \nabla^2 w\end{bmatrix} \qquad \Rightarrow$$

$$[\nabla\cdot\vec{\tau}_1 \quad \nabla\cdot\vec{\tau}_2 \quad \nabla\cdot\vec{\tau}_3]^T = \big(K+\frac{G}{3}\big)\nabla(\nabla\cdot\vec{u}) + G\nabla^2\vec{u} \tag{2.2.41}$$

From (2.2.39)-(2.2.41) and (2.2.14)

$$\big(K+\frac{G}{3}\big)\nabla(\nabla\cdot\vec{u}) + G\nabla^2\vec{u} = \nabla f \qquad \text{in } \Omega\times(0,t_f) \tag{2.2.42}$$

$$\tau\vec{n} = (fI+\hat{\sigma})\vec{n} \qquad \text{on } \Gamma\setminus\Gamma_F\times(0,t_f) \tag{2.2.40}$$

$$\vec{u}\approx 0 \qquad \text{on } \Gamma_F\times(0,t_f) \tag{2.2.14}$$

Differentiating (2.2.42) with respect to time and applying (2.1.30), we obtain the system (2.2.13). On the other hand, since $\dot{\vec{u}}\in V_0$, from (2.2.36) we have

$$a_E(\dot{\vec{u}},\vec{\Phi}) - \int_\Omega f\big(\mathrm{tr}\varepsilon(\vec{\Phi})\big)\,\mathrm{d}\Omega - \int_{\Gamma\setminus\Gamma_F}(\hat{\sigma}\vec{n})\cdot\vec{\Phi}\,\mathrm{d}\Gamma = 0, \qquad \forall\,\vec{\Phi}\in V_0 \tag{2.2.43}$$

Using the same argument as above and applying (2.1.30), we obtain the equivalence of systems (2.2.42), (2.2.40), (2.2.14) and (2.2.13)-(2.2.15).

Remark 2.2. From the condition of mechanical equilibrium given in Remark 2.1, (2.2.24), and (2.1.30), it follows that the system (2.2.42) constitutes the equation of equilibrium.

2.3 Diagonalized TCP System with Homogeneous Boundary Conditions

In the next chapter, we will derive the weak formulation of the TCP initial-boundary value problem. To aid in the formulation of the weak TCP problem, in this section we obtain the diagonalized two- and three-dimensional TCP systems with homogeneous boundary conditions.

First, we obtain the two-dimensional TCP system with homogeneous boundary conditions that is equivalent to (2.1.19)-(2.1.21). To that end, we define the function

$$W(\bar{x},t) = V_B(t) + \frac{|\bar{x}| - R_B}{R_F - R_B}\big(V_F(\bar{x}) - V_B(t)\big) \tag{2.3.1}$$

and the vector

$$V(\bar{x},t) = \bar{V}(\bar{x},t) - W(\bar{x},t) \tag{2.3.2}$$

Then (2.1.19)-(2.1.21) has the form

$$M\dot{V} - A\nabla^2 V = -M\dot{W} + A\nabla^2 W - \vec{b}_0(\nabla \cdot \dot{\vec{u}}) \qquad \text{in } \Omega \times (0, t_f)$$

with boundary conditions

$$V(\bar{x},t) = 0 \qquad \text{on } \Gamma \times [0, t_f)$$

and initial conditions

$$V(\bar{x},0) = V_0(\bar{x}) \qquad \text{in } \Omega$$

where

$$V_0(\bar{x}) = \tilde{V}_I(\bar{x}) - \frac{R_F - |\bar{x}|}{R_F - R_B} V_B(0) - \frac{|\bar{x}| - R_B}{R_F - R_B} V_F(\bar{x}) \tag{2.3.3}$$

Now we obtain the three-dimensional TCP system with homogeneous boundary conditions that is equivalent to (2.2.17)-(2.2.20). Define the function

$$W^0(\bar{x},t) = \frac{R_F - \sqrt{x^2+y^2}}{R_F - R_B} V_B(z,t) + \frac{\sqrt{x^2+y^2} - R_B}{R_F - R_B} V_F(\bar{x}) \tag{2.3.4}$$

and the vector

$$\bar{V}^0(\bar{x},t) = \bar{V}(\bar{x},t) - W^0(\bar{x},t) \tag{2.3.5}$$

Then (2.2.17)-(2.2.20) has the form

$$M\dot{\bar{V}}^0 - A\nabla^2 \bar{V}^0 = -M\dot{W}^0 + A\nabla^2 W^0 - \vec{b}_0(\nabla \cdot \dot{\vec{u}}) \qquad \text{in } \Omega \times (0, t_f)$$

with boundary conditions

$$\bar{V}^0(\bar{x},t) = \begin{cases} 0 & \text{on } (\Gamma_B \cup \Gamma_F) \times [0,t_f) \\ \tilde{V}_T(x,y,t) - W^0(x,y,z_T,t) & \text{on } \tilde{\Gamma}_T \times [0,t_f) \\ \tilde{V}_B(x,y,t) - W^0(x,y,z_B,t) & \text{on } \tilde{\Gamma}_B \times [0,t_f) \end{cases}$$

and initial conditions

$$\bar{V}^0(\bar{x},0) = \tilde{V}_I(\bar{x}) - W^0(\bar{x},0) \qquad \text{in } \Omega$$

Next define

$$\hat{V}_T(x,y,t) = \tilde{V}_T(x,y,t) - W^0(x,y,z_T,t) \tag{2.3.6}$$

$$\hat{V}_B(x,y,t) = \tilde{V}_B(x,y,t) - W^0(x,y,z_B,t) \tag{2.3.7}$$

and

$$W^z(\bar{x},t) = \frac{z - z_T}{z_B - z_T}\hat{V}_B(x,y,t) + \frac{z_B - z}{z_B - z_T}\hat{V}_T(x,y,t) \tag{2.3.8}$$

$$V = \bar{V}^0 - W^z = \bar{V} - W^0 - W^z, \quad \text{so}$$

$$V = \bar{V} - W, \qquad \text{where } \; W = W^0 + W^z \tag{2.3.9}$$

Then (2.2.17)-(2.2.20) has the form

$$M\dot{V} - A\nabla^2 V = -M\dot{W} + A\nabla^2 W - \vec{b}_0(\nabla \cdot \dot{\vec{u}}) \qquad \text{in } \Omega \times (0,t_f)$$

with boundary conditions

$$V(\bar{x},t) = 0 \qquad \text{on } \Gamma \times [0,t_f)$$

and initial conditions

$$V(\bar{x},0) = V_0(\bar{x}) \qquad \text{in } \Omega$$

where

$$V_0(\bar{x}) = \tilde{V}_I(\bar{x}) - W(\bar{x}, 0) \tag{2.3.10}$$

Note that both the two-dimensional TCP system (2.1.19)-(2.1.21) and the three-dimensional TCP system (2.2.17)-(2.2.20) are equivalent to the following parabolic system with homogeneous boundary conditions:

$$M\dot{V} - A\nabla^2 V = -M\dot{W} + A\nabla^2 W - \vec{b}_0(\nabla \cdot \dot{\vec{u}}) \qquad \text{in } \Omega \times (0, t_f) \tag{2.3.11}$$

with boundary conditions

$$V(\bar{x}, t) = 0 \qquad \text{on } \Gamma \times [0, t_f) \tag{2.3.12}$$

and initial conditions

$$V(\bar{x}, 0) = V_0(\bar{x}) \qquad \text{in } \Omega \tag{2.3.13}$$

Now we diagonalize the TCP system (2.3.11)-(2.3.13). To that end, we make the following assumption.

Assumption 2.1. The matrix M is nonsingular and the matrix

$$\mathcal{A} = M^{-1}A \tag{2.3.14}$$

has three distinct positive eigenvalues $\lambda_1 > \lambda_2 > \lambda_3 > 0$.

Remark 2.3. Under Assumption 2.1, the matrix $\mathcal{A}$ is diagonalizable:

$$\mathcal{A} = PDP^{-1} \tag{2.3.15}$$

where

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \qquad P = [\vec{e}_1 \quad \vec{e}_2 \quad \vec{e}_3],$$

$\vec{e}_i$ is the eigenvector of $\mathcal{A}$ corresponding to the eigenvalue λ_i, $||\vec{e}_i|| = 1, \quad 1 \leq i \leq 3$.

Let

$$U = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = P^{-1}V \tag{2.3.16}$$

Using (2.3.14)-(2.3.16), the TCP system (2.3.11)-(2.3.13) has the form

$$\dot{U} - D\nabla^2 U = -P^{-1}\dot{W} + DP^{-1}\nabla^2 W - P^{-1}M^{-1}\vec{b}_0(\nabla \cdot \dot{\vec{u}}) \qquad \text{in } \Omega \times (0, t_f) \tag{2.3.17}$$

with boundary conditions

$$U(\bar{x}, t) = 0 \qquad \text{on } \Gamma \times [0, t_f) \tag{2.3.18}$$

and initial conditions

$$U(\bar{x}, 0) = P^{-1}V_0(\bar{x}) \qquad \text{in } \Omega \tag{2.3.19}$$

Define

$$F_U = -P^{-1}\dot{W} + DP^{-1}\nabla^2 W - P^{-1}M^{-1}\vec{b}_0(\nabla \cdot \dot{\vec{u}}) \tag{2.3.20}$$

$$U_0(\bar{x}) = U(\bar{x}, 0) = P^{-1}V_0(\bar{x}) \tag{2.3.21}$$

Then we have the following diagonalized TCP system:

$$\dot{U} - D\nabla^2 U = F_U \qquad \text{in } \Omega \times (0, t_f) \tag{2.3.22}$$

with homogeneous boundary conditions

$$U(\bar{x}, t) = 0 \qquad \text{on } \Gamma \times [0, t_f) \tag{2.3.23}$$

and initial conditions

$$U(\bar{x}, 0) = U_0(\bar{x}) \qquad \text{in } \Omega \tag{2.3.24}$$

In this chapter, we presented a fully coupled parabolic-elliptic initial-boundary value problem that models fully coupled thermal, chemical, hydraulic, and mechanical processes in homogeneous isotropic chemically active porous media saturated by a fluid. It was shown that, under appropriate assumptions on the matrices of diffusion coefficients, the parabolic initial-boundary value problem is equivalent to the diagonalized one with homogeneous boundary conditions. Using the principle of the minimum total potential energy, we derived the equivalent formulation of the Navier-type elliptic initial-boundary value problem. While the original elliptic system is developed in terms of velocity, the derived system is formulated in terms of the displacement vector and contains compatibility boundary conditions on stresses. The obtained results play a key role in well-posedness analysis presented in the next chapter.

Chapter 3

Well-Posedness of the Coupled TCPu Problem

We begin by defining what we mean by a well-posed problem.

Definition 3.1 (R. Showalter [36])**.** A problem is called well-posed if for each set of data there exists exactly one solution and this dependence of the solution on the data is continuous.

First, we will discuss the well-posedness of the parabolic initial-boundary value problem for the TCP system and the elliptic initial-boundary value problem for the elastic system considering coupling terms as data. Then we will investigate the existence and uniqueness of a solution to the coupled TCPu problem.

3.1 TCP Problem

To aid in giving a mathematical formulation of the problem, we make the following assumption.

Assumption 3.1. $V_B(t)$ and $\dot{V}_B(t)$ are $H^1\big([0,t_f];\mathbb{R}^3\big)$-functions; $V_B(z,t)$ and $\dot{V}_B(z,t)$ are $H^1\big(\Gamma_B\times[0,t_f];\mathbb{R}^3\big)$-functions; $V_F(\bar{x})$ is an $H^2(\Omega;\mathbb{R}^3)$-function; $\tilde{V}_T(\bar{x},t)$ and $\tilde{V}_B(\bar{x},t)$ are L^2-integrable functions from $[0,t_f]$ into $H^2(\Omega;\mathbb{R}^3)$; and $\dot{\tilde{V}}_T(\bar{x},t)$ and $\dot{\tilde{V}}_B(\bar{x},t)$ are L^2-integrable functions from $[0,t_f]$ into $L^2(\Omega;\mathbb{R}^3)$.

We start with the weak formulation of the system (2.3.22)-(2.3.24). Suppose that U is a classical solution of the system (2.3.22)-(2.3.24), say $U\in C^2(\bar{\Omega}\times[0,t_f])$.

Let $\varphi(\bar{x})=\begin{bmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\end{bmatrix}(\bar{x})\in C_0^\infty(\Omega;\mathbb{R}^3)$. Then from (2.3.22) we have

$$\int_\Omega \dot{U}\cdot\varphi\,\mathrm{d}\Omega - \int_\Omega (D\nabla^2 U)\cdot\varphi\,\mathrm{d}\Omega = \int_\Omega F_U\cdot\varphi\,\mathrm{d}\Omega \qquad \Rightarrow$$

$$\int_\Omega \dot{U}\cdot\varphi\,\mathrm{d}\Omega - \sum_{k=1}^{3}\int_\Omega \lambda_k \nabla^2 U_k \varphi_k\,\mathrm{d}\Omega = \int_\Omega F_U\cdot\varphi\,\mathrm{d}\Omega$$

Applying Green's first identity,

$$\int_\Omega \dot{U}\cdot\varphi\,\mathrm{d}\Omega + \sum_{k=1}^{3}\int_\Omega \lambda_k(\nabla U_k\cdot\nabla\varphi_k)\,\mathrm{d}\Omega = \int_\Omega F_U\cdot\varphi\,\mathrm{d}\Omega$$

Since

$$\int_\Omega \dot{U}\cdot\varphi\,\mathrm{d}\Omega = \frac{d}{dt}\int_\Omega U\cdot\varphi\,\mathrm{d}\Omega,$$

it follows that

$$\frac{d}{dt}\int_\Omega U\cdot\varphi\,\mathrm{d}\Omega + \sum_{k=1}^{3}\int_\Omega \lambda_k(\nabla U_k\cdot\nabla\varphi_k)\,\mathrm{d}\Omega = \int_\Omega F_U\cdot\varphi\,\mathrm{d}\Omega \tag{3.1.1}$$

This equation holds $\forall\,\varphi\in C_0^\infty(\Omega;\mathbb{R}^3)$ and, by continuity, it holds $\forall\,\varphi\in H_0^1(\Omega;\mathbb{R}^3)$.

Define the following spaces:

$$\mathcal{H} = L^2(\Omega;\mathbb{R}^3) = \left\{L^2(\Omega)\right\}^3$$

with the scalar product

$$(u,v)_\mathcal{H} = \sum_{k=1}^{3}(u_k,v_k)_{L^2(\Omega)} = \int_\Omega u\cdot v\,\mathrm{d}\Omega, \qquad u = \begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}, v = \begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} \in\mathcal{H} \tag{3.1.2}$$

and the norm $||u||_\mathcal{H} = (u,u)_\mathcal{H}^{1/2}$ and

$$\mathcal{V}_0 = H_0^1(\Omega;\mathbb{R}^3) = \left\{H_0^1(\Omega)\right\}^3$$

with the scalar product

$$(u,v)_{\mathcal{V}_0} = \sum_{k=1}^{3}(u_k,v_k)_{H_0^1(\Omega)} = \sum_{k=1}^{3}\int_\Omega \nabla u_k\cdot\nabla v_k\,\mathrm{d}\Omega, \quad u = \begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}, v = \begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} \in\mathcal{V}_0 \tag{3.1.3}$$

and the norm $||u||_{\mathcal{V}_0} = (u,u)^{1/2}_{\mathcal{V}_0}$.

The space $\mathcal{H}$ is the Cartesian product of three separable Hilbert spaces $L^2(\Omega)$ and the space $\mathcal{V}_0$ is the Cartesian product of three separable Hilbert spaces $H^1_0(\Omega)$, therefore $\mathcal{H}$ and $\mathcal{V}_0$ are also separable Hilbert spaces. Since $H^1_0(\Omega)$ is dense in $L^2(\Omega)$ and the injection $H^1_0(\Omega) \hookrightarrow L^2(\Omega)$ is continuous, it follows that $\mathcal{V}_0$ is dense in $\mathcal{H}$ and the injection $\mathcal{V}_0 \hookrightarrow \mathcal{H}$ is continuous. Also, $L^2(\Omega)$ is identified with its dual space $L^2(\Omega)'$ and $L^2(\Omega)'$ is dense in $H^1_0(\Omega)' = H^{-1}(\Omega)$. It follows that $\mathcal{H}$ is identified with its dual space $\mathcal{H}'$ and $\mathcal{H}'$ is dense in $\mathcal{V}_0' = \left\{H^{-1}(\Omega)\right\}^3 = H^{-1}(\Omega;\mathbb{R}^3)$. We have

$$\mathcal{V}_0 \subset \mathcal{H} = \mathcal{H}' \subset \mathcal{V}_0' \tag{3.1.4}$$

Also, the scalar product in $\mathcal{H}$ of $u \in \mathcal{H}$ and $v \in \mathcal{V}_0$ is the same as the scalar product of u and v in the duality between $\mathcal{V}_0'$ and $\mathcal{V}_0$, denoted by $< u, v >$:

$$< u, v >= (u,v)_{\mathcal{H}}, \qquad u \in \mathcal{H}, v \in \mathcal{V}_0 \tag{3.1.5}$$

Define the bilinear form $a : \mathcal{V}_0 \times \mathcal{V}_0 \to \mathbb{R}$ by

$$a(u,v) = \sum_{k=1}^{3} \lambda_k \int_\Omega \nabla u_k \cdot \nabla v_k \, \mathrm{d}\Omega \tag{3.1.6}$$

$a(\cdot,\cdot)$ is symmetric:

$$a(u,v) = a(v,u) \tag{3.1.7}$$

$$a(u,u) = \sum_{k=1}^{3} \int_\Omega \lambda_k \nabla u_k \cdot \nabla u_k \, \mathrm{d}\Omega = \sum_{k=1}^{3} \int_\Omega \lambda_k |\nabla u_k|^2 \, \mathrm{d}\Omega \geq$$

$$\geq \{\min_k \lambda_k\} \sum_{k=1}^{3} \int_\Omega |\nabla u_k|^2 \, \mathrm{d}\Omega = \lambda_3 ||u||^2_{\mathcal{V}_0}$$

Thus,

$$a(u,u) \geq \lambda_3 ||u||^2_{\mathcal{V}_0}, \qquad \lambda_3 = \min_k \lambda_k > 0 \tag{3.1.8}$$

Also,

$$|a(u,v)| = \left|\sum_{k=1}^{3}\int_{\Omega}\lambda_k \nabla u_k \cdot \nabla v_k \,\mathrm{d}\Omega\right| \leq \left\{\underset{k}{max}\lambda_k\right\}\sum_{k=1}^{3}\left|(u_k, v_k)_{H_0^1(\Omega)}\right|$$

$$\leq \lambda_1 \sum_{k=1}^{3} ||u_k||_{H_0^1(\Omega)}||v_k||_{H_0^1(\Omega)} \leq \lambda_1 \sum_{i=1}^{3}||u_i||_{H_0^1(\Omega)}\sum_{j=1}^{3}||v_j||_{H_0^1(\Omega)}$$

$$\leq 3\lambda_1 ||u||_{\mathcal{V}_0}||v||_{\mathcal{V}_0}, \qquad \text{so}$$

$$|a(u,v)| \leq 3\lambda_1 ||u||_{\mathcal{V}_0}||v||_{\mathcal{V}_0} \tag{3.1.9}$$

From (3.1.7)-(3.1.9) it follows that $a(\cdot,\cdot)$ defines a scalar product on $\mathcal{V}_0$. We denote this scalar product by

$$(u,v)_a = a(u,v), \qquad u, v \in \mathcal{V}_0 \tag{3.1.10}$$

and the corresponding (energy) norm by $||u||_a = (u,u)_a^{1/2}$. Furthermore, the norm $||\cdot||_a$ is equivalent to the norm $||\cdot||_{\mathcal{V}_0}$.

For each $u \in \mathcal{V}_0$, the form $u \in \mathcal{V}_0 \to (u,v)_a \in \mathbb{R}$ is linear and continuous on $\mathcal{V}_0$. Therefore, there exists an element $Au \in \mathcal{V}_0'$ such that

$$(u,v)_a =< Au, v >, \qquad \forall\ v \in \mathcal{V}_0 \tag{3.1.11}$$

From (3.1.9)-(3.1.11), the mapping $u \mapsto Au$ is linear and continuous, that is, $A \in \mathcal{L}(\mathcal{V}_0, \mathcal{V}_0')$. Also, by the Riesz representation theorem, for every $Au \in \mathcal{V}_0'$, there exists a unique $u \in \mathcal{V}_0$ such that (3.1.11) holds. Thus, $A : \mathcal{V}_0 \to \mathcal{V}_0'$ is an isomorphism from $\mathcal{V}_0$ onto $\mathcal{V}_0'$.

Let X be a Hilbert space. We introduce the space $L^2(0, t_f; \mathrm{X})$ of L^2-integrable functions from $[0, t_f]$ into X with the norm

$$||u||_{L^2(0,t_f;\mathrm{X})} = \left[\int_0^{t_f} ||u(t)||_{\mathrm{X}}^2 \,\mathrm{d}t\right]^{1/2}$$

Note that if $u \in L^2(0, t_f; \mathcal{V}_0)$, then $t \mapsto Au(t) \in \mathcal{V}_0'$, that is,

$$A \in \mathcal{L}\big(L^2(0,t_f;\mathcal{V}_0), L^2(0,t_f;\mathcal{V}_0')\big)$$

From Assumption 3.1, (3.1.1), (3.1.2), (3.1.5), (3.1.6), and (3.1.10), we obtain the following weak formulation of the problem (2.3.22)-(2.3.24):

Given $F_U \in L^2(0,t_f;\mathcal{V}_0')$ and $U_0 \in \mathcal{H}$, find $U \in L^2(0,t_f;\mathcal{V}_0)$ such that

$$\frac{d}{dt}(U,\varphi)_{\mathcal{H}} + (U,\varphi)_a =< F_U,\varphi >, \qquad \forall\, \varphi \in \mathcal{V}_0 \tag{3.1.12}$$

and

$$U(0) = U_0 \qquad \text{in } \mathcal{H} \tag{3.1.13}$$

The meaning of (3.1.13) will be explained below.

By (3.1.5) and (3.1.10), the equation (3.1.12) can be written as

$$\frac{d}{dt} < U,\varphi > + < AU,\varphi >=< F_U,\varphi >, \qquad \forall\, \varphi \in \mathcal{V}_0$$

and so

$$\frac{d}{dt} < U,\varphi >=< F_U - AU,\varphi >, \qquad \forall\, \varphi \in \mathcal{V}_0 \tag{3.1.14}$$

Since $F_U \in L^2(0,t_f;\mathcal{V}_0')$ and $AU \in L^2(0,t_f;\mathcal{V}_0')$, it follows that $F_U - AU \in L^2(0,t_f;\mathcal{V}_0')$. From [40],

$$\frac{d}{dt} < U,\varphi >=< \dot{U},\varphi >, \qquad \forall\, \varphi \in \mathcal{V}_0 \tag{3.1.15}$$

in the scalar distribution sense on $(0,t_f)$ and, from (3.1.14) and (3.1.15), $\dot{U} \in L^2(0,t_f;\mathcal{V}_0')$.

We introduce the following Hilbert space [40, 27]:

$$\mathcal{W}(0,t_f;\mathcal{V}_0) = \left\{u : \ u \in L^2(0,t_f;\mathcal{V}_0),\ \dot{u} \in L^2(0,t_f;\mathcal{V}_0')\right\} \tag{3.1.16}$$

with the norm

$$||u||_{\mathcal{W}} = \left[\int_0^{t_f} ||u(t)||^2_{\mathcal{V}_0}\,\mathrm{d}t + \int_0^{t_f} ||\dot{u}(t)||^2_{\mathcal{V}_0'}\,\mathrm{d}t\right]^{1/2} \tag{3.1.17}$$

Remark 3.1 ([40]). Each $u \in \mathcal{W}(0,t_f;\mathcal{V}_0)$ is almost everywhere equal to a function continuous from $[0,t_f]$ into $\mathcal{H}$, and the following inequality holds in the scalar distribution sense on $(0,t_f)$:

$$2 < \dot{u}, u >= \frac{d}{dt}||u||^2_{\mathcal{H}} \tag{3.1.18}$$

The initial condition (3.1.13) now makes sense.

From (3.1.14) and (3.1.15),

$$< \dot{U}, \varphi >=< F_U - AU, \varphi >, \qquad \forall\, \varphi \in \mathcal{V}_0, \qquad \text{so}$$

$$\dot{U} + AU = F_U$$

Thus, the alternative formulation of the weak problem (3.1.12)-(3.1.13) is the following.

Given $F_U \in L^2(0,t_f;\mathcal{V}_0')$ and $U_0 \in \mathcal{H}$, find $U \in \mathcal{W}(0,t_f;\mathcal{V}_0)$ such that

$$\dot{U} + AU = F_U \qquad \text{on } (0,t_f) \tag{3.1.19}$$

and

$$U(0) = U_0 \qquad \text{in } \mathcal{H} \tag{3.1.13}$$

The well-posedness of the problem (3.1.19), (3.1.13) is proved in [40, 27] and we have the following result.

Theorem 3.1. *Problem (3.1.19), (3.1.13) admits a unique solution in* $\mathcal{W}(0,t_f;\mathcal{V}_0)$*. Furthermore, the solution depends continuously on the data: the bilinear map*

$$F_U, U_0 \mapsto U$$

is continuous from $L^2(0,t_f;\mathcal{V}_0') \times \mathcal{H}$ *to* $\mathcal{W}(0,t_f;\mathcal{V}_0)$*.*

Corollary 3.1. *The TCP system (2.1.19)-(2.1.21) admits a unique weak solution in* $L^2\big(0,t_f;H^1(\Omega,\mathbb{R}^3)\big)$ *and this solution depends continuously on the data.*

Proof. The result follows immediately from Assumption 3.1, (2.3.1), (2.3.2), and (2.3.16). □

Corollary 3.2. *The TCP system (2.2.17)-(2.2.19) admits a unique weak solution in* $L^2\big(0, t_f; H^1(\Omega, \mathbb{R}^3)\big)$ *and this solution depends continuously on the data.*

Proof. The result follows immediately from Assumption 3.1, (2.3.4)-(2.3.9), and (2.3.16). □

We finish this section with the following theorem which will be needed later.

Theorem 3.2. *If* $F_U \in L^2(0, t_f; \mathcal{H})$ *and* $U_0 \in \mathcal{V}_0$*, then* $\dot{U} \in L^2(0, t_f, \mathcal{H})$ *and*

$$||\dot{U}||_{L^2(0,t_f,\mathcal{H})} \leq \sqrt{\lambda_1}||U_0||_{\mathcal{V}_0} + ||F_U||_{L^2(0,t_f,\mathcal{H})} \tag{3.1.20}$$

Proof. Since $\mathcal{V}_0$ is separable, there exists a countable set $w_1, \ldots, w_m, \ldots$ which is dense in $\mathcal{V}_0$. For each m, define an approximate solution $U_m(\bar{x}, t)$ as

$$U_m(\bar{x}, t) = \sum_{i=1}^{m} g_{im}(t) w_i(\bar{x}) \tag{3.1.21}$$

where $g_{im}(t)$, $1 \leq i \leq m$, are the scalar functions defined on $[0, t_f]$, $U_m(\bar{x}, t)$ satisfies

$$(\dot{U}_m, w_j)_{\mathcal{H}} + (U_m, w_j)_a =< F_U, w_j >, \qquad j = 1, \ldots, m \tag{3.1.22}$$

$$U_m(\bar{x}, 0) = U_{0_m} = \sum_{i=1}^{m} \xi_{im} w_i(\bar{x}) \underset{m\to\infty}{\longrightarrow} U_0 \qquad \text{in } \mathcal{H} \tag{3.1.23}$$

and

$$U_m(0) = U_{0_m} \tag{3.1.24}$$

where U_{0_m} is the orthogonal projection in $\mathcal{H}$ of U_0 on the space spanned by $w_1, \ldots, w_m$.

We multiply (3.1.22) by $\dot{g}_{jm}(t)$ and add for j:

$$||\dot{U}_m(t)||^2_{\mathcal{H}} + (U_m(t), \dot{U}_m(t))_a =< F_U(t), \dot{U}_m(t) >$$

Using (3.1.23), (3.1.24), and

$$(U_m(t), \dot{U}_m(t))_a = \frac{1}{2}\frac{d}{dt}||U_m(t)||_a^2$$

and integrating over $(0, t_f)$, we have

$$2\int_0^{t_f} ||\dot{U}_m(t)||_{\mathcal{H}}^2 \,\mathrm{dt} + \int_0^{t_f} \frac{d}{dt}||U_m(t)||_a^2 \,\mathrm{dt} = \int_0^{t_f} 2 < F_U(t), \dot{U}_m(t) > \,\mathrm{dt} \qquad \Rightarrow$$

$$2\int_0^{t_f} ||\dot{U}_m(t)||_{\mathcal{H}}^2 \,\mathrm{dt} + ||U_m(t_f)||_a^2 - ||U_m(0)||_a^2 \leq$$

$$\leq \int_0^{t_f} ||F_U(t)||_{\mathcal{H}}^2 \,\mathrm{dt} + \int_0^{t_f} ||\dot{U}_m(t)||_{\mathcal{H}}^2 \,\mathrm{dt} \qquad \Rightarrow$$

$$||\dot{U}_m||_{L^2(0,t_f;\mathcal{H})}^2 \leq ||U_{0_m}||_a^2 + ||F||_{L^2(0,t_f;\mathcal{H})}^2 \tag{3.1.25}$$

Note that

$$||U||_a^2 = a(U,U) = \sum_{k=1}^{3}\int_\Omega \lambda_k |\nabla U_k|^2 \,\mathrm{d}\Omega \leq \{\underset{k}{max}\lambda_k\} \sum_{k=1}^{3}\int_\Omega |\nabla U_k|^2 \,\mathrm{d}\Omega = \lambda_1 ||U||_{\mathcal{V}_0}^2$$

and so

$$||U||_a \leq \sqrt{\lambda_1}||U||_{\mathcal{V}_0} \tag{3.1.26}$$

Applying (3.1.26) to (3.1.25), we obtain

$$||\dot{U}_m||_{L^2(0,t_f;\mathcal{H})}^2 \leq \lambda_1 ||U_0||_{\mathcal{V}_0}^2 + ||F||_{L^2(0,t_f;\mathcal{H})}^2$$

Thus,

$$||\dot{U}_m||_{L^2(0,t_f;\mathcal{H})} \leq \left[\lambda_1 ||U_0||_{\mathcal{V}_0}^2 + ||F||_{L^2(0,t_f;\mathcal{H})}^2\right]^{1/2}$$

This shows that the sequence $\dot{U}_m$ ranges in a bounded set of $L^2(0, t_f; \mathcal{H})$. Therefore, $\dot{U} \in L^2(0, t_f; \mathcal{H})$ and

$$||\dot{U}||_{L^2(0,t_f,\mathcal{H})} \leq \sqrt{\lambda_1}||U_0||_{\mathcal{V}_0} + ||F_U||_{L^2(0,t_f,\mathcal{H})}$$

□

3.2 Elastic Problem

Using notations from Chapter 2, let

$$H = L^2\big(\Omega;\mathbb{R}^n\big) = \big\{L^2\big(\Omega;\mathbb{R}\big)\big\}^n$$

$$V = H^1\big(\Omega;\mathbb{R}^n\big) = \big\{H^1\big(\Omega;\mathbb{R}\big)\big\}^n$$

$$V_0 = \big\{\varphi \in V : \varphi\big|_{\Gamma_F} = 0\big\}$$

where $n = 2, 3$ is the spatial dimensionality of the problem.

We start with the weak formulation of the two-dimensional Navier-type elastic system

$$\big(K + \frac{G}{3}\big)\nabla(\nabla\cdot\vec{u}) + G\nabla^2\vec{u} = \nabla f \qquad \text{in } \Omega\times(0,t_f) \tag{2.1.45}$$

$$\tau\vec{n} = (fI + \hat{\sigma})\vec{n} \qquad \text{on } \Gamma_B\times(0,t_f) \tag{2.1.43}$$

$$\vec{u} \approx 0 \qquad \text{on } \Gamma_F\times(0,t_f) \tag{2.1.14}$$

Take an arbitrary $\vec{\Phi}(\bar{x}) = \phi(\bar{x})\vec{i} + \psi(\bar{x})\vec{j} \in V_0$. From (2.1.27) and (2.1.45) we have

$$\big(K + \frac{G}{3}\big)\int_\Omega \nabla(\nabla\cdot\vec{u})\cdot\vec{\Phi}\,\mathrm{d}\Omega + G\int_\Omega(\nabla^2\vec{u})\cdot\vec{\Phi}\,\mathrm{d}\Omega = \int_\Omega \nabla f\cdot\vec{\Phi}\,\mathrm{d}\Omega \qquad \Rightarrow$$

$$\big(K + \frac{G}{3}\big)\int_\Omega (u_x + v_y)_x\phi + (u_x + v_y)_y\psi\,\mathrm{d}\Omega +$$
$$+G\int_\Omega (u_{xx} + u_{yy})\phi + (v_{xx} + v_{yy})\psi\,\mathrm{d}\Omega = \int_\Omega \nabla f\cdot\vec{\Phi}\,\mathrm{d}\Omega \qquad \Rightarrow$$

$$\int_\Omega \Big[\big(K + \frac{G}{3}\big)u_{xx} + \big(K - \frac{2G}{3}\big)v_{yx} + Gu_{xx} + Gu_{yy} + Gv_{xy}\Big]\phi +$$
$$+\Big[\big(K + \frac{G}{3}\big)v_{yy} + \big(K - \frac{2G}{3}\big)u_{xy} + Gv_{yy} + Gv_{xx} + Gu_{yx}\Big]\psi\,\mathrm{d}\Omega =$$

$$= \int_\Omega \left\{ \Big[\big(K + \frac{G}{3}\big) u_x + \big(K - \frac{2G}{3}\big) v_y + G u_x \Big]_x + G(u_y + v_x)_y \right\} \phi +$$

$$+ \left\{ \Big[\big(K + \frac{G}{3}\big) v_y + \big(K - \frac{2G}{3}\big) u_x + G v_y \Big]_y + G(u_y + v_x)_x \right\} \psi \, \mathrm{d}\Omega =$$

$$= \int_\Omega (\tau_{11_x} + \tau_{12_y})\phi + (\tau_{21_x} + \tau_{22_y})\psi \, \mathrm{d}\Omega = \int_\Omega \nabla f \cdot \vec{\Phi} \, \mathrm{d}\Omega \qquad \Rightarrow$$

$$\int_\Omega (\nabla \cdot \vec{\tau}_1)\phi + (\nabla \cdot \vec{\tau}_2)\psi \, \mathrm{d}\Omega = \int_\Omega \nabla f \cdot \vec{\Phi} \, \mathrm{d}\Omega \qquad \Rightarrow$$

$$\int_\Omega \nabla \cdot (\phi \vec{\tau}_1) - \vec{\tau}_1 \cdot \nabla \phi + \nabla \cdot (\psi \vec{\tau}_2) - \vec{\tau}_2 \cdot \nabla \psi \, \mathrm{d}\Omega = \int_\Omega \nabla \cdot (f \vec{\Phi}) - f \nabla \cdot \vec{\Phi} \, \mathrm{d}\Omega \quad \Rightarrow$$

$$\int_{\Gamma_B} (\phi \vec{\tau}_1 \cdot \vec{n} + \psi \vec{\tau}_2 \cdot \vec{n}) \, \mathrm{d}\Gamma - \int_\Omega \vec{\tau}_1 \cdot \nabla \phi + \vec{\tau}_2 \cdot \nabla \psi \, \mathrm{d}\Omega = \int_{\Gamma_B} f \vec{\Phi} \cdot \vec{n} \, \mathrm{d}\Gamma - \int_\Omega f \nabla \cdot \vec{\Phi} \, \mathrm{d}\Omega$$

$$\Rightarrow \qquad \int_{\Gamma_B} (\tau \vec{n}) \cdot \vec{\Phi} \, \mathrm{d}\Gamma - \int_\Omega (\tau_{11}\phi_x + \tau_{12}\phi_y + \tau_{21}\psi_x + \tau_{22}\psi_y) \, \mathrm{d}\Omega =$$

$$= \int_{\Gamma_B} (f \vec{\Phi}) \cdot \vec{n} \, \mathrm{d}\Gamma - \int_\Omega f(\phi_x + \psi_y) \, \mathrm{d}\Omega \quad \Rightarrow$$

$$\int_\Omega (\tau_{11}\phi_x + \tau_{12}(\phi_y + \psi_x) + \tau_{22}\psi_y) \, \mathrm{d}\Omega - \int_\Omega f(\phi_x + \psi_y) \, \mathrm{d}\Omega +$$

$$\Big[- \int_{\Gamma_B} (\tau \vec{n}) \cdot \vec{\Phi} \, \mathrm{d}\Gamma + \int_{\Gamma_B} (f I \vec{n}) \cdot \vec{\Phi} \, \mathrm{d}\Gamma \Big] = 0$$

Using (2.1.32), (2.1.38) and (2.1.43),

$$\int_\Omega (\tau_{11}\phi_x + \tau_{12}(\phi_y + \psi_x) + \tau_{22}\psi_y) \, \mathrm{d}\Omega - \int_\Omega f(\phi_x + \psi_y) \, \mathrm{d}\Omega -$$

$$- \int_{\Gamma_B} (\hat{\sigma} \vec{n}) \cdot \vec{\Phi} \, \mathrm{d}\Gamma = 0 \quad \Rightarrow$$

$$\int_\Omega \tau_{11}\varepsilon_{11}(\vec{\Phi}) + 2\tau_{12}\varepsilon_{12}(\vec{\Phi}) + \tau_{22}\varepsilon_{22}(\vec{\Phi})\,\mathrm{d}\Omega - \int_\Omega f\,\mathrm{tr}\varepsilon(\vec{\Phi})\,\mathrm{d}\Omega - \int_{\Gamma_B} (\hat{\sigma}\vec{n})\cdot\vec{\Phi}\,\mathrm{d}\Gamma = 0$$

$$\Rightarrow \qquad D\mathcal{V}(\vec{u})\vec{\Phi} = 0, \qquad \forall\, \vec{\Phi} \in V_0$$

Thus,

$$D\mathcal{V}(\vec{u})\vec{\Phi} = 0, \qquad \forall\, \vec{\Phi} \in V_0 \tag{3.2.1}$$

is a weak formulation of the elastic system (2.1.43) and (2.1.45). As before, $D\mathcal{V}(\vec{u})\vec{\Phi}$ is the Gâteaux differential of $\mathcal{V}$ with increment $\vec{\Phi}$.

Now we obtain the weak formulation of the three-dimensional Navier-type elastic system

$$\big(K + \frac{G}{3}\big)\nabla(\nabla\cdot\vec{u}) + G\nabla^2\vec{u} = \nabla f \qquad \text{in } \Omega\times(0,t_f) \tag{2.2.43}$$

$$\tau\vec{n} = (fI + \hat{\sigma})\vec{n} \qquad \text{on } \Gamma\setminus\Gamma_F\times(0,t_f) \tag{2.2.41}$$

$$\vec{u} \approx 0 \qquad \text{on } \Gamma_F\times(0,t_f) \tag{2.2.14}$$

Take an arbitrary $\vec{\Phi}(\bar{x}) = \phi(\bar{x})\vec{i} + \psi(\bar{x})\vec{j} + \omega(\bar{x})\vec{k} \in V_0$. From (2.2.43) we have

$$\big(K + \frac{G}{3}\big)\int_\Omega \nabla(\nabla\cdot\vec{u})\cdot\vec{\Phi}\,\mathrm{d}\Omega + G\int_\Omega (\nabla^2\vec{u})\cdot\vec{\Phi}\,\mathrm{d}\Omega = \int_\Omega \nabla f\cdot\vec{\Phi}\,\mathrm{d}\Omega \qquad \Rightarrow$$

$$\big(K + \frac{G}{3}\big)\int_\Omega (u_x+v_y+w_z)_x\phi + (u_x+v_y+w_z)_y\psi + (u_x+v_y+w_z)_z\omega\,\mathrm{d}\Omega +$$

$$+G\int_\Omega (u_{xx}+u_{yy}+u_{zz})\phi + (v_{xx}+v_{yy}+v_{zz})\psi + (w_{xx}+w_{yy}+w_{zz})\omega\,\mathrm{d}\Omega =$$

$$= \int_\Omega \nabla f\cdot\vec{\Phi}\,\mathrm{d}\Omega \qquad \Rightarrow$$

$$\int_\Omega \Big[\big(K+\frac{G}{3}\big)u_{xx} + \big(K - \frac{2G}{3}\big)(v_y+w_z)_x + G(u_{xx}+v_{xy}+w_{zx}+u_{yy}+u_{zz})\Big]\phi +$$

$$+\Big[\big(K+\frac{G}{3}\big)v_{yy} + \big(K - \frac{2G}{3}\big)(u_x+w_z)_y + G(v_{yy}+u_{xy}+w_{zy}+v_{xx}+v_{zz})\Big]\psi +$$

$$+\Big[\big(K+\frac{G}{3}\big)w_{zz}+\big(K-\frac{2G}{3}\big)(u_x+v_y)_z+G(w_{zz}+u_{xz}+v_{yz}+w_{xx}+w_{zz})\Big]\omega\,\mathrm{d}\Omega=$$

$$\int_\Omega\Bigg\{\Big[\big(K+\frac{G}{3}\big)u_x+\big(K-\frac{2G}{3}\big)(v_y+w_z)+Gu_x\Big]_x+$$

$$+G(u_y+v_x)_y+G(w_x+u_z)_z\Bigg\}\phi+$$

$$+\Bigg\{\Big[\big(K+\frac{G}{3}\big)v_y+\big(K-\frac{2G}{3}\big)(u_x+w_z)+Gv_y\Big]_y+$$

$$+G(u_y+v_x)_x+G(w_y+v_z)_z\Bigg\}\psi+$$

$$+\Bigg\{\Big[\big(K+\frac{G}{3}\big)w_z+\big(K-\frac{2G}{3}\big)(u_x+v_y)+Gw_z\Big]_z+$$

$$+G(u_z+w_x)_x+G(v_z+w_y)_y\Bigg\}\omega\,\mathrm{d}\Omega=$$

$$=\int_\Omega(\tau_{11_x}+\tau_{12_y}+\tau_{13_z})\phi+(\tau_{21_x}+\tau_{22_y}+\tau_{23_z})\psi+(\tau_{31_x}+\tau_{32_y}+\tau_{33_z})\omega\,\mathrm{d}\Omega=$$

$$=\int_\Omega(\nabla\cdot\vec{\tau}_1)\phi+(\nabla\cdot\vec{\tau}_2)\psi+(\nabla\cdot\vec{\tau}_3)\omega\,\mathrm{d}\Omega=\int_\Omega\nabla f\cdot\vec{\Phi}\,\mathrm{d}\Omega\qquad\Rightarrow$$

$$\int_\Omega\nabla\cdot(\phi\vec{\tau}_1)-\vec{\tau}_1\cdot\nabla\phi+\nabla\cdot(\psi\vec{\tau}_2)-\vec{\tau}_2\cdot\nabla\psi+\nabla\cdot(\omega\vec{\tau}_3)-\vec{\tau}_3\cdot\nabla\omega\,\mathrm{d}\Omega=$$

$$=\int_\Omega\nabla\cdot(f\vec{\Phi})-f\nabla\cdot\vec{\Phi}\,\mathrm{d}\Omega\quad\Rightarrow$$

$$\int_{\Gamma\setminus\Gamma_F}(\phi\vec{\tau}_1\cdot\vec{n}+\psi\vec{\tau}_2\cdot\vec{n}+\omega\vec{\tau}_3\cdot\vec{n})\,\mathrm{d}\Gamma-\int_\Omega\vec{\tau}_1\cdot\nabla\phi+\vec{\tau}_2\cdot\nabla\psi+\vec{\tau}_3\cdot\nabla\omega\,\mathrm{d}\Omega=$$

$$=\int_{\Gamma\setminus\Gamma_F}f\vec{\Phi}\cdot\vec{n}\,\mathrm{d}\Gamma-\int_\Omega f\nabla\cdot\vec{\Phi}\,\mathrm{d}\Omega\qquad\Rightarrow$$

$$\int_{\Gamma\setminus\Gamma_F}(\tau\vec{n})\cdot\vec{\Phi}\,\mathrm{d}\Gamma-\int_\Omega(\tau_{11}\phi_x+\tau_{12}\phi_y+\tau_{13}\phi_z+$$

$$+\tau_{21}\psi_x + \tau_{22}\psi_y + \tau_{23}\psi_z + \tau_{31}\omega_x + \tau_{32}\omega_y + \tau_{33}\omega_z)\,\mathrm{d}\Omega =$$

$$= \int_{\Gamma\setminus\Gamma_F} (f\vec{\Phi})\cdot\vec{n}\,\mathrm{d}\Gamma - \int_\Omega f(\phi_x + \psi_y + \omega_z)\,\mathrm{d}\Omega \quad \Rightarrow$$

$$\int_\Omega (\tau_{11}\phi_x + \tau_{12}(\phi_y + \psi_x) + \tau_{22}\psi_y + \tau_{13}(\phi_z + \omega_x) + \tau_{23}(\psi_z + \omega_y) + \tau_{33}\omega_z)\,\mathrm{d}\Omega -$$

$$- \int_\Omega f(\phi_x + \psi_y + \omega_z)\,\mathrm{d}\Omega + \Big[- \int_{\Gamma\setminus\Gamma_F} (\tau\vec{n})\cdot\vec{\Phi}\,\mathrm{d}\Gamma + \int_{\Gamma\setminus\Gamma_F} (fI\vec{n})\cdot\vec{\Phi}\,\mathrm{d}\Gamma \Big] = 0$$

Using (2.2.30), (2.2.35), and (2.2.41),

$$\int_\Omega (\tau_{11}\phi_x + \tau_{12}(\phi_y + \psi_x) + \tau_{22}\psi_y + \tau_{13}(\phi_z + \omega_x) + \tau_{23}(\psi_z + \omega_y) + \tau_{33}\omega_z)\,\mathrm{d}\Omega -$$

$$- \int_\Omega f(\phi_x + \psi_y + \omega_z)\,\mathrm{d}\Omega - \int_{\Gamma\setminus\Gamma_F} (\hat{\sigma}\vec{n})\cdot\vec{\Phi}\,\mathrm{d}\Gamma = 0 \quad \Rightarrow$$

$$\int_\Omega \tau_{11}\varepsilon_{11}(\vec{\Phi}) + 2\tau_{12}\varepsilon_{12}(\vec{\Phi}) + \tau_{22}\varepsilon_{22}(\vec{\Phi}) + 2\tau_{13}\varepsilon_{13}(\vec{\Phi}) + 2\tau_{23}\varepsilon_{23}(\vec{\Phi}) + \tau_{33}\varepsilon_{33}(\vec{\Phi})\,\mathrm{d}\Omega$$

$$- \int_\Omega f\mathrm{tr}\varepsilon(\vec{\Phi})\,\mathrm{d}\Omega - \int_{\Gamma\setminus\Gamma_F} (\hat{\sigma}\vec{n})\cdot\vec{\Phi}\,\mathrm{d}\Gamma = 0$$

$$\Rightarrow \qquad D\mathcal{V}(\vec{u})\vec{\Phi} = 0, \qquad \forall\ \vec{\Phi} \in V_0$$

Thus,

$$D\mathcal{V}(\vec{u})\vec{\Phi} = 0, \qquad \forall\ \vec{\Phi} \in V_0 \tag{3.2.2}$$

is a weak formulation of the elastic system (2.2.41) and (2.2.43). As before, $D\mathcal{V}(\vec{u})\vec{\Phi}$ is the Gâteaux differential of $\mathcal{V}$ with increment $\vec{\Phi}$.

From (2.1.35), (2.2.33), (3.2.1), and (3.2.2), with the assumption that the applied boundary stress $\hat{\sigma}$ is bounded on $\bar{\Omega} \times [0, t_f]$, the weak formulation of the two- and three-dimensional elastic problems has the following form:

Given $f \in L^2\big(0, t_f; H^1(\Omega)\big)$, find $\vec{u} \in L^2(0, t_f; V_0)$ such that $\dot{\vec{u}} \in L^2(0, t_f; V_0)$ and

$$a_E(\vec{u}, \vec{\Phi}) - \int_\Omega f\big(\nabla \cdot \vec{\Phi}\big)\,\mathrm{d}\Omega - \int_{\Gamma\setminus\Gamma_F} (\hat{\sigma}\vec{n}) \cdot \vec{\Phi}\,\mathrm{d}\Gamma = 0, \qquad \forall\, \vec{\Phi} \in V_0 \tag{3.2.3}$$

Next we prove the well-posedness of the elastic system (3.2.3). Define a continuous linear functional F on V by a pairing

$$\vec{\Phi} \mapsto < F, \vec{\Phi} >$$

where

$$< F, \vec{\Phi} >= \int_\Omega f(\nabla \cdot \vec{\Phi})\,\mathrm{d}\Omega + \int_{\Gamma\setminus\Gamma_F} (\hat{\sigma}\vec{n}) \cdot \vec{\Phi}\,\mathrm{d}\Gamma, \qquad \forall\, \vec{\Phi} \in V \tag{3.2.4}$$

Since F is continuous on V, there exists a constant $c_F > 0$ such that

$$| < F, \vec{\Phi} > | \leq c_F ||\vec{\Phi}||_V, \qquad \forall\, \vec{\Phi} \in V \tag{3.2.5}$$

From (3.2.3) and (3.2.4) in follows that

$$a_E(\vec{u}, \vec{\Phi}) =< F, \vec{\Phi} >, \qquad \forall\, \vec{\Phi} \in V \tag{3.2.6}$$

The existence and uniqueness of a solution to the elastic problem is based on Korn's inequality [16]:

Theorem 3.3. *Let Ω be a bounded open set with a regular boundary. Then there exists a constant $c > 0$ that depends on Ω such that*

$$\int_\Omega \varepsilon_{ij}(\vec{v})\varepsilon_{ij}(\vec{v})\,\mathrm{d}\Omega + \int_\Omega v_i^2\,\mathrm{d}\Omega \geq c||\vec{v}||_V^2, \qquad \forall\, \vec{v} \in V$$

The consequence of Theorem 3.3 is the following result [16]:

Theorem 3.4. *Let Ω be a bounded open set with a regular boundary. Let $\Gamma_F \subset \Gamma$ and Γ_F have positive measure. Then there exists a constant $c_E > 0$ such that*

$$a_E(\vec{v}, \vec{v}) \geq c_E||\vec{v}||_V^2, \qquad \forall\, \vec{v} \in V_0 \tag{3.2.7}$$

For the sake of convenience, here and in the following we will take

$$\vec{u} = u_1\vec{i} + u_2\vec{j}, \quad \vec{v} = v_1\vec{i} + v_2\vec{j}, \quad \text{and} \quad \bar{x} = x_1\vec{i} + x_2\vec{j} \in \Omega$$

for the 2D elastic problem and

$$\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}, \quad \vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}, \quad \text{and} \quad \bar{x} = x_1\vec{i} + x_2\vec{j} + x_3\vec{k} \in \Omega$$

for the 3D elastic problem.

Then, from (2.1.23), (2.1.33), and (2.2.31), for every $\vec{u}, \vec{v} \in V$,

$$|a_E(\vec{u},\vec{v})| = \left| \int_\Omega a_{ijkl}\varepsilon_{ij}(\vec{u})\varepsilon_{kl}(\vec{v})\,\mathrm{d}\Omega \right| \le \underset{i,j,k,l}{max}\{a_{ijkl}\} \sum_{i,j,k,l=1}^{n} \left| \int_\Omega \varepsilon_{ij}(\vec{u})\varepsilon_{kl}(\vec{v})\,\mathrm{d}\Omega \right| =$$

$$= \frac{1}{4}\underset{i,j,k,l}{max}\{a_{ijkl}\} \sum_{i,j,k,l=1}^{n} \left| \int_\Omega \Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big)\Big(\frac{\partial v_k}{\partial x_l} + \frac{\partial v_l}{\partial x_k}\Big)\,\mathrm{d}\Omega \right| \le$$

$$\le \frac{1}{4}\underset{i,j,k,l}{max}\{a_{ijkl}\} \sum_{i,j,k,l=1}^{n} \Big\{ \left\|\frac{\partial u_i}{\partial x_j}\right\|_{L^2(\Omega)} \left\|\frac{\partial v_k}{\partial x_l}\right\|_{L^2(\Omega)} + \left\|\frac{\partial u_j}{\partial x_i}\right\|_{L^2(\Omega)} \left\|\frac{\partial v_k}{\partial x_l}\right\|_{L^2(\Omega)} +$$

$$+ \left\|\frac{\partial u_i}{\partial x_j}\right\|_{L^2(\Omega)} \left\|\frac{\partial v_l}{\partial x_k}\right\|_{L^2(\Omega)} + \left\|\frac{\partial u_j}{\partial x_i}\right\|_{L^2(\Omega)} \left\|\frac{\partial v_l}{\partial x_k}\right\|_{L^2(\Omega)} \Big\} =$$

$$= \underset{i,j,k,l}{max}\{a_{ijkl}\} \sum_{i,j=1}^{n} \left\|\frac{\partial u_i}{\partial x_j}\right\|_{L^2(\Omega)} \sum_{k,l=1}^{n} \left\|\frac{\partial v_k}{\partial x_l}\right\|_{L^2(\Omega)} \le$$

$$\le n \cdot \underset{i,j,k,l}{max}\{a_{ijkl}\} \Big[\sum_{i,j=1}^{n} \left\|\frac{\partial u_i}{\partial x_j}\right\|^2_{L^2(\Omega)} \Big]^{\frac{1}{2}} \Big[\sum_{k,l=1}^{n} \left\|\frac{\partial v_k}{\partial x_l}\right\|^2_{L^2(\Omega)} \Big]^{\frac{1}{2}} \le$$

$$\le n \cdot \underset{i,j,k,l}{max}\{a_{ijkl}\}||\vec{u}||_V||\vec{v}||_V = \beta||\vec{u}||_V||\vec{v}||_V, \qquad \beta = n \cdot \underset{i,j,k,l}{max}\{a_{ijkl}\}, \quad n = 2,3.$$

Thus, there is a constant $\beta > 0$ ($\beta = n \cdot \underset{i,j,k,l}{max}\{a_{ijkl}\}$) such that

$$|a_E(\vec{u},\vec{v})| \le \beta||\vec{u}||_V||\vec{v}||_V, \qquad \forall\ \vec{u},\vec{v} \in V \tag{3.2.8}$$

Next we show that $a_E(\cdot,\cdot)$ is symmetric.

$$a_E(\vec{u},\vec{v}) = \int_\Omega a_{ijkl}\varepsilon_{ij}(\vec{u})\varepsilon_{kl}(\vec{v})\,\mathrm{d}\Omega = \int_\Omega a_{klij}\varepsilon_{ij}(\vec{u})\varepsilon_{kl}(\vec{v})\,\mathrm{d}\Omega =$$

$$= \int_\Omega a_{klij}\varepsilon_{kl}(\vec{u})\varepsilon_{ij}(\vec{v})\,\mathrm{d}\Omega = a_E(\vec{v},\vec{u}), \quad \forall\ \vec{u},\vec{v} \in V \tag{3.2.9}$$

From (3.2.7)-(3.2.9) it follows that $a_E(\cdot,\cdot)$ is symmetric and continuous on V_0 and there exist constants $c_E > 0$ and $\beta > 0$ such that

$$c_E||\vec{u}||_V^2 \le a_E(\vec{u},\vec{u}) \le \beta||\vec{u}||_V^2, \qquad \forall\ \vec{u} \in V_0 \tag{3.2.10}$$

Thus, $a_E(\cdot,\cdot)$ is an inner product on V_0 that is associated with a topology equivalent to the usual topology on V_0.

From the Riesz representation theorem and (3.2.5), (3.2.7), we have the following existence and uniqueness result with an a priori estimate.

Theorem 3.5. *There exists a unique function $\vec{u} \in V_0$ such that*

$$a_E(\vec{u},\vec{\Phi}) =< F,\vec{\Phi} >, \qquad \forall\ \vec{\Phi} \in V_0.$$

Furthermore,

$$||\vec{u}||_V \le \frac{c_F}{c_E}$$

3.3 Coupled TCPu Problem

In Section 3.2, for both the two- and three-dimensional elastic problems we obtained the following weak formulation:

$$a_E(\vec{u},\vec{\Phi}) - \int_\Omega f\big(\nabla\cdot\vec{\Phi}\big)\,\mathrm{d}\Omega - \int_{\Gamma\backslash\Gamma_F}(\hat{\sigma}\vec{n})\cdot\vec{\Phi}\,\mathrm{d}\Gamma = 0, \qquad \forall\ \vec{\Phi} \in V_0 \tag{3.2.3}$$

Since $\dot{\vec{u}} \in V_0$, we have

$$a_E(\dot{\vec{u}}, \dot{\vec{u}}) = \int_\Omega \dot{f}(\nabla \cdot \dot{\vec{u}})\, \mathrm{d}\Omega + \int_{\Gamma \backslash \Gamma_F} (\dot{\sigma}\vec{n}) \cdot \dot{\vec{u}}\, \mathrm{d}\Gamma$$

$$\leq ||\dot{f}||_{L^2(\Omega)} ||\nabla \cdot \dot{\vec{u}}||_{L^2(\Omega)} + ||\dot{\sigma}\vec{n}||_{L^2(\Gamma;\mathbb{R}^n)} ||\dot{\vec{u}}||_{L^2(\Gamma;\mathbb{R}^n)}, \quad n = 2, 3. \qquad (3.3.1)$$

Now

$$||\nabla \cdot \dot{\vec{u}}||^2_{L^2(\Omega)} = \int_\Omega \left[\sum_{i=1}^n \frac{\partial \dot{u}_i}{\partial x_i} \right]^2 \mathrm{d}\Omega \leq n \int_\Omega \sum_{i=1}^n \left| \frac{\partial \dot{u}_i}{\partial x_i} \right|^2 \mathrm{d}\Omega \leq n ||\nabla \dot{\vec{u}}||^2_H$$

where

$$||\nabla \dot{\vec{u}}||^2_H = \int_\Omega \sum_{i,j=1}^n \left| \frac{\partial \dot{u}_i}{\partial x_j} \right|^2 \mathrm{d}\Omega, \qquad n = 2, 3.$$

Thus,

$$||\nabla \cdot \dot{\vec{u}}||_{L^2(\Omega)} \leq \sqrt{n} ||\nabla \dot{\vec{u}}||_H \qquad n = 2, 3. \qquad (3.3.2)$$

By the trace theorem [36], there exists $\gamma_1 > 0$ such that

$$||\vec{v}||_{L^2(\Gamma;\mathbb{R}^n)} \leq \gamma_1 ||\vec{v}||_V, \qquad \forall\ \vec{v} \in V \qquad (3.3.3)$$

By Poincare-Friedrichs inequality [15], there exists a constant $\gamma_2 = \gamma_2(\Omega) > 0$ such that

$$||\vec{v}||_H \leq \gamma_2 ||\nabla \vec{v}||_H, \qquad \forall\ \vec{v} \in V_0 \qquad (3.3.4)$$

From (3.3.3) and (3.3.4) it follows that there exists $\gamma = \gamma(\Omega) > 0$ such that

$$||\dot{\vec{u}}||_{L^2(\Gamma;\mathbb{R}^n)} \leq \gamma ||\nabla \dot{\vec{u}}||_H, \qquad n = 2, 3. \qquad (3.3.5)$$

Applying (3.3.2) and (3.3.5) to (3.3.1), we have

$$a_E(\dot{\vec{u}}, \dot{\vec{u}}) \leq \sqrt{n} ||\dot{f}||_{L^2(\Omega)} ||\nabla \dot{\vec{u}}||_H + \gamma ||\dot{\sigma}\vec{n}||_{L^2(\Gamma;\mathbb{R}^n)} ||\nabla \dot{\vec{u}}||_H, \qquad n = 2, 3. \qquad (3.3.6)$$

From (2.1.30)

$$||\dot{f}||_{L^2(\Omega)} = ||\vec{b}_1^T \dot{V}||_{L^2(\Omega)} \leq |\vec{b}_1| \cdot ||\dot{V}||_{\mathcal{H}} \qquad (3.3.7)$$

and from (3.2.10), (3.3.6), and (3.3.7) it follows that

$$c_E||\nabla\dot{\vec{u}}||_H \leq \sqrt{n}|\vec{b}_1|\cdot||\dot{\vec{V}}||_{\mathcal{H}} + \gamma||\dot{\sigma}\vec{n}||_{L^2(\Gamma;\mathbb{R}^n)}, \quad n = 2,3. \tag{3.3.8}$$

The last inequality holds for all $t \in (0, t_f)$ and therefore

$$c_E||\nabla\dot{\vec{u}}||_{L^2(0,t_f;H)} \leq \sqrt{2n}|\vec{b}_1|\cdot||\dot{\vec{V}}||_{L^2(0,t_f;\mathcal{H})} + \sqrt{2}\gamma||\dot{\sigma}\vec{n}||_{L^2(0,t_f;L^2(\Gamma;\mathbb{R}^n))}, \quad n = 2,3. \tag{3.3.9}$$

Now we obtain a priori estimate $||\dot{\vec{V}}||_{L^2(0,t_f;\mathcal{H})}$. Applying (2.3.2) for the 2D problem or (2.3.9) for the 3D problem, (2.3.16), and (2.3.20) to (3.1.20), we have

$$||\dot{\vec{V}}||_{L^2(0,t_f;\mathcal{H})} \leq ||P||\cdot||\dot{U}||_{L^2(0,t_f;\mathcal{H})} + ||\dot{W}||_{L^2(0,t_f;\mathcal{H})} \leq$$

$$\leq ||P||\Big[\sqrt{\lambda_1}||U_0||_{\mathcal{V}_0} + ||P||^{-1}||\dot{W}||_{L^2(0,t_f,\mathcal{H})} + ||DP^{-1}||\cdot||\nabla^2 W||_{L^2(0,t_f,\mathcal{H})} +$$

$$+|P^{-1}M^{-1}\vec{b}_0|\cdot||\nabla\cdot\dot{\vec{u}}||_{L^2(0,t_f,L^2(\Omega))}\Big] + ||\dot{W}||_{L^2(0,t_f,\mathcal{H})} \qquad \Rightarrow$$

$$||\dot{\vec{V}}||_{L^2(0,t_f;\mathcal{H})} \leq ||P||\cdot|P^{-1}M^{-1}\vec{b}_0|\cdot||\nabla\cdot\dot{\vec{u}}||_{L^2(0,t_f,L^2(\Omega))} +$$

$$+||P||\Big[\sqrt{\lambda_1}||U_0||_{\mathcal{V}_0} + ||P||^{-1}||\dot{W}||_{L^2(0,t_f,\mathcal{H})} + ||DP^{-1}||\cdot||\nabla^2 W||_{L^2(0,t_f,\mathcal{H})}\Big] +$$

$$+||\dot{W}||_{L^2(0,t_f,\mathcal{H})}$$

Thus,

$$||\dot{\vec{V}}||_{L^2(0,t_f;\mathcal{H})} \leq ||P||\cdot|P^{-1}M^{-1}\vec{b}_0|\cdot||\nabla\cdot\dot{\vec{u}}||_{L^2(0,t_f,L^2(\Omega))} + \theta \tag{3.3.10}$$

where

$$\theta = ||P||\Big[\sqrt{\lambda_1}||U_0||_{\mathcal{V}_0} + ||P||^{-1}||\dot{W}||_{L^2(0,t_f,\mathcal{H})} + ||DP^{-1}||\cdot||\nabla^2 W||_{L^2(0,t_f,\mathcal{H})}\Big] +$$

$$+||\dot{W}||_{L^2(0,t_f,\mathcal{H})}$$

Applying (3.3.2) and (3.3.10) to (3.3.9), we have

$$c_E||\nabla\dot{\vec{u}}||_{L^2(0,t_f;H)} \leq \sqrt{2}n|\vec{b}_1|\cdot||P||\cdot|P^{-1}M^{-1}\vec{b}_0|\cdot||\nabla\dot{\vec{u}}||_{L^2(0,t_f,H)}+$$

$$+\sqrt{2n}|\vec{b}_1|\theta+\sqrt{2}\gamma||\dot{\hat{\sigma}}\vec{n}||_{L^2(0,t_f;L^2(\Gamma;\mathbb{R}^n))}, \quad n=2,3. \tag{3.3.11}$$

From Corollaries 3.1 and 3.2, Theorem 3.2, Assumption 3.1, (2.3.1), (2.3.2), (2.3.4)-(2.3.9), and (3.3.11), we have the following result:

Theorem 3.6. *The coupled two- and three-dimensional TCPu systems (2.1.19)-(2.1.21), (2.1.43), (2.1.45) and (2.2.17)-(2.2.22), respectively, have unique solutions* $(\bar{V},\vec{u})$, $\bar{V}\in L^2\big(0,t_f;H^1(\Omega,\mathbb{R}^3)\big)$ *and* $\vec{u}\in L^2(0,t_f;V_0)$ *with* $\dot{\bar{V}}\in L^2(0,t_f;\mathcal{H})$ *and* $\dot{\vec{u}}\in L^2(0,t_f;V_0)$ *if*

$$c_E-\sqrt{2}n||P||\cdot|P^{-1}M^{-1}\vec{b}_0|\cdot|\vec{b}_1|>0 \tag{3.3.12}$$

where $c_E>0$ *is the Korn's constant,* $\vec{b}_0$ *and* $\vec{b}_1$ *are coupling vectors given by (2.1.17) and (2.1.30), respectively, and* $M^{-1}A=PDP^{-1}$.

2D Special Case

Remark 3.2. For an annular domain $\Omega\subset\mathbb{R}^2$ of inner radius R_B and outer radius R_F, with $\delta=\dfrac{R_B}{R_F}\leq e^{-1}$, under the following side condition on the displacement $\vec{u}$

$$\int_\Omega\sum_{i,j=1}^{2}\Big(\frac{\partial u_i}{\partial x_j}-\frac{\partial u_j}{\partial x_i}\Big)\,\mathrm{d}\Omega=0 \tag{3.3.13}$$

the following inequality holds [21]:

$$\frac{D(\dot{\vec{u}})}{S(\dot{\vec{u}})}\leq 4\Big[1-\Big(\frac{3}{\delta^{-2}+1+\delta^2}\Big)^{1/2}\Big]^{-1} \tag{3.3.14}$$

where

$$D(\dot{\vec{u}})=\int_\Omega\sum_{i,j=1}^{2}\Big|\frac{\partial\dot{u}_i}{\partial x_j}\Big|^2\,\mathrm{d}\Omega=||\nabla\dot{\vec{u}}||_H^2$$

$$S(\dot{\vec{u}}) = \int_\Omega \varepsilon_{ij}(\dot{\vec{u}})^2 \,\mathrm{d}\Omega$$

It should be noted that the purpose of the condition (3.3.13) is to eliminate pure rotation. From (3.3.14) we have

$$S(\dot{\vec{u}}) \geq \frac{1}{4}\Big[1 - \Big(\frac{3}{\delta^{-2} + 1 + \delta^2}\Big)^{1/2}\Big] ||\nabla \dot{\vec{u}}||_H^2 \tag{3.3.15}$$

For homogeneous isotropic medium, the coefficients of elasticity a_{ijkl}, $i, j, k, l = 1, 2$, in terms of Lamé parameters λ and μ are

$$a_{1111} = a_{2222} = 2\mu + \lambda, \quad a_{1122} = a_{2211} = \lambda, \quad a_{1212} = a_{2121} = a_{1221} = a_{2112} = \mu, \tag{3.3.16}$$

Then from (2.1.23), (2.1.33), and (3.3.16) we have

$$\begin{aligned} a_E(\dot{\vec{u}}, \dot{\vec{u}}) &= \int_\Omega (2\mu + \lambda)\varepsilon_{11}^2(\dot{\vec{u}}) + 2\mu\varepsilon_{12}^2(\dot{\vec{u}}) + 2\mu\varepsilon_{21}^2(\dot{\vec{u}}) \\ &\quad + 2\lambda\varepsilon_{11}(\dot{\vec{u}})\varepsilon_{22}(\dot{\vec{u}}) + (2\mu + \lambda)\varepsilon_{22}^2(\dot{\vec{u}}) \,\mathrm{d}\Omega \\ &= \int_\Omega 2\mu(\varepsilon_{11}^2 + \varepsilon_{12}^2 + \varepsilon_{21}^2 + \varepsilon_{22}^2) + \lambda(\varepsilon_{11} + \varepsilon_{22})^2 \,\mathrm{d}\Omega \\ &\geq 2\mu \int_\Omega \varepsilon_{11}^2 + \varepsilon_{12}^2 + \varepsilon_{21}^2 + \varepsilon_{22}^2 \,\mathrm{d}\Omega = 2\mu S(\dot{\vec{u}}) \end{aligned}$$

Thus,

$$a_E(\dot{\vec{u}}, \dot{\vec{u}}) \geq 2\mu S(\dot{\vec{u}}) \tag{3.3.17}$$

where $\mu = G$ is the shear modulus.

From (3.2.10), (3.3.15), and (3.3.17),

$$c_E = \frac{G}{2}\Big[1 - \Big(\frac{3}{\delta^{-2} + 1 + \delta^2}\Big)^{1/2}\Big] \tag{3.3.18}$$

Applying (3.3.18) to (3.3.12), we obtain the following result.

Corollary 3.3. *For an annular domain* $\Omega \subset \mathbb{R}^2$ *of inner radius* R_B *and outer radius* R_F*, with* $\delta = \dfrac{R_B}{R_F} \leq e^{-1}$ *under side condition (3.3.13), the coupled TCPu system (2.1.19)-(2.1.21), (2.1.43), and (2.1.45) has a unique solution if*

$$\frac{G}{2}\left[1 - \left(\frac{3}{\delta^{-2} + 1 + \delta^2}\right)^{1/2}\right] - 2\sqrt{2}||P|| \cdot |P^{-1}M^{-1}\vec{b}_0| \cdot |\vec{b}_1| > 0 \tag{3.3.19}$$

where G *is the shear modulus,* $\vec{b}_0$ *and* $\vec{b}_1$ *are coupling vectors given by (2.1.17) and (2.1.30), respectively, and* $M^{-1}A = PDP^{-1}$.

In this chapter, we obtained a sufficient condition for the well-posedness in the weak sense of the coupled parabolic-elliptic initial-boundary value problems for the two- and three-dimensional TCPu models. This condition depends on physical parameters of the system, coupling vectors, and the Korn's constant which in turn depends only on the shape of a domain. Evaluation of the Korn's constant becomes necessary for each specific domain and is essential for the proposed well-posedness theory. For a two-dimensional annular region that does not experience pure rotation, the Korn's constant can be expressed explicitly in terms of the ratio of the inner and outer radii of the domain. As a consequence, for such regions, the sufficient condition for the well-posedness of the problem was formulated in terms of physical parameters, coupling vectors, and the ratio of the inner and outer radii.

Chapter 4

Fourier-Finite Element Method for TCPu Problems

In this chapter, we will develop the Fourier-finite element method for the two- and three-dimensional radially nonsymmetric TCPu problems discussed in Chapters 2 and 3. The far-field radius can be specified arbitrarily and therefore we assume that far-field boundary conditions are time-independent and the non-zero displacement in the far-field is allowed with a penalty.

4.1 2D TCPu Problem

As in Section 2.1, let (X, Y) be a borehole Cartesian coordinate system with the x-axis in the direction of the maximum horizontal principal stress σ_H and the y-axis in the direction of the minimum horizontal principal stress σ_h. For the sake of convenience, we will make the transformation from the coordinate system (X, Y) to the borehole polar coordinate system (r, θ) with its origin in the center of the annular region Ω. Then the boundary Γ of Ω is $\Gamma = \Gamma_B \cup \Gamma_F$ with

$$\Gamma_B = \left\{(R_B, \theta) \in \mathbb{R}^2 : 0 \le \theta < 2\pi\right\}$$
$$\Gamma_F = \left\{(R_F, \theta) \in \mathbb{R}^2 : 0 \le \theta < 2\pi\right\}$$

4.1.1 Fourier-type approximation of the TCP sytem

In the borehole polar coordinate system (r, θ), the TCP initial-boundary value problem (2.1.19)-(2.1.21) is given by

$$M\dot{\bar{V}} - A\nabla^2\bar{V} = -\vec{b}_0(\nabla \cdot \dot{\vec{u}}) \qquad \text{in } \Omega \times (0, t_f) \tag{4.1.1}$$

with boundary conditions

$$\bar{V}(r,\theta,t) = \begin{cases} V_B(t) & \text{on } \Gamma_B \times [0,t_f) \\ V_F(\theta) & \text{on } \Gamma_F \times [0,t_f) \end{cases} \tag{4.1.2}$$

and initial conditions

$$\bar{V}(r,\theta,0) = \tilde{V}_I(r,\theta) \tag{4.1.3}$$

where the displacement vector is

$$\vec{u}(r,\theta,t) = u(r,\theta,t)\vec{i}_r + v(r,\theta,t)\vec{i}_\theta$$

The equivalent parabolic TCP-type system with homogeneous boundary conditions, (2.3.11)-(2.3.13), in polar coordinates has the form

$$M\dot{V} - A\nabla^2 V = -M\dot{W} + A\nabla^2 W - \vec{b}_0(\nabla \cdot \dot{\vec{u}}) \qquad \text{in } \Omega \times (0,t_f) \tag{4.1.4}$$

with boundary conditions

$$V(r,\theta,t) = 0 \qquad \text{on } \Gamma \times [0,t_f) \tag{4.1.5}$$

and initial conditions

$$V(r,\theta,0) = \tilde{V}_I(r,\theta) - W(r,\theta,0) \tag{4.1.6}$$

where

$$W(r,\theta,t) = \frac{R_F - r}{R_F - R_B} V_B(t) + \frac{r - R_B}{R_F - R_B} V_F(\theta) \tag{4.1.7}$$

$$V = \bar{V} - W \tag{4.1.8}$$

We introduce the following θ - approximation:

$$V(r,\theta,t) = V_0(r,t) + V_1(r,t)\sin 2\theta + V_2(r,t)\cos 2\theta$$

$$u(r,\theta,t) = u_0(r,t) + u_1(r,t)\sin 2\theta + u_2(r,t)\cos 2\theta$$

$$v(r,\theta,t) = v_0(r,t) + v_1(r,t)\sin 2\theta + v_2(r,t)\cos 2\theta$$

$$V_B(t) = V_{B0}(t)$$

$$V_F(\theta) = V_{F0} + V_{F1}\sin 2\theta + V_{F2}\cos 2\theta$$

$$\tilde{V}_I(r,\theta) = \tilde{V}_{I0}(r) + \tilde{V}_{I1}(r)\sin 2\theta + \tilde{V}_{I2}(r)\cos 2\theta \tag{4.1.9}$$

Then

$$W(r,\theta,t) = W_0(r,t) + W_1(r,t)\sin 2\theta + W_2(r,t)\cos 2\theta \tag{4.1.10}$$

where

$$W_i(r,t) = \frac{R_F - r}{R_F - R_B}V_{Bi}(t) + \frac{r - R_B}{R_F - R_B}V_{Fi}, \qquad i = 0,\dots,2 \tag{4.1.11}$$

and $\quad V_{Bi}(t) = 0, \qquad i = 1,2.$

With this approximation, the system (4.1.4) has the form:

$$M\dot{V} - A\Big[\frac{1}{r}(rV_r)_r + \frac{1}{r^2}V_{\theta\theta}\Big] = -M\dot{W} + A\Big(\frac{1}{r}(rW_r)_r + \frac{1}{r^2}W_{\theta\theta}\Big)$$
$$-\vec{b}_0\Big(\dot{u}_r + \frac{1}{r}(\dot{u} + \dot{v}_\theta)\Big) \tag{4.1.12}$$

where $\quad W$ is given by (4.1.10) and (4.1.11),

$$W_r(r,\theta,t) = W_{0_r}(r,t) + W_{1_r}(r,t)\sin 2\theta + W_{2_r}(r,t)\cos 2\theta \tag{4.1.13}$$

with

$$W_{i_r}(r,t) = \frac{1}{R_F - R_B}\big(V_{Fi} - V_{Bi}(t)\big), \qquad i = 0,\dots,2 \tag{4.1.14}$$

and $\quad V_{Bi}(t) = 0, \qquad i = 1,2.$

$$W_{\theta\theta}(r,\theta,t) = -4W_1(r,t)\sin 2\theta - 4W_2(r,t)\cos 2\theta \tag{4.1.15}$$

The θ-approximation for the initial conditions $V(r,\theta,0)$ is

$$V(r,\theta,0) = V_0(r,0) + V_1(r,0)\sin 2\theta + V_2(r,0)\cos 2\theta$$

where

$$
\begin{aligned}
V_i(r,0) &= \tilde{V}_{Ii}(r) - W_i(r,0) \\
&= \tilde{V}_{Ii}(r) - \frac{R_F - r}{R_F - R_B} V_{Bi}(0) - \frac{r - R_B}{R_F - R_B} V_{Fi}, \qquad i = 0, \dots, 2
\end{aligned} \tag{4.1.16}
$$

and $\quad V_{Bi}(0) = 0, \qquad i = 1, 2.$

Applying (4.1.13)-(4.1.16) to (4.1.12), multiplying by 1, $\sin 2\theta$, and $\cos 2\theta$, and integrating over $(0, 2\pi)$, we obtain the following three systems.

System I:

$$
M\dot{V}_0 - A\frac{1}{r}(rV_{0_r})_r = -M\dot{W}_0 + A\frac{1}{r}(rW_{0_r})_r - \vec{b}_0\Big(\dot{u}_{0_r} + \frac{1}{r}\dot{u}_0\Big) \tag{4.1.17}
$$

with boundary conditions

$$
V_0(r,t) = 0 \qquad \text{on } \Gamma \times [0, t_f) \tag{4.1.18}
$$

and initial conditions

$$
V_0(r,0) = \tilde{V}_{I0}(r) - W_0(r,0) \tag{4.1.19}
$$

System II:

$$
\begin{aligned}
M\dot{V}_1 - A\Big[\frac{1}{r}(rV_{1_r})_r - \frac{4}{r^2}V_1\Big] &= -M\dot{W}_1 + A\Big(\frac{1}{r}(rW_{1_r})_r - \frac{4}{r^2}W_1\Big) \\
&\quad -\vec{b}_0\Big(\dot{u}_{1_r} + \frac{1}{r}(\dot{u}_1 - 2\dot{v}_2)\Big)
\end{aligned} \tag{4.1.20}
$$

with boundary conditions

$$
V_1(r,t) = 0 \qquad \text{on } \Gamma \times [0, t_f) \tag{4.1.21}
$$

and initial conditions

$$
V_1(r,0) = \tilde{V}_{I1}(r) - W_1(r,0) \tag{4.1.22}
$$

System III:

$$M\dot{V}_2 - A\Big[\frac{1}{r}(rV_{2_r})_r - \frac{4}{r^2}V_2\Big] = -M\dot{W}_2 + A\Big(\frac{1}{r}(rW_{2_r})_r - \frac{4}{r^2}W_2\Big)$$
$$-\vec{b}_0\Big(\dot{u}_{2_r} + \frac{1}{r}(\dot{u}_2 + 2\dot{v}_1)\Big) \qquad (4.1.23)$$

with boundary conditions

$$V_2(r,t) = 0 \qquad \text{on } \Gamma \times [0, t_f) \qquad (4.1.24)$$

and initial conditions

$$V_2(r,0) = \tilde{V}_{I2}(r) - W_2(r,0) \qquad (4.1.25)$$

4.1.2 Fourier-type approximation of the elastic system

In the borehole polar coordinate system (r, θ), the displacement is

$$\vec{u}(r,\theta,t) = u(r,\theta,t)\vec{i}_r + v(r,\theta,t)\vec{i}_\theta$$
$$\vec{u} \approx 0 \quad \text{and} \quad \dot{\vec{u}} \approx 0 \qquad \text{on } \Gamma_F \qquad (4.1.26)$$

and the strain tensor has the form

$$\begin{bmatrix} \varepsilon_{rr} & \varepsilon_{r\theta} \\ \varepsilon_{\theta r} & \varepsilon_{\theta\theta} \end{bmatrix}$$

where

$$\varepsilon_{rr} = u_r, \qquad \varepsilon_{r\theta} = \varepsilon_{\theta r} = \frac{1}{2}\big(v_r + \frac{u_\theta - v}{r}\big), \qquad \varepsilon_{\theta\theta} = \frac{u + v_\theta}{r} \qquad (4.1.27)$$

The stress-strain relations for isotropic homogeneous medium in terms of the shear modulus G and the bulk modulus K are

$$\tau = 2G\varepsilon + \big(K - \frac{2}{3}G\big)(\mathrm{tr}\varepsilon)I \qquad (2.1.26)$$

where

$$\tau_{rr} = 2Gu_r + \big(K - \frac{2}{3}G\big)\big(u_r + \frac{u+v_\theta}{r}\big)$$

$$\tau_{\theta\theta} = 2G\frac{u+v_\theta}{r} + \big(K - \frac{2}{3}G\big)\big(u_r + \frac{u+v_\theta}{r}\big)$$

$$\tau_{r\theta} = \tau_{\theta r} = G\big(v_r + \frac{u_\theta - v}{r}\big) \tag{4.1.28}$$

The sum of the total potential energy and the penalty term is

$$\Pi(\vec{u}) = \frac{1}{2}\int_\Omega \Big(\tau_{rr}(\vec{u})\varepsilon_{rr}(\vec{u}) + 2\tau_{r\theta}(\vec{u})\varepsilon_{r\theta}(\vec{u}) + \tau_{\theta\theta}(\vec{u})\varepsilon_{\theta\theta}(\vec{u})\Big)\,\mathrm{d}\Omega$$
$$- \int_\Omega f\big(\mathrm{tr}\varepsilon(\vec{u})\big)\,\mathrm{d}\Omega - \int_\Gamma (\hat{\sigma}\vec{n})\cdot\vec{u}\,\mathrm{d}\Gamma + K_F\int_{\Gamma_F}\vec{u}\cdot\vec{u}\,\mathrm{d}\Gamma \tag{4.1.29}$$

where $f = \vec{b}_1^T\bar{V} = \vec{b}_1^T(V+W)$ is body force, $\vec{b}_1^T = [\tilde{\zeta} \quad -\lambda \quad \tilde{\alpha}]$,

K_F is a penalization constant,

$\hat{\sigma} = \begin{bmatrix} \sigma_r & \sigma_{r\theta} \\ \sigma_{r\theta} & \sigma_\theta \end{bmatrix}$ is applied boundary stress in the system (r,θ).

Next, we describe stress around the borehole.

$\sigma_{XY} = \begin{bmatrix} \sigma_x & \sigma_{xy} \\ \sigma_{xy} & \sigma_y \end{bmatrix} = \begin{bmatrix} \sigma_H & 0 \\ 0 & \sigma_h \end{bmatrix}$ is the stress tensor in (X,Y) coordinate system and

$\sigma(r,\theta) = \begin{bmatrix} \sigma_r & \sigma_{r\theta} \\ \sigma_{r\theta} & \sigma_\theta \end{bmatrix}$ (r,θ) is the stress tensor in (r,θ) coordinate system.

The transformation from σ_{XY} to $\sigma(r,\theta)$ is given by [9]:

$$\sigma_r = \frac{\sigma_x+\sigma_y}{2}\Big(1 - \frac{R_B^2}{r^2}\Big) + p_B\frac{R_B^2}{r^2} + \sigma_{xy}\Big(1 - \frac{4R_B^2}{r^2} + \frac{3R_B^4}{r^4}\Big)\sin 2\theta$$
$$+ \frac{\sigma_x-\sigma_y}{2}\Big(1 - \frac{4R_B^2}{r^2} + \frac{3R_B^4}{r^4}\Big)\cos 2\theta$$

$$\sigma_\theta = \frac{\sigma_x+\sigma_y}{2}\Big(1 + \frac{R_B^2}{r^2}\Big) - p_B\frac{R_B^2}{r^2} - \sigma_{xy}\Big(1 + \frac{3R_B^4}{r^4}\Big)\sin 2\theta$$
$$- \frac{\sigma_x-\sigma_y}{2}\Big(1 + \frac{3R_B^4}{r^4}\Big)\cos 2\theta$$

$$\sigma_{r\theta} = \frac{\sigma_x - \sigma_y}{2}\Big(1 + \frac{2R_B^2}{r^2} - \frac{3R_B^4}{r^4}\Big)\sin 2\theta + \sigma_{xy}\Big(1 + \frac{2R_B^2}{r^2} - \frac{3R_B^4}{r^4}\Big)\cos 2\theta$$

where p_B is the well bore pressure and ν is Poisson's ratio of the formation.

From these equations we obtain applied stress on the boundaries.

On the borehole boundary Γ_B:

$\sigma_r(R_B, t) = p_B(t)$

$\sigma_\theta(R_B, \theta, t) = \sigma_x + \sigma_y - p_B(t) - 4\sigma_{xy}\sin 2\theta - 2\big(\sigma_x - \sigma_y\big)\cos 2\theta$

$\sigma_{r\theta}(R_B, \theta) = 0$

On the far-field boundary Γ_F:

$\sigma_r(R_F, \theta) = \dfrac{\sigma_x + \sigma_y}{2} + \sigma_{xy}\sin 2\theta + \dfrac{\sigma_x - \sigma_y}{2}\cos 2\theta$

$\sigma_\theta(R_F, \theta) = \dfrac{\sigma_x + \sigma_y}{2} - \sigma_{xy}\sin 2\theta - \dfrac{\sigma_x - \sigma_y}{2}\cos 2\theta$

$\sigma_{r\theta}(R_F, \theta) = \dfrac{\sigma_x - \sigma_y}{2}\sin 2\theta + \sigma_{xy}\cos 2\theta$

Let $V = H^1\big(\Omega; \mathbb{R}^2\big)$. Our objective is to find the displacement $\vec{u}$ such that

$$\vec{u} \in L^2(0, t_f; V), \qquad \dot{\vec{u}} \in L^2(0, t_f; V) \qquad \text{and}$$

$$D\Pi(\vec{u})\vec{\Phi} = 0, \qquad \forall\ \vec{\Phi} \in V \tag{4.1.30}$$

where $\vec{u}(r,\theta,t) = u(r,\theta,t)\vec{i}_r + v(r,\theta,t)\vec{i}_\theta$, $\vec{\Phi}(r,\theta) = \varphi(r,\theta)\vec{i}_r + \psi(r,\theta)\vec{i}_\theta$, and $D\Pi(\vec{u})\vec{\Phi}$ is the Gâteaux differential of Π with increment $\vec{\Phi}$.

Applying (4.1.27) and (4.1.28) to (4.1.30), we have

$$\int_0^{2\pi}\int_{R_B}^{R_F}\Big[\Big(2Gu_r + \big(K - \frac{2}{3}G\big)\big(u_r + \frac{u + v_\theta}{r}\big)\Big)\varphi_r + G\big(v_r + \frac{u_\theta - v}{r}\big)\big(\psi_r + \frac{\varphi_\theta - \psi}{r}\big)$$

$$+ \Big(2G\frac{u + v_\theta}{r} + \big(K - \frac{2}{3}G\big)\big(u_r + \frac{u + v_\theta}{r}\big)\Big)\frac{\varphi + \psi_\theta}{r}\Big] r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$-\int_0^{2\pi}\int_{R_B}^{R_F} f\big(\varphi_r + \frac{\varphi + \psi_\theta}{r}\big) r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$+\int_0^{2\pi}\Big(\sigma_r(R_B,\theta)\varphi(R_B,\theta) + \sigma_{r\theta}(R_B,\theta)\psi(R_B,\theta)\Big)R_B\,\mathrm{d}\theta -$$

$$
\begin{aligned}
&- \int_0^{2\pi} \Big(\sigma_r(R_F,\theta)\varphi(R_F,\theta) + \sigma_{r\theta}(R_F,\theta)\psi(R_F,\theta)\Big) R_F \,\mathrm{d}\theta \\
&+ K_F \int_0^{2\pi} \Big(u(R_F,\theta)\varphi(R_F,\theta) + v(R_F,\theta)\psi(R_F,\theta)\Big) R_F \,\mathrm{d}\theta = 0
\end{aligned} \tag{4.1.31}
$$

The first term of the left-hand side of (4.1.31) is

$$
\begin{aligned}
&\int_0^{2\pi} \int_{R_B}^{R_F} \Big[\big(K + \frac{4}{3}G\big) u_r \varphi_r r + \big(K - \frac{2}{3}G\big) u \varphi_r + \big(K - \frac{2}{3}G\big) v_\theta \varphi_r \\
&\quad + G v_r \psi_r r + G v_r \varphi_\theta - G v_r \psi + G u_\theta \psi_r - G v \psi_r + G u_\theta \varphi_\theta \frac{1}{r} - G u_\theta \psi \frac{1}{r} \\
&\quad - G v \varphi_\theta \frac{1}{r} + G v \psi \frac{1}{r} + \big(K + \frac{4}{3}G\big) u \varphi \frac{1}{r} + \big(K + \frac{4}{3}G\big) u \psi_\theta \frac{1}{r} \\
&\quad + \big(K + \frac{4}{3}G\big) v_\theta \varphi \frac{1}{r} + \big(K + \frac{4}{3}G\big) v_\theta \psi_\theta \frac{1}{r} + \big(K - \frac{2}{3}G\big) u_r \varphi + \big(K - \frac{2}{3}G\big) u_r \psi_\theta \,\mathrm{d}r\,\mathrm{d}\theta
\end{aligned} \tag{4.1.32}
$$

Applying θ-approximation (4.1.9) to (4.1.32) and integrating with respect to θ over $(0, 2\pi)$, we have

$$
\begin{aligned}
&\pi \int_{R_B}^{R_F} \Big[\big(K + \frac{4}{3}G\big)\big(2u_{0_r}\varphi_{0_r} r + u_{1_r}\varphi_{1_r} r + u_{2_r}\varphi_{2_r} r\big) \\
&\quad + \big(K - \frac{2}{3}G\big)\big(2u_0\varphi_{0_r} + u_1\varphi_{1_r} + u_2\varphi_{2_r} - 2v_2\varphi_{1_r} + 2v_1\varphi_{2_r}\big) \\
&\quad + G\big(2v_{0_r}\psi_{0_r} r + v_{1_r}\psi_{1_r} r + v_{2_r}\psi_{2_r} r - 2v_{1_r}\varphi_2 + 2v_{2_r} - 2u_2\psi_{1_r} + 2u_1\psi_{2_r}\varphi_1\big) \\
&\quad + G\big(- 2v_{0_r}\psi_0 - v_{1_r}\psi_1 - v_{2_r}\psi_2 - 2v_0\psi_{0_r} - v_1\psi_{1_r} - v_2\psi_{2_r}\big) \\
&\quad + G\big(4u_2\varphi_2\frac{1}{r} + 4u_1\varphi_1\frac{1}{r} + 2u_2\psi_1\frac{1}{r} - 2u_1\psi_2\frac{1}{r}\big) \\
&\quad + G\big(2v_1\varphi_2\frac{1}{r} - 2v_2\varphi_1\frac{1}{r} + 2v_0\psi_0\frac{1}{r} + v_1\psi_1\frac{1}{r} + v_2\psi_2\frac{1}{r}\big) \\
&\quad + \big(K + \frac{4}{3}G\big)\big(2u_0\varphi_0\frac{1}{r} + u_1\varphi_1\frac{1}{r} + u_2\varphi_2\frac{1}{r} - 2u_1\psi_2\frac{1}{r} + 2u_2\psi_1\frac{1}{r}\big) \\
&\quad + \big(K + \frac{4}{3}G\big)\big(- 2v_2\varphi_1\frac{1}{r} + 2v_1\varphi_2\frac{1}{r} + 4v_2\psi_2\frac{1}{r} + 4v_1\psi_1\frac{1}{r}\big) \\
&\quad + \big(K - \frac{2}{3}G\big)\big(2u_{0_r}\varphi_0 + u_{1_r}\varphi_1 + u_{2_r}\varphi_2 - 2u_{1_r}\psi_2 + 2u_{2_r}\psi_1\big)\Big] \,\mathrm{d}r
\end{aligned} \tag{4.1.33}
$$

Combining like terms with respect to the basis functions, (4.1.33) has the following form:

$\varphi_0 :$

$$\int_{R_B}^{R_F} \Big[2\big(K+\frac{4}{3}G\big)u_{0_r}\varphi_{0_r}r + 2\big(K-\frac{2}{3}G\big)(u_0\varphi_{0_r}+u_{0_r}\varphi_0) + 2\big(K+\frac{4}{3}G\big)u_0\varphi_0\frac{1}{r}\Big]\,\mathrm{d}r \tag{4.1.34}$$

$\varphi_1 :$

$$\begin{aligned} + \pi \int_{R_B}^{R_F} \Big[&\big(K+\frac{4}{3}G\big)u_{1_r}\varphi_{1_r}r + \big(K-\frac{2}{3}G\big)u_1\varphi_{1_r} - 2\big(K-\frac{2}{3}G\big)v_2\varphi_{1_r} + 2Gv_{2_r}\varphi_1 \\ &+ \big(K+\frac{16}{3}G\big)u_1\varphi_1\frac{1}{r} - 2\big(K+\frac{7}{3}G\big)v_2\varphi_1\frac{1}{r} + \big(K-\frac{2}{3}G\big)u_{1_r}\varphi_1\Big]\,\mathrm{d}r \end{aligned} \tag{4.1.35}$$

$\varphi_2 :$

$$\begin{aligned} + \pi \int_{R_B}^{R_F} \Big[&\big(K+\frac{4}{3}G\big)u_{2_r}\varphi_{2_r}r + \big(K-\frac{2}{3}G\big)u_2\varphi_{2_r} + 2\big(K-\frac{2}{3}G\big)v_1\varphi_{2_r} - 2Gv_{1_r}\varphi_2 \\ &+ \big(K+\frac{16}{3}G\big)u_2\varphi_2\frac{1}{r} + 2\big(K+\frac{7}{3}G\big)v_1\varphi_2\frac{1}{r} + \big(K-\frac{2}{3}G\big)u_{2_r}\varphi_2\Big]\,\mathrm{d}r \end{aligned} \tag{4.1.36}$$

$\psi_0 :$

$$+ \pi \int_{R_B}^{R_F} \Big[\underbrace{2Gv_{0_r}\psi_{0_r}r - 2Gv_{0_r}\psi_0 - 2Gv_0\psi_{0_r} + 2Gv_0\psi_0\frac{1}{r}}_{2G\left[\left(v_{0_r}-\frac{v_0}{r}\right)\left(\psi_{0_r}-\frac{\psi_0}{r}\right)\right]r}\Big]\,\mathrm{d}r \tag{4.1.37}$$

$\psi_1 :$

$$\begin{aligned} + \pi \int_{R_B}^{R_F} \Big[&Gv_{1_r}\psi_{1_r}r - Gv_{1_r}\psi_1 - 2Gu_2\psi_{1_r} - Gv_1\psi_{1_r} + 2\big(K+\frac{7}{3}G\big)u_2\psi_1\frac{1}{r} \\ &+ \big(4K+\frac{19}{3}G\big)v_1\psi_1\frac{1}{r} + 2\big(K-\frac{2}{3}G\big)u_{2_r}\psi_1\Big]\,\mathrm{d}r \end{aligned} \tag{4.1.38}$$

$\psi_2 :$

$$\begin{aligned} + \pi \int_{R_B}^{R_F} \Big[&Gv_{2_r}\psi_{2_r}r - Gv_{2_r}\psi_2 + 2Gu_1\psi_{2_r} - Gv_2\psi_{2_r} - 2\big(K+\frac{7}{3}G\big)u_1\psi_2\frac{1}{r} \\ &+ \big(4K+\frac{19}{3}G\big)v_2\psi_2\frac{1}{r} - 2\big(K-\frac{2}{3}G\big)u_{1_r}\psi_2\Big]\,\mathrm{d}r \end{aligned} \tag{4.1.39}$$

The σ-approximation of body force f has the form

$$f(r,\theta,t) = f_0(r,t) + f_1(r,t)\sin 2\theta + f_2(r,t)\cos 2\theta \tag{4.1.40}$$

The second term of the left-hand side of (4.1.31) is

$$-\pi\int_{R_B}^{R_F}\big[2f_0\varphi_{0_r}r + f_1\varphi_{1_r}r + f_2\varphi_{2_r}r \\ + 2f_0\varphi_0 + f_1\varphi_1 + f_2\varphi_2 - 2f_1\psi_2 + 2f_2\psi_1\big]\,\mathrm{d}r \tag{4.1.41}$$

The third term of the left-hand side of (4.1.31) is

$$2\pi p_B\varphi_0(R_B)R_B \tag{4.1.42}$$

The fourth term of the left-hand side of (4.1.31) is

$$-\pi\Big[2\frac{\sigma_x+\sigma_y}{2}\varphi_0(R_F) + \sigma_{xy}\varphi_1(R_F) + \frac{\sigma_x-\sigma_y}{2}\varphi_2(R_F) \\ + \frac{\sigma_x-\sigma_y}{2}\psi_1(R_F) + \sigma_{xy}\psi_2(R_F)\Big]R_F \tag{4.1.43}$$

The fifth term of the left-hand side of (4.1.31) is

$$\pi K_F\big[2u_0(R_F)\varphi_0(R_F) + u_1(R_F)\varphi_1(R_F) + u_2(R_F)\varphi_2(R_F) \\ + 2v_0(R_F)\psi_0(R_F) + v_1(R_F)\psi_1(R_F) + v_2(R_F)\psi_2(R_F)\big]R_F \tag{4.1.44}$$

In (4.1.20)-(4.1.39) and (4.1.41)-(4.1.44), we combine terms corresponding to φ_i and ψ_i, $i = 0,\ldots,2$, and equal them to 0.

φ_0 : (u_0)

$$\int_{R_B}^{R_F}\Big[\big(K+\frac{4G}{3}\big)u_{0_r}\varphi_{0_r}r + \big(K-\frac{2G}{3}\big)(u_0\varphi_{0_r} + u_{0_r}\varphi_0) + \big(K+\frac{4G}{3}\big)u_0\varphi_0\frac{1}{r} \\ -\int_{R_B}^{R_F} f_0\big(\varphi_{0_r}r + \varphi_0\big)\,\mathrm{d}r + p_B\varphi_0(R_B)R_B - \frac{\sigma_x+\sigma_y}{2}\varphi_0(R_F)R_F \\ + K_Fu_0(R_F)\varphi_0(R_F)R_F = 0 \tag{4.1.45}$$

$\varphi_1 : \quad (u_1, v_2)$

$$\int_{R_B}^{R_F} \Big[\big(K + \frac{4G}{3}\big) u_{1_r} \varphi_{1_r} r + \big(K - \frac{2G}{3}\big) u_1 \varphi_{1_r} + \big(K - \frac{2G}{3}\big) u_{1_r} \varphi_1$$

$$+ \big(K + \frac{16G}{3}\big) u_1 \varphi_1 \frac{1}{r} - 2\big(K - \frac{2G}{3}\big) v_2 \varphi_{1_r} + 2G v_{2_r} \varphi_1 - 2\big(K + \frac{7G}{3}\big) v_2 \varphi_1 \frac{1}{r}\Big] \, \mathrm{d}r$$

$$- \int_{R_B}^{R_F} f_1 \big(\varphi_{1_r} r + \varphi_1\big) \, \mathrm{d}r - \sigma_{xy} \varphi_1(R_F) R_F + K_F u_1(R_F) \varphi_1(R_F) R_F = 0 \qquad (4.1.46)$$

$\varphi_2 : \quad (u_2, v_1)$

$$\int_{R_B}^{R_F} \Big[\big(K + \frac{4G}{3}\big) u_{2_r} \varphi_{2_r} r + \big(K - \frac{2G}{3}\big) u_2 \varphi_{2_r} + \big(K - \frac{2G}{3}\big) u_{2_r} \varphi_2$$

$$+ \big(K + \frac{16G}{3}\big) u_2 \varphi_2 \frac{1}{r} + 2\big(K - \frac{2G}{3}\big) v_1 \varphi_{2_r} - 2G v_{1_r} \varphi_2 + 2\big(K + \frac{7G}{3}\big) v_1 \varphi_2 \frac{1}{r}\Big] \, \mathrm{d}r$$

$$- \int_{R_B}^{R_F} f_2 \big(\varphi_{2_r} r + \varphi_2\big) \, \mathrm{d}r - \frac{\sigma_x - \sigma_y}{2} \varphi_2(R_F) R_F + K_F u_2(R_F) \varphi_2(R_F) R_F = 0$$

(4.1.47)

$\psi_0 : \quad (v_0)$

$$\int_{R_B}^{R_F} G\Big[\big(v_{0_r} - \frac{v_0}{r}\big)\big(\psi_{0_r} - \frac{\psi_0}{r}\big)\Big] r \, \mathrm{d}r + K_F v_0(R_F) \psi_0(R_F) R_F = 0$$

Take $\psi_0 = v_0$. Then

$$\int_{R_B}^{R_F} G\Big[\big(v_{0_r} - \frac{v_0}{r}\big)^2\Big] r \, \mathrm{d}r + K_F R_F v_0^2(R_F) = 0$$

It follows that

$$v_{0_r} - \frac{v_0}{r} = 0, \quad R_B < r < R_F, \qquad \text{and} \qquad v_0(R_F) = 0$$

and so,

$$v_0(r,t) = C_0(t) r, \quad R_B < r < R_F \qquad (4.1.48)$$

$\psi_1 : \quad (u_2, v_1)$

$$\int_{R_B}^{R_F} \Big[-2Gu_2\psi_{1_r} + 2\big(K - \frac{2G}{3}\big)u_{2_r}\psi_1 + 2\big(K + \frac{7G}{3}\big)u_2\psi_1\frac{1}{r}$$

$$+ Gv_{1_r}\psi_{1_r}r - Gv_1\psi_{1_r} - Gv_{1_r}\psi_1 + \big(4K + \frac{19G}{3}\big)v_1\psi_1\frac{1}{r}\Big]\,\mathrm{d}r$$

$$-\int_{R_B}^{R_F} 2f_2\psi_1\,\mathrm{d}r - \frac{\sigma_x - \sigma_y}{2}\psi_1(R_F)R_F + K_F v_1(R_F)\psi_1(R_F)R_F = 0 \qquad (4.1.49)$$

$\psi_2 : \quad (u_1, v_2)$

$$\int_{R_B}^{R_F} \Big[2Gu_1\psi_{2_r} - 2\big(K - \frac{2G}{3}\big)u_{1_r}\psi_2 - 2\big(K + \frac{7G}{3}\big)u_1\psi_2\frac{1}{r}$$

$$+ Gv_{2_r}\psi_{2_r}r - Gv_2\psi_{2_r} - Gv_{2_r}\psi_2 + \big(4K + \frac{19G}{3}\big)v_2\psi_2\frac{1}{r}\Big]\,\mathrm{d}r$$

$$+\int_{R_B}^{R_F} 2f_1\psi_2\,\mathrm{d}r - \sigma_{xy}\psi_2(R_F)R_F + K_F v_2(R_F)\psi_2(R_F)R_F = 0 \qquad (4.1.50)$$

4.1.3 Finite element approximation

Dirichlet boundary conditions for the TCP system

r-approximation:

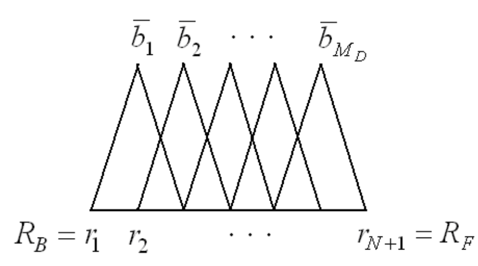

Figure 4.1: r-approximation of 2D Dirichlet boundary conditions

Here N is the number of subintervals; $M_D = N - 1$ is the number of basis functions; $\bar{b}(r) = \Big[\bar{b}_1(r) \quad \dots \quad \bar{b}_{M_D}(r)\Big]^T$ is the vector of basis functions; and

$$B(r) = \begin{bmatrix} \bar{b}(r)^T & 0 & 0 \\ 0 & \bar{b}(r)^T & 0 \\ 0 & 0 & \bar{b}(r)^T \end{bmatrix}_{3\times 3M_D}$$

Using r-approximation, the coefficients of the θ-approximation for $V(r,\theta,t)$ can be represented as

$$V_i(r,t) = B(r)\bar{c}_i(t) \tag{4.1.51}$$

where $\bar{c}_i(t)$ is a $(3M_D \times 1)$ vector.

The coefficients of the θ-approximation (4.1.9) for boundary conditions (4.1.2) and initial condition (4.1.3) can be represented as

$$\begin{aligned} V_{B0}(t) &= \bar{\nu}_{B0}(t) \\ V_{Fi} &= \bar{\nu}_{Fi} \\ \tilde{V}_{Ii}(r) &= B(r)\bar{\nu}_{Ii}, \qquad i = 0,\ldots,2 \end{aligned} \tag{4.1.52}$$

where $\bar{\nu}_{B0}(t)$ and $\bar{\nu}_{Fi}$ are (3×1) vectors and $\bar{\nu}_{Ii}$ is a $(3M_D \times 1)$ vector.

Neumann boundary conditions for the elastic system

r-approximation:

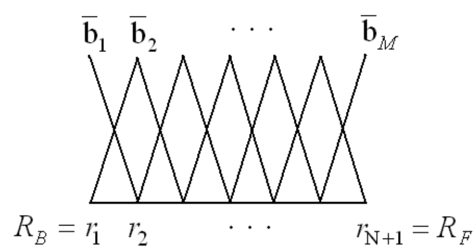

Figure 4.2: r-approximation of 2D Neumann boundary conditions

Here N is the number of subintervals; $M = \mathrm{N}+1$ is the number of basis functions; $\bar{\mathrm{b}}(r) = \begin{bmatrix} \bar{\mathrm{b}}_1(r) & \ldots & \bar{\mathrm{b}}_M(r) \end{bmatrix}^T$ is the vector of basis functions.

Using r-approximation, the coefficients of the θ-approximation (4.1.9) for the displacement coordinates (4.1.51) can be represented as

$$u_i(r,t) = \bar{\mathrm{b}}(r)^T \bar{\mu}_i(t) \tag{4.1.53}$$

$$v_i(r,t) = \bar{\mathrm{b}}(r)^T \bar{\eta}_i(t) \tag{4.1.54}$$

where $\bar{\mu}_i(t)$ and $\bar{\eta}_i(t)$ are $(M \times 1)$ vectors.

Finite element approximation of the elastic system

Define the following $M \times M$ matrices:

$$g_2(r) = \int_{R_B}^{R_F} \bar{\mathrm{b}}_r(r)\, \bar{\mathrm{b}}_r(r)^T r\, \mathrm{d}r$$

$$g_1 = \int_{R_B}^{R_F} \bar{\mathrm{b}}(r)\, \bar{\mathrm{b}}_r(r)^T\, \mathrm{d}r$$

$$g_0\left(\frac{1}{r}\right) = \int_{R_B}^{R_F} \bar{\mathrm{b}}(r)\, \bar{\mathrm{b}}(r)^T \frac{1}{r}\, \mathrm{d}r$$

$$g_F = \bar{\mathrm{b}}(R_F)\, \bar{\mathrm{b}}(R_F)^T$$

Let $P_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$. Then

$$p_B(t) = P_3^T V_B(t) = P_3^T V_{B0}(t) = P_3^T \bar{\nu}_{B0}(t) \tag{4.1.55}$$

Also, let

$$\sigma_R = \frac{\sigma_x + \sigma_y}{2} \qquad \text{and} \qquad \sigma_D = \frac{\sigma_x - \sigma_y}{2} \tag{4.1.56}$$

As before,

$$f(r,\theta,t) = \vec{b}_1^T \bar{V}(r,\theta,t) = \vec{b}_1^T \big(V(r,\theta,t) + W(r,\theta,t)\big)$$

and from (4.1.9)-(4.1.11), (4.1.13), (4.1.51), and (4.1.52) it follows that

$$\begin{aligned} f_i(r,t) &= \vec{b}_1^T \big(V_i(r,t) + W_i(r,t)\big) \\ &= \vec{b}_1^T B(r) \bar{c}_i(t) + \frac{R_F - r}{R_F - R_B} \vec{b}_1^T \bar{\nu}_{Bi}(t) + \frac{r - R_B}{R_F - R_B} \vec{b}_1^T \bar{\nu}_{Fi}, \quad i = 0,1,2 \end{aligned} \tag{4.1.57}$$

and $\bar{\nu}_{Bi}(t) = 0, \quad i = 1, 2.$

From (4.1.45)-(4.1.50) we obtain

$\varphi_0:$

$$\begin{aligned}
&\Big[\big(K + \frac{4G}{3}\big)g_2(r) + \big(K - \frac{2G}{3}\big)\big(g_1^T + g_1\big) + \big(K + \frac{4G}{3}\big)g_0\big(\frac{1}{r}\big) + K_F R_F g_F\Big]\bar{\mu}_0(t) \\
=&\Big[\int_{R_B}^{R_F} \big(r\,\bar{\mathrm{b}}_r(r) + \bar{\mathrm{b}}(r)\big)\vec{b}_1^T B(r)\,\mathrm{d}r\Big]\bar{c}_0(t) \\
+&\Big\{\Big[\int_{R_B}^{R_F} \frac{R_F - r}{R_F - R_B}\big(r\,\bar{\mathrm{b}}_r(r) + \bar{\mathrm{b}}(r)\big)\,\mathrm{d}r\Big]\vec{b}_1^T - R_B\,\bar{\mathrm{b}}(R_B)P_3^T\Big\}\bar{\nu}_{B0}(t) \\
+&\Big[\int_{R_B}^{R_F} \frac{r - R_B}{R_F - R_B}\big(r\,\bar{\mathrm{b}}_r(r) + \bar{\mathrm{b}}(r)\big)\,\mathrm{d}r\Big]\vec{b}_1^T\bar{\nu}_{F0} + R_F\sigma_R\,\bar{\mathrm{b}}(R_F) \qquad (4.1.58)
\end{aligned}$$

$\varphi_1:$

$$\begin{aligned}
&\Big[\big(K + \frac{4G}{3}\big)g_2(r) + \big(K - \frac{2G}{3}\big)\big(g_1^T + g_1\big) + \big(K + \frac{16G}{3}\big)g_0\big(\frac{1}{r}\big) + K_F R_F g_F\Big]\bar{\mu}_1(t) \\
-2&\Big[\big(K - \frac{2G}{3}\big)g_1^T - Gg_1 + \big(K + \frac{7G}{3}\big)g_0\big(\frac{1}{r}\big)\Big]\bar{\eta}_2(t) \\
=&\Big[\int_{R_B}^{R_F} \big(r\,\bar{\mathrm{b}}_r(r) + \bar{\mathrm{b}}(r)\big)\vec{b}_1^T B(r)\,\mathrm{d}r\Big]\bar{c}_1(t) \\
+&\Big[\int_{R_B}^{R_F} \frac{r - R_B}{R_F - R_B}\big(r\,\bar{\mathrm{b}}_r(r) + \bar{\mathrm{b}}(r)\big)\,\mathrm{d}r\Big]\vec{b}_1^T\bar{\nu}_{F1} + R_F\sigma_{xy}\,\bar{\mathrm{b}}(R_F) \qquad (4.1.59)
\end{aligned}$$

$\varphi_2:$

$$\begin{aligned}
&\Big[\big(K + \frac{4G}{3}\big)g_2(r) + \big(K - \frac{2G}{3}\big)\big(g_1^T + g_1\big) + \big(K + \frac{16G}{3}\big)g_0\big(\frac{1}{r}\big) + K_F R_F g_F\Big]\bar{\mu}_2(t) \\
+2&\Big[\big(K - \frac{2G}{3}\big)g_1^T - Gg_1 + \big(K + \frac{7G}{3}\big)g_0\big(\frac{1}{r}\big)\Big]\bar{\eta}_1(t) \\
=&\Big[\int_{R_B}^{R_F} \big(r\,\bar{\mathrm{b}}_r(r) + \bar{\mathrm{b}}(r)\big)\vec{b}_1^T B(r)\,\mathrm{d}r\Big]\bar{c}_2(t) \\
+&\Big[\int_{R_B}^{R_F} \frac{r - R_B}{R_F - R_B}\big(r\,\bar{\mathrm{b}}_r(r) + \bar{\mathrm{b}}(r)\big)\,\mathrm{d}r\Big]\vec{b}_1^T\bar{\nu}_{F2} + R_F\sigma_D\,\bar{\mathrm{b}}(R_F) \qquad (4.1.60)
\end{aligned}$$

$\psi_0:$

$$\Big[Gg_2(r) - G\big(g_1^T + g_1\big) + Gg_0\big(\frac{1}{r}\big) + K_F R_F g_F\Big]\bar{\eta}_0(t) = 0$$

It follows that

$$\bar{\eta}_0(t) = 0$$

ψ_1 :

$$\begin{aligned}
&2\Big[\big(K-\frac{2G}{3}\big)g_1 - Gg_1^T + \big(K+\frac{7G}{3}\big)g_0\big(\frac{1}{r}\big)\Big]\bar{\mu}_2(t) \\
&+\Big[Gg_2(r) - G\big(g_1^T+g_1\big) + \big(4K+\frac{19G}{3}\big)g_0\big(\frac{1}{r}\big) + K_F R_F g_F\Big]\bar{\eta}_1(t) \\
&= 2\Big[\int_{R_B}^{R_F} \bar{\mathrm{b}}(r)\vec{b}_1^T B(r)\,\mathrm{d}r\Big]\bar{c}_2(t) \\
&+2\Big[\int_{R_B}^{R_F} \frac{r-R_B}{R_F-R_B}\,\bar{\mathrm{b}}(r)\,\mathrm{d}r\Big]\vec{b}_1^T\bar{\nu}_{F2} + R_F\sigma_D\,\bar{\mathrm{b}}(R_F)
\end{aligned} \tag{4.1.61}$$

ψ_2 :

$$\begin{aligned}
&-2\Big[\big(K-\frac{2G}{3}\big)g_1 - Gg_1^T + \big(K+\frac{7G}{3}\big)g_0\big(\frac{1}{r}\big)\Big]\bar{\mu}_1(t) \\
&+\Big[Gg_2(r) - G\big(g_1^T+g_1\big) + \big(4K+\frac{19G}{3}\big)g_0\big(\frac{1}{r}\big) + K_F R_F g_F\Big]\bar{\eta}_2(t) \\
&= -2\Big[\int_{R_B}^{R_F} \bar{\mathrm{b}}(r)\vec{b}_1^T B(r)\,\mathrm{d}r\Big]\bar{c}_1(t) \\
&-2\Big[\int_{R_B}^{R_F} \frac{r-R_B}{R_F-R_B}\,\bar{\mathrm{b}}(r)\,\mathrm{d}r\Big]\vec{b}_1^T\bar{\nu}_{F1} + R_F\sigma_{xy}\,\bar{\mathrm{b}}(R_F)
\end{aligned} \tag{4.1.62}$$

Define the following matrices and vectors.

$M \times M$ matrices:

$$\begin{aligned}
\mathcal{M}_0 &= \big(K+\frac{4G}{3}\big)g_2(r) + \big(K-\frac{2G}{3}\big)\big(g_1^T+g_1\big) + \big(K+\frac{4G}{3}\big)g_0\big(\frac{1}{r}\big) + K_F R_F g_F \\
\mathcal{M}_1 &= \big(K+\frac{4G}{3}\big)g_2(r) + \big(K-\frac{2G}{3}\big)\big(g_1^T+g_1\big) + \big(K+\frac{16G}{3}\big)g_0\big(\frac{1}{r}\big) + K_F R_F g_F \\
\mathcal{M}_2 &= Gg_2(r) - G\big(g_1^T+g_1\big) + \big(4K+\frac{19G}{3}\big)g_0\big(\frac{1}{r}\big) + K_F R_F g_F \\
\mathcal{N} &= 2\Big[\big(K-\frac{2G}{3}\big)g_1^T - Gg_1 + \big(K+\frac{7G}{3}\big)g_0\big(\frac{1}{r}\big)\Big]
\end{aligned} \tag{4.1.63}$$

$M \times 3M_D$ matrices:

$$\mathcal{Z}_0 = \int_{R_B}^{R_F} \big(r\,\bar{\mathrm{b}}_r(r) + \bar{\mathrm{b}}(r)\big)\vec{b}_1^T B(r)\,\mathrm{d}r \tag{4.1.64}$$

$$\mathcal{Z}_1 = 2\int_{R_B}^{R_F} \bar{\mathrm{b}}(r)\vec{b}_1^T B(r)\,\mathrm{d}r \tag{4.1.65}$$

$M \times 3$ matrices:

$$Y_{B0} = \Big[\int_{R_B}^{R_F} \frac{R_F - r}{R_F - R_B}\big(r\,\bar{\mathrm{b}}_r(r) + \bar{\mathrm{b}}(r)\big)\,\mathrm{d}r\Big]\vec{b}_1^T - R_B\,\bar{\mathrm{b}}(R_B)P_3^T \tag{4.1.66}$$

$$Y_{F0} = \Big[\int_{R_B}^{R_F} \frac{r - R_B}{R_F - R_B}\big(r\,\bar{\mathrm{b}}_r(r) + \bar{\mathrm{b}}(r)\big)\,\mathrm{d}r\Big]\vec{b}_1^T \tag{4.1.67}$$

$$Y_{F1} = 2\Big[\int_{R_B}^{R_F} \frac{r - R_B}{R_F - R_B}\,\bar{\mathrm{b}}(r)\,\mathrm{d}r\Big]\vec{b}_1^T \tag{4.1.68}$$

$M \times 1$ vectors:

$$S_R = R_F\sigma_R\,\bar{\mathrm{b}}(R_F) \tag{4.1.69}$$

$$S_{xy} = R_F\sigma_{xy}\,\bar{\mathrm{b}}(R_F) \tag{4.1.70}$$

$$S_D = R_F\sigma_D\,\bar{\mathrm{b}}(R_F) \tag{4.1.71}$$

Applying (4.1.63)-(4.1.71) to (4.1.58)-(4.1.62), we obtain the following matrix equations:

$$\mathcal{M}_0\bar{\mu}_0(t) = \mathcal{Z}_0\bar{c}_0(t) + Y_{B0}\bar{\nu}_{B0}(t) + Y_{F0}\bar{\nu}_{F0} + S_R \tag{4.1.72}$$

$$\begin{bmatrix} \mathcal{M}_1 & -\mathcal{N} \\ -\mathcal{N}^T & \mathcal{M}_2 \end{bmatrix} \begin{bmatrix} \bar{\mu}_1(t) \\ \bar{\eta}_2(t) \end{bmatrix} = \begin{bmatrix} \mathcal{Z}_0 \\ -\mathcal{Z}_1 \end{bmatrix} \bar{c}_1(t) + \begin{bmatrix} Y_{F0} \\ -Y_{F1} \end{bmatrix} \bar{\nu}_{F1} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} S_{xy} \tag{4.1.73}$$

$$\begin{bmatrix} \mathcal{M}_1 & \mathcal{N} \\ \mathcal{N}^T & \mathcal{M}_2 \end{bmatrix} \begin{bmatrix} \bar{\mu}_2(t) \\ \bar{\eta}_1(t) \end{bmatrix} = \begin{bmatrix} \mathcal{Z}_0 \\ \mathcal{Z}_1 \end{bmatrix} \bar{c}_2(t) + \begin{bmatrix} Y_{F0} \\ Y_{F1} \end{bmatrix} \bar{\nu}_{F2} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} S_D \tag{4.1.74}$$

Define the following matrices:

$$J = \begin{bmatrix} I_{M\times M} & 0 \\ 0 & -I_{M\times M} \end{bmatrix}$$

$$\mathcal{Q} = \begin{bmatrix} \mathcal{M}_1 & \mathcal{N} \\ \mathcal{N}^T & \mathcal{M}_2 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} \mathcal{M}_1 & -\mathcal{N} \\ -\mathcal{N}^T & \mathcal{M}_2 \end{bmatrix} = J\mathcal{Q}J$$

$$\bar{\mathcal{Z}} = \begin{bmatrix} \mathcal{Z}_0 \\ \mathcal{Z}_1 \end{bmatrix} \quad \Rightarrow \begin{bmatrix} \mathcal{Z}_0 \\ -\mathcal{Z}_1 \end{bmatrix} = J\bar{\mathcal{Z}}$$

$$\bar{Y}_F = \begin{bmatrix} Y_{F0} \\ Y_{F1} \end{bmatrix} \quad \Rightarrow \begin{bmatrix} Y_{F0} \\ -Y_{F1} \end{bmatrix} = J\bar{Y}_F$$

$$\bar{S}_{xy} = \begin{bmatrix} S_{xy} \\ S_{xy} \end{bmatrix}, \qquad \bar{S}_D = \begin{bmatrix} S_D \\ S_D \end{bmatrix}$$

$$P_{10} = \begin{bmatrix} 1 & 0 \end{bmatrix} \otimes I_{M\times M}$$

$$P_{01} = \begin{bmatrix} 0 & 1 \end{bmatrix} \otimes I_{M\times M}$$

From (4.1.72)-(4.1.74) we have

$$\bar{\mu}_0(t) = \mathcal{M}_0^{-1}\Big[\mathcal{Z}_0\bar{c}_0(t) + Y_{B0}\bar{\nu}_{B0}(t) + Y_{F0}\bar{\nu}_{F0} + S_R\Big] \tag{4.1.75}$$

$$\bar{\mu}_1(t) = P_{10}\mathcal{Q}^{-1}\Big[\bar{\mathcal{Z}}\bar{c}_1(t) + \bar{Y}_F\bar{\nu}_{F1} + J\bar{S}_{xy}\Big] \tag{4.1.76}$$

$$\bar{\eta}_2(t) = -P_{01}\mathcal{Q}^{-1}\Big[\bar{\mathcal{Z}}\bar{c}_1(t) + \bar{Y}_F\bar{\nu}_{F1} + J\bar{S}_{xy}\Big] \tag{4.1.77}$$

$$\bar{\mu}_2(t) = P_{10}\mathcal{Q}^{-1}\Big[\bar{\mathcal{Z}}\bar{c}_2(t) + \bar{Y}_F\bar{\nu}_{F2} + \bar{S}_D\Big] \tag{4.1.78}$$

$$\bar{\eta}_1(t) = P_{01}\mathcal{Q}^{-1}\Big[\bar{\mathcal{Z}}\bar{c}_2(t) + \bar{Y}_F\bar{\nu}_{F2} + \bar{S}_D\Big] \tag{4.1.79}$$

Finite element approximation of the TCP system

System I (radially symmetric case)

System I is given by (4.1.17)-(4.1.19):

$$M\dot{V}_0 - A\frac{1}{r}(rV_{0_r})_r = -M\dot{W}_0 + A\frac{1}{r}(rW_{0_r})_r - \vec{b}_0\Big(\dot{u}_{0_r} + \frac{1}{r}\dot{u}_0\Big) \tag{4.1.17}$$

with boundary conditions

$$V_0(r,t) = 0 \qquad \text{on } \Gamma \times [0, t_f) \tag{4.1.18}$$

and initial conditions

$$V_0(r,0) = \tilde{V}_{I0}(r) - W_0(r,0) \tag{4.1.19}$$

Let

$$\mathcal{A} = M^{-1}A$$

Then (4.1.17) has the form

$$\dot{V}_0 - \mathcal{A}\frac{1}{r}(rV_{0_r})_r = -\dot{W}_0 + \mathcal{A}\frac{1}{r}(rW_{0_r})_r - M^{-1}\vec{b}_0\Big(\dot{u}_{0_r} + \frac{1}{r}\dot{u}_0\Big) \tag{4.1.80}$$

Using (4.1.51), the left-hand side of (4.1.80) is

$$B(r)\dot{\bar{c}}_0(t) - \mathcal{A}\frac{1}{r}\Big(rB_r(r)\Big)_r\bar{c}_0(t) \tag{4.1.81}$$

Multiplying (4.1.81) by $B(r)^T$ and integrating with respect to r, we have

$$\Big[\int_{R_B}^{R_F} B(r)^T B(r) r\,\mathrm{d}r\Big]\dot{\bar{c}}_0(t) - \Big[\int_{R_B}^{R_F} B(r)^T\mathcal{A}\frac{1}{r}\Big(rB_r(r)\Big)_r r\,\mathrm{d}r\Big]\bar{c}_0(t)$$

$$=\Big[\int_{R_B}^{R_F} B(r)^T B(r) r\,\mathrm{d}r\Big]\dot{\bar{c}}_0(t) + \Big[\int_{R_B}^{R_F} B_r(r)^T\mathcal{A}B_r(r) r\,\mathrm{d}r\Big]\bar{c}_0(t)$$

$$=\Big[\int_{R_B}^{R_F} B(r)^T B(r) r\,\mathrm{d}r\Big]\dot{\bar{c}}_0(t) + \mathcal{A}\otimes\Big[\int_{R_B}^{R_F} \bar{\mathrm{b}}_r(r)\bar{\mathrm{b}}_r(r)^T r\,\mathrm{d}r\Big]\bar{c}_0(t) \tag{4.1.82}$$

Using (4.1.53) and (4.1.54), the third term of the right-hand side of (4.1.80) is

$$-M^{-1}\vec{b}_0\Big(\dot{u}_{0_r} + \frac{1}{r}\dot{u}_0\Big) = -M^{-1}\vec{b}_0\Big[\bar{\mathrm{b}}_r(r)^T + \frac{1}{r}\bar{\mathrm{b}}(r)^T\Big]\dot{\bar{\mu}}_0(t) \tag{4.1.83}$$

Multiplying (4.1.83) by $B(r)^T$ and integrating with respect to r, we have

$$-\Big[\int_{R_B}^{R_F} B(r)^T M^{-1}\vec{b}_0\big(r\,\bar{\mathrm{b}}_r(r)^T + \bar{\mathrm{b}}(r)^T\big)\,\mathrm{d}r\Big]\dot{\bar{\mu}}_0(t) \tag{4.1.84}$$

From (4.1.11) and (4.1.52)

$$W_i(r,t) = \frac{R_F - r}{R_F - R_B}\bar{\nu}_{Bi}(t) + \frac{r - R_B}{R_F - R_B}\bar{\nu}_{Fi}, \quad i = 0,\dots,2 \tag{4.1.85}$$

and $\bar{\nu}_{Bi}(t) = 0, \quad i = 1,2.$

It follows that

$$\dot{W}_i(r,t) = \frac{R_F - r}{R_F - R_B}\dot{\bar{\nu}}_{Bi}(t), \quad i = 0,\dots,2 \tag{4.1.86}$$

and $\dot{\bar{\nu}}_{Bi}(t) = 0, \quad i = 1,2.$

Multiplying (4.1.86) by $B(r)^T$ and integrating with respect to r, we have

$$\Big[\int_{R_B}^{R_F} B(r)^T \frac{R_F - r}{R_F - R_B} r\,\mathrm{d}r\Big]\dot{\bar{\nu}}_{Bi}(t) \tag{4.1.87}$$

From (4.1.14) and (4.1.52)

$$W_{i_r}(r,t) = -\frac{1}{R_F - R_B}\bar{\nu}_{Bi}(t) + \frac{1}{R_F - R_B}\bar{\nu}_{Fi}, \quad i = 0,\dots,2 \tag{4.1.88}$$

and $\bar{\nu}_{Bi}(t) = 0, \quad i = 1,2.$

Multiplying $\mathcal{A}\frac{1}{r}(rW_{i_r})_r$ by $B(r)^T$ and integrating with respect to r, we have

$$\Big[\int_{R_B}^{R_F} B(r)^T \mathcal{A}\frac{1}{R_F - R_B}\,\mathrm{d}r\Big]\Big(-\bar{\nu}_{Bi}(t) + \bar{\nu}_{Fi}\Big), \quad i = 0,\dots,2 \tag{4.1.89}$$

and $\bar{\nu}_{Bi}(t) = 0, \quad i = 1,2.$

Define the following matrices.

$3M_D \times 3M_D$ matrices:

$$G_0 = \int_{R_B}^{R_F} B(r)^T B(r) r\,\mathrm{d}r$$

$$G_2 = \mathcal{A} \otimes \Big[\int_{R_B}^{R_F} \bar{b}_r(r)\bar{b}_r(r)^T r \,\mathrm{d}r \Big]$$

$3M_D \times M$ matrices:

$$H_0 = \int_{R_B}^{R_F} B(r)^T M^{-1} \vec{b}_0 (r\,\bar{\mathrm{b}}_r(r) + \bar{\mathrm{b}}(r))^T \,\mathrm{d}r$$

$3M_D \times 3$ matrices:

$$\mathcal{R}_B = \int_{R_B}^{R_F} B(r)^T \frac{R_F - r}{R_F - R_B} r \,\mathrm{d}r$$

$$\mathcal{R}_F = \int_{R_B}^{R_F} B(r)^T \frac{r - R_B}{R_F - R_B} r \,\mathrm{d}r$$

$$\mathcal{R} = \Big[\int_{R_B}^{R_F} \frac{1}{R_F - R_B} B(r)^T \,\mathrm{d}r \Big] \mathcal{A}$$

Using these matrices, (4.1.80) multiplied by $B(r)^T$ and integrated with respect to r, has the form

$$G_0 \dot{\bar{c}}_0(t) + G_2 \bar{c}_0(t) = -H_0 \dot{\bar{\mu}}_0(t) - \mathcal{R}_B \dot{\bar{\nu}}_{B0}(t) - \mathcal{R}\bar{\nu}_{B0}(t) + \mathcal{R}\bar{\nu}_{F0} \tag{4.1.90}$$

From (4.1.75)

$$\dot{\bar{\mu}}_0(t) = \mathcal{M}_0^{-1} \Big[\mathcal{Z}_0 \dot{\bar{c}}_0(t) + Y_{B0} \dot{\bar{\nu}}_{B0}(t) \Big] \tag{4.1.91}$$

Applying (4.1.91) to (4.1.90), we have

$$\mathcal{H}_0 \dot{\bar{c}}_0(t) + G_2 \bar{c}_0(t) = \mathcal{D}_{B0} \dot{\bar{\nu}}_{B0}(t) - \mathcal{R}\bar{\nu}_{B0}(t) + \mathcal{R}\bar{\nu}_{F0} \tag{4.1.92}$$

where $\mathcal{H}_0 = G_0 + H_0 \mathcal{M}_0^{-1} \mathcal{Z}_0$

$\mathcal{D}_{B0} = -H_0 \mathcal{M}_0^{-1} Y_{B0} - \mathcal{R}_B$

Now we approximate initial conditions. From (4.1.11), (4.1.19) and (4.1.52), the coefficients $V_i(r, 0)$ of θ-approximation of initial conditions have the form

$$V_i(r, 0) = \tilde{V}_{Ii}(r) - W_i(r, 0)$$

where $\quad \tilde{V}_{Ii}(r) = B(r)\bar{\nu}_{Ii}$

$$W_i(r,0) = \frac{R_F - r}{R_F - R_B}\bar{\nu}_{Bi}(0) + \frac{r - R_B}{R_F - R_B}\bar{\nu}_{Fi}, \quad i = 0, \ldots, 2 \tag{4.1.93}$$

and $\bar{\nu}_{Bi}(0) = 0, \quad i = 1, 2.$

Multiplying $V_i(r,0)$ by $B(r)^T$ and integrating with respect to r, we have

$$\Big[\int_{R_B}^{R_F} B(r)^T B(r) r\,\mathrm{d}r\Big]\bar{c}_i(0) = \Big[\int_{R_B}^{R_F} B(r)^T B(r) r\,\mathrm{d}r\Big]\bar{\nu}_{Ii}$$
$$-\Big[\int_{R_B}^{R_F} B(r)^T \frac{R_F - r}{R_F - R_B} r\,\mathrm{d}r\Big]\bar{\nu}_{Bi}(0) - \Big[\int_{R_B}^{R_F} B(r)^T \frac{r - R_B}{R_F - R_B} r\,\mathrm{d}r\Big]\bar{\nu}_{Fi} \tag{4.1.94}$$

Using matrices defined above, (4.1.94) has the form

$$G_0\bar{c}_i(0) = G_0\bar{\nu}_{Ii} - \mathcal{R}_B\bar{\nu}_{Bi}(0) - \mathcal{R}_F\bar{\nu}_{Fi} \tag{4.1.95}$$

From (4.1.92) and (4.1.95) with $i = 0$ we obtain the following system of ordinary differential equations (ODEs).

System of ODEs for TCP I (radially symmetric case):

$$\mathcal{H}_0\dot{\bar{c}}_0(t) + G_2\bar{c}_0(t) = \mathcal{D}_{B0}\dot{\bar{\nu}}_{B0}(t) - \mathcal{R}\bar{\nu}_{B0}(t) + \mathcal{R}\bar{\nu}_{F0} \tag{4.1.92}$$

with initial conditions

$$\bar{c}_0(0) = \bar{\nu}_{I0} - G_0^{-1}\big[\mathcal{R}_B\bar{\nu}_{B0}(0) + \mathcal{R}_F\bar{\nu}_{F0}\big] \tag{4.1.96}$$

System II (corresponds to $\sin 2\theta$ approximation)

System II is given by (4.1.20)-(4.1.22):

$$M\dot{V}_1 - A\Big[\frac{1}{r}(rV_{1_r})_r - \frac{4}{r^2}V_1\Big] = -M\dot{W}_1 + A\Big(\frac{1}{r}(rW_{1_r})_r - \frac{4}{r^2}W_1\Big)$$
$$-\vec{b}_0\Big(\dot{u}_{1_r} + \frac{1}{r}(\dot{u}_1 - 2\dot{v}_2)\Big) \tag{4.1.20}$$

with boundary conditions

$$V_1(r,t) = 0 \qquad \text{on } \Gamma \times [0, t_f) \tag{4.1.21}$$

and initial conditions

$$V_1(r,0) = \tilde{V}_{I1}(r) - W_1(r,0) \tag{4.1.22}$$

Using $\mathcal{A} = M^{-1}A$, (4.1.20) has the form

$$\dot{V}_1 - \mathcal{A}\Big[\frac{1}{r}(rV_{1_r})_r - \frac{4}{r^2}V_1\Big] = -M^{-1}\vec{b}_0\Big(\dot{u}_{1_r} + \frac{1}{r}\dot{u}_1 - \frac{2}{r}\dot{v}_2\Big)$$
$$-\dot{W}_1 + \mathcal{A}\Big[\frac{1}{r}(rW_{1_r})_r - \frac{4}{r^2}W_1\Big] \tag{4.1.97}$$

Using (4.1.51), the left-hand side of (4.1.97) is

$$B(r)\dot{\bar{c}}_1(t) - \mathcal{A}\frac{1}{r}\Big(rB_r(r)\Big)_r\bar{c}_1(t) + \mathcal{A}\frac{4}{r^2}B(r)\bar{c}_1(t) \tag{4.1.98}$$

Multiplying (4.1.98) by $B(r)^T$ and integrating with respect to r, we have

$$\Big[\int_{R_B}^{R_F} B(r)^T B(r) r \,\mathrm{d}r\Big]\dot{\bar{c}}_1(t)$$
$$+\mathcal{A}\otimes\Big\{\Big[\int_{R_B}^{R_F} \bar{b}_r(r)\bar{b}_r(r)^T r\,\mathrm{d}r\Big] + 4\Big[\int_{R_B}^{R_F} \bar{b}(r)\bar{b}(r)^T\frac{1}{r}\,\mathrm{d}r\Big]\Big\}\bar{c}_1(t) \tag{4.1.99}$$

Using (4.1.53) and (4.1.54), the third term of the right-hand side of (4.1.97) is

$$-M^{-1}\vec{b}_0\Big(\dot{u}_{1_r} + \frac{1}{r}\dot{u}_1 - \frac{2}{r}\dot{v}_2\Big)$$
$$= -M^{-1}\vec{b}_0\Big[\,\bar{\mathrm{b}}_r(r)^T + \frac{1}{r}\,\bar{\mathrm{b}}(r)^T\Big]\dot{\bar{\mu}}_1(t) + M^{-1}\vec{b}_0\frac{2}{r}\,\bar{\mathrm{b}}(r)^T\dot{\bar{\eta}}_2(t)$$

Multiplying by $B(r)^T$ and integrating with respect to r, we have

$$-\Big[\int_{R_B}^{R_F} B(r)^T M^{-1}\vec{b}_0(r\,\bar{\mathrm{b}}_r(r) + \,\bar{\mathrm{b}}(r))^T\,\mathrm{d}r\Big]\dot{\bar{\mu}}_1(t)$$
$$+2\Big[\int_{R_B}^{R_F} B(r)^T M^{-1}\vec{b}_0\,\bar{\mathrm{b}}(r)^T\,\mathrm{d}r\Big]\dot{\bar{\eta}}_2(t) \tag{4.1.100}$$

Multiplying the corresponding term in (4.1.97) by $B(r)^T$ and integrating with respect to r, we have the following results for $\dot{W}_1$: (4.1.87) with $\dot{\bar{\nu}}_{B1}(t) = 0$ and

for $\mathcal{A}\frac{1}{r}(rW_{1_r})_r$: (4.1.89) with $\bar{\nu}_{B1}(t) = 0$.

$W_1(r,t)$ is given by (4.1.85) with $\bar{\nu}_{B1}(t) = 0$.

Multiplying $\mathcal{A}\frac{4}{r^2}W_i$ by $B(r)^T$ and integrating with respect to r, we have

$$\Big[\int_{R_B}^{R_F} B(r)^T \mathcal{A}\frac{4}{r^2}\frac{r-R_B}{R_F-R_B} r\,\mathrm{d}r\Big]\bar{\nu}_{Fi}$$

$$=4\Big[\int_{R_B}^{R_F} B(r)^T \frac{r-R_B}{R_F-R_B}\frac{1}{r}\,\mathrm{d}r\Big]\mathcal{A}\bar{\nu}_{Fi}, \qquad i=1,2. \tag{4.1.101}$$

Define the following matrices.

$3M_D \times 3M_D$ matrix:

$$G_0\big(\frac{1}{r}\big) = \mathcal{A} \otimes \Big[\int_{R_B}^{R_F} \bar{b}(r)\bar{b}(r)^T \frac{1}{r}\,\mathrm{d}r\Big]$$

$3M_D \times M$ matrix:

$$H_0(r) = \int_{R_B}^{R_F} B(r)^T M^{-1}\vec{b}_0\,\bar{\mathrm{b}}(r)^T\,\mathrm{d}r$$

$3M_D \times 3$ matrix:

$$\mathcal{R}_F^{(a)}\big(\frac{1}{r}\big) = \Big[\int_{R_B}^{R_F} B(r)^T \frac{r-R_B}{R_F-R_B}\frac{1}{r}\,\mathrm{d}r\Big]\mathcal{A}$$

Using these matrices together with matrices defined for System I, (4.1.97) multiplied by $B(r)^T$ and integrated with respect to r, has the form

$$G_0\dot{\bar{c}}_1(t) + \Big[G_2 + 4G_0\big(\frac{1}{r}\big)\Big]\bar{c}_1(t) = -H_0\dot{\bar{\mu}}_1(t) + 2H_0(r)\dot{\bar{\eta}}_2(t) + \Big[\mathcal{R} - 4\mathcal{R}_F^{(a)}\big(\frac{1}{r}\big)\Big]\bar{\nu}_{F1} \tag{4.1.102}$$

From (4.1.76) and (4.1.77)

$$\dot{\bar{\mu}}_1(t) = P_{10}\mathcal{Q}^{-1}\bar{\mathcal{Z}}\dot{\bar{c}}_1(t) \tag{4.1.103}$$

$$\dot{\bar{\eta}}_2(t) = -P_{01}\mathcal{Q}^{-1}\bar{\mathcal{Z}}\dot{\bar{c}}_1(t) \tag{4.1.104}$$

Applying (4.1.103) and (4.1.104) to (4.1.102), we have

$$\mathcal{H}_1\dot{\bar{c}}_1(t) + \mathcal{G}_1\bar{c}_1(t) = \mathcal{F}_{F1}\bar{\nu}_{F1} \tag{4.1.105}$$

where

$$\begin{aligned}\mathcal{H}_1 &= G_0 + \big(H_0P_{10} + 2H_0(r)P_{01}\big)\mathcal{Q}^{-1}\bar{\mathcal{Z}}\\ \mathcal{G}_1 &= G_2 + 4G_0\big(\frac{1}{r}\big)\\ \mathcal{F}_{F1} &= \Big(\mathcal{R} - 4\mathcal{R}_F^{(a)}\big(\frac{1}{r}\big)\Big)\end{aligned} \tag{4.1.106}$$

Initial conditions are given by (4.1.95) with $i = 1$ and $\bar{\nu}_{B1}(0) = 0$. From (4.1.106) and (4.1.95) with $i = 1$ and $\bar{\nu}_{B1}(0) = 0$ we obtain the following system.

System of ODEs for TCP II (corresponds to $\sin 2\theta$ approximation):

$$\mathcal{H}_1\dot{\bar{c}}_1(t) + \mathcal{G}_1\bar{c}_1(t) = \mathcal{F}_{F1}\bar{\nu}_{F1} \tag{4.1.105}$$

with initial conditions

$$\bar{c}_1(0) = \bar{\nu}_{I1} - G_0^{-1}\mathcal{R}_F\bar{\nu}_{F1} \tag{4.1.107}$$

System III (corresponds to $\cos 2\theta$ approximation)

System III is given by (4.1.23)-(4.1.25):

$$\begin{aligned}M\dot{V}_2 - A\Big[\frac{1}{r}(rV_{2_r})_r - \frac{4}{r^2}V_2\Big] &= -M\dot{W}_2 + A\Big(\frac{1}{r}(rW_{2_r})_r - \frac{4}{r^2}W_2\Big)\\ &\quad -\vec{b}_0\Big(\dot{u}_{2_r} + \frac{1}{r}(\dot{u}_2 + 2\dot{v}_1)\Big)\end{aligned} \tag{4.1.23}$$

with boundary conditions

$$V_2(r,t) = 0 \qquad \text{on } \Gamma \times (0, t_f) \tag{4.1.24}$$

and initial conditions

$$V_2(r,0) = \tilde{V}_{I2}(r) - W_2(r,0) \tag{4.1.25}$$

Using $\mathcal{A} = M^{-1}A$, (4.1.23) has the form

$$\dot{V}_2 - \mathcal{A}\Big[\frac{1}{r}(rV_{2_r})_r - \frac{4}{r^2}V_2\Big] = -M^{-1}\vec{b}_0\Big(\dot{u}_{2_r} + \frac{1}{r}\dot{u}_2 + \frac{2}{r}\dot{v}_1\Big)$$

$$-\dot{W}_2 + \mathcal{A}\Big[\frac{1}{r}(rW_{2_r})_r - \frac{4}{r^2}W_2\Big] \qquad (4.1.108)$$

Using (4.1.51), the left-hand side of (4.1.108) is

$$B(r)\dot{\bar{c}}_2(t) - \mathcal{A}\frac{1}{r}\Big(rB_r(r)\Big)_r\bar{c}_2(t) + \mathcal{A}\frac{4}{r^2}B(r)\bar{c}_2(t) \qquad (4.1.109)$$

Multiplying (4.1.109) by $B(r)^T$ and integrating with respect to r, we have

$$\Big[\int_{R_B}^{R_F} B(r)^T B(r) r \,\mathrm{d}r\Big]\dot{\bar{c}}_2(t)$$

$$+\mathcal{A}\otimes\Big\{\Big[\int_{R_B}^{R_F} \bar{b}_r(r)\bar{b}_r(r)^T r\,\mathrm{d}r\Big] + 4\Big[\int_{R_B}^{R_F} \bar{b}(r)\bar{b}(r)^T\frac{1}{r}\,\mathrm{d}r\Big]\Big\}\bar{c}_2(t) \qquad (4.1.110)$$

Using (4.1.53) and (4.1.54), the first term of the right-hand side of (4.1.108) is

$$-M^{-1}\vec{b}_0\Big(\dot{u}_{2_r} + \frac{1}{r}\dot{u}_2 + \frac{2}{r}\dot{v}_1\Big)$$

$$= -M^{-1}\vec{b}_0\Big[\bar{\mathrm{b}}_r(r)^T + \frac{1}{r}\bar{\mathrm{b}}(r)^T\Big]\dot{\bar{\mu}}_2(t) - M^{-1}\vec{b}_0\frac{2}{r}\bar{\mathrm{b}}(r)^T\dot{\bar{\eta}}_1(t)$$

Multiplying by $B(r)^T$ and integrating with respect to r, we have

$$-\Big[\int_{R_B}^{R_F} B(r)^T M^{-1}\vec{b}_0(r\,\bar{\mathrm{b}}_r(r) + \bar{\mathrm{b}}(r))^T\,\mathrm{d}r\Big]\dot{\bar{\mu}}_2(t)$$

$$-2\Big[\int_{R_B}^{R_F} B(r)^T M^{-1}\vec{b}_0\,\bar{\mathrm{b}}(r)^T\,\mathrm{d}r\Big]\dot{\bar{\eta}}_1(t) \qquad (4.1.111)$$

Multiplying the corresponding term in (4.1.108) by $B(r)^T$ and integrating with respect to r, we have the following results

for $\dot{W}_2$: (4.1.87) with $\dot{\bar{\nu}}_{B2}(t) = 0$

for $\mathcal{A}\frac{1}{r}(rW_{2_r})_r$: (4.1.89) with $\bar{\nu}_{B2}(t) = 0$.

for $\mathcal{A}\frac{4}{r^2}W_2$: (4.1.101)

Using the matrices defined for Systems I and II, (4.1.108) multiplied by $B(r)^T$ and integrated with respect to r, has the form

$$G_0\dot{\bar{c}}_2(t) + \Big[G_2 + 4G_0\big(\frac{1}{r}\big)\Big]\bar{c}_2(t) = -H_0\dot{\bar{\mu}}_2(t) - 2H_0(r)\dot{\bar{\eta}}_1(t) + \Big[\mathcal{R} - 4\mathcal{R}_F^{(a)}\big(\frac{1}{r}\big)\Big]\bar{\nu}_{F2} \tag{4.1.112}$$

From (4.1.78) and (4.1.79)

$$\dot{\bar{\mu}}_2(t) = P_{10}\mathcal{Q}^{-1}\bar{\mathcal{Z}}\dot{\bar{c}}_2(t) \tag{4.1.113}$$

$$\dot{\bar{\eta}}_1(t) = P_{01}\mathcal{Q}^{-1}\bar{\mathcal{Z}}\dot{\bar{c}}_2(t) \tag{4.1.114}$$

Applying (4.1.113) and (4.1.114) to (4.1.112), we have

$$\mathcal{H}_1\dot{\bar{c}}_2(t) + \mathcal{G}_1\bar{c}_2(t) = \mathcal{F}_{F1}\bar{\nu}_{F2} \tag{4.1.115}$$

where the matrices $\mathcal{H}_1$, $\mathcal{G}_1$, and $\mathcal{F}_{F1}$ are given by (4.1.106)
Initial conditions are given by (4.1.95) with $i = 2$ and $\bar{\nu}_{B2}(0) = 0$. From (4.1.115) and (4.1.95) with $i = 2$ and $\bar{\nu}_{B2}(0) = 0$ we obtain the following system.

System of ODEs for TCP III (corresponds to $\cos 2\theta$ approximation):

$$\mathcal{H}_1\dot{\bar{c}}_2(t) + \mathcal{G}_1\bar{c}_2(t) = \mathcal{F}_{F1}\bar{\nu}_{F2} \tag{4.1.115}$$

with initial conditions

$$\bar{c}_2(0) = \bar{\nu}_{I2} - G_0^{-1}\mathcal{R}_F\bar{\nu}_{F2} \tag{4.1.116}$$

We have the following three systems of ODEs:

System of ODEs for TCP I (radially symmetric case):

$$\mathcal{H}_0\dot{\bar{c}}_0(t) + G_2\bar{c}_0(t) = \mathcal{D}_{B0}\dot{\bar{\nu}}_{B0}(t) - \mathcal{R}\bar{\nu}_{B0}(t) + \mathcal{R}\bar{\nu}_{F0} \tag{4.1.92}$$

with initial conditions

$$\bar{c}_0(0) = \bar{\nu}_{I0} - G_0^{-1}\big[\mathcal{R}_B\bar{\nu}_{B0}(0) + \mathcal{R}_F\bar{\nu}_{F0}\big] \tag{4.1.96}$$

System I gives coefficients V_0 and u_0 of the θ-approximation (4.1.9).

System of ODEs for TCP II (corresponds to $\sin 2\theta$ approximation):

$$\mathcal{H}_1\dot{\bar{c}}_1(t) + \mathcal{G}_1\bar{c}_1(t) = \mathcal{F}_{F1}\bar{\nu}_{F1} \tag{4.1.105}$$

with initial conditions

$$\bar{c}_1(0) = \bar{\nu}_{I1} - G_0^{-1}\mathcal{R}_F\bar{\nu}_{F1} \tag{4.1.107}$$

System II gives $\sin 2\theta$-coefficients V_1, u_1 and $\cos 2\theta$-coefficient v_2 of the θ-approximation (4.1.9).

System of ODEs for TCP III (corresponds to $\cos 2\theta$ approximation):

$$\mathcal{H}_1\dot{\bar{c}}_2(t) + \mathcal{G}_1\bar{c}_2(t) = \mathcal{F}_{F1}\bar{\nu}_{F2} \tag{4.1.115}$$

with initial conditions

$$\bar{c}_2(0) = \bar{\nu}_{I2} - G_0^{-1}\mathcal{R}_F\bar{\nu}_{F2} \tag{4.1.116}$$

System III gives $\sin 2\theta$-coefficient v_1 and $\cos 2\theta$-coefficients V_2, u_2 of the θ-approximation (4.1.9).

4.2 3D TCPu Problem for an Inclined Borehole

As in Section 2.2, let (X, Y, Z) be a borehole Cartesian coordinate system with the x -axis in the direction of the maximum horizontal principal stress σ_H , the y -axis in the direction of the minimum horizontal principal stress σ_h, and the z -axis in the direction of overburden stress σ_V and let (X_b, Y_b, Z_b) be the borehole Cartesian coordinate system. For the sake of convenience, we will make the transformation from the coordinate system (X_b, Y_b, Z_b) to the borehole cylindrical coordinate system (r, θ, z) with the z -axis passing through the central axis of the cylindrical region Ω.

Then the boundary Γ of Ω is $\Gamma = \Gamma_B \cup \Gamma_F \cup \tilde{\Gamma}_T \cup \tilde{\Gamma}_B$:

$$\Gamma_B = \left\{(R_B, \theta, z) \in \mathbb{R}^3 : 0 \leq \theta < 2\pi, z_T \leq z \leq z_B\right\}$$

$$\Gamma_F = \left\{(R_F, \theta, z) \in \mathbb{R}^3 : 0 \leq \theta < 2\pi, z_T \leq z \leq z_B\right\}$$

$$\tilde{\Gamma}_T = \left\{(r, \theta, z_T) \in \mathbb{R}^3 : R_B \leq r \leq R_F, 0 \leq \theta < 2\pi\right\}$$

$$\tilde{\Gamma}_B = \left\{(r, \theta, z_B) \in \mathbb{R}^3 : R_B \leq r \leq R_F, 0 \leq \theta < 2\pi\right\}$$

4.2.1 Fourier-type approximation of the TCP sytem

In the borehole cylindrical coordinate system (r, θ, z) the TCP initial-boundary value problem (2.2.17)-(2.2.20) is given by

$$M\dot{\bar{V}} - A\nabla^2\bar{V} = -\vec{b}_0(\nabla \cdot \dot{\vec{u}}) \qquad \text{in } \Omega \times (0, t_f) \tag{4.2.1}$$

with boundary conditions

$$\bar{V}(r, \theta, z, t) = \begin{cases} V_B(z, t) & \text{on } \Gamma_B \times [0, t_f) \\ V_F(\theta, z) & \text{on } \Gamma_F \times [0, t_f) \\ \tilde{V}_T(r, \theta, t) & \text{on } \tilde{\Gamma}_T \times [0, t_f) \\ \tilde{V}_B(r, \theta, t) & \text{on } \tilde{\Gamma}_B \times [0, t_f) \end{cases} \tag{4.2.2}$$

initial conditions

$$\bar{V}(r, \theta, z, 0) = \tilde{V}_I(r, \theta, z) \tag{4.2.3}$$

and compatibility conditions:

$$\tilde{V}_T(R_B, \theta, t) = V_B(z_T, t) \tag{4.2.4a}$$

$$\tilde{V}_T(R_F, \theta, t) = V_F(\theta, z_T) \tag{4.2.4b}$$

$$\tilde{V}_B(R_B, \theta, t) = V_B(z_B, t) \tag{4.2.4c}$$

$$\tilde{V}_B(R_F, \theta, t) = V_F(\theta, z_B) \tag{4.2.4d}$$

where the displacement vector is

$$\vec{u}(r,\theta,z,t) = u(r,\theta,z,t)\vec{i}_r + v(r,\theta,z,t)\vec{i}_\theta + w(r,\theta,z,t)\vec{i}_z$$

To obtain the equivalent parabolic TCP-type system with homogeneous boundary conditions (2.3.11)-(2.3.13), we first define

$$W^r(r,\theta,z,t) = \frac{R_F - r}{R_F - R_B} V_B(z,t) + \frac{r - R_B}{R_F - R_B} V_F(\theta, z) \tag{4.2.5}$$

$$\bar{V}^0 = \bar{V} - W^r \tag{4.2.6}$$

Then (4.2.1)-(4.2.3) has the form

$$M\dot{\bar{V}}^0 - A\nabla^2 \bar{V}^0 = -M\dot{W}^r + A\nabla^2 W^r - \vec{b}_0(\nabla \cdot \dot{\vec{u}}) \qquad \text{in } \Omega \times (0, t_f) \tag{4.2.7}$$

with boundary conditions

$$\bar{V}^0(r,\theta,z,t) = \begin{cases} 0 & \text{on } \Gamma_B \times [0,t_f) \\ 0 & \text{on } \Gamma_F \times [0,t_f) \\ \tilde{V}_T(r,\theta,t) - W^r(r,\theta,z_T,t) & \text{on } \tilde{\Gamma}_T \times [0,t_f) \\ \tilde{V}_B(r,\theta,t) - W^r(r,\theta,z_B,t) & \text{on } \tilde{\Gamma}_B \times [0,t_f) \end{cases} \tag{4.2.8}$$

and initial conditions

$$\bar{V}^0(r,\theta,z,0) = \tilde{V}_I(r,\theta,z) - W^r(r,\theta,z,0) \tag{4.2.9}$$

Now define

$$\hat{V}_T(r,\theta,t) = \tilde{V}_T(r,\theta,t) - W^r(r,\theta,z_T,t) \tag{4.2.10}$$

$$\hat{V}_B(r,\theta,t) = \tilde{V}_B(r,\theta,t) - W^r(r,\theta,z_B,t) \tag{4.2.11}$$

and

$$W^z(r,\theta,z,t) = \frac{z - z_T}{z_B - z_T} \hat{V}_B(r,\theta,t) + \frac{z_B - z}{z_B - z_T} \hat{V}_T(r,\theta,t) \tag{4.2.12}$$

$$V = \bar{V}^0 - W^z = \bar{V} - W^r - W^z \tag{4.2.13}$$

$$V = \bar{V} - W(r,\theta,z,t), \qquad \text{where } W = W^r + W^z \tag{4.2.14}$$

Then (4.2.1)-(4.2.3) has the form

$$M\dot{V} - A\nabla^2 V = -M\dot{W} + A\nabla^2 W - \vec{b}_0(\nabla \cdot \dot{\vec{u}}) \qquad \text{in } \Omega \times (0, t_f) \tag{4.2.15}$$

with boundary conditions

$$V(r,\theta,z,t) = 0 \qquad \text{on } \Gamma \times [0, t_f) \tag{4.2.16}$$

and initial conditions

$$V(r,\theta,z,0) = \tilde{V}_I(r,\theta,z) - W(r,\theta,z,0) \tag{4.2.17}$$

We introduce the following θ - approximation:

$$V(r,\theta,z,t) = V_0(r,z,t) + V_1(r,z,t)\sin\theta + V_2(r,z,t)\cos\theta$$
$$+ V_3(r,z,t)\sin 2\theta + V_4(r,z,t)\cos 2\theta$$

$$u(r,\theta,z,t) = u_0(r,z,t) + u_1(r,z,t)\sin\theta + u_2(r,z,t)\cos\theta$$
$$+ u_3(r,z,t)\sin 2\theta + u_4(r,z,t)\cos 2\theta$$

$$v(r,\theta,z,t) = v_0(r,z,t) + v_1(r,z,t)\sin\theta + v_2(r,z,t)\cos\theta$$
$$+ v_3(r,z,t)\sin 2\theta + v_4(r,z,t)\cos 2\theta$$

$$w(r,\theta,z,t) = w_0(r,z,t) + w_1(r,z,t)\sin\theta + w_2(r,z,t)\cos\theta$$
$$+ w_3(r,z,t)\sin 2\theta + w_4(r,z,t)\cos 2\theta$$

$$V_B(z,t) = V_{B0}(z,t)$$

$$V_F(\theta,z) = V_{F0}(z) + V_{F1}(z)\sin\theta + V_{F2}(z)\cos\theta$$
$$+ V_{F3}(z)\sin 2\theta + V_{F4}(z)\cos 2\theta$$

$$\tilde{V}_T(r,\theta,t) = \tilde{V}_{T0}(r,t) + \tilde{V}_{T1}(r,t)\sin\theta + \tilde{V}_{T2}(r,t)\cos\theta$$
$$+ \tilde{V}_{T3}(r,t)\sin 2\theta + \tilde{V}_{T4}(r,t)\cos 2\theta$$

$$\tilde{V}_B(r,\theta,t) = \tilde{V}_{B0}(r,t) + \tilde{V}_{B1}(r,t)\sin\theta + \tilde{V}_{B2}(r,t)\cos\theta$$
$$+ \tilde{V}_{B3}(r,t)\sin 2\theta + \tilde{V}_{B4}(r,t)\cos 2\theta$$

$$\tilde{V}_I(r,\theta,z) = \tilde{V}_{I0}(r,z) + \tilde{V}_{I1}(r,z)\sin\theta + \tilde{V}_{I2}(r,z)\cos\theta$$
$$+ \tilde{V}_{I3}(r,z)\sin 2\theta + \tilde{V}_{I4}(r,z)\cos 2\theta \qquad (4.2.18)$$

Then

$$W(r,\theta,z,t) = W_0(r,z,t) + W_1(r,z,t)\sin\theta + W_2(r,z,t)\cos\theta$$
$$+ W_3(r,z,t)\sin 2\theta + W_4(r,z,t)\cos 2\theta \qquad (4.2.19)$$

where

$$W_i(r,z,t) = \frac{R_F - r}{R_F - R_B}V_{Bi}(z,t) + \frac{r - R_B}{R_F - R_B}V_{Fi}(z) +$$
$$+\frac{z - z_T}{z_B - z_T}\Big[\tilde{V}_{Bi}(r,t) - \frac{R_F - r}{R_F - R_B}V_{Bi}(z_B,t) - \frac{r - R_B}{R_F - R_B}V_{Fi}(z_B)\Big]$$
$$+\frac{z_B - z}{z_B - z_T}\Big[\tilde{V}_{Ti}(r,t) - \frac{R_F - r}{R_F - R_B}V_{Bi}(z_T,t) - \frac{r - R_B}{R_F - R_B}V_{Fi}(z_T)\Big], \qquad i = 0,\ldots,4$$
$$(4.2.20)$$

and $\quad V_{Bi}(z,t) = 0, \qquad z_T \le z \le z_B, \qquad i = 1,\ldots,4.$

With this approximation, the system (4.2.15) has the form:

$$M\dot{V} - A\Big[\frac{1}{r}(rV_r)_r + \frac{1}{r^2}V_{\theta\theta} + V_{zz}\Big] = -M\dot{W} + A\Big(\frac{1}{r}(rW_r)_r + \frac{1}{r^2}W_{\theta\theta} + W_{zz}\Big)$$
$$-\vec{b}_0\Big(\dot{u}_r + \frac{1}{r}(\dot{u} + \dot{v}_\theta) + \dot{w}_z\Big) \qquad (4.2.21)$$

where W is given by (4.2.19) and (4.2.20),

$$W_r(r,\theta,z,t) = W_{0_r}(r,z,t) + W_{1_r}(r,z,t)\sin\theta + W_{2_r}(r,z,t)\cos\theta$$
$$+ W_{3_r}(r,z,t)\sin 2\theta + W_{4_r}(r,z,t)\cos 2\theta \qquad (4.2.22)$$

with

$$W_{i_r}(r,z,t) = \frac{1}{R_F - R_B}\big(V_{Fi}(z) - V_{Bi}(z,t)\big)$$

$$+\frac{z - z_T}{z_B - z_T}\Big[\tilde{V}_{Bi_r}(r,t) + \frac{1}{R_F - R_B}\big(V_{Bi}(z_B,t) - V_{Fi}(z_B)\big)\Big]$$

$$+\frac{z_B - z}{z_B - z_T}\Big[\tilde{V}_{Ti_r}(r,t) + \frac{1}{R_F - R_B}\big(V_{Bi}(z_T,t) - V_{Fi}(z_T)\big)\Big], \qquad i = 0,\dots,4 \quad (4.2.23)$$

and $\quad V_{Bi}(z,t) = 0, \qquad z_T \le z \le z_B, \qquad i = 1,\dots,4;$

$$W_{\theta\theta}(r,\theta,z,t) = -W_1(r,z,t)\sin\theta - W_2(r,z,t)\cos\theta$$

$$-4W_3(r,z,t)\sin 2\theta - 4W_4(r,z,t)\cos 2\theta \qquad (4.2.24)$$

$$W_z(r,\theta,z,t) = W_{0_z}(r,z,t) + W_{1_z}(r,z,t)\sin\theta + W_{2_z}(r,z,t)\cos\theta$$

$$+W_{3_z}(r,z,t)\sin 2\theta + W_{4_z}(r,z,t)\cos 2\theta \qquad (4.2.25)$$

with

$$W_{i_z}(r,z,t) = \frac{R_F - r}{R_F - R_B}V_{Bi_z}(z,t) + \frac{r - R_B}{R_F - R_B}V_{Fi_z}(z)$$

$$+\frac{1}{z_B - z_T}\Big[\tilde{V}_{Bi}(r,t) - \tilde{V}_{Ti}(r,t) - \frac{R_F - r}{R_F - R_B}\big(V_{Bi}(z_B,t) - V_{Bi}(z_T,t)\big)$$

$$-\frac{r - R_B}{R_F - R_B}\Big(V_{Fi}(z_B) - V_{Fi}(z_T)\Big)\Big], \qquad i = 0,\dots,4 \qquad (4.2.26)$$

and $\quad V_{Bi_z}(z,t) = 0, \qquad z_T \le z \le z_B, \qquad i = 1,\dots,4.$

The θ-approximation for the initial conditions $V(r,\theta,z,0)$ (4.2.17) is

$$V(r,\theta,z,0) = V_0(r,z,0) + V_1(r,z,0)\sin\theta + V_2(r,z,0)\cos\theta$$

$$+V_3(r,z,0)\sin 2\theta + V_4(r,z,0)\cos 2\theta$$

where

$$V_i(r,z,0) = \tilde{V}_{Ii}(r,z) - W_i(r,z,0) =$$

$$= \tilde{V}_{Ii}(r,z) - \frac{R_F - r}{R_F - R_B} V_{Bi}(z,0) - \frac{r - R_B}{R_F - R_B} V_{Fi}(z)$$

$$- \frac{z - z_T}{z_B - z_T}\Big(\tilde{V}_{Bi}(r,0) - \frac{R_F - r}{R_F - R_B} V_{Bi}(z_B,0) - \frac{r - R_B}{R_F - R_B} V_{Fi}(z_B)\Big) -$$

$$- \frac{z_B - z}{z_B - z_T}\Big(\tilde{V}_{Ti}(r,0) - \frac{R_F - r}{R_F - R_B} V_{Bi}(z_T,0) - \frac{r - R_B}{R_F - R_B} V_{Fi}(z_T)\Big), \qquad i = 0, \dots, 4 \tag{4.2.27}$$

and $\quad V_{Bi}(z,0) = 0, \qquad z_T \leq z \leq z_B, \qquad i = 1, \dots, 4.$

Applying (4.2.22)-(4.2.27) to (4.2.21), multiplying by $1, \sin\theta, \cos\theta, \sin 2\theta, \text{and} \cos 2\theta$, and integrating over $(0, 2\pi)$, we obtain five systems.

System I:

$$M\dot{V}_0 - A\Big[\frac{1}{r}(rV_{0_r})_r + (V_{0_z})_z\Big] = -M\dot{W}_0 + A\Big(\frac{1}{r}(rW_{0_r})_r + (W_{0_z})_z\Big)$$

$$-\vec{b}_0\Big(\dot{u}_{0_r} + \frac{1}{r}\dot{u}_0 + \dot{w}_{0_z}\Big) \tag{4.2.28}$$

with boundary conditions

$$V_0(r,z,t) = 0 \qquad \text{on } \Gamma \times [0, t_f) \tag{4.2.29}$$

and initial conditions

$$V_0(r,z,0) = \tilde{V}_{I0}(r,z) - W_0(r,z,0) \tag{4.2.30}$$

System II:

$$M\dot{V}_1 - A\Big[\frac{1}{r}(rV_{1_r})_r - \frac{1}{r^2}V_1 + (V_{1_z})_z\Big] = -M\dot{W}_1 + A\Big(\frac{1}{r}(rW_{1_r})_r - \frac{1}{r^2}W_1 + (W_{1_z})_z\Big)$$

$$-\vec{b}_0\Big(\dot{u}_{1_r} + \frac{1}{r}(\dot{u}_1 - \dot{v}_2) + \dot{w}_{1_z}\Big) \tag{4.2.31}$$

with boundary conditions

$$V_1(r,z,t) = 0 \qquad \text{on } \Gamma \times [0, t_f) \tag{4.2.32}$$

and initial conditions

$$V_1(r,z,0) = \tilde{V}_{I1}(r,z) - W_1(r,z,0) \tag{4.2.33}$$

System III:

$$M\dot{V}_2 - A\Big[\frac{1}{r}(rV_{2_r})_r - \frac{1}{r^2}V_2 + (V_{2_z})_z\Big] = -M\dot{W}_2 + A\Big(\frac{1}{r}(rW_{2_r})_r - \frac{1}{r^2}W_2 + (W_{2_z})_z\Big)$$
$$-\vec{b}_0\Big(\dot{u}_{2_r} + \frac{1}{r}(\dot{u}_2 + \dot{v}_1) + \dot{w}_{2_z}\Big) \tag{4.2.34}$$

with boundary conditions

$$V_2(r,z,t) = 0 \qquad \text{on } \Gamma \times [0,t_f) \tag{4.2.35}$$

and initial conditions

$$V_2(r,z,0) = \tilde{V}_{I2}(r,z) - W_2(r,z,0) \tag{4.2.36}$$

System IV:

$$M\dot{V}_3 - A\Big[\frac{1}{r}(rV_{3_r})_r - \frac{4}{r^2}V_3 + (V_{3_z})_z\Big] = -M\dot{W}_3 + A\Big(\frac{1}{r}(rW_{3_r})_r - \frac{4}{r^2}W_3 + (W_{3_z})_z\Big)$$
$$-\vec{b}_0\Big(\dot{u}_{3_r} + \frac{1}{r}(\dot{u}_3 - 2\dot{v}_4) + \dot{w}_{3_z}\Big) \tag{4.2.37}$$

with boundary conditions

$$V_3(r,z,t) = 0 \qquad \text{on } \Gamma \times [0,t_f) \tag{4.2.38}$$

and initial conditions

$$V_3(r,z,0) = \tilde{V}_{I3}(r,z) - W_3(r,z,0) \tag{4.2.39}$$

System V:

$$M\dot{V}_4 - A\Big[\frac{1}{r}(rV_{4_r})_r - \frac{4}{r^2}V_4 + (V_{4_z})_z\Big] = -M\dot{W}_4 + A\Big(\frac{1}{r}(rW_{4_r})_r - \frac{4}{r^2}W_4 + (W_{4_z})_z\Big)$$
$$-\vec{b}_0\Big(\dot{u}_{4_r} + \frac{1}{r}(\dot{u}_4 + 2\dot{v}_3) + \dot{w}_{4_z}\Big) \tag{4.2.40}$$

with boundary conditions

$$V_4(r,z,t) = 0 \qquad \text{on } \Gamma \times [0, t_f) \tag{4.2.41}$$

and initial conditions

$$V_4(r,z,0) = \tilde{V}_{I4}(r,z) - W_4(r,z,0) \tag{4.2.42}$$

4.2.2 Fourier-type approximation of the elastic system

In the borehole cylindrical coordinate system (r, θ, z), the displacement is

$$\vec{u}(r,\theta,z,t) = u(r,\theta,z,t)\vec{i}_r + v(r,\theta,z,t)\vec{i}_\theta + w(r,\theta,z,t)\vec{i}_z$$

$$\vec{u} \approx 0 \quad \text{and} \quad \dot{\vec{u}} \approx 0 \qquad \text{on } \Gamma_F$$

and the strain tensor has the form

$$\begin{bmatrix} \varepsilon_{rr} & \varepsilon_{r\theta} & \varepsilon_{rz} \\ \varepsilon_{\theta r} & \varepsilon_{\theta\theta} & \varepsilon_{\theta z} \\ \varepsilon_{zr} & \varepsilon_{z\theta} & \varepsilon_{zz} \end{bmatrix}$$

where

$$\varepsilon_{rr} = u_r, \qquad \varepsilon_{r\theta} = \varepsilon_{\theta r} = \frac{1}{2}\big(v_r + \frac{u_\theta - v}{r}\big), \qquad \varepsilon_{\theta\theta} = \frac{u + v_\theta}{r}$$

$$\varepsilon_{rz} = \varepsilon_{zr} = \frac{1}{2}\big(u_z + w_r\big), \qquad \varepsilon_{\theta z} = \varepsilon_{z\theta} = \frac{1}{2}\big(\frac{w_\theta}{r} + v_z\big), \qquad \varepsilon_{zz} = w_z \tag{4.2.43}$$

The stress-strain relations for isotropic homogeneous medium in terms of the shear modulus G and the bulk modulus K are

$$\tau = 2G\varepsilon + \big(K - \frac{2}{3}G\big)(\mathrm{tr}\varepsilon)I \tag{2.2.23}$$

where

$$\tau_{rr} = 2Gu_r + \big(K - \frac{2}{3}G\big)\big(u_r + \frac{u + v_\theta}{r} + w_z\big)$$

$$\tau_{\theta\theta} = 2G\frac{u+v_\theta}{r} + \big(K - \frac{2}{3}G\big)\big(u_r + \frac{u+v_\theta}{r} + w_z\big)$$

$$\tau_{zz} = 2Gw_z + \big(K - \frac{2}{3}G\big)\big(u_r + \frac{u+v_\theta}{r} + w_z\big)$$

$$\tau_{r\theta} = \tau_{\theta r} = G\big(v_r + \frac{u_\theta - v}{r}\big)$$

$$\tau_{rz} = \tau_{zr} = G\big(u_z + w_r\big)$$

$$\tau_{\theta z} = \tau_{z\theta} = G\big(\frac{w_\theta}{r} + v_z\big) \tag{4.2.44}$$

The sum of total potential energy and penalty term is

$$\begin{aligned}\Pi(\vec{u}) =& \frac{1}{2}\int_\Omega \Big(\tau_{rr}(\vec{u})\varepsilon_{rr}(\vec{u}) + 2\tau_{r\theta}(\vec{u})\varepsilon_{r\theta}(\vec{u}) + 2\tau_{rz}(\vec{u})\varepsilon_{rz}(\vec{u}) \\ &+ \tau_{\theta\theta}(\vec{u})\varepsilon_{\theta\theta}(\vec{u}) + 2\tau_{\theta z}(\vec{u})\varepsilon_{\theta z}(\vec{u}) + 2\tau_{zz}(\vec{u})\varepsilon_{zz}(\vec{u})\Big)\,\mathrm{d}\Omega \\ &- \int_\Omega f\big(\mathrm{tr}\varepsilon(\vec{u})\big)\,\mathrm{d}\Omega - \int_\Gamma (\hat{\sigma}\vec{n})\cdot\vec{u}\,\mathrm{d}\Gamma + K_F\int_{\Gamma_F} \vec{u}\cdot\vec{u}\,\mathrm{d}\Gamma\end{aligned} \tag{4.2.45}$$

where $f = \vec{b}_1^T\bar{V} = \vec{b}_1^T(V+W)$ is body force, $\vec{b}_1^T = [\tilde{\zeta} \quad -\lambda \quad \tilde{\alpha}]$,

$\hat{\sigma} = \begin{bmatrix} \sigma_r & \sigma_{r\theta} & \sigma_{rz} \\ \sigma_{r\theta} & \sigma_\theta & \sigma_{\theta z} \\ \sigma_{rz} & \sigma_{\theta z} & \sigma_z \end{bmatrix}$ is applied boundary stress in the system (r, θ, Z_b), and

K_F is a penalization constant.

Next, we describe stress around the borehole. Let

$$\sigma_{XYZ}(z) = \begin{bmatrix} \sigma_H(z) & 0 & 0 \\ 0 & \sigma_h(z) & 0 \\ 0 & 0 & \sigma_V \end{bmatrix}, \qquad \sigma_{XYZ}^b(z) = \begin{bmatrix} \sigma_x & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_y & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix}(z)$$

$$\sigma(r,\theta,z) = \begin{bmatrix} \sigma_r & \sigma_{r\theta} & \sigma_{rz} \\ \sigma_{r\theta} & \sigma_\theta & \sigma_{\theta z} \\ \sigma_{rz} & \sigma_{\theta z} & \sigma_z \end{bmatrix}(r,\theta,z)$$

be the stress tensors in the (X,Y,Z), (X_b,Y_b,Z_b), and (r,θ,z) coordinate systems, respectively.

The transformation from $\sigma_{XYZ}(z)$ to $\sigma^b_{XYZ}(z)$ is given by [22]:

$$\sigma^b_{XYZ}(z) = L_\sigma \cdot \sigma_{XYZ}(z) \cdot L_\sigma^T$$

where

$$L_\sigma = \begin{bmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{bmatrix} = \begin{bmatrix} \cos\varphi_x \cos\varphi_z & \sin\varphi_x \cos\varphi_z & -\sin\varphi_z \\ -\sin\varphi_x & \cos\varphi_x & 0 \\ \cos\varphi_x \sin\varphi_z & \sin\varphi_x \sin\varphi_z & \cos\varphi_z \end{bmatrix}$$

is the transformation matrix, φ_x is the horizontal angle between the x-axis and the borehole axis measures counterclockwise, and φ_z is the vertical angle between the z-axis and the borehole axis measures clockwise.

Equivalently,

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{xz} \end{bmatrix} (z) = T_\sigma \begin{bmatrix} \sigma_H(z) \\ \sigma_h(z) \\ \sigma_V \end{bmatrix} (z), \qquad \text{where } T_\sigma = \begin{bmatrix} l_{11}^2 & l_{12}^2 & l_{13}^2 \\ l_{21}^2 & l_{22}^2 & l_{23}^2 \\ l_{31}^2 & l_{32}^2 & l_{33}^2 \\ l_{11}l_{12} & l_{12}l_{22} & l_{13}l_{23} \\ l_{21}l_{31} & l_{22}l_{32} & l_{23}l_{33} \\ l_{11}l_{31} & l_{12}l_{32} & l_{13}l_{33} \end{bmatrix}$$

The transformation from $\sigma^b_{XYZ}(z)$ to $\sigma(r, \theta, z)$ is given by [9]:

$$\sigma_r = \frac{\sigma_x + \sigma_y}{2}\Big(1 - \frac{R_B^2}{r^2}\Big) + p_B\frac{R_B^2}{r^2} + \sigma_{xy}\Big(1 - \frac{4R_B^2}{r^2} + \frac{3R_B^4}{r^4}\Big)\sin 2\theta$$
$$+ \frac{\sigma_x - \sigma_y}{2}\Big(1 - \frac{4R_B^2}{r^2} + \frac{3R_B^4}{r^4}\Big)\cos 2\theta$$

$$\sigma_\theta = \frac{\sigma_x + \sigma_y}{2}\Big(1 + \frac{R_B^2}{r^2}\Big) - p_B\frac{R_B^2}{r^2} - \sigma_{xy}\Big(1 + \frac{3R_B^4}{r^4}\Big)\sin 2\theta$$
$$- \frac{\sigma_x - \sigma_y}{2}\Big(1 + \frac{3R_B^4}{r^4}\Big)\cos 2\theta$$

$$\sigma_z = \sigma_{zz} - 4\nu\sigma_{xy}\frac{R_B^2}{r^2}\sin 2\theta - 2\nu(\sigma_x - \sigma_y)\frac{R_B^2}{r^2}\cos 2\theta$$

$$\sigma_{r\theta} = \frac{\sigma_x - \sigma_y}{2}\Big(1 + \frac{2R_B^2}{r^2} - \frac{3R_B^4}{r^4}\Big)\sin 2\theta + \sigma_{xy}\Big(1 + \frac{2R_B^2}{r^2} - \frac{3R_B^4}{r^4}\Big)\cos 2\theta$$

$$\sigma_{\theta z} = (-\sigma_{xz}\sin\theta + \sigma_{yz}\cos\theta)\Big(1 + \frac{R_B^2}{r^2}\Big)$$

$$\sigma_{rz} = (\sigma_{yz}\sin\theta + \sigma_{xz}\cos\theta)\Big(1 - \frac{R_B^2}{r^2}\Big)$$

where p_B is well bore pressure and ν is Poisson's ratio of the formation.

From these equations we obtain applied stress on the boundaries.

On the borehole boundary Γ_B:

$$\sigma_r(R_B, z, t) = p_B(z, t)$$

$$\sigma_\theta(R_B, \theta, z, t) = \sigma_x(z) + \sigma_y(z) - p_B(z, t) - 4\sigma_{xy}(z)\sin 2\theta - 2\big(\sigma_x(z) - \sigma_y(z)\big)\cos 2\theta$$

$$\sigma_z(R_B, \theta, z) = \sigma_{zz}(z) - 4\nu\sigma_{xy}(z)\sin 2\theta - 2\nu\big(\sigma_x(z) - \sigma_y(z)\big)\cos 2\theta$$

$$\sigma_{r\theta}(R_B, \theta, z) = 0$$

$$\sigma_{rz}(R_B, \theta, z) = 0$$

$$\sigma_{\theta z}(R_B, \theta, z) = 2\big(-\sigma_{xz}(z)\sin\theta + \sigma_{yz}(z)\cos\theta\big)$$

On the far-field boundary Γ_F:

$$\sigma_r(R_F, \theta, z) = \frac{\sigma_x(z) + \sigma_y(z)}{2} + \sigma_{xy}(z)\sin 2\theta + \frac{\sigma_x(z) - \sigma_y(z)}{2}\cos 2\theta$$

$$\sigma_\theta(R_F, \theta, z) = \frac{\sigma_x(z) + \sigma_y(z)}{2} - \sigma_{xy}(z)\sin 2\theta - \frac{\sigma_x(z) - \sigma_y(z)}{2}\cos 2\theta$$

$$\sigma_z(R_F, \theta, z) = \sigma_{zz}(z)$$

$$\sigma_{r\theta}(R_F, \theta, z) = \frac{\sigma_x(z) - \sigma_y(z)}{2}\sin 2\theta + \sigma_{xy}(z)\cos 2\theta$$

$$\sigma_{rz}(R_F, \theta, z) = \sigma_{yz}(z)\sin\theta + \sigma_{xz}(z)\cos\theta$$

$$\sigma_{\theta z}(R_F, \theta, z) = -\sigma_{xz}(z)\sin\theta + \sigma_{yz}(z)\cos\theta$$

On the top boundary $\tilde{\Gamma}_T$ and the bottom boundary $\tilde{\Gamma}_B$:

$$\sigma_r(r, \theta, z_i, t) = \frac{\sigma_x(z_i) + \sigma_y(z_i)}{2}\Big(1 - \frac{R_B^2}{r^2}\Big) + p_B(z_i, t)\frac{R_B^2}{r^2} + \Big(1 - \frac{4R_B^2}{r^2} + \frac{3R_B^4}{r^4}\Big)\Big(\sigma_{xy}(z_i)\sin 2\theta + \frac{\sigma_x(z_i) - \sigma_y(z_i)}{2}\cos 2\theta\Big)$$

$$\sigma_\theta(r,\theta,z_i,t) = \frac{\sigma_x(z_i)+\sigma_y(z_i)}{2}\Big(1+\frac{R_B^2}{r^2}\Big) - p_B(z_i,t)\frac{R_B^2}{r^2}$$
$$- \Big(1+\frac{3R_B^4}{r^4}\Big)\Big(\sigma_{xy}(z_i)\sin 2\theta + \frac{\sigma_x(z_i)-\sigma_y(z_i)}{2}\cos 2\theta\Big)$$

$$\sigma_z(r,\theta,z_i) = \sigma_{zz}(z_i) - 4\nu\frac{R_B^2}{r^2}\Big(\sigma_{xy}(z_i)\sin 2\theta + \frac{\sigma_x(z_i)-\sigma_y(z_i)}{2}\cos 2\theta\Big)$$

$$\sigma_{r\theta}(r,\theta,z_i) = \Big(1+\frac{2R_B^2}{r^2}-\frac{3R_B^4}{r^4}\Big)\Big(\frac{\sigma_x(z)-\sigma_y(z)}{2}\sin 2\theta + \sigma_{xy}(z)\cos 2\theta\Big)$$

$$\sigma_{rz}(r,\theta,z_i) = \Big(1-\frac{R_B^2}{r^2}\Big)\big(\sigma_{yz}(z_i)\sin\theta + \sigma_{xz}(z_i)\cos\theta\big)$$

$$\sigma_{\theta z}(r,\theta,z_i) = \Big(1+\frac{R_B^2}{r^2}\Big)\big(-\sigma_{xz}(z_i)\sin\theta + \sigma_{yz}(z_i)\cos\theta\big) \qquad (4.2.46)$$

where $\quad i = T$ (top) or $i = B$ (bottom).

Let $V = H^1(\Omega;\mathbb{R}^3)$. Our objective is to find the displacement $\vec{u}$ such that

$$\vec{u} \in L^2(0,t_f;V), \qquad \dot{\vec{u}} \in L^2(0,t_f;V) \qquad \text{and}$$

$$D\Pi(\vec{u})\vec{\Phi} = 0, \qquad \forall\ \vec{\Phi} \in V \qquad (4.2.47)$$

where $\vec{u}(r,\theta,z,t) = u(r,\theta,z,t)\vec{i}_r + v(r,\theta,z,t)\vec{i}_\theta + w(r,\theta,z,t)\vec{i}_z,$

$\vec{\Phi}(r,\theta,z) = \varphi(r,\theta,z)\vec{i}_r + \psi(r,\theta,z)\vec{i}_\theta + \omega(r,\theta,z)\vec{i}_z,$

$D\Pi(\vec{u})\vec{\Phi}$ is the Gâteaux differential of Π with increment $\vec{\Phi}$.

Then (4.2.47) has the form

$$\int_{z_T}^{z_B}\int_0^{2\pi}\int_{R_B}^{R_F}\Big[\Big(2Gu_r + \big(K-\frac{2G}{3}\big)\big(u_r+\frac{u+v_\theta}{r}+w_z\big)\Big)\varphi_r$$
$$+ G\big(v_r+\frac{u_\theta-v}{r}\big)\big(\psi_r+\frac{\varphi_\theta-\psi}{r}\big) + G(u_z+w_r)(\varphi_z+\omega_r)$$
$$+ \Big(2G\frac{u+v_\theta}{r} + \big(K-\frac{2G}{3}\big)\big(u_r+\frac{u+v_\theta}{r}+w_z\big)\Big)\frac{\varphi+\psi_\theta}{r}$$
$$+ G\big(\frac{w_\theta}{r}+v_z\big)\big(\frac{\omega_\theta}{r}+\psi_z\big) + \Big(2Gw_z + \big(K-\frac{2G}{3}\big)\big(u_r+\frac{u+v_\theta}{r}+w_z\big)\Big)\omega_z\Big] r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$$

$$
\begin{aligned}
&- \int_{z_T}^{z_B} \int_0^{2\pi} \int_{R_B}^{R_F} f\big(\varphi_r + \frac{\varphi + \psi_\theta}{r} + \omega_z\big) r \,\mathrm{d}r \,\mathrm{d}\theta \,\mathrm{d}z \\
&+ \int_{z_T}^{z_B} \int_0^{2\pi} \Big(\sigma_r(R_B,\theta,z)\varphi(R_B,\theta,z) + \sigma_{r\theta}(R_B,\theta,z)\psi(R_B,\theta,z) \\
&\qquad + \sigma_{rz}(R_B,\theta,z)\omega(R_B,\theta,z)\Big) R_B \,\mathrm{d}\theta \,\mathrm{d}z \\
&- \int_{z_T}^{z_B} \int_0^{2\pi} \Big(\sigma_r(R_F,\theta,z)\varphi(R_F,\theta,z) + \sigma_{r\theta}(R_F,\theta,z)\psi(R_F,\theta,z) \\
&\qquad + \sigma_{rz}(R_F,\theta,z)\omega(R_F,\theta,z)\Big) R_F \,\mathrm{d}\theta \,\mathrm{d}z \\
&+ \int_0^{2\pi} \int_{R_B}^{R_F} \Big(\sigma_{rz}(r,\theta,z_T)\varphi(r,\theta,z_T) + \sigma_{\theta z}(r,\theta,z_T)\psi(r,\theta,z_T) \\
&\qquad + \sigma_z(r,\theta,z_T)\omega(r,\theta,z_T)\Big) r \,\mathrm{d}r \,\mathrm{d}\theta \\
&- \int_0^{2\pi} \int_{R_B}^{R_F} \Big(\sigma_{rz}(r,\theta,z_B)\varphi(r,\theta,z_B) + \sigma_{\theta z}(r,\theta,z_B)\psi(r,\theta,z_B) \\
&\qquad + \sigma_z(r,\theta,z_B)\omega(r,\theta,z_B)\Big) r \,\mathrm{d}r \,\mathrm{d}\theta \\
&+ K_F \int_{z_T}^{z_B} \int_0^{2\pi} \Big(u(R_F,\theta,z)\varphi(R_F,\theta,z) + v(R_F,\theta,z)\psi(R_F,\theta,z) \\
&\qquad + w(R_F,\theta,z)\omega(R_F,\theta,z)\Big) R_F \,\mathrm{d}\theta \,\mathrm{d}z = 0 \qquad (4.2.48)
\end{aligned}
$$

The first term of the left-hand side of (4.2.48) is

$$
\begin{aligned}
&\int_{z_T}^{z_B} \int_0^{2\pi} \int_{R_B}^{R_F} \Big[\big(K+\frac{4G}{3}\big)u_r\varphi_r r+\big(K-\frac{2G}{3}\big)u\varphi_r+\big(K-\frac{2G}{3}\big)v_\theta\varphi_r+\big(K-\frac{2G}{3}\big)w_z\varphi_r r \\
&\quad + Gv_r\psi_r r + Gv_r\varphi_\theta - Gv_r\psi + Gu_\theta\psi_r - Gv\psi_r + Gu_\theta\varphi_\theta\frac{1}{r} - Gu_\theta\psi\frac{1}{r} \\
&\quad - Gv\varphi_\theta\frac{1}{r} + Gv\psi\frac{1}{r} + Gu_z\varphi_z r + Gu_z\omega_r r + Gw_r\varphi_z r + Gw_r\omega_r r \\
&\quad + \big(K + \frac{4G}{3}\big)u\varphi\frac{1}{r} + \big(K + \frac{4G}{3}\big)u\psi_\theta\frac{1}{r} + \big(K + \frac{4G}{3}\big)v_\theta\varphi\frac{1}{r} + \big(K + \frac{4G}{3}\big)v_\theta\psi_\theta\frac{1}{r} \\
&\quad + \big(K - \frac{2G}{3}\big)u_r\varphi + \big(K - \frac{2G}{3}\big)u_r\psi_\theta + \big(K - \frac{2G}{3}\big)w_z\varphi + \big(K - \frac{2G}{3}\big)w_z\psi_\theta \\
&\quad + Gw_\theta\omega_\theta\frac{1}{r} + Gw_\theta\psi_z + Gv_z\omega_\theta + Gv_z\psi_z r + \big(K + \frac{4G}{3}\big)w_z\omega_z r \\
&\quad + \big(K - \frac{2G}{3}\big)u_r\omega_z r + \big(K - \frac{2G}{3}\big)u\omega_z + \big(K - \frac{2G}{3}\big)v_\theta\omega_z\Big] \,\mathrm{d}r \,\mathrm{d}\theta \,\mathrm{d}z \qquad (4.2.49)
\end{aligned}
$$

Applying θ-approximation (4.2.18) to (4.2.49), integrating with respect to θ over $(0, 2\pi)$, and combining like terms with respect to the basis functions, we have

φ_0 :

$$\pi \int_{z_T}^{z_B} \int_{R_B}^{R_F} \Big[2\big(K + \frac{4}{3}G\big) u_{0_r} \varphi_{0_r} r + 2\big(K - \frac{2}{3}G\big) u_0 \varphi_{0_r} + 2\big(K - \frac{2}{3}G\big) w_{0_z} \varphi_{0_r} r + 2G u_{0_z} \varphi_{0_z} r + 2G w_{0_r} \varphi_{0_z} r + 2\big(K + \frac{4}{3}G\big) u_0 \varphi_0 \frac{1}{r} + 2\big(K - \frac{2}{3}G\big) u_{0_r} \varphi_0 + 2\big(K - \frac{2}{3}G\big) w_{0_z} \varphi_0 \Big] \,\mathrm{d}r \,\mathrm{d}z \quad (4.2.50)$$

φ_1 :

$$+ \pi \int_{z_T}^{z_B} \int_{R_B}^{R_F} \Big[\big(K + \frac{4}{3}G\big) u_{1_r} \varphi_{1_r} r + \big(K - \frac{2}{3}G\big) u_1 \varphi_{1_r} - \big(K - \frac{2}{3}G\big) v_2 \varphi_{1_r} + \big(K - \frac{2}{3}G\big) w_{1_z} \varphi_{1_r} r + G v_{2_r} \varphi_1 + \big(K + \frac{7}{3}G\big) u_1 \varphi_1 \frac{1}{r} + G u_{1_z} \varphi_{1_z} r + G w_{1_r} \varphi_{1_z} r - \big(K + \frac{7}{3}G\big) v_2 \varphi_1 \frac{1}{r} + \big(K - \frac{2}{3}G\big) u_{1_r} \varphi_1 + \big(K - \frac{2}{3}G\big) w_{1_z} \varphi_1 \Big] \,\mathrm{d}r \,\mathrm{d}z \quad (4.2.51)$$

φ_2 :

$$+ \pi \int_{z_T}^{z_B} \int_{R_B}^{R_F} \Big[\big(K + \frac{4}{3}G\big) u_{2_r} \varphi_{2_r} r + \big(K - \frac{2}{3}G\big) u_2 \varphi_{2_r} + \big(K - \frac{2}{3}G\big) v_1 \varphi_{2_r} + \big(K - \frac{2}{3}G\big) w_{2_z} \varphi_{2_r} r - G v_{1_r} \varphi_2 + \big(K + \frac{7}{3}G\big) u_2 \varphi_2 \frac{1}{r} + G u_{2_z} \varphi_{2_z} r + G w_{2_r} \varphi_{2_z} r + \big(K + \frac{7}{3}G\big) v_1 \varphi_2 \frac{1}{r} + \big(K - \frac{2}{3}G\big) u_{2_r} \varphi_2 + \big(K - \frac{2}{3}G\big) w_{2_z} \varphi_2 \Big] \,\mathrm{d}r \,\mathrm{d}z \quad (4.2.52)$$

φ_3 :

$$+ \pi \int_{z_T}^{z_B} \int_{R_B}^{R_F} \Big[\big(K + \frac{4}{3}G\big) u_{3_r} \varphi_{3_r} r + \big(K - \frac{2}{3}G\big) u_3 \varphi_{3_r} - 2\big(K - \frac{2}{3}G\big) v_4 \varphi_{3_r} + \big(K - \frac{2}{3}G\big) w_{3_z} \varphi_{3_r} r + 2G v_{4_r} \varphi_3 + \big(K + \frac{16}{3}G\big) u_3 \varphi_3 \frac{1}{r} + G u_{3_z} \varphi_{3_z} r + G w_{3_r} \varphi_{3_z} r - 2\big(K + \frac{7}{3}G\big) v_4 \varphi_3 \frac{1}{r} + \big(K - \frac{2}{3}G\big) u_{3_r} \varphi_3 + \big(K - \frac{2}{3}G\big) w_{3_z} \varphi_3 \Big] \,\mathrm{d}r \,\mathrm{d}z \quad (4.2.53)$$

$\varphi_4:$

$$+\pi\int_{z_T}^{z_B}\int_{R_B}^{R_F}\Big[\big(K+\frac{4}{3}G\big)u_{4_r}\varphi_{4_r}r+\big(K-\frac{2}{3}G\big)u_4\varphi_{4_r}+2\big(K-\frac{2}{3}G\big)v_3\varphi_{4_r}$$
$$+\big(K-\frac{2}{3}G\big)w_{4_z}\varphi_{4_r}r-2Gv_{3_r}\varphi_4+\big(K+\frac{16}{3}G\big)u_4\varphi_4\frac{1}{r}+Gu_{4_z}\varphi_{4_z}r+Gw_{4_r}\varphi_{4_z}r$$
$$+2\big(K+\frac{7}{3}G\big)v_3\varphi_4\frac{1}{r}+\big(K-\frac{2}{3}G\big)u_{4_r}\varphi_4+\big(K-\frac{2}{3}G\big)w_{4_z}\varphi_4\Big]\,\mathrm{d}r\,\mathrm{d}z \quad (4.2.54)$$

$\psi_0:$

$$+\pi\int_{z_T}^{z_B}\int_{R_B}^{R_F}\Big[2G\Big(\big(v_{0_r}-\frac{v_0}{r}\big)\big(\psi_{0_r}-\frac{\psi_0}{r}\big)+v_{0_z}\psi_{0_z}\Big)r\Big]\,\mathrm{d}r\,\mathrm{d}z \quad (4.2.55)$$

$\psi_1:$

$$+\pi\int_{z_T}^{z_B}\int_{R_B}^{R_F}\Big[Gv_{1_r}\psi_{1_r}r-Gv_{1_r}\psi_1-Gu_2\psi_{1_r}-Gv_1\psi_{1_r}+\big(K+\frac{7}{3}G\big)u_2\psi_1\frac{1}{r}$$
$$+\big(K+\frac{7}{3}G\big)v_1\psi_1\frac{1}{r}+\big(K-\frac{2}{3}G\big)u_{2_r}\psi_1+\big(K-\frac{2}{3}G\big)w_{2_z}\psi_1$$
$$-Gw_2\psi_{1_z}+Gv_{1_z}\psi_{1_z}r\Big]\,\mathrm{d}r\,\mathrm{d}z \quad (4.2.56)$$

$\psi_2:$

$$+\pi\int_{z_T}^{z_B}\int_{R_B}^{R_F}\Big[Gv_{2_r}\psi_{2_r}r-Gv_{2_r}\psi_2+Gu_1\psi_{2_r}-Gv_2\psi_{2_r}-\big(K+\frac{7}{3}G\big)u_1\psi_2\frac{1}{r}$$
$$+\big(K+\frac{7}{3}G\big)v_2\psi_2\frac{1}{r}-\big(K-\frac{2}{3}G\big)u_{1_r}\psi_2-\big(K-\frac{2}{3}G\big)w_{1_z}\psi_2$$
$$+Gw_1\psi_{2_z}+Gv_{2_z}\psi_{2_z}r\Big]\,\mathrm{d}r\,\mathrm{d}z \quad (4.2.57)$$

$\psi_3:$

$$+\pi\int_{z_T}^{z_B}\int_{R_B}^{R_F}\Big[Gv_{3_r}\psi_{3_r}r-Gv_{3_r}\psi_3-2Gu_4\psi_{3_r}-Gv_3\psi_{3_r}+2\big(K+\frac{7}{3}G\big)u_4\psi_3\frac{1}{r}$$
$$+\big(4K+\frac{19}{3}G\big)v_3\psi_3\frac{1}{r}+2\big(K-\frac{2}{3}G\big)u_{4_r}\psi_3+2\big(K-\frac{2}{3}G\big)w_{4_z}\psi_3$$
$$-2Gw_4\psi_{3_z}+Gv_{3_z}\psi_{3_z}r\Big]\,\mathrm{d}r\,\mathrm{d}z \quad (4.2.58)$$

ψ_4 :

$$+\,\pi\int_{z_T}^{z_B}\int_{R_B}^{R_F}\Big[Gv_{4_r}\psi_{4_r}r - Gv_{4_r}\psi_4 + 2Gu_3\psi_{4_r} - Gv_4\psi_{4_r} - 2\big(K+\frac{7}{3}G\big)u_3\psi_4\frac{1}{r} + \big(4K+\frac{19}{3}G\big)v_4\psi_4\frac{1}{r} - 2\big(K-\frac{2}{3}G\big)u_{3_r}\psi_4 - 2\big(K-\frac{2}{3}G\big)w_{3_z}\psi_4 + 2Gw_3\psi_{4_z} + Gv_{4_z}\psi_{4_z}r\Big]\,\mathrm{d}r\,\mathrm{d}z \quad (4.2.59)$$

ω_0 :

$$+\,\pi\int_{z_T}^{z_B}\int_{R_B}^{R_F}\Big[2Gu_{0_z}\omega_{0_r}r + 2Gw_{0_r}\omega_{0_r}r + 2\big(K+\frac{4}{3}G\big)w_{0_z}\omega_{0_z}r + 2\big(K-\frac{2}{3}G\big)u_{0_r}\omega_{0_z}r + 2\big(K-\frac{2}{3}G\big)u_0\omega_{0_z}\Big]\,\mathrm{d}r\,\mathrm{d}z \quad (4.2.60)$$

ω_1 :

$$+\,\pi\int_{z_T}^{z_B}\int_{R_B}^{R_F}\Big[Gu_{1_z}\omega_{1_r}r + Gw_{1_r}\omega_{1_r}r + Gw_1\omega_1\frac{1}{r} + Gv_{2_z}\omega_1 + \big(K+\frac{4}{3}G\big)w_{1_z}\omega_{1_z}r + \big(K-\frac{2}{3}G\big)u_{1_r}\omega_{1_z}r + \big(K-\frac{2}{3}G\big)u_1\omega_{1_z} - \big(K-\frac{2}{3}G\big)v_2\omega_{1_z}\Big]\,\mathrm{d}r\,\mathrm{d}z \quad (4.2.61)$$

ω_2 :

$$+\,\pi\int_{z_T}^{z_B}\int_{R_B}^{R_F}\Big[Gu_{2_z}\omega_{2_r}r + Gw_{2_r}\omega_{2_r}r + Gw_2\omega_2\frac{1}{r} - Gv_{1_z}\omega_2 + \big(K+\frac{4}{3}G\big)w_{2_z}\omega_{2_z}r + \big(K-\frac{2}{3}G\big)u_{2_r}\omega_{2_z}r + \big(K-\frac{2}{3}G\big)u_2\omega_{2_z} + \big(K-\frac{2}{3}G\big)v_1\omega_{2_z}\Big]\,\mathrm{d}r\,\mathrm{d}z \quad (4.2.62)$$

ω_3 :

$$+\,\pi\int_{z_T}^{z_B}\int_{R_B}^{R_F}\Big[Gu_{3_z}\omega_{3_r}r + Gw_{3_r}\omega_{3_r}r + 4Gw_3\omega_3\frac{1}{r} + 2Gv_{4_z}\omega_3 + \big(K+\frac{4}{3}G\big)w_{3_z}\omega_{3_z}r + \big(K-\frac{2}{3}G\big)u_{3_r}\omega_{3_z}r + \big(K-\frac{2}{3}G\big)u_3\omega_{3_z} - 2\big(K-\frac{2}{3}G\big)v_4\omega_{3_z}\Big]\,\mathrm{d}r\,\mathrm{d}z \quad (4.2.63)$$

ω_4 :

$$+\pi\int_{z_T}^{z_B}\int_{R_B}^{R_F}\Big[Gu_{4_z}\omega_{4_r}r+Gw_{4_r}\omega_{4_r}r+4Gw_4\omega_4\frac{1}{r}-2Gv_{3_z}\omega_4+\big(K+\frac{4}{3}G\big)w_{4_z}\omega_{4_z}r$$
$$+\big(K-\frac{2}{3}G\big)u_{4_r}\omega_{4_z}r+\big(K-\frac{2}{3}G\big)u_4\omega_{4_z}+2\big(K-\frac{2}{3}G\big)v_3\omega_{4_z}\Big]\,\mathrm{d}r\,\mathrm{d}z \quad (4.2.64)$$

The σ-approximation of body force f has the form

$$f(r,\theta,z,t)=f_0(r,z,t)+f_1(r,z,t)\sin\theta+f_2(r,z,t)\cos\theta$$
$$=f_3(r,z,t)\sin 2\theta+f_4(r,z,t)\cos 2\theta \quad (4.2.65)$$

The second term of the left-hand side of (4.2.48) is

$$-\pi\int_{z_T}^{z_B}\int_{R_B}^{R_F}\big[2f_0\varphi_{0_r}r+f_1\varphi_{1_r}r+f_2\varphi_{2_r}r+f_3\varphi_{3_r}r+f_4\varphi_{4_r}r$$
$$+2f_0\varphi_0+f_1\varphi_1+f_2\varphi_2+f_3\varphi_3+f_4\varphi_4-f_1\psi_2+f_2\psi_1-2f_3\psi_4+2f_4\psi_3$$
$$+2f_0\omega_{0_z}r+f_1\omega_{1_z}r+f_2\omega_{2_z}r+f_3\omega_{3_z}r+f_4\omega_{4_z}r\big]\,\mathrm{d}r\,\mathrm{d}z \quad (4.2.66)$$

The third term of the left-hand side of (4.2.48) is

$$\pi\int_{z_T}^{z_B}2p_B(z)\varphi_0(R_B,z)R_B\,\mathrm{d}z \quad (4.2.67)$$

The fourth term of the left-hand side of (4.2.48) is

$$-\pi\int_{z_T}^{z_B}\Big[2\frac{\sigma_x(z)+\sigma_y(z)}{2}\varphi_0(R_F,z)+\sigma_{xy}(z)\varphi_3(R_F,z)+\frac{\sigma_x(z)-\sigma_y(z)}{2}\varphi_4(R_F,z)$$
$$+\frac{\sigma_x(z)-\sigma_y(z)}{2}\psi_3(R_F,z)+\sigma_{xy}(z)\psi_4(R_F,z)$$
$$+\sigma_{yz}(z)\omega_1(R_F,z)+\sigma_{xz}(z)\omega_2(R_F,z)\Big]R_F\,\mathrm{d}z \quad (4.2.68)$$

The fifth term of the left-hand side of (4.2.48) is

$$\pi\int_{R_B}^{R_F}\Big[\big(1-\frac{R_B^2}{r^2}\big)r\big(\sigma_{yz}(z_T)\varphi_1(r,z_T)+\sigma_{xz}(z_T)\varphi_2(r,z_T)\big)$$
$$+\big(1+\frac{R_B^2}{r^2}\big)r\big(-\sigma_{xz}(z_T)\psi_1(r,z_T)+\sigma_{yz}(z_T)\psi_2(r,z_T)\big)+2\sigma_{zz}(z_T)\omega_0(r,z_T)r-$$

$$- 4\nu \frac{R_B^2}{r}\sigma_{xy}(z_T)\omega_3(r, z_T) - 4\nu \frac{R_B^2}{r}\frac{\sigma_x(z_T) - \sigma_y(z_T)}{2}\omega_4(r, z_T)\Big]\,\mathrm{d}r \quad (4.2.69)$$

The sixth term of the left-hand side of (4.2.48) is

$$\begin{aligned} - \pi \int_{R_B}^{R_F} \Big[&\big(1 - \frac{R_B^2}{r^2}\big) r \big(\sigma_{yz}(z_B)\varphi_1(r, z_B) + \sigma_{xz}(z_B)\varphi_2(r, z_B)\big) \\ &+ \big(1 + \frac{R_B^2}{r^2}\big) r \big(- \sigma_{xz}(z_B)\psi_1(r, z_B) + \sigma_{yz}(z_B)\psi_2(r, z_B)\big) + 2\sigma_{zz}(z_B)\omega_0(r, z_B) r \\ &- 4\nu \frac{R_B^2}{r}\sigma_{xy}(z_B)\omega_3(r, z_B) - 4\nu \frac{R_B^2}{r}\frac{\sigma_x(z_B) - \sigma_y(z_B)}{2}\omega_4(r, z_B)\Big]\,\mathrm{d}r \quad (4.2.70) \end{aligned}$$

The seventh term of the left-hand side of (4.2.48) is

$$\begin{aligned} K_F \pi \int_{z_T}^{z_B} \big[&2u_0(R_F, z)\varphi_0(R_F, z) + u_1(R_F, z)\varphi_1(R_F, z) + u_2(R_F, z)\varphi_2(R_F, z) \\ &+ u_3(R_F, z)\varphi_3(R_F, z) + u_4(R_F, z)\varphi_4(R_F, z) + 2v_0(R_F, z)\psi_0(R_F, z) \\ &+ v_1(R_F, z)\psi_1(R_F, z) + v_2(R_F, z)\psi_2(R_F, z) + v_3(R_F, z)\psi_3(R_F, z) \\ &+ v_4(R_F, z)\psi_4(R_F, z) + 2w_0(R_F, z)\omega_0(R_F, z) + w_1(R_F, z)\omega_1(R_F, z) \\ &+ w_2(R_F, z)\omega_2(R_F, z) + w_3(R_F, z)\omega_3(R_F, z) + w_4(R_F, z)\omega_4(R_F, z)\big] R_F\,\mathrm{d}z \quad (4.2.71) \end{aligned}$$

In (4.2.50)-(4.2.64) and (4.2.66)-(4.2.71), we combine terms corresponding to φ_i, ψ_i, ω_i, $i = 0, \dots, 4$, and equal them to 0.

$\varphi_0 :$ (u_0, w_0)

$$\begin{aligned} &\int_{z_T}^{z_B}\int_{R_B}^{R_F} \Big[\big(K + \frac{4G}{3}\big)u_{0_r}\varphi_{0_r} r + \big(K - \frac{2G}{3}\big)u_0\varphi_{0_r} + \big(K - \frac{2G}{3}\big)u_{0_r}\varphi_0 \\ &\quad + \big(K + \frac{4G}{3}\big)u_0\varphi_0\frac{1}{r} + G u_{0_z}\varphi_{0_z} r + \big(K - \frac{2G}{3}\big)w_{0_z}\varphi_{0_r} r + G w_{0_r}\varphi_{0_z} r \\ &\quad + \big(K - \frac{2G}{3}\big)w_{0_z}\varphi_0\Big]\,\mathrm{d}r\,\mathrm{d}z - \int_{z_T}^{z_B}\int_{R_B}^{R_F} f_0\big(\varphi_{0_r} r + \varphi_0\big)\,\mathrm{d}r\,\mathrm{d}z \\ &+ \int_{z_T}^{z_B} p_B(z)\varphi_0(R_B, z) R_B\,\mathrm{d}z - \int_{z_T}^{z_B} \frac{\sigma_x(z) + \sigma_y(z)}{2}\varphi_0(R_F, z) R_F\,\mathrm{d}z \\ &+ K_F\int_{z_T}^{z_B} u_0(R_F, z)\varphi_0(R_F, z) R_F\,\mathrm{d}z = 0 \qquad (4.2.72) \end{aligned}$$

$\varphi_1 : \quad (u_1, v_2, w_1)$

$$\int_{z_T}^{z_B}\int_{R_B}^{R_F}\Big[\big(K+\frac{4G}{3}\big)u_{1_r}\varphi_{1_r}r+\big(K-\frac{2G}{3}\big)u_1\varphi_{1_r}+\big(K-\frac{2G}{3}\big)u_{1_r}\varphi_1+Gw_{1_r}\varphi_{1_z}r$$
$$+\big(K+\frac{7G}{3}\big)u_1\varphi_1\frac{1}{r}+Gu_{1_z}\varphi_{1_z}r-\big(K-\frac{2G}{3}\big)v_2\varphi_{1_r}+Gv_{2_r}\varphi_1$$
$$-\big(K+\frac{7G}{3}\big)v_2\varphi_1\frac{1}{r}+\big(K-\frac{2G}{3}\big)w_{1_z}\varphi_{1_r}r+\big(K-\frac{2G}{3}\big)w_{1_z}\varphi_1\Big]\,\mathrm{d}r\,\mathrm{d}z$$
$$-\int_{z_T}^{z_B}\int_{R_B}^{R_F}f_1\big(\varphi_{1_r}r+\varphi_1\big)\,\mathrm{d}r\,\mathrm{d}z+\int_{R_B}^{R_F}\big(1-\frac{R_B^2}{r^2}\big)r\sigma_{yz}(z_T)\varphi_1(r,z_T)\,\mathrm{d}r$$
$$-\int_{R_B}^{R_F}\big(1-\frac{R_B^2}{r^2}\big)r\sigma_{yz}(z_B)\varphi_1(r,z_B)\,\mathrm{d}r+K_F\int_{z_T}^{z_B}u_1(R_F,z)\varphi_1(R_F,z)R_F\,\mathrm{d}z=0 \tag{4.2.73}$$

$\varphi_2 : \quad (u_2, v_1, w_2)$

$$\int_{z_T}^{z_B}\int_{R_B}^{R_F}\Big[\big(K+\frac{4G}{3}\big)u_{2_r}\varphi_{2_r}r+\big(K-\frac{2G}{3}\big)u_2\varphi_{2_r}+\big(K-\frac{2G}{3}\big)u_{2_r}\varphi_2+Gu_{2_z}\varphi_{2_z}r$$
$$+\big(K+\frac{7G}{3}\big)u_2\varphi_2\frac{1}{r}+\big(K-\frac{2G}{3}\big)v_1\varphi_{2_r}-Gv_{1_r}\varphi_2+\big(K+\frac{7G}{3}\big)v_1\varphi_2\frac{1}{r}$$
$$+Gw_{2_r}\varphi_{2_z}r+\big(K-\frac{2G}{3}\big)w_{2_z}\varphi_{2_r}r+\big(K-\frac{2G}{3}\big)w_{2_z}\varphi_2\Big]\,\mathrm{d}r\,\mathrm{d}z$$
$$-\int_{z_T}^{z_B}\int_{R_B}^{R_F}f_2\big(\varphi_{2_r}r+\varphi_2\big)\,\mathrm{d}r\,\mathrm{d}z+\int_{R_B}^{R_F}\big(1-\frac{R_B^2}{r^2}\big)r\sigma_{xz}(z_T)\varphi_2(r,z_T)\,\mathrm{d}r$$
$$-\int_{R_B}^{R_F}\big(1-\frac{R_B^2}{r^2}\big)r\sigma_{xz}(z_B)\varphi_2(r,z_B)\,\mathrm{d}r+K_F\int_{z_T}^{z_B}u_2(R_F,z)\varphi_2(R_F,z)R_F\,\mathrm{d}z=0 \tag{4.2.74}$$

$\varphi_3 : \quad (u_3, v_4, w_3)$

$$\int_{z_T}^{z_B}\int_{R_B}^{R_F}\Big[\big(K+\frac{4G}{3}\big)u_{3_r}\varphi_{3_r}r+\big(K-\frac{2G}{3}\big)u_3\varphi_{3_r}+\big(K-\frac{2G}{3}\big)u_{3_r}\varphi_3+Gu_{3_z}\varphi_{3_z}r$$
$$+\big(K+\frac{16G}{3}\big)u_3\varphi_3\frac{1}{r}-2\big(K-\frac{2G}{3}\big)v_4\varphi_{3_r}-2\big(K+\frac{7G}{3}\big)v_4\varphi_3\frac{1}{r}$$
$$+2Gv_{4_r}\varphi_3+Gw_{3_r}\varphi_{3_z}r+\big(K-\frac{2G}{3}\big)w_{3_z}\varphi_{3_r}r+\big(K-\frac{2G}{3}\big)w_{3_z}\varphi_3\Big]\,\mathrm{d}r\,\mathrm{d}z$$

$$
-\int_{z_T}^{z_B}\int_{R_B}^{R_F} f_3\big(\varphi_{3_r} r+\varphi_3\big)\,\mathrm{d}r\,\mathrm{d}z-\int_{z_B}^{z_T}\sigma_{xy}(z)\varphi_3(R_F,z)R_F\,\mathrm{d}z
$$

$$
+K_F\int_{z_T}^{z_B} u_3(R_F,z)\varphi_3(R_F,z)R_F\,\mathrm{d}z=0 \tag{4.2.75}
$$

$\varphi_4 :\quad (u_4, v_3, w_4)$

$$
\int_{z_T}^{z_B}\int_{R_B}^{R_F}\Big[\big(K+\frac{4G}{3}\big)u_{4_r}\varphi_{4_r}r+\big(K-\frac{2G}{3}\big)u_4\varphi_{4_r}+\big(K-\frac{2G}{3}\big)u_{4_r}\varphi_4+Gu_{4_z}\varphi_{4_z}r
$$

$$
+\big(K+\frac{16G}{3}\big)u_4\varphi_4\frac{1}{r}+2\big(K-\frac{2G}{3}\big)v_3\varphi_{4_r}+2\big(K+\frac{7G}{3}\big)v_3\varphi_4\frac{1}{r}
$$

$$
-2Gv_{3_r}\varphi_4+Gw_{4_r}\varphi_{4_z}r+\big(K-\frac{2G}{3}\big)w_{4_z}\varphi_{4_r}r+\big(K-\frac{2G}{3}\big)w_{4_z}\varphi_4\Big]\,\mathrm{d}r\,\mathrm{d}z
$$

$$
-\int_{z_T}^{z_B}\int_{R_B}^{R_F} f_4\big(\varphi_{4_r} r+\varphi_4\big)\,\mathrm{d}r\,\mathrm{d}z-\int_{z_B}^{z_T}\frac{\sigma_x(z)-\sigma_y(z)}{2}\varphi_4(R_F,z)R_F\,\mathrm{d}z
$$

$$
+K_F\int_{z_T}^{z_B} u_4(R_F,z)\varphi_4(R_F,z)R_F\,\mathrm{d}z=0 \tag{4.2.76}
$$

$\psi_0 :\quad (v_0)$

$$
\int_{z_T}^{z_B}\int_{R_B}^{R_F} G\Big[\big(v_{0_r}-\frac{v_0}{r}\big)\big(\psi_{0_r}-\frac{\psi_0}{r}\big)+v_{0_z}\psi_{0_z}\Big]r\,\mathrm{d}r\,\mathrm{d}z
$$

$$
+K_F\int_{z_T}^{z_B} v_0(R_F,z)\psi_0(R_F,z)R_F\,\mathrm{d}z=0
$$

Take $\psi_0 = v_0$. Then

$$
\int_{z_T}^{z_B}\int_{R_B}^{R_F} G\Big[\big(v_{0_r}-\frac{v_0}{r}\big)^2+v_{0_z}^2\Big]r\,\mathrm{d}r\,\mathrm{d}z+K_F R_F\int_{z_T}^{z_B} v_0^2(R_F,z)\,\mathrm{d}z=0
$$

It follows that

$$
v_{0_r}-\frac{v_0}{r}=0\quad\text{and}\quad v_{0_z}=0,\qquad z_T<z<z_B,\quad R_B<r<R_F
$$

$$
v_0(R_F,z)=0,\quad z_T<z<z_B
$$

and so,

$$
v_0(r,z,t)=C_0(t)r,\quad R_B<r<R_F \tag{4.2.77}
$$

$\psi_1: \quad (u_2, v_1, w_2)$

$$\int_{z_T}^{z_B}\int_{R_B}^{R_F}\Big[-Gu_2\psi_{1_r} + \big(K - \frac{2G}{3}\big)u_{2_r}\psi_1 + \big(K + \frac{7G}{3}\big)u_2\psi_1\frac{1}{r} + Gv_{1_r}\psi_{1_r}r - Gv_1\psi_{1_r}$$

$$-Gv_{1_r}\psi_1 + \big(K + \frac{7G}{3}\big)v_1\psi_1\frac{1}{r} + Gv_{1_z}\psi_{1_z}r - Gw_2\psi_{1_z} + \big(K - \frac{2G}{3}\big)w_{2_z}\psi_1\Big]\,\mathrm{d}r\,\mathrm{d}z$$

$$-\int_{z_T}^{z_B}\int_{R_B}^{R_F} f_2\psi_1\,\mathrm{d}r\,\mathrm{d}z - \int_{R_B}^{R_F}\big(1 + \frac{R_B^2}{r^2}\big)r\sigma_{xz}(z_T)\psi_1(r, z_T)\,\mathrm{d}r$$

$$+\int_{R_B}^{R_F}\big(1 + \frac{R_B^2}{r^2}\big)r\sigma_{xz}(z_B)\psi_1(r, z_B)\,\mathrm{d}r + K_F\int_{z_T}^{z_B} v_1(R_F, z)\psi_1(R_F, z)R_F\,\mathrm{d}z = 0$$

(4.2.78)

$\psi_2: \quad (u_1, v_2, w_1)$

$$\int_{z_T}^{z_B}\int_{R_B}^{R_F}\Big[Gu_1\psi_{2_r} - \big(K - \frac{2G}{3}\big)u_{1_r}\psi_2 - \big(K + \frac{7G}{3}\big)u_1\psi_2\frac{1}{r} + Gv_{2_r}\psi_{2_r}r - Gv_2\psi_{2_r}$$

$$-Gv_{2_r}\psi_2 + \big(K + \frac{7G}{3}\big)v_2\psi_2\frac{1}{r} + Gv_{2_z}\psi_{2_z}r + Gw_1\psi_{2_z} - \big(K - \frac{2G}{3}\big)w_{1_z}\psi_2\Big]\,\mathrm{d}r\,\mathrm{d}z +$$

$$+\int_{z_T}^{z_B}\int_{R_B}^{R_F} f_1\psi_2\,\mathrm{d}r\,\mathrm{d}z + \int_{R_B}^{R_F}\big(1 + \frac{R_B^2}{r^2}\big)r\sigma_{yz}(z_T)\psi_2(r, z_T)\,\mathrm{d}r$$

$$-\int_{R_B}^{R_F}\big(1 + \frac{R_B^2}{r^2}\big)r\sigma_{yz}(z_B)\psi_2(r, z_B)\,\mathrm{d}r + K_F\int_{z_T}^{z_B} v_2(R_F, z)\psi_2(R_F, z)R_F\,\mathrm{d}z = 0$$

(4.2.79)

$\psi_3: \quad (u_4, v_3, w_4)$

$$\int_{z_T}^{z_B}\int_{R_B}^{R_F}\Big[-2Gu_4\psi_{3_r} + 2\big(K - \frac{2G}{3}\big)u_{4_r}\psi_3 + 2\big(K + \frac{7G}{3}\big)u_4\psi_3\frac{1}{r} + Gv_{3_r}\psi_{3_r}r$$

$$-Gv_3\psi_{3_r} - Gv_{3_r}\psi_3 + \big(4K + \frac{19G}{3}\big)v_3\psi_3\frac{1}{r} + Gv_{3_z}\psi_{3_z}r - 2Gw_4\psi_{3_z}$$

$$+2\big(K - \frac{2G}{3}\big)w_{4_z}\psi_3\Big]\,\mathrm{d}r\,\mathrm{d}z - \int_{z_T}^{z_B}\int_{R_B}^{R_F} 2f_4\psi_3\,\mathrm{d}r\,\mathrm{d}z$$

$$-\int_{z_T}^{z_B}\frac{\sigma_x(z) - \sigma_y(z)}{2}\psi_3(R_F, z)R_F\,\mathrm{d}z + K_F\int_{z_T}^{z_B} v_3(R_F, z)\psi_3(R_F, z)R_F\,\mathrm{d}z = 0$$

(4.2.80)

$\psi_4: \quad (u_3, v_4, w_3)$

$$\int_{z_T}^{z_B}\int_{R_B}^{R_F}\Big[2Gu_3\psi_{4_r} - 2\big(K-\frac{2G}{3}\big)u_{3_r}\psi_4 - 2\big(K+\frac{7G}{3}\big)u_3\psi_4\frac{1}{r} + Gv_{4_r}\psi_{4_r}r$$
$$- Gv_4\psi_{4_r} - Gv_{4_r}\psi_4 + \big(4K+\frac{19G}{3}\big)v_4\psi_4\frac{1}{r} + Gv_{4_z}\psi_{4_z}r + 2Gw_3\psi_{4_z}$$
$$- 2\big(K-\frac{2G}{3}\big)w_{3_z}\psi_4\Big]\,\mathrm{d}r\,\mathrm{d}z + \int_{z_T}^{z_B}\int_{R_B}^{R_F} 2f_3\psi_4\,\mathrm{d}r\,\mathrm{d}z$$
$$- \int_{z_T}^{z_B}\sigma_{xy}(z)\psi_4(R_F,z)R_F\,\mathrm{d}z + K_F\int_{z_T}^{z_B} v_4(R_F,z)\psi_4(R_F,z)R_F\,\mathrm{d}z = 0 \qquad (4.2.81)$$

$\omega_0: \quad (u_0, w_0)$

$$\int_{z_T}^{z_B}\int_{R_B}^{R_F}\Big[Gu_{0_z}\omega_{0_r}r + \big(K-\frac{2G}{3}\big)u_{0_r}\omega_{0_z}r + \big(K-\frac{2G}{3}\big)u_0\omega_{0_z}$$
$$+ \big(K+\frac{4G}{3}\big)w_{0_z}\omega_{0_z}r\Big]\,\mathrm{d}r\,\mathrm{d}z$$
$$- \int_{z_T}^{z_B}\int_{R_B}^{R_F} f_0\omega_{0_z}r\,\mathrm{d}r\,\mathrm{d}z + \int_{F_B}^{R_F}\sigma_{zz}(z_T)\omega_0(r,z_T)r\,\mathrm{d}r - \int_{F_B}^{R_F}\sigma_{zz}(z_B)\omega_0(r,z_B)r\,\mathrm{d}r$$
$$+K_F\int_{z_T}^{z_B} w_0(R_F,z)\omega_0(R_F,z)R_F\,\mathrm{d}z = 0 \qquad (4.2.82)$$

$\omega_1: \quad (u_1, v_2, w_1)$

$$\int_{z_T}^{z_B}\int_{R_B}^{R_F}\Big[Gu_{1_z}\omega_{1_r}r + \big(K-\frac{2G}{3}\big)u_{1_r}\omega_{1_z}r + \big(K-\frac{2G}{3}\big)u_1\omega_{1_z} + Gv_{2_z}\omega_1$$
$$- \big(K-\frac{2G}{3}\big)v_2\omega_{1_z} + Gw_{1_r}\omega_{1_r}r + Gw_1\omega_1\frac{1}{r} + \big(K+\frac{4G}{3}\big)w_{1_z}\omega_{1_z}r\Big]\,\mathrm{d}r\,\mathrm{d}z$$
$$- \int_{z_T}^{z_B}\int_{R_B}^{R_F} f_1\omega_{1_z}r\,\mathrm{d}r\,\mathrm{d}z - \int_{z_T}^{z_B}\sigma_{yz}(z)\omega_1(R_F,z)R_F\,\mathrm{d}z$$
$$+K_F\int_{z_T}^{z_B} w_1(R_F,z)\omega_1(R_F,z)R_F\,\mathrm{d}z = 0 \qquad (4.2.83)$$

$\omega_2: \quad (u_2, v_1, w_2)$

$$\int_{z_T}^{z_B}\int_{R_B}^{R_F}\Big[Gu_{2_z}\omega_{2_r}r + \big(K-\frac{2G}{3}\big)u_{2_r}\omega_{2_z}r + \big(K-\frac{2G}{3}\big)u_2\omega_{2_z} - Gv_{1_z}\omega_2 + \big(K-\frac{2G}{3}\big)v_1\omega_{2_z} + Gw_{2_r}\omega_{2_r}r + Gw_2\omega_2\frac{1}{r} + \big(K+\frac{4G}{3}\big)w_{2_z}\omega_{2_z}r\Big]\,\mathrm{d}r\,\mathrm{d}z - \int_{z_T}^{z_B}\int_{R_B}^{R_F} f_2\omega_{2_z}r\,\mathrm{d}r\,\mathrm{d}z - \int_{z_T}^{z_B}\sigma_{xz}(z)\omega_2(R_F,z)R_F\,\mathrm{d}z + K_F\int_{z_T}^{z_B} w_2(R_F,z)\omega_2(R_F,z)R_F\,\mathrm{d}z = 0 \tag{4.2.84}$$

$\omega_3: \quad (u_3, v_4, w_3)$

$$\int_{z_T}^{z_B}\int_{R_B}^{R_F}\Big[Gu_{3_z}\omega_{3_r}r + \big(K-\frac{2G}{3}\big)u_{3_r}\omega_{3_z}r + \big(K-\frac{2G}{3}\big)u_3\omega_{3_z} + 2Gv_{4_z}\omega_3 - 2\big(K-\frac{2G}{3}\big)v_4\omega_{3_z} + Gw_{3_r}\omega_{3_r}r + 4Gw_3\omega_3\frac{1}{r} + \big(K+\frac{4G}{3}\big)w_{3_z}\omega_{3_z}r\Big]\,\mathrm{d}r\,\mathrm{d}z - \int_{z_T}^{z_B}\int_{R_B}^{R_F} f_3\omega_{3_z}r\,\mathrm{d}r\,\mathrm{d}z - \int_{R_B}^{R_F} 4\nu\frac{R_B^2}{r}\sigma_{xy}(z_T)\omega_3(r,z_T)\,\mathrm{d}r + \int_{R_B}^{R_F} 4\nu\frac{R_B^2}{r}\sigma_{xy}(z_B)\omega_3(r,z_B)\,\mathrm{d}r + K_F\int_{z_T}^{z_B} w_3(R_F,z)\omega_3(R_F,z)R_F\,\mathrm{d}z = 0 \tag{4.2.85}$$

$\omega_4: \quad (u_4, v_3, w_4)$

$$\int_{z_T}^{z_B}\int_{R_B}^{R_F}\Big[Gu_{4_z}\omega_{4_r}r + \big(K-\frac{2G}{3}\big)u_{4_r}\omega_{4_z}r + \big(K-\frac{2G}{3}\big)u_4\omega_{4_z} - 2Gv_{3_z}\omega_4 + 2\big(K-\frac{2G}{3}\big)v_3\omega_{4_z} + Gw_{4_r}\omega_{4_r}r + 4Gw_4\omega_4\frac{1}{r} + \big(K+\frac{4G}{3}\big)w_{4_z}\omega_{4_z}r\Big]\,\mathrm{d}r\,\mathrm{d}z - \int_{z_T}^{z_B}\int_{R_B}^{R_F} f_4\omega_{4_z}r\,\mathrm{d}r\,\mathrm{d}z - \int_{R_B}^{R_F} 4\nu\frac{R_B^2}{r}\frac{\sigma_x(z_T)-\sigma_y(z_T)}{2}\omega_4(r,z_T)\,\mathrm{d}r + \int_{R_B}^{R_F} 4\nu\frac{R_B^2}{r}\frac{\sigma_x(z_B)-\sigma_y(z_B)}{2}\omega_4(r,z_B)\,\mathrm{d}r + K_F\int_{z_T}^{z_B} w_4(R_F,z)\omega_4(R_F,z)R_F\,\mathrm{d}z = 0 \tag{4.2.86}$$

4.2.3 Finite element approximation

Dirichlet boundary conditions for the TCP system

r-approximation:

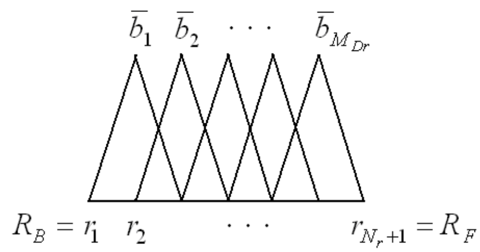

Figure 4.3: r-approximation of 3D Dirichlet boundary conditions

Here N_r is the number of subintervals; $M_{Dr} = N_r - 1$ is the number of basis functions $\bar{b}(r) = \begin{bmatrix} \bar{b}_1(r) & \dots & \bar{b}_{M_Dr}(r) \end{bmatrix}^T$ is the vector of basis functions; and

$$B(r) = \begin{bmatrix} \bar{b}(r)^T & 0 & 0 \\ 0 & \bar{b}(r)^T & 0 \\ 0 & 0 & \bar{b}(r)^T \end{bmatrix}_{3\times 3M_{Dr}}$$

z-approximation:

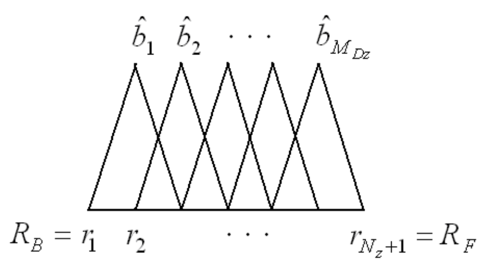

Figure 4.4: z-approximation of 3D Dirichlet boundary conditions

Here N_z is the number of subintervals; $M_{Dz} = N_z - 1$ is the number of basis functions $\hat{b}(z) = \begin{bmatrix} \hat{b}_1(z) & \dots & \hat{b}_{M_Dz}(z) \end{bmatrix}^T$ is the vector of basis functions; and

$$\hat{B}(z) = \begin{bmatrix} \hat{b}^T(z) & 0 & 0 \\ 0 & \hat{b}(z)^T & 0 \\ 0 & 0 & \hat{b}(z)^T \end{bmatrix}_{3\times 3M_{Dz}} = I_{3\times 3} \otimes \hat{b}(z)^T$$

Using r-and z-approximations, the coefficients of the θ-approximation for $V(r, \theta, z, t)$ can be represented as

$$V_i(r, z, t) = \hat{B}(z) \cdot \begin{bmatrix} \bar{b}(r)^T & \dots & 0 \\ & \dots & \\ 0 & \dots & \hat{b}(r)^T \end{bmatrix}_{3M_{Dz} \times (M_{Dr} \cdot 3M_{Dz})} \cdot \bar{c}_i(t)_{(M_{Dr} \cdot 3M_{Dz}) \times 1}$$

$$i = 0, \dots, 4 \quad (4.2.87)$$

We can rewrite (4.2.87) as

$$V_i(r, z, t) = \left(\hat{b}(z)^T \otimes B(r)\right) \bar{c}_i(t) \quad (4.2.88)$$

where $\bar{c}_i(t)$ is a $(3M_{Dr}M_{Dz} \times 1)$ vector and $\hat{b}(z)^T \otimes B(r)$ is a $(3 \times 3M_{Dr}M_{Dz})$ matrix. The coefficients of the θ-approximation for boundary conditions (4.2.2) and initial condition (4.2.3) can be represented as

$$\begin{aligned} V_{B0}(z, t) &= \hat{B}(z) \bar{\nu}_{B0}(t) \\ V_{Fi}(z) &= \hat{B}(z) \bar{\nu}_{Fi} \\ \tilde{V}_{Ti}(r, t) &= B(r) \bar{\nu}_{\tilde{T}i}(t) \\ \tilde{V}_{Bi}(r, t) &= B(r) \bar{\nu}_{\tilde{B}i}(t) \\ \tilde{V}_{Ii}(r, z) &= \left(\hat{b}(z)^T \otimes B(r)\right) \bar{\nu}_{Ii}, \qquad i = 0, \dots, 4 \end{aligned} \quad (4.2.89)$$

where $\bar{\nu}_{B0}(t)$ and $\bar{\nu}_{Fi}$ are $(3M_{Dz} \times 1)$ vectors; $\bar{\nu}_{\tilde{B}i}(t)$ and $\bar{\nu}_{\tilde{T}i}(t)$ are $(3M_{Dr} \times 1)$ vectors; and $\bar{\nu}_{Ii}$ is a $(3M_{Dr}M_{Dz} \times 1)$ vector.

Neumann boundary conditions for the elastic system

r-approximation:

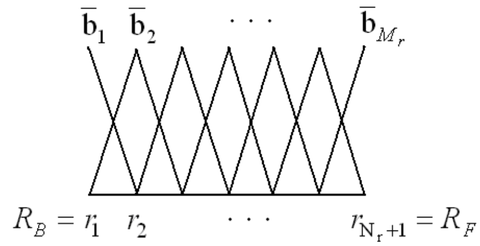

Figure 4.5: r-approximation of 3D Neumann boundary conditions

N_r is the number of subintervals; $M_r = N_r + 1$ is the number of basis functions $\bar{b}(r) = \begin{bmatrix} \bar{b}_1(r) & \dots & \bar{b}_{M_r}(r) \end{bmatrix}^T$ is the vector of basis functions.

z-approximation:

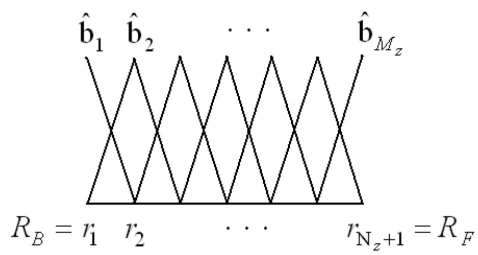

Figure 4.6: z-approximation of 3D Neumann boundary conditions

N_z is the number of subintervals; $M_z = N_z + 1$ is the number of basis functions $\hat{b}(z) = \begin{bmatrix} \hat{b}_1(z) & \dots & \hat{b}_{M_z}(z) \end{bmatrix}^T$ is the vector of basis functions.

Using r-and z-approximations, the coefficients of the θ-approximation for the displacement coordinates can be represented as

$$u_i(r, z, t) = \left(\hat{b}(z)^T \otimes \bar{b}(r)^T\right)\bar{\mu}_i(t) \tag{4.2.90}$$

$$v_i(r, z, t) = \left(\hat{b}(z)^T \otimes \bar{b}(r)^T\right)\bar{\eta}_i(t) \tag{4.2.91}$$

$$w_i(r, z, t) = \left(\hat{b}(z)^T \otimes \bar{b}(r)^T\right)\bar{\lambda}_i(t) \tag{4.2.92}$$

where $\hat{b}(z)^T \otimes \bar{b}(r)^T$ is a $(1 \times M_r M_z)$ vector and $\bar{\mu}_i(t), \bar{\eta}_i(t), \bar{\lambda}_i(t)$ are $(M_r M_z \times 1)$ vectors.

Define the following matrices.

$M_r \times M_r$ matrices:

$$g_2(r) = \int_{R_B}^{R_F} \bar{\mathrm{b}}_r(r)\,\bar{\mathrm{b}}_r(r)^T r\,\mathrm{d}r$$

$$g_1^{(r)} = \int_{R_B}^{R_F} \bar{\mathrm{b}}(r)\,\bar{\mathrm{b}}_r(r)^T\,\mathrm{d}r$$

$$g_1(r) = \int_{R_B}^{R_F} \bar{\mathrm{b}}(r)\,\bar{\mathrm{b}}_r(r)^T r\,\mathrm{d}r$$

$$g_0\big(\frac{1}{r}\big) = \int_{R_B}^{R_F} \bar{\mathrm{b}}(r)\,\bar{\mathrm{b}}(r)^T \frac{1}{r}\,\mathrm{d}r$$

$$g_0(r) = \int_{R_B}^{R_F} \bar{\mathrm{b}}(r)\,\bar{\mathrm{b}}(r)^T r\,\mathrm{d}r$$

$$g_0^{(r)} = \int_{R_B}^{R_F} \bar{\mathrm{b}}(r)\,\bar{\mathrm{b}}(r)^T\,\mathrm{d}r$$

$M_z \times M_z$ matrices:

$$g_2^{(z)} = \int_{z_T}^{z_B} \hat{\mathrm{b}}_z(z)\,\hat{\mathrm{b}}_z(z)^T\,\mathrm{d}z$$

$$g_1^{(z)} = \int_{z_T}^{z_B} \hat{\mathrm{b}}(z)\,\hat{\mathrm{b}}_z(z)^T\,\mathrm{d}z$$

$$g_0^{(z)} = \int_{z_T}^{z_B} \hat{\mathrm{b}}(z)\,\hat{\mathrm{b}}(z)^T\,\mathrm{d}z$$

$M_rM_z \times M_rM_z$ matrix:

$$g_F = \int_{z_T}^{z_B} \Big(\hat{\mathrm{b}}(z) \otimes \bar{\mathrm{b}}(R_F)\Big)\Big(\hat{\mathrm{b}}(z) \otimes \bar{\mathrm{b}}(R_F)\Big)^T\,\mathrm{d}z$$

Let $P_3 = \begin{bmatrix}0 & 0 & 1\end{bmatrix}^T$. Then

$$\begin{aligned} p_B(z,t) &= P_3^T V_B(z,t) = P_3^T V_{B0}(z,t) \\ &= P_3^T \hat{B}(z)\bar{\nu}_{B0}(t) \end{aligned} \tag{4.2.93}$$

Also, let

$$\begin{aligned}\sigma_R(z) &= \frac{\sigma_x(z)+\sigma_y(z)}{2} = \hat{\mathrm{b}}(z)^T\bar{\sigma}_R \\ \sigma_D(z) &= \frac{\sigma_x(z)-\sigma_y(z)}{2} = \hat{\mathrm{b}}(z)^T\bar{\sigma}_D \\ \sigma_{xy}(z) &= \hat{\mathrm{b}}(z)^T\bar{\sigma}_{xy}\end{aligned} \tag{4.2.94}$$

As before,

$$f(r,\theta,z,t) = \vec{b}_1^T\bar{V}(r,\theta,z,t) = \vec{b}_1^T(V(r,\theta,z,t)+W(r,\theta,z,t))$$

and from (4.2.18)-(4.2.20), (4.2.65), (4.2.88), and (4.2.89) it follows that

$$\begin{aligned}f_i(r,z,t) &= \vec{b}_1^T\big(V_i(r,z,t)+W_i(r,z,t)\big) \\ &= \vec{b}_1^T\Big(\hat{b}(z)^T\otimes B(r)\Big)\bar{c}_i(t) \\ &+ \frac{R_F-r}{R_F-R_B}\vec{b}_1^T\Big(\hat{B}(z)-\frac{z-z_T}{z_B-z_T}\hat{B}(z_B)-\frac{z_B-z}{z_B-z_T}\hat{B}(z_T)\Big)\bar{\nu}_{Bi}(t) \\ &+ \frac{r-R_B}{R_F-R_B}\vec{b}_1^T\Big(\hat{B}(z)-\frac{z-z_T}{z_B-z_T}\hat{B}(z_B)-\frac{z_B-z}{z_B-z_T}\hat{B}(z_T)\Big)\bar{\nu}_{Fi} \\ &+ \frac{z-z_T}{z_B-z_T}\vec{b}_1^T B(r)\bar{\nu}_{\tilde{B}i}(t) + \frac{z_B-z}{z_B-z_T}\vec{b}_1^T B(r)\bar{\nu}_{\tilde{T}i}(t), \quad i=0,\dots,4\end{aligned} \tag{4.2.95}$$

and $\bar{\nu}_{Bi}(t)=0, \quad i=1,\dots,4.$

From (4.2.72)-(4.2.86) we obtain

$\varphi_0:$

$$\begin{aligned}&\Big[\big(K+\frac{4G}{3}\big)g_0^{(z)}\otimes g_2(r) + \big(K-\frac{2G}{3}\big)g_0^{(z)}\otimes\big(g_1^{(r)T}+g_1^{(r)}\big) \\ &+ \big(K+\frac{4G}{3}\big)g_0^{(z)}\otimes g_0\big(\frac{1}{r}\big) + Gg_2^{(z)}\otimes g_0(r) + K_F R_F g_F\Big]\bar{\mu}_0(t) \\ &+\Big[\big(K-\frac{2G}{3}\big)g_1^{(z)}\otimes g_1(r)^T + Gg_1^{(z)T}\otimes g_1(r) + \big(K-\frac{2G}{3}\big)g_1^{(z)}\otimes g_0^{(r)}\Big]\bar{\lambda}_0(t) \\ &= \int_{z_T}^{z_B}\int_{R_B}^{R_F} f_0\big[\,\hat{\mathrm{b}}(z)\otimes(r\,\bar{\mathrm{b}}_r(r)+\,\bar{\mathrm{b}}(r))\big]\,\mathrm{d}r\,\mathrm{d}z-\end{aligned}$$

$$-R_B\Big[\int_{z_T}^{z_B}(\,\hat{\mathrm{b}}(z)\otimes\,\bar{\mathrm{b}}(R_B))P_3^T\hat{B}(z)\,\mathrm{d}z\Big]\bar{\nu}_{B0}(t)+R_F\Big(g_0^{(z)}\otimes\,\bar{\mathrm{b}}(R_F)\Big)\bar{\sigma}_R \qquad (4.2.96)$$

φ_1 :

$$\begin{aligned}
&\Big[\big(K+\frac{4G}{3}\big)g_0^{(z)}\otimes g_2(r)+\big(K-\frac{2G}{3}\big)g_0^{(z)}\otimes\big(g_1^{(r)T}+g_1^{(r)}\big)\\
&+\big(K+\frac{7G}{3}\big)g_0^{(z)}\otimes g_0\big(\frac{1}{r}\big)+Gg_2^{(z)}\otimes g_0(r)+K_FR_Fg_F\Big]\bar{\mu}_1(t)\\
-&\Big[\big(K-\frac{2G}{3}\big)g_0^{(z)}\otimes g_1^{(r)T}-Gg_0^{(z)}\otimes g_1^{(r)}+\big(K+\frac{7G}{3}\big)g_0^{(z)}\otimes g_0\big(\frac{1}{r}\big)\Big]\bar{\eta}_2(t)\\
+&\Big[\big(K-\frac{2G}{3}\big)g_1^{(z)}\otimes g_1(r)^T+Gg_1^{(z)T}\otimes g_1(r)+\big(K-\frac{2G}{3}\big)g_1^{(z)}\otimes g_0^{(r)}\Big]\bar{\lambda}_1(t)\\
=&\int_{z_T}^{z_B}\int_{R_B}^{R_F}f_1\big[\,\hat{\mathrm{b}}(z)\otimes(r\,\bar{\mathrm{b}}_r(r)+\,\bar{\mathrm{b}}(r))\big]\,\mathrm{d}r\,\mathrm{d}z\\
+&\Big\{\Big(\,\hat{\mathrm{b}}(z_B)\,\hat{\mathrm{b}}(z_B)^T-\,\hat{\mathrm{b}}(z_T)\,\hat{\mathrm{b}}(z_T)^T\Big)\otimes\Big[\int_{R_B}^{R_F}\big(1-\frac{R_B^2}{r^2}\big)r\,\bar{\mathrm{b}}(r)\,\mathrm{d}r\Big]\Big\}\bar{\sigma}_{yz} \qquad (4.2.97)
\end{aligned}$$

φ_2 :

$$\begin{aligned}
&\Big[\big(K+\frac{4G}{3}\big)g_0^{(z)}\otimes g_2(r)+\big(K-\frac{2G}{3}\big)g_0^{(z)}\otimes\big(g_1^{(r)T}+g_1^{(r)}\big)\\
&+\big(K+\frac{7G}{3}\big)g_0^{(z)}\otimes g_0\big(\frac{1}{r}\big)+Gg_2^{(z)}\otimes g_0(r)+K_FR_Fg_F\Big]\bar{\mu}_2(t)\\
+&\Big[\big(K-\frac{2G}{3}\big)g_0^{(z)}\otimes g_1^{(r)T}-Gg_0^{(z)}\otimes g_1^{(r)}+\big(K+\frac{7G}{3}\big)g_0^{(z)}\otimes g_0\big(\frac{1}{r}\big)\Big]\bar{\eta}_1(t)\\
+&\Big[\big(K-\frac{2G}{3}\big)g_1^{(z)}\otimes g_1(r)^T+Gg_1^{(z)T}\otimes g_1(r)+\big(K-\frac{2G}{3}\big)g_1^{(z)}\otimes g_0^{(r)}\Big]\bar{\lambda}_2(t)\\
=&\int_{z_T}^{z_B}\int_{R_B}^{R_F}f_2\big[\,\hat{\mathrm{b}}(z)\otimes(r\,\bar{\mathrm{b}}_r(r)+\,\bar{\mathrm{b}}(r))\big]\,\mathrm{d}r\,\mathrm{d}z\\
+&\Big\{\Big(\,\hat{\mathrm{b}}(z_B)\,\hat{\mathrm{b}}(z_B)^T-\,\hat{\mathrm{b}}(z_T)\,\hat{\mathrm{b}}(z_T)^T\Big)\otimes\Big[\int_{R_B}^{R_F}\big(1-\frac{R_B^2}{r^2}\big)r\,\bar{\mathrm{b}}(r)\,\mathrm{d}r\Big]\Big\}\bar{\sigma}_{xz} \qquad (4.2.98)
\end{aligned}$$

φ_3 :

$$\Big[\big(K+\frac{4G}{3}\big)g_0^{(z)}\otimes g_2(r)+\big(K-\frac{2G}{3}\big)g_0^{(z)}\otimes\big(g_1^{(r)T}+g_1^{(r)}\big)$$

$$+\big(K+\frac{16G}{3}\big)g_0^{(z)}\otimes g_0\big(\frac{1}{r}\big)+Gg_2^{(z)}\otimes g_0(r)+K_FR_Fg_F\Big]\bar{\mu}_3(t)$$

$$-2\Big[\big(K-\frac{2G}{3}\big)g_0^{(z)}\otimes g_1^{(r)T}-Gg_0^{(z)}\otimes g_1^{(r)}+\big(K+\frac{7G}{3}\big)g_0^{(z)}\otimes g_0\big(\frac{1}{r}\big)\Big]\bar{\eta}_4(t)$$

$$+\Big[\big(K-\frac{2G}{3}\big)g_1^{(z)}\otimes g_1(r)^T+Gg_1^{(z)T}\otimes g_1(r)+\big(K-\frac{2G}{3}\big)g_1^{(z)}\otimes g_0^{(r)}\Big]\bar{\lambda}_3(t)$$

$$=\int_{z_T}^{z_B}\int_{R_B}^{R_F}f_3\big[\,\hat{\mathrm{b}}(z)\otimes(r\,\bar{\mathrm{b}}_r(r)+\,\bar{\mathrm{b}}(r))\big]\,\mathrm{d}r\,\mathrm{d}z+R_F\Big(g_0^{(z)}\otimes\,\bar{\mathrm{b}}(R_F)\Big)\bar{\sigma}_{xy}\quad(4.2.99)$$

φ_4 :

$$\Big[\big(K+\frac{4G}{3}\big)g_0^{(z)}\otimes g_2(r)+\big(K-\frac{2G}{3}\big)g_0^{(z)}\otimes\big(g_1^{(r)T}+g_1^{(r)}\big)$$

$$+\big(K+\frac{16G}{3}\big)g_0^{(z)}\otimes g_0\big(\frac{1}{r}\big)+Gg_2^{(z)}\otimes g_0(r)+K_FR_Fg_F\Big]\bar{\mu}_4(t)$$

$$+2\Big[\big(K-\frac{2G}{3}\big)g_0^{(z)}\otimes g_1^{(r)T}-Gg_0^{(z)}\otimes g_1^{(r)}+\big(K+\frac{7G}{3}\big)g_0^{(z)}\otimes g_0\big(\frac{1}{r}\big)\Big]\bar{\eta}_3(t)$$

$$+\Big[\big(K-\frac{2G}{3}\big)g_1^{(z)}\otimes g_1(r)^T+Gg_1^{(z)T}\otimes g_1(r)+\big(K-\frac{2G}{3}\big)g_1^{(z)}\otimes g_0^{(r)}\Big]\bar{\lambda}_4(t)$$

$$=\int_{z_T}^{z_B}\int_{R_B}^{R_F}f_4\big[\,\hat{\mathrm{b}}(z)\otimes(r\,\bar{\mathrm{b}}_r(r)+\,\bar{\mathrm{b}}(r))\big]\,\mathrm{d}r\,\mathrm{d}z+R_F\Big(g_0^{(z)}\otimes\,\bar{\mathrm{b}}(R_F)\Big)\bar{\sigma}_D\quad(4.2.100)$$

ψ_0 :

$$\Big[Gg_0^{(z)}\otimes g_2(r)-Gg_0^{(z)}\otimes\big(g_1^{(r)T}+g_1^{(r)}\big)+Gg_0^{(z)}\otimes g_0\big(\frac{1}{r}\big)+Gg_2^{(z)}\otimes g_0(r)$$

$$+K_FR_Fg_F\Big]\bar{\eta}_0(t)=0$$

It follows that

$$\bar{\eta}_0(t)=0$$

ψ_1 :

$$\Big[\big(K-\frac{2G}{3}\big)g_0^{(z)}\otimes g_1^{(r)} - Gg_0^{(z)}\otimes g_1^{(r)T} + \big(K+\frac{7G}{3}\big)g_0^{(z)}\otimes g_0\big(\frac{1}{r}\big)\Big]\bar{\mu}_2(t)$$

$$+\Big[Gg_0^{(z)}\otimes g_2(r) - Gg_0^{(z)}\otimes\big(g_1^{(r)T}+g_1^{(r)}\big) + \big(K+\frac{7G}{3}\big)g_0^{(z)}\otimes g_0\big(\frac{1}{r}\big)$$

$$+Gg_2^{(z)}\otimes g_0(r) + K_F R_F g_F\Big]\bar{\eta}_1(t)$$

$$+\Big[-Gg_1^{(z)T}\otimes g_0^{(r)} + \big(K-\frac{2G}{3}\big)g_1^{(z)}\otimes g_0^{(r)}\Big]\bar{\lambda}_2(t)$$

$$=\int_{z_T}^{z_B}\int_{R_B}^{R_F} f_2\big[\,\hat{\mathrm{b}}(z)\otimes\,\bar{\mathrm{b}}(r)\big]\,\mathrm{d}r\,\mathrm{d}z$$

$$-\Big\{\Big(\hat{\mathrm{b}}(z_B)\,\hat{\mathrm{b}}(z_B)^T - \,\hat{\mathrm{b}}(z_T)\,\hat{\mathrm{b}}(z_T)^T\Big)\otimes\Big[\int_{R_B}^{R_F}\big(1+\frac{R_B^2}{r^2}\big)r\,\bar{\mathrm{b}}(r)\,\mathrm{d}r\Big]\Big\}\bar{\sigma}_{xz} \quad (4.2.101)$$

ψ_2 :

$$-\Big[\big(K-\frac{2G}{3}\big)g_0^{(z)}\otimes g_1^{(r)} - Gg_0^{(z)}\otimes g_1^{(r)T} + \big(K+\frac{7G}{3}\big)g_0^{(z)}\otimes g_0\big(\frac{1}{r}\big)\Big]\bar{\mu}_1(t)$$

$$+\Big[Gg_0^{(z)}\otimes g_2(r) - Gg_0^{(z)}\otimes\big(g_1^{(r)T}+g_1^{(r)}\big) + \big(K+\frac{7G}{3}\big)g_0^{(z)}\otimes g_0\big(\frac{1}{r}\big)$$

$$+Gg_2^{(z)}\otimes g_0(r) + K_F R_F g_F\Big]\bar{\eta}_2(t)$$

$$-\Big[-Gg_1^{(z)T}\otimes g_0^{(r)} + \big(K-\frac{2G}{3}\big)g_1^{(z)}\otimes g_0^{(r)}\Big]\bar{\lambda}_1(t)$$

$$=-\int_{z_T}^{z_B}\int_{R_B}^{R_F} f_1\big[\,\hat{\mathrm{b}}(z)\otimes\,\bar{\mathrm{b}}(r)\big]\,\mathrm{d}r\,\mathrm{d}z$$

$$+\Big\{\Big(\hat{\mathrm{b}}(z_B)\,\hat{\mathrm{b}}(z_B)^T - \,\hat{\mathrm{b}}(z_T)\,\hat{\mathrm{b}}(z_T)^T\Big)\otimes\Big[\int_{R_B}^{R_F}\big(1+\frac{R_B^2}{r^2}\big)r\,\bar{\mathrm{b}}(r)\,\mathrm{d}r\Big]\Big\}\bar{\sigma}_{yz} \quad (4.2.102)$$

ψ_3 :

$$2\Big[\big(K-\frac{2G}{3}\big)g_0^{(z)}\otimes g_1^{(r)} - Gg_0^{(z)}\otimes g_1^{(r)T} + \big(K+\frac{7G}{3}\big)g_0^{(z)}\otimes g_0\big(\frac{1}{r}\big)\Big]\bar{\mu}_4(t)$$

$$+\Big[Gg_0^{(z)}\otimes g_2(r) - Gg_0^{(z)}\otimes\big(g_1^{(r)T}+g_1^{(r)}\big) + \big(4K+\frac{19G}{3}\big)g_0^{(z)}\otimes g_0\big(\frac{1}{r}\big)+$$

$$+ Gg_2^{(z)} \otimes g_0(r) + K_F R_F g_F \Big] \bar{\eta}_3(t)$$

$$+2\Big[- Gg_1^{(z)T} \otimes g_0^{(r)} + \big(K - \frac{2G}{3}\big) g_1^{(z)} \otimes g_0^{(r)} \Big] \bar{\lambda}_4(t)$$

$$= 2 \int_{z_T}^{z_B} \int_{R_B}^{R_F} f_4 \big[\, \hat{\mathrm{b}}(z) \otimes \bar{\mathrm{b}}(r) \big] \,\mathrm{d}r \,\mathrm{d}z + R_F \Big(g_0^{(z)} \otimes \bar{\mathrm{b}}(R_F) \Big) \bar{\sigma}_D \tag{4.2.103}$$

ψ_4 :

$$-2\Big[\big(K - \frac{2G}{3}\big) g_0^{(z)} \otimes g_1^{(r)} - Gg_0^{(z)} \otimes g_1^{(r)T} + \big(K + \frac{7G}{3}\big) g_0^{(z)} \otimes g_0\big(\frac{1}{r}\big) \Big] \bar{\mu}_3(t)$$

$$+\Big[Gg_0^{(z)} \otimes g_2(r) - Gg_0^{(z)} \otimes \big(g_1^{(r)T} + g_1^{(r)}\big) + \big(4K + \frac{19G}{3}\big) g_0^{(z)} \otimes g_0\big(\frac{1}{r}\big)$$

$$+ Gg_2^{(z)} \otimes g_0(r) + K_F R_F g_F \Big] \bar{\eta}_4(t)$$

$$-2\Big[- Gg_1^{(z)T} \otimes g_0^{(r)} + \big(K - \frac{2G}{3}\big) g_1^{(z)} \otimes g_0^{(r)} \Big] \bar{\lambda}_3(t)$$

$$= -2 \int_{z_T}^{z_B} \int_{R_B}^{R_F} f_3 \big[\, \hat{\mathrm{b}}(z) \otimes \bar{\mathrm{b}}(r) \big] \,\mathrm{d}r \,\mathrm{d}z + R_F \Big(g_0^{(z)} \otimes \bar{\mathrm{b}}(R_F) \Big) \bar{\sigma}_{xy} \tag{4.2.104}$$

ω_0 :

$$\Big[\big(K - \frac{2G}{3}\big) g_1^{(z)T} \otimes g_1(r) + Gg_1^{(z)} \otimes g_1(r)^T + \big(K - \frac{2G}{3}\big) g_1^{(z)T} \otimes g_0^{(r)} \Big] \bar{\mu}_0(t)$$

$$+\Big[Gg_0^{(z)} \otimes g_2(r) + \big(K + \frac{4G}{3}\big) g_2^{(z)} \otimes g_0(r) + K_F R_F g_F \Big] \bar{\lambda}_0(t)$$

$$= \int_{z_T}^{z_B} \int_{R_B}^{R_F} f_0 \big[\, \hat{\mathrm{b}}_z(z) \otimes r\, \bar{\mathrm{b}}(r) \big] \,\mathrm{d}r \,\mathrm{d}z$$

$$+\Big\{ \Big(\hat{\mathrm{b}}(z_B)\, \hat{\mathrm{b}}(z_B)^T - \hat{\mathrm{b}}(z_T)\, \hat{\mathrm{b}}(z_T)^T \Big) \otimes \Big[\int_{R_B}^{R_F} r\, \bar{\mathrm{b}}(r) \,\mathrm{d}r \Big] \Big\} \bar{\sigma}_{zz} \tag{4.2.105}$$

ω_1 :

$$\Big[\big(K - \frac{2G}{3}\big) g_1^{(z)T} \otimes g_1(r) + Gg_1^{(z)} \otimes g_1(r)^T + \big(K - \frac{2G}{3}\big) g_1^{(z)T} \otimes g_0^{(r)} \Big] \bar{\mu}_1(t) -$$

$$
-\Big[-Gg_1^{(z)} \otimes g_0^{(r)} + \big(K - \frac{2G}{3}\big) g_1^{(z)T} \otimes g_0^{(r)} \Big] \bar{\eta}_2(t)
$$

$$
+\Big[Gg_0^{(z)} \otimes g_2(r) + Gg_0^{(z)} \otimes g_0\big(\frac{1}{r}\big) + \big(K + \frac{4G}{3}\big) g_2^{(z)} \otimes g_0(r) + K_F R_F g_F \Big] \bar{\lambda}_1(t)
$$

$$
= \int_{z_T}^{z_B} \int_{R_B}^{R_F} f_1 \big[\hat{\mathrm{b}}_z(z) \otimes r\, \bar{\mathrm{b}}(r) \big] \, \mathrm{d}r \, \mathrm{d}z + R_F \Big(g_0^{(z)} \otimes \bar{\mathrm{b}}(R_F) \Big) \bar{\sigma}_{yz} \qquad (4.2.106)
$$

ω_2 :

$$
\Big[\big(K - \frac{2G}{3}\big) g_1^{(z)T} \otimes g_1(r) + Gg_1^{(z)} \otimes g_1(r)^T + \big(K - \frac{2G}{3}\big) g_1^{(z)T} \otimes g_0^{(r)} \Big] \bar{\mu}_2(t)
$$

$$
+\Big[-Gg_1^{(z)} \otimes g_0^{(r)} + \big(K - \frac{2G}{3}\big) g_1^{(z)T} \otimes g_0^{(r)} \Big] \bar{\eta}_1(t)
$$

$$
+\Big[Gg_0^{(z)} \otimes g_2(r) + Gg_0^{(z)} \otimes g_0\big(\frac{1}{r}\big) + \big(K + \frac{4G}{3}\big) g_2^{(z)} \otimes g_0(r) + K_F R_F g_F \Big] \bar{\lambda}_2(t)
$$

$$
= \int_{z_T}^{z_B} \int_{R_B}^{R_F} f_2 \big[\hat{\mathrm{b}}_z(z) \otimes r\, \bar{\mathrm{b}}(r) \big] \, \mathrm{d}r \, \mathrm{d}z + R_F \Big(g_0^{(z)} \otimes \bar{\mathrm{b}}(R_F) \Big) \bar{\sigma}_{xz} \qquad (4.2.107)
$$

ω_3 :

$$
\Big[\big(K - \frac{2G}{3}\big) g_1^{(z)T} \otimes g_1(r) + Gg_1^{(z)} \otimes g_1(r)^T + \big(K - \frac{2G}{3}\big) g_1^{(z)T} \otimes g_0^{(r)} \Big] \bar{\mu}_3(t)
$$

$$
-2\Big[-Gg_1^{(z)} \otimes g_0^{(r)} + \big(K - \frac{2G}{3}\big) g_1^{(z)T} \otimes g_0^{(r)} \Big] \bar{\eta}_4(t)
$$

$$
+\Big[Gg_0^{(z)} \otimes g_2(r) + 4Gg_0^{(z)} \otimes g_0\big(\frac{1}{r}\big) + \big(K + \frac{4G}{3}\big) g_2^{(z)} \otimes g_0(r) + K_F R_F g_F \Big] \bar{\lambda}_3(t)
$$

$$
= \int_{z_T}^{z_B} \int_{R_B}^{R_F} f_3 \big[\hat{\mathrm{b}}_z(z) \otimes r\, \bar{\mathrm{b}}(r) \big] \, \mathrm{d}r \, \mathrm{d}z
$$

$$
-4\nu R_B^2 \Big\{ \Big(\hat{\mathrm{b}}(z_B)\, \hat{\mathrm{b}}(z_B)^T - \hat{\mathrm{b}}(z_T)\, \hat{\mathrm{b}}(z_T)^T \Big) \otimes \Big[\int_{R_B}^{R_F} \frac{1}{r}\, \bar{\mathrm{b}}(r) \, \mathrm{d}r \Big] \Big\} \bar{\sigma}_{xy} \qquad (4.2.108)
$$

ω_4 :

$$
\Big[\big(K - \frac{2G}{3}\big) g_1^{(z)T} \otimes g_1(r) + Gg_1^{(z)} \otimes g_1(r)^T + \big(K - \frac{2G}{3}\big) g_1^{(z)T} \otimes g_0^{(r)} \Big] \bar{\mu}_4(t) +
$$

$$+2\Big[-Gg_1^{(z)}\otimes g_0^{(r)}+\big(K-\frac{2G}{3}\big)g_1^{(z)T}\otimes g_0^{(r)}\Big]\bar{\eta}_3(t)$$

$$+\Big[Gg_0^{(z)}\otimes g_2(r)+4Gg_0^{(z)}\otimes g_0\big(\frac{1}{r}\big)+\big(K+\frac{4G}{3}\big)g_2^{(z)}\otimes g_0(r)+K_F R_F g_F\Big]\bar{\lambda}_4(t)$$

$$=\int_{z_T}^{z_B}\int_{R_B}^{R_F} f_4\big[\,\hat{\mathrm{b}}_z(z)\otimes r\,\bar{\mathrm{b}}(r)\big]\,\mathrm{d}r\,\mathrm{d}z$$

$$-4\nu R_B^2\Big\{\Big(\hat{\mathrm{b}}(z_B)\,\hat{\mathrm{b}}(z_B)^T-\hat{\mathrm{b}}(z_T)\,\hat{\mathrm{b}}(z_T)^T\Big)\otimes\Big[\int_{R_B}^{R_F}\frac{1}{r}\,\bar{\mathrm{b}}(r)\,\mathrm{d}r\Big]\Big\}\bar{\sigma}_D \qquad (4.2.109)$$

Define the following matrices.

$M_r M_z \times M_r M_z$ matrices:

$$\mathcal{M}_0=\big(K+\frac{4G}{3}\big)g_0^{(z)}\otimes g_2(r)+\big(K-\frac{2G}{3}\big)g_0^{(z)}\otimes\big(g_1^{(r)T}+g_1^{(r)}\big)$$
$$+\big(K+\frac{4G}{3}\big)g_0^{(z)}\otimes g_0\big(\frac{1}{r}\big)+Gg_2^{(z)}\otimes g_0(r)+K_F R_F g_F$$

$$\mathcal{M}_1=\big(K+\frac{4G}{3}\big)g_0^{(z)}\otimes g_2(r)+\big(K-\frac{2G}{3}\big)g_0^{(z)}\otimes\big(g_1^{(r)T}+g_1^{(r)}\big)$$
$$+\big(K+\frac{7G}{3}\big)g_0^{(z)}\otimes g_0\big(\frac{1}{r}\big)+Gg_2^{(z)}\otimes g_0(r)+K_F R_F g_F$$

$$\mathcal{M}_2=\big(K+\frac{4G}{3}\big)g_0^{(z)}\otimes g_2(r)+\big(K-\frac{2G}{3}\big)g_0^{(z)}\otimes\big(g_1^{(r)T}+g_1^{(r)}\big)$$
$$+\big(K+\frac{16G}{3}\big)g_0^{(z)}\otimes g_0\big(\frac{1}{r}\big)+Gg_2^{(z)}\otimes g_0(r)+K_F R_F g_F$$

$$\mathcal{M}_3=\big(K-\frac{2G}{3}\big)g_1^{(z)}\otimes g_1(r)^T+Gg_1^{(z)T}\otimes g_1(r)+\big(K-\frac{2G}{3}\big)g_1^{(z)}\otimes g_0^{(r)}$$

$$\mathcal{N}_1=\big(K-\frac{2G}{3}\big)g_0^{(z)}\otimes g_1^{(r)T}-Gg_0^{(z)}\otimes g_1^{(r)}+\big(K+\frac{7G}{3}\big)g_0^{(z)}\otimes g_0\big(\frac{1}{r}\big)$$

$$\mathcal{N}_2=Gg_0^{(z)}\otimes g_2(r)-Gg_0^{(z)}\otimes\big(g_1^{(r)T}+g_1^{(r)}\big)+\big(K+\frac{7G}{3}\big)g_0^{(z)}\otimes g_0\big(\frac{1}{r}\big)$$
$$+Gg_2^{(z)}\otimes g_0(r)+K_F R_F g_F$$

$$\mathcal{N}_3 = Gg_0^{(z)} \otimes g_2(r) - Gg_0^{(z)} \otimes \big(g_1^{(r)T} + g_1^{(r)}\big) + \big(4K + \frac{19G}{3}\big)g_0^{(z)} \otimes g_0\big(\frac{1}{r}\big)$$
$$+ Gg_2^{(z)} \otimes g_0(r) + K_F R_F g_F$$

$$\mathcal{L}_0 = Gg_0^{(z)} \otimes g_2(r) + \big(K + \frac{4G}{3}\big)g_2^{(z)} \otimes g_0(r) + K_F R_F g_F$$

$$\mathcal{L}_1 = -Gg_1^{(z)T} \otimes g_0^{(r)} + \big(K - \frac{2G}{3}\big)g_1^{(z)} \otimes g_0^{(r)}$$

$$\mathcal{L}_2 = Gg_0^{(z)} \otimes g_2(r) + Gg_0^{(z)} \otimes g_0\big(\frac{1}{r}\big) + \big(K + \frac{4G}{3}\big)g_2^{(z)} \otimes g_0(r) + K_F R_F g_F$$

$$\mathcal{L}_3 = Gg_0^{(z)} \otimes g_2(r) + 4Gg_0^{(z)} \otimes g_0\big(\frac{1}{r}\big) + \big(K + \frac{4G}{3}\big)g_2^{(z)} \otimes g_0(r) + K_F R_F g_F \tag{4.2.110}$$

$M_r M_z \times 3M_{Dr} M_{Dz}$ matrices:

$$\mathcal{Z}_0 = \int_{z_T}^{z_B} \int_{R_B}^{R_F} \big[\hat{\mathrm{b}}(z) \otimes \big(r\,\bar{\mathrm{b}}_r(r) + \bar{\mathrm{b}}(r)\big)\big]\vec{b}_1^T \big(\hat{b}(z)^T \otimes B(r)\big)\,\mathrm{d}r\,\mathrm{d}z$$
$$= \Big[\int_{z_T}^{z_B} \hat{\mathrm{b}}(z)\hat{b}(z)^T\,\mathrm{d}z\Big] \otimes \Big[\int_{R_B}^{R_F} \big(r\,\bar{\mathrm{b}}_r(r) + \bar{\mathrm{b}}(r)\big)\vec{b}_1^T B(r)\,\mathrm{d}r\Big] \tag{4.2.111}$$

$$\mathcal{Z}_1 = \int_{z_T}^{z_B} \int_{R_B}^{R_F} \big[\hat{\mathrm{b}}(z) \otimes \bar{\mathrm{b}}(r)\big]\vec{b}_1^T \big(\hat{b}(z)^T \otimes B(r)\big)\,\mathrm{d}r\,\mathrm{d}z$$
$$= \Big[\int_{z_T}^{z_B} \hat{\mathrm{b}}(z)\hat{b}(z)^T\,\mathrm{d}z\Big] \otimes \Big[\int_{R_B}^{R_F} \bar{\mathrm{b}}(r)\vec{b}_1^T B(r)\,\mathrm{d}r\Big] \tag{4.2.112}$$

$$\mathcal{Z}_2 = \int_{z_T}^{z_B} \int_{R_B}^{R_F} \big[\hat{\mathrm{b}}_z(z) \otimes r\,\bar{\mathrm{b}}(r)\big]\vec{b}_1^T \big(\hat{b}(z)^T \otimes B(r)\big)\,\mathrm{d}r\,\mathrm{d}z$$
$$= \Big[\int_{z_T}^{z_B} \hat{\mathrm{b}}_z(z)\hat{b}(z)^T\,\mathrm{d}z\Big] \otimes \Big[\int_{R_B}^{R_F} r\,\bar{\mathrm{b}}(r)\vec{b}_1^T B(r)\,\mathrm{d}r\Big] \tag{4.2.113}$$

$M_r M_z \times 3M_{Dz}$ matrices:

$$Y_{B0} = \int_{z_T}^{z_B} \int_{R_B}^{R_F} \left[\hat{\mathrm{b}}(z) \otimes \left(r\,\bar{\mathrm{b}}_r(r) + \bar{\mathrm{b}}(r) \right) \right] \frac{R_F - r}{R_F - R_B} \vec{b}_1^T$$

$$\cdot \left[\hat{B}(z) - \frac{z - z_T}{z_B - z_T} \hat{B}(z_B) - \frac{z_B - z}{z_B - z_T} \hat{B}(z_T) \right] \mathrm{d}r\,\mathrm{d}z$$

$$= \left[\int_{z_T}^{z_B} \hat{\mathrm{b}}(z) \vec{b}_1^T \left(\hat{B}(z) - \frac{z - z_T}{z_B - z_T} \hat{B}(z_B) - \frac{z_B - z}{z_B - z_T} \hat{B}(z_T) \right) \mathrm{d}z \right]$$

$$\otimes \left[\int_{R_B}^{R_F} \left(r\,\bar{\mathrm{b}}_r(r) + \bar{\mathrm{b}}(r) \right) \frac{R_F - r}{R_F - R_B} \,\mathrm{d}r \right] \tag{4.2.114}$$

Similarly,

$$Y_{F0} = \left[\int_{z_T}^{z_B} \hat{\mathrm{b}}(z) \vec{b}_1^T \left(\hat{B}(z) - \frac{z - z_T}{z_B - z_T} \hat{B}(z_B) - \frac{z_B - z}{z_B - z_T} \hat{B}(z_T) \right) \mathrm{d}z \right]$$

$$\otimes \left[\int_{R_B}^{R_F} \left(r\,\bar{\mathrm{b}}_r(r) + \bar{\mathrm{b}}(r) \right) \frac{r - R_B}{R_F - R_B} \,\mathrm{d}r \right] \tag{4.2.115}$$

$$Y_{B1} = \left[\int_{z_T}^{z_B} \hat{\mathrm{b}}(z) \vec{b}_1^T \left(\hat{B}(z) - \frac{z - z_T}{z_B - z_T} \hat{B}(z_B) - \frac{z_B - z}{z_B - z_T} \hat{B}(z_T) \right) \mathrm{d}z \right]$$

$$\otimes \left[\int_{R_B}^{R_F} \bar{\mathrm{b}}(r) \frac{R_F - r}{R_F - R_B} \,\mathrm{d}r \right] \tag{4.2.116}$$

$$Y_{F1} = \left[\int_{z_T}^{z_B} \hat{\mathrm{b}}(z) \vec{b}_1^T \left(\hat{B}(z) - \frac{z - z_T}{z_B - z_T} \hat{B}(z_B) - \frac{z_B - z}{z_B - z_T} \hat{B}(z_T) \right) \mathrm{d}z \right]$$

$$\otimes \left[\int_{R_B}^{R_F} \bar{\mathrm{b}}(r) \frac{r - R_B}{R_F - R_B} \,\mathrm{d}r \right] \tag{4.2.117}$$

$$Y_{B2} = \left[\int_{z_T}^{z_B} \hat{\mathrm{b}}_z(z) \vec{b}_1^T \left(\hat{B}(z) - \frac{z - z_T}{z_B - z_T} \hat{B}(z_B) - \frac{z_B - z}{z_B - z_T} \hat{B}(z_T) \right) \mathrm{d}z \right]$$

$$\otimes \left[\int_{R_B}^{R_F} r\,\bar{\mathrm{b}}(r) \frac{R_F - r}{R_F - R_B} \,\mathrm{d}r \right] \tag{4.2.118}$$

$$Y_{F2} = \left[\int_{z_T}^{z_B} \hat{\mathrm{b}}_z(z) \vec{b}_1^T \Big(\hat{B}(z) - \frac{z - z_T}{z_B - z_T} \hat{B}(z_B) - \frac{z_B - z}{z_B - z_T} \hat{B}(z_T) \Big) \, \mathrm{d}z \right]$$

$$\otimes \left[\int_{R_B}^{R_F} r \, \bar{\mathrm{b}}(r) \frac{r - R_B}{R_F - R_B} \, \mathrm{d}r \right] \tag{4.2.119}$$

$$\begin{aligned} S_0 &= R_B \int_{z_T}^{z_B} \big(\hat{\mathrm{b}}(z) \otimes \bar{\mathrm{b}}(R_B) P_3^T \hat{B}(z) \, \mathrm{d}z \\ &= R_B \left[\int_{z_T}^{z_B} \hat{\mathrm{b}}(z) P_3^T \hat{B}(z) \, \mathrm{d}z \right] \otimes \bar{\mathrm{b}}(R_B) \end{aligned} \tag{4.2.120}$$

$M_r M_z \times 3M_{Dr}$ matrices:

$$\begin{aligned} \tilde{Y}_{B0} &= \int_{z_T}^{z_B} \int_{R_B}^{R_F} \big[\hat{\mathrm{b}}(z) \otimes \big(r \, \bar{\mathrm{b}}_r(r) + \bar{\mathrm{b}}(r) \big) \big] \frac{z - z_T}{z_B - z_T} \vec{b}_1^T B(r) \, \mathrm{d}r \, \mathrm{d}z \\ &= \left[\int_{z_T}^{z_B} \hat{\mathrm{b}}(z) \frac{z - z_T}{z_B - z_T} \, \mathrm{d}z \right] \otimes \left[\int_{R_B}^{R_F} \big(r \, \bar{\mathrm{b}}_r(r) + \bar{\mathrm{b}}(r) \big) \vec{b}_1^T B(r) \, \mathrm{d}r \right] \end{aligned} \tag{4.2.121}$$

Similarly,

$$\tilde{Y}_{T0} = \left[\int_{z_T}^{z_B} \hat{\mathrm{b}}(z) \frac{z_B - z}{z_B - z_T} \, \mathrm{d}z \right] \otimes \left[\int_{R_B}^{R_F} \big(r \, \bar{\mathrm{b}}_r(r) + \bar{\mathrm{b}}(r) \big) \vec{b}_1^T B(r) \, \mathrm{d}r \right] \tag{4.2.122}$$

$$\tilde{Y}_{B1} = \left[\int_{z_T}^{z_B} \hat{\mathrm{b}}(z) \frac{z - z_T}{z_B - z_T} \, \mathrm{d}z \right] \otimes \left[\int_{R_B}^{R_F} \bar{\mathrm{b}}(r) \vec{b}_1^T B(r) \, \mathrm{d}r \right] \tag{4.2.123}$$

$$\tilde{Y}_{T1} = \left[\int_{z_T}^{z_B} \hat{\mathrm{b}}(z) \frac{z_B - z}{z_B - z_T} \, \mathrm{d}z \right] \otimes \left[\int_{R_B}^{R_F} \bar{\mathrm{b}}(r) \vec{b}_1^T B(r) \, \mathrm{d}r \right] \tag{4.2.124}$$

$$\tilde{Y}_{B2} = \left[\int_{z_T}^{z_B} \hat{\mathrm{b}}_z(z) \frac{z - z_T}{z_B - z_T} \, \mathrm{d}z \right] \otimes \left[\int_{R_B}^{R_F} r \, \bar{\mathrm{b}}_r(r) \vec{b}_1^T B(r) \, \mathrm{d}r \right] \tag{4.2.125}$$

$$\tilde{Y}_{T2} = \left[\int_{z_T}^{z_B} \hat{\mathrm{b}}_z(z) \frac{z_B - z}{z_B - z_T} \, \mathrm{d}z \right] \otimes \left[\int_{R_B}^{R_F} r \, \bar{\mathrm{b}}_r(r) \vec{b}_1^T B(r) \, \mathrm{d}r \right] \tag{4.2.126}$$

$M_r M_z \times M_{Dz}$ matrices:

$$S_{RF} = R_F g_0^{(z)} \otimes \bar{\mathrm{b}}(R_F) \tag{4.2.127}$$

$$S_1 = \Big(\hat{\mathrm{b}}(z_B)\,\hat{\mathrm{b}}(z_B)^T - \hat{\mathrm{b}}(z_T)\,\hat{\mathrm{b}}(z_T)^T \Big) \otimes \Big[\int_{R_B}^{R_F} \big(1 - \frac{R_B^2}{r^2}\big) r\,\bar{\mathrm{b}}(r)\,\mathrm{d}r \Big] \tag{4.2.128}$$

$$S_2 = -\Big(\hat{\mathrm{b}}(z_B)\,\hat{\mathrm{b}}(z_B)^T - \hat{\mathrm{b}}(z_T)\,\hat{\mathrm{b}}(z_T)^T \Big) \otimes \Big[\int_{R_B}^{R_F} \big(1 + \frac{R_B^2}{r^2}\big) r\,\bar{\mathrm{b}}(r)\,\mathrm{d}r \Big] \tag{4.2.129}$$

$$S_3 = \Big(\hat{\mathrm{b}}(z_B)\,\hat{\mathrm{b}}(z_B)^T - \hat{\mathrm{b}}(z_T)\,\hat{\mathrm{b}}(z_T)^T \Big) \otimes \Big[\int_{R_B}^{R_F} r\,\bar{\mathrm{b}}(r)\,\mathrm{d}r \Big] \tag{4.2.130}$$

$$S_4 = -4\nu R_B^2 \Big(\hat{\mathrm{b}}(z_B)\,\hat{\mathrm{b}}(z_B)^T - \hat{\mathrm{b}}(z_T)\,\hat{\mathrm{b}}(z_T)^T \Big) \otimes \Big[\int_{R_B}^{R_F} r\,\bar{\mathrm{b}}(r)\,\mathrm{d}r \Big] \tag{4.2.131}$$

Applying (4.2.110)-(4.2.131) to (4.2.96)-(4.2.109), we obtain the following matrix equations:

$$\begin{bmatrix} \mathcal{M}_0 & \mathcal{M}_3 \\ \mathcal{M}_3^T & \mathcal{L}_0 \end{bmatrix} \begin{bmatrix} \bar{\mu}_0(t) \\ \bar{\lambda}_0(t) \end{bmatrix} = \begin{bmatrix} \mathcal{Z}_0 \\ \mathcal{Z}_2 \end{bmatrix} \bar{c}_0(t) + \begin{bmatrix} S_{RF} \\ 0 \end{bmatrix} \bar{\sigma}_R + \begin{bmatrix} 0 \\ S_3 \end{bmatrix} \bar{\sigma}_{zz}$$

$$+ \begin{bmatrix} Y_{B0} - S_0 \\ Y_{B2} \end{bmatrix} \bar{\nu}_{B0}(t) + \begin{bmatrix} Y_{F0} \\ Y_{F2} \end{bmatrix} \bar{\nu}_{F0} + \begin{bmatrix} \tilde{Y}_{B0} \\ \tilde{Y}_{B2} \end{bmatrix} \bar{\nu}_{\tilde{B}0}(t) + \begin{bmatrix} \tilde{Y}_{T0} \\ \tilde{Y}_{T2} \end{bmatrix} \bar{\nu}_{\tilde{T}0}(t) \tag{4.2.132}$$

$$\begin{bmatrix} \mathcal{M}_1 & -\mathcal{N}_1 & \mathcal{M}_3 \\ -\mathcal{N}_1^T & \mathcal{N}_2 & -\mathcal{L}_1 \\ \mathcal{M}_3^T & -\mathcal{L}_1^T & \mathcal{L}_2 \end{bmatrix} \begin{bmatrix} \bar{\mu}_1(t) \\ \bar{\eta}_2(t) \\ \bar{\lambda}_1(t) \end{bmatrix} = \begin{bmatrix} \mathcal{Z}_0 \\ -\mathcal{Z}_1 \\ \mathcal{Z}_2 \end{bmatrix} \bar{c}_1(t) + \begin{bmatrix} S_1 \\ -S_2 \\ S_{RF} \end{bmatrix} \bar{\sigma}_{yz}$$

$$+ \begin{bmatrix} Y_{F0} \\ -Y_{F1} \\ Y_{F2} \end{bmatrix} \bar{\nu}_{F1} + \begin{bmatrix} \tilde{Y}_{B0} \\ -\tilde{Y}_{B1} \\ \tilde{Y}_{B2} \end{bmatrix} \bar{\nu}_{\tilde{B}1}(t) + \begin{bmatrix} \tilde{Y}_{T0} \\ -\tilde{Y}_{T1} \\ \tilde{Y}_{T2} \end{bmatrix} \bar{\nu}_{\tilde{T}1}(t) \tag{4.2.133}$$

$$\begin{bmatrix} \mathcal{M}_1 & \mathcal{N}_1 & \mathcal{M}_3 \\ \mathcal{N}_1^T & \mathcal{N}_2 & \mathcal{L}_1 \\ \mathcal{M}_3^T & \mathcal{L}_1^T & \mathcal{L}_2 \end{bmatrix} \begin{bmatrix} \bar{\mu}_2(t) \\ \bar{\eta}_1(t) \\ \bar{\lambda}_2(t) \end{bmatrix} = \begin{bmatrix} \mathcal{Z}_0 \\ \mathcal{Z}_1 \\ \mathcal{Z}_2 \end{bmatrix} \bar{c}_2(t) + \begin{bmatrix} S_1 \\ S_2 \\ S_{RF} \end{bmatrix} \bar{\sigma}_{xz}$$

$$+ \begin{bmatrix} Y_{F0} \\ Y_{F1} \\ Y_{F2} \end{bmatrix} \bar{\nu}_{F2} + \begin{bmatrix} \tilde{Y}_{B0} \\ \tilde{Y}_{B1} \\ \tilde{Y}_{B2} \end{bmatrix} \bar{\nu}_{\tilde{B}2}(t) + \begin{bmatrix} \tilde{Y}_{T0} \\ \tilde{Y}_{T1} \\ \tilde{Y}_{T2} \end{bmatrix} \bar{\nu}_{\tilde{T}2}(t) \tag{4.2.134}$$

$$\begin{bmatrix} \mathcal{M}_2 & -2\mathcal{N}_1 & \mathcal{M}_3 \\ -2\mathcal{N}_1^T & \mathcal{N}_3 & -2\mathcal{L}_1 \\ \mathcal{M}_3^T & -2\mathcal{L}_1^T & \mathcal{L}_3 \end{bmatrix} \begin{bmatrix} \bar{\mu}_3(t) \\ \bar{\eta}_4(t) \\ \bar{\lambda}_3(t) \end{bmatrix} = \begin{bmatrix} \mathcal{Z}_0 \\ -2\mathcal{Z}_1 \\ \mathcal{Z}_2 \end{bmatrix} \bar{c}_3(t) + \begin{bmatrix} S_{RF} \\ S_{RF} \\ S_4 \end{bmatrix} \bar{\sigma}_{xy}$$

$$+ \begin{bmatrix} Y_{F0} \\ -2Y_{F1} \\ Y_{F2} \end{bmatrix} \bar{\nu}_{F3} + \begin{bmatrix} \tilde{Y}_{B0} \\ -2\tilde{Y}_{B1} \\ \tilde{Y}_{B2} \end{bmatrix} \bar{\nu}_{\tilde{B}3}(t) + \begin{bmatrix} \tilde{Y}_{T0} \\ -2\tilde{Y}_{T1} \\ \tilde{Y}_{T2} \end{bmatrix} \bar{\nu}_{\tilde{T}3}(t) \tag{4.2.135}$$

$$\begin{bmatrix} \mathcal{M}_2 & 2\mathcal{N}_1 & \mathcal{M}_3 \\ 2\mathcal{N}_1^T & \mathcal{N}_3 & 2\mathcal{L}_1 \\ \mathcal{M}_3^T & 2\mathcal{L}_1^T & \mathcal{L}_3 \end{bmatrix} \begin{bmatrix} \bar{\mu}_4(t) \\ \bar{\eta}_3(t) \\ \bar{\lambda}_4(t) \end{bmatrix} = \begin{bmatrix} \mathcal{Z}_0 \\ 2\mathcal{Z}_1 \\ \mathcal{Z}_2 \end{bmatrix} \bar{c}_4(t) + \begin{bmatrix} S_{RF} \\ S_{RF} \\ S_4 \end{bmatrix} \bar{\sigma}_D$$

$$+ \begin{bmatrix} Y_{F0} \\ 2Y_{F1} \\ Y_{F2} \end{bmatrix} \bar{\nu}_{F4} + \begin{bmatrix} \tilde{Y}_{B0} \\ 2\tilde{Y}_{B1} \\ \tilde{Y}_{B2} \end{bmatrix} \bar{\nu}_{\tilde{B}4}(t) + \begin{bmatrix} \tilde{Y}_{T0} \\ 2\tilde{Y}_{T1} \\ \tilde{Y}_{T2} \end{bmatrix} \bar{\nu}_{\tilde{T}4}(t) \tag{4.2.136}$$

Define the following matrices:

$$\mathcal{Q}_0 = \begin{bmatrix} \mathcal{M}_0 & \mathcal{M}_3 \\ \mathcal{M}_3^T & \mathcal{L}_0 \end{bmatrix}$$

$$J = \begin{bmatrix} I_{M_r M_z} & 0 & 0 \\ 0 & -I_{M_r M_z} & 0 \\ 0 & 0 & I_{M_r M_z} \end{bmatrix}$$

$$\mathcal{Q}_1 = \begin{bmatrix} \mathcal{M}_1 & \mathcal{N}_1 & \mathcal{M}_3 \\ \mathcal{N}_1^T & \mathcal{N}_2 & \mathcal{L}_1 \\ \mathcal{M}_3^T & \mathcal{L}_1^T & \mathcal{L}_2 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} \mathcal{M}_1 & -\mathcal{N}_1 & \mathcal{M}_3 \\ -\mathcal{N}_1^T & \mathcal{N}_2 & -\mathcal{L}_1 \\ \mathcal{M}_3^T & -\mathcal{L}_1^T & \mathcal{L}_2 \end{bmatrix} = J\mathcal{Q}_1 J$$

$$\mathcal{Q}_2 = \begin{bmatrix} \mathcal{M}_2 & 2\mathcal{N}_1 & \mathcal{M}_3 \\ 2\mathcal{N}_1^T & \mathcal{N}_3 & 2\mathcal{L}_1 \\ \mathcal{M}_3^T & 2\mathcal{L}_1^T & \mathcal{L}_3 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} \mathcal{M}_2 & -2\mathcal{N}_1 & \mathcal{M}_3 \\ -2\mathcal{N}_1^T & \mathcal{N}_3 & -2\mathcal{L}_1 \\ \mathcal{M}_3^T & -2\mathcal{L}_1^T & \mathcal{L}_3 \end{bmatrix} = J\mathcal{Q}_2 J$$

$$\bar{\mathcal{Z}}_0 = \begin{bmatrix} \mathcal{Z}_0 \\ \mathcal{Z}_2 \end{bmatrix}, \qquad \bar{\mathcal{Z}}_1 = \begin{bmatrix} \mathcal{Z}_0 \\ \mathcal{Z}_1 \\ \mathcal{Z}_2 \end{bmatrix}, \qquad \bar{\mathcal{Z}}_2 = \begin{bmatrix} \mathcal{Z}_0 \\ 2\mathcal{Z}_1 \\ \mathcal{Z}_2 \end{bmatrix} \qquad \Rightarrow$$

$$\begin{bmatrix} \mathcal{Z}_0 \\ -\mathcal{Z}_1 \\ \mathcal{Z}_2 \end{bmatrix} = J\bar{\mathcal{Z}}_1, \qquad \begin{bmatrix} \mathcal{Z}_0 \\ -2\mathcal{Z}_1 \\ \mathcal{Z}_2 \end{bmatrix} = J\bar{\mathcal{Z}}_2$$

$$X_{B0} = \begin{bmatrix} Y_{B0} - S_0 \\ Y_{B2} \end{bmatrix}, \quad X_{F0} = \begin{bmatrix} Y_{F0} \\ Y_{F2} \end{bmatrix}, \quad X_{F1} = \begin{bmatrix} Y_{F0} \\ Y_{F1} \\ Y_{F2} \end{bmatrix}, \quad X_{F2} = \begin{bmatrix} Y_{F0} \\ 2Y_{F1} \\ Y_{F2} \end{bmatrix} \quad \Rightarrow$$

$$\begin{bmatrix} Y_{F0} \\ -Y_{F1} \\ Y_{F2} \end{bmatrix} = JX_{F1}, \qquad \begin{bmatrix} Y_{F0} \\ -2Y_{F1} \\ Y_{F2} \end{bmatrix} = JX_{F2}$$

$$\tilde{X}_{B0} = \begin{bmatrix} \tilde{Y}_{B0} \\ \tilde{Y}_{B2} \end{bmatrix}, \qquad \tilde{X}_{B1} = \begin{bmatrix} \tilde{Y}_{B0} \\ \tilde{Y}_{B1} \\ \tilde{Y}_{B2} \end{bmatrix}, \qquad \tilde{X}_{B2} = \begin{bmatrix} \tilde{Y}_{B0} \\ 2\tilde{Y}_{B1} \\ \tilde{Y}_{B2} \end{bmatrix} \qquad \Rightarrow$$

$$\begin{bmatrix} \tilde{Y}_{B0} \\ -\tilde{Y}_{B1} \\ \tilde{Y}_{B2} \end{bmatrix} = J\tilde{X}_{B1}, \qquad \begin{bmatrix} \tilde{Y}_{B0} \\ -2\tilde{Y}_{B1} \\ \tilde{Y}_{B2} \end{bmatrix} = J\tilde{X}_{B2}$$

$$\tilde{X}_{T0} = \begin{bmatrix} \tilde{Y}_{T0} \\ \tilde{Y}_{T2} \end{bmatrix}, \qquad \tilde{X}_{T1} = \begin{bmatrix} \tilde{Y}_{T0} \\ \tilde{Y}_{T1} \\ \tilde{Y}_{T2} \end{bmatrix}, \qquad \tilde{X}_{T2} = \begin{bmatrix} \tilde{Y}_{T0} \\ 2\tilde{Y}_{T1} \\ \tilde{Y}_{T2} \end{bmatrix} \qquad \Rightarrow$$

$$\begin{bmatrix} \tilde{Y}_{T0} \\ -\tilde{Y}_{T1} \\ \tilde{Y}_{T2} \end{bmatrix} = J\tilde{X}_{T1}, \qquad \begin{bmatrix} \tilde{Y}_{T0} \\ -2\tilde{Y}_{T1} \\ \tilde{Y}_{T2} \end{bmatrix} = J\tilde{X}_{T2}$$

$$\bar{S}_0 = \begin{bmatrix} S_{RF} \\ 0 \end{bmatrix}, \quad \bar{S}_1 = \begin{bmatrix} S_1 \\ S_2 \\ S_{RF} \end{bmatrix} \Rightarrow \begin{bmatrix} S_1 \\ -S_2 \\ S_{RF} \end{bmatrix} = J\bar{S}_1, \quad \bar{S}_2 = \begin{bmatrix} S_{RF} \\ S_{RF} \\ S_4 \end{bmatrix}, \quad \bar{S}_3 = \begin{bmatrix} 0 \\ S_3 \end{bmatrix}$$

$$P_{100} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \otimes I_{M_rM_z \times M_rM_z}$$

$$P_{010} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \otimes I_{M_rM_z \times M_rM_z}$$

$$P_{001} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \otimes I_{M_rM_z \times M_rM_z}$$

$$P_{10} = \begin{bmatrix} 1 & 0 \end{bmatrix} \otimes I_{M_rM_z \times M_rM_z}$$

$$P_{01} = \begin{bmatrix} 0 & 1 \end{bmatrix} \otimes I_{M_rM_z \times M_rM_z}$$

From (4.2.132)-(4.2.136) we have

$$\mathcal{Q}_0 \begin{bmatrix} \bar{\mu}_0(t) \\ \bar{\lambda}_0(t) \end{bmatrix} = \bar{\mathcal{Z}}_0\bar{c}_0(t) + \bar{S}_0\bar{\sigma}_R + \bar{S}_3\bar{\sigma}_{zz} + X_{B0}\bar{\nu}_{B0}(t) + X_{F0}\bar{\nu}_{F0} + \tilde{X}_{B0}\bar{\nu}_{\tilde{B}0}(t) + \tilde{X}_{T0}\bar{\nu}_{\tilde{T}0}(t) \qquad (4.2.137)$$

$$J\mathcal{Q}_1 J \begin{bmatrix} \bar{\mu}_1(t) \\ \bar{\eta}_2(t) \\ \bar{\lambda}_1(t) \end{bmatrix} = J\bar{\mathcal{Z}}_1\bar{c}_1(t) + J\bar{S}_1\bar{\sigma}_{yz} + JX_{F1}\bar{\nu}_{F1} + J\tilde{X}_{B1}\bar{\nu}_{\tilde{B}1}(t) + J\tilde{X}_{T1}\bar{\nu}_{\tilde{T}1}(t) \tag{4.2.138}$$

$$\mathcal{Q}_1 \begin{bmatrix} \bar{\mu}_2(t) \\ \bar{\eta}_1(t) \\ \bar{\lambda}_2(t) \end{bmatrix} = \bar{\mathcal{Z}}_1\bar{c}_2(t) + \bar{S}_1\bar{\sigma}_{xz} + X_{F1}\bar{\nu}_{F2} + \tilde{X}_{B1}\bar{\nu}_{\tilde{B}2}(t) + \tilde{X}_{T1}\bar{\nu}_{\tilde{T}2}(t) \tag{4.2.139}$$

$$J\mathcal{Q}_2 J \begin{bmatrix} \bar{\mu}_3(t) \\ \bar{\eta}_4(t) \\ \bar{\lambda}_3(t) \end{bmatrix} = J\bar{\mathcal{Z}}_2\bar{c}_3(t) + JJ\bar{S}_2\bar{\sigma}_{xy} + JX_{F2}\bar{\nu}_{F3} + J\tilde{X}_{B2}\bar{\nu}_{\tilde{B}3}(t) + J\tilde{X}_{T2}\bar{\nu}_{\tilde{T}3}(t) \tag{4.2.140}$$

$$\mathcal{Q}_2 \begin{bmatrix} \bar{\mu}_4(t) \\ \bar{\eta}_3(t) \\ \bar{\lambda}_4(t) \end{bmatrix} = \bar{\mathcal{Z}}_2\bar{c}_4(t) + \bar{S}_2\bar{\sigma}_{D} + X_{F2}\bar{\nu}_{F4} + \tilde{X}_{B2}\bar{\nu}_{\tilde{B}4}(t) + \tilde{X}_{T2}\bar{\nu}_{\tilde{T}4}(t) \tag{4.2.141}$$

From (4.2.137)-(4.2.141) we obtain

$$\bar{\mu}_0(t) = P_{10}\mathcal{Q}_0^{-1}\Big[\bar{\mathcal{Z}}_0\bar{c}_0(t) + \bar{S}_0\bar{\sigma}_R + \bar{S}_3\bar{\sigma}_{zz} + X_{B0}\bar{\nu}_{B0}(t) + X_{F0}\bar{\nu}_{F0} + \tilde{X}_{B0}\bar{\nu}_{\tilde{B}0}(t) + \tilde{X}_{T0}\bar{\nu}_{\tilde{T}0}(t)\Big] \tag{4.2.142}$$

$$\bar{\lambda}_0(t) = P_{01}\mathcal{Q}_0^{-1}\Big[\bar{\mathcal{Z}}_0\bar{c}_0(t) + \bar{S}_0\bar{\sigma}_R + \bar{S}_3\bar{\sigma}_{zz} + X_{B0}\bar{\nu}_{B0}(t) + X_{F0}\bar{\nu}_{F0} + \tilde{X}_{B0}\bar{\nu}_{\tilde{B}0}(t) + \tilde{X}_{T0}\bar{\nu}_{\tilde{T}0}(t)\Big] \tag{4.2.143}$$

$$\bar{\mu}_1(t) = P_{100}\mathcal{Q}_1^{-1}\Big[\bar{\mathcal{Z}}_1\bar{c}_1(t) + \bar{S}_1\bar{\sigma}_{yz} + X_{F1}\bar{\nu}_{F1} + \tilde{X}_{B1}\bar{\nu}_{\tilde{B}1}(t) + \tilde{X}_{T1}\bar{\nu}_{\tilde{T}1}(t)\Big] \tag{4.2.144}$$

$$\bar{\eta}_2(t) = -P_{010}\mathcal{Q}_1^{-1}\Big[\bar{\mathcal{Z}}_1\bar{c}_1(t) + \bar{S}_1\bar{\sigma}_{yz} + X_{F1}\bar{\nu}_{F1} + \tilde{X}_{B1}\bar{\nu}_{\tilde{B}1}(t) + \tilde{X}_{T1}\bar{\nu}_{\tilde{T}1}(t)\Big] \tag{4.2.145}$$

$$\bar{\lambda}_1(t) = P_{001}\mathcal{Q}_1^{-1}\Big[\bar{\mathcal{Z}}_1\bar{c}_1(t) + \bar{S}_1\bar{\sigma}_{yz} + X_{F1}\bar{\nu}_{F1} + \tilde{X}_{B1}\bar{\nu}_{\tilde{B}1}(t) + \tilde{X}_{T1}\bar{\nu}_{\tilde{T}1}(t)\Big] \tag{4.2.146}$$

$$\bar{\mu}_2(t) = P_{100}\mathcal{Q}_1^{-1}\Big[\bar{\mathcal{Z}}_1\bar{c}_2(t) + \bar{S}_1\bar{\sigma}_{xz} + X_{F1}\bar{\nu}_{F2} + \tilde{X}_{B1}\bar{\nu}_{\tilde{B}2}(t) + \tilde{X}_{T1}\bar{\nu}_{\tilde{T}2}(t)\Big] \tag{4.2.147}$$

$$\bar{\eta}_1(t) = P_{010}\mathcal{Q}_1^{-1}\Big[\bar{\mathcal{Z}}_1\bar{c}_2(t) + \bar{S}_1\bar{\sigma}_{xz} + X_{F1}\bar{\nu}_{F2} + \tilde{X}_{B1}\bar{\nu}_{\tilde{B}2}(t) + \tilde{X}_{T1}\bar{\nu}_{\tilde{T}2}(t)\Big] \tag{4.2.148}$$

$$\bar{\lambda}_2(t) = P_{001}\mathcal{Q}_1^{-1}\Big[\bar{\mathcal{Z}}_1\bar{c}_2(t) + \bar{S}_1\bar{\sigma}_{xz} + X_{F1}\bar{\nu}_{F2} + \tilde{X}_{B1}\bar{\nu}_{\tilde{B}2}(t) + \tilde{X}_{T1}\bar{\nu}_{\tilde{T}2}(t)\Big] \tag{4.2.149}$$

$$\bar{\mu}_3(t) = P_{100}\mathcal{Q}_2^{-1}\Big[\bar{\mathcal{Z}}_2\bar{c}_3(t) + J\bar{S}_2\bar{\sigma}_{xy} + X_{F2}\bar{\nu}_{F3} + \tilde{X}_{B2}\bar{\nu}_{\tilde{B}3}(t) + \tilde{X}_{T2}\bar{\nu}_{\tilde{T}3}(t)\Big] \tag{4.2.150}$$

$$\bar{\eta}_4(t) = -P_{010}\mathcal{Q}_2^{-1}\Big[\bar{\mathcal{Z}}_2\bar{c}_3(t) + J\bar{S}_2\bar{\sigma}_{xy} + X_{F2}\bar{\nu}_{F3} + \tilde{X}_{B2}\bar{\nu}_{\tilde{B}3}(t) + \tilde{X}_{T2}\bar{\nu}_{\tilde{T}3}(t)\Big] \tag{4.2.151}$$

$$\bar{\lambda}_3(t) = P_{001}\mathcal{Q}_2^{-1}\Big[\bar{\mathcal{Z}}_2\bar{c}_3(t) + J\bar{S}_2\bar{\sigma}_{xy} + X_{F2}\bar{\nu}_{F3} + \tilde{X}_{B2}\bar{\nu}_{\tilde{B}3}(t) + \tilde{X}_{T2}\bar{\nu}_{\tilde{T}3}(t)\Big] \tag{4.2.152}$$

$$\bar{\mu}_4(t) = P_{100}\mathcal{Q}_2^{-1}\Big[\bar{\mathcal{Z}}_2\bar{c}_4(t) + \bar{S}_2\bar{\sigma}_{D} + X_{F2}\bar{\nu}_{F4} + \tilde{X}_{B2}\bar{\nu}_{\tilde{B}4}(t) + \tilde{X}_{T2}\bar{\nu}_{\tilde{T}4}(t)\Big] \tag{4.2.153}$$

$$\bar{\eta}_3(t) = P_{010}\mathcal{Q}_2^{-1}\Big[\bar{\mathcal{Z}}_2\bar{c}_4(t) + \bar{S}_2\bar{\sigma}_{D} + X_{F2}\bar{\nu}_{F4} + \tilde{X}_{B2}\bar{\nu}_{\tilde{B}4}(t) + \tilde{X}_{T2}\bar{\nu}_{\tilde{T}4}(t)\Big] \tag{4.2.154}$$

$$\bar{\lambda}_4(t) = P_{001}\mathcal{Q}_2^{-1}\Big[\bar{\mathcal{Z}}_2\bar{c}_4(t) + \bar{S}_2\bar{\sigma}_{D} + X_{F2}\bar{\nu}_{F4} + \tilde{X}_{B2}\bar{\nu}_{\tilde{B}4}(t) + \tilde{X}_{T2}\bar{\nu}_{\tilde{T}4}(t)\Big] \tag{4.2.155}$$

Finite element approximation of the TCP system

System I (radially symmetric case)

System I is given by (4.2.28)-(4.2.30):

$$M\dot{V}_0 - A\Big[\frac{1}{r}(rV_{0_r})_r + (V_{0_z})_z\Big] = -M\dot{W}_0 + A\Big(\frac{1}{r}(rW_{0_r})_r + (W_{0_z})_z\Big) - \vec{b}_0\Big(\dot{u}_{0_r} + \frac{1}{r}\dot{u}_0 + \dot{w}_{0_z}\Big) \tag{4.2.28}$$

with boundary conditions

$$V_0(r,z,t) = 0 \qquad \text{on } \Gamma \times [0, t_f) \tag{4.2.29}$$

and initial conditions

$$V_0(r,z,0) = \tilde{V}_{I0}(r,z) - W_0(r,z,0) \tag{4.2.30}$$

Let

$$\mathcal{A} = M^{-1}A$$

Then (4.2.28) has the form

$$\dot{V}_0 - \mathcal{A}\Big[\frac{1}{r}(rV_{0_r})_r + (V_{0_z})_z\Big] = -\dot{W}_0 + \mathcal{A}\Big[\frac{1}{r}(rW_{0_r})_r + (W_{0_z})_z\Big] - M^{-1}\vec{b}_0\Big(\dot{u}_{0_r} + \frac{1}{r}\dot{u}_0 + \dot{w}_{0_z}\Big) \tag{4.2.156}$$

Using (4.2.88), the left-hand side of (4.2.156) is

$$\Big(\hat{b}(z)^T \otimes B(r)\Big)\dot{\bar{c}}_0(t) - \mathcal{A}\frac{1}{r}\Big(\hat{b}(z)^T \otimes rB_r(r)\Big)_r \bar{c}_0(t) - \mathcal{A}\Big(\hat{b}_z(z)^T \otimes B(r)\Big)_z \bar{c}_0(t) \tag{4.2.157}$$

Multiplying (4.2.157) by $\Big(\hat{b}(z)^T \otimes B(r)\Big)^T = \Big(\hat{b}(z) \otimes B(r)^T\Big)$ and integrating with respect to r and z, we have

$$\Big[\int_{z_T}^{z_B}\int_{R_B}^{R_F}\Big(\hat{b}(z) \otimes B(r)^T\Big)\Big(\hat{b}(z)^T \otimes B(r)\Big) r\,\mathrm{d}r\,\mathrm{d}z\Big]\dot{\bar{c}}_0(t)$$

$$-\Big[\int_{z_T}^{z_B}\int_{R_B}^{R_F}\Big(\hat{b}(z)\otimes B(r)^T\Big)\mathcal{A}\frac{1}{r}\Big(\hat{b}(z)^T\otimes rB_r(r)\Big)_r r\,\mathrm{d}r\,\mathrm{d}z\Big]\bar{c}_0(t)$$

$$-\Big[\int_{z_T}^{z_B}\int_{R_B}^{R_F}\Big(\hat{b}(z)\otimes B(r)^T\Big)\mathcal{A}\Big(\hat{b}_z(z)^T\otimes B(r)\Big)_z r\,\mathrm{d}r\,\mathrm{d}z\Big]\bar{c}_0(t)$$

$$=\Big\{\Big[\int_{z_T}^{z_B}\hat{b}(z)\hat{b}(z)^T\,\mathrm{d}z\Big]\otimes\Big[\int_{R_B}^{R_F}B(r)^TB(r)r\,\mathrm{d}r\Big]\Big\}\dot{\bar{c}}_0(t)$$

$$+\Big[\int_{z_T}^{z_B}\int_{R_B}^{R_F}\Big(\hat{b}(z)\otimes B_r(r)^T\Big)\mathcal{A}\Big(\hat{b}(z)^T\otimes rB_r(r)^T\Big)\,\mathrm{d}r\,\mathrm{d}z\Big]\bar{c}_0(t)$$

$$+\Big[\int_{z_T}^{z_B}\int_{R_B}^{R_F}\Big(\hat{b}_z(z)\otimes B(r)^T\Big)\mathcal{A}\Big(\hat{b}_z(z)^T\otimes B_r(r)^T\Big)r\,\mathrm{d}r\,\mathrm{d}z\Big]\bar{c}_0(t)$$

$$=\Big\{\Big[\int_{z_T}^{z_B}\hat{b}(z)\hat{b}(z)^T\,\mathrm{d}z\Big]\otimes\Big[\int_{R_B}^{R_F}B(r)^TB(r)r\,\mathrm{d}r\Big]\Big\}\dot{\bar{c}}_0(t)$$

$$+\Big\{\Big[\int_{z_T}^{z_B}\hat{b}(z)\hat{b}(z)^T\,\mathrm{d}z\Big]\otimes\Big[\int_{R_B}^{R_F}B_r(r)^T\mathcal{A}B_r(r)r\,\mathrm{d}r\Big]\Big\}\bar{c}_0(t)$$

$$+\Big\{\Big[\int_{z_T}^{z_B}\hat{b}_z(z)\hat{b}_z(z)^T\,\mathrm{d}z\Big]\otimes\Big[\int_{R_B}^{R_F}B(r)^T\mathcal{A}B(r)r\,\mathrm{d}r\Big]\Big\}\bar{c}_0(t)$$

$$=\Big\{\Big[\int_{z_T}^{z_B}\hat{b}(z)\hat{b}(z)^T\,\mathrm{d}z\Big]\otimes\Big[\int_{R_B}^{R_F}B(r)^TB(r)r\,\mathrm{d}r\Big]\Big\}\dot{\bar{c}}_0(t)$$

$$+\Big\{\Big[\int_{z_T}^{z_B}\hat{b}(z)\hat{b}(z)^T\,\mathrm{d}z\Big]\otimes\mathcal{A}\otimes\Big[\int_{R_B}^{R_F}\bar{b}_r(r)\bar{b}_r(r)^Tr\,\mathrm{d}r\Big]\Big\}\bar{c}_0(t)$$

$$+\Big\{\Big[\int_{z_T}^{z_B}\hat{b}_z(z)\hat{b}_z(z)^T\,\mathrm{d}z\Big]\otimes\mathcal{A}\otimes\Big[\int_{R_B}^{R_F}\bar{b}(r)\bar{b}(r)^Tr\,\mathrm{d}r\Big]\Big\}\bar{c}_0(t) \qquad (4.2.158)$$

Using (4.2.90)-(4.2.92), the last term of the right-hand side of (4.2.156) is

$$-M^{-1}\vec{b}_0\Big(\dot{u}_{0_r}+\frac{1}{r}\dot{u}_0+\dot{w}_{0_z}\Big)$$

$$=-M^{-1}\vec{b}_0\Big[\Big(\hat{\mathrm{b}}(z)^T\otimes\bar{\mathrm{b}}_r(r)^T\Big)\dot{\mu}_0(t)+\Big(\hat{\mathrm{b}}(z)^T\otimes\frac{1}{r}\bar{\mathrm{b}}(r)^T\Big)\dot{\mu}_0(t)$$

$$+\Big(\hat{\mathrm{b}}_z(z)^T\otimes\bar{\mathrm{b}}(r)^T\Big)\dot{\lambda}_0(t)\Big]$$

Multiplying by $\left(\hat{b}(z) \otimes B(r)^T\right)$ and integrating with respect to r and z, we have

$$
\begin{aligned}
&-\Big[\int_{z_T}^{z_B}\int_{R_B}^{R_F}\left(\hat{b}(z)\otimes B(r)^T\right)M^{-1}\vec{b}_0\left(\hat{\mathrm{b}}(z)^T\otimes\bar{\mathrm{b}}_r(r)^T\right)r\,\mathrm{d}r\,\mathrm{d}z\\
&\quad+\int_{z_T}^{z_B}\int_{R_B}^{R_F}\left(\hat{b}(z)\otimes B(r)^T\right)M^{-1}\vec{b}_0\left(\hat{\mathrm{b}}(z)^T\otimes\frac{1}{r}\bar{\mathrm{b}}(r)^T\right)r\,\mathrm{d}r\,\mathrm{d}z\Big]\dot{\bar{\mu}}_0(t)\\
&-\Big[\int_{z_T}^{z_B}\int_{R_B}^{R_F}\left(\hat{b}(z)\otimes B(r)^T\right)M^{-1}\vec{b}_0\left(\hat{\mathrm{b}}_z(z)^T\otimes\bar{\mathrm{b}}(r)^T\right)r\,\mathrm{d}r\,\mathrm{d}z\Big]\dot{\bar{\lambda}}_0(t)
\end{aligned}
$$

$$
\begin{aligned}
=&-\Big\{\Big[\int_{z_T}^{z_B}\hat{b}(z)\,\hat{\mathrm{b}}(z)^T\,\mathrm{d}z\Big]\otimes\Big[\int_{R_B}^{R_F}B(r)^TM^{-1}\vec{b}_0(r\,\bar{\mathrm{b}}_r(r)+\bar{\mathrm{b}}(r))^T\,\mathrm{d}r\Big]\Big\}\dot{\bar{\mu}}_0(t)\\
&-\Big\{\Big[\int_{z_T}^{z_B}\hat{b}(z)\,\hat{\mathrm{b}}_z(z)^T\,\mathrm{d}z\Big]\otimes\Big[\int_{R_B}^{R_F}B(r)^TM^{-1}\vec{b}_0\,\bar{\mathrm{b}}(r)^Tr\,\mathrm{d}r\Big]\Big\}\dot{\bar{\lambda}}_0(t) \qquad (4.2.159)
\end{aligned}
$$

From (4.2.20) and (4.2.89)

$$
\begin{aligned}
W_i(r,z,t) =& \frac{R_F-r}{R_F-R_B}\hat{B}(z)\bar{\nu}_{Bi}(t)+\frac{r-R_B}{R_F-R_B}\hat{B}(z)\bar{\nu}_{Fi}\\
&+\frac{z-z_T}{z_B-z_T}\Big(B(r)\bar{\nu}_{\tilde{B}i}(t)-\frac{R_F-r}{R_F-R_B}\hat{B}(z_B)\bar{\nu}_{Bi}(t)-\frac{r-R_B}{R_F-R_B}\hat{B}(z_B)\bar{\nu}_{Fi}\Big)\\
&+\frac{z_B-z}{z_B-z_T}\Big(B(r)\bar{\nu}_{\tilde{T}i}(t)-\frac{R_F-r}{R_F-R_B}\hat{B}(z_T)\bar{\nu}_{Bi}(t)-\frac{r-R_B}{R_F-R_B}\hat{B}(z_T)\bar{\nu}_{Fi}\Big)
\end{aligned}
$$

$$
\begin{aligned}
=& \frac{R_F-r}{R_F-R_B}\Big(\hat{B}(z)-\frac{z-z_T}{z_B-z_T}\hat{B}(z_B)-\frac{z_B-z}{z_B-z_T}\hat{B}(z_T)\Big)\bar{\nu}_{Bi}(t)\\
&+\frac{r-R_B}{R_F-R_B}\Big(\hat{B}(z)-\frac{z-z_T}{z_B-z_T}\hat{B}(z_B)-\frac{z_B-z}{z_B-z_T}\hat{B}(z_T)\Big)\bar{\nu}_{Fi}\\
&+\frac{z-z_T}{z_B-z_T}B(r)\bar{\nu}_{\tilde{B}i}(t)+\frac{z_B-z}{z_B-z_T}B(r)\bar{\nu}_{\tilde{T}i}(t), \quad i=0,\dots,4 \qquad (4.2.160)
\end{aligned}
$$

and $\bar{\nu}_{Bi}(t)=0, \quad i=1,\dots,4.$

Multiplying (4.2.160) by $\left(\hat{b}(z)\otimes B(r)^T\right)=\left(\hat{b}(z)\otimes I_{3\times 3}\right)\otimes\bar{b}(r)$ and integrating with respect to r and z, we have

$$
\begin{aligned}
&\Big[\int_{z_T}^{z_B}\int_{R_B}^{R_F}\Big(\hat{b}(z)\otimes B(r)^T\Big)\frac{R_F-r}{R_F-R_B}\Big(\hat{B}(z)-\frac{z-z_T}{z_B-z_T}\hat{B}(z_B) \\
&\qquad -\frac{z_B-z}{z_B-z_T}\hat{B}(z_T)\Big)r\,\mathrm{d}r\,\mathrm{d}z\Big]\bar{\nu}_{Bi}(t) \\
&+\Big[\int_{z_T}^{z_B}\int_{R_B}^{R_F}\Big(\hat{b}(z)\otimes B(r)^T\Big)\frac{r-R_B}{R_F-R_B}\Big(\hat{B}(z)-\frac{z-z_T}{z_B-z_T}\hat{B}(z_B) \\
&\qquad -\frac{z_B-z}{z_B-z_T}\hat{B}(z_T)\Big)r\,\mathrm{d}r\,\mathrm{d}z\Big]\bar{\nu}_{Fi} \\
&+\Big[\int_{z_T}^{z_B}\int_{R_B}^{R_F}\Big(\hat{b}(z)\otimes B(r)^T\Big)\frac{z-z_T}{z_B-z_T}B(r)r\,\mathrm{d}r\,\mathrm{d}z\Big]\bar{\nu}_{\tilde{B}i}(t) \\
&+\Big[\int_{z_T}^{z_B}\int_{R_B}^{R_F}\Big(\hat{b}(z)\otimes B(r)^T\Big)\frac{z_B-z}{z_B-z_T}B(r)r\,\mathrm{d}r\,\mathrm{d}z\Big]\bar{\nu}_{\tilde{T}i}(t)
\end{aligned}
$$

$$
\begin{aligned}
=&\Big[\int_{z_T}^{z_B}\hat{b}(z)\otimes\Big(\hat{B}(z)-\frac{z-z_T}{z_B-z_T}\hat{B}(z_B)-\frac{z_B-z}{z_B-z_T}\hat{B}(z_T)\Big)\,\mathrm{d}z\Big] \\
&\otimes\Big\{\Big[\int_{R_B}^{R_F}\bar{b}(r)\frac{R_F-r}{R_F-R_B}r\,\mathrm{d}r\Big]\bar{\nu}_{Bi}(t)+\Big[\int_{R_B}^{R_F}\bar{b}(r)\frac{r-R_B}{R_F-R_B}r\,\mathrm{d}r\Big]\bar{\nu}_{Fi}\Big\} \\
&+\Big\{\Big[\int_{z_T}^{z_B}\hat{b}(z)\frac{z-z_T}{z_B-z_T}\,\mathrm{d}z\Big]\otimes\Big[\int_{R_B}^{R_F}B(r)^TB(r)r\,\mathrm{d}r\Big]\Big\}\bar{\nu}_{\tilde{B}i}(t) \\
&+\Big\{\Big[\int_{z_T}^{z_B}\hat{b}(z)\frac{z_B-z}{z_B-z_T}\,\mathrm{d}z\Big]\otimes\Big[\int_{R_B}^{R_F}B(r)^TB(r)r\,\mathrm{d}r\Big]\Big\}\bar{\nu}_{\tilde{T}i}(t) \qquad (4.2.161)
\end{aligned}
$$

From (4.2.23) and (4.2.89)

$$
\begin{aligned}
W_{i_r}(r,z,t)=&-\frac{1}{R_F-R_B}\Big(\hat{B}(z)-\frac{z-z_T}{z_B-z_T}\hat{B}(z_B)-\frac{z_B-z}{z_B-z_T}\hat{B}(z_T)\Big)\bar{\nu}_{Bi}(t) \\
&+\frac{1}{R_F-R_B}\Big(\hat{B}(z)-\frac{z-z_T}{z_B-z_T}\hat{B}(z_B)-\frac{z_B-z}{z_B-z_T}\hat{B}(z_T)\Big)\bar{\nu}_{Fi} \\
&+\frac{z-z_T}{z_B-z_T}B_r(r)\bar{\nu}_{\tilde{B}i}(t)+\frac{z_B-z}{z_B-z_T}B_r(r)\bar{\nu}_{\tilde{T}i}(t),\quad i=0,\ldots,4 \qquad (4.2.162)
\end{aligned}
$$

and $\bar{\nu}_{Bi}(t)=0,\quad i=1,\ldots,4.$

Multiplying $\mathcal{A}\frac{1}{r}(rW_{i_r})_r$ by $\Big(\hat{b}(z)\otimes B(r)^T\Big) = \Big(\hat{b}(z)\otimes I_{3\times 3}\Big)\otimes \bar{b}(r)$ and integrating with respect to r and z, we have

$$\Big[\int_{z_T}^{z_B}\int_{R_B}^{R_F}\Big(\hat{b}(z)\otimes B(r)^T\Big)\mathcal{A}\frac{1}{R_F-R_B}\Big(\hat{B}(z)-\frac{z-z_T}{z_B-z_T}\hat{B}(z_B)$$

$$-\frac{z_B-z}{z_B-z_T}\hat{B}(z_T)\Big)\,\mathrm{d}r\,\mathrm{d}z\Big]\big(-\bar{\nu}_{Bi}(t)+\bar{\nu}_{Fi}\big)$$

$$-\Big[\int_{z_T}^{z_B}\int_{R_B}^{R_F}\Big(\hat{b}(z)\otimes B_r(r)^T\Big)\mathcal{A}\frac{z-z_T}{z_B-z_T}B_r(r)r\,\mathrm{d}r\,\mathrm{d}z\Big]\bar{\nu}_{\tilde{B}i}(t)$$

$$-\Big[\int_{z_T}^{z_B}\int_{R_B}^{R_F}\Big(\hat{b}(z)\otimes B_r(r)^T\Big)\mathcal{A}\frac{z_B-z}{z_B-z_T}B_r(r)r\,\mathrm{d}r\,\mathrm{d}z\Big]\bar{\nu}_{\tilde{T}i}(t)$$

$$=\Bigg\{\Big[\int_{z_T}^{z_B}\hat{b}(z)\otimes\Big[\mathcal{A}\Big(\hat{B}(z)-\frac{z-z_T}{z_B-z_T}\hat{B}(z_B)-\frac{z_B-z}{z_B-z_T}\hat{B}(z_T)\Big)\Big]\,\mathrm{d}z\Big]$$

$$\otimes\Big[\int_{R_B}^{R_F}\bar{b}(r)\frac{1}{R_F-R_B}\,\mathrm{d}r\Big]\Bigg\}\big(-\bar{\nu}_{Bi}(t)+\bar{\nu}_{Fi}\big)$$

$$-\Bigg\{\Big[\int_{z_T}^{z_B}\hat{b}(z)\frac{z-z_T}{z_B-z_T}\,\mathrm{d}z\Big]\otimes\mathcal{A}\otimes\Big[\int_{R_B}^{R_F}\bar{b}_r(r)\bar{b}_r(r)^T r\,\mathrm{d}r\Big]\Bigg\}\bar{\nu}_{\tilde{B}i}(t)$$

$$-\Bigg\{\Big[\int_{z_T}^{z_B}\hat{b}(z)\frac{z_B-z}{z_B-z_T}\,\mathrm{d}z\Big]\otimes\mathcal{A}\otimes\Big[\int_{R_B}^{R_F}\bar{b}_r(r)\bar{b}_r(r)^T r\,\mathrm{d}r\Big]\Bigg\}\bar{\nu}_{\tilde{T}i}(t),\quad i=0,\dots,4 \tag{4.2.163}$$

and $\bar{\nu}_{Bi}(t)=0,\quad i=1,\dots,4.$

From (4.2.26) and (4.2.89)

$$W_{i_z}(r,z,t)=\frac{R_F-r}{R_F-R_B}\Big(\hat{B}_z(z)-\frac{1}{z_B-z_T}\hat{B}(z_B)+\frac{1}{z_B-z_T}\hat{B}(z_T)\Big)\bar{\nu}_{Bi}(t)$$

$$+\frac{r-R_B}{R_F-R_B}\Big(\hat{B}_z(z)-\frac{1}{z_B-z_T}\hat{B}(z_B)+\frac{1}{z_B-z_T}\hat{B}(z_T)\Big)\bar{\nu}_{Fi}$$

$$+\frac{1}{z_B-z_T}B(r)\big(\bar{\nu}_{\tilde{B}i}(t)-\bar{\nu}_{\tilde{T}i}(t)\big),\quad i=0,\dots,4 \tag{4.2.164}$$

and $\bar{\nu}_{Bi}(t)=0,\quad i=1,\dots,4.$

Multiplying $\mathcal{A}(W_{i_z})_z$ by $\left(\hat{b}(z) \otimes B(r)^T\right) = \left(\hat{b}(z) \otimes I_{3\times 3}\right) \otimes \bar{b}(r)$ and integrating with respect to r and z, we have

$$
\begin{aligned}
&- \left[\int_{z_T}^{z_B} \int_{R_B}^{R_F} \left(\hat{b}_z(z) \otimes B(r)^T\right) \mathcal{A} \frac{R_F - r}{R_F - R_B} \left(\hat{B}_z(z) - \frac{1}{z_B - z_T} \hat{B}(z_B) \right.\right. \\
&\qquad\qquad \left.\left. + \frac{1}{z_B - z_T} \hat{B}(z_T)\right) r \, \mathrm{d}r \, \mathrm{d}z \right] \bar{\nu}_{Bi}(t) \\
&- \left[\int_{z_T}^{z_B} \int_{R_B}^{R_F} \left(\hat{b}_z(z) \otimes B(r)^T\right) \mathcal{A} \frac{r - R_B}{R_F - R_B} \left(\hat{B}_z(z) - \frac{1}{z_B - z_T} \hat{B}(z_B) \right.\right. \\
&\qquad\qquad \left.\left. + \frac{1}{z_B - z_T} \hat{B}(z_T)\right) r \, \mathrm{d}r \, \mathrm{d}z \right] \bar{\nu}_{Fi} \\
&- \left[\int_{z_T}^{z_B} \int_{R_B}^{R_F} \left(\hat{b}_z(z) \otimes B(r)^T\right) \mathcal{A} \frac{1}{z_B - z_T} B(r) r \, \mathrm{d}r \, \mathrm{d}z \right] \bar{\nu}_{\tilde{B}i}(t) \\
&+ \left[\int_{z_T}^{z_B} \int_{R_B}^{R_F} \left(\hat{b}_z(z) \otimes B(r)^T\right) \mathcal{A} \frac{1}{z_B - z_T} B_r(r) r \, \mathrm{d}r \, \mathrm{d}z \right] \bar{\nu}_{\tilde{T}i}(t) \\
\\
&= - \left[\int_{z_T}^{z_B} \hat{b}_z(z) \otimes \left[\mathcal{A} \left(\hat{B}_z(z) - \frac{1}{z_B - z_T} \hat{B}(z_B) + \frac{1}{z_B - z_T} \hat{B}(z_T) \right) \right] \mathrm{d}z \right] \\
&\qquad \otimes \left\{ \left[\int_{R_B}^{R_F} \bar{b}(r) \frac{R_F - r}{R_F - R_B} r \, \mathrm{d}r \right] \bar{\nu}_{Bi}(t) + \left[\int_{R_B}^{R_F} \bar{b}(r) \frac{r - R_B}{R_F - R_B} r \, \mathrm{d}r \right] \bar{\nu}_{Fi} \right\} \\
&\quad - \left\{ \left[\int_{z_T}^{z_B} \hat{b}_z(z) \frac{1}{z_B - z_T} \, \mathrm{d}z \right] \otimes \mathcal{A} \otimes \left[\int_{R_B}^{R_F} \bar{b}(r) \bar{b}(r)^T r \, \mathrm{d}r \right] \right\} \left(\bar{\nu}_{\tilde{B}i}(t) - \bar{\nu}_{\tilde{T}i}(t) \right)
\end{aligned}
\tag{4.2.165}
$$

Define the following matrices:

$3M_{Dr}M_{Dz} \times 3M_{Dr}M_{Dz}$ matrices:

$$G_0 = \left[\int_{z_T}^{z_B} \hat{b}(z) \hat{b}(z)^T \, \mathrm{d}z \right] \otimes \left[\int_{R_B}^{R_F} B(r)^T B(r) r \, \mathrm{d}r \right]$$

$$G_2(r) = \left[\int_{z_T}^{z_B} \hat{b}(z) \hat{b}(z)^T \, \mathrm{d}z \right] \otimes \mathcal{A} \otimes \left[\int_{R_B}^{R_F} \bar{b}_r(r) \bar{b}_r(r)^T r \, \mathrm{d}r \right]$$

$$G_2(z) = \left[\int_{z_T}^{z_B} \hat{b}_z(z) \hat{b}_z(z)^T \, \mathrm{d}z \right] \otimes \mathcal{A} \otimes \left[\int_{R_B}^{R_F} \bar{b}(r) \bar{b}(r)^T r \, \mathrm{d}r \right]$$

$3M_{Dr}M_{Dz} \times M_r M_z$ matrices:

$$H_0 = \Big[\int_{z_T}^{z_B} \hat{b}(z)\, \hat{\mathrm{b}}(z)^T \,\mathrm{d}z \Big] \otimes \Big[\int_{R_B}^{R_F} B(r)^T M^{-1} \vec{b}_0 (r\, \bar{\mathrm{b}}_r(r) + \bar{\mathrm{b}}(r))^T \,\mathrm{d}r \Big]$$

$$H_1 = \Big[\int_{z_T}^{z_B} \hat{b}(z)\, \hat{\mathrm{b}}_z(z)^T \,\mathrm{d}z \Big] \otimes \Big[\int_{R_B}^{R_F} B(r)^T M^{-1} \vec{b}_0\, \bar{\mathrm{b}}(r)^T r \,\mathrm{d}r \Big]$$

$3M_{Dr}M_{Dz} \times 3M_{Dz}$ matrices:

$$\mathcal{R}_B = \int_{z_T}^{z_B} \hat{b}(z) \otimes \Big(\hat{B}(z) - \frac{z - z_T}{z_B - z_T} \hat{B}(z_B) - \frac{z_B - z}{z_B - z_T} \hat{B}(z_T) \Big) \,\mathrm{d}z \Big]$$
$$\otimes \Big[\int_{R_B}^{R_F} \bar{b}(r) \frac{R_F - r}{R_F - R_B} r \,\mathrm{d}r \Big]$$

$$\mathcal{R}_F = \int_{z_T}^{z_B} \hat{b}(z) \otimes \Big(\hat{B}(z) - \frac{z - z_T}{z_B - z_T} \hat{B}(z_B) - \frac{z_B - z}{z_B - z_T} \hat{B}(z_T) \Big) \,\mathrm{d}z \Big]$$
$$\otimes \Big[\int_{R_B}^{R_F} \bar{b}(r) \frac{r - R_B}{R_F - R_B} r \,\mathrm{d}r \Big]$$

$$\mathcal{R}_F(a) = \Big[\int_{z_T}^{z_B} \hat{b}(z) \otimes \Big[\mathcal{A} \Big(\hat{B}(z) - \frac{z - z_T}{z_B - z_T} \hat{B}(z_B) - \frac{z_B - z}{z_B - z_T} \hat{B}(z_T) \Big) \Big] \,\mathrm{d}z \Big]$$
$$\otimes \Big[\int_{R_B}^{R_F} \bar{b}(r) \frac{1}{R_F - R_B} \,\mathrm{d}r \Big]$$

$$\mathcal{R}_B(z) = \Big[\int_{z_T}^{z_B} \hat{b}_z(z) \otimes \Big[\mathcal{A} \Big(\hat{B}_z(z) - \frac{1}{z_B - z_T} \hat{B}(z_B) + \frac{1}{z_B - z_T} \hat{B}(z_T) \Big) \Big] \,\mathrm{d}z \Big]$$
$$\otimes \Big[\int_{R_B}^{R_F} \bar{b}(r) \frac{R_F - r}{R_F - R_B} r \,\mathrm{d}r \Big]$$

$$\mathcal{R}_F(z) = \Big[\int_{z_T}^{z_B} \hat{b}_z(z) \otimes \Big[\mathcal{A} \Big(\hat{B}_z(z) - \frac{1}{z_B - z_T} \hat{B}(z_B) + \frac{1}{z_B - z_T} \hat{B}(z_T) \Big) \Big] \,\mathrm{d}z \Big]$$
$$\otimes \Big[\int_{R_B}^{R_F} \bar{b}(r) \frac{r - R_B}{R_F - R_B} r \,\mathrm{d}r \Big]$$

$3M_{Dr}M_{Dz} \times 3M_{Dr}$ matrices:

$$\tilde{\mathcal{R}}_B = \Big[\int_{z_T}^{z_B} \hat{b}(z)\frac{z - z_T}{z_B - z_T}\,\mathrm{d}z\Big] \otimes \Big[\int_{R_B}^{R_F} B(r)^T B(r) r\,\mathrm{d}r\Big]$$

$$\tilde{\mathcal{R}}_T = \Big[\int_{z_T}^{z_B} \hat{b}(z)\frac{z_B - z}{z_B - z_T}\,\mathrm{d}z\Big] \otimes \Big[\int_{R_B}^{R_F} B(r)^T B(r) r\,\mathrm{d}r\Big]$$

$$\tilde{\mathcal{R}}_B(a) = \Big[\int_{z_T}^{z_B} \hat{b}(z)\frac{z - z_T}{z_B - z_T}\,\mathrm{d}z\Big] \otimes \mathcal{A} \otimes \Big[\int_{R_B}^{R_F} \bar{b}_r(r)\bar{b}_r(r)^T r\,\mathrm{d}r\Big]$$

$$\tilde{\mathcal{R}}_T(a) = \Big[\int_{z_T}^{z_B} \hat{b}(z)\frac{z_B - z}{z_B - z_T}\,\mathrm{d}z\Big] \otimes \mathcal{A} \otimes \Big[\int_{R_B}^{R_F} \bar{b}_r(r)\bar{b}_r(r)^T r\,\mathrm{d}r\Big]$$

$$\tilde{\mathcal{R}}_0(z) = \Big[\int_{z_T}^{z_B} \hat{b}_z(z)\frac{1}{z_B - z_T}\,\mathrm{d}z\Big] \otimes \mathcal{A} \otimes \Big[\int_{R_B}^{R_F} \bar{b}(r)\bar{b}(r)^T r\,\mathrm{d}r\Big]$$

Using these matrices, (4.2.156) multiplied by $\Big(\hat{b}(z) \otimes B(r)^T\Big)$ and integrated with respect to r and z, has the form

$$\begin{aligned} G_0\dot{\bar{c}}_0(t) + \big(G_2(r) + G_2(z)\big)\bar{c}_0(t) = &-H_0\dot{\bar{\mu}}_0(t) - H_1\dot{\bar{\lambda}}_0(t) \\ &- \mathcal{R}_B\dot{\bar{\nu}}_{B0}(t) - \tilde{\mathcal{R}}_B\dot{\bar{\nu}}_{\tilde{B}0}(t) - \tilde{\mathcal{R}}_T\dot{\bar{\nu}}_{\tilde{T}0}(t) \\ &- \mathcal{R}_F(a)\bar{\nu}_{B0}(t) + \mathcal{R}_F(a)\bar{\nu}_{F0} - \tilde{\mathcal{R}}_B(a)\bar{\nu}_{\tilde{B}0}(t) - \tilde{\mathcal{R}}_T(a)\bar{\nu}_{\tilde{T}0}(t) \\ &- \mathcal{R}_B(z)\bar{\nu}_{B0}(t) - \mathcal{R}_F(z)\bar{\nu}_{F0} - \tilde{\mathcal{R}}_0(z)\bar{\nu}_{\tilde{B}0}(t) + \tilde{\mathcal{R}}_0(z)\bar{\nu}_{\tilde{T}0}(t) \end{aligned} \tag{4.2.166}$$

From (4.2.142), (4.2.143)

$$\dot{\bar{\mu}}_0(t) = P_{10}\mathcal{Q}_0^{-1}\Big[\bar{\mathcal{Z}}_0\dot{\bar{c}}_0(t) + X_{B0}\dot{\bar{\nu}}_{B0}(t) + \tilde{X}_{B0}\dot{\bar{\nu}}_{\tilde{B}0}(t) + \tilde{X}_{T0}\dot{\bar{\nu}}_{\tilde{T}0}(t)\Big] \tag{4.2.167}$$

$$\dot{\bar{\lambda}}_0(t) = P_{01}\mathcal{Q}_0^{-1}\Big[\bar{\mathcal{Z}}_0\dot{\bar{c}}_0(t) + X_{B0}\dot{\bar{\nu}}_{B0}(t) + \tilde{X}_{B0}\dot{\bar{\nu}}_{\tilde{B}0}(t) + \tilde{X}_{T0}\dot{\bar{\nu}}_{\tilde{T}0}(t)\Big] \tag{4.2.168}$$

Applying (4.2.167) and (4.2.168) to (4.2.166), we have

$$\begin{aligned} \mathcal{H}_0\dot{\bar{c}}_0(t) + \mathcal{G}_0\bar{c}_0(t) = &\mathcal{D}_{B0}\dot{\bar{\nu}}_{B0}(t) + \mathcal{D}_{\tilde{B}0}\dot{\bar{\nu}}_{\tilde{B}0}(t) + \mathcal{D}_{\tilde{T}0}\dot{\bar{\nu}}_{\tilde{T}0}(t) \\ &+ \mathcal{F}_{B0}\bar{\nu}_{B0}(t) + \mathcal{F}_{F0}\bar{\nu}_{F0} + \mathcal{F}_{\tilde{B}0}\bar{\nu}_{\tilde{B}0}(t) + \mathcal{F}_{\tilde{T}0}\bar{\nu}_{\tilde{T}0}(t) \end{aligned} \tag{4.2.169}$$

where $\mathcal{H}_0 = G_0 + \big(H_0P_{10} + H_1P_{01}\big)\mathcal{Q}_0^{-1}\bar{\mathcal{Z}}_0$

$$\mathcal{G}_0 = G_2(r) + G_2(z)$$
$$\mathcal{D}_{B0} = -\big(H_0P_{10} + H_1P_{01}\big)\mathcal{Q}_0^{-1}X_{B0} - \mathcal{R}_B$$
$$\mathcal{D}_{\tilde{B}0} = -\big(H_0P_{10} + H_1P_{01}\big)\mathcal{Q}_0^{-1}\tilde{X}_{B0} - \tilde{\mathcal{R}}_B$$
$$\mathcal{D}_{\tilde{T}0} = -\big(H_0P_{10} + H_1P_{01}\big)\mathcal{Q}_0^{-1}\tilde{X}_{T0} - \tilde{\mathcal{R}}_T$$
$$\mathcal{F}_{B0} = -\mathcal{R}_F(a) - \mathcal{R}_B(z)$$
$$\mathcal{F}_{F0} = \mathcal{R}_F(a) - \mathcal{R}_F(z)$$
$$\mathcal{F}_{\tilde{B}0} = -\tilde{\mathcal{R}}_B(a) - \tilde{\mathcal{R}}_0(z)$$
$$\mathcal{F}_{\tilde{T}0} = -\tilde{\mathcal{R}}_T(a) + \tilde{\mathcal{R}}_0(z)$$

Now we approximate initial conditions. From (4.2.20), (4.2.30) and (4.2.89), the coefficients $V_i(r,z,0)$ of θ-approximation of initial conditions have the form

$$V_i(r,z,0) = \tilde{V}_{Ii}(r,z) - W_i(r,z,0)$$

where $\tilde{V}_{Ii}(r,z) = \Big(\hat{b}(z)^T \otimes B(r)\Big)\bar{\nu}_{Ii}$

$$\begin{aligned} W_i(r,z,0) = &\frac{R_F - r}{R_F - R_B}\Big(\hat{B}(z) - \frac{z - z_T}{z_B - z_T}\hat{B}(z_B) - \frac{z_B - z}{z_B - z_T}\hat{B}(z_T)\Big)\bar{\nu}_{Bi}(0) \\ &+ \frac{r - R_B}{R_F - R_B}\Big(\hat{B}(z) - \frac{z - z_T}{z_B - z_T}\hat{B}(z_B) - \frac{z_B - z}{z_B - z_T}\hat{B}(z_T)\Big)\bar{\nu}_{Fi} \\ &+ \frac{z - z_T}{z_B - z_T}B(r)\bar{\nu}_{\tilde{B}i}(0) + \frac{z_B - z}{z_B - z_T}B(r)\bar{\nu}_{\tilde{T}i}(0), \quad i = 0,\dots,4 \end{aligned} \tag{4.2.170}$$

and $\bar{\nu}_{Bi}(0) = 0, \quad i = 1,\dots,4.$

Multiplying $V_i(r,z,0)$ by $\hat{b}(z) \otimes B(r)^T$ and integrating with respect to r and z, we have

$$\begin{aligned} &\Big\{\Big[\int_{z_T}^{z_B} \hat{b}(z)\hat{b}(z)^T\,\mathrm{d}z\Big] \otimes \Big[\int_{R_B}^{R_F} B(r)^T B(r) r\,\mathrm{d}r\Big]\Big\}\bar{c}_i(0) \\ =&\Big\{\Big[\int_{z_T}^{z_B} \hat{b}(z)\hat{b}(z)^T\,\mathrm{d}z\Big] \otimes \Big[\int_{R_B}^{R_F} B(r)^T B(r) r\,\mathrm{d}r\Big]\Big\}\bar{\nu}_{Ii} \\ -&\Big[\int_{z_T}^{z_B} \hat{b}(z) \otimes \Big(\hat{B}(z) - \frac{z - z_T}{z_B - z_T}\hat{B}(z_B) - \frac{z_B - z}{z_B - z_T}\hat{B}(z_T)\Big)\,\mathrm{d}z\Big] \end{aligned}$$

$$\otimes\left\{\Big[\int_{R_B}^{R_F}\bar{b}(r)\frac{R_F-r}{R_F-R_B}r\,\mathrm{d}r\Big]\bar{\nu}_{Bi}(0)+\Big[\int_{R_B}^{R_F}\bar{b}(r)\frac{r-R_B}{R_F-R_B}r\,\mathrm{d}r\Big]\bar{\nu}_{Fi}\right\}$$
$$-\left\{\Big[\int_{z_T}^{z_B}\hat{b}(z)\frac{z-z_T}{z_B-z_T}\,\mathrm{d}z\Big]\otimes\Big[\int_{R_B}^{R_F}B(r)^TB(r)r\,\mathrm{d}r\Big]\right\}\bar{\nu}_{\tilde{B}i}(0)$$
$$-\left\{\Big[\int_{z_T}^{z_B}\hat{b}(z)\frac{z_B-z}{z_B-z_T}\,\mathrm{d}z\Big]\otimes\Big[\int_{R_B}^{R_F}B(r)^TB(r)r\,\mathrm{d}r\Big]\right\}\bar{\nu}_{\tilde{T}i}(0) \tag{4.2.171}$$

Using matrices defined above, (4.2.171) multiplied by $\Big(\hat{b}(z)\otimes B(r)^T\Big)$ and integrated with respect to r and z, has the form

$$G_0\bar{c}_i(0)=G_0\bar{\nu}_{Ii}-\mathcal{R}_B\bar{\nu}_{Bi}(0)-\mathcal{R}_F\bar{\nu}_{Fi}-\tilde{\mathcal{R}}_B\bar{\nu}_{\tilde{B}i}(0)-\tilde{\mathcal{R}}_T\bar{\nu}_{\tilde{T}i}(0) \tag{4.2.172}$$

From (4.2.169) and (4.2.172) with $i=0$ we obtain the following system.

System of ODEs for TCP I (radially symmetric case):

$$\mathcal{H}_0\dot{\bar{c}}_0(t)+\mathcal{G}_0\bar{c}_0(t)=\mathcal{D}_{B0}\dot{\bar{\nu}}_{B0}(t)+\mathcal{D}_{\tilde{B}0}\dot{\bar{\nu}}_{\tilde{B}0}(t)+\mathcal{D}_{\tilde{T}0}\dot{\bar{\nu}}_{\tilde{T}0}(t)$$
$$+\mathcal{F}_{B0}\bar{\nu}_{B0}(t)+\mathcal{F}_{F0}\bar{\nu}_{F0}+\mathcal{F}_{\tilde{B}0}\bar{\nu}_{\tilde{B}0}(t)+\mathcal{F}_{\tilde{T}0}\bar{\nu}_{\tilde{T}0}(t) \tag{4.2.169}$$

with initial conditions

$$\bar{c}_0(0)=\bar{\nu}_{I0}-G_0^{-1}\big[\mathcal{R}_B\bar{\nu}_{B0}(0)+\mathcal{R}_F\bar{\nu}_{F0}+\tilde{\mathcal{R}}_B\bar{\nu}_{\tilde{B}0}(0)+\tilde{\mathcal{R}}_T\bar{\nu}_{\tilde{T}0}(0)\big] \tag{4.2.173}$$

System II (corresponds to $\sin\theta$ approximation)

System II is given by (4.2.31)-(4.2.33):

$$M\dot{V}_1-A\Big[\frac{1}{r}(rV_{1_r})_r-\frac{1}{r^2}V_1+(V_{1_z})_z\Big]=-M\dot{W}_1+A\Big(\frac{1}{r}(rW_{1_r})_r-\frac{1}{r^2}W_1+(W_{1_z})_z\Big)$$
$$-\vec{b}_0\Big(\dot{u}_{1_r}+\frac{1}{r}(\dot{u}_1-\dot{v}_2)+\dot{w}_{1_z}\Big) \tag{4.2.31}$$

with boundary conditions

$$V_1(r,z,t)=0 \qquad \text{on } \Gamma\times[0,t_f) \tag{4.2.32}$$

and initial conditions

$$V_1(r,z,0) = \tilde{V}_{I1}(r,z) - W_1(r,z,0) \tag{4.2.33}$$

Using $\mathcal{A} = M^{-1}A$, (4.2.31) has the form

$$\dot{V}_1 - \mathcal{A}\Big[\frac{1}{r}(rV_{1_r})_r - \frac{1}{r^2}V_1 + (V_{1_z})_z\Big] = -M^{-1}\vec{b}_0\Big(\dot{u}_{1_r} + \frac{1}{r}\dot{u}_1 - \frac{1}{r}\dot{v}_2 + \dot{w}_{1_z}\Big)$$
$$-\dot{W}_1 + \mathcal{A}\Big[\frac{1}{r}(rW_{1_r})_r - \frac{1}{r^2}W_1 + (W_{1_z})_z\Big] \tag{4.2.174}$$

Using (4.2.87), the left-hand side of (4.2.174) is

$$\Big(\hat{b}(z)^T \otimes B(r)\Big)\dot{\bar{c}}_1(t) - \mathcal{A}\frac{1}{r}\Big(\hat{b}(z)^T \otimes rB_r(r)\Big)_r\bar{c}_1(t) + \mathcal{A}\frac{1}{r^2}\Big(\hat{b}(z)^T \otimes B(r)\Big)\bar{c}_1(t)$$
$$-\mathcal{A}\Big(\hat{b}_z(z)^T \otimes B(r)\Big)_z\bar{c}_0(t) \tag{4.2.175}$$

Multiplying (4.2.175) by $\Big(\hat{b}(z) \otimes B(r)^T\Big)$ and integrating with respect to r and z we have

$$\Big\{\Big[\int_{z_T}^{z_B} \hat{b}(z)\hat{b}(z)^T \,\mathrm{d}z\Big] \otimes \Big[\int_{R_B}^{R_F} B(r)^T B(r) r \,\mathrm{d}r\Big]\Big\}\dot{\bar{c}}_1(t)$$
$$+\Big\{\Big[\int_{z_T}^{z_B} \hat{b}(z)\hat{b}(z)^T \,\mathrm{d}z\Big] \otimes \mathcal{A} \otimes \Big(\Big[\int_{R_B}^{R_F} \bar{b}_r(r)\bar{b}_r(r)^T r \,\mathrm{d}r\Big] + \Big[\int_{R_B}^{R_F} \bar{b}(r)\bar{b}(r)^T \frac{1}{r} \,\mathrm{d}r\Big]\Big)$$
$$+\Big[\int_{z_T}^{z_B} \hat{b}_z(z)\hat{b}_z(z)^T \,\mathrm{d}z\Big] \otimes \mathcal{A} \otimes \Big[\int_{R_B}^{R_F} \bar{b}(r)\bar{b}(r)^T r \,\mathrm{d}r\Big]\Big\}\bar{c}_1(t) \tag{4.2.176}$$

Using (4.2.90)-(4.2.92), the first term of the right-hand side of (4.2.174) is

$$-M^{-1}\vec{b}_0\Big[\Big(\hat{\mathrm{b}}(z)^T \otimes \bar{\mathrm{b}}_r(r)^T\Big)\dot{\bar{\mu}}_1(t) + \Big(\hat{\mathrm{b}}(z)^T \otimes \frac{1}{r}\bar{\mathrm{b}}(r)^T\Big)\dot{\bar{\mu}}_1(t)$$
$$-\Big(\hat{\mathrm{b}}(z)^T \otimes \frac{1}{r}\bar{\mathrm{b}}(r)^T\Big)\dot{\bar{\eta}}_2(t) + \Big(\hat{\mathrm{b}}_z(z)^T \otimes \bar{\mathrm{b}}(r)^T\Big)\dot{\bar{\lambda}}_1(t)\Big]$$

Multiplying by $\Big(\hat{b}(z) \otimes B(r)^T\Big)$ and integrating with respect to r and z, we have

$$-\Big\{\Big[\int_{z_T}^{z_B} \hat{b}(z)\,\hat{\mathrm{b}}(z)^T \,\mathrm{d}z\Big] \otimes \Big[\int_{R_B}^{R_F} B(r)^T M^{-1}\vec{b}_0(r\,\bar{\mathrm{b}}_r(r) + \bar{\mathrm{b}}(r))^T \,\mathrm{d}r\Big]\Big\}\dot{\bar{\mu}}_1(t)$$

$$+\left\{\left[\int_{z_T}^{z_B}\hat{b}(z)\,\hat{\mathrm{b}}(z)^T\,\mathrm{d}z\right]\otimes\left[\int_{R_B}^{R_F}B(r)^TM^{-1}\vec{b}_0\,\bar{\mathrm{b}}(r)^T\,\mathrm{d}r\right]\right\}\dot{\bar{\eta}}_2(t)$$

$$-\left\{\left[\int_{z_T}^{z_B}\hat{b}(z)\,\hat{\mathrm{b}}_z(z)^T\,\mathrm{d}z\right]\otimes\left[\int_{R_B}^{R_F}B(r)^TM^{-1}\vec{b}_0\,\bar{\mathrm{b}}(r)^Tr\,\mathrm{d}r\right]\right\}\dot{\bar{\lambda}}_1(t) \qquad (4.2.177)$$

Multiplying the corresponding term in (4.2.174) by $\left(\hat{b}(z)\otimes B(r)^T\right)$ and integrating with respect to r and z, we have the following results

for $\dot{W}_1$: (4.2.161) with $\dot{\bar{\nu}}_{B1}(t)=0$

for $\mathcal{A}\frac{1}{r}(rW_{1_r})_r$: (4.2.163) with $\bar{\nu}_{B1}(t)=0$

for $\mathcal{A}(rW_{1_z})_z$: (4.2.165) with $\bar{\nu}_{B1}(t)=0$.

$W_i(r,z,t)$ is given by (4.2.160) with $\bar{\nu}_{Bi}(t)=0, \quad i=1,\ldots,4$.

Multiplying $\mathcal{A}\frac{1}{r^2}W_i$ by $\left(\hat{b}(z)\otimes B(r)^T\right)$ and integrating with respect to r and z, we have

$$\Bigg[\int_{z_T}^{z_B}\int_{R_B}^{R_F}\left(\hat{b}(z)\otimes B(r)^T\right)\mathcal{A}\frac{1}{r^2}\frac{r-R_B}{R_F-R_B}\Bigg(\hat{B}(z)-\frac{z-z_T}{z_B-z_T}\hat{B}(z_B)$$

$$-\frac{z_B-z}{z_B-z_T}\hat{B}(z_T)\Bigg)r\,\mathrm{d}r\,\mathrm{d}z\Bigg]\bar{\nu}_{Fi}$$

$$+\left[\int_{z_T}^{z_B}\int_{R_B}^{R_F}\left(\hat{b}(z)\otimes B(r)^T\right)\mathcal{A}\frac{1}{r^2}\frac{z-z_T}{z_B-z_T}B(r)r\,\mathrm{d}r\,\mathrm{d}z\right]\bar{\nu}_{\tilde{B}i}(t)$$

$$+\left[\int_{z_T}^{z_B}\int_{R_B}^{R_F}\left(\hat{b}(z)\otimes B(r)^T\right)\mathcal{A}\frac{1}{r^2}\frac{z_B-z}{z_B-z_T}B(r)r\,\mathrm{d}r\,\mathrm{d}z\right]\bar{\nu}_{\tilde{T}i}(t)$$

$$=\left[\int_{z_T}^{z_B}\hat{b}(z)\otimes\left(\mathcal{A}\left(\hat{B}(z)-\frac{z-z_T}{z_B-z_T}\hat{B}(z_B)-\frac{z_B-z}{z_B-z_T}\hat{B}(z_T)\right)\right)\mathrm{d}z\right]$$

$$\otimes\left[\int_{R_B}^{R_F}\bar{b}(r)\frac{r-R_B}{R_F-R_B}\frac{1}{r}\,\mathrm{d}r\right]\bar{\nu}_{Fi}$$

$$+\left\{\left[\int_{z_T}^{z_B}\hat{b}(z)\frac{z-z_T}{z_B-z_T}\,\mathrm{d}z\right]\otimes\mathcal{A}\otimes\left[\int_{R_B}^{R_F}\bar{b}(r)\bar{b}(r)^T\frac{1}{r}\,\mathrm{d}r\right]\right\}\bar{\nu}_{\tilde{B}i}(t)$$

$$+\left\{\left[\int_{z_T}^{z_B}\hat{b}(z)\frac{z_B-z}{z_B-z_T}\,\mathrm{d}z\right]\otimes\mathcal{A}\otimes\left[\int_{R_B}^{R_F}\bar{b}(r)\bar{b}(r)^T\frac{1}{r}\,\mathrm{d}r\right]\right\}\bar{\nu}_{\tilde{B}i}(t), \qquad i=1,\ldots,4.$$

(4.2.178)

Define the following matrices:

$$3M_{Dr}M_{Dz} \times 3M_{Dr}M_{Dz} \text{ matrix:}$$

$$G_0\big(\frac{1}{r}\big) = \Big[\int_{z_T}^{z_B} \hat{b}(z)\hat{b}(z)^T\,\mathrm{d}z\Big] \otimes \mathcal{A} \otimes \Big[\int_{R_B}^{R_F} \bar{b}(r)\bar{b}(r)^T \frac{1}{r}\,\mathrm{d}r\Big]$$

$$3M_{Dr}M_{Dz} \times M_r M_z \text{ matrix:}$$

$$H_0(r) = \Big[\int_{z_T}^{z_B} \hat{b}(z)\,\hat{\mathrm{b}}(z)^T\,\mathrm{d}z\Big] \otimes \Big[\int_{R_B}^{R_F} B(r)^T M^{-1}\vec{b}_0\,\bar{\mathrm{b}}(r)^T\,\mathrm{d}r\Big]$$

$$3M_{Dr}M_{Dz} \times 3M_{Dz} \text{ matrix:}$$

$$\mathcal{R}_F^{(a)}\big(\frac{1}{r}\big) = \Big\{\int_{z_T}^{z_B} \hat{b}(z) \otimes \Big[\mathcal{A}\Big(\hat{B}(z) - \frac{z-z_T}{z_B-z_T}\hat{B}(z_B) - \frac{z_B-z}{z_B-z_T}\hat{B}(z_T)\Big)\Big]\,\mathrm{d}z\Big\}$$
$$\otimes \Big[\int_{R_B}^{R_F} \bar{b}(r)\frac{r-R_B}{R_F-R_B}\frac{1}{r}\,\mathrm{d}r\Big]$$

$$3M_{Dr}M_{Dz} \times 3M_{Dr} \text{ matrices:}$$

$$\tilde{\mathcal{R}}_B^{(a)}\big(\frac{1}{r}\big) = \Big[\int_{z_T}^{z_B} \hat{b}(z)\frac{z-z_T}{z_B-z_T}\,\mathrm{d}z\Big] \otimes \mathcal{A} \otimes \Big[\int_{R_B}^{R_F} \bar{b}(r)\bar{b}(r)^T\frac{1}{r}\,\mathrm{d}r\Big]$$
$$\tilde{\mathcal{R}}_T^{(a)}\big(\frac{1}{r}\big) = \Big[\int_{z_T}^{z_B} \hat{b}(z)\frac{z_B-z}{z_B-z_T}\,\mathrm{d}z\Big] \otimes \mathcal{A} \otimes \Big[\int_{R_B}^{R_F} \bar{b}(r)\bar{b}(r)^T\frac{1}{r}\,\mathrm{d}r\Big]$$

Using these matrices together with matrices defined for System I, (4.2.174) multiplied by $\Big(\hat{b}(z) \otimes B(r)^T\Big)$ and integrated with respect to r and z, has the form

$$\begin{aligned} G_0\dot{\bar{c}}_1(t) + &\big(G_2(r) + G_0\big(\frac{1}{r}\big) + G_2(z)\big)\bar{c}_1(t) = \\ &= -H_0\dot{\bar{\mu}}_1(t) + H_0(r)\dot{\bar{\eta}}_2(t) - H_1\dot{\bar{\lambda}}_1(t) - \tilde{\mathcal{R}}_B\dot{\bar{\nu}}_{\tilde{B}1}(t) - \tilde{\mathcal{R}}_T\dot{\bar{\nu}}_{\tilde{T}1}(t) \\ &\quad + \mathcal{R}_F(a)\bar{\nu}_{F1} - \tilde{\mathcal{R}}_B(a)\bar{\nu}_{\tilde{B}1}(t) - \tilde{\mathcal{R}}_T(a)\bar{\nu}_{\tilde{T}1}(t) \\ &\quad - \mathcal{R}_F^{(a)}\big(\frac{1}{r}\big)\bar{\nu}_{F1} - \tilde{\mathcal{R}}_B^{(a)}\big(\frac{1}{r}\big)\bar{\nu}_{\tilde{B}1}(t) - \tilde{\mathcal{R}}_T^{(a)}\big(\frac{1}{r}\big)\bar{\nu}_{\tilde{T}1}(t) \\ &\quad - \mathcal{R}_F(z)\bar{\nu}_{F1} - \tilde{\mathcal{R}}_0(z)\bar{\nu}_{\tilde{B}1}(t) + \tilde{\mathcal{R}}_0(z)\bar{\nu}_{\tilde{T}1}(t) \end{aligned} \tag{4.2.179}$$

From (4.2.144)-(4.2.146)

$$\dot{\bar{\mu}}_1(t) = P_{100}\mathcal{Q}_1^{-1}\Big[\bar{\mathcal{Z}}_1\dot{\bar{c}}_1(t) + \tilde{X}_{B1}\dot{\bar{\nu}}_{\tilde{B}1}(t) + \tilde{X}_{T1}\dot{\bar{\nu}}_{\tilde{T}1}(t)\Big] \quad (4.2.180)$$

$$\dot{\bar{\eta}}_2(t) = -P_{010}\mathcal{Q}_1^{-1}\Big[\bar{\mathcal{Z}}_1\dot{\bar{c}}_1(t) + \tilde{X}_{B1}\dot{\bar{\nu}}_{\tilde{B}1}(t) + \tilde{X}_{T1}\dot{\bar{\nu}}_{\tilde{T}1}(t)\Big] \quad (4.2.181)$$

$$\dot{\bar{\lambda}}_1(t) = P_{001}\mathcal{Q}_1^{-1}\Big[\bar{\mathcal{Z}}_1\dot{\bar{c}}_1(t) + \tilde{X}_{B1}\dot{\bar{\nu}}_{\tilde{B}1}(t) + \tilde{X}_{T1}\dot{\bar{\nu}}_{\tilde{T}1}(t)\Big] \quad (4.2.182)$$

Applying (4.2.180)-(4.2.182) to (4.2.179), we have

$$\begin{aligned}\mathcal{H}_1\dot{\bar{c}}_1(t) + \mathcal{G}_1\bar{c}_1(t) &= \mathcal{D}_{\tilde{B}1}\dot{\bar{\nu}}_{\tilde{B}1}(t) + \mathcal{D}_{\tilde{T}1}\dot{\bar{\nu}}_{\tilde{T}1}(t)\\ &\quad + \mathcal{F}_{F1}\bar{\nu}_{F1} + \mathcal{F}_{\tilde{B}1}\bar{\nu}_{\tilde{B}1}(t) + \mathcal{F}_{\tilde{T}1}\bar{\nu}_{\tilde{T}1}(t) \end{aligned} \quad (4.2.183)$$

where

$$\begin{aligned}\mathcal{H}_1 &= G_0 + \big(H_0P_{100} + H_0(r)P_{010} + H_1P_{001}\big)\mathcal{A}_1^{-1}\bar{\mathcal{Z}}_1\\
\mathcal{G}_1 &= G_2(r) + G_0\big(\frac{1}{r}\big) + G_2(z)\\
\mathcal{D}_{\tilde{B}1} &= -\big(H_0P_{100} + H_0(r)P_{010} + H_1P_{001}\big)\mathcal{A}_1^{-1}\tilde{X}_{B1} - \tilde{\mathcal{R}}_B\\
\mathcal{D}_{\tilde{T}1} &= -\big(H_0P_{100} + H_0(r)P_{010} + H_1P_{001}\big)\mathcal{A}_1^{-1}\tilde{X}_{T1} - \tilde{\mathcal{R}}_T\\
\mathcal{F}_{F1} &= -\big(-\mathcal{R}_F(a) + \mathcal{R}_F^{(a)}\big(\frac{1}{r}\big) + \mathcal{R}_F(z)\big)\\
\mathcal{F}_{\tilde{B}1} &= -\big(\mathcal{R}_B(a) + \mathcal{R}_B^{(a)}\big(\frac{1}{r}\big) + \tilde{\mathcal{R}}_0(z)\big)\\
\mathcal{F}_{\tilde{T}1} &= -\big(\mathcal{R}_T(a) + \mathcal{R}_T^{(a)}\big(\frac{1}{r}\big) + \tilde{\mathcal{R}}_0(z)\big)\end{aligned} \quad (4.2.184)$$

Initial conditions are given by (4.2.172) with $i = 1$ and $\bar{\nu}_{B1}(0) = 0$.

From (4.2.172) and (4.2.183) with $i = 1$ and $\bar{\nu}_{B1}(0) = 0$ we obtain the following system.

System of ODEs for TCP II (corresponds to $\sin\theta$ approximation):

$$\begin{aligned}\mathcal{H}_1\dot{\bar{c}}_1(t) + \mathcal{G}_1\bar{c}_1(t) &= \mathcal{D}_{\tilde{B}1}\dot{\bar{\nu}}_{\tilde{B}1}(t) + \mathcal{D}_{\tilde{T}1}\dot{\bar{\nu}}_{\tilde{T}1}(t)\\ &\quad + \mathcal{F}_{F1}\bar{\nu}_{F1} + \mathcal{F}_{\tilde{B}1}\bar{\nu}_{\tilde{B}1}(t) + \mathcal{F}_{\tilde{T}1}\bar{\nu}_{\tilde{T}1}(t) \end{aligned} \quad (4.2.183)$$

with initial conditions

$$\bar{c}_1(0) = \bar{\nu}_{I1} + G_0^{-1}\big[\mathcal{R}_F\bar{\nu}_{F1} + \tilde{\mathcal{R}}_B\bar{\nu}_{\tilde{B}1}(0) + \tilde{\mathcal{R}}_T\bar{\nu}_{\tilde{T}1}(0)\big] \tag{4.2.185}$$

System III (corresponds to $\cos\theta$ approximation)

System III is given by (4.2.34)-(4.2.36):

$$M\dot{V}_2 - A\Big[\frac{1}{r}(rV_{2_r})_r - \frac{1}{r^2}V_2 + (V_{2_z})_z\Big] = -M\dot{W}_2 + A\Big(\frac{1}{r}(rW_{2_r})_r - \frac{1}{r^2}W_2 + (W_{2_z})_z\Big)$$
$$-\vec{b}_0\Big(\dot{u}_{2_r} + \frac{1}{r}(\dot{u}_2 + \dot{v}_1) + \dot{w}_{2_z}\Big) \tag{4.2.34}$$

with boundary conditions

$$V_2(r,z,t) = 0 \qquad \text{on } \Gamma \times [0,t_f) \tag{4.2.35}$$

and initial conditions

$$V_2(r,z,0) = \tilde{V}_{I2}(r,z) - W_2(r,z,0) \tag{4.2.36}$$

Using $\mathcal{A} = M^{-1}A$, (4.2.34) has the form

$$\dot{V}_2 - \mathcal{A}\Big[\frac{1}{r}(rV_{2_r})_r - \frac{1}{r^2}V_2 + (V_{2_z})_z\Big] = -M^{-1}\vec{b}_0\Big(\dot{u}_{2_r} + \frac{1}{r}\dot{u}_2 + \frac{1}{r}\dot{v}_1 + \dot{w}_{2_z}\Big)$$
$$-\dot{W}_2 + \mathcal{A}\Big[\frac{1}{r}(rW_{2_r})_r - \frac{1}{r^2}W_2 + (W_{2_z})_z\Big] \tag{4.2.186}$$

Applying (4.2.87) to the left-hand side of (4.2.186), multiplying the result by $\Big(\hat{b}(z) \otimes B(r)^T\Big)$ and integrating with respect to r and z, we obtain

$$\Big\{\Big[\int_{z_T}^{z_B} \hat{b}(z)\hat{b}(z)^T\,\mathrm{d}z\Big] \otimes \Big[\int_{R_B}^{R_F} B(r)^T B(r) r\,\mathrm{d}r\Big]\Big\}\dot{\bar{c}}_2(t)$$
$$+\Big\{\Big[\int_{z_T}^{z_B} \hat{b}(z)\hat{b}(z)^T\,\mathrm{d}z\Big] \otimes \mathcal{A} \otimes \Big(\Big[\int_{R_B}^{R_F} \bar{b}_r(r)\bar{b}_r(r)^T r\,\mathrm{d}r\Big] + \Big[\int_{R_B}^{R_F} \bar{b}(r)\bar{b}(r)^T \frac{1}{r}\,\mathrm{d}r\Big]\Big)$$
$$+\Big[\int_{z_T}^{z_B} \hat{b}_z(z)\hat{b}_z(z)^T\,\mathrm{d}z\Big] \otimes \mathcal{A} \otimes \Big[\int_{R_B}^{R_F} \bar{b}(r)\bar{b}(r)^T r\,\mathrm{d}r\Big]\Big\}\bar{c}_2(t) \tag{4.2.187}$$

Using (4.2.90)-(4.2.92), the last term of the right-hand side of (4.2.186) is

$$\begin{aligned} & - M^{-1}\vec{b}_0\Big(\dot{u}_{2_r} + \frac{1}{r}\dot{u}_2 + \frac{1}{r}\dot{v}_1 + \dot{w}_{2_z}\Big) \\ = & - M^{-1}\vec{b}_0\Big[\Big(\hat{\mathrm{b}}(z)^T \otimes \bar{\mathrm{b}}_r(r)^T\Big)\dot{\bar{\mu}}_2(t) + \Big(\hat{\mathrm{b}}(z)^T \otimes \frac{1}{r}\,\bar{\mathrm{b}}(r)^T\Big)\dot{\bar{\mu}}_2(t) \\ & + \Big(\hat{\mathrm{b}}(z)^T \otimes \frac{1}{r}\,\bar{\mathrm{b}}(r)^T\Big)\dot{\bar{\eta}}_1(t) + \Big(\hat{\mathrm{b}}_z(z)^T \otimes \bar{\mathrm{b}}(r)^T\Big)\dot{\bar{\lambda}}_2(t)\Big] \end{aligned}$$

Multiplying by $\Big(\hat{b}(z) \otimes B(r)^T\Big)$ and integrating with respect to r and z, we have

$$\begin{aligned} & -\Big\{\Big[\int_{z_T}^{z_B} \hat{b}(z)\,\hat{\mathrm{b}}(z)^T\,\mathrm{d}z\Big] \otimes \Big[\int_{R_B}^{R_F} B(r)^T M^{-1}\vec{b}_0(r\,\bar{\mathrm{b}}_r(r) + \bar{\mathrm{b}}(r))^T\,\mathrm{d}r\Big]\Big\}\dot{\bar{\mu}}_2(t) \\ & -\Big\{\Big[\int_{z_T}^{z_B} \hat{b}(z)\,\hat{\mathrm{b}}(z)^T\,\mathrm{d}z\Big] \otimes \Big[\int_{R_B}^{R_F} B(r)^T M^{-1}\vec{b}_0\,\bar{\mathrm{b}}(r)^T\,\mathrm{d}r\Big]\Big\}\dot{\bar{\eta}}_1(t) \\ & -\Big\{\Big[\int_{z_T}^{z_B} \hat{b}(z)\,\hat{\mathrm{b}}_z(z)^T\,\mathrm{d}z\Big] \otimes \Big[\int_{R_B}^{R_F} B(r)^T M^{-1}\vec{b}_0\,\bar{\mathrm{b}}(r)^T r\,\mathrm{d}r\Big]\Big\}\dot{\bar{\lambda}}_2(t) \end{aligned} \tag{4.2.188}$$

Multiplying the corresponding term in (4.2.186) by $\Big(\hat{b}(z) \otimes B(r)^T\Big)$ and integrating with respect to r and z, we have the following results

for $\dot{W}_2$: (4.2.161) with $\dot{\bar{\nu}}_{B2}(t) = 0$

for $\mathcal{A}\frac{1}{r}(rW_{2_r})_r$: (4.2.163) with $\bar{\nu}_{B2}(t) = 0$

for $\mathcal{A}(rW_{2_z})_z$: (4.2.165) with $\bar{\nu}_{B2}(t) = 0$

for $\mathcal{A}\frac{1}{r^2}W_2$: (4.2.178).

Using matrices defined for Systems I and II, (4.2.186) multiplied by $\Big(\hat{b}(z) \otimes B(r)^T\Big)$ and integrated with respect to r and z, has the form

$$\begin{aligned} & G_0\dot{\bar{c}}_2(t) + \big(G_2(r) + G_0(\tfrac{1}{r}) + G_2(z)\big)\bar{c}_2(t) = \\ & = - H_0\dot{\bar{\mu}}_2(t) - H_0(r)\dot{\bar{\eta}}_1(t) - H_1\dot{\bar{\lambda}}_2(t) - \tilde{\mathcal{R}}_B\dot{\bar{\nu}}_{\tilde{B}2}(t) - \tilde{\mathcal{R}}_T\dot{\bar{\nu}}_{\tilde{T}2}(t) \\ & \quad + \mathcal{R}_F(a)\bar{\nu}_{F2} - \tilde{\mathcal{R}}_B(a)\bar{\nu}_{\tilde{B}2}(t) - \tilde{\mathcal{R}}_T(a)\bar{\nu}_{\tilde{T}2}(t) \\ & \quad - \mathcal{R}_F^{(a)}(\tfrac{1}{r})\bar{\nu}_{F2} - \tilde{\mathcal{R}}_B^{(a)}(\tfrac{1}{r})\bar{\nu}_{\tilde{B}2}(t) - \tilde{\mathcal{R}}_T^{(a)}(\tfrac{1}{r})\bar{\nu}_{\tilde{T}2}(t) \\ & \quad - \mathcal{R}_F(z)\bar{\nu}_{F2} - \tilde{\mathcal{R}}_0(z)\bar{\nu}_{\tilde{B}2}(t) + \tilde{\mathcal{R}}_0(z)\bar{\nu}_{\tilde{T}2}(t) \end{aligned} \tag{4.2.189}$$

From (4.2.147)-(4.2.149)

$$\dot{\bar{\mu}}_2(t) = P_{100}\mathcal{Q}_1^{-1}\Big[\bar{\mathcal{Z}}_1\dot{\bar{c}}_2(t) + \tilde{X}_{B1}\dot{\bar{\nu}}_{\tilde{B}2}(t) + \tilde{X}_{T1}\dot{\bar{\nu}}_{\tilde{T}2}(t)\Big] \tag{4.2.190}$$

$$\dot{\bar{\eta}}_1(t) = P_{010}\mathcal{Q}_1^{-1}\Big[\bar{\mathcal{Z}}_1\dot{\bar{c}}_2(t) + \tilde{X}_{B1}\dot{\bar{\nu}}_{\tilde{B}2}(t) + \tilde{X}_{T1}\dot{\bar{\nu}}_{\tilde{T}2}(t)\Big] \tag{4.2.191}$$

$$\dot{\bar{\lambda}}_2(t) = P_{001}\mathcal{Q}_1^{-1}\Big[\bar{\mathcal{Z}}_1\dot{\bar{c}}_2(t) + \tilde{X}_{B1}\dot{\bar{\nu}}_{\tilde{B}2}(t) + \tilde{X}_{T1}\dot{\bar{\nu}}_{\tilde{T}2}(t)\Big] \tag{4.2.192}$$

Applying (4.2.190)-(4.2.192) to (4.2.189), we have

$$\begin{aligned}\mathcal{H}_1\dot{\bar{c}}_2(t) + \mathcal{G}_1\bar{c}_2(t) &= \mathcal{D}_{\tilde{B}1}\dot{\bar{\nu}}_{\tilde{B}2}(t) + \mathcal{D}_{\tilde{T}1}\dot{\bar{\nu}}_{\tilde{T}2}(t)\\ &\quad + \mathcal{F}_{F1}\bar{\nu}_{F2} + \mathcal{F}_{\tilde{B}1}\bar{\nu}_{\tilde{B}2}(t) + \mathcal{F}_{\tilde{T}1}\bar{\nu}_{\tilde{T}2}(t)\end{aligned} \tag{4.2.193}$$

where matrices $\mathcal{H}_1, \mathcal{G}_1, \mathcal{D}_{\tilde{B}1}, \mathcal{D}_{\tilde{T}1}, \mathcal{F}_{F1}, \mathcal{F}_{\tilde{B}1}, \mathcal{F}_{\tilde{T}1}$ are given by (4.2.183).

Initial conditions are given by (4.2.172) with $i = 2$ and $\bar{\nu}_{B2}(0) = 0$.

From (4.2.193) and (4.2.172) with $i = 2$ and $\bar{\nu}_{B2}(0) = 0$ we obtain the following system.

System of ODEs for TCP III (corresponds to $\cos\theta$ approximation):

$$\begin{aligned}\mathcal{H}_1\dot{\bar{c}}_2(t) + \mathcal{G}_1\bar{c}_2(t) &= \mathcal{D}_{\tilde{B}1}\dot{\bar{\nu}}_{\tilde{B}2}(t) + \mathcal{D}_{\tilde{T}1}\dot{\bar{\nu}}_{\tilde{T}2}(t)\\ &\quad + \mathcal{F}_{F1}\bar{\nu}_{F2} + \mathcal{F}_{\tilde{B}1}\bar{\nu}_{\tilde{B}2}(t) + \mathcal{F}_{\tilde{T}1}\bar{\nu}_{\tilde{T}2}(t)\end{aligned} \tag{4.2.193}$$

with initial conditions

$$\bar{c}_2(0) = \bar{\nu}_{I2} + G_0^{-1}\big[\mathcal{R}_F\bar{\nu}_{F2} + \tilde{\mathcal{R}}_B\bar{\nu}_{\tilde{B}2}(0) + \tilde{\mathcal{R}}_T\bar{\nu}_{\tilde{T}2}(0)\big] \tag{4.2.194}$$

System IV (corresponds to $\sin 2\theta$ approximation)

System IV is given by (4.2.37)-(4.2.39):

$$\begin{aligned}M\dot{V}_3 - A\Big[\frac{1}{r}(rV_{3_r})_r - \frac{4}{r^2}V_3 + (V_{3_z})_z\Big] &= -M\dot{W}_3 + A\Big(\frac{1}{r}(rW_{3_r})_r - \frac{4}{r^2}W_3 + (W_{3_z})_z\Big)\\ &\quad -\vec{b}_0\Big(\dot{u}_{3_r} + \frac{1}{r}(\dot{u}_3 - 2\dot{v}_4) + \dot{w}_{3_z}\Big)\end{aligned} \tag{4.2.37}$$

with boundary conditions

$$V_3(r,z,t) = 0 \qquad \text{on } \Gamma \times [0, t_f) \tag{4.2.38}$$

and initial conditions

$$V_3(r,z,0) = \tilde{V}_{I3}(r,z) - W_3(r,z,0) \tag{4.2.39}$$

Using $\mathcal{A} = M^{-1}A$, (4.2.37) has the form

$$\begin{aligned}\dot{V}_3 - \mathcal{A}\Big[\frac{1}{r}(rV_{3_r})_r - \frac{4}{r^2}V_3 + (V_{3_z})_z\Big] &= -M^{-1}\vec{b}_0\Big(\dot{u}_{3_r} + \frac{1}{r}\dot{u}_3 - \frac{2}{r}\dot{v}_4 + \dot{w}_{3_z}\Big)\\ &\quad -\dot{W}_3 + \mathcal{A}\Big[\frac{1}{r}(rW_{3_r})_r - \frac{4}{r^2}W_3 + (W_{3_z})_z\Big]\end{aligned} \tag{4.2.195}$$

Applying (4.2.87) to the left-hand side of (4.2.195), multiplying the result by $\Big(\hat{b}(z) \otimes B(r)^T\Big)$ and integrating with respect to r and z, we obtain

$$\begin{aligned}&\Big\{\Big[\int_{z_T}^{z_B} \hat{b}(z)\hat{b}(z)^T \,\mathrm{d}z\Big] \otimes \Big[\int_{R_B}^{R_F} B(r)^T B(r) r \,\mathrm{d}r\Big]\Big\}\dot{\bar{c}}_3(t)\\ &+\Big\{\Big[\int_{z_T}^{z_B} \hat{b}(z)\hat{b}(z)^T \,\mathrm{d}z\Big] \otimes \mathcal{A} \otimes \Big(\Big[\int_{R_B}^{R_F} \bar{b}_r(r)\bar{b}_r(r)^T r \,\mathrm{d}r\Big] + 4\Big[\int_{R_B}^{R_F} \bar{b}(r)\bar{b}(r)^T \frac{1}{r} \,\mathrm{d}r\Big]\Big)\\ &\quad + \Big[\int_{z_T}^{z_B} \hat{b}_z(z)\hat{b}_z(z)^T \,\mathrm{d}z\Big] \otimes \mathcal{A} \otimes \Big[\int_{R_B}^{R_F} \bar{b}(r)\bar{b}(r)^T r \,\mathrm{d}r\Big]\Big\}\bar{c}_2(t)\end{aligned} \tag{4.2.196}$$

Using (4.2.90)-(4.2.92), the first term of the right-hand side of (4.2.195) is

$$\begin{aligned}&- M^{-1}\vec{b}_0\Big(\dot{u}_{3_r} + \frac{1}{r}\dot{u}_3 - \frac{2}{r}\dot{v}_4 + \dot{w}_{3_z}\Big)\\ &= - M^{-1}\vec{b}_0\Big[\Big(\hat{\mathrm{b}}(z)^T \otimes \bar{\mathrm{b}}_r(r)^T\Big)\dot{\mu}_3(t) + \Big(\hat{\mathrm{b}}(z)^T \otimes \frac{1}{r}\,\bar{\mathrm{b}}(r)^T\Big)\dot{\mu}_3(t)\\ &\qquad - \Big(\hat{\mathrm{b}}(z)^T \otimes \frac{2}{r}\,\bar{\mathrm{b}}(r)^T\Big)\dot{\eta}_4(t) + \Big(\hat{\mathrm{b}}_z(z)^T \otimes \bar{\mathrm{b}}(r)^T\Big)\dot{\lambda}_3(t)\Big]\end{aligned}$$

Multiplying by $\Big(\hat{b}(z) \otimes B(r)^T\Big)$ and integrating with respect to r and z, we have

$$-\Big\{\Big[\int_{z_T}^{z_B} \hat{b}(z)\,\hat{\mathrm{b}}(z)^T \,\mathrm{d}z\Big] \otimes \Big[\int_{R_B}^{R_F} B(r)^T M^{-1}\vec{b}_0(r\,\bar{\mathrm{b}}_r(r) + \bar{\mathrm{b}}(r))^T \,\mathrm{d}r\Big]\Big\}\dot{\mu}_3(t)$$

$$+2\Big\{\Big[\int_{z_T}^{z_B}\hat{b}(z)\,\hat{\mathrm{b}}(z)^T\,\mathrm{d}z\Big]\otimes\Big[\int_{R_B}^{R_F}B(r)^TM^{-1}\vec{b}_0\,\bar{\mathrm{b}}(r)^T\,\mathrm{d}r\Big]\Big\}\dot{\bar{\eta}}_4(t)$$

$$-\Big\{\Big[\int_{z_T}^{z_B}\hat{b}(z)\,\hat{\mathrm{b}}_z(z)^T\,\mathrm{d}z\Big]\otimes\Big[\int_{R_B}^{R_F}B(r)^TM^{-1}\vec{b}_0\,\bar{\mathrm{b}}(r)^Tr\,\mathrm{d}r\Big]\Big\}\dot{\bar{\lambda}}_3(t) \qquad (4.2.197)$$

Multiplying the corresponding term in (4.2.195) by $\Big(\hat{b}(z)\otimes B(r)^T\Big)$ and integrating with respect to r and z, we have the following results

for $\dot{W}_3$: (4.2.161) with $\dot{\bar{\nu}}_{B3}(t)=0$

for $\mathcal{A}\dfrac{1}{r}(rW_{3_r})_r$: (4.2.163) with $\bar{\nu}_{B3}(t)=0$

for $\mathcal{A}(rW_{3_z})_z$: (4.2.165) with $\bar{\nu}_{B3}(t)=0$

for $\mathcal{A}\dfrac{4}{r^2}W_3=4\mathcal{A}\dfrac{1}{r^2}W_3$: (4.2.178).

Using matrices defined for Systems I and II, (4.2.195) multiplied by $\Big(\hat{b}(z)\otimes B(r)^T\Big)$ and integrated with respect to r and z, has the form

$$\begin{aligned}G_0\dot{\bar{c}}_3(t)+&\big(G_2(r)+4G_0\big(\frac{1}{r}\big)+G_2(z)\big)\bar{c}_3(t)=\\ &=-H_0\dot{\bar{\mu}}_3(t)+2H_0(r)\dot{\bar{\eta}}_4(t)-H_1\dot{\bar{\lambda}}_3(t)-\tilde{\mathcal{R}}_B\dot{\bar{\nu}}_{\tilde{B}3}(t)-\tilde{\mathcal{R}}_T\dot{\bar{\nu}}_{\tilde{T}3}(t)\\ &\quad+\mathcal{R}_F(a)\bar{\nu}_{F3}-\tilde{\mathcal{R}}_B(a)\bar{\nu}_{\tilde{B}3}(t)-\tilde{\mathcal{R}}_T(a)\bar{\nu}_{\tilde{T}3}(t)\\ &\quad-4\mathcal{R}_F^{(a)}\big(\frac{1}{r}\big)\bar{\nu}_{F3}-4\tilde{\mathcal{R}}_B^{(a)}\big(\frac{1}{r}\big)\bar{\nu}_{\tilde{B}3}(t)-4\tilde{\mathcal{R}}_T^{(a)}\big(\frac{1}{r}\big)\bar{\nu}_{\tilde{T}3}(t)\\ &\quad-\mathcal{R}_F(z)\bar{\nu}_{F3}-\tilde{\mathcal{R}}_0(z)\bar{\nu}_{\tilde{B}3}(t)+\tilde{\mathcal{R}}_0(z)\bar{\nu}_{\tilde{T}3}(t)\end{aligned} \qquad (4.2.198)$$

From (4.2.150)-(4.2.152)

$$\dot{\bar{\mu}}_3(t)=P_{100}\mathcal{Q}_2^{-1}\Big[\bar{\mathcal{Z}}_2\dot{\bar{c}}_3(t)+\tilde{X}_{B2}\dot{\bar{\nu}}_{\tilde{B}3}(t)+\tilde{X}_{T2}\dot{\bar{\nu}}_{\tilde{T}3}(t)\Big] \qquad (4.2.199)$$

$$\dot{\bar{\eta}}_4(t)=-P_{010}\mathcal{Q}_2^{-1}\Big[\bar{\mathcal{Z}}_2\dot{\bar{c}}_3(t)+\tilde{X}_{B2}\dot{\bar{\nu}}_{\tilde{B}3}(t)+\tilde{X}_{T2}\dot{\bar{\nu}}_{\tilde{T}3}(t)\Big] \qquad (4.2.200)$$

$$\dot{\bar{\lambda}}_3(t)=P_{001}\mathcal{Q}_2^{-1}\Big[\bar{\mathcal{Z}}_2\dot{\bar{c}}_3(t)+\tilde{X}_{B2}\dot{\bar{\nu}}_{\tilde{B}3}(t)+\tilde{X}_{T2}\dot{\bar{\nu}}_{\tilde{T}3}(t)\Big] \qquad (4.2.201)$$

Applying (4.2.199)-(4.2.201) to (4.2.198), we have

$$\begin{aligned}\mathcal{H}_2\dot{\bar{c}}_3(t)+\mathcal{G}_2\bar{c}_3(t)&=\mathcal{D}_{\tilde{B}2}\dot{\bar{\nu}}_{\tilde{B}3}(t)+\mathcal{D}_{\tilde{T}2}\dot{\bar{\nu}}_{\tilde{T}3}(t)\\ &\quad+\mathcal{F}_{F2}\bar{\nu}_{F3}+\mathcal{F}_{\tilde{B}2}\bar{\nu}_{\tilde{B}3}(t)+\mathcal{F}_{\tilde{T}2}\bar{\nu}_{\tilde{T}3}(t)\end{aligned} \qquad (4.2.202)$$

where

$$\begin{aligned}
\mathcal{H}_2 &= G_0 + \big(H_0 P_{100} + 2H_0(r)P_{010} + H_1 P_{001}\big)\mathcal{A}_2^{-1}\bar{\mathcal{Z}}_2 \\
\mathcal{G}_2 &= G_2(r) + 4G_0\big(\frac{1}{r}\big) + G_2(z) \\
\mathcal{D}_{\tilde{B}2} &= -\big(H_0 P_{100} + 2H_0(r)P_{010} + H_1 P_{001}\big)\mathcal{A}_2^{-1}\tilde{X}_{B2} - \tilde{\mathcal{R}}_B \\
\mathcal{D}_{\tilde{T}2} &= -\big(H_0 P_{100} + 2H_0(r)P_{010} + H_1 P_{001}\big)\mathcal{A}_2^{-1}\tilde{X}_{T2} - \tilde{\mathcal{R}}_T \\
\mathcal{F}_{F2} &= -\big(- \mathcal{R}_F(a) + 4\mathcal{R}_F^{(a)}\big(\frac{1}{r}\big) + \mathcal{R}_F(z)\big) \\
\mathcal{F}_{\tilde{B}2} &= -\big(\mathcal{R}_B(a) + 4\mathcal{R}_B^{(a)}\big(\frac{1}{r}\big) + \tilde{\mathcal{R}}_0(z)\big) \\
\mathcal{F}_{\tilde{T}2} &= -\big(\mathcal{R}_T(a) + 4\mathcal{R}_T^{(a)}\big(\frac{1}{r}\big) + \tilde{\mathcal{R}}_0(z)\big) \qquad (4.2.203)
\end{aligned}$$

Initial conditions are given by (4.2.172) with $i = 3$ and $\bar{\nu}_{B3}(0) = 0$. From (4.2.202) and (4.2.172) with $i = 3$ and $\bar{\nu}_{B3}(0) = 0$ we obtain the following system.

System of ODEs for TCP IV (corresponds to $\sin 2\theta$ approximation):

$$\begin{aligned}
\mathcal{H}_2\dot{\bar{c}}_3(t) + \mathcal{G}_2\bar{c}_3(t) &= \mathcal{D}_{\tilde{B}2}\dot{\bar{\nu}}_{\tilde{B}3}(t) + \mathcal{D}_{\tilde{T}2}\dot{\bar{\nu}}_{\tilde{T}3}(t) \\
&\quad + \mathcal{F}_{F2}\bar{\nu}_{F3} + \mathcal{F}_{\tilde{B}2}\bar{\nu}_{\tilde{B}3}(t) + \mathcal{F}_{\tilde{T}2}\bar{\nu}_{\tilde{T}3}(t) \qquad (4.2.202)
\end{aligned}$$

with initial conditions

$$\bar{c}_3(0) = \bar{\nu}_{I3} + G_0^{-1}\big[\mathcal{R}_F\bar{\nu}_{F3} + \tilde{\mathcal{R}}_B\bar{\nu}_{\tilde{B}3}(0) + \tilde{\mathcal{R}}_T\bar{\nu}_{\tilde{T}3}(0)\big] \qquad (4.2.204)$$

System V (corresponds to $\cos 2\theta$ approximation)

System V is given by (4.2.40)-(4.2.42):

$$\begin{aligned}
M\dot{V}_4 - A\Big[\frac{1}{r}(rV_{4_r})_r - \frac{4}{r^2}V_4 + (V_{4_z})_z\Big] &= -M\dot{W}_4 + A\Big(\frac{1}{r}(rW_{4_r})_r - \frac{4}{r^2}W_4 + (W_{4_z})_z\Big) \\
&\quad -\vec{b}_0\Big(\dot{u}_{4_r} + \frac{1}{r}(\dot{u}_4 + 2\dot{v}_3) + \dot{w}_{4_z}\Big) \qquad (4.2.40)
\end{aligned}$$

with boundary conditions

$$V_4(r,z,t)=0 \qquad \text{on } \Gamma\times[0,t_f) \tag{4.2.41}$$

and initial conditions

$$V_4(r,z,0)=\tilde{V}_{I4}(r,z)-W_4(r,z,0) \tag{4.2.42}$$

Using $\mathcal{A}=M^{-1}A$, (4.2.40) has the form

$$\begin{aligned}\dot{V}_4-\mathcal{A}\Big[\frac{1}{r}(rV_{4_r})_r-\frac{4}{r^2}V_4+(V_{4_z})_z\Big] &= -M^{-1}\vec{b}_0\Big(\dot{u}_{4_r}+\frac{1}{r}\dot{u}_4+\frac{2}{r}\dot{v}_3+\dot{w}_{4_z}\Big)\\ &\quad -\dot{W}_4+\mathcal{A}\Big[\frac{1}{r}(rW_{4_r})_r-\frac{4}{r^2}W_4+(W_{4_z})_z\Big]\end{aligned} \tag{4.2.205}$$

Applying (4.2.87) to the left-hand side of (4.2.205), multiplying the result by $\Big(\hat{b}(z)\otimes B(r)^T\Big)$ and integrating with respect to r and z, we obtain

$$\begin{aligned}&\Big\{\Big[\int_{z_T}^{z_B}\hat{b}(z)\hat{b}(z)^T\,\mathrm{d}z\Big]\otimes\Big[\int_{R_B}^{R_F}B(r)^TB(r)r\,\mathrm{d}r\Big]\Big\}\dot{\bar{c}}_4(t)\\ &+\Big\{\Big[\int_{z_T}^{z_B}\hat{b}(z)\hat{b}(z)^T\,\mathrm{d}z\Big]\otimes\mathcal{A}\otimes\Big(\Big[\int_{R_B}^{R_F}\bar{b}_r(r)\bar{b}_r(r)^Tr\,\mathrm{d}r\Big]+4\Big[\int_{R_B}^{R_F}\bar{b}(r)\bar{b}(r)^T\frac{1}{r}\,\mathrm{d}r\Big]\Big)\\ &\quad+\Big[\int_{z_T}^{z_B}\hat{b}_z(z)\hat{b}_z(z)^T\,\mathrm{d}z\Big]\otimes\mathcal{A}\otimes\Big[\int_{R_B}^{R_F}\bar{b}(r)\bar{b}(r)^Tr\,\mathrm{d}r\Big]\Big\}\bar{c}_4(t)\end{aligned} \tag{4.2.206}$$

Using (4.2.90)-(4.2.92), the last term of the right-hand side of (4.2.205) is

$$\begin{aligned}&-M^{-1}\vec{b}_0\Big(\dot{u}_{4_r}+\frac{1}{r}\dot{u}_4+\frac{2}{r}\dot{v}_3+\dot{w}_{4_z}\Big)\\ =&-M^{-1}\vec{b}_0\Big[\Big(\hat{\mathrm{b}}(z)^T\otimes\bar{\mathrm{b}}_r(r)^T\Big)\dot{\bar{\mu}}_4(t)+\Big(\hat{\mathrm{b}}(z)^T\otimes\frac{1}{r}\,\bar{\mathrm{b}}(r)^T\Big)\dot{\bar{\mu}}_4(t)\\ &\quad-\Big(\hat{\mathrm{b}}(z)^T\otimes\frac{2}{r}\,\bar{\mathrm{b}}(r)^T\Big)\dot{\bar{\eta}}_3(t)+\Big(\hat{\mathrm{b}}_z(z)^T\otimes\bar{\mathrm{b}}(r)^T\Big)\dot{\bar{\lambda}}_4(t)\Big]\end{aligned}$$

Multiplying by $\Big(\hat{b}(z)\otimes B(r)^T\Big)$ and integrating with respect to r and z, we have

$$-\Big\{\Big[\int_{z_T}^{z_B}\hat{b}(z)\,\hat{\mathrm{b}}(z)^T\,\mathrm{d}z\Big]\otimes\Big[\int_{R_B}^{R_F}B(r)^TM^{-1}\vec{b}_0(r\,\bar{\mathrm{b}}_r(r)+\bar{\mathrm{b}}(r))^T\,\mathrm{d}r\Big]\Big\}\dot{\bar{\mu}}_4(t)$$

$$-2\left\{\Big[\int_{z_T}^{z_B}\hat{b}(z)\,\hat{\mathrm{b}}(z)^T\,\mathrm{d}z\Big]\otimes\Big[\int_{R_B}^{R_F}B(r)^TM^{-1}\vec{b}_0\,\bar{\mathrm{b}}(r)^T\,\mathrm{d}r\Big]\right\}\dot{\bar{\eta}}_3(t)$$

$$-\left\{\Big[\int_{z_T}^{z_B}\hat{b}(z)\,\hat{\mathrm{b}}_z(z)^T\,\mathrm{d}z\Big]\otimes\Big[\int_{R_B}^{R_F}B(r)^TM^{-1}\vec{b}_0\,\bar{\mathrm{b}}(r)^T r\,\mathrm{d}r\Big]\right\}\dot{\bar{\lambda}}_4(t) \qquad (4.2.207)$$

Multiplying the corresponding term in (4.2.205) by $\Big(\hat{b}(z)\otimes B(r)^T\Big)$ and integrating with respect to r and z, we have the following results

for $\dot{W}_4$: (4.2.161) with $\dot{\bar{\nu}}_{B4}(t)=0$

for $\mathcal{A}\frac{1}{r}(rW_{4_r})_r$: (4.2.163) with $\bar{\nu}_{B4}(t)=0$

for $\mathcal{A}(rW_{4_z})_z$: (4.2.165) with $\bar{\nu}_{B4}(t)=0$

for $\mathcal{A}\frac{4}{r^2}W_4=4\mathcal{A}\frac{1}{r^2}W_4$: (4.2.178).

Using matrices defined for Systems I and II, (4.2.195) multiplied by $\Big(\hat{b}(z)\otimes B(r)^T\Big)$ and integrated with respect to r and z, has the form

$$\begin{aligned}G_0\dot{\bar{c}}_4(t)+&\big(G_2(r)+4G_0\big(\frac{1}{r}\big)+G_2(z)\big)\bar{c}_4(t)=\\&=-H_0\dot{\bar{\mu}}_4(t)-2H_0(r)\dot{\bar{\eta}}_3(t)-H_1\dot{\bar{\lambda}}_4(t)-\tilde{\mathcal{R}}_B\dot{\bar{\nu}}_{\tilde{B}4}(t)-\tilde{\mathcal{R}}_T\dot{\bar{\nu}}_{\tilde{T}4}(t)\\&\quad+\mathcal{R}_F(a)\bar{\nu}_{F4}-\tilde{\mathcal{R}}_B(a)\bar{\nu}_{\tilde{B}4}(t)-\tilde{\mathcal{R}}_T(a)\bar{\nu}_{\tilde{T}4}(t)\\&\quad-4\mathcal{R}_F^{(a)}\big(\frac{1}{r}\big)\bar{\nu}_{F4}-4\tilde{\mathcal{R}}_B^{(a)}\big(\frac{1}{r}\big)\bar{\nu}_{\tilde{B}4}(t)-4\tilde{\mathcal{R}}_T^{(a)}\big(\frac{1}{r}\big)\bar{\nu}_{\tilde{T}4}(t)\\&\quad-\mathcal{R}_F(z)\bar{\nu}_{F4}-\tilde{\mathcal{R}}_0(z)\bar{\nu}_{\tilde{B}4}(t)+\tilde{\mathcal{R}}_0(z)\bar{\nu}_{\tilde{T}4}(t)\end{aligned}\qquad (4.2.208)$$

From (4.2.153)-(4.2.155)

$$\dot{\bar{\mu}}_4(t)=P_{100}\mathcal{Q}_2^{-1}\Big[\bar{\mathcal{Z}}_2\dot{\bar{c}}_4(t)+\tilde{X}_{B2}\dot{\bar{\nu}}_{\tilde{B}4}(t)+\tilde{X}_{T2}\dot{\bar{\nu}}_{\tilde{T}4}(t)\Big] \qquad (4.2.209)$$

$$\dot{\bar{\eta}}_3(t)=-P_{010}\mathcal{Q}_2^{-1}\Big[\bar{\mathcal{Z}}_2\dot{\bar{c}}_4(t)+\tilde{X}_{B2}\dot{\bar{\nu}}_{\tilde{B}4}(t)+\tilde{X}_{T2}\dot{\bar{\nu}}_{\tilde{T}4}(t)\Big] \qquad (4.2.210)$$

$$\dot{\bar{\lambda}}_4(t)=P_{001}\mathcal{Q}_2^{-1}\Big[\bar{\mathcal{Z}}_2\dot{\bar{c}}_4(t)+\tilde{X}_{B2}\dot{\bar{\nu}}_{\tilde{B}4}(t)+\tilde{X}_{T2}\dot{\bar{\nu}}_{\tilde{T}4}(t)\Big] \qquad (4.2.211)$$

Applying (4.2.209)-(4.2.211) to (4.2.208), we have

$$\begin{aligned}\mathcal{H}_2\dot{\bar{c}}_4(t)+\mathcal{G}_2\bar{c}_4(t)=&\,\mathcal{D}_{\tilde{B}2}\dot{\bar{\nu}}_{\tilde{B}4}(t)+\mathcal{D}_{\tilde{T}2}\dot{\bar{\nu}}_{\tilde{T}4}(t)\\&+\mathcal{F}_{F2}\bar{\nu}_{F4}+\mathcal{F}_{\tilde{B}2}\bar{\nu}_{\tilde{B}4}(t)+\mathcal{F}_{\tilde{T}2}\bar{\nu}_{\tilde{T}4}(t)\end{aligned}\qquad (4.2.212)$$

where matrices $\mathcal{H}_2, \mathcal{G}_2, \mathcal{D}_{\tilde{B}2}, \mathcal{D}_{\tilde{T}2}, \mathcal{F}_{F2}, \mathcal{F}_{\tilde{B}2}, \mathcal{F}_{\tilde{T}2}$ are given by (4.2.203).

Initial conditions are given by (4.2.172) with $i = 4$ and $\bar{\nu}_{B4}(0) = 0$.

From (4.2.212) and (4.2.172) with $i = 4$ and $\bar{\nu}_{B4}(0) = 0$ we obtain the following system.

System of ODEs for TCP V (corresponds to $\cos 2\theta$ approximation):

$$\mathcal{H}_2\dot{\bar{c}}_4(t) + \mathcal{G}_2\bar{c}_4(t) = \mathcal{D}_{\tilde{B}2}\dot{\bar{\nu}}_{\tilde{B}4}(t) + \mathcal{D}_{\tilde{T}2}\dot{\bar{\nu}}_{\tilde{T}4}(t) + \mathcal{F}_{F2}\bar{\nu}_{F4} + \mathcal{F}_{\tilde{B}2}\bar{\nu}_{\tilde{B}4}(t) + \mathcal{F}_{\tilde{T}2}\bar{\nu}_{\tilde{T}4}(t) \tag{4.2.212}$$

with initial conditions

$$\bar{c}_4(0) = \bar{\nu}_{I4} + G_0^{-1}\big[\mathcal{R}_F\bar{\nu}_{F4} + \tilde{\mathcal{R}}_B\bar{\nu}_{\tilde{B}4}(0) + \tilde{\mathcal{R}}_T\bar{\nu}_{\tilde{T}4}(0)\big] \tag{4.2.213}$$

We have the following five systems of ODEs:

System of ODEs for TCP I (radially symmetric case):

$$\mathcal{H}_0\dot{\bar{c}}_0(t) + \mathcal{G}_0\bar{c}_0(t) = \mathcal{D}_{B0}\dot{\bar{\nu}}_{B0}(t) + \mathcal{D}_{\tilde{B}0}\dot{\bar{\nu}}_{\tilde{B}0}(t) + \mathcal{D}_{\tilde{T}0}\dot{\bar{\nu}}_{\tilde{T}0}(t) + \mathcal{F}_{B0}\bar{\nu}_{B0}(t) + \mathcal{F}_{F0}\bar{\nu}_{F0} + \mathcal{F}_{\tilde{B}0}\bar{\nu}_{\tilde{B}0}(t) + \mathcal{F}_{\tilde{T}0}\bar{\nu}_{\tilde{T}0}(t) \tag{4.2.169}$$

with initial conditions

$$\bar{c}_0(0) = \bar{\nu}_{I0} + G_0^{-1}\big[\mathcal{R}_B\bar{\nu}_{B0}(0) + \mathcal{R}_F\bar{\nu}_{F0} + \tilde{\mathcal{R}}_B\bar{\nu}_{\tilde{B}0}(0) + \tilde{\mathcal{R}}_T\bar{\nu}_{\tilde{T}0}(0)\big] \tag{4.2.173}$$

System I gives coefficients V_0 , u_0, ω_0 of the θ-approximation (4.2.18).

System of ODEs for TCP II (corresponds to $\sin\theta$ approximation):

$$\mathcal{H}_1\dot{\bar{c}}_1(t) + \mathcal{G}_1\bar{c}_1(t) = \mathcal{D}_{\tilde{B}1}\dot{\bar{\nu}}_{\tilde{B}1}(t) + \mathcal{D}_{\tilde{T}1}\dot{\bar{\nu}}_{\tilde{T}1}(t) + \mathcal{F}_{F1}\bar{\nu}_{F1} + \mathcal{F}_{\tilde{B}1}\bar{\nu}_{\tilde{B}1}(t) + \mathcal{F}_{\tilde{T}1}\bar{\nu}_{\tilde{T}1}(t) \tag{4.2.183}$$

with initial conditions

$$\bar{c}_1(0) = \bar{\nu}_{I1} + G_0^{-1}\big[\mathcal{R}_F\bar{\nu}_{F1} + \tilde{\mathcal{R}}_B\bar{\nu}_{\tilde{B}1}(0) + \tilde{\mathcal{R}}_T\bar{\nu}_{\tilde{T}1}(0)\big] \tag{4.2.185}$$

System II gives $\sin\theta$-coefficients V_1, u_1, w_1 and $\cos\theta$-coefficient v_2 of the θ-approximation (4.2.18).

System of ODEs for TCP III (corresponds to $\cos\theta$ approximation):

$$\begin{aligned}\mathcal{H}_1\dot{\bar{c}}_2(t) + \mathcal{G}_1\bar{c}_2(t) &= \mathcal{D}_{\tilde{B}1}\dot{\bar{\nu}}_{\tilde{B}2}(t) + \mathcal{D}_{\tilde{T}1}\dot{\bar{\nu}}_{\tilde{T}2}(t)\\ &\quad + \mathcal{F}_{F1}\bar{\nu}_{F2} + \mathcal{F}_{\tilde{B}1}\bar{\nu}_{\tilde{B}2}(t) + \mathcal{F}_{\tilde{T}1}\bar{\nu}_{\tilde{T}2}(t)\end{aligned} \tag{4.2.193}$$

with initial conditions

$$\bar{c}_2(0) = \bar{\nu}_{I2} + G_0^{-1}\big[\mathcal{R}_F\bar{\nu}_{F2} + \tilde{\mathcal{R}}_B\bar{\nu}_{\tilde{B}2}(0) + \tilde{\mathcal{R}}_T\bar{\nu}_{\tilde{T}2}(0)\big] \tag{4.2.194}$$

System III gives $\sin\theta$-coefficient v_1 and $\cos\theta$-coefficients V_2, u_2, w_2 of the θ-approximation (4.2.18).

System of ODEs for TCP IV (corresponds to $\sin 2\theta$ approximation):

$$\begin{aligned}\mathcal{H}_2\dot{\bar{c}}_3(t) + \mathcal{G}_2\bar{c}_3(t) &= \mathcal{D}_{\tilde{B}2}\dot{\bar{\nu}}_{\tilde{B}3}(t) + \mathcal{D}_{\tilde{T}2}\dot{\bar{\nu}}_{\tilde{T}3}(t)\\ &\quad + \mathcal{F}_{F2}\bar{\nu}_{F3} + \mathcal{F}_{\tilde{B}2}\bar{\nu}_{\tilde{B}3}(t) + \mathcal{F}_{\tilde{T}2}\bar{\nu}_{\tilde{T}3}(t)\end{aligned} \tag{4.2.202}$$

with initial conditions

$$\bar{c}_3(0) = \bar{\nu}_{I3} + G_0^{-1}\big[\mathcal{R}_F\bar{\nu}_{F3} + \tilde{\mathcal{R}}_B\bar{\nu}_{\tilde{B}3}(0) + \tilde{\mathcal{R}}_T\bar{\nu}_{\tilde{T}3}(0)\big] \tag{4.2.204}$$

System IV gives $\sin 2\theta$-coefficients V_3, u_3, w_3 and $\cos 2\theta$-coefficient v_4 of the θ-approximation (4.2.18).

System of ODEs for TCP V (corresponds to $\cos 2\theta$ approximation):

$$\mathcal{H}_2\dot{\bar{c}}_4(t) + \mathcal{G}_2\bar{c}_4(t) = \mathcal{D}_{\tilde{B}2}\dot{\bar{\nu}}_{\tilde{B}4}(t) + \mathcal{D}_{\tilde{T}2}\dot{\bar{\nu}}_{\tilde{T}4}(t) + \mathcal{F}_{F2}\bar{\nu}_{F4} + \mathcal{F}_{\tilde{B}2}\bar{\nu}_{\tilde{B}4}(t) + \mathcal{F}_{\tilde{T}2}\bar{\nu}_{\tilde{T}4}(t) \quad (4.2.212)$$

with initial conditions

$$\bar{c}_4(0) = \bar{\nu}_{I4} + G_0^{-1}\big[\mathcal{R}_F\bar{\nu}_{F4} + \tilde{\mathcal{R}}_B\bar{\nu}_{\tilde{B}4}(0) + \tilde{\mathcal{R}}_T\bar{\nu}_{\tilde{T}4}(0)\big] \quad (4.2.213)$$

System V gives $\sin 2\theta$-coefficient v_3 and $\cos 2\theta$-coefficients V_4, u_4, w_4 of the θ-approximation (4.2.18).

Thus, in this chapter, numerical methods for solving fully coupled parabolic-elliptic initial-boundary value problems for two- and three-dimensional TCPu processes, including radially non-symmetric processes in an inclined borehole were developed. These methods are based on a hybrid Fourier-finite-element approximation technique proposed in this research. The technique also involves a boundary penalization strategy and captures borehole geometry and mechanics. The proposed numerical methods reduce the solution of fully coupled parabolic-elliptic initial-boundary value problems to the solution of systems of time-dependent ODEs supplemented by initial conditions. The systems of ODEs explicitly contain data of the underlying boundary conditions, thereby allowing numerical analysis of a control problem in which the temperature and pressure on the borehole boundary are considered as control parameters used to achieve desirable stresses in the neighborhood of the borehole.

Chapter 5

Experimental Validation of the Fourier-Finite Element Method

In this chapter, we experimentally validate the Fourier-finite element method for the TCPu models developed in Chapter 4. Numerical experiments are performed using a MATLAB code that we have written to implement the above numerical method. In this code, the backward Euler approximation is used to discretize the temporal domain. The validation is done by comparing experimental results to other available analytical and previously validated numerical solutions for thermo-chemo-poroelastic wellbore problems.

For our numerical experiments, we focus on the two-dimensional radially symmetric fully coupled TCPu model. In this case, the temperature T, the solute mass fraction C, and the pore pressure p on the far-field boundary are considered as independent of the polar angle θ. However, applied far-field boundary stresses remain θ-dependent in accordance with realistic geomechanical behavior around the borehole. We assume that an annular region exterior to the borehole has the inner (wellbore) radius $R_B = 0.1$ m and the outer (far-field) radius $R_F = 1$ m. The values of physical constants used in the computations are presented in Appendix A.

Below, we present the experimental results for the temperature T, the solute mass fraction C, and the pore pressure p, as well as for total radial stress τ_r and total tangential stress τ_θ at different polar angles θ. Each figure contains four solution curves that correspond to time intervals of 1 hour, 12 hours, 24 hours, and 120 hours.

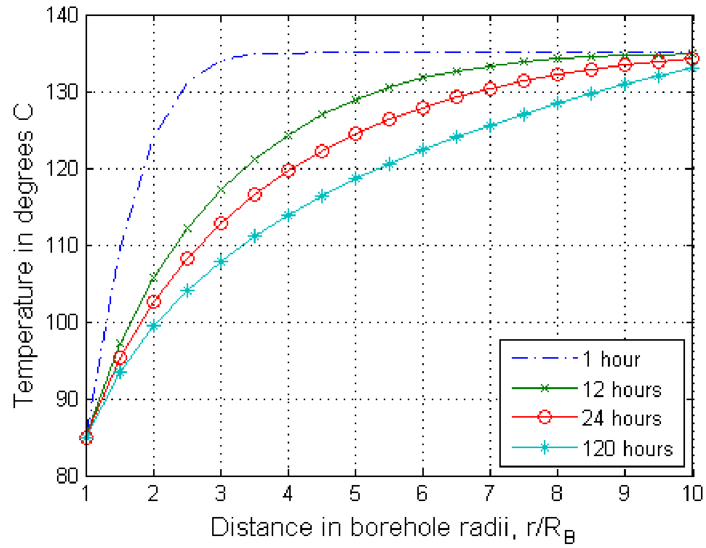

Figure 5.1: Temperature distribution around the borehole

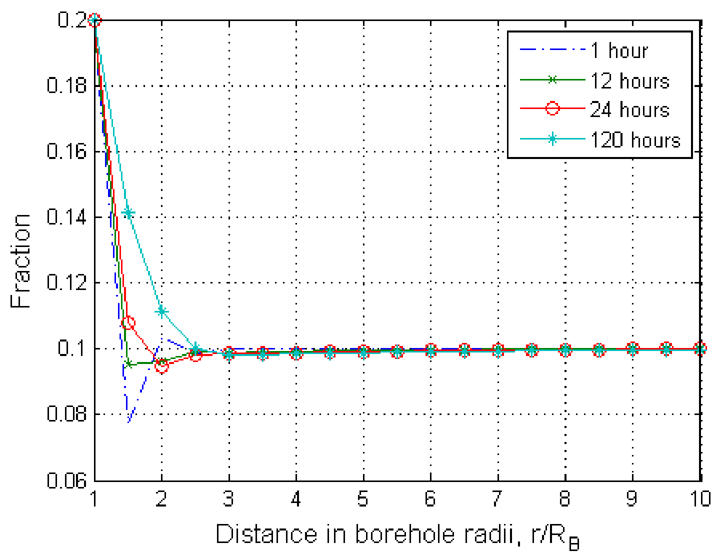

Figure 5.2: Solute mass fraction distribution around the borehole

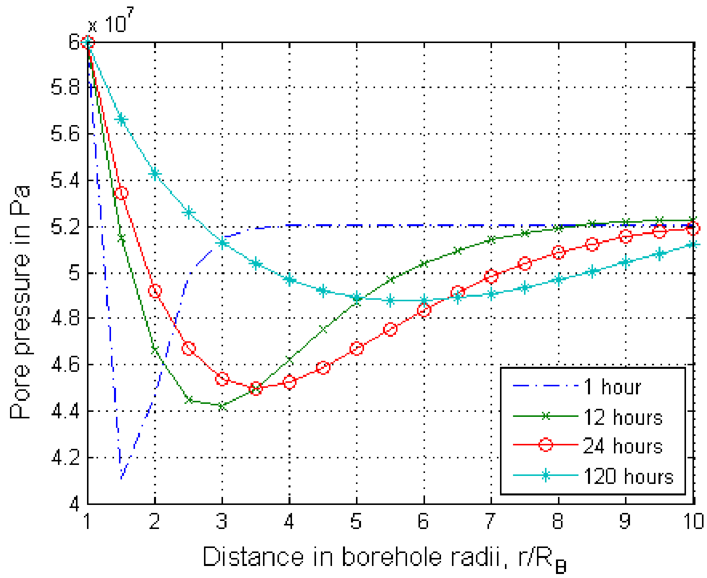

Figure 5.3: Pore pressure distribution around the borehole

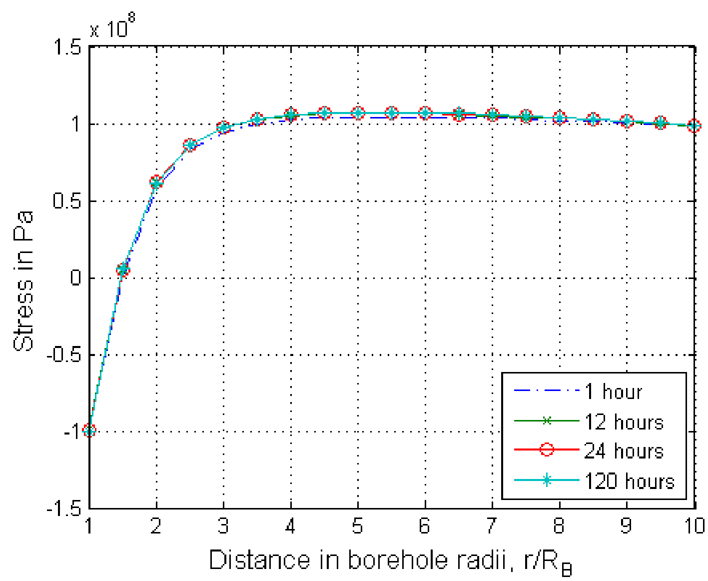

Figure 5.4: Total radial stress distribution around the borehole at the polar angle of $\theta = 0$

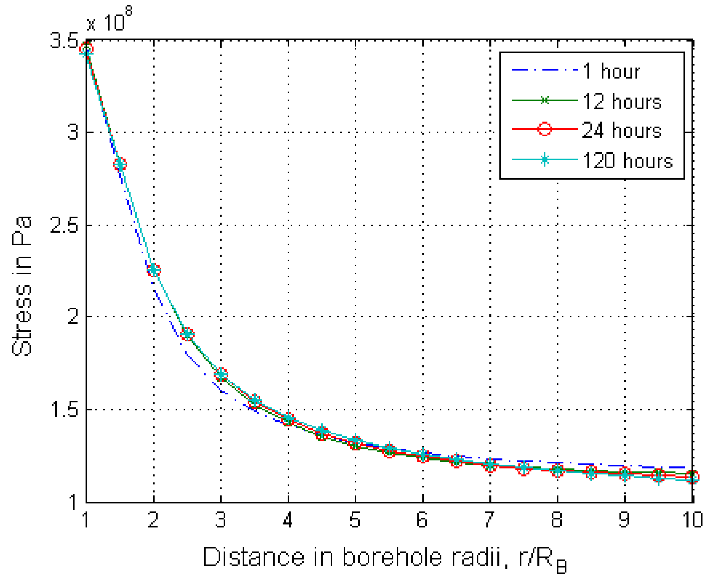

Figure 5.5: Total tangential stress distribution around the borehole at the polar angle of $\theta = 0$

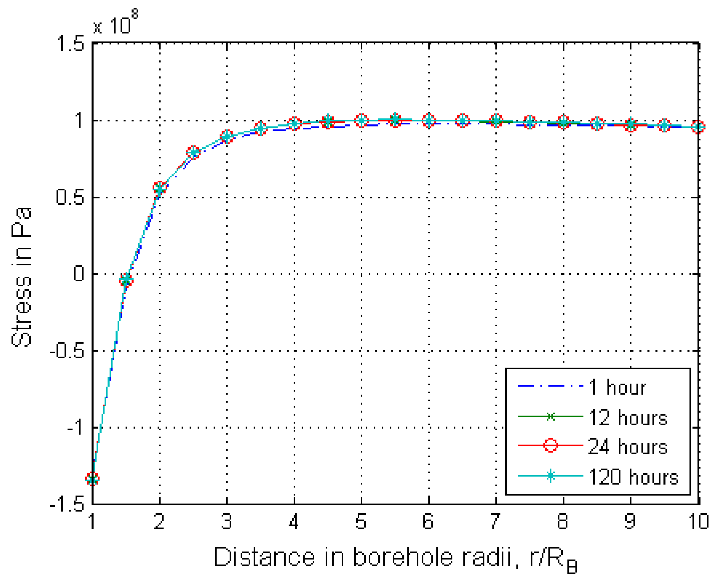

Figure 5.6: Total radial stress distribution around the borehole at the polar angle of $\theta = 30$

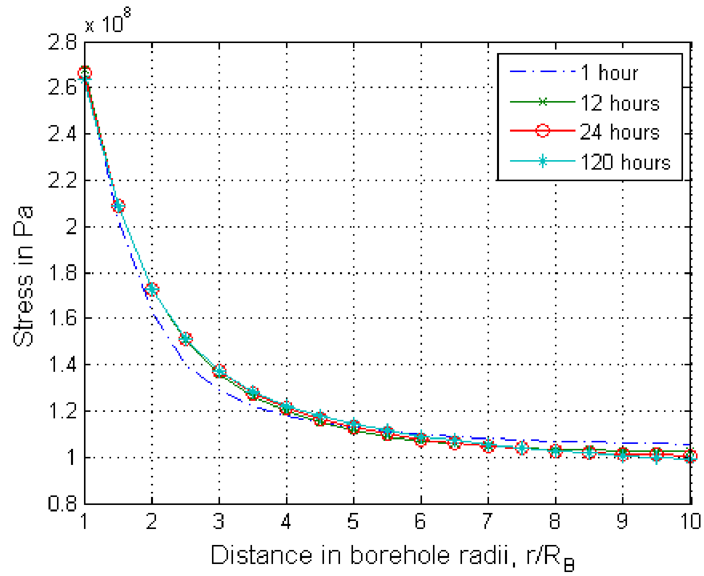

Figure 5.7: Total tangential stress distribution around the borehole at the polar angle of $\theta = 30$

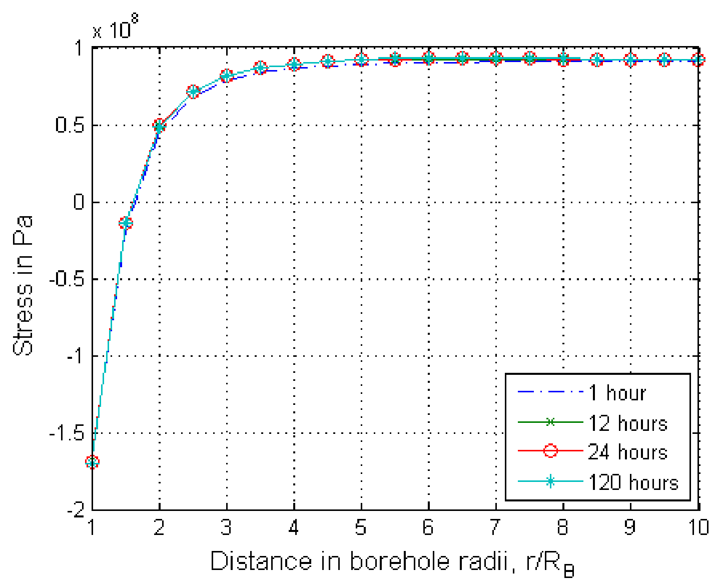

Figure 5.8: Total radial stress distribution around the borehole at the polar angle of $\theta = 45$

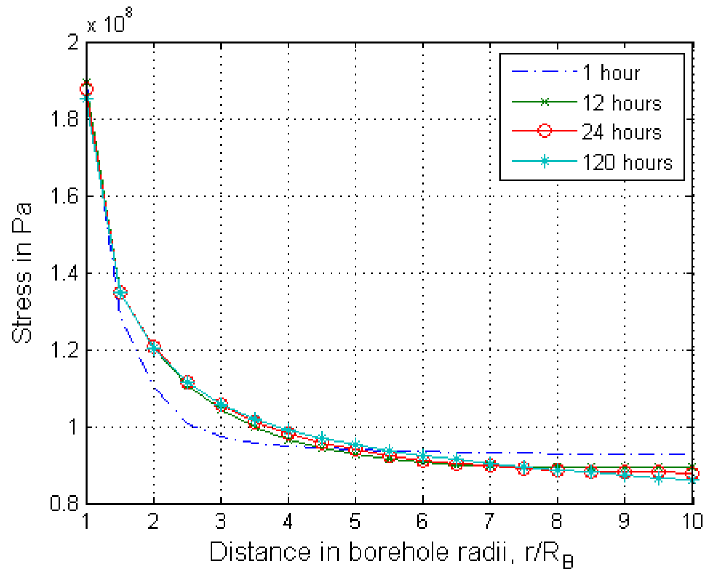

Figure 5.9: Total tangential stress distribution around the borehole at the polar angle of $\theta = 45$

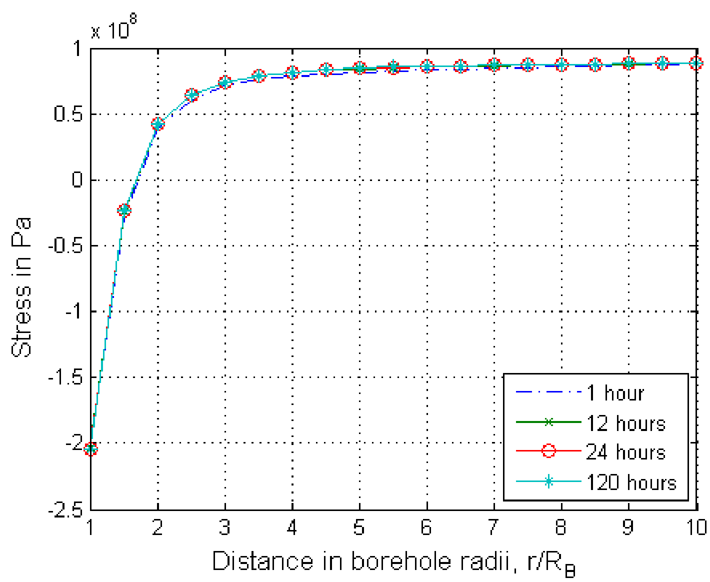

Figure 5.10: Total radial stress distribution around the borehole at the polar angle of $\theta = 60$

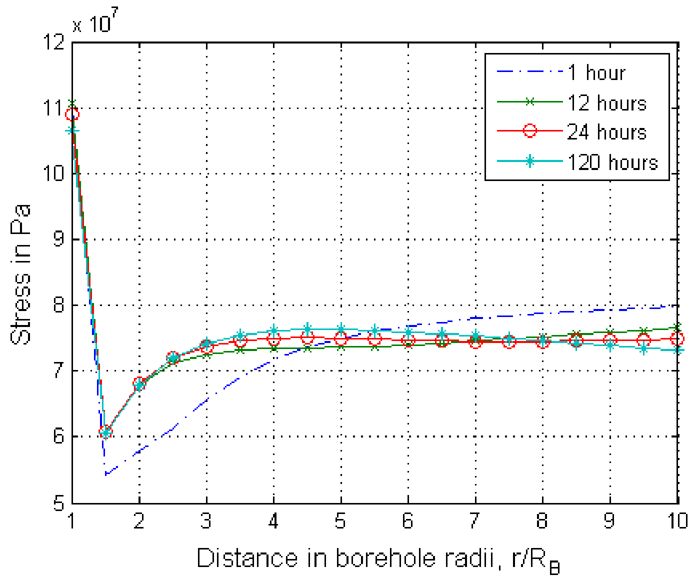

Figure 5.11: Total tangential stress distribution around the borehole at the polar angle of $\theta = 60$

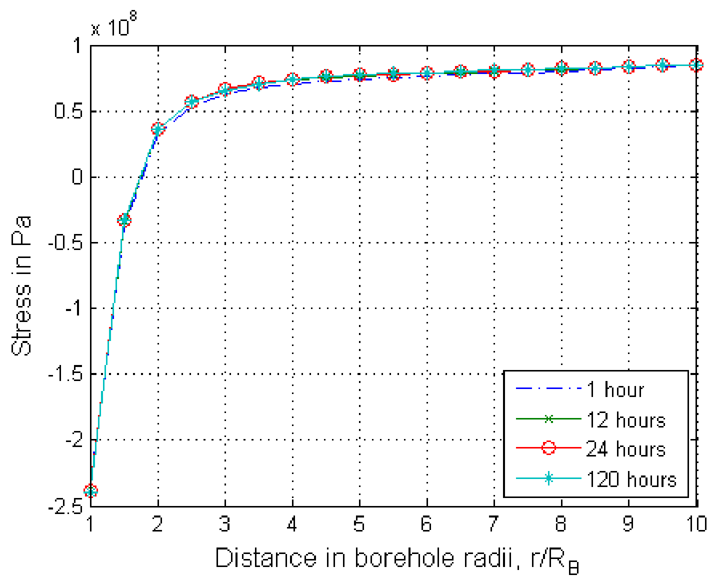

Figure 5.12: Total radial stress distribution around the borehole at the polar angle of $\theta = 90$

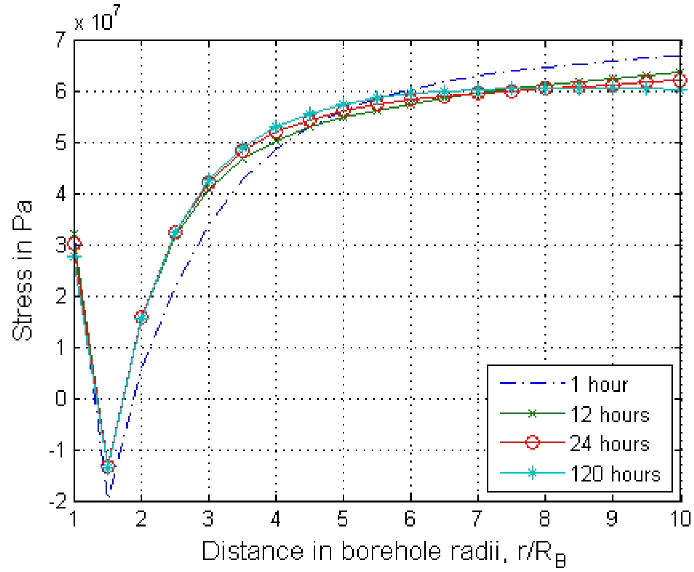

Figure 5.13: Total tangential stress distribution around the borehole at the polar angle of $\theta = 90$

The obtained experimental results are consistent with available analytical and previously validated numerical solutions for thermo-chemo-poroelastic wellbore problems presented, for instance, in Zhou and Ghassemi [46] or Rafieepour at el. [34]. The existing discrepancy between our results and those presented in [34, 46] is due to the essential difference between the TCPu model and the models considered in [34, 46]. The TCPu model includes all coupling mechanisms, whereas both models [34, 46] deal with fully decoupled thermal diffusion, and [34] neglects couplings in the solute diffusion equation, as well. This allows us to conclude that the developed Fourier-finite element method provides correct and satisfactory numerical approximation to the coupled TCPu models.

Chapter 6

Conclusions and Future Work

This dissertation has addressed the questions of well-posedness and approximation for a specific class of coupled parabolic-elliptic initial-boundary value problems with applications to geomechanics. The underlying thermo-chemo-poroelastic (TCPu) model constitutes the general theory of chemical thermo-poro-elasticity that fully couples thermal, chemical, hydrolic, and mechanical processes in homogeneous isotropic fluid-saturated porous media. The main results obtained within the dissertation research are summarized below. These results are fundamental to posing optimal control problems using borehole parameters as controls to achieve desired stress distributions in the neighborhood of the borehole.

A sufficient condition for well-posedness in the weak sense of the coupled parabolic-elliptic initial-boundary value problems for the two-and three-dimensional TCPu models has been established. This condition depends on physical parameters of the system, coupling vectors, and the Korn's constant. For a two-dimensional annular region that does not experience pure rotation and has the ratio of inner to outer radii not exceeding e^{-1}, the sufficient condition for the well-posedness of the problem has been formulated in terms of physical parameters, coupling vectors, and the ratio of the inner and outer radii. The obtained results have provided a mathematical basis for the general theory of fully coupled chemo-thermo-poroelasticity in homogeneous isotropic porous media saturated by a fluid.

Numerical methods for solving fully coupled parabolic-elliptic initial-boundary value problems for two- and three-dimensional TCPu processes, including radially non-symmetric processes in an inclined borehole have been developed. These methods are based on a hybrid Fourier-finite-element approximation technique proposed in this research. The technique also involves a boundary penalization strategy and captures borehole geometry and mechanics. The developed numerical methods allow numerical analysis of a control problem in which the temperature and pressure on the borehole boundary are considered as control parameters used to achieve desirable stresses in the neighborhood of the borehole.

The experimental validation of the developed numerical method was performed in MATLAB using the two-dimensional radially symmetric fully coupled TCPu model with real data. The experimental results have confirmed that the hybrid Fourier-finite-element method provides correct and satisfactory numerical approximation to the coupled TCPu models for the purpose of boundary control and comprehensive numerical investigation of the corresponding fully coupled reservoir-geomechanical processes that impact borehole stability.

There are several promising directions for further research on coupled parabolic-elliptic systems with applications to geomechanics:

1. An open question that immediately arises from this dissertation is establishing necessary conditions for well-posedness of the coupled parabolic-elliptic initial-boundary value problems for the two- and three-dimensional TCPu models.

2. The second direction is the development of a well-posedness theory and numerical methods for coupled parabolic-elliptic systems with variable coefficients. The variability of coefficients results in nonlinearity of the coupled system but allows modeling fully coupled reservoir-geomechanical processes in inhomogeneous fluid-saturated porous media.

3. The other project is the development of the optimal boundary control theory for systems governed by coupled linear and nonlinear parabolic-elliptic partial

differential equations. This research direction is motivated by geomechanics and wellbore stability problems in which the temperature and pressure at the borehole are considered as control parameters used to obtain desirable stresses in the neighborhood of the borehole.

4. Another project is also dictated by the geomechanical nature of the problem and includes the development of the mathematical theory of fully coupled thermo-chemo-poroelastic fracture initiation and propagation processes in fluid-saturated porous media.

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Appendix A

Physical Parameters of the Coupled TCPu System

The coefficients in the matrices and vectors (2.1.17) and (2.1.30),

$$M = \begin{bmatrix} \Lambda & \Sigma & \Phi \\ 0 & \phi & 0 \\ \Gamma & \chi & \Psi \end{bmatrix}, \qquad A = \begin{bmatrix} \dfrac{k^T}{T_F} & \dfrac{\rho_f R T_F D^T}{M^s(1-C_F)} & -K^T \\ C_F D^T & D & -\dfrac{k\mathfrak{R}}{\eta} \\ -K^T & -\dfrac{\rho_f \mathfrak{R}\Omega k}{\eta} & \dfrac{k}{\eta} \end{bmatrix}$$

$$\vec{b}_0 = \begin{bmatrix} \zeta \\ 0 \\ \alpha \end{bmatrix}, \qquad \vec{b}_1 = \begin{bmatrix} \tilde{\zeta} \\ -\xi \\ \tilde{\alpha} \end{bmatrix}$$

are the following:

$$\Lambda = \frac{c}{T_F} - s_F\beta_s\tilde{\omega}, \qquad c = (1-\phi)\rho_s c_s + \phi\rho_f c_f, \quad s_F = c_f - \frac{\beta_f p_F}{\rho_f}, \quad \tilde{\omega} = \frac{\omega}{1-C_F}$$

$$\Sigma = \xi\beta_s, \qquad \xi = \frac{RT_F\tilde{\omega}(1-2C_F)}{M^s C_F}$$

$$\Phi = \Xi - \phi\beta_f + \frac{\beta_s\tilde{\omega}}{\rho_f}, \qquad \Xi = (\alpha-\phi)\beta_s, \qquad \alpha = 1 - \frac{K}{K_s}$$

$$\Gamma = \Xi - \phi\beta_f - \frac{s_F\tilde{\omega}}{K_s}, \qquad \chi = -\frac{\xi}{K_s}$$

$$\Psi = \psi + \frac{\phi}{K_f} - \frac{\tilde{\omega}}{\rho_f K_s}, \qquad \psi = \frac{\alpha-\phi}{K_s}$$

$$\Omega = \frac{RT_F}{M^s(1-C_F)C_F}$$

$$\zeta = K\beta_s, \qquad \tilde{\zeta} = \zeta + s_F\tilde{\omega}, \qquad \tilde{\alpha} = \alpha - \frac{\tilde{\omega}}{\rho_f}$$

where s_F is specific fluid entropy, respectively, R is the universal gas constant, α is Biot effective stress coefficient, and K is the rock's bulk modulus. Other physical properties of the rock/fluid system and input parameters for examples are given in Table A.1 and Table A.2, respectively.

Table A.1: Physical properties of the rock/fluid system

Drained elastic modulus, E	45×10^9 Pa
Solid bulk modulus, K_s	65×10^9 Pa
Drained Poisson's ratio, ν	0.27
Permeability/Fluid viscosity ratios, k/η	10^{-17} m^2/(Pa·s)
Reference porosity, ϕ	0.15
Fluid density, ρ_f	1111 kg/m^3
Fluid specific heat capacity, c_f	4186 J/(Kg°K)
Fluid bulk modulus, K_f	3.3×10^9 Pa
Fluid volumetric thermal expansion coefficient, β_f	3×10^{-4} °K^{-1}
Solid density, ρ_s	2.83×10^3 kg/m^3
Solid specific heat capacity, c_s	920 J/(Kg°K)
Solid volumetric thermal expansion coefficient, β_s	2.4×10^{-5} °K^{-1}
Rock thermal conductivity coefficient, k^T	4 W/(m°K)
Thermal osmosis coefficient, K^T	10^{-11} m^2/(s°K)
Solute molar mass, M^s	0.1111 kg/m^3
Solute reflected fraction, $\mathfrak{R}$	0.01
Solute chemical diffusion coefficient, D	10^{-9} m^2/s
Solute thermal diffusion coefficient, D^T	10^{-10} m^2/(s°K)
Chemical stress coupling parameter, ω	100 kg/m^3
Universal gas constant, R	8.3 JK^{-1}/mol^{-1}

Table A.2: Input parameters for experimental validation

In situ hydrostatic stress, $\sigma_x = \sigma_y$	91MPa (Therm. ex.) 88.5 MPa (Chem. ex.)
In situ non-hydrostatic stress	$\sigma_x = 91$ MPa $\sigma_y = 86$ MPa
In situ formation pore pressure, p_F	52 MPa
Mud pressure, p_B	60 MPa
Mud solute mass fraction, C_B	0.2
Reference formation solute mass fraction, C_F	0.1
In situ formation temperature, T_F	135°C
Mud temperature, T_B	85 °C
Wellbore radius, R_B	0.1 m