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COMPLEX P-LAPLACIAN ON KÄHLER MANIFOLDS
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BY

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Abstract

From a geometric point of view, we use coordinates as the main tool to define the holomorphic gradient $\nabla^+ f$, the antiholomorphic gradient $\nabla^- f$, and the complex gradient $\nabla^c f$ of a complex-valued function f on Kähler manifolds. Then we define the holomorphic Laplacian $\Delta^+ f$, the antiholomorphic Laplacian $\Delta^- f$, and the complex Laplacian $\Delta^c f$ of a real-valued function f. For a C^2 function f, the holomorphic Laplacian $\Delta^+ f$ actually coincides with the well-known complex Laplacian $\Box f$ since under holomorphic normal coordinates $\Delta^+ f = \sum_{i=1}^n \frac{\partial}{\partial z^i}\frac{\partial f}{\partial \bar{z}^i} = \frac{1}{2}\Delta f = \Box f$. For the first time, we introduce the holomorphic p-Laplacian $\Delta_p^+ f$, the antiholomorphic p-Laplacian $\Delta_p^- f$, and the complex p-Laplacian $\Delta_p^c f$, and we find the relationship among them. We also find a relationship between $\Delta_p^c f$ and $\Delta_p f$. Finally, based on this relationship, we make global integral estimates on complete noncompact Kähler manifolds as an application of $\Delta_p^c f$, $\Delta_p^+ f$, and $\Delta_p^- f$.

Chapter 1

Introduction

In this paper, we will find a reasonable way to define and then calculate the complex p-Laplacian on Kähler manifolds.

An *almost complex structure* on a differentiable manifold M is a real differentiable tensor field J of rank (1,1) with the property

$$J(J(\xi)) = -\xi$$

for any differentiable vector field ξ; that is, a real tensor J whose (real) components $j_a^{\ b}$ satisfy

$$j_a^{\ s} j_s^{\ b} = -\delta_a^{\ b}.$$

A manifold M with an almost complex structure J is even-dimensional. A manifold which admits an almost complex structure is called an *almost complex manifold*.

If M is a complex manifold, then the holomorphic charts identify each tangent space T_pM with $\mathbf{C}^n$, so we can define $J(v) = \sqrt{-1}v$ for every $v \in T_pM$, which gives an almost complex structure J. The fact that the transition functions are holomorphic means precisely that multiplication by $\sqrt{-1}$ is compatible under the different identifications of T_pM with $\mathbf{C}^n$ using different charts. Since the transition maps between charts are biholomorphic, complex manifolds are, in particular, smooth.

If $z^1, ..., z^n$ are holomorphic coordinates and $z^i = x^i + \sqrt{-1}y^i$ for real functions

x^i, y^i, then we can also write

$$J(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial y^i},\ J(\frac{\partial}{\partial y^i}) = -\frac{\partial}{\partial x^i}.$$

An almost complex structure is called *integrable*, if it arises from holomorphic charts as in the previous way.

An alternative and equivalent definition of an integrable almost complex structure is as follows (see [10]):
Let M be an almost complex manifold with almost complex structure J. We define the *torsion* of J to be the tensor field N of type (1,2) given by

$$N(X, Y) = 2\{[JX, JY] - [X, Y] - J[X, JY] - J[JX, Y]\} \text{ for } X, Y \in \mathfrak{X}(M).$$

An almost complex structure is said to be *integrable* if it has not torsion.

Theorem 1.1. *(Theorem 2.5, Chapter IX, [10]). An almost complex structure is a complex structure if and only if it has no torsion.*

A *Hermitian metric* on an almost complex manifold M is a Riemannian metric g invariant by the almost complex structure J, i.e.,

$$g(JX, JY) = g(X, Y) \quad \text{for any vector fields } X \text{ and } Y.$$

An almost complex manifold (resp. a complex manifold) with a Hermitian metric is called an *almost Hermitian manifold* (resp. a *Hermitian manifold*).

Theorem 1.2. *(Proposition 4.1, Chapter IX, [10]). Every almost complex manifold admits a Hermitian metric provided it is paracompact.*

The *fundamental 2-form* Φ of an almost Hermitian manifold M with almost complex structure J and metric g is defined by

$$\Phi(X,Y) = g(X, JY) \text{ for all vector fields } X \text{ and } Y.$$

A Hermitian metric on an almost complex manifold is called a *Kähler metric* if the fundamental 2-form is closed.

An almost complex manifold (resp. a complex manifold) with a Kähler metric is called an *almost Kähler manifold* (resp. a *Kähler manifold*).

On complex manifolds it is convenient to work with the complexified tangent bundle

$$T^C M = TM \otimes_{\mathbb{R}} \mathbb{C}.$$

In terms of local holomorphic coordinates it is convenient to use the basis

$$\{\frac{\partial}{\partial z^1}, ..., \frac{\partial}{\partial z^n}, \frac{\partial}{\partial \bar{z}^1}, ..., \frac{\partial}{\partial \bar{z}^n}\},$$

where in terms of the real and imaginary parts $z^i = x^i + \sqrt{-1}y^i$ we have

$$\frac{\partial}{\partial z^i} = \frac{1}{2}(\frac{\partial}{\partial x^i} - \sqrt{-1}\frac{\partial}{\partial y^i}), \text{ and } \frac{\partial}{\partial \bar{z}^i} = \frac{1}{2}(\frac{\partial}{\partial x^i} + \sqrt{-1}\frac{\partial}{\partial y^i}).$$

The endomorphism J extends to a complex linear endomorphism of $T^C M$, and induces a decomposition of this bundle pointwise into the $\sqrt{-1}$ and $-\sqrt{-1}$ eigenspaces

$$T^C M = T^{(1,0)}M \oplus T^{(0,1)}M.$$

In terms of local holomorphic coordinates $T^{(1,0)}M$ is spanned by the $\frac{\partial}{\partial z^i}$ while $T^{(0,1)}M$ is spanned by the $\frac{\partial}{\partial \bar{z}^i}$.

In local coordinates $z^1, ..., z^n$ a Hermitian metric is determined by the components

$$g_{j\bar{k}} = g(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial \bar{z}^k}),$$

and we are extending g to complex tangent vectors by complex linearity in both entries. The Hermitian condition implies that for any j, k we have

$$g(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^k}) = g(\frac{\partial}{\partial \bar{z}^j}, \frac{\partial}{\partial \bar{z}^k}) = 0.$$

Theorem 1.3. *(Proposition 1.14, [15]). (Normal coordinates). If (M, g) is a Kähler manifold, then around any point $p \in M$ we can choose holomorphic coordinates $z^1, ..., z^n$ such that the components of g at the point p satisfy*

$$g_{j\bar{k}}(p) = \delta_{jk} \textit{ and } \frac{\partial}{\partial z^i} g_{j\bar{k}}(p) = \frac{\partial}{\partial \bar{z}^i} g_{j\bar{k}}(p) = 0,$$

where δ_{jk} is the identity matrix, i.e. $\delta_{jk} = 0$ if $j \neq k$, and $\delta_{jk} = 1$ if $j = k$.

The following formulas (1)-(12) are from [10].

Let M be an n-dimensional complex manifold and $z^1, ..., z^n$ a complex local coordinate system in M. Unless otherwise stated, Greek indices α, β, γ,... run from 1 to n, while Latin capitals A, B, C,... run through 1, ..., n, $\bar{1}$,..., $\bar{n}$. We set

$$(1) \quad Z_\alpha = \frac{\partial}{\partial z^\alpha}, \quad Z_{\bar{\alpha}} = \bar{Z}_\alpha = \frac{\partial}{\partial \bar{z}^\alpha}.$$

Given a Hermitian metric g on M, we extend the Hermitian inner product in each tangent space $T_x(M)$ defined by g to a unique complex symmetric bilinear form in the complex tangent space $T_x^c(M)$ (cf. Proposition 1.10 in [10]) and set

$$(2) \quad g_{AB} = g(Z_A, Z_B).$$

Then, by Proposition 1.10 in [10],

$$(3)\quad g_{\alpha\beta} = g_{\bar{\alpha}\bar{\beta}} = 0$$

and $(g_{\alpha\bar{\beta}})$ is an $n \times n$ Hermitian matrix. It is then customary to write

$$(4)\quad ds^2 = 2\sum_{\alpha,\beta} g_{\alpha\bar{\beta}} dz^\alpha d\bar{z}^\beta$$

for the metric g. By Proposition 1.12 in [10], the fundamental 2-form is given by

$$(5)\quad \Phi = -2i \sum_{\alpha,\beta} g_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta.$$

A necessary and sufficient condition for g to be a Kähler metric is given by

$$(6)\quad \frac{\partial g_{\alpha\bar{\beta}}}{\partial z^\gamma} = \frac{\partial g_{\gamma\bar{\beta}}}{\partial z^\alpha} \text{ or } \frac{\partial g_{\alpha\bar{\beta}}}{\partial z^{\bar{\gamma}}} = \frac{\partial g_{\alpha\bar{\gamma}}}{\partial z^{\bar{\beta}}}.$$

Given any affine connection with covariant differentiation ∇ on M, we set

$$(7)\quad \nabla_{Z_B} Z_C = \sum_A \Gamma^A_{BC} Z_A.$$

The covariant differentiation, which is originally defined for real vector fields, is extended by complex linearity to act on complex vector fields. Then

$$(8)\quad \bar{\Gamma}^A_{BC} = \Gamma^{\bar{A}}_{\bar{B}\bar{C}}$$

with the convention that $\bar{\bar{\alpha}} = \alpha$. From the fact that $JZ_\alpha = iZ_\alpha$ and $JZ_{\bar{\alpha}} = -iZ_{\bar{\alpha}}$, it follows that the connection is almost complex (see [10]) if and only if

$$(9) \quad \Gamma^{\alpha}_{B\bar{\gamma}} = \Gamma^{\bar{\alpha}}_{B\gamma} = 0.$$

By direct calculation we see that an almost complex connection has no torsion if and only if

$$(10) \quad \Gamma^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\gamma\beta}, \quad \Gamma^{\bar{\alpha}}_{\bar{\beta}\bar{\gamma}} = \Gamma^{\bar{\alpha}}_{\bar{\gamma}\bar{\beta}}$$

and

$$(11) \quad \text{other } \Gamma^{A}_{BC} = 0.$$

In particular, (8), (9), (10), and (11) hold for every Kähler manifold. For a Kähler manifold, the Γ^{A}_{BC}'s are determined by the metric as follows:

$$(12) \quad \sum_{\alpha} g_{\alpha\bar{\varepsilon}} \Gamma^{\alpha}_{\beta\gamma} = \frac{\partial g_{\bar{\varepsilon}\beta}}{\partial z^{\gamma}}, \quad \sum_{\alpha} g_{\bar{\alpha}\varepsilon} \Gamma^{\bar{\alpha}}_{\bar{\beta}\bar{\gamma}} = \frac{\partial g_{\varepsilon\bar{\beta}}}{\partial \bar{z}^{\gamma}}.$$

Theorem 1.4. *If (M, g) is a Kähler manifold, then under holomorphic normal coordinates around a point $p \in M$,*

$$\Gamma^{A}_{BC}(p) = 0,$$

where $A, B, C,\ldots$ run through 1, ..., n, $\bar{1}$,..., $\bar{n}$.

Proof. From (8), (9), (10), (11), (12), and Theorem 1.3., we can obtain the result in this theorem.

On a complex manifold M, in local holomorphic coordinates, we have the 1-forms

$$dz^j = dx^j + idy^j, dz^{\bar{k}} = dx^k - idy^k.$$

Decompose the space Ω^k of k-forms into subspaces $\Omega^{p,q}$ with $p+q=k$. Namely, $\Omega^{p,q}$ is locally spanned by forms of the type

$$\omega(z) = \eta(z)dz^{i_1} \wedge ... \wedge dz^{i_p} \wedge dz^{\bar{j}_1} \wedge ... \wedge dz^{\bar{j}_q}.$$

Thus

$$\Omega^k(M) = \sum_{p+q=k} \Omega^{p,q}(M).$$

We can then let the differential operators

$$\partial = \frac{1}{2}(\frac{\partial}{\partial x^j} - i\frac{\partial}{\partial y^j})(dx^j + idy^j)$$

and

$$\bar{\partial} = \frac{1}{2}(\frac{\partial}{\partial x^j} + i\frac{\partial}{\partial y^j})(dx^j - idy^j)$$

operate on such a form by

$$\partial\omega = \frac{\partial\eta}{\partial z^i} dz^i \wedge dz^{i_1} \wedge ... \wedge dz^{\bar{j}_1} \wedge ... \wedge dz^{\bar{j}_q},$$

and

$$\bar{\partial}\omega = \frac{\partial\eta}{\partial z^{\bar{j}}} dz^{\bar{j}} \wedge dz^{i_1} \wedge ... \wedge dz^{\bar{j}_1} \wedge ... \wedge dz^{\bar{j}_q}.$$

Lemma 1.5. *The exterior derivative d, ∂, and $\bar{\partial}$ satisfies*

$$(1)\quad d = \partial + \bar{\partial},$$

$$(2)\quad \partial\partial = 0,\ \bar{\partial}\bar{\partial} = 0,$$

$$(3)\quad \partial\bar{\partial} = -\bar{\partial}\partial.$$

Proof. We have

$$\partial+\bar{\partial} = \frac{1}{2}(\frac{\partial}{\partial x^j} - i\frac{\partial}{\partial y^j})(dx^j + i dy^j) + \frac{1}{2}(\frac{\partial}{\partial x^j} + i\frac{\partial}{\partial y^j})(dx^j - i dy^j) = \frac{\partial}{\partial x^j}dx^j + \frac{\partial}{\partial y^j}dy^j = d.$$

Therefore,

$$0 = d^2 = (\partial + \bar{\partial})(\partial + \bar{\partial}) = \partial^2 + \partial\bar{\partial} + \bar{\partial}\partial + \bar{\partial}^2$$

and decomposing into types yields (2) and (3).

Chapter 2

Coordinates

Theorem 2.1. *On an n-dim Hermitian manifold M, we have* $\frac{\partial}{\partial y^j} = J(\frac{\partial}{\partial x^j})$, $\frac{\partial}{\partial z^j} = \frac{1}{2}(\frac{\partial}{\partial x^j} - \sqrt{-1}\frac{\partial}{\partial y^j})$, $\frac{\partial}{\partial \bar{z}^j} = \frac{1}{2}(\frac{\partial}{\partial x^j} + \sqrt{-1}\frac{\partial}{\partial y^j})$. *Then*

$$g(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial y^k}) = 0,\ g(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}) = 2\delta_{jk}$$

if and only if

$$g(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^k}) = g(\frac{\partial}{\partial \bar{z}^j}, \frac{\partial}{\partial \bar{z}^k}) = 0,\ \textit{and}\ g(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial \bar{z}^k}) = \delta_{jk}.$$

Proof. According to [10], we can choose $\{\frac{\partial}{\partial x^1}, J(\frac{\partial}{\partial x^1}), \frac{\partial}{\partial x^2}, J(\frac{\partial}{\partial x^2}), ..., \frac{\partial}{\partial x^n}, J(\frac{\partial}{\partial x^n})\}$ such that $g(\frac{\partial}{\partial x^j}, J(\frac{\partial}{\partial x^k})) = 0,\ g(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}) = 2\delta_{jk}$. Then

$$\begin{aligned}
& g(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^k}) \\
&= g(\frac{1}{2}(\frac{\partial}{\partial x^j} - \sqrt{-1}\frac{\partial}{\partial y^j}), \frac{1}{2}(\frac{\partial}{\partial x^k} - \sqrt{-1}\frac{\partial}{\partial y^k})) && (2.1) \\
&= \frac{1}{4}[g(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}) - \sqrt{-1}g(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial y^k}) - \sqrt{-1}g(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial x^k}) + (\sqrt{-1})^2 g(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^k})] && (2.2) \\
&= \frac{1}{4}[2\delta_{jk} - \sqrt{-1}\cdot 0 - \sqrt{-1}\cdot 0 - g(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k})] && (2.3) \\
&= \frac{1}{4}[2\delta_{jk} - 2\delta_{jk}] && (2.4) \\
&= 0.
\end{aligned}$$

$$g(\frac{\partial}{\partial \bar{z}^j}, \frac{\partial}{\partial \bar{z}^k}) = \overline{g(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^k})} = \bar{0} = 0.$$

$$
\begin{aligned}
& g_{j\bar{k}} \\
&= g(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial \bar{z}^k}) \\
&= g(\frac{1}{2}(\frac{\partial}{\partial x^j} - \sqrt{-1}\frac{\partial}{\partial y^j}), \frac{1}{2}(\frac{\partial}{\partial x^k} + \sqrt{-1}\frac{\partial}{\partial y^k})) \qquad (2.5) \\
&= \frac{1}{4}[g(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}) + \sqrt{-1}g(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial y^k}) - \sqrt{-1}g(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial x^k}) - (\sqrt{-1})^2 g(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^k})] \qquad (2.6) \\
&= \frac{1}{4}[2\delta_{jk} + \sqrt{-1}\cdot 0 - \sqrt{-1}\cdot 0 + g(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k})] \qquad (2.7) \\
&= \frac{1}{4}[2\delta_{jk} + 2\delta_{jk}] \qquad (2.8) \\
&= \delta_{jk}.
\end{aligned}
$$

With the properties $g(JX, JY) = g(X, Y)$ and $J^2 = -I$, it is not hard to show the other direction of the conclusion of this theorem.

Theorem 2.2. *On an n-dim Hermitian manifold M, we have* $\frac{\partial}{\partial y^j} = J(\frac{\partial}{\partial x^j})$, $\frac{\partial}{\partial z^j} = \frac{1}{2}(\frac{\partial}{\partial x^j} - \sqrt{-1}\frac{\partial}{\partial y^j})$, $\frac{\partial}{\partial \bar{z}^j} = \frac{1}{2}(\frac{\partial}{\partial x^j} + \sqrt{-1}\frac{\partial}{\partial y^j})$. *Then*

$$\frac{\partial g_{AB}}{\partial x^C}(p) = 0$$

if and only if

$$\frac{\partial}{\partial z^i} g_{j\bar{k}}(p) = \frac{\partial}{\partial \bar{z}^i} g_{j\bar{k}}(p) = 0.$$

Proof. Here $g_{AB} = g(\frac{\partial}{\partial x^A}, \frac{\partial}{\partial x^B})$, $A, B, C = 1, ..., 2n$, and

$$\frac{\partial}{\partial x^{(j+n)}} = \frac{\partial}{\partial y^j}, \quad j = 1, ..., n.$$

$$
\begin{aligned}
&g_{j\bar{k}}\\
&= g(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial \bar{z}^k})\\
&= \frac{1}{4}[g(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}) + g(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^k}) + \sqrt{-1}g(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial y^k}) - \sqrt{-1}g(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial x^k})].
\end{aligned}
\tag{2.9}
$$

$\frac{\partial}{\partial z^i} + \frac{\partial}{\partial \bar{z}^i} = \frac{\partial}{\partial x^i}$, so if $\frac{\partial}{\partial z^i} g_{j\bar{k}}(p) = \frac{\partial}{\partial \bar{z}^i} g_{j\bar{k}}(p) = 0$, then

$$
\frac{\partial}{\partial x^i} g_{j\bar{k}} = (\frac{\partial}{\partial z^i} + \frac{\partial}{\partial \bar{z}^i}) g_{j\bar{k}} = 0,
$$

$$
\frac{\partial}{\partial x^i} g_{j\bar{k}} = \frac{1}{4}\frac{\partial}{\partial x^i}[g(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}) + g(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^k}) + \sqrt{-1}g(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial y^k}) - \sqrt{-1}g(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial x^k})] = 0,
$$

$$
\frac{\partial}{\partial x^i}[g(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}) + g(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^k})] = 0,
$$

$$
\frac{\partial}{\partial x^i}[g(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial y^k}) - g(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial x^k})] = 0.
$$

Also, since $g(JX, JY) = g(X, Y)$, $J^2 = -I$, and $\frac{\partial}{\partial y^j} = J(\frac{\partial}{\partial x^j})$, we get

$$
\begin{aligned}
&\frac{\partial}{\partial x^i}[g(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}) + g(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^k})]\\
&= \frac{\partial}{\partial x^i}[g(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}) + g(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k})] && (2.10)\\
&= 2\frac{\partial}{\partial x^i} g(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}) && (2.11)\\
&= 0,
\end{aligned}
$$

so

$$
\frac{\partial}{\partial x^i} g(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}) = 0;
$$

$$\begin{aligned}
&\frac{\partial}{\partial x^i}[g(\frac{\partial}{\partial x^j},\frac{\partial}{\partial y^k})-g(\frac{\partial}{\partial y^j},\frac{\partial}{\partial x^k})\\
&=\frac{\partial}{\partial x^i}[g(\frac{\partial}{\partial x^j},\frac{\partial}{\partial y^k})+g(\frac{\partial}{\partial x^j},\frac{\partial}{\partial y^k})] \qquad (2.12)\\
&=2\frac{\partial}{\partial x^i}g(\frac{\partial}{\partial x^j},\frac{\partial}{\partial y^k}) \qquad (2.13)\\
&=0,
\end{aligned}$$

so $\frac{\partial}{\partial x^i}g(\frac{\partial}{\partial x^j},\frac{\partial}{\partial y^k})=0$.

Similarly, since $\frac{\partial}{\partial \bar{z}^i}-\frac{\partial}{\partial z^i}=\sqrt{-1}\frac{\partial}{\partial y^i}$, we can obtain $\frac{\partial}{\partial y^i}g(\frac{\partial}{\partial x^j},\frac{\partial}{\partial x^k})=0$, and $\frac{\partial}{\partial y^i}g(\frac{\partial}{\partial x^j},\frac{\partial}{\partial y^k})=0$.

Since $g(JX,JY)=g(X,Y)$ and $g(X,Y)=g(Y,X)$, we get $\frac{\partial g_{AB}}{\partial x^C}=0$.

It is a little similar and easier to show the other direction of the conclusion of this theorem.

Theorem 2.3. *For n-dim Kähler manifold (M,g), the Riemannian $\Gamma^{(R)}=0$ iff the complexified $\Gamma^{(C)}=0$.*

Proof. In terms of a local coordinate system $x^1,...,x^{2n}$, the components Γ^I_{JK} of the Riemannian connection are given by

$$\sum_L g_{LK}\Gamma^L_{JI}=\frac{1}{2}(\frac{\partial g_{KI}}{\partial x^J}+\frac{\partial g_{JK}}{\partial x^I}-\frac{\partial g_{JI}}{\partial x^K})$$

(see [9]).

For a Kähler manifold, the complexified Γ^A_{BC}'s are determined by the metric as follows: $\sum_\alpha g_{\alpha\bar{\varepsilon}}\Gamma^\alpha_{\beta\gamma}=\frac{\partial g_{\bar{\varepsilon}\beta}}{\partial z^\gamma}$, $\quad \sum_\alpha g_{\bar{\alpha}\varepsilon}\Gamma^{\bar{\alpha}}_{\bar{\beta}\bar{\gamma}}=\frac{\partial g_{\varepsilon\bar{\beta}}}{\partial \bar{z}^\gamma}$, $\alpha,\beta,\gamma,\varepsilon=1,...,n$ (see [10]).

By Theorem 4.2., for a Kähler manifold, $\Gamma^{(R)}=0$ iff $\Gamma^{(C)}=0$.

Chapter 3

Complex gradient

Let M be an n-dim Kähler manifold with Kähler metric g and $f : M \to \mathbb{C}$ be some complex-valued function. We define the *holomorphic gradient* vector field $\nabla^+ f$ with respect to metric g as follows:

$$\bar{\partial} f(X) = g(\nabla^+ f, X) \quad \text{for every complexified vector field } X.$$

Remark 3.1. The holomorphic gradient $\nabla^+ f$ can also be defined on a complex manifold with a metric g since $\bar{\partial}$ is defined on complex manifolds. Here we define the holomorphic gradient on a Kähler manifold in order to use holomorphic normal coordinates to simplify the calculation.

$g(\nabla^+ f, X) = \bar{\partial} f(X) = \frac{\partial f}{\partial \bar{z}^j} d\bar{z}^j(X)$ for any complexified vector field X, so

$$g(\nabla^+ f, \frac{\partial}{\partial z^i}) = \frac{\partial f}{\partial \bar{z}^j} d\bar{z}^j(\frac{\partial}{\partial z^i}) = 0,$$

$$g(\nabla^+ f, \frac{\partial}{\partial \bar{z}^i}) = \frac{\partial f}{\partial \bar{z}^j} d\bar{z}^j(\frac{\partial}{\partial \bar{z}^i}) = \frac{\partial f}{\partial \bar{z}^j} \delta_i^j = \frac{\partial f}{\partial \bar{z}^i}.$$

Let $\nabla^+ f = f^j \frac{\partial}{\partial z^j} + f^{\bar{j}} \frac{\partial}{\partial \bar{z}^j}$, then

$$g(\nabla^+ f, \frac{\partial}{\partial z^i}) = f^j g_{ji} + f^{\bar{j}} g_{\bar{j}i} = 0,$$

$$g(\nabla^+ f, \frac{\partial}{\partial \bar{z}^i}) = f^j g_{j\bar{i}} + f^{\bar{j}} g_{\bar{j}\bar{i}} = \frac{\partial f}{\partial \bar{z}^j} \delta_i^j = \frac{\partial f}{\partial \bar{z}^i}.$$

$$\begin{pmatrix} f^1 & f^2 & \dots & f^n & f^{\bar{1}} & f^{\bar{2}} & \dots & f^{\bar{n}} \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} & \cdots & g_{1n} & g_{1\bar{1}} & g_{1\bar{2}} & \cdots & g_{1\bar{n}} \\ g_{21} & g_{22} & \cdots & g_{2n} & g_{2\bar{1}} & g_{2\bar{2}} & \cdots & g_{2\bar{n}} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ g_{n1} & g_{n2} & \cdots & g_{nn} & g_{n\bar{1}} & g_{n\bar{2}} & \cdots & g_{n\bar{n}} \\ g_{\bar{1}1} & g_{\bar{1}2} & \cdots & g_{\bar{1}n} & g_{\bar{1}\bar{1}} & g_{\bar{1}\bar{2}} & \cdots & g_{\bar{1}\bar{n}} \\ g_{\bar{2}1} & g_{\bar{2}2} & \cdots & g_{\bar{2}n} & g_{\bar{2}\bar{1}} & g_{\bar{2}\bar{2}} & \cdots & g_{\bar{2}\bar{n}} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ g_{\bar{n}1} & g_{\bar{n}2} & \cdots & g_{\bar{n}n} & g_{\bar{n}\bar{1}} & g_{\bar{n}\bar{2}} & \cdots & g_{\bar{n}\bar{n}} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & \dots & 0 & \frac{\partial f}{\partial \bar{z}^1} & \frac{\partial f}{\partial \bar{z}^2} & \dots & \frac{\partial f}{\partial \bar{z}^n} \end{pmatrix},$$

so

$$\begin{pmatrix} f^1 & f^2 & \dots & f^n & f^{\bar{1}} & f^{\bar{2}} & \dots & f^{\bar{n}} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \dots & 0 & \frac{\partial f}{\partial \bar{z}^1} & \frac{\partial f}{\partial \bar{z}^2} & \dots & \frac{\partial f}{\partial \bar{z}^n} \end{pmatrix} \begin{pmatrix} g_{ij} & g_{i\bar{j}} \\ g_{\bar{i}j} & g_{\bar{i}\bar{j}} \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 0 & 0 & \dots & 0 & \frac{\partial f}{\partial \bar{z}^1} & \frac{\partial f}{\partial \bar{z}^2} & \dots & \frac{\partial f}{\partial \bar{z}^n} \end{pmatrix} \begin{pmatrix} 0 & g^{j\bar{i}} \\ g^{\bar{j}i} & 0 \end{pmatrix},$$

since Hermitian condition implies that for any j, k, we have

$$g(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^k}) = g(\frac{\partial}{\partial \bar{z}^j}, \frac{\partial}{\partial \bar{z}^k}) = 0,$$

that is, $g_{ij} = g_{\bar{i}\bar{j}} = 0$. The symmetry of g implies that $\overline{g_{j\bar{k}}} = g_{\bar{j}k} = g_{k\bar{j}}$.

$$\begin{pmatrix} g_{ij} & g_{i\bar{j}} \\ g_{\bar{i}j} & g_{\bar{i}\bar{j}} \end{pmatrix} = \begin{pmatrix} 0 & g_{i\bar{j}} \\ g_{\bar{i}j} & 0 \end{pmatrix}$$

$$\begin{pmatrix} g_{ij} & g_{i\bar{j}} \\ g_{\bar{i}j} & g_{\bar{i}\bar{j}} \end{pmatrix}^{-1} = \begin{pmatrix} 0 & g_{i\bar{j}} \\ g_{\bar{i}j} & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & (g_{\bar{i}j})^{-1} \\ (g_{i\bar{j}})^{-1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & g^{j\bar{i}} \\ g^{\bar{j}i} & 0 \end{pmatrix},$$

since $(g^{\bar{j}i}) := (g_{i\bar{j}})^{-1}$, and $(g^{j\bar{i}}) := (g_{\bar{i}j})^{-1}$. $g^{\bar{i}j} = g^{j\bar{i}}$.

Thus, $\begin{pmatrix} f^1 & f^2 & ... & f^n & f^{\bar{1}} & f^{\bar{2}} & ... & f^{\bar{n}} \end{pmatrix} = (\frac{\partial f}{\partial \bar{z}^j} g^{\bar{j}i}, 0)$,

$$f^i = g^{\bar{j}i} \frac{\partial f}{\partial \bar{z}^j},\ f^{\bar{i}} = 0,\ i = 1,\ ...\ n.$$

So

$$\nabla^+ f = f^i \frac{\partial}{\partial z^i} + f^{\bar{i}} \frac{\partial}{\partial \bar{z}^i} = g^{\bar{j}i} \frac{\partial f}{\partial \bar{z}^j} \frac{\partial}{\partial z^i}.$$

Similarly, for $f : M \to \mathbb{C}$, M a Kähler manifold with Kähler metric g, define the *antiholomorphic gradient* $\nabla^- f$ as:

$$g(\nabla^- f, X) = \partial f(X) = \frac{\partial f}{\partial z^j} dz^j(X),$$

for any complex vector field X. Then we can get

$$\nabla^- f = g^{j\bar{i}} \frac{\partial f}{\partial z^j} \frac{\partial}{\partial \bar{z}^i}.$$

Under holomorphic normal coordinates, $\begin{pmatrix} g_{ij} & g_{i\bar{j}} \\ g_{\bar{i}j} & g_{\bar{i}\bar{j}} \end{pmatrix} = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} = \begin{pmatrix} g_{ij} & g_{i\bar{j}} \\ g_{\bar{i}j} & g_{\bar{i}\bar{j}} \end{pmatrix}^{-1}$,

$$\nabla^+ f = \sum_{j=1}^{n} \frac{\partial f}{\partial \bar{z}^j} \frac{\partial}{\partial z^j},$$

and

$$\nabla^- f = \sum_{j=1}^{n} \frac{\partial f}{\partial z^j} \frac{\partial}{\partial \bar{z}^j}.$$

The exterior derivative on a complex manifold is $d = \partial + \bar{\partial}$, so $df = \partial f + \bar{\partial} f$. For $f : M \to \mathbb{C}$, M a Kähler manifold with Kähler metric g, define the *complex* *gradient* of f, $\nabla^c f$, as follows:

$$g(\nabla^c f, X) = df(X)$$

for any complex vector field X. Then we can get

$$\nabla^c f = g^{\bar{j}i}\frac{\partial f}{\partial \bar{z}^j}\frac{\partial}{\partial z^i} + g^{j\bar{i}}\frac{\partial f}{\partial z^j}\frac{\partial}{\partial \bar{z}^i} = \nabla^+ f + \nabla^- f.$$

The usual Riemannian gradient of f is

$$\nabla f = g^{JI}\frac{\partial f}{\partial x^J}\frac{\partial}{\partial x^I},$$

$I, J = 1, ..., 2n$.

Remark 3.2. $\nabla^+ f$ is the projection of complex gradient $\nabla^c f$ onto $T^{(1,0)}M$, which is a holomorphic vector field.

Remark 3.3. $f : M \to \mathbb{C}$ is holomorphic if and only if $\nabla^+ f = 0$.

So we call $\nabla^+ f$ the *holomorphic gradient* of f.

Chapter 4

Complex Laplacian

Let M be an n-dim Kähler manifold with Kähler metric g, and $f : M \to \mathbb{R}$ be a real-valued function.

For any real vector field X on M, we define the usual Riemannian *divergence* of X, denoted by $\text{div}X$, to be the function such that at each point x of M

$$(\text{div}X)_x = \text{trace of the endomorphism } V \to \nabla_V X \text{ of } T_xM,$$

where ∇ is the connection in the Riemannian manifold. In terms of local coordinates $(x^1, ..., x^n, y^1, ..., y^n)$, let $x^{\bar{i}}$ denote y^i where $\bar{i} = i + n$, and let Latin capitals $A, B, C, ..., H, I, J, K, ...$ run through 1, ..., $2n$. Then we have

$$df = \sum_{I=1}^{2n} f_I dx^I, \text{ where } f_I = \frac{\partial f}{\partial x^I}$$

and

$$\text{grad}f = \sum_{I=1}^{2n} f^I \frac{\partial}{\partial x^I}, \text{ where } f^I = \sum_{J=1}^{2n} g^{IJ} f_J,$$

where $(g^{IJ}) = (g_{IJ})^{-1}$, i.e. $\sum_J g_{IJ} g^{JK} = \delta_I^K$. If $X = \sum_{I=1}^{2n} \xi^I \frac{\partial}{\partial x^I}$, then

$$\text{div}X = \sum_{I=1}^{2n} \xi^I_{\ ;I},$$

where $\xi^I_{\ ;J} = \frac{\partial \xi^I}{\partial x^J} + \sum_K \Gamma^I_{JK} \xi^K$. (see [9] and [10]).

For any complexified vector field X on M, we define the *complex divergence* of X, denoted also by div$X$, to be the function such that at each point x of M

$$(\mathrm{div}X)_x = \text{trace of the endomorphism } V \to \nabla_V X \text{ of } T_x^c M,$$

where ∇ is the complex linear extension of the connection in the Riemannian manifold. If $X = X^i \frac{\partial}{\partial z^i} + X^{\bar{i}} \frac{\partial}{\partial \bar{z}^i}$, then

$$\nabla X = X^A_{;B} \frac{\partial}{\partial z^A} \otimes dz^B = (\frac{\partial X^A}{\partial z^B} + X^C \Gamma^A_{CB}) \frac{\partial}{\partial z^A} \otimes dz^B,$$

where $i, j, k = 1, 2, ..., n$, $\bar{i} = i + n$, and $A, B, C = 1, 2, ..., 2n$. Thus,

$$\mathrm{div}X = \frac{\partial X^A}{\partial z^A} + X^C \Gamma^A_{CA}.$$

Since on a Kähler manifold, $\Gamma^k_{ij} = \Gamma^k_{ji}, \Gamma^{\bar{k}}_{\bar{i}\bar{j}} = \Gamma^{\bar{k}}_{\bar{j}\bar{i}} = \overline{\Gamma^k_{ij}}$, other $\Gamma^A_{BC} = 0$, thus, on a Kähler manifold,

$$\mathrm{div}X = \frac{\partial X^A}{\partial z^A} + X^k \Gamma^i_{ki} + X^{\bar{k}} \Gamma^{\bar{i}}_{\bar{k}\bar{i}},$$

so

$$\Delta^+ f := \mathrm{div}(\nabla^+ f) = \frac{\partial}{\partial z^i}(g^{\bar{j}i} \frac{\partial f}{\partial \bar{z}^j}) + g^{\bar{j}k} \frac{\partial f}{\partial \bar{z}^j} \Gamma^i_{ki}.$$

We call $\Delta^+ f$ the *holomorphic Laplacian* of f.

Similarly, for Kähler manifold (M, g) and $f : M \to \mathbb{R}$, we can define the *antiholomorphic Laplacian* of f as:

$$\Delta^- f := \mathrm{div}(\nabla^- f) = \frac{\partial}{\partial \bar{z}^i}(g^{j\bar{i}} \frac{\partial f}{\partial z^j}) + g^{j\bar{k}} \frac{\partial f}{\partial z^j} \Gamma^{\bar{i}}_{\bar{k}\bar{i}},$$

and the *complex Laplacian* of f as:

$$\Delta^c f := \mathrm{div}(\nabla^c f) = \mathrm{div}(\nabla^+ f + \nabla^- f) = \Delta^+ f + \Delta^- f.$$

The usual Riemannian Laplacian of f is

$$\Delta f := \operatorname{div}(\nabla f) = \frac{\partial}{\partial x^I}(g^{JI}\frac{\partial f}{\partial x^J}) + g^{JK}\frac{\partial f}{\partial x^J}\Gamma^I_{KI}.$$

For a Kähler manifold (M, g) and a C^2 function f on M, under holomorphic normal coordinates,

$$\Delta^+ f = \sum_{i=1}^{n}\frac{\partial}{\partial z^i}\frac{\partial f}{\partial \bar{z}^i} = \sum_{i=1}^{n}\frac{1}{4}(\frac{\partial^2 f}{\partial x^i \partial x^i} + \frac{\partial^2 f}{\partial y^i \partial y^i}) = \frac{1}{4}\sum_{I=1}^{2n}\frac{\partial^2 f}{(\partial x^I)^2}.$$

The usual Riemannian gradient of f is: $\nabla f = g^{JI}\frac{\partial f}{\partial x^J}\frac{\partial}{\partial x^I}$. The usual Riemannian Laplacian of f is Δf:

$$\Delta f := \operatorname{div}(\nabla f) = \frac{\partial}{\partial x^I}(g^{JI}\frac{\partial f}{\partial x^J}) + g^{JK}\frac{\partial f}{\partial x^J}\Gamma^I_{KI}.$$

Under the real coordinates corresponding to the holomorphic normal coordinates,

$$\Delta f = \sum_{I=1}^{2n}(\frac{1}{2}\frac{\partial^2 f}{(\partial x^I)^2} + 0) = \frac{1}{2}\sum_{I=1}^{2n}\frac{\partial^2 f}{(\partial x^I)^2}.$$

Thus,

$$\Delta^+ f = \frac{1}{2}\Delta f.$$

Similarly, $\Delta^- f = \frac{1}{2}\Delta f$. So $\Delta^+ f = \Delta^- f$.

$$\Delta^c f = \operatorname{div}(\nabla^+ f + \nabla^- f) = \operatorname{div}(\nabla^+ f) + \operatorname{div}(\nabla^- f) = \Delta^+ f + \Delta^- f = \Delta f.$$

Now let us take a look at the well-known complex Laplacian of f on M. Let $*$ be the Hodge $*$-operator, and $\bar{\partial}^* := -* \circ \partial \circ *$. Then the well-known complex Laplacian of f is

$$\Box f = (\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial})f.$$

On Kähler manifold (M, g), $\Box f = \frac{1}{2}\Delta f$ (see [6], [4], and [19]).

We have $\Delta^+ f = \frac{1}{2}\Delta f = \Box f$. Thus, our holomorphic Laplacian $\Delta^+ f$ coincides with the well-known complex Laplacian $\Box f$ on Kähler manifolds.

Chapter 5

Complex p-Laplacian

On an n-dim Kähler manifold (M, g), for $f : M \to \mathbb{R}$ and $p > 1$, the usual Riemannian p-Laplacian of f is $\Delta_p f$:

$$\Delta_p f = \operatorname{div}(|\nabla f|^{p-2} \nabla f),$$

where

$$\begin{aligned}
&|\nabla f| \\
&= [g(\nabla f, \nabla f)]^{\frac{1}{2}} \\
&= [g(g^{IJ} \frac{\partial f}{\partial x^J} \frac{\partial}{\partial x^I}, g^{KL} \frac{\partial f}{\partial x^L} \frac{\partial}{\partial x^K})]^{\frac{1}{2}} \\
&= [g^{IJ} g^{KL} \frac{\partial f}{\partial x^J} \frac{\partial f}{\partial x^L} g_{IK}]^{\frac{1}{2}},
\end{aligned}$$

$$|\nabla f|^{p-2} \nabla f = (g_{IK} g^{IJ} g^{KL} \frac{\partial f}{\partial x^J} \frac{\partial f}{\partial x^L})^{\frac{p-2}{2}} g^{ST} \frac{\partial f}{\partial x^T} \frac{\partial}{\partial x^S}.$$

Under the real coordinates corresponding to the holomorphic normal coordinates,

$$g^{IJ} = \frac{1}{2} \delta_{IJ},$$

$$
\begin{aligned}
&\Delta_p f \\
&= \operatorname{div}(|\nabla f|^{p-2}\nabla f) \\
&= \sum_H \frac{\partial}{\partial x^H}[(g_{IK}g^{IJ}g^{KL}\frac{\partial f}{\partial x^J}\frac{\partial}{\partial x^L})^{\frac{p-2}{2}}g^{HT}\frac{\partial f}{\partial x^T}] && (5.1)\\
&= \sum_H \frac{\partial}{\partial x^H}[(g^{JL}\frac{\partial f}{\partial x^J}\frac{\partial}{\partial x^L})^{\frac{p-2}{2}}g^{HT}\frac{\partial f}{\partial x^T}] && (5.2)\\
&= \sum_H \frac{p-2}{2}(g^{JL}\frac{\partial f}{\partial x^J}\frac{\partial f}{\partial x^L})^{\frac{p-2}{2}-1}g^{HT}\frac{\partial f}{\partial x^T}\frac{\partial}{\partial x^H}(g^{JL}\frac{\partial f}{\partial x^J}\frac{\partial f}{\partial x^L}) \\
&\quad + \sum_H (g^{JL}\frac{\partial f}{\partial x^J}\frac{\partial}{\partial x^L})^{\frac{p-2}{2}}\frac{\partial}{\partial x^H}(g^{HT}\frac{\partial f}{\partial x^T}) && (5.3)\\
&= \sum_H \frac{p-2}{2}(g^{JL}\frac{\partial f}{\partial x^J}\frac{\partial f}{\partial x^L})^{\frac{p-4}{2}}g^{HT}\frac{\partial f}{\partial x^T}g^{JL}\frac{\partial}{\partial x^H}(\frac{\partial f}{\partial x^J}\frac{\partial f}{\partial x^L}) \\
&\quad + \sum_H (g^{JL}\frac{\partial f}{\partial x^J}\frac{\partial f}{\partial x^L})^{\frac{p-2}{2}}g^{HT}\frac{\partial^2 f}{\partial x^H\partial x^T} && (5.4)\\
&= \sum_H \frac{p-2}{2}(\frac{1}{2}\delta_{JL}\frac{\partial f}{\partial x^J}\frac{\partial f}{\partial x^L})^{\frac{p-4}{2}}\frac{1}{2}\delta_{HT}\frac{\partial f}{\partial x^T}\frac{\partial}{\partial x^H}(\frac{1}{2}\delta_{JL}\frac{\partial f}{\partial x^J}\frac{\partial f}{\partial x^L}) \\
&\quad + \sum_H (\frac{1}{2}\delta_{JL}\frac{\partial f}{\partial x^J}\frac{\partial f}{\partial x^L})^{\frac{p-2}{2}}\frac{1}{2}\delta_{HT}\frac{\partial^2 f}{\partial x^H\partial x^T} && (5.5)\\
&= \sum_H \frac{p-2}{2}(\frac{1}{2})^{\frac{p}{2}}[\sum_J(\frac{\partial f}{\partial x^J})^2]^{(\frac{p-4}{2})}\frac{\partial f}{\partial x^H}\frac{\partial}{\partial x^H}(\sum_J(\frac{\partial f}{\partial x^J})^2) \\
&\quad + \sum_H (\frac{1}{2})^{\frac{p}{2}}[\sum_J(\frac{\partial f}{\partial x^J})^2]^{\frac{p-2}{2}}\frac{\partial^2 f}{\partial x^H\partial x^H}. && (5.6)
\end{aligned}
$$

For $X \in T_x^c M$, $X = X^i \frac{\partial}{\partial z^i} + X^{\bar{j}} \frac{\partial}{\partial \bar{z}^j}$, define

$$|X| := \sqrt{g(X, \overline{X})}.$$

$$
\begin{aligned}
&g(X,\overline{X}) \\
&= g(X^i\frac{\partial}{\partial z^i} + X^{\bar{j}}\frac{\partial}{\partial \bar{z}^j}, \overline{X^k\frac{\partial}{\partial z^k} + X^{\bar{h}}\frac{\partial}{\partial \bar{z}^h}}) && (5.7)\\
&= g(X^i\frac{\partial}{\partial z^i} + X^{\bar{j}}\frac{\partial}{\partial \bar{z}^j}, \overline{X^k}\frac{\partial}{\partial \bar{z}^k} + \overline{X^{\bar{h}}}\frac{\partial}{\partial z^h}) && (5.8)\\
&= g(X^i\frac{\partial}{\partial z^i}, \overline{X^k}\frac{\partial}{\partial \bar{z}^k}) + g(X^i\frac{\partial}{\partial z^i}, \overline{X^{\bar{h}}}\frac{\partial}{\partial z^h}) + g(X^{\bar{j}}\frac{\partial}{\partial \bar{z}^j}, \overline{X^k}\frac{\partial}{\partial \bar{z}^k}) + g(X^{\bar{j}}\frac{\partial}{\partial \bar{z}^j}, \overline{X^{\bar{h}}}\frac{\partial}{\partial z^h}) && (5.9)\\
&= X^i\overline{X^k}g_{i\bar{k}} + X^i\overline{X^{\bar{h}}}g(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^h}) + X^{\bar{j}}\overline{X^k}g_{\bar{j}\bar{k}} + X^{\bar{j}}\overline{X^{\bar{h}}}g_{\bar{j}h} && (5.10)\\
&= X^i\overline{X^k}g_{i\bar{k}} + X^{\bar{j}}\overline{X^{\bar{h}}}g_{\bar{j}h}. && (5.11)
\end{aligned}
$$

Under holomorphic normal coordinates, $g_{j\bar{k}} = \delta_{jk}$, so

$$g(X,\overline{X}) = \sum_{i=1}^{n} X^i\overline{X^i} + \sum_{j=1}^{n} X^{\bar{j}}\overline{X^{\bar{j}}} = \sum_{A=1}^{2n} X^A\overline{X^A} \geqslant 0,$$

$$g(\nabla^+ f, \overline{\nabla^+ f}) = \sum_{i=1}^{n}\frac{\partial f}{\partial z^i}\frac{\partial f}{\partial \bar{z}^i} = \sum_{i=1}^{n}|\frac{\partial f}{\partial z^i}|^2 = \sum_{i=1}^{n}|\frac{\partial f}{\partial \bar{z}^i}|^2$$

for $f : M \to \mathbb{R}$, $\overline{\frac{\partial f}{\partial \bar{z}^j}} = \frac{\partial f}{\partial z^j}$.

For $p > 1$, similar to how we define the usual Riemannian p-Laplacian of f, we can define the *holomorphic p-Laplacian* of f as:

$$\Delta_p^+ f := \mathrm{div}(|\nabla^+ f|^{p-2}\nabla^+ f),$$

define the *antiholomorphic p-Laplacian* of f as:

$$\Delta_p^- f := \mathrm{div}(|\nabla^- f|^{p-2}\nabla^- f),$$

and define the *complex p-Laplacian* of f as:

$$\Delta_p^c f := \mathrm{div}(|\nabla^c f|^{p-2}\nabla^c f).$$

Remark 5.1.

$$\frac{\partial f}{\partial \bar{z}^k}\frac{\partial f}{\partial z^k} = \frac{1}{2}(\frac{\partial}{\partial x^k} + \sqrt{-1}\frac{\partial}{\partial y^k})f\,\frac{1}{2}(\frac{\partial}{\partial x^k} - \sqrt{-1}\frac{\partial}{\partial y^k})f = \frac{1}{4}[(\frac{\partial f}{\partial x^k})^2 + (\frac{\partial f}{\partial y^k})^2].$$

If f is C^2,

$$\frac{\partial^2 f}{\partial z^j \partial \bar{z}^j} = \frac{1}{2}(\frac{\partial}{\partial x^j} - \sqrt{-1}\frac{\partial}{\partial y^j})\frac{1}{2}(\frac{\partial}{\partial x^j} + \sqrt{-1}\frac{\partial}{\partial y^j})f = \frac{1}{4}(\frac{\partial^2 f}{\partial x^j \partial x^j} + \frac{\partial^2 f}{\partial y^j \partial y^j}).$$

Theorem 5.2. *For a C^2 function f on n-dim Kähler manifold (M, g), under holomorphic normal coordinates,*

$$\begin{aligned} Re\Delta_p^+ f = \frac{p-2}{2}(\sum_k \frac{\partial f}{\partial \bar{z}^k}\frac{\partial f}{\partial z^k})^{\frac{p-4}{2}} \times \frac{1}{8}\sum_{k,j}(\frac{\partial f}{\partial x^k}\frac{\partial^2 f}{\partial x^j \partial x^k}\frac{\partial f}{\partial x^j} \\ + \frac{\partial f}{\partial x^j}\frac{\partial^2 f}{\partial x^j \partial y^k}\frac{\partial f}{\partial y^k} + \frac{\partial f}{\partial x^k}\frac{\partial^2 f}{\partial y^j \partial x^k}\frac{\partial f}{\partial y^j} + \frac{\partial f}{\partial y^k}\frac{\partial^2 f}{\partial y^j \partial y^k}\frac{\partial f}{\partial y^j}) \\ + (\sum_k \frac{\partial f}{\partial \bar{z}^k}\frac{\partial f}{\partial z^k})^{\frac{p-2}{2}}\sum_j \frac{\partial^2 f}{\partial z^j \partial \bar{z}^j}. \end{aligned} \quad (5.12)$$

Proof. Under holomorphic normal coordinates,

$$|\nabla^+ f|^2 = g(\nabla^+ f, \overline{\nabla^+ f}) = \sum_j \frac{\partial f}{\partial \bar{z}^j}\frac{\partial f}{\partial z^j},$$

$$
\begin{aligned}
&\Delta_p^+ f \\
&= \mathrm{div}(|\nabla^+ f|^{p-2}\nabla^+ f) \\
&= \sum_j \frac{\partial(|\nabla^+ f|^{p-2}\frac{\partial f}{\partial \bar{z}^j})}{\partial z^j} && (5.13) \\
&= \sum_j \frac{\partial(|\nabla^+ f|^{p-2})}{\partial z^j}\frac{\partial f}{\partial \bar{z}^j} + \sum_j |\nabla^+ f|^{p-2}\frac{\partial^2 f}{\partial z^j \partial \bar{z}^j} && (5.14) \\
&= \sum_j \frac{\partial[(\sum_k \frac{\partial f}{\partial \bar{z}^k}\frac{\partial f}{\partial z^k})^{\frac{p-2}{2}}]}{\partial z^j}\frac{\partial f}{\partial \bar{z}^j} + (\sum_k \frac{\partial f}{\partial \bar{z}^k}\frac{\partial f}{\partial z^k})^{\frac{p-2}{2}}\sum_j \frac{\partial^2 f}{\partial z^j \partial \bar{z}^j} && (5.15) \\
&= \frac{p-2}{2}(\sum_k \frac{\partial f}{\partial \bar{z}^k}\frac{\partial f}{\partial z^k})^{\frac{p-4}{2}}\sum_{k,j}\frac{\partial}{\partial z^j}(\frac{\partial f}{\partial \bar{z}^k}\frac{\partial f}{\partial z^k})\frac{\partial f}{\partial \bar{z}^j} + (\sum_k \frac{\partial f}{\partial \bar{z}^k}\frac{\partial f}{\partial z^k})^{\frac{p-2}{2}}\sum_j \frac{\partial^2 f}{\partial z^j \partial \bar{z}^j}. \\
& && (5.16)
\end{aligned}
$$

$$\frac{\partial}{\partial z^j}(\frac{\partial f}{\partial \bar{z}^k}\frac{\partial f}{\partial z^k})\frac{\partial f}{\partial \bar{z}^j} \tag{5.17}$$

$$= \frac{\partial^2 f}{\partial z^j \partial \bar{z}^k}\frac{\partial f}{\partial z^k}\frac{\partial f}{\partial \bar{z}^j} + \frac{\partial f}{\partial \bar{z}^k}\frac{\partial^2 f}{\partial z^j \partial z^k}\frac{\partial f}{\partial \bar{z}^j} \tag{5.18}$$

$$= \frac{1}{2}(\frac{\partial}{\partial x^j} - \sqrt{-1}\frac{\partial}{\partial y^j})\frac{1}{2}(\frac{\partial}{\partial x^k} + \sqrt{-1}\frac{\partial}{\partial y^k})f\frac{1}{2}(\frac{\partial}{\partial x^k} - \sqrt{-1}\frac{\partial}{\partial y^k})f\,\frac{1}{2}(\frac{\partial}{\partial x^j} + \sqrt{-1}\frac{\partial}{\partial y^j})f$$
$$+ \frac{1}{2}(\frac{\partial}{\partial x^k} + \sqrt{-1}\frac{\partial}{\partial y^k})f\frac{1}{2}(\frac{\partial}{\partial x^j} - \sqrt{-1}\frac{\partial}{\partial y^j})\frac{1}{2}(\frac{\partial}{\partial x^k} - \sqrt{-1}\frac{\partial}{\partial y^k})f\frac{1}{2}(\frac{\partial}{\partial x^j} + \sqrt{-1}\frac{\partial}{\partial y^j})f \tag{5.19}$$

$$= \frac{1}{16}[\frac{\partial^2 f}{\partial x^j \partial x^k}\frac{\partial f}{\partial x^k}\frac{\partial f}{\partial x^j} + \frac{\partial^2 f}{\partial x^j \partial x^k}\frac{\partial f}{\partial y^k}\frac{\partial f}{\partial y^j} - \frac{\partial^2 f}{\partial x^j \partial y^k}\frac{\partial f}{\partial x^k}\frac{\partial f}{\partial y^j} + \frac{\partial^2 f}{\partial x^j \partial y^k}\frac{\partial f}{\partial y^k}\frac{\partial f}{\partial x^j}$$
$$+ \frac{\partial^2 f}{\partial y^j \partial x^k}\frac{\partial f}{\partial x^k}\frac{\partial f}{\partial y^j} - \frac{\partial^2 f}{\partial y^j \partial x^k}\frac{\partial f}{\partial y^k}\frac{\partial f}{\partial x^j} + \frac{\partial^2 f}{\partial y^j \partial y^k}\frac{\partial f}{\partial x^k}\frac{\partial f}{\partial x^j} + \frac{\partial^2 f}{\partial y^j \partial y^k}\frac{\partial f}{\partial y^k}\frac{\partial f}{\partial y^j}$$
$$+ \frac{\partial f}{\partial x^k}\frac{\partial^2 f}{\partial x^j \partial x^k}\frac{\partial f}{\partial x^j} + \frac{\partial f}{\partial x^k}\frac{\partial^2 f}{\partial x^j \partial y^k}\frac{\partial f}{\partial y^j} + \frac{\partial f}{\partial x^k}\frac{\partial^2 f}{\partial y^j \partial x^k}\frac{\partial f}{\partial y^j} - \frac{\partial f}{\partial x^k}\frac{\partial^2 f}{\partial y^j \partial y^k}\frac{\partial f}{\partial x^j}$$
$$- \frac{\partial f}{\partial y^k}\frac{\partial^2 f}{\partial x^j \partial x^k}\frac{\partial f}{\partial y^j} + \frac{\partial f}{\partial y^k}\frac{\partial^2 f}{\partial x^j \partial y^k}\frac{\partial f}{\partial x^j} + \frac{\partial f}{\partial y^k}\frac{\partial^2 f}{\partial y^j \partial x^k}\frac{\partial f}{\partial x^j} + \frac{\partial f}{\partial y^k}\frac{\partial^2 f}{\partial y^j \partial y^k}\frac{\partial f}{\partial y^j}$$
$$+ \sqrt{-1}(\frac{\partial^2 f}{\partial x^j \partial x^k}\frac{\partial f}{\partial x^k}\frac{\partial f}{\partial y^j} - \frac{\partial^2 f}{\partial x^j \partial x^k}\frac{\partial f}{\partial y^k}\frac{\partial f}{\partial x^j} + \frac{\partial^2 f}{\partial x^j \partial y^k}\frac{\partial f}{\partial x^k}\frac{\partial f}{\partial x^j} + \frac{\partial^2 f}{\partial x^j \partial y^k}\frac{\partial f}{\partial y^k}\frac{\partial f}{\partial y^j}$$
$$- \frac{\partial^2 f}{\partial y^j \partial x^k}\frac{\partial f}{\partial x^k}\frac{\partial f}{\partial x^j} - \frac{\partial^2 f}{\partial y^j \partial x^k}\frac{\partial f}{\partial y^k}\frac{\partial f}{\partial y^j} + \frac{\partial^2 f}{\partial y^j \partial y^k}\frac{\partial f}{\partial x^k}\frac{\partial f}{\partial y^j} - \frac{\partial^2 f}{\partial y^j \partial y^k}\frac{\partial f}{\partial y^k}\frac{\partial f}{\partial x^j}$$
$$+ \frac{\partial f}{\partial x^k}\frac{\partial^2 f}{\partial x^j \partial x^k}\frac{\partial f}{\partial y^j} - \frac{\partial f}{\partial x^k}\frac{\partial^2 f}{\partial x^j \partial y^k}\frac{\partial f}{\partial x^j} - \frac{\partial f}{\partial x^k}\frac{\partial^2 f}{\partial y^j \partial x^k}\frac{\partial f}{\partial x^j} - \frac{\partial f}{\partial x^k}\frac{\partial^2 f}{\partial y^j \partial y^k}\frac{\partial f}{\partial y^j}$$
$$+ \frac{\partial f}{\partial y^k}\frac{\partial^2 f}{\partial x^j \partial x^k}\frac{\partial f}{\partial x^j} + \frac{\partial f}{\partial y^k}\frac{\partial^2 f}{\partial x^j \partial y^k}\frac{\partial f}{\partial y^j} + \frac{\partial f}{\partial y^k}\frac{\partial^2 f}{\partial y^j \partial x^k}\frac{\partial f}{\partial y^j} - \frac{\partial f}{\partial y^k}\frac{\partial^2 f}{\partial y^j \partial y^k}\frac{\partial f}{\partial x^j})] \tag{5.20}$$

$$= \frac{1}{16}[2\frac{\partial^2 f}{\partial x^j \partial x^k}\frac{\partial f}{\partial x^k}\frac{\partial f}{\partial x^j} + 0\frac{\partial^2 f}{\partial x^j \partial x^k}\frac{\partial f}{\partial y^k}\frac{\partial f}{\partial y^j} + 0\frac{\partial^2 f}{\partial x^j \partial y^k}\frac{\partial f}{\partial x^k}\frac{\partial f}{\partial y^j} + 2\frac{\partial^2 f}{\partial x^j \partial y^k}\frac{\partial f}{\partial y^k}\frac{\partial f}{\partial x^j}$$
$$+ 2\frac{\partial^2 f}{\partial y^j \partial x^k}\frac{\partial f}{\partial x^k}\frac{\partial f}{\partial y^j} + 0\frac{\partial^2 f}{\partial y^j \partial x^k}\frac{\partial f}{\partial y^k}\frac{\partial f}{\partial x^j} + 0\frac{\partial^2 f}{\partial y^j \partial y^k}\frac{\partial f}{\partial x^k}\frac{\partial f}{\partial x^j} + 2\frac{\partial^2 f}{\partial y^j \partial y^k}\frac{\partial f}{\partial y^k}\frac{\partial f}{\partial y^j}$$
$$+ \sqrt{-1}(2\frac{\partial^2 f}{\partial x^j \partial x^k}\frac{\partial f}{\partial x^k}\frac{\partial f}{\partial y^j} + 0\frac{\partial^2 f}{\partial x^j \partial x^k}\frac{\partial f}{\partial y^k}\frac{\partial f}{\partial x^j} + 0\frac{\partial^2 f}{\partial x^j \partial y^k}\frac{\partial f}{\partial x^k}\frac{\partial f}{\partial x^j} + 2\frac{\partial^2 f}{\partial x^j \partial y^k}\frac{\partial f}{\partial y^k}\frac{\partial f}{\partial y^j}$$
$$- 2\frac{\partial^2 f}{\partial y^j \partial x^k}\frac{\partial f}{\partial x^k}\frac{\partial f}{\partial x^j} + 0\frac{\partial^2 f}{\partial y^j \partial x^k}\frac{\partial f}{\partial y^k}\frac{\partial f}{\partial y^j} + 0\frac{\partial^2 f}{\partial y^j \partial y^k}\frac{\partial f}{\partial x^k}\frac{\partial f}{\partial y^j} - 2\frac{\partial^2 f}{\partial y^j \partial y^k}\frac{\partial f}{\partial y^k}\frac{\partial f}{\partial x^j})] \tag{5.21}$$

$$= \frac{1}{8}[\frac{\partial^2 f}{\partial x^j \partial x^k}\frac{\partial f}{\partial x^k}\frac{\partial f}{\partial x^j} + \frac{\partial^2 f}{\partial x^j \partial y^k}\frac{\partial f}{\partial y^k}\frac{\partial f}{\partial x^j} + \frac{\partial^2 f}{\partial y^j \partial x^k}\frac{\partial f}{\partial x^k}\frac{\partial f}{\partial y^j} + \frac{\partial^2 f}{\partial y^j \partial y^k}\frac{\partial f}{\partial y^k}\frac{\partial f}{\partial y^j}$$
$$+ \sqrt{-1}(\frac{\partial^2 f}{\partial x^j \partial x^k}\frac{\partial f}{\partial x^k}\frac{\partial f}{\partial y^j} + \frac{\partial^2 f}{\partial x^j \partial y^k}\frac{\partial f}{\partial y^k}\frac{\partial f}{\partial y^j} - \frac{\partial^2 f}{\partial y^j \partial x^k}\frac{\partial f}{\partial x^k}\frac{\partial f}{\partial x^j} - \frac{\partial^2 f}{\partial y^j \partial y^k}\frac{\partial f}{\partial y^k}\frac{\partial f}{\partial x^j})]. \tag{5.22}$$

$$
\begin{aligned}
&\Delta_p^+ f\\
&= \frac{p-2}{2}(\sum_k \frac{\partial f}{\partial \bar z^k}\frac{\partial f}{\partial z^k})^{\frac{p-4}{2}} \sum_{k,j} \frac{\partial}{\partial z^j}(\frac{\partial f}{\partial \bar z^k}\frac{\partial f}{\partial z^k})\frac{\partial f}{\partial \bar z^j} + (\sum_k \frac{\partial f}{\partial \bar z^k}\frac{\partial f}{\partial z^k})^{\frac{p-2}{2}} \sum_j \frac{\partial^2 f}{\partial z^j \partial \bar z^j}\\
&= \frac{p-2}{2}(\sum_k \frac{\partial f}{\partial \bar z^k}\frac{\partial f}{\partial z^k})^{\frac{p-4}{2}} \frac{1}{8}\sum_{k,j}[\frac{\partial^2 f}{\partial x^j \partial x^k}\frac{\partial f}{\partial x^k}\frac{\partial f}{\partial x^j} + \frac{\partial^2 f}{\partial x^j \partial y^k}\frac{\partial f}{\partial y^k}\frac{\partial f}{\partial x^j}\\
&+ \frac{\partial^2 f}{\partial y^j \partial x^k}\frac{\partial f}{\partial x^k}\frac{\partial f}{\partial y^j} + \frac{\partial^2 f}{\partial y^j \partial y^k}\frac{\partial f}{\partial y^k}\frac{\partial f}{\partial y^j}\\
&+ \sqrt{-1}(\frac{\partial^2 f}{\partial x^j \partial x^k}\frac{\partial f}{\partial x^k}\frac{\partial f}{\partial y^j} + \frac{\partial^2 f}{\partial x^j \partial y^k}\frac{\partial f}{\partial y^k}\frac{\partial f}{\partial y^j} - \frac{\partial^2 f}{\partial y^j \partial x^k}\frac{\partial f}{\partial x^k}\frac{\partial f}{\partial x^j} - \frac{\partial^2 f}{\partial y^j \partial y^k}\frac{\partial f}{\partial y^k}\frac{\partial f}{\partial x^j})]\\
&+ (\sum_k \frac{\partial f}{\partial \bar z^k}\frac{\partial f}{\partial z^k})^{\frac{p-2}{2}} \sum_j \frac{\partial^2 f}{\partial z^j \partial \bar z^j}. \qquad (5.23)
\end{aligned}
$$

By Remark 5.1, we get

$$
\begin{aligned}
\mathrm{Re}\Delta_p^+ f = &\frac{p-2}{2}(\sum_k \frac{\partial f}{\partial \bar z^k}\frac{\partial f}{\partial z^k})^{\frac{p-4}{2}} \times \frac{1}{8}\sum_{k,j}(\frac{\partial f}{\partial x^k}\frac{\partial^2 f}{\partial x^j \partial x^k}\frac{\partial f}{\partial x^j}\\
&+ \frac{\partial f}{\partial x^j}\frac{\partial^2 f}{\partial x^j \partial y^k}\frac{\partial f}{\partial y^k} + \frac{\partial f}{\partial x^k}\frac{\partial^2 f}{\partial y^j \partial x^k}\frac{\partial f}{\partial y^j} + \frac{\partial f}{\partial y^k}\frac{\partial^2 f}{\partial y^j \partial y^k}\frac{\partial f}{\partial y^j})\\
&+ (\sum_k \frac{\partial f}{\partial \bar z^k}\frac{\partial f}{\partial z^k})^{\frac{p-2}{2}} \sum_j \frac{\partial^2 f}{\partial z^j \partial \bar z^j}.
\end{aligned}
$$

You can see that $\Delta_p^+ f$ is not necessarily real-valued.

Theorem 5.3. *If M is a Kähler manifold, $p > 1$, and $f : M \to \mathbb{R}$ is C^2, then*

$$|\nabla^+ f| = |\nabla^- f|,$$

$$\Delta_p^- f = \overline{\Delta_p^+ f},$$

and

$$\Delta_p^c f = 2^{\frac{p-2}{2}}(\Delta_p^+ f + \Delta_p^- f) = 2^{\frac{p}{2}} Re\Delta_p^+ f.$$

Proof. Under holomorphic normal coordinates,

$$|\nabla^+ f|^2 = g(\nabla^+ f, \overline{\nabla^+ f}) = \sum_j \frac{\partial f}{\partial \bar{z}^j}\frac{\partial f}{\partial z^j},$$

$$|\nabla^- f|^2 = g(\nabla^- f, \overline{\nabla^- f}) = \sum_j \frac{\partial f}{\partial z^j}\frac{\partial f}{\partial \bar{z}^j},$$

so

$$|\nabla^+ f| = |\nabla^- f|.$$

$$\Delta_p^+ f = \mathrm{div}(|\nabla^+ f|^{p-2}\nabla^+ f) = \sum_j \frac{\partial(|\nabla^+ f|^{p-2}\frac{\partial f}{\partial \bar{z}^j})}{\partial z^j} \tag{5.24}$$

$$= \sum_j \frac{\partial(|\nabla^+ f|^{p-2})}{\partial z^j}\frac{\partial f}{\partial \bar{z}^j} + |\nabla^+ f|^{p-2}\sum_j \frac{\partial^2 f}{\partial z^j \partial \bar{z}^j}, \tag{5.25}$$

$$\Delta_p^- f = \mathrm{div}(|\nabla^- f|^{p-2}\nabla^- f) = \sum_j \frac{\partial(|\nabla^- f|^{p-2}\frac{\partial f}{\partial z^j})}{\partial \bar{z}^j} \tag{5.26}$$

$$= \sum_j \frac{\partial(|\nabla^- f|^{p-2})}{\partial \bar{z}^j}\frac{\partial f}{\partial z^j} + |\nabla^- f|^{p-2}\sum_j \frac{\partial^2 f}{\partial \bar{z}^j \partial z^j}. \tag{5.27}$$

Since $|\nabla^+ f|^{p-2}$, f and $\frac{\partial^2 f}{\partial z^j \partial \bar{z}^j}$ are real-valued, $\frac{\partial^2 f}{\partial \bar{z}^j \partial z^j} = \frac{\partial^2 f}{\partial z^j \partial \bar{z}^j}$ for the fact that f is C^2, and $|\nabla^+ f| = |\nabla^- f|$, thus $\Delta_p^- f = \overline{\Delta_p^+ f}$.

$$\begin{aligned} &|\nabla^c f|^2 \\ &= g(\nabla^c f, \overline{\nabla^c f}) \\ &= g(\nabla^+ f + \nabla^- f, \overline{\nabla^+ f + \nabla^- f}) && (5.28) \\ &= g(\nabla^+ f + \nabla^- f, \overline{\nabla^+ f} + \overline{\nabla^- f}) && (5.29) \\ &= |\nabla^+ f|^2 + |\nabla^- f|^2 + g(\nabla^+ f, \overline{\nabla^- f}) + g(\nabla^- f, \overline{\nabla^+ f}). && (5.30) \end{aligned}$$

Since $\nabla^+ f \in T^{(1,0)}M$, $\nabla^- f \in T^{(0,1)}M$, $T^{(0,1)}M = \overline{T^{(1,0)}M}$, and $g_{jk} = g_{\bar{j}\bar{k}} = 0$, thus

$$g(\nabla^+ f, \overline{\nabla^- f}) = 0, g(\nabla^- f, \overline{\nabla^+ f}) = 0,$$

and

$$|\nabla^c f|^2 = |\nabla^+ f|^2 + |\nabla^- f|^2 = 2|\nabla^+ f|^2.$$

$$\begin{aligned}
&\Delta_p^c f \\
&= \mathrm{div}(|\nabla^c f|^{p-2} \nabla^c f) \\
&= \mathrm{div}([2^{1/2}|\nabla^+ f|]^{p-2} \nabla^c f) && (5.31) \\
&= (2^{1/2})^{p-2} \mathrm{div}(|\nabla^+ f|^{p-2} \nabla^c f) && (5.32) \\
&= 2^{\frac{p-2}{2}} (\Delta_p^+ f + \Delta_p^- f) && (5.33) \\
&= 2^{\frac{p-2}{2}} (\Delta_p^+ f + \overline{\Delta_p^+ f}) && (5.34) \\
&= 2^{\frac{p-2}{2}} 2 \mathrm{Re} \Delta_p^+ f && (5.35) \\
&= 2^{\frac{p}{2}} \mathrm{Re} \Delta_p^+ f, && (5.36)
\end{aligned}$$

since $\nabla^c f = \nabla^+ f + \nabla^- f$ and $|\nabla^+ f| = |\nabla^- f|$.

Question to think about: What difference can the imaginary part of $\Delta_p^+ f$ make?

Theorem 5.4. *If M is a Kähler manifold, and $f : M \to \mathbb{R}$ is C^2, then $\Delta_p^c f = \Delta_p f$.*

Proof.

$$
\begin{aligned}
&\Delta_p^c f \\
&= 2^{\frac{p-2}{2}}(\Delta_p^+ f + \Delta_p^- f) && (5.37)\\
&= 2^{\frac{p-2}{2}} 2\mathrm{Re}\Delta_p^+ f && (5.38)\\
&= 2^{\frac{p}{2}}\mathrm{Re}\Delta_p^+ f && (5.39)\\
&= 2^{\frac{p}{2}}[\frac{p-2}{2}(\sum_k \frac{\partial f}{\partial \bar{z}^k}\frac{\partial f}{\partial z^k})^{\frac{p-4}{2}} \times \frac{1}{8}\sum_{k,j}(\frac{\partial f}{\partial x^k}\frac{\partial^2 f}{\partial x^j \partial x^k}\frac{\partial f}{\partial x^j} + \frac{\partial f}{\partial x^j}\frac{\partial^2 f}{\partial x^j \partial y^k}\frac{\partial f}{\partial y^k}\\
&+ \frac{\partial f}{\partial x^k}\frac{\partial^2 f}{\partial y^j \partial x^k}\frac{\partial f}{\partial y^j} + \frac{\partial f}{\partial y^k}\frac{\partial^2 f}{\partial y^j \partial y^k}\frac{\partial f}{\partial y^j}) + (\sum_k \frac{\partial f}{\partial \bar{z}^k}\frac{\partial f}{\partial z^k})^{\frac{p-2}{2}}\sum_j \frac{\partial^2 f}{\partial z^j \partial \bar{z}^j}] && (5.40)\\
&= 2^{\frac{p}{2}}\{\frac{p-2}{2}(\sum_k \frac{1}{4}[(\frac{\partial f}{\partial x^k})^2 + (\frac{\partial f}{\partial y^k})^2])^{\frac{p-4}{2}} \times \frac{1}{8}\sum_{k,j}(\frac{\partial f}{\partial x^j}\frac{\partial f}{\partial x^k}\frac{\partial^2 f}{\partial x^j \partial x^k}\\
&+ \frac{\partial f}{\partial x^j}\frac{\partial f}{\partial y^k}\frac{\partial^2 f}{\partial x^j \partial y^k} + \frac{\partial f}{\partial y^j}\frac{\partial f}{\partial x^k}\frac{\partial^2 f}{\partial y^j \partial x^k} + \frac{\partial f}{\partial y^j}\frac{\partial f}{\partial y^k}\frac{\partial^2 f}{\partial y^j \partial y^k})\\
&+ (\sum_k \frac{1}{4}[(\frac{\partial f}{\partial x^k})^2 + (\frac{\partial f}{\partial y^k})^2])^{\frac{p-2}{2}}\sum_j \frac{1}{4}(\frac{\partial^2 f}{\partial x^j \partial x^j} + \frac{\partial^2 f}{\partial y^j \partial y^j})\} && (5.41)\\
&= 2^{\frac{p}{2}}\{\frac{p-2}{2}(\frac{1}{4})^{\frac{p-4}{2}}(\sum_J (\frac{\partial f}{\partial x^J})^2)^{\frac{p-4}{2}} \times \frac{1}{8}[\sum_{J,H}(\frac{\partial f}{\partial x^H}\frac{\partial f}{\partial x^J}\frac{\partial^2 f}{\partial x^H \partial x^J})]\\
&+ (\frac{1}{4})^{\frac{p-2}{2}}\frac{1}{4}[\sum_J (\frac{\partial f}{\partial x^J})^2]^{\frac{p-2}{2}}(\sum_H \frac{\partial^2 f}{\partial x^H \partial x^H})\} && (5.42)\\
&= 2^{\frac{p}{2}}\{\frac{p-2}{2}(\frac{1}{4})^{\frac{p-4}{2}}(\sum_J (\frac{\partial f}{\partial x^J})^2)^{\frac{p-4}{2}} \times \frac{1}{8}[\sum_H \frac{1}{2}\frac{\partial f}{\partial x^H}\frac{\partial}{\partial x^H}(\sum_J (\frac{\partial f}{\partial x^J})^2)]\\
&+ (\frac{1}{4})^{\frac{p}{2}}[\sum_J (\frac{\partial f}{\partial x^J})^2]^{\frac{p-2}{2}}(\sum_H \frac{\partial^2 f}{\partial x^H \partial x^H})\} && (5.43)\\
&= \frac{p-2}{2}(\frac{1}{2})^{\frac{p}{2}}[\sum_J (\frac{\partial f}{\partial x^J})^2]^{(\frac{p-4}{2})} \times (\sum_H \frac{\partial f}{\partial x^H}\frac{\partial}{\partial x^H}(\sum_J (\frac{\partial f}{\partial x^J})^2))\\
&+ (\frac{1}{2})^{\frac{p}{2}}[\sum_J (\frac{\partial f}{\partial x^J})^2]^{\frac{p-2}{2}}(\sum_H \frac{\partial^2 f}{\partial x^H \partial x^H}) && (5.44)\\
&= \Delta_p f.
\end{aligned}
$$

Chapter 6

Applications

There can be many applications of the holomorphic p-Laplacian $\Delta_p^+ f$, the antiholomorphic p-Laplacian $\Delta_p^- f$, and the complex p-Laplacian $\Delta_p^c f$. Here, we give one example of their applications.

Throughout this chapter, we let $p > 1$, $\{r_j\}$ be an unbounded sequence of strictly increasing positive numbers, M be a complete noncompact Kähler manifold, and f be a real-valued function on M.

Theorems 6.1-6.5 are from [18].

Theorem 6.1. *(Theorem 2.1, [18]). Let f be a nonconstant C^2 function. Suppose that $q > p-1$ and $f\Delta_p f \geq 0$. Then f has p-infinite growth. That is, for every $x_0 \in M$,*

$$\lim_{r\to\infty} \frac{1}{r^p} \int_{B(x_0;r)} |f|^q dv = \infty.$$

Theorem 6.2. *(Theorem 2.2, [18]). Let f be a nonconstant C^2 function. Suppose that $q > p-1$ and $f\Delta_p f \geq 0$. Then f has p-severe growth. That is, for every $x_0 \in M$, there exists a constant $a > 0$ such that for every unbounded strictly increasing sequence $\{r_j\}_0^\infty$, and every $r_{l_0} > a$,*

$$\sum_{j=l_0}^{\infty} \left(\frac{(r_{j+1}-r_j)^p}{\int_{B(x_0;r_{j+1})\setminus B(x_0;r_j)} |f|^q dv}\right)^{\frac{1}{p-1}} < \infty.$$

Theorem 6.3. *(Theorem 2.3, [18]). Let f be a nonconstant C^2 function. Suppose that $q > p-1$ and $f\Delta_p f \geq 0$. Then f has p-acute growth. That is, for* *every* $x_0 \in M$*, there exists a constant* $a > 0$ *such that*

$$\int_a^\infty \left(\frac{1}{\int_{\partial B(x_0;r)} |f|^q dv}\right)^{\frac{1}{p-1}} dr < \infty.$$

Theorem 6.4. *(Theorem 2.4, [18]). Let* f *be a nonconstant* C^2 *function. Suppose that* $q > p-1$ *and* $f\Delta_p f \geq 0$*. Then* f *has p-immoderate growth. That is, for every* $x_0 \in M$*, and every* $F(r) \in \mathcal{F}$ *where* $\mathcal{F} = \{F : [a,\infty) \to (0,\infty) | \int_a^\infty \frac{dr}{rF(r)} = \infty$ *for some* $a \geq 0\}$*,*

$$\limsup_{r\to\infty} \frac{1}{r^p F^{p-1}(r)} \int_{B(x_0;r)} |f|^q dv = \infty.$$

Theorem 6.5. *(Theorem 2.5, [18]). Let* f *be a nonconstant* C^2 *function. Suppose that* $q > p-1$ *and* $f\Delta_p f \geq 0$*. Then* f *has p-large growth. That is, for every* $x_0 \in M$*, there exists a constant* $a > 0$*, such that*

$$\int_a^\infty \left(\frac{r}{\int_{B(x_0;r)} |f|^q dv}\right)^{\frac{1}{p-1}} dr < \infty.$$

Theorems 6.1-6.5 can be condensed into one theorem because of the following two theorems:

Theorem 6.6. *(Proposition 2.1, [18]).* f *is p-immoderate if and only if* f *is p-large. Therefore, Theorems 6.4 and 6.5 are equivalent.*

Theorem 6.7. *(Proposition 2.2. (i), [18]). If* f *is p-acute, then* f *is p-severe. If* f *is p-severe, then* f *is p-immoderate and p-large.*

So Theorems 6.1-6.5 can be condensed into the following theorem:

Theorem 6.8. *Let* f *be a nonconstant* C^2 *function. Suppose that* $q > p-1$ *and* $f\Delta_p f \geq 0$*. Then* f *has p-infinite growth and p-acute growth.*

Theorem 6.9 is an application of the complex p-Laplacian of f.

Theorem 6.9. *Let f be a nonconstant C^2 function. Suppose that $q > p-1$ and $f\Delta_p^c f \geq 0$. Then f has p-infinite growth and p-acute growth.*

Theorem 6.10 is an application of the holomorphic p-Laplacian of f.

Theorem 6.10. *Let f be a nonconstant C^2 function. Suppose that $q > p-1$ and $f Re\Delta_p^+ f \geq 0$. Then f has p-infinite growth and p-acute growth.*

Theorem 6.11 is an application of the antiholomorphic p-Laplacian of f.

Theorem 6.11. *Let f be a nonconstant C^2 function. Suppose that $q > p-1$ and $f Re\Delta_p^- f \geq 0$. Then f has p-infinite growth and p-acute growth.*

Proof of Theorems 6.9-6.11. By Theorem 5.4 in Chapter 5, we get $\Delta_p^c f = \Delta_p f$. By Theorem 5.3 in Chapter 5, we get $\Delta_p^c f = 2^{\frac{p}{2}}\mathrm{Re}\Delta_p^+ f = 2^{\frac{p}{2}}\mathrm{Re}\Delta_p^- f$. By Theorems 6.8, immediately we can get Theorems 6.9-6.11.

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