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CHAPTER I

INTRODUCTION

§1.1 Historical Background

The last fifteen years have been a period of rapid progress in the field of algebraic topology. The important contributions of Eilenberg, Cartan, Leray, Serre, Steenrod, and Grothendieck have not only added greatly to the store of knowledge of the modern algebraic topologist, but also radically changed his point of view, as is graphically demonstrated by a comparison of Eilenberg and Steenrod's modern treatise [6] with the "classical" work of Lefschetz [8] which antedates it by less than a decade. It will be assumed that the reader is to some extent already familiar with this "modern viewpoint," (and in particular with the book by Eilenberg and Steenrod just cited), although we will devote the next few pages to a brief outline of some of the recent concepts and theories which are related to this thesis.

The notions of category and functor first acquired formal status in a paper by Eilenberg and MacLane [11] . A <u>category</u> is a collection of "objects" and "mappings" of one object into another, subject to the condition that the composition of two maps in the category, and the identity map of any object, should be in the category. The general nature of a category is indicated by the examples; "groups and homomorphisms," "topological spaces and continuous mappings," and "vector spaces and linear transformations." A functor is a mapping of one category into another which takes objects into objects, maps into maps of the corresponding objects, and preserves identities and compositions. That is, denoting our functor by T, we must have that $T(f \circ g) = T(f) \circ T(g)$ and if i is the identity map of A, then T(i) is the identity map of T(A). If $f:A \to B$ we may have either $T(f): T(A) \to T(B)$ or $T(f): T(B) \to T(A)$. In the first case the functor is said to be covariant, in the second case contravariant. A recent book in which these concepts are further explored and exhaustively applied to problems in pure algebra is [2].

An important technique of the "modern school" of topologists is the use of "diagrams," which are networks whose nodes represent objects of a category, the nodes being joined by arrows which represent mappings of the category. The diagram is said to be commutative if all paths between a given pair of nodes represent the same map. It is clear that the functorial image of a commutative diagram is an identical commutative diagram.

In [6], Eilenberg and Steenrod define a homology theory as a covariant functor from a category of pairs of topological spaces into the category of graded abelian groups, which satisfies five axioms,[1] and they showed that on a reasonably large category of pairs of spaces the axioms are categorical. Previously a large number of ways of constructing homology groups had been suggested, each having its advantages, and an

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[1]Eilenberg and Steenrod list 7 axioms--the first two, however, are merely the statement that a homology theory is a covariant functor.

enormous amount of work was done in establishing the relationship between these various theories. Now it is necessary only to verify that a particular homology theory satisfies these axioms, and it is immediate that all of the theorems established for other theories, hold for it as well.

The "classical" methods of constructing homology theories can be interpreted as "factoring" the homology functor into two "sub functors," with the intermediate category being the category of differential-graded abelian groups (or <u>chain groups</u>) and their homomorphisms which preserve gradation and commute with the differential operator.[1] The distinguishing characteristic of a homology theory is the way in which the first functor (from pairs of spaces to d-g-groups) is defined. This functor will be called the <u>chain functor</u> and the other will be called the <u>Mayer functor</u>, after the man who established most of its properties. Chapter 5 of [6] gives an account of the Mayer functor, which is constructed in the same manner for all theories.

Originally, a space was "triangulated" to get a simplicial complex, and the chain groups were free groups whose generators were the simplexes of that complex. Although it has been established, with some difficulty, that this composite homology functor is independent of the triangulation used in the construction of the chain functor, this approach still suffers from the defect that not enough spaces are triangulable, nor can the triangulable spaces be characterized in terms of other more useful topological properties. The <u>Čech</u> approach was to use the nerves of coverings of the space as the simplicial complex and to form homology

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[1]The abstract properties of graded differential algebras are discussed by Chevalley in [4].

groups, as before, for each covering. The Čech homology functor is defined as the inverse limit of this spectrum of functors while the Čech cohomology functor is the direct limit of the spectrum of cohomology functors. The singular chain groups are free groups whose generators are mapping of fixed standard simplexes into a topological space. These constructions are also given in [6]. Still another approach was that taken by Alexander, who defined cohomology groups by taking as M-cochains, functions of M + 1 variables modulo the functions which vanished whenever all their arguments were sufficiently close together. The Alexander theory was put in its present form by Spanier in [17].

All of these theories are defined with respect to a fixed group--called the coefficient group. One way to generalize this situation is to allow the coefficient group or even the construction of the chain groups to vary continuously in some manner from point to point of the space. This is the situation which is studied in sheaf theory--the sheaf is, of course, this continuously varying collection of algebraic structures defined on a topological space.

Sheaves were first mentioned in 1946 by Leray in a brief paragraph in the Comptes Rendus of the French Academy of Sciences and they have been defined and redefined several times since then--it is, in fact, quite possible that this concept has still not found its final form. In 1950 appeared Leray's important paper [15] which introduced spectral sequences and contained the first detailed account of sheaves. Leray's definition of a sheaf was what would now be called a presheaf indexed by the closed sets of a space. Shortly thereafter in [3], sheaf theory was redone by Cartan using a "topological" definition of a sheaf which he credited to Lazard. The notion of a fine sheaf, which is essentially a sheaf of algebraic structures with a unit that can be expressed as the sum of arbitrarily small "local units," is instrumental in both of these theories. Cartan's approach to cohomology with coefficients in a sheaf F of modules over a principal ring R, was to form a fundamental sheaf which is a resolution of the constant sheaf R and tensor it with F.

In 1955, Serre [16] introduced coherent sheaves and constructed a cohomology theory in a manner analogous to the classical Čech approach. He used local coefficients for his Čech cochains, namely the module of local sections of the sheaf F, which is defined for every open set. Using an appropriately defined boundary operator he thus obtained a cohomology module for each cover of a space, and by taking a direct limit he obtained the cohomology of the space.

In the last two years still another different approach to sheaf theory was proposed by Godement [7] and Grothendieck [13]. In these works, cohomology was defined in terms of a special, canonical, resolution of a sheaf, and a sheaf is defined merely as a presheaf which satisfies a pair of additional axioms. The topology of a sheaf plays a subordinate role in this approach--Lazard's sheaves are Godement's associated "espaces etalés." In this thesis the approach of Godement and Grothendieck will be carried to a natural conclusion and the sheaf topology will be suppressed entirely.

§1.2 The Hurewicz Theorem

The starting point for the investigations in this thesis is a well-known theorem of Hurewicz, which is unfortunately not accessible in the standard mathematical literature. In this section we will give a proof of the Hurewicz theorem and indicate some of the questions raised by it.

A <u>generalized complex</u> $\{A, \mathfrak{A}\}$ consists of a lattice $\mathfrak{A}$, a differential R-module A, and a family of submodules $\{A_\alpha\}$ of A, indexed by $\mathfrak{A}$, which satisfies the conditions:

GC(1) $d(A_\alpha) \subset A_\alpha$ for each $\alpha \in \mathfrak{A}$

GC(2) $A_{\alpha \cap \beta} = A_\alpha \cap A_\beta$ and $A_{\alpha \cup \beta} = A_\alpha + A_\beta$ for all $\alpha, \beta \in \mathfrak{A}$.

A <u>map</u> $\{f, \phi\} : \{A, \mathfrak{A}\} \to \{B, \mathfrak{B}\}$ of one generalized complex into another consists of a lattice homomorphism $\phi : \mathfrak{A} \to \mathfrak{B}$ and a set of module homomorphisms $f_\alpha : A_\alpha \to B_{\phi(\alpha)}$.

One observes readily that "generalized complexes and their maps" form a category. We obtain a homology theory on this category by applying the Mayer functor to each submodule A_α with its induced differential d_α, to get the groups $H_\alpha(A) = \operatorname{Ker} d_\alpha / \operatorname{Im} d_\alpha$, and the induced maps $f^*_\alpha : H_\alpha(A) \to H_{\phi(\alpha)}(B)$.

<u>Theorem 1.2.1 (Hurewicz)</u>. Let $\{f, \phi\} : \{A, \mathfrak{A}\} \to \{B, \mathfrak{B}\}$ be a map of generalized complexes. If the induced map $f^*_\alpha : H_\alpha(A) \to H_{\phi(\alpha)}(B)$ is an isomorphism for each join irreducible element, and if the lattice $\mathfrak{A}$ satisfies the descending chain condition, then $\{f^*, \phi\} : \{H(A), \mathfrak{A}\} \to \{H(B), \mathfrak{B}\}$ is an isomorphism.

<u>Proof</u>. For every pair of lattice elements α, β we have an associated "Mayer-Vietoris" couple:

$$0 \longrightarrow A_{\alpha \cap \beta} \xrightarrow{\Phi} A_\alpha \oplus A_\beta \xrightarrow{\Psi} A_{\alpha \cup \beta} \longrightarrow 0$$

where the maps Φ and Ψ are defined as follows:

$$\Phi(a) = (a,a) \quad \text{and} \quad \Psi(a,b) = a-b$$

It is clear from the definitions of Φ and Ψ, and the fact that $A_{\alpha \cup \beta} = A_\alpha + A_\beta$ that the couple is exact. Moreover our map f induces a mapping of Mayer-Vietoris couples with the following commutative diagram. (For convenience, we denote $\phi(\alpha)$ by $\bar{\alpha}$.)

$$\begin{tikzcd}
0 \arrow[r] & A_{\alpha \cap \beta} \arrow[r] \arrow[d, "f_{\alpha \cap \beta}"] & A_\alpha \oplus A_\beta \arrow[r] \arrow[d, "f'_{\alpha,\beta}"] & A_{\alpha \cup \beta} \arrow[r] \arrow[d, "f_{\alpha \cup \beta}"] & 0 \\
0 \arrow[r] & B_{\bar{\alpha} \cap \bar{\beta}} \arrow[r] & B_{\bar{\alpha}} \oplus B_{\bar{\beta}} \arrow[r] & B_{\overline{\alpha \cup \beta}} \arrow[r] & 0
\end{tikzcd}$$

The map $f'_{\alpha,\beta}$ takes (a,b) into $(f_\alpha(a), f_\beta(b))$. On the homology level we have the usual induced diagram:

$$\begin{tikzcd}
H(A_{\alpha \cap \beta}) \arrow[r, "\Phi^*"] \arrow[d, "f^*_{\alpha \cap \beta}"] & H(A_\alpha) \oplus H(A_\beta) \arrow[r, "\Psi^*"] \arrow[d, "f'^*_{\alpha,\beta}"] & H(A_{\alpha \cup \beta}) \arrow[d, "f^*_{\alpha \cup \beta}"] \arrow[ll, bend right, "\Delta"'] \\
H(B_{\bar{\alpha} \cap \bar{\beta}}) \arrow[r, "\Phi^*"] & H(B_{\bar{\alpha}}) \oplus H(B_{\bar{\beta}}) \arrow[r, "\Psi^*"] & H(B_{\overline{\alpha \cup \beta}}) \arrow[ll, bend left, "\Delta"]
\end{tikzcd}$$

From the construction of the Mayer functor, it follows that $f'^*_{\alpha,\beta}(a,b) = (f^*_\alpha(a), f^*_\beta(b))$ whence if both f^*_α and f^*_β are isomorphisms, so is $f'^*_{\alpha,\beta}$. If in addition, $f^*_{\alpha \cap \beta}$ is an isomorphism, it follows from the five lemma[1] that $f^*_{\alpha \cup \beta}$ is also an isomorphism.

The descending chain condition implies that any subset of $\mathfrak{A}$ must have a minimal element or be empty,[2] whence let us suppose that f^*_α

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[1] See [6], page 16.

[2] See [1], page 37.

is not an isomorphism, and that α is minimal with respect to that property. By hypothesis α cannot be irreducible, so that $\alpha = \beta \cup \gamma$ and the minimality of α implies that f^*_β, f^*_γ and $f^*_{\beta \cap \gamma}$ are all isomorphisms. But we have just shown that this implies that f^*_α is an isomorphism so that we conclude that the set of α in $\mathfrak{A}$ for which f^*_α is not an isomorphism is empty, proving the theorem.

It is clear that a corresponding theorem can be proven if the lattice satisfies the ascending chain condition and we have an isomorphism on the meet irreducible elements. The proof, which is entirely analogous, can be easily supplied by the reader.

This theorem is quite useful in establishing isomorphism theorems such as the equivalence of the homology theories obtained from the ordered and oriented chain functors, and can be used to simplify the proof of the Poincaré duality theorem. In the first of these applications one considers the lattice of subcomplexes of a finite simplicial complex, (which clearly satisfies the descending chain condition), and associates with each subcomplex the oriented chain groups A and the ordered chain groups B. The join irreducible elements here are the simplices, which of course have trivial homology in both theories, and the mapping f is the natural mapping of an oriented chain onto itself, considered as an ordered chain. The announced isomorphism follows immediately.

Although we can extend the isomorphism between these homology functors to include infinite complexes (utilizing the fact that chains are carried by finite subcomplexes), and even show, with a certain amount of difficulty, that the singular and simplicial theories are isomorphic on triangulable pairs, there are still a wide variety of isomorphism theorems of this type which are known to be true, but which cannot be demonstrated using the Hurewicz theorem without some weakening of its hypotheses.

On considering the question of how to extend this result, a number of lines of investigation suggested themselves. In proving the theorem one notes that only the first four of the Eilenberg-Steenrod axioms were used--no mention was made of the homotopy, excision[1] and dimension axioms--a fact which is suggestive of the Cartan uniqueness theorem for cohomology with coefficients in a sheaf, as this theory satisfies similar axioms.[2] This analogy is further emphasized by restating the Hurewicz theorem in some form like: "A homology functor on the category of generalized complexes over a lattice $\mathfrak{A}$, which satisfies the first four Eilenberg-Steenrod axioms, is uniquely determined by its values on the irreducible elements." It is, in fact, the exploitation of this analogy which occupies a major portion of this thesis--Chapters 3 and 4 and the following section of this chapter.

In establishing the isomorphism of two homology functors, a recent and quite useful technique is the utilization of the acyclic model theorem of Eilenberg and MacLane [10]. This would amount to exchanging the descending chain condition for an assumption that the homology functor is acyclic on models. The problem is then to define the proper set of models, either the irreducible elements themselves or perhaps certain simple sublattices, and to show from the axioms that the homology

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[1]However, in more general situations, some sort of an excision axiom would be needed to establish that the Mayer-Vietoris triad is proper.

[2]See [7], page 183, for a discussion of those axioms.

functor is representable without using the descending chain condition. Unfortunately, however, this line of attack has thus far not been successful, although it does seem to be a promising direction for future research.

Certain superficial similarities between the proof of the Hurewicz theorem and theorems involving double complexes, led to a consideration of that technique. This investigation, while not of direct significance, proved of use in giving a relatively simple proof of the uniqueness theorem in Chapter 4 without the introduction of spectral sequences, and also a well-known theorem of Dowker follows readily from this construction. These results are reported in Chapter 2.

Still another interesting comment about the Hurewicz theorem is that in the construction a gradation is not required on the generalized complex, although the homology groups are graded. This means that the gradation of the homology groups is acquired through the structure of the lattice and dimension changing mappings of the exact homology sequence. Since our homology theory is uniquely determined by the groups associated with the irreducible elements, we should be able to assign groups to these elements and give a constructive process for obtaining the groups of the remainder of the lattice. This is presented also as a promising line of future research, although thus far no results have been obtained in this direction.

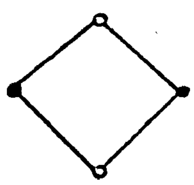
§1.3 The Lattice Approach to Topology

The process of topologizing a set S consists of assigning to it a lattice $\mathfrak{O}$, and to each element of S a proper prime filter of that lattice. The elements of $\mathfrak{O}$ are called "open sets," and the prime filter A(s) is called the "neighborhood system" of the "point" s. Since an open set is uniquely determined by listing the points which belong to it, our lattice must have the property that no two of its elements are contained in precisely the same set of prime filters, and it should clearly have both a minimal element $\emptyset$ and a maximal element M which correspond to the "empty set" and the entire set S. It should be distributive, since lattice of sets have this property, and it should be "semi-complete" in the sense that every family of lattice elements should have a least upper bound (but not necessarily a greatest lower bound), since the union of an arbitrary family of open sets is open. We thus define a <u>regular lattice</u> to be a distributive lattice with a minimal and a maximal element, satisfying: (1) no two elements of $\mathfrak{A}$ are contained in the same set of prime filters, and (2) every family of elements of $\mathfrak{A}$ has a l.u.b. In the following chapters it is understood that every lattice considered is regular, unless explicitly stated otherwise.

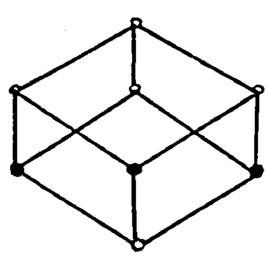
Additional axioms which serve to further restrict the type of topological space under discussion can be easily formulated in terms of lattices. For example: a <u>connected</u> lattice is one in which for <u>no</u> pair α, β of lattice elements is it true that $\alpha \cup \beta = M$ and $\alpha \cap \beta = \emptyset$; a Hausdorff lattice has the property that any two prime filters A and A' contain elements α and α' respectively with $\alpha \cap \alpha' = \emptyset$; and a <u>compact</u> lattice is one in which any family of lattice elements whose l.u.b. is M contains a finite subfamily whose l.u.b. is M.

The concept of a continuous mapping of a topological space X into a topological space Y also has a ready interpretation in this context, since by definition it induces a lattice homomorphism of the open sets of Y into the open sets of X which takes the empty set into the empty set. We will call such lattice homomorphisms <u>$\emptyset$-preserving</u>. It is of interest that one can also carry out the converse of this construction. That is, given a pair of regular lattices $\mathfrak{A}$ and $\mathfrak{A}'$, and a $\emptyset$-preserving homomorphism $h:\mathfrak{A}\to\mathfrak{A}'$, we can define an "associated" pair of topological "spaces" X and X', and an "associated" continuous mapping $h:X'\to X$. X and X' are, of course, the sets of prime filters of $\mathfrak{A}$ and $\mathfrak{A}'$, and h is defined by specifying that $h^{-1}(A)$ $(A \in X)$ is the set of prime filters of $\mathfrak{A}'$ which contain $h(A)$. To show that h is a mapping we must show that each $A' \in X'$ has a unique image $h(A')$ in X, or equivalently that if A and B are two prime filters of $\mathfrak{A}$ then the union of the sets $h(A)$ and $h(B)$ in $\mathfrak{A}'$ is contained in no prime filter. This is, however, clear, for there are elements $a \in A$ and $b \in B$ such that $a \cap b = \emptyset$ (since A and B are both prime) whence $h(a) \cap h(b) = \emptyset$ and any prime filter containing $h(A)$ and $h(B)$ would thus contain $\emptyset$ and be improper.

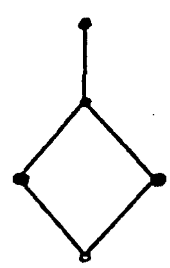
We have thus exhibited a certain equivalence between the category of spaces and continuous maps and the category of regular lattices and $\emptyset$-preserving homomorphisms, although it should be emphasized that these lattice spaces are, in general, not true topological spaces. This correspondence is contravariant and homomorphic spaces have the same associated lattices. The importance of this point of view lies in the character of its "simple examples." Consider the following finite lattices:



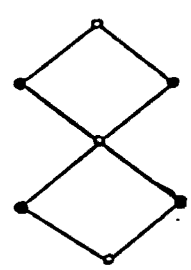
(a)



(b)



(c)



(d)

From lattice theory we know that in a distributive lattice which satisfies the descending chain condition (hence a priori in a finite lattice), every prime filter is principal and its generator is a non-minimal, join irreducible element.[1] (The black dots in the above diagrams represent such elements.) A quick examination of the above diagrams verifies that they represent regular lattices, and that the "spaces" we would associate to (c) and (d) on the basis of the foregoing construction are far from simple. In fact, they have the property that some "points" are subsets of other "points." Lattices (a) and (b) are the lattice of open sets for the two and three point discrete spaces, while (c) and (d) come from the simple cell decomposition of the closed line segment and the circle.

In the following chapters a method for associating cohomology groups with lattices will be defined, and it will be shown that the above lattices have the "right" groups. This will perhaps give a new approach to the problem of decomposing a space in some manner in which the cohomology groups become readily computible, and showing that the computation process gives the right groups. We will have a uniform computation process since we are dealing with regular lattices throughout. The problem, in a sense, is how to characterize simple regular sublattices which give the same cohomology groups.

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[1]Birkhoff - See [1], page 142. The statement follows readily from exercise 4(a) and Theorem 9.

CHAPTER II

DOUBLE COMPLEXES

§2.1 Basic Definitions

A double complex K of degree r is a system $\{K_{p,q}\}$ of R-modules (R a principal ideal ring) indexed by pairs of integers, together with two homomorphisms, d_1 and d_2, which satisfy:

DC(1) $d_1: K_{p,q} \to K_{p+r,q}$; $d_2: K_{p,q} \to K_{p,q+r}$

DC(2) $d_2 \circ d_2 = d_1 \circ d_1 = 0$

DC(3) $d_1 \circ d_2 = d_2 \circ d_1$

If r = +1, K will be called a cochain complex and if r = -1, K will be called a chain complex.

The formalism of double complexes seems to have been introduced by Serre in [16]. The usual double complex axioms replace DC(3) by the anticommutivity rule $d_2 \circ d_1 = -d_1 \circ d_2$. This is because people are usually interested in the associated "single" complex $K_n = \sum K_{p,q} (p+q=n)$ with the "diagonal" boundary operator d, which for anticommutative complexes has the simple form $d = d_1 + d_2$, whereas in our notation $d = d_1 + (-1)^p d_2$ when applied to an element in $K_{p,q}$. However, as we are not concerned with the complex K_n axiom DC(3) is slightly more convenient. It should be clear that since our theories involve only the kernels and images of d_1 and d_2, that they are equally valid in the anti-commutative case.

A subcomplex L is a collection of submodules $\{L_{p,q}\}$ of the modules $\{K_{p,q}\}$. L is said to be stable if for all p and q, $d_1(L_{p,q}) \subset L_{p+r,q}$ and $d_2(L_{p,q}) \subset L_{p,q+r}$. The subcomplexes ${}^iZ = \text{Ker } d_i$ and ${}^iB = \text{Im } d_i (i=1,2)$ are readily verified to be stable. We carry through the calculation for ${}^1Z$ and ${}^1B$. Let $a \in {}^1Z$. Then $d_1(a) = 0 \in {}^1Z$ and $d_1(d_2 a) = d_2(d_1 a) = d_2(0) = 0$ whence $d_2(a) \in {}^1Z$ also. Let $b = d_1(c) \in {}^1B$. Then $d_1(b) = 0 \in {}^1B$ and $d_2(b) = d_2(d_1 c) = d_1(d_2 c) \in {}^1B$ also.

We may thus define the quotient double complexes ${}^iH = {}^iZ/{}^iB$. As is customary, we will write p and q as subscripts when speaking of chain complexes, and as superscripts when speaking of cochain complexes.

§2.2 The Isomorphism Theorems

In this section we establish a pair of isomorphism theorems--one for homology, the other for cohomology. The homology theorem is, essentially, a theorem stated first in a paper by Kelly and Pitcher [14] , reformulated by Floyd [12] , and proved here in a slightly different setting.

Theorem 2.2.1. Let $K_{p,q}$ be a chain complex with $K_{p,q} = 0$ for $q \leqq -1$ or $p \leqq -1$.

If $${}^1H_{p,q}(K) = \begin{cases} H_p(X) & \text{for } q = -1 \\ 0 & \text{for } q > -1 \end{cases} \quad \text{and} \quad {}^2H_{p,q}(K) = \begin{cases} H_q(Y) & \text{for } p = -1 \\ 0 & \text{for } p > -1 \end{cases}$$

then $H_p(X) = H_p(Y)$.

We prove first a lemma:

Lemma 2.2.1. If the hypotheses of the theorem are satisfied, and $p,q \neq -1$, then ${}^1H_{p,-1}({}^2Z) \approx {}^1H_{p,-1}(K) = H_p(X)$ and ${}^1H_{-1,q}({}^2Z) \approx {}^2H_{-1,q}(K) = H_q(Y)$.

Proof. The first of the above statements is trivial, for since $d_2(K_{p,-1}) = 0$, $K_{p,-1}$ and ${}^2Z_{p,-1}$ are identical. The second follows from a consideration of the diagram at the right. By definition, ${}^1H_{-1,q}({}^2Z) = {}^2Z_{-1,q}/\mathrm{Im}\ d_1^{\,q}$ and ${}^2H_{-1,q}(K) = {}^2Z_{-1,q}/\mathrm{Im}\ d_2^{\,-1}$. From the hypotheses it follows that both $d_1^{\,q+1}$ and $d_2^{\,o}$ are surjective, and since the diagram is commutative, we must have $\mathrm{Im}\ d_1^{\,q} = \mathrm{Im}\ d_2^{\,-1}$, which proves the lemma.

$$\begin{tikzcd} 0 & K_{-1,q+1} \arrow[l] \arrow[d, "d_2^{-1}"] & K_{o,q+1} \arrow[l, "d_1^{q+1}"'] \arrow[d, "d_2^{o}"] \\ 0 & {}^2Z_{-1,q} \arrow[l] & {}^2Z_{o,q} \arrow[l, "d_1^{q}"'] \end{tikzcd}$$

To prove the theorem, we first note that since ${}^2H_{p,q}(K) = 0$ for $p \geqq 0$ and all q, it follows that for such p and q the couples

$$0 \longrightarrow {}^2Z_{p,q} \xrightarrow{\ i\ } K_{p,q} \xrightarrow{\ d_2\ } Z_{p,q-1} \longrightarrow 0$$

are exact. We have then for all q the exact homology sequences:

$$\longrightarrow {}^1H_{p,q}({}^2Z) \xrightarrow{i^*} {}^1H_{p,q}(K) \xrightarrow{d_2^*} {}^1H_{p,q-1}({}^2Z) \xrightarrow{\Delta} {}^1H_{p-1,q}({}^2Z) \longrightarrow \cdots$$

$$\longrightarrow {}^1H_{0,q}({}^2Z) \xrightarrow{i^*} {}^1H_{0,q}(K) \xrightarrow{d_2^*} {}^1H_{0,q-1}({}^2Z) \xrightarrow{\Delta} {}^1H_{-1,q}({}^2Z) \xrightarrow{i^*} {}^1H_{-1,q}(K).$$

The series terminates with ${}^1H_{-1,q}(K)$ since for $p = -1$ the mapping d_2 is in general not surjective. Assume now that $q > -1$, whence for all p, ${}^1H_{p,q}(K) = 0$, and therefore $\Delta : {}^1H_{p,q-1}({}^2Z) \to {}^1H_{p-1,q}({}^2Z)$ is an isomorphism for all p. Thus we have

$$H_q(X) \approx {}^1H_{q,-1}({}^2Z) \approx {}^1H_{q-1,0} \approx \cdots \approx {}^1H_{-1,q}({}^2Z) \approx H_q(Y)$$

when $q = -1$, $K_{-1,-1} \equiv {}^1Z_{-1,-1} \equiv {}^2Z_{-1,-1}$ and

$${}^1B_{-1,-1} = d_1(K_{0,-1}) = d_1d_2(K_{0,0}) = d_2d_1(K_{-1,0}) = {}^2B_{-1,-1}$$

whence $H_{-1}(X) = {}^1Z_{-1,-1}/{}^1B_{-1,-1} = {}^2Z_{-1,-1}/{}^2B_{-1,-1} = H_{-1}(Y)$

This completes the proof of the theorem.

We might prove the analogous theorem for cohomology:

Theorem 2.2.2. Let $K^{p,q}$ be a cochain complex with $K^{p,q} = 0$ for $q \leq -1$ or $p \leq -1$.

If ${}^1H^{p,q}(K) = \begin{cases} H^p(X) & \text{for } q = -1 \\ 0 & \text{for } q > -1 \end{cases}$ and ${}^2H^{p,q}(K) = \begin{cases} H^q(Y) & \text{for } p = -1 \\ 0 & \text{for } p > -1 \end{cases}$

then $H^p(X) = H^p(Y)$.

Again we first prove a lemma:

Lemma 2.2.2. If the hypotheses of the theorem are satisfied, and $p,q \neq -1$, then ${}^1H^{p,-1}(K) \approx {}^1H^{p,0}({}^2B) = H^p(X)$ and ${}^2H^{-1,q}(K) \approx {}^1H^{0,q}({}^2B) = H^q(Y)$.

Proof. The first statement is trivial, for $d_2\colon K_{p,-1} \to {}^2B_{p,0}$ is injective since $K_{p,-2} = 0$. To prove the second we must show that ${}^2H^{-1,q}(K) = {}^2Z^{-1,q}/{}^2B^{-1,q}$ is isomorphic to ${}^1Z^{0,q} \cap {}^2B^{0,q}/d_1({}^2B^{-1,q}) = {}^1H^{-1,q}({}^2B)$. Since ${}^1H^{-1,q}(K) = 0$, for $q \neq -1$ it follows that $d_1\colon K^{-1,q} \to K^{0,q}$ is injective, whence ${}^2B^{-1,q} = d_1({}^2B^{-1,q})$ and we need only show that ${}^2Z^{-1,q} \approx {}^1Z^{0,q} \cap {}^2B^{0,q}$. This is trivial since d_1 maps ${}^2Z^{-1,q}$ isomorphically onto ${}^1Z^{0,q} \cap {}^2B^{0,q}$ and both ${}^1Z^{0,q} = {}^1B^{0,q}$ and ${}^2Z^{0,q} = {}^2B^{0,q}$. This proves the lemma.

To establish the theorem we note that if $p \geqq 0$ the couples

$$0 \longrightarrow {}^2B^{p,q-1} \xrightarrow{\ i\ } K^{p,q-1} \xrightarrow{\ d_2\ } {}^2B^{p,q} \longrightarrow 0$$

are exact. We have then for each q the exact cohomology sequence:

$${}^1H^{0,q-1}(K) \xrightarrow{d_2^*} {}^1H^{0,q}({}^2B) \longrightarrow {}^1H^{1,q-1}({}^2B) \xrightarrow{i^*} \cdots$$

$$\cdots \longrightarrow {}^1H^{p,q-1}(K) \xrightarrow{d_2^*} {}^1H^{p,q}({}^2B) \longrightarrow {}^1H^{p+1,q-1}({}^2B) \longrightarrow \cdots$$

where we must start at ${}^1H^{0,q-1}(K)$ since for $p = 0$ Im $i \neq$ Ker d_2 in general. Assume $q \neq -1$, and as before we have that for all p, Δ: ${}^1H^{p,q+1}({}^2B) \to {}^1H^{p+1,q}({}^2B)$ is an isomorphism. Thus we have

$$H^q(Y) \approx {}^1H^{-1,q}({}^2B) \approx \cdots \approx {}^1H^{q,-1}({}^2B) \approx H^q(X)$$

establishing the theorem except when $q = -1$. In this case we have ${}^1B^{-1,-1} = {}^2B^{-1,-1} = 0$. Examining the diagram at the right, we note that since the right column and top row are exact, an element in $K^{-1,-1}$ maps into 0 in $K^{0,0}$ if and only if it maps into 0 in both $K^{-1,0}$ and $K^{0,-1}$.

$$\begin{tikzcd}
0 \arrow[r] & K^{-1,0} \arrow[r, "d_1"] & K^{0,0} \\
0 \arrow[r] & K^{-1,-1} \arrow[r, "d_1"] \arrow[u, "d^2"'] & K^{0,-1} \arrow[u, "d^2"'] \\
 & 0 \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

Thus ${}^1Z^{-1,-1} = {}^2Z^{-1,-1}$, proving the theorem in this case too.

It should be noted that all of the constructions which one carries out for ordinary chain and cochain complexes can be carried out for double complexes. The proofs are carried out in the same manner--one has only the notational inconvenience of an added subscript. For example, we define a <u>map of double complexes</u> $f: K \to L$ to be a collection of homomorphisms which preserve bidegree (i.e., $f_{p,q}: K_{p,q} \to L_{p,q}$) and commute with both d_1 and d_2. One easily verifies that "double complexes and maps" form an abstract category, on which we have a pair of homology functors ${}^1H$ and ${}^2H$. We now define "exact couples" of double complexes as exact sequences $0 \to K' \xrightarrow{\Phi} K \xrightarrow{\Psi} K'' \to 0$ and each exact couple induces an exact homology sequence in the usual manner. We define direct and inverse systems as usual, and by taking limits arrive at Čech type functors ${}^i\check{H}^{p,q} = \mathrm{DirLim}\ {}^iH^{p,q}$ and ${}^i\check{H}_{p,q} = \mathrm{InvLim}\ {}^iH_{p,q}$. It follows then that if the theorems of this section hold for all of the ${}^iH_{p,q}$ in a direct or inverse system, they must hold in the limit as well.

§2.3 Application to Homology Groups of Relations

In this section we will use the isomorphism theorems of the previous section to give a new proof of a well-known theorem on the homology and cohomology groups of relations. Our main application, that the cohomology of a sheaf is determined by any flasque resolution, must wait until Chapter 4. With the background of Chapter 3, this will follow immediately from Theorem 2.2.2.

Let X and Y be two sets. A relation from X to Y is merely a subset R of X x Y. We say that xRy or x "is related to y" whenever (x,y) is an element of R. We define an n-simplex in X to be a set of n+1 elements of X which are all related to a common y of Y, and similarly an n-simplex in Y is a set of n+1 elements of Y which are all related to a common x of X. It is easily verified that every "face" (i.e., proper subset) of a simplex is again a simplex, and that the intersection of two simplices is a face of both, whence both X and Y are sets of vertices of simplicial complexes, which we will denote $\tilde{X}$ and $\tilde{Y}$.

We next form chain and cochain modules with respect to a fixed coefficient ring Λ (Λ is a principal ideal ring) in the usual manner. That is, $C_q(X)$ and $C_q(Y)$ are the sets of finite linear combinations of q-simplices with coefficients in Λ, and $C^q(X)$ and $C^q(Y)$ are the sets of linear functions on the sets of q-simplices with values in Λ. The boundary operators, δ for cochains and ∂ for chains, are defined as usual by:

$$\partial(x_0, \ldots x_n) = \Sigma(-1)^i (x_0, \ldots \hat{x}_i \ldots x_n) \quad \text{and}$$
$$\delta f(x_0, \ldots x_{n+1}) = \Sigma(-1)^i f(x_0, \ldots \hat{x}_i, \ldots x_{n+1}).$$

We augment the complexes $\tilde{X}$ and $\tilde{Y}$ by adding to each a unique, -1 dimensional simplex, denoted respectively by S_{-1} and T_{-1}. We further define for any q-simplex S_q of X, the subcomplex $\sigma(S_q)$ of $\tilde{Y}$ which is spanned by those elements of Y which are related to <u>all</u> of the x_i which make up S_q. (By "spanned" we mean the subcomplex of $\tilde{Y}$ which consists of all finite subsets of this set of y's.) Similarly, $\sigma(T_q)$ is the subcomplex of $\tilde{X}$ spanned by those x's which are related to all of the y_i in T_q. We define $\sigma(T_{-1})$ to be all of $\tilde{X}$ and $\sigma(S_{-1})$ to be all of $\tilde{Y}$.

Let Ω be the set of all pairs (S_p, T_q) with S_p in $\tilde{X}$ and T_q in $\tilde{Y}$, subject to the condition that $S_p \subset \sigma(T_q)$ and $T_q \subset \sigma(S_p)$. We define the double chain complex $K_{p,q}$ as that subset of the tensor product $\tilde{X} \otimes \tilde{Y}$ in which summands $S_p \otimes T_q$ are allowed <u>only</u> if (S_p, T_q) is in Ω. Each chain is thus a finite linear combination of the form $\Sigma \lambda_i S_p^{\,i} \otimes T_q^{\,i}$ with coefficients λ_i in Λ. If we have chains $A_p = \Sigma \lambda_i S_p^{\,i}$ in $C_p(X)$ and $B_q = \Sigma \mu_j T_q^{\,j}$ in $C_q(Y)$ the product $A_p \otimes B_q = \Sigma \lambda_i \mu_j S_p^{\,i} \otimes T_q^{\,j}$ is defined whenever all of the $S_p^{\,i}$ are related to each of the $T_q^{\,j}$ and vice-versa. The boundary operators are defined pointwise--that is, $d_1(A_p \otimes B_q) = \partial A_p \otimes B_q$ and $d_2(A_p \otimes B_q) = A_p \otimes \partial B_q$. We have that $K_{-1,q} \approx C_q(Y)$ and $K_{p,-1} \approx C_p(X)$ under the obvious correspondences $B_q \longleftrightarrow S_{-1} \otimes B_q$ and $A_p \longleftrightarrow A_p \otimes T_{-1}$, and this induces the isomorphisms ${}^1H_{p,-1}(K) \approx H_p(X)$ and ${}^2H_{-1,q}(K) \approx H_q(Y)$.

It is clear that we need only show that ${}^1H_{p,q}(K) = 0$ for $q \geqq 0$ and ${}^2H_{p,q}(K) = 0$ for $p \geqq 0$, in order to conclude from the theorem of the previous section that $H_p(X)$ and $H_p(Y)$ are isomorphic. In fact, since our construction was completely symmetric we need only show that ${}^1H_{p,q}(K) = 0$ for $p \geqq 0$.

For cohomology, we define the cochain complex $K^{p,q}$ as that subset of Hom $(\widetilde{X} \otimes \widetilde{Y}, \Lambda)$ which contains only functions which vanish whenever (S_p, T_q) is not in Ω. If we have cochains $A^p(S)$ and $B^q(T)$, their "product" $A^p \cdot B^q$ $(S \times T)$ is that function which takes the value $A^p(S) \cdot B^q(T)$ whenever (S,T) is in Ω. The coboundary operators d^1 and d^2 are also defined pointwise and, as before, we have isomorphisms ${}^1H^{p,-1}(K) \approx H^p(X)$ and ${}^2H^{-1,q}(K) \approx H^q(Y)$. Here again the proof of the theorem depends on showing that ${}^1H^{p,q}(K) = 0$ for $p \geqq 0$.

A typical chain in $K_{p,q}$ can be written

$$A_p^{\,1} \otimes T_q^{\,1} + A_p^{\,2} \otimes T_q^{\,2} + \ldots + A_p^{\,n} \otimes T_q^{\,n}$$

by grouping all the summands that contain a particular $T_q^{\,i}$, where $A_q^{\,i}$ is a chain in $C_q(X)$. Since the boundary operator operates pointwise, it is easily seen that

$${}^1H_{p,q}(K) = \sum H_p(\sigma(T_q)) \quad \text{(direct sum over all } T_q \in \widetilde{Y})$$

and an analogous representation of the typical cochain in $K^{p,q}$ shows that:

$$^{1}H^{p,q}(K) = \Pi\, H^{p}(\sigma(T_q)) \quad \text{(direct product over all } T_q \in \tilde{Y})$$

Thus the proof of the theorem has been reduced to a demonstration that $\sigma(T_q)$ has trivial groups in all dimensions. But $\sigma(T_q)$ is just an infinite set, every finite subset of which is a simplex, and in addition $\sigma(T_q)$ has been augmented by the addition of the unique -1 dimensional simplex S_{-1}. The proof that this is an acyclic complex is the standard argument for cone complexes. Let $S_p = (x_0,\ldots x_p)$ and define the p+1 simplex $x \cdot S$ as $(x,x_0,\ldots,x_p)$, and extend this operator to chains A_p by linearity. It is clear that $\partial x \cdot A_p = A_p - x \cdot \partial A_p$, whence if A_p is a cycle it is the boundary of the chain $x \cdot A_p$. The corresponding argument for cohomology completes the proof of the theorem.

This theorem, that "related" sets have isomorphic homology and cohomology, is due to Dowker [9], who used it to establish that on the category of compact pairs, the Čech and Alexander cohomology groups and the Čech and Vietoris homology groups are isomorphic. In this application X is the set of points of a space, Y is the set of open sets in a particular cover, and R is the membership relation. The result follows after taking limits over the directed set of coverings.

CHAPTER III

SHEAVES OVER LATTICES

In this chapter we discuss sheaves and presheaves over regular lattices. The development is similar in many respects to that in Godement [7] .

§3.1 Basic Definitions

A presheaf of R-modules over a regular lattice $\mathfrak{A}$ consists of a collection of R-modules $\{G_\alpha\}$ indexed by $\mathfrak{A}$ and a collection of module homomorphisms $\{\rho_\beta{}^\alpha\}$ (where $\rho_\beta{}^\alpha: G_\alpha \to G_\beta$), defined whenever $\alpha \geq \beta$, subject to the restrictions:

P(1) $\rho_\beta{}^\alpha \circ \rho_\alpha{}^\gamma = \rho_\beta{}^\gamma$ and $\rho_\alpha{}^\alpha$ is the identity map.

P(2) $G_\emptyset = 0$, the trivial, one-element module.

If we regard $\mathfrak{A}$ as an abstract category whose "objects" are the lattice elements and whose "maps" are the relations $\alpha \geq \beta$, a presheaf can be concisely defined as a covariant functor to the category of R-modules and homomorphisms, which is trivial on the minimal element.

We will say that $\bar{\mathfrak{G}} = \{\bar{G}_\alpha, \bar{\rho}_\beta{}^\alpha\}$ is a subpresheaf of $\mathfrak{G} = \{G_\alpha, \rho_\beta{}^\alpha\}$ if $\bar{G}_\alpha \subset G_\alpha$, $\rho_\beta{}^\alpha(\bar{G}_\alpha) \subset \bar{G}_\beta$, and $\bar{\rho}_\beta{}^\alpha$ is the restriction of $\rho_\beta{}^\alpha$ to $\bar{G}_\alpha$. Let $\mathfrak{F}$ and $\mathfrak{G}$ be two presheaves over the same lattice $\mathfrak{A}$. A presheaf homomorphism $f: \mathfrak{F} \to \mathfrak{G}$, is a collection of module homomorphisms $f_\alpha: F_\alpha \to G_\alpha$, indexed by $\mathfrak{A}$, which commute with the $\{\rho_\beta{}^\alpha\}$. If $\mathfrak{F}$ and $\mathfrak{G}$ are thought of as functors on $\mathfrak{A}$, then f is a natural transformation. f gives rise to the subpresheaves Ker f and Im f defined for each α by $(\text{Ker } f)_\alpha = \text{Ker } f_\alpha$ and $(\text{Im } f)_\alpha = \text{Im } f_\alpha$. The mapping f is said to be surjective or injective if each of the f_α is respectively surjective or injective. We define a quotient presheaf $\mathfrak{G}/\bar{\mathfrak{G}}$ at each α by $(\mathfrak{G}/\bar{\mathfrak{G}})_\alpha = G_\alpha/\bar{G}_\alpha$ with the maps $\bar{\bar{\rho}}_\beta{}^\alpha$ induced by the $\rho_\beta{}^\alpha$ in the usual manner. Thus we may make the general statement that the restriction of a presheaf diagram to the modules and maps indexed by a single lattice element α, has the same "exactness properties" at each point as the original presheaf diagram. The corresponding statement for sheaves is, however, false.

If $\rho_\beta{}^\alpha(g_\alpha) = g_\beta$, we will say that g_α is an <u>ancestor</u> of g_β in G_α and that g_β is a <u>descendant</u> of g_α in G_β. A <u>clan</u> is a collection of elements $g_{\alpha_i} \in G_{\alpha_i}$ with the property that $\rho_{\alpha_i \cap \alpha_j}{}^{\alpha_i}(g_{\alpha_i}) = \rho_{\alpha_i \cap \alpha_j}{}^{\alpha_j}(g_{\alpha_j})$ for every i and j. We now define a <u>sheaf</u> over $\mathfrak{A}$, as a presheaf which satisfies the following pair of axioms:

S(1) If a clan $\{g_{\alpha_i}\}$ has a common ancestor in G_α $(\alpha = \cup \alpha_i)$, then it is unique.

S(2) Every clan $\{g_{\alpha_i}\}$ has a common ancestor in G_α $(\alpha = \cup \alpha_i)$.

It is clear that by the insertion of the word "unique" in S(2) we could eliminate the necessity for axiom S(1). The reason for not doing this is that presheaves which satisfy only S(1) have some nice properties. In particular, any subpresheaf of a presheaf satisfying S(1) also satisfies S(1), that is axiom S(1) is "hereditary." Axiom S(2) is not hereditary in general, since the common ancestor need not be in the sub-presheaf, even though the clan is.

It is clear from the definitions that if $f: \mathfrak{F} \to \mathfrak{G}$ is a presheaf homomorphism then the image under f of a clan is again a clan, whereas the inverse image of a clan is <u>not</u> in general a clan.

We now define several types of sheaves and presheaves which will be considered extensively in the following sections. A sheaf or presheaf will be called a <u>flasque</u> if all of the $\rho_\beta{}^\alpha$ are surjective, and <u>constant</u> if all of the $\rho_\beta{}^\alpha$ are isomorphisms. The constant sheaf in which each of the modules G_α is the trivial, one-element module is called the <u>zero sheaf</u>.

§3.2 <u>The Canonical Embedding</u>

In this section we will consider the question of how a presheaf can be made into a sheaf. It is clear from the axioms that two sorts of trouble can arise. If S(1) fails to be satisfied, some elements have too many ancestors and we must remove this indeterminacy by putting the multiple ancestors into a single equivalence class. If S(2) fails to be satisfied, we will have some ancestorless clans which must be supplied with unique ancestors in a natural manner. Finally, our constructive process for obtaining a sheaf from a presheaf should, in case it were applied to a sheaf, yield an isomorphic image of that sheaf. We will give in this section a natural method for obtaining a sheaf from a presheaf, which satisfies these criteria.

Our first step is to define the <u>flasque extension</u> $\tilde{\mathfrak{F}}$ of a presheaf $\mathfrak{F}$. We note that the subset of $\mathfrak{F}$ containing those modules F_α and maps $\rho_\beta{}^\alpha$ which are indexed by elements contained in a fixed prime filter A of $\mathfrak{a}$ is a direct system of modules. We form the direct limit module for each prime filter A which we denote by F_A, and we denote the usual projections into the direct limit by $\rho_A^{\alpha}: F_{\alpha} \to F_A$. We define $\tilde{F}_{\alpha}$ to be the direct product of F_A over all prime filters containing α. Thus each element of $\tilde{F}_{\alpha}$ is a function on the set of prime filters containing α with its value at A in F_A. If $\beta \leq \alpha$ there are fewer prime filters containing β, hence the domain of the functions in $\tilde{F}_{\beta}$ is a subset of the domain of the functions in $\tilde{F}_{\alpha}$, and we may define $\tilde{\rho}_{\beta}^{\alpha}$ to be the "domain restricting" homomorphism. One readily verifies that $\tilde{\mathfrak{F}} = \{\tilde{F}_{\alpha}, \tilde{\rho}_{\beta}^{\alpha}\}$ is in fact a flasque sheaf.

We define next the <u>natural mapping</u> $i: \mathfrak{F} \to \tilde{\mathfrak{F}}$ of a presheaf into its flasque extension as follows: if $g \in F_{\alpha}$, then i(g) is that function on the prime filters containing α whose value at A is $\rho_A^{\alpha}(g)$. The verification that i is a presheaf homomorphism is trivial. Now suppose that $\mathfrak{F}$ satisfies S(1), and that i(g) = 0. We must then have that $\rho_A^{\alpha}(g) = 0$ for all A containing α, and by the usual properties of direct limits, for each A there is a $\beta(A)$ with $\rho_{\beta(A)}^{\alpha}(g) = 0$. The collection of zeros of the modules $F_{\beta(A)}$ is thus a clan with both g and 0 as common ancestors in F_{α}, whence we conclude from S(1) that g = 0. Moreover, any two elements g and g' in F_{α} which are both ancestors of the same clan must clearly project into the same element of F_A for every prime filter A containing α, hence i(g) and i(g') are the same function. We state these observations in the form of a theorem:

<u>Theorem 3.2.1</u>. Let i be the natural mapping of a presheaf $\mathfrak{F}$ into its flasque extension. $i(\mathfrak{F})$ is a presheaf satisfying S(1), and if $\mathfrak{F}$ satisfies S(1) then i is injective.

The problem now is to enlarge $i(\mathfrak{F})$ to a sheaf. Every clan in $\mathfrak{F}$ maps under i into a clan of functions in $\tilde{\mathfrak{F}}$ which agree whenever their domains overlap, thus uniquely defining a function on the union of their domains which is the ancestor of the clan. This ancestor can be characterized as the function whose values are the projections of the clan into the modules F_A. If $\mathfrak{F}$ satisfies S(2) it is clear that $i(\mathfrak{F})$ must consist of precisely these functions, which suggests defining as our associated sheaf, the set of functions in $\tilde{\mathfrak{F}}$ of this type. The trouble with this is that without some further assumptions, this set of functions does not necessarily form a sheaf, since the unique ancestor of a clan of such functions may not itself be a function of this type. A slight generalization of this idea, however, gives the "right" functions in the flasque extension.

A function $\tilde{g}_\alpha$ in $\tilde{F}_\alpha$ will be called a germ if its values are given locally by projections of elements in the presheaf. More precisely, we must have for each prime filter A containing α, a lattice element $\beta(A)$ and an element g_β in F_β such that $\rho_X{}^\beta(g_\beta) = g_\alpha(X)$ for every prime filter X which contains β. The element g_β will be called a precursor of the germ at A. It is clear that any descendent of a precursor at A is again a precursor at A. This implies that a linear combination of germs is again a germ, for if $\tilde{g}$ and $\tilde{h}$ are germs with precursors g_α and h_β at A, then $c_1 \cdot \rho^\alpha_{\alpha\cap\beta}(g_\alpha) + c_2 \cdot \rho^\beta_{\alpha\cap\beta}(g_\beta)$ is a precursor of $c_1 \cdot \tilde{g} + c_2 \cdot \tilde{h}$ at A.

Let us denote by $\bar{F}_\alpha$ the subset of germs of $\tilde{F}_\alpha$, and by $\bar{\rho}_\beta{}^\alpha$ the maps induced by the $\tilde{\rho}_\beta{}^\alpha$. It is clear that $\{\bar{F}_\alpha, \bar{\rho}_\beta{}^\alpha\}$ is a presheaf of modules which satisfies the hereditary property S(1). Since the unique ancestor of any clan of germs has, as precursor at A, any precursor of a clan element with A in its domain, it follows that clan ancestors are germs, and hence $\overline{\mathfrak{F}}$ is a sheaf. We thus define the <u>sheaf generated by a presheaf</u> $\mathfrak{F}$ to be the subsheaf of germs of the flasque extension of $\overline{\mathfrak{F}}$. In particular, a <u>quotient sheaf</u> is a sheaf generated by a quotient presheaf, and a <u>simple sheaf</u> is a sheaf generated by a constant presheaf.

It is obvious that any clan is the set of precursors of a germ, but the converse is not in general true. Suppose, however, that the sheaf satisfies S(1), and let $\{ g_{\beta(A)} \}$ be a set of precursors for the germ $\tilde{g}_\alpha$. We will assume that $\beta(A_1) \cap \beta(A_2) \neq \emptyset$ and denote for convenience, $\beta(A_1)$ by β_1, $\beta(A_2)$ by β_2, and $\beta(A_1) \cap \beta(A_2)$ by β_{12}. By hypothesis

$$\rho_X^{\beta_1}(g_{\beta_1}) = \rho_X^{\beta_2}(g_{\beta_2}) = \tilde{g}_\alpha(X)$$

whence

$$\rho_X^{\beta_{12}}\left(\rho_{\beta_{12}}^{\beta_1}(g_{\beta_1}) - \rho_{\beta_{12}}^{\beta_2}(g_{\beta_2})\right) = 0$$

for all prime filters X containing β_{12}. As we have already shown, the only element of $G_{\beta_{12}}$ which is in the kernel of all of the $\rho_X^{\beta_{12}}$ is the 0 element, whence

$$\rho_{\beta_{12}}^{\beta_1}(g_{\beta_1}) = \rho_{\beta_{12}}^{\beta_2}(g_{\beta_2})$$

and we have established that:

<u>Theorem 3.2.2</u>. If $\mathfrak{F}$ is a presheaf satisfying S(1), all of the germs in $\overline{\mathfrak{F}}$ are functions whose values are the set of projections of a clan.

An immediate corollary of this pair of theorems is:

Corollary 3.2.1. If $\mathfrak{F}$ is a sheaf, i is an isomorphism of $\mathfrak{F}$ onto the associated sheaf of germs $\overline{\mathfrak{F}}$.

§3.3 Sheaf Homomorphisms

In the first section of this chapter we defined a homomorphism $f: \mathfrak{F} \to \mathfrak{G}$, where $\mathfrak{F}$ and $\mathfrak{G}$ are presheaves over a regular lattice $\mathfrak{A}$. If it should happen that both $\mathfrak{F}$ and $\mathfrak{G}$ are sheaves, f will be called a sheaf homomorphism, without any other condition being imposed. We define the image and kernel of a sheaf homomorphism f, (denoted by $\overline{\text{Im } f}$ and $\overline{\text{Ker } f}$), as the sheaves generated by the presheaves Im f and Ker f. One sees easily that Ker f is already a sheaf, hence isomorphic to $\overline{\text{Ker } f}$. To establish this we note that since clans are preserved by f, any clan in Ker f maps into the zero clan, and its ancestor maps into the unique ancestor of the zero clan, i.e. zero, hence is in Ker f also. On the other hand, since the inverse image of a clan is not, in general, a clan, one cannot say the same thing about Im f, although, by construction, Im f $\subset \overline{\text{Im } f}$.

Consider now the following sequence of sheaves and sheaf homomorphisms:

$$(*) \qquad \cdots \longrightarrow \mathfrak{G}^{n-1} \xrightarrow{\phi^{n-1}} \mathfrak{G}^{n} \xrightarrow{\phi^{n}} \mathfrak{G}^{n+1} \longrightarrow \cdots$$

This sequence is said to be exact at n if $\overline{\text{Im } \phi^{n-1}} = \text{Ker } \phi^n$, that is if the subsheaf Ker ϕ^n is the sheaf generated by the presheaf Im ϕ^{n-1}. Thus to establish exactness at n, it is necessary and sufficient to show that the ancestor of any clan in Im ϕ^{n-1} is in Ker ϕ^n. If it should happen that Im ϕ^{n-1} is already a sheaf, then we have Im $\phi^{n-1} =$ Ker ϕ^n, and in this case we will say that the sequence is <u>totally exact at n</u>. A sequence is said to be exact (or totally exact), if it possesses that property for each n. The sequence (*) induces for each α in $\mathfrak{A}$ a corresponding sequence of modules

$$\cdots \longrightarrow G_\alpha^{n-1} \xrightarrow{\phi_\alpha^{n-1}} G_\alpha^{n} \xrightarrow{\phi_\alpha^{n}} G_\alpha^{n+1} \longrightarrow \cdots$$

which is of order two if (*) is exact, and exact if (*) is totally exact.

To put the above statements in functorial language, we will define the "restriction functor" Γ_α which associates G_α with $\mathcal{G}$ and f_α with f for every sheaf $\mathcal{G}$ over $\mathfrak{A}$ and every sheaf homomorphism f. Suppose we have an "exact couple" of sheaves over :

$$0 \longrightarrow \mathcal{G}' \xrightarrow{i} \mathcal{G} \xrightarrow{\pi} \mathcal{G}'' \longrightarrow 0.$$

We have total exactness at $\mathcal{G}'$ since the zero presheaf is a sheaf, whence i is an isomorphism into. Since Im i is isomorphic to $\mathcal{G}'$ it is a sheaf and we have total exactness at $\mathcal{G}$, but of π we can say only that $\overline{\text{Im}\ \pi} = \mathcal{G}''$. Thus our exact couple maps under Γ_α into the "left exact couple"

$$0 \longrightarrow G'_\alpha \xrightarrow{i_\alpha} G_\alpha \xrightarrow{\pi_\alpha} G''_\alpha$$

In the next section we will see that this left exact functor Γ_α is exact if $\mathcal{G}'$ is a flasque sheaf.

Identifying our sheaves $\mathcal{G}'$, $\mathcal{G}$, and $\mathcal{G}''$ momentarily with their associated sheaves of germs, we can easily see why π is, in general, not surjective. We are requiring only that for each prime filter A, some precursor of $\pi g(A)$ be the image of a precursor of $g(A)$. In other words, each prime filter A contains an element α such that Γ_β is exact whenever $\beta \leq \alpha$. This, of course, implies that the direct limit functor Γ_A is exact for each A.

The correspondence which associates to each sheaf $\mathfrak{F}$ over $\mathfrak{A}$ its flasque extension $\tilde{\mathfrak{F}}$ and to each sheaf homomorphism $f\colon \mathfrak{F} \to \mathfrak{G}$ the homomorphism $\tilde{f}\colon \tilde{\mathfrak{F}} \to \tilde{\mathfrak{G}}$ defined by $\tilde{f}(\phi(A)) = f_A(\phi(A))$ (where f_A is the direct limit homomorphism), is easily verified to be functorial. It is, in fact, the direct product of the functors Γ_A over the set of prime filters, and since the Γ_A are all exact functors, this functor is exact too.

A <u>resolution</u> of a sheaf $\mathfrak{F}$ is any exact sequence

$$0 \longrightarrow \mathfrak{F} \longrightarrow \mathfrak{G}^0 \longrightarrow \mathfrak{G}^1 \longrightarrow \cdots \longrightarrow \mathfrak{G}^n \longrightarrow$$

The <u>canonical resolution</u> is defined as follows: Let $\mathcal{C}^0 = \tilde{\mathfrak{F}}$ be the flasque extension, and let Z^1 be the quotient sheaf $\tilde{\mathfrak{F}}/\mathfrak{F}$. Let $\mathcal{C}^1 = \tilde{Z}^1$ and define Z^2 as the quotient sheaf $\tilde{Z}^1/Z^1$. Proceeding in this manner, we obtain a sequence of exact couples:

$$0 \longrightarrow \mathfrak{F} \xrightarrow{i} \mathcal{C}^0 \xrightarrow{\pi^0} Z^1 \longrightarrow 0$$
$$0 \longrightarrow Z^1 \xrightarrow{i^1} \mathcal{C}^1 \xrightarrow{\pi^1} Z^2 \longrightarrow 0$$
$$\cdots\cdots\cdots$$
$$0 \longrightarrow Z^n \xrightarrow{i^n} \mathcal{C}^n \xrightarrow{\pi^n} Z^{n+1} \longrightarrow 0$$

where i is the natural injection into the flasque extension, and π is the surjection onto the quotient sheaf. We define $d^n = i^{n+1} \circ \pi^n$ and obtain the sequence

$$0 \longrightarrow \mathfrak{F} \xrightarrow{i} \mathcal{C}^0 \xrightarrow{d^0} \mathcal{C}^1 \xrightarrow{d^1} \mathcal{C}^2 \longrightarrow \cdots$$

which is clearly exact, and made up of flasque sheaves. Moreover, whenever we have a sheaf homomorphism $f\colon \mathfrak{F} \to \mathfrak{G}$, we have automatically a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{F} & \xrightarrow{i} & \tilde{\mathfrak{F}} & \xrightarrow{\pi} & \tilde{\mathfrak{F}}/\mathfrak{F} & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle \tilde{f}} & & \Big\downarrow{\scriptstyle f'} & & \\ 0 & \longrightarrow & \mathfrak{G} & \xrightarrow{i} & \tilde{\mathfrak{G}} & \xrightarrow{\pi} & \tilde{\mathfrak{G}}/\mathfrak{G} & \longrightarrow & 0 \end{array}$$

where f' is the map induced by the flasque extension map $\tilde{f}$. Upon iterating this construction, we obtain a sequence of maps of the canonical resolution of $\mathfrak{F}$ into the canonical resolution of $\mathfrak{G}$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{F} & \xrightarrow{i} & \mathcal{C}^0(\mathfrak{F}) & \xrightarrow{d^0} & \mathcal{C}^1(\mathfrak{F}) & \longrightarrow & \cdots \\ & & \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle \tilde{f}^0} & & \Big\downarrow{\scriptstyle \tilde{f}^1} & & \\ 0 & \longrightarrow & \mathfrak{G} & \xrightarrow{i} & \mathcal{C}^0(\mathfrak{G}) & \xrightarrow{d^0} & \mathcal{C}^1(\mathfrak{G}) & \longrightarrow & \cdots \end{array}$$

with a commutative diagram. This sequence of maps $\{\tilde{f}^n\}$ will be called a <u>resolution</u> <u>map</u>.

The correspondence which assigns to each sheaf its canonical resolution and to each map the corresponding resolution map is functorial, and this "resolution functor" will be denoted by $\mathcal{R}$.

Given an exact couple

$$0 \longrightarrow \mathfrak{F}' \xrightarrow{\phi} \mathfrak{F} \xrightarrow{\psi} \mathfrak{F}'' \longrightarrow 0$$

we have already seen that the induced couple of flasque extensions is exact, and the induced quotient couple is readily verified to be exact. This then implies that the canonical resolution functor $\mathcal{R}$ is an exact functor.

§3.4 Sheaves Over Locally Paracompact Lattices

In this section, we consider further a question which was raised in §3.2, that is, when are all the germs in an associated sheaf $\overline{\mathfrak{F}}$ projections of clans in the presheaf $\mathfrak{F}$? We have already shown that this is the case if $\mathfrak{F}$ satisfies S(1), and in this section we will show that it is also the case if the lattice $\mathfrak{A}$ is locally paracompact and Hausdorff. The theorems of this section are restatements of well-known theorems about paracompact and normal spaces, and their proofs require, at most, minor modifications of the usual proofs, even though the terminology may be somewhat different.

Let $\mathfrak{A}$ be a regular lattice, and let $\{a_i\}$ $(i \varepsilon I)$ be a family of lattice elements. $\{b_j\}$ $(j \varepsilon J)$ is said to be a <u>refinement</u> of this family, if the following conditions are satisfied:

R(1) $\bigcup_J b_j = \bigcup_I \alpha_i$

R(2) There is a mapping ϕ: $J \to I$ with $b_j \leq a_{\phi(j)}$

For convenience, throughout this section we will use the set theoretic terminology "α meets β" for $\alpha \cap \beta \neq \emptyset$ and "α and β are disjoint" for $\alpha \cap \beta = \emptyset$. A family $\{a_i\}$ will be called <u>filter finite</u> if every prime filter contains only a finite number of the $\{a_i\}$ and <u>locally finite</u> if each prime filter contains an element which meets only a finite number of the $\{a_i\}$.

A Hausdorff lattice in which every family has a locally finite refinement is called a <u>locally paracompact</u> lattice.

A prime filter A will be said to <u>almost contain</u> $\underline{\alpha}$ if every β in

A meets α. We will denote by S_α the set of prime filters which contain α, and by $\bar{S}_\alpha$ the set of prime filters which almost contain α. It is clear that $\bar{S}_\alpha \supset S_\alpha$. We will denote by S^*_α the set of prime filters which do not contain α.

One readily verifies that $\bar{S}_\alpha \cup \bar{S}_\beta = \bar{S}_{\alpha\cup\beta}$ and $\bigcup S_{\alpha_i} = S_{\cup\alpha_i}$ for an arbitrary family $\{\alpha_i\}$. These identities follow directly from the regularity of $\mathfrak{A}$ and the fact that these are sets of <u>prime</u> filters. One final fact which is necessary for the theorems in this section is that each $\bar{S}_\alpha$ is, at the same time, S^*_β for some β. β is, of course, the l.u.b of the set of lattice elements which are not contained in any filter in $\bar{S}_\alpha$. For the rest we note that if A is a prime filter which is not in $\bar{S}_\alpha$, then it contains a γ which does not meet α, and as $\beta \geqq \gamma$, we must have that $A \in S_\beta$. Conversely, if A almost contains α, it cannot contain β, since $S_\beta = \bigcup S_{\beta_i}$ over the set of β_i which are not contained in any filter in $\bar{S}_\alpha$.

Suppose that for every pair of lattice elements α, β for which the sets S^*_α, S^*_β are disjoint, there exist disjoint α' and β' with $S^*_\alpha \subset S_{\alpha'}$ and $S^*_\beta \subset S_{\beta'}$. A lattice which satisfies this condition is called <u>normal</u>, and if for every α the regular sublattice whose maximal element is α also satisfies this condition, the lattice is called <u>locally normal</u>.

<u>Theorem 3.4.1</u>. Let $\mathfrak{A}$ be a locally paracompact, Hausdorff lattice, and let there be given a set of prime filters S^*_α and a filter A containing α. Then there exist disjoint elements α' and β' such that $S_{\alpha'} \supset S^*_\alpha$ and $A \in S_{\beta'}$.

<u>Proof</u>. By the Hausdorff property, for every filter X in S^*_α we

may find disjoint elements $\alpha(X)$ and $\beta(X)$, contained respectively in X and A. The family $\{\alpha(X), \alpha\}$, as X ranges over S_α^*, has a locally finite refinement $R = \{\alpha', \alpha'(X)\}$, hence for some γ in A, only the elements $\alpha'(X_1), \ldots \alpha'(X_n)$ meet γ. Then the g.l.b. of the corresponding set of $\beta(X_i)$ (which will be our β') does not meet any of the $\alpha'(X)$, whence we may take $\alpha' = \bigcup \alpha'(X)$ $(X \varepsilon S_\alpha^*)$. This establishes the theorem, which holds without modification when $\mathfrak{A}$ is replaced by the sublattice whose maximal element is α.

Theorem 3.4.2. A locally paracompact, Hausdorff lattice is locally normal.

Proof. Given disjoint sets of prime filters S_α^* and S_β^*, we may find by the previous theorem, for each filter X in S_α^*, elements $\alpha(X)$ and $\beta(X)$ such that $X \varepsilon S_{\alpha(X)}$ and $S_{\beta(X)} \supset S_\beta^*$. The family $\{\alpha, \alpha(X)\}$, as before, has a locally finite refinement $R = \{\alpha', \alpha'(X)\}$. Let α' be the l.u.b. of that subfamily of R made up of elements which are contained in some filter in S_α^*. Clearly, $S_{\alpha'} \supset S_\alpha^*$. We will show that there is a β' disjoint from α' such that $S_{\beta'} \supset S_\beta^*$. For each filter Y in S_β^* there is, by the local finiteness of R, a lattice element $\lambda(Y)$ which meets only a finite number of the elements in R, say $\alpha'(X_1) \ldots \alpha'(X_n)$. Each of these $\alpha'(X_i)$ which are contained in a filter of S_α^* are, by definition, less than some $\alpha(X_i)$ contained in the filter X_i of S_α^*. Let $\gamma(Y)$ be the g.l.b. of $\lambda(Y)$ and $\beta(X_i)$ corresponding to these $\alpha(X_i)$. $S_{\gamma(Y)}$ contains Y and $\gamma(Y)$ does not meet α'. Let β' be the l.u.b. of the $\gamma(Y)$ as Y ranges over S_β^*. As α' and β' have the required properties, we have established that $\mathfrak{A}$ is normal. The proof of the locally normal property is exactly the same--we merely reduce the domain of discourse to the sublattice with α as maximal element.

A family of lattice elements $\{\alpha_i\}$ is called <u>shrinkable</u> if there is another family $\{\beta_i\}$, indexed by the same set I, with $\overline{S}_{\beta_i} \subset S_{\alpha_i}$ and $\cup\beta_i = \cup\alpha_i = \alpha$.

<u>Theorem 3.4.3</u>. Every filter finite family in a locally normal lattice is shrinkable. (Hence a priori, any finite or locally finite family.)

<u>Proof</u>. Let Ω be the set of all families $\{\gamma_i\}$ which (a) are indexed by I, (b) have the same l.u.b. (that is, $\cup\gamma_i = \alpha$), (c) for each i, either $\overline{S}_{\gamma_i} \subset S_{\alpha_i}$ or $\gamma_i = \alpha_i$. We will partly order Ω by agreeing that $\{\gamma_i'\}$ is "larger" than $\{\gamma_i\}$ if it is different from $\{\alpha_i\}$ in more places, but unchanged wherever $\{\gamma_i\}$ was already different from $\{\alpha_i\}$. Each "chain," i.e. linearly ordered subset of Ω, has a sup, $\{\Gamma_i\}$ which is α_i if every family in the chain is α_i for that i, and otherwise is equal to the γ_i which appears suddenly at some point in the chain and thereafter remains constant. Each filter A containing α, contains only a finite number of the α_i, whence there is some point in the chain such that all of <u>these</u> which will eventually be modified, have already been modified. It is thus clear that A contains $\cup\Gamma_i$, and since A is arbitrary, $\Gamma_i = \alpha$, whence $\{\Gamma_i\}$ is in Ω. We now apply Zorn's lemma, and conclude there exists a maximal family $\{\beta_i\}$. To complete the proof we need only show that β_i is not equal to α_i for any i. Assume the contrary, that $\beta_{i_0} = \alpha_{i_0}$, and let $\beta' = \cup\beta_i$ over the index set $I - \{i_0\}$. Let S' be the set of filters $S_\alpha \cap S^*_{\beta'}$, and S" be the set of filters $S_\alpha \cap S^*_{\alpha_{i_0}}$. S" and S' are disjoint, since no filter can contain $\alpha = \alpha_{i_0} \cup \beta'$, yet neither α_{i_0} or β'. By local normality we can find a γ

$S'' \subset S_\gamma$ and a γ' with $S' \subset S_{\gamma'}$ which implies that $\overline{S}_\gamma \subset S_{\alpha_{i_0}}$. This implies that $\{\beta_i\}$ was not maximal, a contradiction which establishes the theorem.

<u>Theorem 3.4.4</u>. Let $\{\alpha_i\}$ be a locally finite family in a normal lattice, and suppose that we are given for every pair (i,j) such that α_i meets α_j, and every prime filter A containing $\alpha_i \cap \alpha_j$, an element $\beta_{ij}(A) \leq \alpha_i \cap \alpha_j$. We can find for each prime filter A an element $\gamma(A)$ such that

(1) If $\alpha_i \cap \alpha_j \in A$, then $\gamma(A) \leq \beta_{ij}(A)$

(2) If $\gamma(A) \cap \gamma(B) \neq \emptyset$, then for some i, $\gamma(A)$ and $\gamma(B)$ are both less than α_i.

<u>Proof</u>. By local finiteness, for each A there is an element δ in A such that only a finite number of the $\beta_{ij}(A)$ meet δ, so it is trivial to find a family $\gamma(A)$ which satisfies (1). Having done this, we shrink $\{\alpha_i\}$ to $\{\alpha_i'\}$ and impose the additional condition that $\gamma(A) < \alpha_i'$ if A contains α_i. This condition is also easily satisfied by the local finiteness of $\{\alpha_i'\}$. Finally, we suppose that if A is contained in S_{α_i} but not in $\overline{S}_{\alpha_i'}$, then $\gamma(A) \cap \alpha_i' = \emptyset$. To see that this last condition can also be satisfied, note that for each A there are only finitely many of these $\overline{S}_{\alpha_i'}$ and their union is the set $\overline{S}_{\cup\alpha_i'}$, which is S_β^* for some β, as we have already remarked--the rest follows from Theorem 3.4.1.

These three conditions imply that $\gamma(A)$ meets α_i' if and only if A is in $\overline{S}_{\alpha_i'}$. Now suppose that $\gamma(A) \cap \gamma(B) \neq \emptyset$. A contains some α_i' whence $\gamma(A) \cap \gamma(B) < \gamma(A) < \alpha_i'$. This says that $\gamma(B)$ meets α_i'; hence B is in $\overline{S}_{\alpha_i'}$

and therefore $\gamma(B) < \alpha_i$. Since both $\gamma(A)$ and $\gamma(B)$ are less than α_i, the theorem is proved.

<u>Theorem 3.4.5</u>. If $\mathfrak{F}$ is a presheaf over a locally paracompact, Hausdorff lattice $\mathfrak{A}$, every germ in $\overline{\mathfrak{F}}$ is given by the set of projections of a clan in $\mathfrak{F}$.

<u>Proof</u>. Suppose $\overline{g}_\alpha$ is a germ whose values are the projections of a set of precursors $\{g_\alpha\}$. $\{\alpha_i\}$ has a locally finite refining family $\{\delta_j\}$ and $\{g_{\delta_j} = \rho^{\alpha_{\phi(j)}}_{\delta_j}(g_{\alpha_i})\}$ is also a set of precursors for $\overline{g}_\alpha$. For each A containing $\delta_i \cap \delta_j$ we have

$$\rho^{\delta_i}_A(g_{\delta_i}) = \rho^{\delta_j}_A(g_{\delta_j}) = \overline{g}_{\alpha(A)}$$

whence there is an element $\beta_{ij}(A)$ such that

$$\rho^{\delta_i}_{\beta_{ij}(A)}(g_{\delta_i}) = \rho^{\delta_j}_{\beta_{ij}(A)}(g_{\delta_j})$$

Now construct the $\{\gamma(A)\}$ whose existence is asserted by the previous theorem, and note that $\{g_{\gamma(A)} = \rho^{\alpha_i}_{\gamma(A)}(g_{\alpha_i})\}$ is again a set of precursors for $\overline{g}_\alpha$. It is, in fact, a clan, since if $\gamma(A) \quad \gamma(B) \neq \emptyset$ then $\gamma(A)$ and $\gamma(B)$ are both $< \alpha_i$, whence

$$\rho^{\alpha_i}_{\gamma(A)\cap\gamma(B)}(g_{\alpha_i}) = \rho^{\gamma(A)}_{\gamma(A)\cap\gamma(B)}(g_{\gamma(A)}) = \rho^{\gamma(B)}_{\gamma(A)\cap\gamma(B)}(g_{\gamma(B)}).$$

This proves the theorem.

In §3.2 we noted that if $\mathfrak{F}$ was a presheaf satisfying S(2), its image under the natural mapping i is the set of germs whose values are projections of a clan. We thus have the following corollary.

<u>Corollary 3.4.5</u>. If $\mathfrak{F}$ is a presheaf satisfying S(2) over a locally paracompact, Hausdorff lattice $\mathfrak{A}$, then the natural mapping $i: \mathfrak{F} \to \overline{\mathfrak{F}}$ is surjective.

§3.5 Theorems on Flasque Sheaves

Theorem 3.5.1. If $0 \longrightarrow \mathcal{G}' \xrightarrow{i} \mathcal{G} \xrightarrow{\pi} \mathcal{G}'' \longrightarrow 0$ is exact and $\mathcal{G}'$ is flasque, then the associated couples of modules $0 \longrightarrow G'_\alpha \xrightarrow{i_\alpha} G_\alpha \xrightarrow{\pi_\alpha} G''_\alpha \longrightarrow 0$ are exact for each α.

Proof. We must show that π_α is a homomorphism onto. Pick an element s" in G''_α. $\mathcal{G}''$ is the sheaf generated by the presheaf $\{\operatorname{Im} \pi_\alpha\}$, hence from Theorem 3.2.2 it follows that s" is a function on the prime filters containing α whose values are given by the projections of a clan of elements in Im π.

Consider the set S of elements in $\mathcal{G}$ which map under π onto some descendent of s", and partially order it by agreeing that $s_\beta > s_\gamma$ iff s_β is an ancestor of s_γ. Any linearly ordered subset of S is also a clan, and therefore has a unique l.u.b., namely the ancestor of that clan. Applying Zorn's lemma, we conclude that there is a maximal element s_β in $\mathcal{G}$ which is mapped by π onto a descendent of s".

If $\beta = \alpha$, then the proof is complete, so assume $\beta < \alpha$. We then can find a prime filter A_0 which contains α but not β, an element γ in A_0, and an element s_γ in G_γ which is mapped by π onto a descendent of s". We must have that $\rho^\beta_{\beta\cap\gamma}(s_\beta) - \rho^\gamma_{\beta\cap\gamma}(s_\gamma)$ is in Ker $\pi_{\beta\cap\gamma}$ which, by exactness, is Im $i_{\beta\cap\gamma}$, whence there is a unique element $s'_{\beta\cap\gamma}$ in $G'_{\beta\cap\gamma}$ which is mapped onto it by $i_{\beta\cap\gamma}$. Since $\mathcal{G}'$ is flasque, $s'_{\beta\cap\gamma}$ has an ancestor s'_γ in G'_γ, and we will denote $i_\gamma(s'_\gamma)$ by $\bar{s}_\gamma$. It is clear that $\rho^\gamma_{\beta\cap\gamma}(\bar{s}_\gamma + s_\gamma) = \rho^\beta_{\beta\cap\gamma}(s_\beta)$, whence $\bar{s}_\gamma + s_\gamma$ and s_β form a clan, and therefore have a common ancestor $s_{\gamma\cup\beta}$. As $s_{\gamma\cup\beta}$ must be mapped by $\pi_{\gamma\cup\beta}$ onto a descendent of s", the maximality of s_β is contradicted, and the theorem is established.

<u>Corollary 3.5.1</u>. If $0 \longrightarrow \mathcal{G}' \xrightarrow{i} \mathcal{G} \xrightarrow{\pi} \mathcal{G}'' \longrightarrow 0$ is exact and both $\mathcal{G}'$ and $\mathcal{G}$ are flasque, then $\mathcal{G}''$ is flasque, too.

<u>Proof</u>. Any s''_β in G''_β is the image under π of an element s_β in G_β. s_β has an ancestor s_α in G_α, and $\pi(s_\alpha)$ is an ancestor of s''_β in G''_α. Q.E.D.

The developments of the previous section lead to the following useful theorem about flasque presheaves.

<u>Theorem 3.5.2</u>. If $\mathfrak{F}$ is a flasque presheaf over a locally paracompact, Hausdorff lattice which satisfies S(2), then the associated sheaf $\overline{\mathfrak{F}}$ is also flasque.

<u>Proof</u>. This is an immediate consequence of the fact that i is surjective under these hypotheses.

CHAPTER IV

COHOMOLOGY THEORY

§4.1 <u>The Cohomology Presheaf and the Uniqueness Theorem</u>

Suppose we are given a class of sheaves $\mathfrak{Z}$ which possesses the following two properties:

Z(1) If $0 \longrightarrow \mathfrak{F}' \longrightarrow \mathfrak{F} \longrightarrow \mathfrak{F}'' \longrightarrow 0$ is an exact couple with $\mathfrak{F}'$ and $\mathfrak{F}$ in $\mathfrak{Z}$, then $\mathfrak{F}''$ is in $\mathfrak{Z}$ and Γ_α applied to this couple is exact.

Z(2) There is an exact resolution functor $\mathcal{R}_{\mathfrak{Z}}$, such that all of the sheaves in the resolution are in the class $\mathfrak{Z}$.

These axioms are certainly not incompatible, since from the definition of the canonical resolution functor in §3.3 and the theorems in §3.5, we know that the class of flasque sheaves satisfies these axioms. Moreover, the class of fine sheaves over locally paracompact lattices satisfies these axioms also, and was, in fact, the class of sheaves used in the original version of the Cartan uniqueness theorem.[1]

We construct the cohomology presheaf in the following manner. Given any sheaf $\mathfrak{F}$ we apply first the resolution functor $\mathcal{R}_{\mathfrak{Z}}$, then the functor Γ_α to obtain for each α a sequence of modules:

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[1]See [3] , Chapter 15.

$$0 \longrightarrow F_\alpha \xrightarrow{i_\alpha} C_\alpha{}^0(\mathfrak{F}) \xrightarrow{d_\alpha{}^0} C_\alpha{}^1(\mathfrak{F}) \xrightarrow{d_\alpha{}^1} C_\alpha{}^2(\mathfrak{F}) \longrightarrow \cdots$$

which is of order two since Γ_α is left exact. Regarding $\{C_\alpha{}^n\}$ as a cochain complex with coboundary operator d_α, we may apply the Mayer functor to obtain for each α the sequence of modules $\{H_\alpha{}^n(\mathfrak{F})\}$ with the following properties:

U(1) $H_\alpha{}^0(\mathfrak{F}) = F_\alpha$

U(2) $H_\alpha{}^n(\mathfrak{F}) = 0$ for $n > 0$ if $\mathfrak{F}$ is in the class $\mathfrak{Z}$

U(3) Every exact couple $0 \longrightarrow \mathfrak{F}' \xrightarrow{\phi} \mathfrak{F} \xrightarrow{\psi} \mathfrak{F}'' \longrightarrow 0$ induces for each α an exact cohomology sequence

$$0 \longrightarrow H_\alpha{}^0(\mathfrak{F}') \xrightarrow{\phi^*} H_\alpha{}^0(\mathfrak{F}) \xrightarrow{\psi^*} H_\alpha{}^0(\mathfrak{F}'') \xrightarrow{\Delta} H_\alpha{}^1(\mathfrak{F}') \longrightarrow \cdots$$

and the mapping of sequences induced by a map of couples has a commutative diagram.

The proof that U(1) is satisfied is a trivial consequence of the left exactness of Γ_α and U(3) is merely a statement of a property of the Mayer functor as applied to any cochain complex. U(2) follows from Z(1), Z(2), and the following lemma.

<u>Lemma 4.1.1</u>. The functor $\Gamma_\alpha \circ \mathcal{R}_{\mathfrak{Z}}$ is exact when applied to a sheaf $\mathfrak{F}$ in $\mathfrak{Z}$.

<u>Proof</u>. Split the sequence $0 \longrightarrow \mathfrak{F} \xrightarrow{i} C_\alpha{}^0(\mathfrak{F}) \xrightarrow{d_\alpha{}^0} C_\alpha{}^1(\mathfrak{F}) \xrightarrow{d_\alpha{}^1}$ . . . into the exact couples:

$$0 \longrightarrow \mathfrak{F} \xrightarrow{i} C_\alpha{}^0(\mathfrak{F}) \longrightarrow \operatorname{Im} d_\alpha{}^0 \longrightarrow 0$$

$$0 \longrightarrow \operatorname{Im} d_\alpha{}^0 \longrightarrow C_\alpha{}^1(\mathfrak{F}) \longrightarrow \operatorname{Im} d_\alpha{}^1 \longrightarrow 0$$

. . . . .

$$0 \longrightarrow \mathrm{Im}\ d_\alpha^{\,n-1} \longrightarrow C_\alpha^{\,n}(\mathfrak{F}) \longrightarrow \mathrm{Im}\ d_\alpha^{\,n} \longrightarrow 0$$

From Z(1) and Z(2) we see that all of the sheaves which appear in these couples are in $\mathcal{J}$, and Γ_α is exact on each couple, whence clearly Γ_α is exact on the original sequence. This proves the lemma.

We prove now an analogue of the Cartan uniqueness theorem.

Theorem 4.1.1. All cohomology functors which satisfy U(1) - U(3) are isomorphic.

Proof. Let H be the functor obtained by our constructive process and let $\hat{H}$ be any other functor which satisfies U(1) - U(3). Let $0 \longrightarrow \mathfrak{F} \xrightarrow{\ i\ } \mathcal{C}^0(\mathfrak{F}) \xrightarrow{\ \pi\ } Z^1(\mathfrak{F}) \longrightarrow 0$ be the exact couple obtained from the resolution functor as follows: $\mathcal{C}^0(\mathfrak{F})$ is the first sheaf in the resolution, and $Z^1(\mathfrak{F})$ is the quotient sheaf $\mathcal{C}^0(\mathfrak{F})/\mathfrak{F}$. Applying H and $\hat{H}$ to this couple, we obtain the commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & F_\alpha \arrow[r] \arrow[d] & C_\alpha^{\,0}(\mathfrak{F}) \arrow[r] \arrow[d] & Z_\alpha^{\,1}(\mathfrak{F}) \arrow[r, "\Delta"] \arrow[d] & H_\alpha^{\,1}(\mathfrak{F}) \arrow[r] & 0 \\
0 \arrow[r] & F_\alpha \arrow[r] & \hat{C}_\alpha^{\,0}(\mathfrak{F}) \arrow[r] & \hat{Z}_\alpha^{\,1}(\mathfrak{F}) \arrow[r] & \hat{H}_\alpha^{\,1}(\mathfrak{F}) \arrow[r] & 0
\end{tikzcd}$$

The diagram is, of course, the first few terms of the exact cohomology sequence whose existence is asserted in U(3). U(1) tells us that the functors H and $\hat{H}$ are already isomorphic in dimension zero, these isomorphisms being the three vertical arrows in the diagram. The zeros on the right come from U(2) and the fact that $\mathcal{C}^0(\mathfrak{F})$, as the first sheaf of the resolution, is in $\mathcal{J}$. It is clear that there is a unique isomorphism of $H_\alpha^{\,1}(\mathfrak{F})$ with $\hat{H}_\alpha^{\,1}(\mathfrak{F})$. Since $\mathfrak{F}$ is arbitrary, H and $\hat{H}$ are isomorphic in dimension one. Continuing by induction, we assume H and $\hat{H}$ are isomorphic up to dimension n. From our cohomology sequence we obtain the diagram

$$\begin{tikzcd}
0 \arrow[r] & H_\alpha^{n-1}(Z^1(\mathfrak{F})) \arrow[r, "\Delta"] \arrow[d] & H_\alpha^{n}(\mathfrak{F}) \arrow[r] & 0 \\
0 \arrow[r] & \hat{H}_\alpha^{n-1}(Z^1(\mathfrak{F})) \arrow[r, "\Delta"] & \hat{H}_\alpha^{n}(\mathfrak{F}) \arrow[r] & 0
\end{tikzcd}$$

where the vertical arrow is the isomorphism asserted by the inductive hypothesis. As this clearly induces an isomorphism in dimension n, we have established the theorem.

A sheaf $\mathfrak{F}$ for which $H^n(\mathfrak{F}) = 0$ for $n>0$ is called <u>acyclic</u>. The next theorem asserts that although we obtained the functor H from a particular resolution, there is a rather wide class of resolutions which would have done just as well--in particular, any resolution by means of flasque sheaves.

<u>Theorem 4.1.2</u>. Let $0 \longrightarrow \mathfrak{F} \xrightarrow{i} \mathfrak{G}^0 \xrightarrow{\delta^0} \mathfrak{G}^1 \xrightarrow{\delta^1} \mathfrak{G}^2 \longrightarrow \cdots$ be a resolution of $\mathfrak{F}$ for which all of the $\mathfrak{G}^i$ are acyclic. Then the cohomology obtained from this resolution by applying the functor Γ_α and then the Mayer functor is isomorphic to the cohomology obtained from the "standard" resolution $\mathcal{R}_{\mathfrak{Z}}$.

<u>Proof</u>. As $\mathcal{R}_{\mathfrak{Z}}$ is exact and Γ_α is exact on $\mathfrak{Z}$, an application of the functor $\Gamma_\alpha \circ \mathcal{R}_{\mathfrak{Z}}$ to the above exact sequence, yields the following double-complex:

$$\begin{tikzcd}
 & & & & \\
0 \arrow[r] & G_\alpha^1 \arrow[r] \arrow[u] & C_\alpha^0(\mathfrak{G}^1) \arrow[r] \arrow[u] & C_\alpha^1(\mathfrak{G}^1) \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & G_\alpha^0 \arrow[r] \arrow[u] & C_\alpha^0(\mathfrak{G}^0) \arrow[r] \arrow[u] & C_\alpha^1(\mathfrak{G}^0) \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & F_\alpha \arrow[r] \arrow[u] & C_\alpha^0(\mathfrak{F}) \arrow[r] \arrow[u] & C_\alpha^1(\mathfrak{F}) \arrow[r] \arrow[u] & \cdots \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

in which all of the rows and columns are exact except for the bottom row and the first column, since all of the $C_\alpha{}^n(\mathcal{G}_\eta^k)$ are in $\mathfrak{J}$ and all of the $G_\alpha{}^n$ are assumed acyclic. This is, however, precisely the situation of Theorem 2.2.2, and an application of that theorem completes the proof.

§4.2 Antipresheaves and the Alexander Resolution

An antipresheaf $\mathcal{A} = \{A_\alpha, i_\beta{}^\alpha\}$, is a collection of sets A_α indexed by a regular lattice $\mathfrak{A}$, and injections $i_\beta{}^\alpha: A_\alpha \to A_\beta$ defined whenever $\alpha \leq \beta$, such that $i_\beta{}^\alpha \circ i_\alpha{}^\gamma = i_\gamma{}^\alpha$. One notes that the maps $i_\beta{}^\alpha$ are all in the "wrong" direction, and the sets A_α are not assumed to have any algebraic or topological structure.

In general, given a mapping of sets $f: A \to B$, and an arbitrary algebraic structure K, one can always define the sets K^A and K^B of all functions on A (respectively B) having values in K, and the "dual map" $f^*: K^B \to K^A$, which takes ϕ into $\phi \circ f$. It is clear that K^A and K^B have the same algebraic structure as K, and if f is injective then f^* is surjective, since any function ϕ from A to K can be regarded as already defined on the subset Im f of B and extended to the rest of B arbitrarily. Any such extension will be mapped onto ϕ by f^*.

Let us assume that K is a principal ideal ring, and denote $K^{A\alpha}$ by $A_\alpha^*(K)$ and the dual maps $i_\beta{}^{\alpha *}$ by $\rho_\beta{}^\alpha$. Since $\mathcal{A}$ is a functor on the lattice $\mathfrak{A}$, $\mathcal{A}^*$ is the "dual functor" whence $\rho_\alpha{}^\gamma \circ \rho_\gamma{}^\beta = \rho_\alpha{}^\beta$, and $\rho_\alpha{}^\alpha$ is the identity. Thus $\mathcal{A}^* = \{A_\alpha^*(K), \rho_\beta{}^\alpha\}$ is a flasque presheaf of K modules. Moreover, A^* satisfies sheaf axiom S(2), since if $\{f_{\alpha_i}\}$ is a clan of functions mapping A_{α_i} into K, the restrictions $f_{\alpha_i} | A_{\alpha_i \cap \alpha_j}$ and $f_{\alpha_j} | A_{\alpha_i \cap \alpha_j}$ must be identical. Identifying a subset of A_α with each A_{α_i}, it is clear that the clan defines uniquely a function on the part of A_α which is the union of the images of the $i_\alpha^{\alpha_i}$. No matter how we define the function on the remainder of A_α, it is clear that it will be an ancestor of our clan. The ancestor will be unique, and therefore $\mathcal{A}^*$ will be a sheaf if, and only if, for every pair α, β in $\mathfrak{A}$, $A_{\alpha \cup \beta} = A_\alpha \cup A_\beta$.

In any case, if $\mathfrak{A}$ is locally paracompact and Hausdorff, the associated sheaf $\mathcal{A}^*$ will be flasque, and hence cohomologically trivial. The importance of this fact is that any resolution of a sheaf $\mathfrak{F}$ which uses only sheaves constructed in this manner from antipresheaves, must give the same cohomology sheaf as the canonical resolution, in virtue of the theorems of the last section.

Before defining the Alexander resolution we prove a theorem which materially simplifies the task of verifying exactness for a sequence of sheaves constructed from antipresheaves. We assume as before, that $\mathfrak{A}$ is locally paracompact and Hausdorff.

<u>Theorem 4.2.1</u>. If $0 \longrightarrow \mathcal{A}^0 \xrightarrow{f^0} \mathcal{A}^1 \xrightarrow{f^1} \cdots$ is an exact sequence of presheaves which satisfy S(2), then the associated sequence $0 \longrightarrow \bar{\mathcal{A}}^0 \xrightarrow{\bar{f}^0} \bar{\mathcal{A}}^1 \xrightarrow{\bar{f}^1} \bar{\mathcal{A}}^2 \longrightarrow \cdots$ of sheaves and sheaf homomorphisms is exact, where $\bar{\mathcal{A}}^n$ is the sheaf generated by $\mathcal{A}^n$, and $\bar{f}^n$ is induced by f^n.

<u>Proof</u>. We show first that $\bar{f}^n$ is well defined. Since S(2) is satisfied, the natural mapping i can be regarded as identifying multiple ancestors, and by Corollary 3.4.5, i_α is surjective for each α. Thus $\bar{f}^n$ is defined for an

$$\begin{tikzcd} \mathcal{A}^n \arrow[r, "f^n"] \arrow[d, "i"] & \mathcal{A}^{n+1} \arrow[d, "i"] \\ \bar{\mathcal{A}}^n \arrow[r, "\bar{f}^n"] & \bar{\mathcal{A}}^{n+1} \end{tikzcd}$$

element $\bar{a} \in \bar{A}_\alpha^n$ as $i \circ f^n(a)$, where a is any element of $i^{-1}(\bar{a})$. If a' is any other element of $i^{-1}(\bar{a})$, then a - a' is an ancestor of a zero clan, hence is mapped into an ancestor of a zero clan by f^n, and i $f^n(a-a') = 0$, showing that $\bar{f}^n$ is well defined.

We wish now to show that if Im $f_\alpha{}^n$ = Ker $f_\alpha{}^n$ for all α, then Im $\bar{f}^n$ generates the sheaf Ker $\bar{f}^{n+1}$. We must show that if $\bar{f}_\alpha{}^{n+1}(\bar{a}) = 0$, then for every prime filter A containing α there is a $\beta(A)$ such that $\bar{\rho}_\beta{}^\alpha(\bar{a}) = \bar{f}_\beta{}^n(\bar{b})$ for some $\bar{b}$ in $\bar{A}_\beta{}^n$. One sees readily from the diagram at the right that this is indeed the case.

$$\begin{tikzcd}
 & a'_\alpha \arrow[r] \arrow[d] & c'_\alpha \arrow[d] \\
b_\beta \arrow[r] \arrow[d] & a''_\beta \arrow[r] \arrow[d] & 0_\beta \arrow[d] \\
\bar{b}_\alpha \arrow[r] & \bar{a}_\alpha \arrow[r] & 0
\end{tikzcd}$$

We define the Alexander anti-presheaf $\mathcal{A}^n = \{A_\alpha{}^n, i_\beta{}^{(n)\alpha}\}$ as follows: $A_\alpha{}^n$ is the product of the set S_α of prime filters containing α with itself n+1 times, and the maps $i_\beta{}^{(n)\alpha}$ are the natural inclusions. The elements of the presheaves $\mathcal{A}^{n*}$ generated by the $\mathcal{A}^n$ are seen to be functions of n+1 variables with values in K, each variable having S_α as its domain. $\mathcal{A}^{0*}$ is just the flasque extension of K. The coboundary operator $\delta_\alpha{}^n$ is defined, as usual, by

$$\delta f(X_0, \ldots X_{n+1}) = \sum (-1)^i f(X_0, \ldots \hat{X}_i, \ldots X_{n+1})$$

whence we have the sequence of presheaves

$$0 \longrightarrow K \xrightarrow{i} \mathcal{A}^{0*} \xrightarrow{\delta^0} \mathcal{A}^{1*} \xrightarrow{\delta^1} \cdots$$

For any f in $\mathcal{A}^{n*}$ we define the function $Y \cdot f \in \mathcal{A}^{n-1*}$ by $Y \cdot f(X_0, \ldots X_{n-1}) = f(Y, X_0, \ldots X_{n-1})$ and verify that $\delta Y \cdot f = f - Y \cdot \delta f$. Thus if $\delta f = 0$ then $Y \cdot \delta f = 0$ and $f = \delta Y \cdot f$, whence this sequence of presheaves is exact. In view of the previous theorem, the associated sequence of sheaves is seen to be a flasque resolution of the simple sheaf generated by K, which we will call the Alexander resolution. If we consider $\mathfrak{A}$ to be the lattice of open sets of a topological space, it is clear that the groups $H_\alpha(\overline{K})$ of the cohomology sheaf are the Alexander cohomology groups of open set α, with the relative topology. Thus the cohomology groups of the space are the groups $H_M(\overline{K})$, where M is the maximal element of the lattice.

Since we obtain the "right" groups in this case by resolving the simple sheaf $\overline{K}$, it seems reasonable to suppose that resolving $\overline{K}$ will also give us the "right" groups in the case of the four finite lattices of Chapter I. We will show in the following section that this is indeed the case.

As a final illustration of this technique, we will give a brief indication of how a singular theory can be generated by antipresheaves. For this we define a sequence $\{\mathfrak{M}_n\}$ of "model lattices," and denote by $\mathfrak{A}_\alpha$ the sublattice of $\mathfrak{A}$ whose maximal element is α. We define $A_\alpha^{\,n}$ to be the set of $\emptyset$-preserving lattice homomorphisms of $\mathfrak{A}_\alpha$ into $\mathfrak{M}_n$, and $i_\beta^{\,\alpha}$ to be the "homomorphism extending" maps which take $f\colon \mathfrak{A}_\alpha \to \mathfrak{M}_n$ into the map $i_\beta^{\,\alpha}(f)\colon \mathfrak{A}_\beta \to \mathfrak{M}_n$ defined by $i_\beta^{\,\alpha} f(\gamma) = f(\gamma \cap \alpha)$. We further suppose that we are given n+1 lattice homomorphisms $D_k\colon \mathfrak{M}_n \to \mathfrak{M}_{n-1}$ $(k=0,\ldots n)$, which we use to define our coboundary operator in the usual manner:

$$\delta f(\emptyset) = \Sigma\, (-1)^k\, f(D_k \circ \emptyset)$$

We have already noted that the $\mathcal{A}^{n*}$ are flasque presheaves whenever $\mathfrak{A}$ is locally paracompact, hence we need only show that the sequence of presheaves

$$0 \longrightarrow K \longrightarrow \mathcal{A}^{0*} \longrightarrow \mathcal{A}^{1*} \longrightarrow \cdots$$

is exact to conclude that the associated sequence of sheaves gives the same cohomology as the canonical resolution. However, it is almost certainly not true that this sequence of presheaves is always exact, since we have allowed quite general model lattices. In fact, even if we restrict ourselves to acyclic models it is possible that some further restriction on the underlying lattice would still have to be made to obtain exactness. It is hoped that further study will clarify this point.

§4.3 The "Classical" Cohomology Groups and the Finite Test Lattices

The object of this section is to show that if we take as our sheaf the simple sheaf generated by a constant presheaf K, the finite test lattices of Chapter 1 have the "right" cohomology groups. For these simple lattices we can compute the groups directly from the canonical resolution--a procedure which will serve to illustrate some of the theory of the preceding sections.

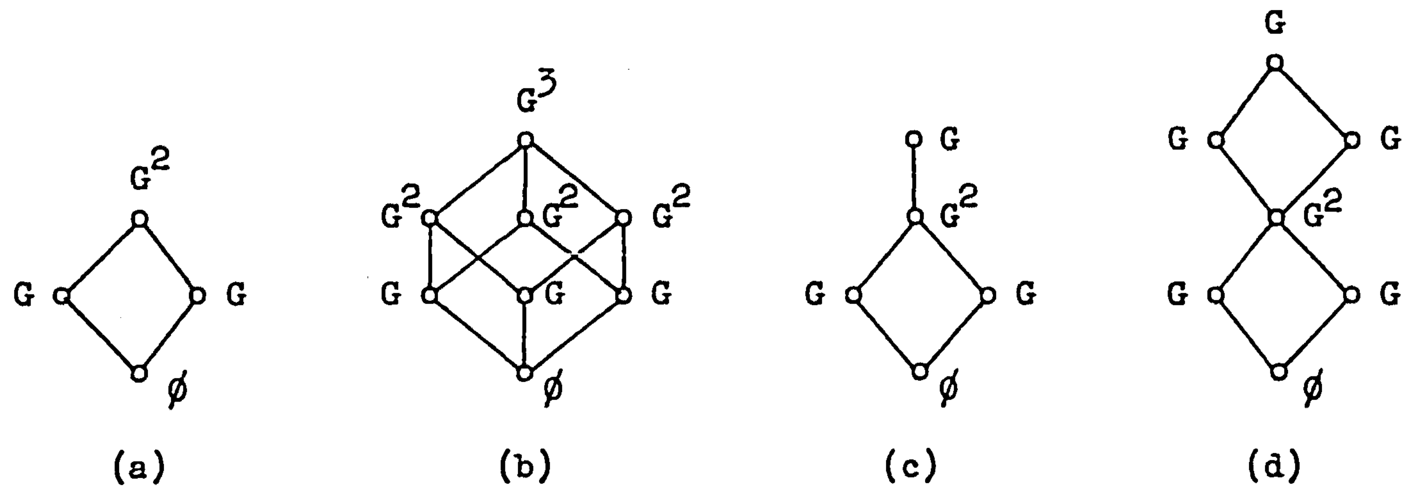
We will begin by making some general statements about sheaves over finite lattices--all of which are trivial to verify.

For a finite lattice the "direct limit" modules $\{F_A\}$ associated with the prime filters, are merely those modules F_α where α is join irreducible.

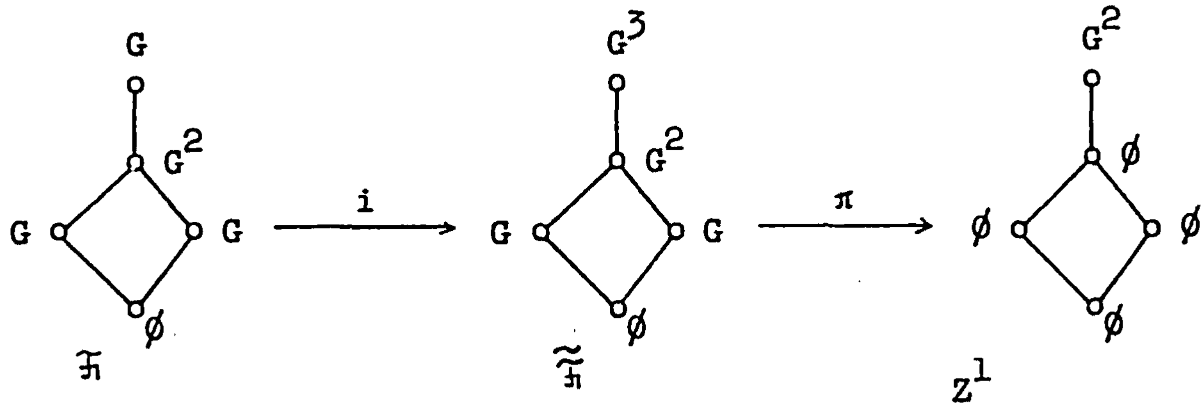
$H_\alpha^0(\mathfrak{F}) = F_\alpha$ by U(1). We may characterize the higher dimensional groups by $H_\alpha^n(\mathfrak{F}) = Z_\alpha^n/(\mathrm{Im}\ \pi_\alpha^{n-1})$ where the notation refers to the canonical resolution, §3.3. One further notes that $\{\mathrm{Im}\ \pi_\alpha^{n-1}\}$ is the presheaf whose associated sheaf is $\{Z_\alpha{}^n\}$.

If for some n, $Z^n = \tilde{Z}^n$, then $Z^k = 0$ for all $k > n$, and the cohomology sheaf is the zero sheaf in dimension k.

The constant presheaf is not a sheaf for any of the lattices of Chapter I. The associated simple sheaves are

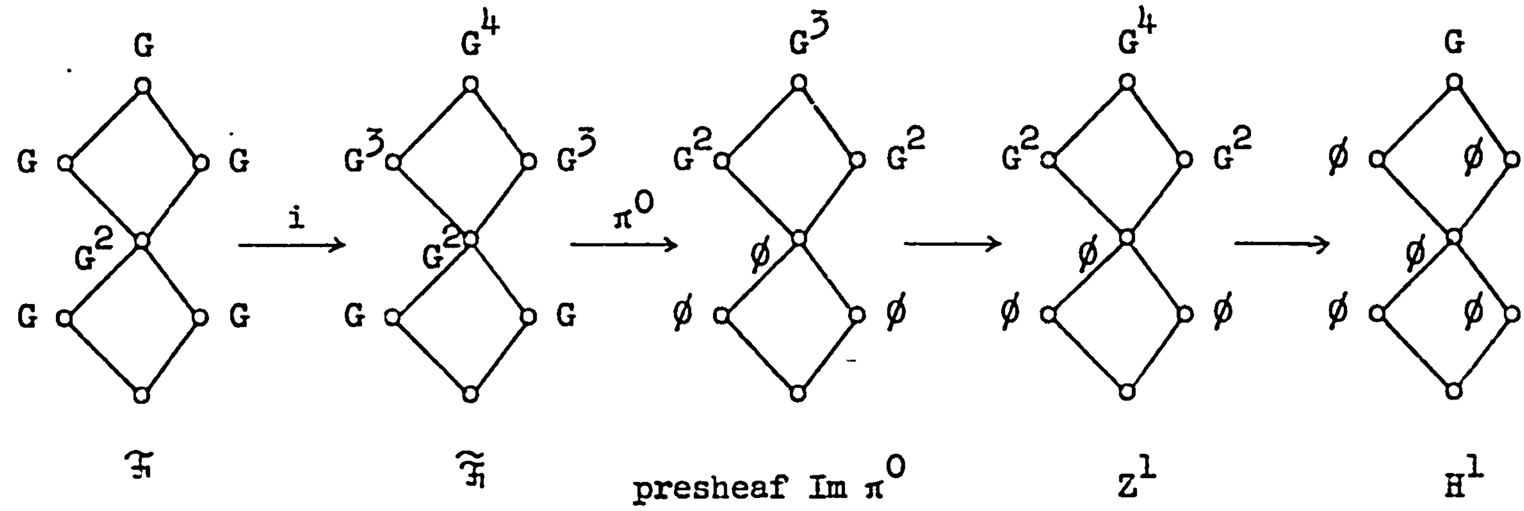


where G^n denotes the direct product of G with itself n times. These diagrams also represent the zero dimensional cohomology of these lattices. For lattices (a) and (b) there is nothing more since (a) and (b) are flasque. Neither (c) nor (d) is flasque, but even so, (c) has no higher dimensional cohomology. Since the presheaf $\{\operatorname{Im} \pi_\alpha{}^0\}$



is already a sheaf, there is no 1-dimensional cohomology, and since

$Z^1 = \widetilde{Z}^1$ there is no cohomology of dimension higher than 1. Only for lattice (d) do we get something in dimension 1, as the diagram below illustrates.



CHAPTER V

CONCLUSIONS

With the exception of a brief digression in Chapter 2, where we gave an alternative proof of Dowker's theorem on the cohomology groups of relations, the preceding chapters have been devoted to the building up of a cohomology theory for lattices with coefficients in a sheaf. The properties of sheaves which are necessary to develop the theory were elucidated in Chapter 3, and in Chapter 4 the cohomology functor was defined and certain of its properties established. In particular, we have shown that this theory gives the "right" results when the underlying lattice is the lattice of open sets of a locally paracompact Hausdorff space, as well as for certain finite lattices which are also associated in a natural manner with topological spaces. Notable in this approach is the uniformity achieved by associating lattices with both the topology of a space and the decompositions used for the purpose of computing the cohomology groups. Also notable is the fact that the sheaves were endowed with no topological structure, but treated throughout from a purely algebraic viewpoint. A final comment is that in this thesis we considered a cohomology presheaf rather than the usual cohomology module (which is, of course, the maximal element of the presheaf), and in this sense we have a sort of rudimentary "relative" theory in which we have simultaneously represented the various stages in the assembly of the cohomology of a space from the cohomology of the "atoms" of a particular decomposition.

Although the original problem of placing the Hurewicz theorem in its "proper" setting cannot be considered solved, a certain amount of progress has been made in that direction and a number of related questions have been answered. The proper analogue of the Hurewicz generalized complex seems to be the flasque cochain sheaf $\{C_\alpha^{\ n}\}$ of §4.1, although it is not clear why one deals with homology in one case and cohomology in the other. We may, in fact, write down the corresponding Mayer-Vietoris couple

$$0 \longrightarrow C_{\alpha \cup \beta} \xrightarrow{\ \phi\ } C_\alpha \oplus C_\beta \xrightarrow{\ \psi\ } C_{\alpha \cap \beta} \longrightarrow 0$$

where

$$\phi(g) = (\rho_\alpha^{\ \alpha\cup\beta}(g),\ \rho_\beta^{\ \alpha\cup\beta}(g)) \quad \text{and} \quad \psi(g_1,g_2) = \rho^\alpha_{\alpha\cap\beta}(g_1) - \rho^\beta_{\alpha\cap\beta}(g_2)$$

and it follows readily that the couple is exact. In fact, $\psi\phi = 0$ since $\rho_\gamma^{\ \beta} \circ \rho_\beta^{\ \alpha} = \rho_\gamma^{\ \alpha}$, and if (g_1, g_2) is in Ker ψ, then g_1 and g_2 form a clan and have a unique common ancestor. This shows Im ϕ = Ker ψ, and ancestral uniqueness shows that ϕ is injective. ψ is surjective, since C is flasque. On the cohomology level, therefore, we have an induced exact triangle

$$\begin{tikzcd} H_{\alpha\cup\beta} \arrow[r, "\phi*"] & H_\alpha \oplus H_\beta \arrow[r, "\psi*"] & H_{\alpha\cap\beta} \arrow[ll, bend left, "\Delta"] \end{tikzcd}$$

We can thus state the following obvious analogue to the Hurewicz theorem: If $f: \mathfrak{F} \to \mathfrak{G}$ is a presheaf homomorphism for which the induced map $\hat{f}: \mathcal{C}(\mathfrak{F}) \longrightarrow \mathcal{C}(\mathfrak{G})$ of cochain sheaves is an isomorphism on the irreducible elements, and if $\mathfrak{A}$ satisfies the descending chain condition, then the sheaves $\mathfrak{F}$ and $\mathfrak{G}$ have the same cohomology. The proof is trivial.

In this light, the Hurewicz theorem is seen as an attack on the problem of determining when two sheaves $\mathfrak{F}$ and $\mathfrak{G}$ over the same lattice have the same cohomology presheaf, while the Cartan theorem deals with the question of whether two cohomology functors H and H' assign the same cohomology presheaf to a particular sheaf.

In addition to the problem of extending the Hurewicz theorem, there are a number of questions raised by this thesis which would seem to be worthy of further research. Although all of the usual decompositions of topological spaces can be seen to give rise to regular lattices, it might still be of interest to know to what extent the assumption of regularity could be weakened without invalidating the theory we have built up.

One might also investigate how the theory can be extended if the underlying lattice fails to be locally paracompact. The "classical" approach to this problem was to introduce a "paracompactifying" family Φ of closed sets, and consider only sections with supports in this family. The corresponding idea in this case is actually much simpler. Translating the axioms for the family Φ, first into axioms about the complements of sets in Φ, then into general lattice terminology, we see that this amounts to picking a cofinal filter Φ in the lattice $\mathfrak{A}$, and defining the subsheaf $\mathfrak{F}_\Phi$ of $\mathfrak{F}$ by setting $F_\alpha = 0$ and $\rho_\beta{}^\alpha = 0$ whenever

$\alpha \varepsilon \Phi$. It may happen that the sheaf $\mathfrak{H}$ can be sufficiently "thinned out" by this process, that all clans will have locally finite refining clans with the same ancestor. An investigation of the relationship between the cohomology of $\mathfrak{H}$ and $\mathfrak{H}_{\Phi}$ would seem to be quite difficult, but any results in this direction should be of interest.

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