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Abstract

From a geometric point of view, we use coordinates as the main tool to define the holomorphic gradient $\nabla^+ f$, the antiholomorphic gradient $\nabla^- f$, and the complex gradient $\nabla^c f$ of a complex-valued function $f$ on Kähler manifolds. Then we define the holomorphic Laplacian $\Delta^+ f$, the antiholomorphic Laplacian $\Delta^- f$, and the complex Laplacian $\Delta^c f$ of a real-valued function $f$. For a $C^2$ function $f$, the holomorphic Laplacian $\Delta^+ f$ actually coincides with the well-known complex Laplacian $\Box f$ since under holomorphic normal coordinates $\Delta^+ f = \sum_{i=1}^{n} \frac{\partial}{\partial z^i} \frac{\partial f}{\partial \bar{z}^i} = \frac{1}{2} \Delta f = \Box f$. For the first time, we introduce the holomorphic $p$-Laplacian $\Delta^+_p f$, the antiholomorphic $p$-Laplacian $\Delta^-_p f$, and the complex $p$-Laplacian $\Delta^c_p f$, and we find the relationship among them. We also find a relationship between $\Delta^c_p f$ and $\Delta_p f$. Finally, based on this relationship, we make global integral estimates on complete noncompact Kähler manifolds as an application of $\Delta^c_p f$, $\Delta^+_p f$, and $\Delta^-_p f$. 
Chapter 1

Introduction

In this paper, we will find a reasonable way to define and then calculate the complex $p$-Laplacian on Kähler manifolds.

An *almost complex structure* on a differentiable manifold $M$ is a real differentiable tensor field $J$ of rank (1,1) with the property

$$J(J(\xi)) = -\xi$$

for any differentiable vector field $\xi$; that is, a real tensor $J$ whose (real) components $j^b_a$ satisfy

$$j^s_a j^b_s = -\delta^b_a.$$ 

A manifold $M$ with an almost complex structure $J$ is even-dimensional. A manifold which admits an almost complex structure is called an *almost complex manifold*.

If $M$ is a complex manifold, then the holomorphic charts identify each tangent space $T_p M$ with $\mathbb{C}^n$, so we can define $J(v) = \sqrt{-1} v$ for every $v \in T_p M$, which gives an almost complex structure $J$. The fact that the transition functions are holomorphic means precisely that multiplication by $\sqrt{-1}$ is compatible under the different identifications of $T_p M$ with $\mathbb{C}^n$ using different charts. Since the transition maps between charts are biholomorphic, complex manifolds are, in particular, smooth.

If $z^1, ..., z^n$ are holomorphic coordinates and $z^i = x^i + \sqrt{-1} y^i$ for real functions...
\( x^i, y^i, \) then we can also write

\[
J\left( \frac{\partial}{\partial x^i} \right) = \frac{\partial}{\partial y^i}, \quad J\left( \frac{\partial}{\partial y^i} \right) = -\frac{\partial}{\partial x^i}.
\]

An almost complex structure is called *integrable*, if it arises from holomorphic charts as in the previous way.

An alternative and equivalent definition of an integrable almost complex structure is as follows (see [10]):

Let \( M \) be an almost complex manifold with almost complex structure \( J \). We define the *torsion* of \( J \) to be the tensor field \( N \) of type \((1,2)\) given by

\[
N(X,Y) = 2\{[JX,JY] - [X,Y] - J[X,JY] - J[JX,Y]\} \text{ for } X,Y \in \mathfrak{X}(M).
\]

An almost complex structure is said to be *integrable* if it has not torsion.

**Theorem 1.1.** (*Theorem 2.5, Chapter IX, [10]*). An almost complex structure is a complex structure if and only if it has no torsion.

A *Hermitian metric* on an almost complex manifold \( M \) is a Riemannian metric \( g \) invariant by the almost complex structure \( J \), i.e.,

\[
g(JX,JY) = g(X,Y) \quad \text{for any vector fields } X \text{ and } Y.
\]

An almost complex manifold (resp. a complex manifold) with a Hermitian metric is called an *almost Hermitian manifold* (resp. a *Hermitian manifold*).

**Theorem 1.2.** (*Proposition 4.1, Chapter IX, [10]*). Every almost complex manifold admits a Hermitian metric provided it is paracompact.
The fundamental 2-form $\Phi$ of an almost Hermitian manifold $M$ with almost complex structure $J$ and metric $g$ is defined by

$$
\Phi(X, Y) = g(X, JY) \text{ for all vector fields } X \text{ and } Y.
$$

A Hermitian metric on an almost complex manifold is called a Kähler metric if the fundamental 2-form is closed.

An almost complex manifold (resp. a complex manifold) with a Kähler metric is called an almost Kähler manifold (resp. a Kähler manifold).

On complex manifolds it is convenient to work with the complexified tangent bundle

$$
T^C M = TM \otimes_{\mathbb{R}} \mathbb{C}.
$$

In terms of local holomorphic coordinates it is convenient to use the basis

$$
\left\{ \frac{\partial}{\partial z^1}, \ldots, \frac{\partial}{\partial z^n}, \frac{\partial}{\partial \bar{z}^1}, \ldots, \frac{\partial}{\partial \bar{z}^n} \right\},
$$

where in terms of the real and imaginary parts $z^i = x^i + \sqrt{-1} y^i$ we have

$$
\frac{\partial}{\partial z^i} = \frac{1}{2} (\frac{\partial}{\partial x^i} - \sqrt{-1} \frac{\partial}{\partial y^i}) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}^i} = \frac{1}{2} (\frac{\partial}{\partial x^i} + \sqrt{-1} \frac{\partial}{\partial y^i}).
$$

The endomorphism $J$ extends to a complex linear endomorphism of $T^C M$, and induces a decomposition of this bundle pointwise into the $\sqrt{-1}$ and $-\sqrt{-1}$ eigenspaces

$$
T^C M = T^{(1,0)} M \oplus T^{(0,1)} M.
$$

In terms of local holomorphic coordinates $T^{(1,0)} M$ is spanned by the $\frac{\partial}{\partial z^i}$ while $T^{(0,1)} M$ is spanned by the $\frac{\partial}{\partial \bar{z}^i}$.

In local coordinates $z^1, ..., z^n$ a Hermitian metric is determined by the com-
ponents
\[ g_{j\bar{k}} = g\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_k}\right), \]

and we are extending \( g \) to complex tangent vectors by complex linearity in both entries. The Hermitian condition implies that for any \( j, k \) we have
\[ g\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_k}\right) = g\left(\frac{\partial}{\partial \bar{z}_j}, \frac{\partial}{\partial z_k}\right) = 0. \]

**Theorem 1.3.** (Proposition 1.14, [15]). (Normal coordinates). If \((M, g)\) is a Kähler manifold, then around any point \( p \in M \) we can choose holomorphic coordinates \( z^1, ..., z^n \) such that the components of \( g \) at the point \( p \) satisfy
\[ g_{j\bar{k}}(p) = \delta_{jk} \text{ and } \frac{\partial}{\partial z^i} g_{j\bar{k}}(p) = \frac{\partial}{\partial \bar{z}^i} g_{j\bar{k}}(p) = 0, \]
where \( \delta_{jk} \) is the identity matrix, i.e. \( \delta_{jk} = 0 \) if \( j \neq k \), and \( \delta_{jk} = 1 \) if \( j = k \).

The following formulas (1)-(12) are from [10].

Let \( M \) be an \( n \)-dimensional complex manifold and \( z^1, ..., z^n \) a complex local coordinate system in \( M \). Unless otherwise stated, Greek indices \( \alpha, \beta, \gamma, ... \) run from 1 to \( n \), while Latin capitals \( A, B, C, ... \) run through 1, ..., \( n \), \( \bar{1}, ..., \bar{n} \). We set
\[ (1) \quad Z_\alpha = \frac{\partial}{\partial z^\alpha}, \quad Z_{\bar{\alpha}} = \frac{\partial}{\partial \bar{z}^\alpha}. \]

Given a Hermitian metric \( g \) on \( M \), we extend the Hermitian inner product in each tangent space \( T_x(M) \) defined by \( g \) to a unique complex symmetric bilinear form in the complex tangent space \( T^c_x(M) \) (cf. Proposition 1.10 in [10]) and set
\[ (2) \quad g_{AB} = g(Z_A, Z_B). \]
Then, by Proposition 1.10 in [10],

\[ g_{\alpha\beta} = g_{\bar{\alpha}\bar{\beta}} = 0 \]

and \((g_{\alpha\beta})\) is an \(n \times n\) Hermitian matrix. It is then customary to write

\[ ds^2 = 2 \sum_{\alpha,\beta} g_{\alpha\bar{\beta}} dz^\alpha d\bar{z}^\beta \]

for the metric \(g\). By Proposition 1.12 in [10], the fundamental 2-form is given by

\[ \Phi = -2i \sum_{\alpha,\beta} g_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta. \]

A necessary and sufficient condition for \(g\) to be a Kähler metric is given by

\[ \frac{\partial g_{\alpha\beta}}{\partial z^\gamma} = \frac{\partial g_{\gamma\bar{\beta}}}{\partial z^\alpha} \quad \text{or} \quad \frac{\partial g_{\alpha\beta}}{\partial z^\bar{\alpha}} = \frac{\partial g_{\alpha\bar{\gamma}}}{\partial z^\beta}. \]

Given any affine connection with covariant differentiation \(\nabla\) on \(M\), we set

\[ \nabla_Z \Gamma_{BC}^A = \sum_A \Gamma_{BC}^A Z_A. \]

The covariant differentiation, which is originally defined for real vector fields, is extended by complex linearity to act on complex vector fields. Then

\[ \Gamma_{BC}^A = \Gamma_{\bar{B}\bar{C}}^\bar{A}. \]
with the convention that $\bar{\alpha} = \alpha$. From the fact that $JZ_\alpha = iZ_\alpha$ and $JZ_{\bar{\alpha}} = -iZ_{\bar{\alpha}}$, it follows that the connection is almost complex (see [10]) if and only if

$$\tag{9} \Gamma^\alpha_{B\gamma} = \Gamma^{\bar{\alpha}}_{B\gamma} = 0.$$ 

By direct calculation we see that an almost complex connection has no torsion if and only if

$$\tag{10} \Gamma^\alpha_{\beta\gamma} = \Gamma^{\bar{\alpha}}_{\gamma\beta}, \quad \Gamma^{\bar{\alpha}}_{\beta\bar{\gamma}} = \Gamma^\alpha_{\bar{\gamma}\beta} \quad \text{and}$$

$$\tag{11} \text{other } \Gamma^A_{BC} = 0.$$ 

In particular, (8), (9), (10), and (11) hold for every Kähler manifold. For a Kähler manifold, the $\Gamma^A_{BC}$’s are determined by the metric as follows:

$$\tag{12} \sum_{\alpha} g_{\alpha\bar{\epsilon}} \Gamma^{\alpha}_{\beta\gamma} = \frac{\partial g_{\bar{\epsilon}\bar{\beta}}}{\partial z^\gamma}, \quad \sum_{\alpha} g_{\alpha\bar{\epsilon}} \Gamma^{\bar{\alpha}}_{\beta\bar{\gamma}} = \frac{\partial g_{\bar{\epsilon}\bar{\beta}}}{\partial \bar{z}^\gamma}.$$ 

**Theorem 1.4.** If $(M, g)$ is a Kähler manifold, then under holomorphic normal coordinates around a point $p \in M$,

$$\Gamma^A_{BC}(p) = 0,$$

where $A, B, C, ...$ run through $1, ..., n, \bar{1}, ..., \bar{n}$.

**Proof.** From (8), (9), (10), (11), (12), and Theorem 1.3., we can obtain the result in this theorem.
On a complex manifold $M$, in local holomorphic coordinates, we have the 1-forms
\[ dz^j = dx^j + idy^j, \quad dz^k = dx^k - idy^k. \]
Decompose the space $\Omega^k$ of $k$-forms into subspaces $\Omega^{p,q}$ with $p + q = k$. Namely, $\Omega^{p,q}$ is locally spanned by forms of the type
\[ \omega(z) = \eta(z)dz^{i_1} \wedge ... \wedge dz^{i_p} \wedge dz^{\bar{j_1}} \wedge ... \wedge dz^{\bar{j_q}}. \]
Thus
\[ \Omega^k(M) = \sum_{p+q=k} \Omega^{p,q}(M). \]
We can then let the differential operators
\[ \partial = \frac{1}{2}(\frac{\partial}{\partial x^j} - i \frac{\partial}{\partial y^j})(dx^j + idy^j) \]
and
\[ \bar{\partial} = \frac{1}{2}(\frac{\partial}{\partial x^j} + i \frac{\partial}{\partial y^j})(dx^j - idy^j) \]
operate on such a form by
\[ \partial \omega = \frac{\partial \eta}{\partial z^j} dz^j \wedge dz^{i_1} \wedge ... \wedge dz^{\bar{j_1}} \wedge ... \wedge dz^{\bar{j_q}}, \]
and
\[ \bar{\partial} \omega = \frac{\partial \eta}{\partial z^j} dz^j \wedge dz^{i_1} \wedge ... \wedge dz^{\bar{j_1}} \wedge ... \wedge dz^{\bar{j_q}}. \]
Lemma 1.5. The exterior derivative $d$, $\partial$, and $\bar\partial$ satisfies

\begin{align*}
(1) \quad d &= \partial + \bar\partial, \\
(2) \quad \partial\partial &= 0, \quad \bar\partial\bar\partial = 0, \\
(3) \quad \partial\bar\partial &= -\bar\partial\partial.
\end{align*}

Proof. We have

$$
\partial + \bar\partial = \frac{1}{2} (\frac{\partial}{\partial x^j} - i \frac{\partial}{\partial y^j})(dx^j +idy^j) + \frac{1}{2} (\frac{\partial}{\partial x^j} + i \frac{\partial}{\partial y^j})(dx^j -idy^j) = \frac{\partial}{\partial x^j} dx^j + \frac{\partial}{\partial y^j} dy^j = d.
$$

Therefore,

$$
0 = d^2 = (\partial + \bar\partial)(\partial + \bar\partial) = \partial^2 + \partial\bar\partial + \bar\partial\partial + \bar\partial^2
$$

and decomposing into types yields (2) and (3).
Chapter 2

Coordinates

**Theorem 2.1.** On an $n$-dim Hermitian manifold $M$, we have $\frac{\partial}{\partial y_j} = J(\frac{\partial}{\partial x_j}), \frac{\partial}{\partial z_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} - \sqrt{-1}\frac{\partial}{\partial y_j}), \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} + \sqrt{-1}\frac{\partial}{\partial y_j})$. Then

$$g\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_k}\right) = 0, \quad g\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right) = 2\delta_{jk}$$

if and only if

$$g\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}\right) = g\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_k}\right) = 0, \quad g\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_k}\right) = \delta_{jk}.$$

**Proof.** According to [10], we can choose \{\frac{\partial}{\partial x_1}, J(\frac{\partial}{\partial x_1}), \frac{\partial}{\partial x_2}, J(\frac{\partial}{\partial x_2}), \ldots, \frac{\partial}{\partial x_n}, J(\frac{\partial}{\partial x_n})\} such that $g\left(\frac{\partial}{\partial x_j}, J\left(\frac{\partial}{\partial x_j}\right)\right) = 0, g\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right) = 2\delta_{jk}$. Then

$$g\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}\right) = g\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_k}\right) = 0, \quad g\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_k}\right) = \delta_{jk}.$$
\[ g(\frac{\partial}{\partial \bar{z}^j}, \frac{\partial}{\partial \bar{z}^k}) = g(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^k}) = 0 = 0. \]

\[ g_{jk} = g(\frac{\partial}{\partial \bar{z}^j}, \frac{\partial}{\partial \bar{z}^k}) \]
\[ = g(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^k}) \]
\[ = g(\frac{1}{2}(\frac{\partial}{\partial x^j} - \sqrt{-1} \frac{\partial}{\partial y^j}), \frac{1}{2}(\frac{\partial}{\partial x^k} + \sqrt{-1} \frac{\partial}{\partial y^k})) \]
\[ = \frac{1}{4}[g(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}) + \sqrt{-1}g(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial y^k}) - \sqrt{-1}g(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial x^k}) - (\sqrt{-1})^2g(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^k})] \]
\[ = \frac{1}{4}[2\delta_{jk} + \sqrt{-1} \cdot 0 - \sqrt{-1} \cdot 0 + g(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k})] \]
\[ = \frac{1}{4}[2\delta_{jk} + 2\delta_{jk}] \]
\[ = \delta_{jk}. \]

With the properties \( g(JX, JY) = g(X, Y) \) and \( J^2 = -I \), it is not hard to show the other direction of the conclusion of this theorem.

**Theorem 2.2.** On an \( n \)-dim Hermitian manifold \( M \), we have \( \frac{\partial}{\partial y^j} = J(\frac{\partial}{\partial x^j}) \),
\[ \frac{\partial}{\partial \bar{z}^j} = \frac{1}{2}(\frac{\partial}{\partial x^j} - \sqrt{-1} \frac{\partial}{\partial y^j}), \quad \frac{\partial}{\partial \bar{z}^j} = \frac{1}{2}(\frac{\partial}{\partial x^j} + \sqrt{-1} \frac{\partial}{\partial y^j}). \] Then
\[ \frac{\partial g_{AB}}{\partial x^C}(p) = 0 \]
if and only if
\[ \frac{\partial}{\partial \bar{z}^i}g_{jk}(p) = \frac{\partial}{\partial \bar{z}^i}g_{jk}(p) = 0. \]

**Proof.** Here \( g_{AB} = g(\frac{\partial}{\partial x^A}, \frac{\partial}{\partial x^B}) \), \( A, B, C = 1, ..., 2n \), and
\[ \frac{\partial}{\partial x^{(j+n)}} = \frac{\partial}{\partial y^j}, \quad j = 1, ..., n. \]
\[
g_{jk} = g(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_k}) = \frac{1}{4}[g(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}) + g(\frac{\partial}{\partial y_j}, \frac{\partial}{\partial y_k}) + \sqrt{-1}g(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_k}) - \sqrt{-1}g(\frac{\partial}{\partial y_j}, \frac{\partial}{\partial x_k})] .
\]

(2.9)

\[
\frac{\partial}{\partial z_i} + \frac{\partial}{\partial \bar{z}_i} = \frac{\partial}{\partial x_i}, \text{ so if } \frac{\partial}{\partial z_i}g_{jk}(p) = \frac{\partial}{\partial \bar{z}_i}g_{jk}(p) = 0, \text{ then }
\]

\[
\frac{\partial}{\partial x_i}g_{jk} = \left(\frac{\partial}{\partial z_i} + \frac{\partial}{\partial \bar{z}_i}\right)g_{jk} = 0,
\]

(2.10)

\[
\frac{\partial}{\partial x_i}g_{jk} = \frac{1}{4}\frac{\partial}{\partial x^i}[g(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}) + g(\frac{\partial}{\partial y_j}, \frac{\partial}{\partial y_k}) + \sqrt{-1}g(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_k}) - \sqrt{-1}g(\frac{\partial}{\partial y_j}, \frac{\partial}{\partial x_k})] = 0,
\]

(2.11)

\[
\frac{\partial}{\partial x_i}[g(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k})] + g(\frac{\partial}{\partial y_j}, \frac{\partial}{\partial y_k})] = 0,
\]

(2.10)

\[
\frac{\partial}{\partial x_i}[g(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k})] - g(\frac{\partial}{\partial y_j}, \frac{\partial}{\partial x_k})] = 0.
\]

Also, since \(g(JX, JY) = g(X, Y), J^2 = -I, \) and \(\frac{\partial}{\partial y^i} = J(\frac{\partial}{\partial x^i}), \) we get

\[
\frac{\partial}{\partial x^i}[g(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k})] + g(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^k})] = 0,
\]

(2.10)

\[
2\frac{\partial}{\partial x^i}g(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}) = 0,
\]

(2.11)

so

\[
\frac{\partial}{\partial x^i}g(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}) = 0;
\]

11
\[
\frac{\partial}{\partial x^i} [g\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial y^k}\right) - g\left(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial x^k}\right)] = \frac{\partial}{\partial x^i} [g\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial y^k}\right) + g\left(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial x^k}\right)]
\]

(2.12)

\[
= 2 \frac{\partial}{\partial x^i} g\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial y^k}\right)
\]

(2.13)

\[
= 0,
\]

so \( \frac{\partial}{\partial x^i} g\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial y^k}\right) = 0. \)

Similarly, since \( \frac{\partial}{\partial z^i} - \frac{\partial}{\partial \bar{z}^i} = \sqrt{-1} \frac{\partial}{\partial y^i} \), we can obtain \( \frac{\partial}{\partial y^i} g\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right) = 0 \), and \( \frac{\partial}{\partial y^i} g\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial y^k}\right) = 0. \)

Since \( g(JX, JY) = g(X, Y) \) and \( g(X, Y) = g(Y, X) \), we get \( \frac{\partial g}{\partial x^i} = 0. \)

It is a little similar and easier to show the other direction of the conclusion of this theorem.

**Theorem 2.3.** For n-dim Kähler manifold \((M, g)\), the Riemannian \( \Gamma^{(R)} = 0 \) iff the complexified \( \Gamma^{(C)} = 0. \)

Proof. In terms of a local coordinate system \( x^1, ..., x^{2n} \), the components \( \Gamma^I_{JK} \) of the Riemannian connection are given by

\[
\sum_L g_{LK} \Gamma^L_{JI} = \frac{1}{2} \left( \frac{\partial g_{KI}}{\partial x^J} + \frac{\partial g_{JK}}{\partial x^I} - \frac{\partial g_{JI}}{\partial x^K} \right)
\]

(see [9]).

For a Kähler manifold, the complexified \( \Gamma^A_{BC} \)'s are determined by the metric as follows: \( \sum _\alpha g_{\alpha \bar{e}} \Gamma^\alpha_{\beta \gamma} = \frac{\partial g_{z^3}}{\partial z^7}, \sum _\alpha g_{\alpha \bar{e}} \Gamma^\alpha_{\bar{\beta} \bar{\gamma}} = \frac{\partial g_{z^3}}{\partial \bar{z}^7}, \alpha, \beta, \gamma, \bar{e} = 1, ..., n \) (see [10]).

By Theorem 4.2., for a Kähler manifold, \( \Gamma^{(R)} = 0 \) iff \( \Gamma^{(C)} = 0. \)
Chapter 3

Complex gradient

Let $M$ be an $n$-dim Kähler manifold with Kähler metric $g$ and $f : M \to \mathbb{C}$ be some complex-valued function. We define the holomorphic gradient vector field $\nabla^+ f$ with respect to metric $g$ as follows:

$$\bar{\partial} f(X) = g(\nabla^+ f, X)$$

for every complexified vector field $X$.

Remark 3.1. The holomorphic gradient $\nabla^+ f$ can also be defined on a complex manifold with a metric $g$ since $\bar{\partial}$ is defined on complex manifolds. Here we define the holomorphic gradient on a Kähler manifold in order to use holomorphic normal coordinates to simplify the calculation.

$$g(\nabla^+ f, X) = \bar{\partial} f(X) = \frac{\partial f}{\partial \bar{z}^j} d\bar{z}^j(X)$$

for any complexified vector field $X$, so

$$g(\nabla^+ f, \frac{\partial}{\partial \bar{z}^j}) = \frac{\partial f}{\partial \bar{z}^j} d\bar{z}^j \left( \frac{\partial}{\partial \bar{z}^i} \right) = 0,$$

$$g(\nabla^+ f, \frac{\partial}{\partial z^i}) = \frac{\partial f}{\partial z^j} d\bar{z}^j \left( \frac{\partial}{\partial z^i} \right) = \partial f \delta^j_i = \frac{\partial f}{\partial \bar{z}^i}.$$

Let $\nabla^+ f = f^i \frac{\partial}{\partial \bar{z}^i} + f^\bar{i} \frac{\partial}{\partial z^i}$, then

$$g(\nabla^+ f, \frac{\partial}{\partial \bar{z}^i}) = f^i g_{\bar{i}i} + f^\bar{i} g_{\bar{i}i} = 0,$$

$$g(\nabla^+ f, \frac{\partial}{\partial z^i}) = f^i g_{\bar{i}i} + f^\bar{i} g_{\bar{i}i} = \frac{\partial f}{\partial \bar{z}^i} \delta^j_i = \frac{\partial f}{\partial \bar{z}^i}.$$
\[
\left( f^1, f^2, \ldots, f^n, f^1, f^2, \ldots, f^n \right) = \begin{pmatrix}
0 & 0 & \cdots & 0 & \frac{\partial f}{\partial z^1} & \frac{\partial f}{\partial z^2} & \cdots & \frac{\partial f}{\partial z^n}
\end{pmatrix},
\]
so

\[
\left( f^1, f^2, \ldots, f^n, f^1, f^2, \ldots, f^n \right) = \begin{pmatrix}
0 & 0 & \cdots & 0 & \frac{\partial f}{\partial z^1} & \frac{\partial f}{\partial z^2} & \cdots & \frac{\partial f}{\partial z^n}
\end{pmatrix} \begin{pmatrix}
g_{ij} & g_{ij} \\
g_{\bar{i}j} & g_{\bar{i}j}
\end{pmatrix}^{-1}
\]

\[
= \begin{pmatrix}
0 & 0 & \cdots & 0 & \frac{\partial f}{\partial z^1} & \frac{\partial f}{\partial z^2} & \cdots & \frac{\partial f}{\partial z^n}
\end{pmatrix} \begin{pmatrix}
0 & g^{\bar{i}j} \\
g^{\bar{i}j}
\end{pmatrix},
\]

since Hermitian condition implies that for any \( j, k \), we have

\[
g(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^k}) = g(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^k}) = 0,
\]
that is, \( g_{ij} = g_{\bar{i}j} = 0 \). The symmetry of \( g \) implies that \( g_{jk} = g_{kj} = g_{k\bar{j}} \).

\[
\begin{pmatrix}
g_{ij} & g_{ij} \\
g_{\bar{i}j} & g_{\bar{i}j}
\end{pmatrix} = \begin{pmatrix}
0 & g_{\bar{i}j} \\
g_{ij} & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
g_{ij} & g_{ij} \\
g_{\bar{i}j} & g_{\bar{i}j}
\end{pmatrix}^{-1} = \begin{pmatrix}
0 & g_{\bar{i}j}^{-1} \\
g_{ij} & 0
\end{pmatrix} = \begin{pmatrix}
0 & (g_{ij})^{-1} \\
(g_{ij})^{-1} & 0
\end{pmatrix} = \begin{pmatrix}
0 & g^{\bar{i}j} \\
g^{\bar{i}j} & 0
\end{pmatrix},
\]
since \((g^i_j) := (g_{ij})^{-1}\), and \((g^{i\bar{j}}) := (g_{i\bar{j}})^{-1}\).

Thus, \((f_1 \ f^2 \ \ldots \ f^n \ f^1 \ f^2 \ \ldots \ f^n) = (\frac{\partial f}{\partial z^j} g^{i\bar{j}}, 0),\)

\[ f^i = g^{i\bar{j}} \frac{\partial f}{\partial z^j}, \quad f^{i\bar{j}} = 0, \quad i = 1, \ldots, n. \]

So

\[ \nabla^+ f = f^i \frac{\partial}{\partial z^i} + f^{i\bar{j}} \frac{\partial}{\partial z^{i\bar{j}}} = g^{i\bar{j}} \frac{\partial f}{\partial z^j} \frac{\partial}{\partial z^i}. \]

Similarly, for \(f : M \to \mathbb{C}, M\) a Kähler manifold with Kähler metric \(g\), define the antiholomorphic gradient \(\nabla^- f\) as:

\[ g(\nabla^- f, X) = \partial f(X) = \frac{\partial f}{\partial z^j} dz^j(X), \]

for any complex vector field \(X\). Then we can get

\[ \nabla^- f = g^{i\bar{j}} \frac{\partial f}{\partial z^j} \frac{\partial}{\partial z^{i\bar{j}}}. \]

Under holomorphic normal coordinates,

\[
\begin{pmatrix}
g_{ij} & g_{i\bar{j}} \\
g_{i\bar{j}} & g_{j\bar{i}}
\end{pmatrix} = \begin{pmatrix}0 & I_n \\ I_n & 0\end{pmatrix} = \begin{pmatrix}g_{ij} & g_{i\bar{j}} \\
g_{i\bar{j}} & g_{j\bar{i}}\end{pmatrix}^{-1},
\]

\[ \nabla^+ f = \sum_{j=1}^n \frac{\partial f}{\partial z^j} \frac{\partial}{\partial z^j}, \]

and

\[ \nabla^- f = \sum_{j=1}^n \frac{\partial f}{\partial z^j} \frac{\partial}{\partial z^{i\bar{j}}}. \]

The exterior derivative on a complex manifold is \(d = \partial + \bar{\partial}\), so \(df = \partial f + \bar{\partial} f\). For \(f : M \to \mathbb{C}, M\) a Kähler manifold with Kähler metric \(g\), define the complex
gradient of $f$, $\nabla^c f$, as follows:

$$g(\nabla^c f, X) = df(X)$$

for any complex vector field $X$. Then we can get

$$\nabla^c f = g^{ji} \frac{\partial f}{\partial \bar{z}^j} \frac{\partial}{\partial z^i} + g^{ji} \frac{\partial f}{\partial z^j} \frac{\partial}{\partial \bar{z}^i} = \nabla^+ f + \nabla^- f.$$

The usual Riemannian gradient of $f$ is

$$\nabla f = g^{IJ} \frac{\partial f}{\partial x^I} \frac{\partial}{\partial x^J},$$

$I, J = 1, ..., 2n$.

**Remark 3.2.** $\nabla^+ f$ is the projection of complex gradient $\nabla^c f$ onto $T^{(1,0)}M$, which is a holomorphic vector field.

**Remark 3.3.** $f : M \to \mathbb{C}$ is holomorphic if and only if $\nabla^+ f = 0$.

So we call $\nabla^+ f$ the holomorphic gradient of $f$. 
Chapter 4

Complex Laplacian

Let $M$ be an $n$-dim Kähler manifold with Kähler metric $g$, and $f : M \rightarrow \mathbb{R}$ be a real-valued function.

For any real vector field $X$ on $M$, we define the usual Riemannian divergence of $X$, denoted by $\text{div}X$, to be the function such that at each point $x$ of $M$

$$(\text{div}X)_x = \text{trace of the endomorphism } V \mapsto \nabla_V X \text{ of } T_x M,$$

where $\nabla$ is the connection in the Riemannian manifold. In terms of local coordinates $(x^1, ..., x^n, y^1, ..., y^n)$, let $x^\bar{i}$ denote $y^i$ where $\bar{i} = i + n$, and let Latin capitals $A, B, C, ..., H, I, J, K, ...$ run through 1, ..., $2n$. Then we have

$$df = \sum_{I=1}^{2n} f_I dx^I, \text{ where } f_I = \frac{\partial f}{\partial x^I}$$

and

$$\text{grad} f = \sum_{I=1}^{2n} f^I \frac{\partial}{\partial x^I}, \text{ where } f^I = \sum_{J=1}^{2n} g^{IJ} f_J,$$

where $(g^{IJ}) = (g_{IJ})^{-1}$, i.e. $\sum_I g_{IJ} g^{JK} = \delta^K_I$. If $X = \sum_{I=1}^{2n} \xi^I \frac{\partial}{\partial x^I}$, then

$$\text{div}X = \sum_{I=1}^{2n} \xi^I,$$

where $\xi^I = \frac{\partial \xi^I}{\partial x^I} + \sum_K \Gamma^I_{JK} \xi^K$. (see [9] and [10]).

For any complexified vector field $X$ on $M$, we define the complex divergence
of $X$, denoted also by $\text{div}X$, to be the function such that at each point $x$ of $M$

$$(\text{div}X)_x = \text{trace of the endomorphism } V \to \nabla_V X \text{ of } T^c_x M,$$

where $\nabla$ is the complex linear extension of the connection in the Riemannian manifold. If $X = X^i \frac{\partial}{\partial z^i} + X^{\bar{i}} \frac{\partial}{\partial \bar{z}^i}$, then

$$\nabla X = X^A_{;B} \frac{\partial}{\partial z^A} \otimes dz^B = (\frac{\partial X^A}{\partial z^B} + X^C \Gamma^A_{CB}) \frac{\partial}{\partial z^A} \otimes dz^B,$$

where $i, j, k = 1, 2, ..., n$, $\bar{i} = i + n$, and $A, B, C = 1, 2, ..., 2n$. Thus,

$$\text{div}X = \frac{\partial X^A}{\partial z^A} + X^C \Gamma^A_{CA}.$$  

Since on a Kähler manifold, $\Gamma^k_{ij} = \Gamma^k_{ji}$, $\Gamma^{\bar{k}}_{ij} = \Gamma^{\bar{k}}_{ji}$, other $\Gamma^A_{BC} = 0$, thus, on a Kähler manifold,

$$\text{div}X = \frac{\partial X^A}{\partial z^A} + X^k \Gamma^i_{ki} + X^{\bar{k}} \Gamma^{\bar{i}}_{\bar{k}i},$$

so

$$\Delta^+ f := \text{div}(\nabla^+ f) = \frac{\partial}{\partial z^i} (g^{\bar{j}i} \frac{\partial f}{\partial \bar{z}^j}) + g^{\bar{j}k} \frac{\partial f}{\partial \bar{z}^j} \Gamma^i_{ki}.$$  

We call $\Delta^+ f$ the holomorphic Laplacian of $f$.

Similarly, for Kähler manifold $(M, g)$ and $f : M \to \mathbb{R}$, we can define the antiholomorphic Laplacian of $f$ as:

$$\Delta^- f := \text{div}(\nabla^- f) = \frac{\partial}{\partial \bar{z}^i} (g^{ij} \frac{\partial f}{\partial z^j}) + g^{jk} \frac{\partial f}{\partial z^j} \Gamma^i_{ki},$$

and the complex Laplacian of $f$ as:

$$\Delta^c f := \text{div}(\nabla^c f) = \text{div}(\nabla^+ f + \nabla^- f) = \Delta^+ f + \Delta^- f.$$
The usual Riemannian Laplacian of $f$ is
\[ \Delta f := \text{div}(\nabla f) = \frac{\partial}{\partial x^I} (g^{IJ} \frac{\partial f}{\partial x^J}) + g^{JK} \frac{\partial f}{\partial x^J} \Gamma^I_{KJ}. \]

For a Kähler manifold $(M, g)$ and a $C^2$ function $f$ on $M$, under holomorphic normal coordinates,
\[ \Delta^+ f = \sum_{i=1}^n \frac{1}{\partial z^i \partial \bar{z}^i} = \sum_{i=1}^n \frac{1}{4} \left( \frac{\partial^2 f}{\partial x^i \partial x^i} + \frac{\partial^2 f}{\partial y^i \partial y^i} \right) = \frac{1}{4} \sum_{i=1}^{2n} \frac{\partial^2 f}{(\partial x^i)^2}. \]

The usual Riemannian gradient of $f$ is: $\nabla f = g^{IJ} \frac{\partial f}{\partial x^J} \frac{\partial}{\partial x^I}$. The usual Riemannian Laplacian of $f$ is $\Delta f$:
\[ \Delta f := \text{div}(\nabla f) = \frac{\partial}{\partial x^I} (g^{IJ} \frac{\partial f}{\partial x^J}) + g^{JK} \frac{\partial f}{\partial x^J} \Gamma^I_{KJ}. \]

Under the real coordinates corresponding to the holomorphic normal coordinates,
\[ \Delta f = \sum_{i=1}^{2n} \left( \frac{1}{2} \frac{\partial^2 f}{(\partial x^I)^2} + 0 \right) = \frac{1}{2} \sum_{i=1}^{2n} \frac{\partial^2 f}{(\partial x^I)^2}. \]

Thus,
\[ \Delta^+ f = \frac{1}{2} \Delta f. \]

Similarly, $\Delta^- f = \frac{1}{2} \Delta f$. So $\Delta^+ f = \Delta^- f$.

$\Delta^c f = \text{div}(\nabla^+ f + \nabla^- f) = \text{div}(\nabla^+ f) + \text{div}(\nabla^- f) = \Delta^+ f + \Delta^- f = \Delta f$.

Now let us take a look at the well-known complex Laplacian of $f$ on $M$. Let $\ast$ be the Hodge $\ast$-operator, and $\bar{\partial}^{\ast} := - \ast \circ \partial \circ \ast$. Then the well-known complex Laplacian of $f$ is
\[ \Box f = (\bar{\partial} \bar{\partial}^{\ast} + \bar{\partial}^{\ast} \bar{\partial}) f. \]
On Kähler manifold \((M, g)\), \(\Box f = \frac{1}{2} \Delta f\) (see [6], [4], and [19]).

We have \(\Delta^+ f = \frac{1}{2} \Delta f = \Box f\). Thus, our holomorphic Laplacian \(\Delta^+ f\) coincides with the well-known complex Laplacian \(\Box f\) on Kähler manifolds.
Chapter 5

Complex $p$-Laplacian

On an $n$-dim Kähler manifold $(M, g)$, for $f : M \to \mathbb{R}$ and $p > 1$, the usual Riemannian $p$-Laplacian of $f$ is $\Delta_p f$:

$$\Delta_p f = \text{div}(|\nabla f|^{p-2} \nabla f),$$

where

$$|\nabla f| = [g(\nabla f, \nabla f)]^{\frac{1}{2}}$$

$$= [g(g^{IJ} \frac{\partial f}{\partial x^J}, g^{KL} \frac{\partial f}{\partial x^L} \frac{\partial}{\partial x^K})]^{\frac{1}{2}}$$

$$= [g^{IJ} g^{KL} \frac{\partial f}{\partial x^J} \frac{\partial f}{\partial x^L} g_{IK}]^{\frac{1}{2}},$$

$$|\nabla f|^{p-2} \nabla f = (g_{IK} g^{IJ} g^{KL} \frac{\partial f}{\partial x^J} \frac{\partial f}{\partial x^L})^{\frac{1-p}{2}} g^{ST} \frac{\partial f}{\partial x^T} \frac{\partial}{\partial x^S}. $$

Under the real coordinates corresponding to the holomorphic normal coordinates,

$$g'^{IJ} = \frac{1}{2} \delta_{IJ},$$

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\[ \Delta_p f \]
\[ = \text{div}(|\nabla f|^{p-2} \nabla f) \]
\[ = \sum_H \frac{\partial}{\partial x^H} \left[ (g^L_i g^J_k g^{KL} \frac{\partial f}{\partial x^T}, \frac{\partial f}{\partial x^T}) \right] \]
\[ = \sum_H \frac{\partial}{\partial x^H} \left[ (g^{JL} \frac{\partial f}{\partial x^T}, \frac{\partial f}{\partial x^T}) \right] \]
\[ = \sum_H \left\{ \frac{p-2}{2} (g^{JL} \frac{\partial f}{\partial x^T}, \frac{\partial f}{\partial x^T}) \right\}^{\frac{p-2}{2}} \frac{\partial f}{\partial x^T} \]
\[ + \sum_H \left\{ (g^{JL} \frac{\partial f}{\partial x^T}, \frac{\partial f}{\partial x^T}) \right\}^{\frac{p-2}{2}} \frac{\partial^2 f}{\partial x^T \partial x^T} \]
\[ = \sum_H \left\{ \frac{1}{2} \frac{\partial f}{\partial x^T}, \frac{\partial f}{\partial x^T} \right\}^{\frac{p-2}{2}} \frac{\partial^2 f}{\partial x^T \partial x^T} \]

For \( X \in T^c_x M \), \( X = X^i \frac{\partial}{\partial z^i} + X^j \frac{\partial}{\partial \bar{z}^j} \), define

\[ |X| := \sqrt{g(X, \overline{X})}. \]
\[ g(X, \overline{X}) \]
\[ = g(X^i \frac{\partial}{\partial z^i} + X^j \frac{\partial}{\partial z^j}, X^k \frac{\partial}{\partial z^k} + X^h \frac{\partial}{\partial z^h}) \]  
(5.7)
\[ = g(X^i \frac{\partial}{\partial z^i} + X^j \frac{\partial}{\partial z^j}, \overline{X}^k \frac{\partial}{\partial \overline{z}^k} + \overline{X}^h \frac{\partial}{\partial \overline{z}^h}) \]  
(5.8)
\[ = g(X^i \frac{\partial}{\partial z^i}, \overline{X}^k \frac{\partial}{\partial \overline{z}^k}) + g(X^i \frac{\partial}{\partial z^i}, \overline{X}^h \frac{\partial}{\partial \overline{z}^h}) + g(X^j \frac{\partial}{\partial z^j}, \overline{X}^k \frac{\partial}{\partial \overline{z}^k}) + g(X^j \frac{\partial}{\partial z^j}, \overline{X}^h \frac{\partial}{\partial \overline{z}^h}) \]  
(5.9)
\[ = X^i \overline{X}^k g_{ik} + X^j \overline{X}^h g(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \overline{z}^h}) + X^j \overline{X}^k g_{jk} + X^j \overline{X}^h g_{jh} \]  
(5.10)
\[ = X^i \overline{X}^k g_{ik} + X^j \overline{X}^h g_{jh}. \]  
(5.11)

Under holomorphic normal coordinates, \( g_{jk} = \delta_{jk} \), so

\[ g(X, \overline{X}) = \sum_{i=1}^{n} X^i \overline{X}^i + \sum_{j=1}^{n} X^j \overline{X}^j = \sum_{A=1}^{2n} X^A \overline{X}^A \geq 0, \]

\[ g(\nabla^+ f, \overline{\nabla}^+ f) = \sum_{i=1}^{n} \frac{\partial f}{\partial z^i} \frac{\partial f}{\partial \overline{z}^i} = \sum_{i=1}^{n} |\frac{\partial f}{\partial z^i}|^2 = \sum_{i=1}^{n} |\frac{\partial f}{\partial \overline{z}^i}|^2 \]

for \( f : M \to \mathbb{R}, \overline{\frac{\partial f}{\partial \overline{z}^i}} = \frac{\partial f}{\partial z^i} \).

For \( p > 1 \), similar to how we define the usual Riemannian \( p \)-Laplacian of \( f \), we can define the holomorphic \( p \)-Laplacian of \( f \) as:

\[ \Delta^+_p f := \text{div}(|\nabla^+ f|^{p-2} \nabla^+ f), \]

define the antiholomorphic \( p \)-Laplacian of \( f \) as:

\[ \Delta^-_p f := \text{div}(|\nabla^- f|^{p-2} \nabla^- f), \]

and define the complex \( p \)-Laplacian of \( f \) as:

\[ \Delta^c_p f := \text{div}(|\nabla^c f|^{p-2} \nabla^c f). \]
Remark 5.1.

\[
\frac{\partial f}{\partial \bar{z}^k} \frac{\partial f}{\partial z^k} = \frac{1}{2} \left( \frac{\partial}{\partial x^k} + \sqrt{-1} \frac{\partial}{\partial y^k} \right) f \left( \frac{1}{2} \frac{\partial}{\partial x^k} - \sqrt{-1} \frac{\partial}{\partial y^k} \right) f = \frac{1}{4} \left( \left( \frac{\partial f}{\partial x^k} \right)^2 + \left( \frac{\partial f}{\partial y^k} \right)^2 \right).
\]

If \( f \) is \( C^2 \),

\[
\frac{\partial^2 f}{\partial z^j \partial \bar{z}^j} = \frac{1}{2} \left( \frac{\partial}{\partial x^j} - \sqrt{-1} \frac{\partial}{\partial y^j} \right) \frac{1}{2} \left( \frac{\partial}{\partial x^j} + \sqrt{-1} \frac{\partial}{\partial y^j} \right) f = \frac{1}{4} \left( \frac{\partial^2 f}{\partial x^j \partial x^j} + \frac{\partial^2 f}{\partial y^j \partial y^j} \right).
\]

Theorem 5.2. For a \( C^2 \) function \( f \) on \( n \)-dim Kähler manifold \( (M,g) \), under holomorphic normal coordinates,

\[
\text{Re} \Delta^+_p f = \frac{p-2}{2} \sum_k \left( \frac{\partial f}{\partial \bar{z}^k} \frac{\partial f}{\partial z^k} \right)^{\frac{p-4}{2}} \times \frac{1}{8} \sum_{k,j} \left( \frac{\partial f}{\partial x^k} \frac{\partial^2 f}{\partial x^j \partial x^k} \frac{\partial f}{\partial x^j} \right) \left( \frac{\partial f}{\partial y^k} \frac{\partial^2 f}{\partial y^j \partial y^k} \frac{\partial f}{\partial y^j} \right) + \left( \sum_k \frac{\partial f}{\partial \bar{z}^k} \frac{\partial f}{\partial z^k} \right)^{\frac{p-4}{2}} \sum_j \frac{\partial^2 f}{\partial z^j \partial \bar{z}^j}. \tag{5.12}
\]

Proof. Under holomorphic normal coordinates,

\[
|\nabla^+ f|^2 = g(\nabla^+ f, \nabla^+ f) = \sum_j \frac{\partial f}{\partial \bar{z}^j} \frac{\partial f}{\partial z^j},
\]
\[ \Delta^+_p f \]
\[ = \text{div}(\|\nabla^+ f\|^{p-2} \nabla^+ f) \]
\[ = \sum_j \frac{\partial(\|\nabla^+ f\|^{p-2} \frac{\partial f}{\partial \bar{z}^j})}{\partial z^j} \]  
\[ = \sum_j \frac{\partial(\|\nabla^+ f\|^{p-2})}{\partial z^j} \frac{\partial f}{\partial \bar{z}^j} + \sum_j \|\nabla^+ f\|^{p-2} \frac{\partial^2 f}{\partial z^j \partial \bar{z}^j} \]  
\[ = \sum_j \frac{\partial[(\sum_k \frac{\partial f}{\partial \bar{z}^k} \frac{\partial f}{\partial z^k})^{p-2}]}{\partial z^j} + (\sum_k \frac{\partial f}{\partial \bar{z}^k} \frac{\partial f}{\partial z^k})^{p-2} \sum_j \frac{\partial^2 f}{\partial z^j \partial \bar{z}^j} \]  
\[ = \frac{p-2}{2} (\sum_k \frac{\partial f}{\partial \bar{z}^k} \frac{\partial f}{\partial z^k})^{p-2} \sum_{k,j} \frac{\partial}{\partial \bar{z}^j} (\frac{\partial f}{\partial \bar{z}^k} \frac{\partial f}{\partial z^k}) \frac{\partial f}{\partial \bar{z}^j} + (\sum_k \frac{\partial f}{\partial \bar{z}^k} \frac{\partial f}{\partial z^k})^{p-2} \sum_j \frac{\partial^2 f}{\partial z^j \partial \bar{z}^j}. \]  

\[ (5.13) \]
\[ (5.14) \]
\[ (5.15) \]
\[ (5.16) \]
\[
\frac{\partial}{\partial z^j} \left( \frac{\partial f}{\partial z^k} \frac{\partial f}{\partial z^l} \right) \frac{\partial f}{\partial z^m} = \frac{\partial^2 f}{\partial z^j \partial z^k} \frac{\partial f}{\partial z^l} + \frac{\partial f}{\partial z^j} \frac{\partial^2 f}{\partial z^k \partial z^l} \frac{\partial f}{\partial z^m} 
\]

(5.17)

\[
= \frac{1}{2} \left( \frac{\partial}{\partial x^j} - \sqrt{-1} \frac{\partial}{\partial y^j} \right) \frac{1}{2} \left( \frac{\partial}{\partial x^k} + \sqrt{-1} \frac{\partial}{\partial y^k} \right) f \left( \frac{\partial}{\partial x^l} - \sqrt{-1} \frac{\partial}{\partial y^l} \right) f \left( \frac{\partial}{\partial x^m} + \sqrt{-1} \frac{\partial}{\partial y^m} \right) f 
\]

(5.18)

\[
= \frac{1}{2} \left( \frac{\partial}{\partial x^j} + \sqrt{-1} \frac{\partial}{\partial y^j} \right) f \left( \frac{\partial}{\partial x^k} - \sqrt{-1} \frac{\partial}{\partial y^k} \right) f \left( \frac{\partial}{\partial x^l} - \sqrt{-1} \frac{\partial}{\partial y^l} \right) f \left( \frac{\partial}{\partial x^m} + \sqrt{-1} \frac{\partial}{\partial y^m} \right) f 
\]

(5.19)

\[
= \frac{1}{16} \left[ \frac{\partial^2 f}{\partial x^j \partial x^k} \frac{\partial f}{\partial x^l} - \frac{\partial^2 f}{\partial x^j \partial x^l} \frac{\partial f}{\partial x^k} - \frac{\partial^2 f}{\partial x^j \partial y^k} \frac{\partial f}{\partial x^l} + \frac{\partial^2 f}{\partial x^j \partial y^l} \frac{\partial f}{\partial x^k} \right] 
\]

(5.20)

\[
= \frac{1}{16} \left[ 2 \frac{\partial^2 f}{\partial x^j \partial x^k} \frac{\partial f}{\partial x^l} + \frac{\partial^2 f}{\partial x^j \partial x^l} \frac{\partial f}{\partial x^k} + \frac{\partial^2 f}{\partial x^j \partial y^k} \frac{\partial f}{\partial x^l} + \frac{\partial^2 f}{\partial x^j \partial y^l} \frac{\partial f}{\partial x^k} + 2 \frac{\partial^2 f}{\partial x^j \partial y^k} \frac{\partial f}{\partial x^l} + \frac{\partial^2 f}{\partial x^j \partial y^l} \frac{\partial f}{\partial x^k} \right] 
\]

(5.21)

\[
= \frac{1}{16} \left[ \frac{\partial^2 f}{\partial x^j \partial x^k} \frac{\partial f}{\partial x^l} + \frac{\partial^2 f}{\partial x^j \partial x^l} \frac{\partial f}{\partial x^k} + \frac{\partial^2 f}{\partial x^j \partial y^k} \frac{\partial f}{\partial x^l} + \frac{\partial^2 f}{\partial x^j \partial y^l} \frac{\partial f}{\partial x^k} + \frac{\partial^2 f}{\partial x^j \partial y^k} \frac{\partial f}{\partial x^l} + \frac{\partial^2 f}{\partial x^j \partial y^l} \frac{\partial f}{\partial x^k} \right] 
\]

(5.22)
\[
\Delta_p^+ f = \frac{p - 2}{2} \left( \sum_k \frac{\partial f}{\partial z^k} \frac{\partial f}{\partial z^k} \right)^{\frac{p-4}{2}} \sum_{k,j} \frac{\partial}{\partial z^j} \left( \frac{\partial f}{\partial z^k} \frac{\partial f}{\partial z^k} \right) \frac{\partial f}{\partial z^j} + \left( \sum_k \frac{\partial f}{\partial z^k} \frac{\partial f}{\partial z^k} \right)^{\frac{p-4}{2}} \sum_{j} \frac{\partial^2 f}{\partial z^j \partial \bar{z}^j}
\]

\[
= \frac{p - 2}{2} \left( \sum_k \frac{\partial f}{\partial z^k} \frac{\partial f}{\partial z^k} \right)^{\frac{p-4}{2}} \frac{1}{8} \sum_{k,j} \left[ \frac{\partial^2 f}{\partial x^j \partial x^k} \frac{\partial f}{\partial x^k} \frac{\partial f}{\partial x^j} + \frac{\partial^2 f}{\partial x^j \partial y^k} \frac{\partial f}{\partial y^k} \frac{\partial f}{\partial x^j} \right]
\]

\[
+ \frac{\partial^2 f}{\partial y^i \partial x^k} \frac{\partial f}{\partial x^k} \frac{\partial f}{\partial y^i} + \frac{\partial^2 f}{\partial y^i \partial y^k} \frac{\partial f}{\partial y^k} \frac{\partial f}{\partial y^i}
\]

\[
+ \sqrt{-1} \left( \frac{\partial^2 f}{\partial x^j \partial x^k} \frac{\partial f}{\partial x^k} \frac{\partial f}{\partial y^j} + \frac{\partial^2 f}{\partial x^j \partial y^k} \frac{\partial f}{\partial y^k} \frac{\partial f}{\partial x^j} - \frac{\partial^2 f}{\partial y^i \partial x^k} \frac{\partial f}{\partial x^k} \frac{\partial f}{\partial y^i} - \frac{\partial^2 f}{\partial y^i \partial y^k} \frac{\partial f}{\partial y^k} \frac{\partial f}{\partial y^i} \right)
\]

\[
= \frac{2}{2} \left( \sum_k \frac{\partial f}{\partial z^k} \frac{\partial f}{\partial z^k} \right)^{\frac{p-4}{2}} \sum_{j} \frac{\partial^2 f}{\partial z^j \partial \bar{z}^j}.
\]

By Remark 5.1, we get

\[
\text{Re} \Delta_p^+ f = \frac{p - 2}{2} \left( \sum_k \frac{\partial f}{\partial z^k} \frac{\partial f}{\partial z^k} \right)^{\frac{p-4}{2}} \times \frac{1}{8} \sum_{k,j} \left( \frac{\partial f}{\partial x^j} \frac{\partial^2 f}{\partial x^j \partial x^k} \frac{\partial f}{\partial x^k} + \frac{\partial f}{\partial y^j} \frac{\partial^2 f}{\partial y^j \partial y^k} \frac{\partial f}{\partial y^k} \right)
\]

\[
+ \frac{\partial f}{\partial x^j} \frac{\partial^2 f}{\partial x^j \partial y^k} \frac{\partial f}{\partial y^k} + \frac{\partial f}{\partial y^j} \frac{\partial^2 f}{\partial y^j \partial x^k} \frac{\partial f}{\partial x^k} + \frac{\partial f}{\partial y^j} \frac{\partial^2 f}{\partial y^j \partial y^k} \frac{\partial f}{\partial y^k}
\]

\[
+ \left( \sum_k \frac{\partial f}{\partial z^k} \frac{\partial f}{\partial z^k} \right)^{\frac{p-4}{2}} \sum_{j} \frac{\partial^2 f}{\partial z^j \partial \bar{z}^j}.
\]

You can see that \(\Delta_p^+ f\) is not necessarily real-valued.

**Theorem 5.3.** If \(M\) is a Kähler manifold, \(p > 1\), and \(f : M \to \mathbb{R}\) is \(C^2\), then

\[
|\nabla^+ f| = |\nabla^- f|,
\]

\[
\Delta_p^- f = \overline{\Delta_p^+ f},
\]

and

\[
\Delta_p^\ell f = 2^{\frac{p-2}{2}} (\Delta_p^+ f + \Delta_p^- f) = 2^\frac{p}{2} \text{Re} \Delta_p^+ f.
\]
Proof. Under holomorphic normal coordinates,

\[ |\nabla^+ f|^2 = g(\nabla^+ f, \nabla^+ f) = \sum_j \frac{\partial f}{\partial \bar{z}^j} \frac{\partial f}{\partial z^j}, \]

\[ |\nabla^- f|^2 = g(\nabla^- f, \nabla^- f) = \sum_j \frac{\partial f}{\partial z^j} \frac{\partial f}{\partial \bar{z}^j}, \]

so

\[ |\nabla^+ f| = |\nabla^- f|. \]

\[ \Delta^+_p f = \text{div}(|\nabla^+ f|^{p-2} \nabla^+ f) = \sum_j \frac{\partial (|\nabla^+ f|^{p-2} \frac{\partial f}{\partial \bar{z}^j})}{\partial z^j} \]

\[ = \sum_j \frac{\partial (|\nabla^+ f|^{p-2})}{\partial \bar{z}^j} \frac{\partial f}{\partial z^j} + |\nabla^+ f|^{p-2} \sum_j \frac{\partial^2 f}{\partial z^j \partial \bar{z}^j}, \quad (5.24) \]

\[ \Delta^-_p f = \text{div}(|\nabla^- f|^{p-2} \nabla^- f) = \sum_j \frac{\partial (|\nabla^- f|^{p-2} \frac{\partial f}{\partial z^j})}{\partial \bar{z}^j} \]

\[ = \sum_j \frac{\partial (|\nabla^- f|^{p-2})}{\partial z^j} \frac{\partial f}{\partial \bar{z}^j} + |\nabla^- f|^{p-2} \sum_j \frac{\partial^2 f}{\partial \bar{z}^j \partial z^j}. \quad (5.27) \]

Since \( |\nabla^+ f|^{p-2} \), \( f \) and \( \frac{\partial^2 f}{\partial z^j \partial \bar{z}^j} \) are real-valued, \( \frac{\partial^2 f}{\partial z^j \partial \bar{z}^j} = \frac{\partial^2 f}{\partial \bar{z}^j \partial z^j} \) for the fact that \( f \)

is \( C^2 \), and \( |\nabla^+ f| = |\nabla^- f| \), thus \( \Delta^-_p f = \Delta^+_p f \).

\[ |\nabla^c f|^2 \]

\[ = g(\nabla^c f, \nabla^c f) \]

\[ = g(\nabla^+ f + \nabla^- f, \nabla^+ f + \nabla^- f) \]

\[ = g(\nabla^+ f + \nabla^- f, \nabla^+ f + \nabla^- f) \]

\[ = g(\nabla^+ f, \nabla^- f) + g(\nabla^+ f, \nabla^- f) + g(\nabla^- f, \nabla^+ f). \quad (5.30) \]

Since \( \nabla^+ f \in T^{(1,0)} M, \nabla^- f \in T^{(0,1)} M, T^{(0,1)} M = \overline{T^{(1,0)} M} \), and \( g_{jk} = g_{j\bar{k}} = 0, \)
thus
\[ g(\nabla^+ f, \nabla^- f) = 0, g(\nabla^- f, \nabla^+ f) = 0, \]

and
\[ |\nabla^c f|^2 = |\nabla^+ f|^2 + |\nabla^- f|^2 = 2|\nabla^+ f|^2. \]

\[ \Delta_c f = \mathrm{div}(|\nabla^c f|^{p-2}\nabla^c f) \]
\[ = \mathrm{div}([2^{1/2}|\nabla^+ f|]^{p-2}\nabla^c f) \]
\[ = (2^{1/2})^{p-2}\mathrm{div}(|\nabla^+ f|^{p-2}\nabla^c f) \]
\[ = 2^{\frac{p-2}{2}}(\Delta_p^+ f + \Delta_p^- f) \]
\[ = 2^{\frac{p-2}{2}}(\Delta_p^+ f + \overline{\Delta_p^+ f}) \]
\[ = 2^{\frac{p-2}{2}}2\mathrm{Re}\Delta_p^+ f \]
\[ = 2^{\frac{p}{2}}\mathrm{Re}\Delta_p^+ f, \]

since \( \nabla^c f = \nabla^+ f + \nabla^- f \) and \( |\nabla^+ f| = |\nabla^- f| \).

Question to think about: What difference can the imaginary part of \( \Delta_p^+ f \) make?

**Theorem 5.4.** If \( M \) is a Kähler manifold, and \( f : M \to \mathbb{R} \) is \( C^2 \), then \( \Delta_c^+ f = \Delta_p f \).
Proof.

\[ \Delta_p f \]

\[ = 2^{\frac{p-2}{2}} (\Delta_p^+ f + \Delta_p^- f) \]  \hfill (5.37)

\[ = 2^{\frac{p-2}{2}} 2 \text{Re} \Delta_p^+ f \]  \hfill (5.38)

\[ = 2^{\frac{p}{2}} \text{Re} \Delta_p^+ f \]  \hfill (5.39)

\[ = 2^{\frac{p}{2}} \left\{ \frac{p-2}{2} \left( \sum_k \frac{\partial f}{\partial x_k} \frac{\partial f}{\partial x_k} \right)^{\frac{p-4}{2}} \times \frac{1}{8} \sum_{k,j} \left( \frac{\partial f}{\partial x_k} \frac{\partial^2 f}{\partial x_k \partial x_j} + \frac{\partial f}{\partial x_j} \frac{\partial^2 f}{\partial x_k \partial x_j} \right) \right\} \]  \hfill (5.40)

\[ = 2^{\frac{p}{2}} \left\{ \frac{p-2}{2} \left( \sum_k \frac{1}{4} \left[ \left( \frac{\partial f}{\partial x_k} \right)^2 + \left( \frac{\partial f}{\partial y_k} \right)^2 \right] \right)^{\frac{p-4}{2}} \times \frac{1}{8} \sum_{k,j} \left( \frac{\partial f}{\partial x_k} \frac{\partial f}{\partial x_j} \frac{\partial^2 f}{\partial x_k \partial x_j} \right) \right\} \]  \hfill (5.41)

\[ = 2^{\frac{p}{2}} \left\{ \frac{p-2}{2} \left( \sum_k \frac{1}{4} \left( \frac{\partial f}{\partial x_k} \right)^2 \right)^{\frac{p-4}{2}} \times \frac{1}{8} \left( \sum_{k} \frac{\partial f}{\partial x_k} \frac{\partial f}{\partial x_k} \frac{\partial^2 f}{\partial x_k^2} \right) \right\} \]  \hfill (5.42)

\[ = 2^{\frac{p}{2}} \left\{ \frac{p-2}{2} \left( \sum_k \frac{1}{4} \left( \frac{\partial f}{\partial x_k} \right)^2 \right)^{\frac{p-4}{2}} \times \frac{1}{8} \left( \sum_{k} \frac{\partial f}{\partial x_k} \frac{\partial f}{\partial x_k} \frac{\partial^2 f}{\partial x_k^2} \right) \right\} \]  \hfill (5.43)

\[ = p-2 \left( \sum_{k} \frac{\partial f}{\partial x_k} \right)^2 \left( \sum_{k} \frac{\partial f}{\partial x_k} \frac{\partial^2 f}{\partial x_k^2} \right) \]  \hfill (5.44)

\[ = \Delta_p f. \]
Chapter 6

Applications

There can be many applications of the holomorphic $p$-Laplacian $\Delta^+_p f$, the antiholomorphic $p$-Laplacian $\Delta^-_p f$, and the complex $p$-Laplacian $\Delta^c_p f$. Here, we give one example of their applications.

Throughout this chapter, we let $p > 1$, $\{r_j\}$ be an unbounded sequence of strictly increasing positive numbers, $M$ be a complete noncompact Kähler manifold, and $f$ be a real-valued function on $M$.

Theorems 6.1-6.5 are from [18].

**Theorem 6.1.** (Theorem 2.1, [18]). Let $f$ be a nonconstant $C^2$ function. Suppose that $q > p - 1$ and $f \Delta_p f \geq 0$. Then $f$ has $p$-infinite growth. That is, for every $x_0 \in M$,

$$\lim_{r \to \infty} \frac{1}{r^p} \int_{B(x_0,r)} |f|^q dv = \infty.$$ 

**Theorem 6.2.** (Theorem 2.2, [18]). Let $f$ be a nonconstant $C^2$ function. Suppose that $q > p - 1$ and $f \Delta_p f \geq 0$. Then $f$ has $p$-severe growth. That is, for every $x_0 \in M$, there exists a constant $a > 0$ such that for every unbounded strictly increasing sequence $\{r_j\}_{j=0}^\infty$, and every $r_{l_0} > a$,

$$\sum_{j=l_0}^\infty \frac{(r_{j+1} - r_j)^p}{\int_{B(x_0;r_{j+1})\setminus B(x_0;r_j)} |f|^q dv}^{\frac{1}{p-1}} < \infty.$$ 

**Theorem 6.3.** (Theorem 2.3, [18]). Let $f$ be a nonconstant $C^2$ function. Suppose that $q > p - 1$ and $f \Delta_p f \geq 0$. Then $f$ has $p$-acute growth. That is, for
every \(x_0 \in M\), there exists a constant \(a > 0\) such that

\[
\int_a^\infty \left( \frac{1}{\int_{\partial B(x_0,r)} |f|^q dv} \right)^{\frac{1}{p-1}} dr < \infty.
\]

**Theorem 6.4.** (Theorem 2.4, [18]). Let \(f\) be a nonconstant \(C^2\) function. Suppose that \(q > p - 1\) and \(f \Delta_p f \geq 0\). Then \(f\) has \(p\)-immoderate growth. That is, for every \(x_0 \in M\), and every \(F(r) \in \mathcal{F}\) where \(\mathcal{F} = \{F : [a, \infty) \to (0, \infty)] \int_a^\infty \frac{dr}{r F(r)} = \infty \) for some \(a \geq 0\),

\[
\limsup_{r \to \infty} \frac{1}{r^p F^{p-1}(r)} \int_{B(x_0,r)} |f|^q dv = \infty.
\]

**Theorem 6.5.** (Theorem 2.5, [18]). Let \(f\) be a nonconstant \(C^2\) function. Suppose that \(q > p - 1\) and \(f \Delta_p f \geq 0\). Then \(f\) has \(p\)-large growth. That is, for every \(x_0 \in M\), there exists a constant \(a > 0\), such that

\[
\int_a^\infty \left( \frac{r}{\int_{B(x_0,r)} |f|^q dv} \right)^{\frac{1}{p-1}} dr < \infty.
\]

Theorems 6.1-6.5 can be condensed into one theorem because of the following two theorems:

**Theorem 6.6.** (Proposition 2.1, [18]). \(f\) is \(p\)-immoderate if and only if \(f\) is \(p\)-large. Therefore, Theorems 6.4 and 6.5 are equivalent.

**Theorem 6.7.** (Proposition 2.2. (i), [18]). If \(f\) is \(p\)-acute, then \(f\) is \(p\)-severe. If \(f\) is \(p\)-severe, then \(f\) is \(p\)-immoderate and \(p\)-large.

So Theorems 6.1-6.5 can be condensed into the following theorem:

**Theorem 6.8.** Let \(f\) be a nonconstant \(C^2\) function. Suppose that \(q > p - 1\) and \(f \Delta_p f \geq 0\). Then \(f\) has \(p\)-infinite growth and \(p\)-acute growth.
Theorem 6.9 is an application of the complex $p$-Laplacian of $f$.

**Theorem 6.9.** Let $f$ be a nonconstant $C^2$ function. Suppose that $q > p - 1$ and $f \Delta^c_p f \geq 0$. Then $f$ has $p$-infinite growth and $p$-acute growth.

Theorem 6.10 is an application of the holomorphic $p$-Laplacian of $f$.

**Theorem 6.10.** Let $f$ be a nonconstant $C^2$ function. Suppose that $q > p - 1$ and $f \Re \Delta^+_p f \geq 0$. Then $f$ has $p$-infinite growth and $p$-acute growth.

Theorem 6.11 is an application of the antiholomorphic $p$-Laplacian of $f$.

**Theorem 6.11.** Let $f$ be a nonconstant $C^2$ function. Suppose that $q > p - 1$ and $f \Re \Delta^-_p f \geq 0$. Then $f$ has $p$-infinite growth and $p$-acute growth.

**Proof of Theorems 6.9-6.11.** By Theorem 5.4 in Chapter 5, we get $\Delta^c_p f = \Delta_p f$. By Theorem 5.3 in Chapter 5, we get $\Delta^c_p f = 2 \frac{q}{2} \Re \Delta^+_p f = 2 \frac{q}{2} \Re \Delta^-_p f$. By Theorems 6.8, immediately we can get Theorems 6.9-6.11.
Bibliography


