DEFAULT THEORY: AN ALTERNATIVE APPROACH

By

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1984

Submitted to the Faculty of the
Graduate College of the
Oklahoma State University
in partial fulfillment of
the requirements for
the Degree of
MASTER OF SCIENCE
July, 1993

OKLAHOMA STATE UNIVERSITY

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ACKNOWLEDGMENTS

Alhamdulillah. Thanks to God.

I would like to extend my appreciations and thanks to many people without whom this thesis would have never existed.

Dr. Blayne E. Mayfield, the head of my thesis adviser, provided invaluable guidance, support, critiques, time, patience, and advice from the early until the late stages of the works. Dr. Paul Benjamin, my advisory committee member, reviewed and examined the proofs of the theorems in the report and gave some suggestions. Dr. John P. Chandler, my advisory committee member and my academic adviser, gave some comments on the report. Dr. D. Miller allocated his time in the very early stages of the work.

Most of the references I needed have been provided by the Interlibrary Loan Department of Edmond Low Library, Oklahoma State University.

I would like to thank my host mother, Mrs. L. Higginbotham, and friends of mine, R. Retnowulan, S. Silvey, I. Firdaus, C. Marpaung, I.B.M.P. Jandhana, T. Rahardjo, E. Purba, D. Gumilang, etc., who gave me support, motivation and material. Also, thanks Ms. F. Griffin for proofreading.

Also, I would like to thank the Government of Indonesia, which provided funds for my graduate study, through the Minister for Research and Technology/the Agency for the Assessment and Application of Technology in the framework of the implementation of the Science and Technology for Industrial Development Program. Special thanks to Prof. Dr. Ing. B.J. Habibie, Minister for Research and Technology, Dr. Ing. Wardiman Djojonegoro, the Chairman of the Science and Technology for Industrial Development Program, Prof. Dr. S. Sapiie, Senior EVP Engineering and Technology of Indonesian Aircraft Industries, Inc., Mrs. Ir. Ina Juniarti, Vice President of the Computing Center of Indonesian Aircraft Industries, Inc., and Mr. Ir. Muriawan A. Kadir, the head of Research and Development of the Computing Center of Indonesian Aircraft Industries, Inc.

Finally, I would like to thank my family, my wife Sri R.M. Andayani and my son Hasyantha A. Sinaga, for their generous love, support, patience, and encouragement during my graduate study, to whom this thesis is dedicated. Special thanks are due to my parents, my late father-in-law, and my mother-in-law for their love, patience, and support. At last, thanks to my sister, Nurati Sinaga.

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LIST OF SYMBOLS

Symbols	Meanings
$\in$	a member of
$\notin$	not a member of
$\subseteq$	a subset of
$\not\subset$	not a subset of
$\supseteq$	a superset of
$\exists$	an existential quantifier
$\forall$	a universal quantifier
$\neg$	a negation
$\wedge$	a conjunction
$\vee$	a disjunction
$\rightarrow$	implication
$\leftrightarrow$	equivalence
τ	verum
$\perp$	falsum
$\cup$	a union of
▒	end of definitions, theorems, examples, or lemmas

CHAPTER I

INTRODUCTION

1.1 Background

Much current research deals with incomplete knowledge using classical logics such as First Order Logics. One problem with classical logics is their monotonicity properties. In order to deal with this problem, there have been several proposals for augmenting classical logics to allow nonmonotonic reasoning. Some of these proposals are Nonmonotonic Logics [26], Default Logics [39], Circumscription [30], Autoepistemic Logics [31], Probabilistic Logics [33]. Default Logics, developed by Reiter in 1980, are based on an extension of classical first order logics. In this approach, first order logics are augmented with plausible inference rules, called defaults. However, these logics have several disadvantages. In order to remedy these disadvantages, several authors conducted some research to improve the original Reiter default logics for example [1, 2, 5, 8, 18, 24, 47].

The motivation behind the development of classical logics some time ago such as first order logics was to put mathematical reasoning on a precise, formal foundation. The main reason was to establish a mathematical reason for us to deal with. However, such reasoning is based on precise and complete knowledge. In real life, humans rarely have precise and complete knowledge.

The main problem of first order logics in dealing with incomplete knowledge is its monotonicity property. Monotonicity properties can be summed up as adding new information to a system employing first order logics never shrinking the system. For example, if a fact P is derivable from a set of premises S, then P is also derivable from each super set of S. Therefore, adding a new fact to the system never invalidates already derivable facts.

Dealing with incomplete knowledge problems requires another approach. Incomplete knowledge reasoning is usually not monoton. In many situations, humans are forced to draw conclusions even though they have little information. The classic example is the flying ability of birds. If we are given information that Tweety is a bird, we tend to conclude that Tweety can fly since a bird typically flies. If further information is given that Tweety is a penguin, we certainly withdraw our former conclusion, but without withdrawing any of our former premises. However, we still have the ability to deduce that other birds can fly. Such forms of reasoning in which the system has the ability to invalidate conclusions whenever further information is given are called nonmonotonic.

1.2 Nonmonotonic Logics

Since monotonicity properties are fundamental to the concept of first order logics, a new approach should be taken to deal with nonmonotonicity. Several logics have been developed to deal with nonmonotonicity properties. Those logics have some things in common. They are based on deriving conclusions in the absence of some information to the contrary. These kinds of inferences are called plausible inferences. Furthermore, when facts that are contrary to facts derived using plausible inferences are found and added to a system, the only facts "removed" from the systems are those that are contradictory. Other facts are not affected.

1.2.1 Reiter Default Logics

Default Logics were initially developed by Reiter [39]. The purpose of Default Logics is to extend first order logics to deal with incomplete knowledge. Furthermore, these logics have nonmonotonicity properties.

Default logics are more attractive because of their simplicity and the naturalness of their underlying ideas. This term is related to the process of inducing conclusions based upon patterns of inferences of the form "in the absence of any information to the contrary, assume . . ." [38]. These patterns represent forms of plausible inferences. These patterns are needed to reason with incomplete knowledge about a world being modeled.

The plausible inference rules in a default logic are of the form

$$\frac{A:B}{C}.$$

These forms are called defaults, noted δ. Such rules are read as, "If A is provable and it is consistent to assume B, then infer C." These plausible inference rules may be applied by inferring C in absence of information refuting B, that is, if we do not have any evidence to believe $\neg B$. Therefore, using these kinds of forms of plausible inference rules, it is legitimate to infer C for the given A.

A, B, and C in a default δ are closed well-formed formulae. They are called the prerequisite, the justification, and the consequent of the default, respectively. If C is logically equivalent with B, default d is called normal; otherwise default δ is called non-normal. If C can be logically inferred from B, default δ is also called semi-normal.

Reiter and Criscuolo [42] investigated some phenomena that can be represented using defaults. They conclude that typical facts can be represented using normal defaults [42]. For example, the fact "typically, birds fly" can be represented using a normal default d of the following,

$$\frac{\text{bird}(x){:}\text{fly}(x)}{\text{fly}(x)}.$$

This default is read as, "If a thing is a bird, and it is consistent to believe that it can fly, infer that it can fly." Therefore, in the absence of information to the contrary, if Tweety is a bird, then it is legitimate to conclude that Tweety can fly. Furthermore, when we find that Tweety is a penguin, and penguins cannot fly, then the plausible inference rule cannot be applied. So, we cannot infer that Tweety can fly.

Formally, a default logic consists of a set of defaults and a set of closed well-formed formulae of first order logics. The set of well-formed formulae consists of facts known to be believed and called hard facts. A default logic is called normal if all defaults are normal; otherwise it is called non-normal.

A default logic may result in sets of acceptable beliefs called extensions. An extension for a default logic is defined using a fixed point of an operator. The operator is defined in terms of three conditions: deductively closed in terms of first order logic; a superset of the hard fact set; and plausible inference using defaults. Moreover, it corresponds to a maximal consistent augmentation; i.e., no additional defaults can be applied without violating the consistency of the extension. Since several extensions may satisfy the three conditions and a maximal consistency, a consequence of a default logic is any well-formed formula that belongs to an extension. However, there is no mechanism for choosing among extensions for a default logic, if they exist.

Reiter found a class of default logics that is very attractive [39]. These are normal default logics. He found that an extension for a normal default theory is guaranteed to exist. Furthermore, he also proved that these logics are semi-monoton; that is, adding defaults to default theory may enlarge its extensions but adding hard facts may not enlarge extensions.

However, default logics have some disadvantages, such as lacking an extension. Most arbitrary default logics do not have an extension, for example, non-normal default logics.

A theory that does not have an extension represents the worst form of inconsistency since it does not define the notion of theorems or consequents. Furthermore, an extension for a default logic is a set of beliefs that one may hold due to incomplete knowledge of the world being modeled. So, an extension can be viewed as a solution to an incomplete knowledge problem. Therefore, an extension for default logics is essential.

Reiter [39] conjectured that all naturally occurring default logics are normal. However, as Reiter and Criscuolo [42] pointed out, this guess seems not to be true. They found that there are natural non-normal default theories. Additionally, most of these theories do not have any extensions.

Several variations to Reiter default logics have been introduced such as Free Default Logics [2], Conditional Default Logics [8], Modified Default Logics [24], and Cumulative Default Logics [3].

1.2.2 Lukaszewicz Default Logics

Lukaszewicz developed another approach to building an extension [24]. The main idea is to make an arbitrary default logic have an extension. Building an extension, called a modified extension for default logics in the Lukaszewicz approach is formally more complex than in the Reiter approach. Lukaszewicz defined a modified extension for a default logic using two operators. The first two conditions of one operator for the Lukaszewicz versions are the same as for Reiter's. The main difference is in the third condition, that is, in applying a default. The second operator is used to control an application of a default to build the extension.

Lukaszewicz also proved that modified extensions for arbitrary default theories have similar properties to the Reiter normal default theories. Two of them are existence of extensions and semi-monotonicity.

1.2.3 Disjunctive Fact Problems

Lukaszewicz proves that the new version has at least one modified extension. Furthermore, he also shows that it has semi-monotonicity properties.

There are some problems in using both the Reiter and the Lukaszewicz approaches to build extensions. They fail to deal with disjunctive facts concerning prerequisites and justifications of defaults. Disjunctive facts consist of two or more facts which we do not have sufficient evidence to believe except in their disjunctive form. The first problem is disjunctive facts concerning prerequisites. For example, suppose that we are given two birds and one of them has broken wings. This example is formalized using a default logic with hard facts "two birds" and "one of them has broken wings" and a default "typically, a bird can fly in absence of evidence that it cannot fly and its wings are not broken," that is,

$$\frac{\mathrm{bird}(x):\mathrm{fly}(x) \land \neg \mathrm{broken\text{-}wing}(x)}{\mathrm{fly}(x)}.$$

This problem has both an extension and a modified extension but both the extension and the modified extension contain "both birds can fly." This is counter-intuitive since we know that one of them has broken wings.

Another problem is that both Reiter and Lukaszewicz default logics cannot deal with default logics that have disjunctive hard facts in which the applicability of defaults is blocked due to no evidence to believe any of the prerequisites of the defaults. These defaults have the same justification. For example, suppose that we believe that either "John was born in the US," or "John was naturalized by the US government" but we do not have sufficient evidence to believe either of them.

Formalizing this problem with two prototypical facts, "Typically, if a person is born in the US, the person is a US citizen" and "Typically, if a person is naturalized by the US, the person is a US citizen" using defaults,

$$\frac{\text{born - in - US(john):US - citizen(john)}}{\text{US - citizen(john)}}$$

and

$$\frac{\text{naturalized - by - US(john):US - citizen(john)}}{\text{US - citizen(john)}},$$

we cannot infer that "John is a US citizen," using either the Reiter or the Lukaszewicz approaches.

1.2.4 Related Works

Much research has been done in order to improve the original Reiter work, for example Free Default Logics [2], Conditional Default Logics [8], Ordered Default Theory [14], Possibility Default Reasoning [11], Cumulative Default Logics [3], and Disjunctive Defaults [18]. Free Default Logics developed by Besnard supposedly deal with modus tollens. These logics let the system admit inferences similar to contra-positive in First Order Logics, but they are accomplished by using defaults, free defaults to be precise. However, these logics have undesirable features, namely deriving facts without supporting facts.

In 1987, Etherington conducted research in order to find a sufficient condition for semi-normal default theories to have extensions [14]. First, he defined an order to check whether "circularities" exist within semi-normal default theories. He found that semi-normal default theories have at least one extension if they are ordered; i.e., no circularity occurs. He also presented a procedure to generate extensions based on a series of successive approximations. As Reiter showed, Etherington also found that this procedure may or may not converge to an extension. Furthermore, this procedure may not be computable due to first order provability. However, he proved that the procedure will converge to an extension for a class of default theories [14].

In 1991, Chen figured out some disadvantages of the Etherington approach [5]. He found that the Etherington original work for orderedness of a default theory depends on a particular representation of both the hard fact set and the default set. To remedy these problems, first he defined a minimal form, and then he defined a new approach to order a semi-normal default theory. In his approach, a default theory is represented using a "canonical" form. A default theory $\mathcal{D}' = (\mathcal{F}', \Delta')$ is a canonical form of a default theory $\mathcal{D} = (\mathcal{F}, \Delta)$ if its hard fact set is represented using a minimal form, and if for each default $\delta = \frac{A:B}{C} \in \Delta$, there is one $\delta' = \frac{A':B'}{C'} \in \Delta'$, where A', B', and C' are minimal forms of A, B, and C, respectively. For semi-normal default theory which is in the canonical form, he presented a well-defined order relation similar to the Etherington approach. He also proved that every semi-normal default theory in the canonical form has an extension. Furthermore, he also provided an algorithm that converges to generate an extension for any ordered, finite, semi-normal default theory.

Brewka addressed the cumulativity of default theories [3]. The Reiter Default Logics are usually not cumulative. Brewka developed logics called Cumulative Default Logics in order make default theories cumulative. The Brewka approach dealt with assertions instead of simple well-formed formulae. (An assertion is a pair of well-formed formulae.) In doing so, he defined the notion of derivability using assertions as an extension of first order derivability. He found that the new approach has semi-monotonicity properties. In addition, he also addressed inconsistency between justifications of defaults that may lead to a counter-intuitive result [3].

In 1991, Gelfond et al. developed a totally new logic, called Disjunctive

Default Logics, to deal with "disjunctive information" [18]. The new approach not only changes a default theory, the way of deriving extensions, but also changes the way of representing defaults. The motivation behind this logic is that a difficulty was found in attempts to use defaults in the presence of disjunctive information as observed by Lin and Shoham in [22]. To handle disjunctive information, Gelfond et. al. used a disjunctive default of the form

$$\frac{A:B}{C_1|C_2|\ \ldots\ |C_n},$$

instead of the Reiter simple default. A disjunctive default theory is a set of disjunctive defaults. They defined an extension for disjunctive default theories as follows. A set of sentences E is an extension for a disjunctive default theory $\mathcal{D}$ "if it is one of the minimal deductively closed sets of sentences E' satisfying the condition: For any ground instance of any [disjunctive] default in $\mathcal{D}$," if $A \in$ E' and $\neg B \notin$ E, then $C_i \in$ E' for some i, where $1 \leq i \leq n$. They demonstrated that a related disjunctive default theory has a Reiter default theory. However, an extension for a disjunctive default theory is not generally equivalent with an extension for the related Reiter default theory. In other words, it is possible that a disjunctive default theory has an extension but the related Reiter default theory does not have an extension. So, the classes of extensions for both approaches are not similar. In addition, there is an unintended result using this new approach as we see in Chapter 3.

Besides default theories, there are several theories to deal with nonmonotonicity properties. One of them is Nonmonotonic Logics [26]. In these logics, NMLs for short, a language of first order theory is augmented with a modal operator $\mathcal{M}$ to capture the notion of nonmonotonicity. It is read as "is consistent." So, an expression $\mathcal{M}A$ is read as "A is consistent." For example, an expression

$$\forall x\ \text{bird}(x) \wedge \mathcal{M}\text{fly}(x) \rightarrow \text{fly}(x),$$

is read as "For all x, if x is a bird and if the fact that x is able to fly is consistent with everything else that is believed, then we infer that x is able to fly." One of the main differences between NMLs and Default Logics is that in Default Logics, such nonmonotonic expressions, called defaults, are rules of inferences, whereas in NMLs, such a nonmonotonic expression is in the language. McDermott and Doyle defined an extension for an NML, i.e., a set of all theorems, as the intersection of all its fixed points. That is, if $\mathcal{N}$ is an NML, then S is a fixed point of $\mathcal{N}$ if

$$S = \mathrm{Th}(\mathcal{N} \cup \{\mathcal{M}A \mid \neg A \notin S\}).$$

The intuition behind this is to capture the notion that if $\neg A$ is not derivable, then infer $\mathcal{M}A$ which intendedly means "A is consistent." Some problems can be solved using NMLs. However, this approach has some disadvantages. One of the main problems is that the operator $\mathcal{M}$ can be miscarried to capture the intuitive notion of consistency. For example, an NML $\mathcal{N} = \{\mathcal{M}P, \neg P\}$ is consistent [40].

To remedy these problems, McDermott attempted to develop several stronger approaches. These approaches are based on various standard modal logics as opposed to the previous approach which is based on a first order logic. Unfortunately, these attempts seem to result in a weak characterization for operator $\mathcal{M}$ based on T and S4 and to result in a monotonic logic based on S5 [40].

1.3 Objectives

The main objective of this research is to develop a new approach to solving problems with disjunctive facts concerning prerequisite and justification of defaults. To solve disjunctive fact problems concerning justifications, we will develop two kinds of sets of beliefs. The first is a consistent set consisting of (1) hard facts, (2) all justifications and conclusions of applicable defaults, and (3) all derivable facts in terms of first order logics. It is an assumption set for applying defaults and is called an assumption set. The second set is a subset of the first set but without justifications. This set is related to a Reiter's belief set, that is, an extension containing all facts that are to be believed. Assuming that the set of hard facts is consistent, then both an assumption set and extension are consistent.

To solve disjunctive hard fact problems concerning prerequisites, we will build sets of equivalence classes of defaults having the same justification. For each set, we associate a set whose elements are disjunctive forms consisting of prerequisites of the defaults. The applicability of defaults is associated to the sets. For example, a default can be applied if its related set is not disjoint with an extension. So, applicability of defaults does not depend on individual defaults but on the sets of classes built from defaults having the same justifications. Therefore, their applicability is not restricted to their prerequisites, but to the sets. Hence, these defaults are more applicable than to those of both the Reiter and the Lukaszewicz approaches. This new approach can be achieved by strictly following the above development of the two sets: an assumption set and an extension.

In adition, a survey of complexity analysis of the new approach will be conducted.

The main benefit of the new approach is that every default logic has an extension. Furthermore, the new logics have semi-monotonicity properties. In addition, the logics can deal with disjunctive fact problems adequately.

CHAPTER II

OVERVIEW OF DEFAULT THEORIES

2.1 Introduction

Default theories are nonmonotonic theories developed by Reiter. In general, the purpose of default theories is to enhance derivability of first order theories to deal with incomplete knowledge. These theories extend the logical language by using defaults as additional inference rules inducing the so-called extensions of classical logical theories. The defaults are used to extend a logical knowledge base to belief sets, called extensions, containing non-logically derivable facts in terms of the first order theories from the "known-to-be-true" knowledge base [39].

Default theories introduce kinds of non-logical derivable inference rules called defaults. Such inference rules, sometimes called plausible inference rules, are imbedded in first order theories to deal with incomplete knowledge. For example, the problem "the ability of birds to fly" is represented using a form

$$\frac{\mathrm{bird}(x):\mathrm{fly}(x)}{\mathrm{fly}(x)}.$$

which is to be interpreted as: "If a thing is a bird and it is consistent to believe that it can fly, we can infer that it can fly." The phrase "it is consistent to believe that" is read as "there is no proof to the contrary." Therefore, if Tweety is a bird and there is no proof in the knowledge base that Tweety cannot fly, the inference rule can be applied by inferring that it can fly [39]. However, when the fact "Tweety cannot fly" is derivable, for example by adding facts "Tweety is a penguin" and "penguins cannot fly" to the system, the plausible inference rule cannot be applied any more. So, the fact "Tweety can fly" is not derivable further. This means that adding facts to the knowledge base may invalidate some inferred facts using defaults; i.e., this logic is not monoton.

Reiter finds an interesting class of default theories, called normal default theories. He proves that every normal default theory has at least one extension. Furthermore, he also shows that normal default theories have a semi-monotonicity property. That is, adding some defaults to the system may augment extensions, but it never shrinks the previous extensions.

In general, default theories, for example non-normal default theories, need not have an extension. In order to make an arbitrary default theory have an extension, Lukaszewicz restricts the applicability of defaults [24]. He proposes an alternative approach to building an extension so that every default theory always has an extension. He calls it a modified extension. He proves that modified extensions have semi-monotonicity properties.

In the next two sections we will present and review both the Reiter and the Lukaszewicz versions. Materials found in the next two sections are mostly taken from the original Reiter and Lukaszewicz works [24, 39]. Furthermore, all proofs to the theorems can be found in the references.

2.2 Reiter Default Theories

In this section, we present some definitions used to develop the formal default theories in Reiter's original works. As Reiter found, a special class of default theories will be identified. Furthermore, we also present some properties that belong to this class such as the existence of extensions and semi-monotonicity.

2.2.1 Formal Definitions

As previously mentioned, the purpose of default theories is to extend the derivability of first order theories. In order to do that, the language used in default theories is a first order language. All conventions in the first order language are used. For example, plain capital characters like P and Q belong to the first order language. They are well-formed formulae. Meanwhile, italic capital characters like *A* and *B* belong to "meta-variables," denoting well-formed formulae. Furthermore, terms in the first order theory are referred to, such as closed well-formed formulae, subformulae, logical validity, logical implication, consistency. In addition, $Th_{\mathcal{L}}(S)$ denotes the well-formed formulae in a first order language $\mathcal{L}$ that are consequences of or inferable from the set S. See the appendix for more detail.

Definition: Default theory language

A default theory language is any first order language $\mathcal{L}$. All terms such as terms and well-formed formulae are included in a default theory language.

In default theories, defaults are used to extend non-logical derivable inference rules. Defaults are defined by the following.

Definition: Defaults

Defaults are of the forms:

$$\delta = \frac{A(x) : B(x)}{C(x)},$$

where $A(x)$, $B(x)$, and $C(x)$ are closed well-formed formulae of a first order language $\mathcal{L}$, called a prerequisite, a justification, and a consequent of default δ, respectively.

A default $\delta = \frac{A(\mathrm{x}):B(\mathrm{x})}{C(\mathrm{x})}$ is interpreted as: "If A(x) is believed and it is consistent to believe B(x), then infer C(x)."

Example 2.2.1:

The ability of birds to fly can be represented using a default δ, where

$$\delta = \frac{\mathrm{bird(Tweety)}:\mathrm{fly(Tweety)}}{\mathrm{fly(tweety)}}.$$

The default δ is read as: "If Tweety is a bird and it is consistent to believe that it can fly, then infer that it can fly."

In the definition of defaults, their prerequisites, justifications, and consequents are closed well-formed formulae. For convenience, we may need to "classify" some defaults into a set of the same nature in forms of defaults. For example, if we have some birds, we may want to abbreviate the default example above as a group of defaults of the same nature as the following. Suppose that there are some birds: Tweety, Clyde, etc. A default set for these birds is:

$$\{\delta_1 = \frac{\mathrm{bird(Tweety)}:\mathrm{fly(Tweety)}}{\mathrm{fly(Tweety)}}, \delta_2 = \frac{\mathrm{bird(Clyde)}:\mathrm{fly(Clyde)}}{\mathrm{fly(Clyde)}}, \ldots\}.$$

These defaults can be represented using a set of defaults $\bar{\delta}$,

$$\bar{\delta} = \{\frac{\mathrm{bird(x)}:\mathrm{fly(x)}}{\mathrm{fly(x)}} \mid \mathrm{x = Tweety, x = Clyde, \ldots}\},$$

where x is a symbol that does not belong to the first order language. Of course, the symbol x does not appear in any well-formed formulae A, B, and C. For short, this set of defaults is abbreviated using,

$$\bar{\delta} = \frac{\mathrm{bird(x)}:\mathrm{fly(x)}}{\mathrm{fly(x)}}$$

and we will write $\bar{\delta}$ as a simple δ.

From now on, by "a default" we mean a set of defaults unless otherwise specified. However, when applying a default, the resulting well-formed formula is closed.

In general, defaults can be classified into two classes: normal defaults and non-normal defaults. A default δ is called normal if its justification is logically equivalent with its consequent, that is, $C(x) = B(x)$. Otherwise, it is called non-normal. For example, the default to represent the ability of flying birds above is normal. A subclass of non-normal defaults can also be identified; if a consequent of a default can be logically implied from its prerequisite, that is, $B(x)$ logically implies $C(x)$. These defaults are called semi-normal.

Example 2.2.2: Non-normal defaults

A default δ_1, where

$$\delta_1 = \frac{P:Q}{R},$$

is non-normal, whereas a default δ_2, where

$$\delta_2 = \frac{\mathrm{adult}(x):\mathrm{student}(x) \wedge \neg\mathrm{married}(x)}{\neg\mathrm{married}(x)},$$

is semi-normal.

According to Reiter and Criscuolo [42], typical facts such as "prototypical facts" can be represented using normal defaults. For example, facts such as "Typically, birds fly" known as the ability of birds to fly can be represented using a normal default δ, where

$$\delta = \frac{\mathrm{bird}(x):\mathrm{fly}(x)}{\mathrm{fly}(x)}.$$

A default theory consists of two sets. The first is a set of closed well-formed formulae $\mathcal{F}$ of a first order language $\mathcal{L}$. This set describes all facts that are known about the world being modeled. It is called a hard fact set and is intended to represent the "known-to-be-believed" knowledge. The second is a set of countable, possibly finite, defaults Δ to be used to infer facts beyond first order derivability. It is intended to express potential beliefs. The prerequisites, justifications, and consequents of all defaults in the set Δ are closed well-formed formulae of the same first order language $\mathcal{L}$. The following is the definition of a default theory.

Definition: Default theories

A default theory $\mathcal{D}$ *is a pair* $(\mathcal{F}, \Delta)$ *of a set of closed well-formed formulae* $\mathcal{F}$ *of first order language* $\mathcal{L}$ *and a countable, possibly finite, set of defaults* Δ *whose prerequisites, justifications, and consequents are well-formed formulae in the same first order language* $\mathcal{L}$.

A default theory is called normal if and only if its default set consists only of normal defaults. Otherwise, it is non-normal. Normal default theories are an important class of default theories. They have some interesting properties, as we see later on.

In the Reiter approach, a default theory may result in a set. This set consists of all facts derivable using first order theories and defaults. Reiter defines this set in terms of a fixed point of a certain operator based on three conditions. Furthermore, it is the smallest set satisfying the three conditions. The set is called an extension for a default theory.

An extension is defined as follows.

Definition: Extensions for default theories [39]

Let $\mathcal{D} = (\mathcal{F}, \Delta)$ *be a default theory and* S *be a closed well-formed formula set of a first order language* $\mathcal{L}$. *Suppose that* $\Gamma(\mathrm{S})$ *denotes the smallest set* *satisfying the three following conditions*:

D1 $\mathfrak{F} \subseteq \Gamma(S)$;

D2 $Th_{\mathfrak{L}}(\Gamma(S)) = \Gamma(S)$; *and*

D3 if $\delta = \dfrac{A(x){:}B(x)}{C(x)} \in \Delta$, $A(x) \in \Gamma(S)$, *and* $\neg B(x) \notin S$, *then* $C(x) \in \Gamma(S)$.

A closed well-formed formula set $\mathfrak{E}$ *is an extension for* $\mathfrak{D}$ *if and only if* $\mathfrak{E} = \Gamma(\mathfrak{E})$. ∎

This definition provides criteria for a closed well-formed formula set $\mathfrak{E}$ to be an extension. The first criterion is that what is known about the world being modeled is in any extension. The second criterion is that any extension is closed under first order theories; i.e., it is deductively closed or $Th_{\mathfrak{L}}(\mathfrak{E}) = \mathfrak{E}$. The third is related to applicability of a default δ. It says that if A is in an extension $\mathfrak{E}$ and it is consistent to believe B (i.e., $\neg B \notin \mathfrak{E}$), then C must be in the extension $\mathfrak{E}$. In this case, the default δ is called applicable. In addition, the smallness of the extensions ensures that no fact is added into any extension for no reason, i.e., without violating the three conditions.

Example 2.2.3:

Let $\mathfrak{D} = (\mathfrak{F}, \Delta = \{\delta\})$ be a default theory, where

$\mathfrak{F} = \{\text{bird(Tweety)}\}$,

and

$$\delta = \frac{\text{bird(x):fly(x)}}{\text{fly(x)}}.$$

That is, all we know is that "Tweety is a bird," and we have only one default, a typical fact, that is, "if a thing is a bird and it is consistent to believe that it can fly, infer that it can fly." Therefore, an extension for $\mathfrak{D}$ must contain "bird(Tweety)." Furthermore, since δ can be applied, it also contains "fly(Tweety)." An extension for the default theory $\mathfrak{D}$ is

$$\mathcal{E} = Th_{\mathcal{L}}(\mathfrak{F} \cup \{fly(Tweety)\}).$$

Now, we prove that $\mathcal{E}$ is the only extension for $\mathcal{D}$. Let E be another extension for $\mathcal{D}$. Then $\mathfrak{F} \subseteq E$, and fly(Tweety) $\in$ E since δ is applicable. Furthermore, no other default is applicable. So, E = $\mathcal{E}$.

$\blacksquare$

Furthermore, a fact that is believed, is in an extension, if and only if it is either a hard fact, a consequent of applicable defaults, or monotonically derivable from the set of hard facts and consequents of applicable defaults. In case of an empty default set, an extension is the first order derivability. That is, $\mathcal{E} = Th_{\mathcal{L}}(\mathfrak{F})$ is an extension for default theory $(\mathfrak{F}, \Delta)$.

The idea of the notion of extensions for default theories is that defaults may result in some facts that cannot be derived in terms of first order theory due to incomplete knowledge. In the definition above, there is no mechanism to determine which defaults should be applied first. The applications of defaults in different orders may result in different extensions. Therefore, an extension for a default theory may not be unique. Furthermore, it will be interpreted as an acceptable set of beliefs that one may believe about the incomplete knowledge world being modeled [39].

Example 2.2.4:

Let $\mathcal{D} = (\mathfrak{F}, \Delta = \{\delta_1, \delta_2\})$ be a default theory, where

$\mathfrak{F}$ = {student(john), adult(john)},

$$\delta_1 = \frac{student(x){:}\neg married(x)}{\neg married(x)},$$

and

$$\delta_2 = \frac{adult(x){:}married(x)}{married(x)}.$$

That is, the facts are "John is a student" and "John is also an adult" and defaults are

"Typically, students are not married" and "Typically, adults are married." It can be shown that this default has two extensions, that is

$$\mathfrak{E}_1 = Th_{\mathfrak{L}}(\mathfrak{F} \cup \{\neg married(john)\}),$$

and

$$\mathfrak{E}_2 = Th_{\mathfrak{L}}(\mathfrak{F} \cup \{married(john)\}).$$

In $\mathfrak{E}_1$, default δ_2 is not applicable; i.e., it is not consistent to believe "married(john)" since "¬married(john)" is derivable in $\mathfrak{E}_1$. A similar reason is also applied to $\mathfrak{E}_2$.

∎

2.2.2 General Properties of Default Theories

Reiter presents a way of characterizing a closed well-formed formula set to be an extension for a default theory. It can be used to characterize an extension. However, this characterization cannot be used to construct an extension. It is given in the following theorem.

Theorem 2.2.1: Characterization of Extensions [39]

Let $\mathfrak{D} = (\mathfrak{F}, \Delta)$ *be a default theory and* $\mathfrak{E}$ *be a set of closed well-formed formulae of a first order language* $\mathfrak{L}$. *Suppose that* $E_0, E_1, E_2, \ldots$ *is a sequence of closed well-formed formula sets such that*

$$E_0 = \mathfrak{F},$$

and, for $i > 0$

$$E_i = Th_{\mathfrak{L}}(E_{i-1}) \cup \{C(x) \mid \delta = \frac{A(x):B(x)}{C(x)} \in \Delta, A(x) \in E_{i-1} \text{ and } \neg B(x) \notin \mathfrak{E}\}.$$

The set $\mathfrak{E}$ *is an extension for* $\mathfrak{D}$ *if and only if*

$$\mathfrak{E} = \bigcup_{i=0}^{\infty} E_i .$$

∎

The beauty of this theorem is that it gives a more intuitive characterization of an extension for a default theory. For a given closed well-formed formula set, we can construct a sequence of closed well-formed formula sets. To check whether the given set is an extension for the default theory, we need only to check the union of all sets in the sequence. If the given set is identical with the union, then the given set is an extension for the default theory.

The candidate set for an extension is involved in constructing the sequence of well-formed formula sets, . That is, the candidate set is referred to in the construction. (It can be seen in the occurrence of $\mathfrak{E}$ in the construction of E_i.) Therefore, this theorem cannot be used to construct an extension since we need to have an initial candidate set.

Example 2.2.5:

Let $\mathfrak{D} = (\mathfrak{F}, \Delta = \{\delta\})$ be a default theory, where $\mathfrak{F} = \{\text{bird(Tweety)}\}$ and $\delta = \dfrac{\text{bird}(x):\text{fly}(x)}{\text{fly}(x)}$. As previously mentioned, this default theory has exactly one extension $\mathfrak{E}$,

$$\mathfrak{E} = Th_{\mathfrak{L}}(\mathfrak{F} \cup \{\text{fly(Tweety)}\}).$$

The sequence for this extension is $E_0, E_1, E_2, \ldots$, where

$$E_0 = \{\text{bird(Tweety)}\},$$
$$E_1 = Th_{\mathfrak{L}}(E_0) \cup \{\text{fly(Tweety)}\},$$
$$E_2 = Th_{\mathfrak{L}}(E_1) \cup \{\text{fly(Tweety)}\} = Th_{\mathfrak{L}}(E_1), \text{ and}$$
$$E_i = Th_{\mathfrak{L}}(E_2), \text{ for } i \geq 3.$$

Therefore, verifying the result, we get

$$\bigcup_{i=0}^{\infty} E_i = Th_{\mathfrak{L}}(E_2) = \mathfrak{E}.$$

In general, default theories are not monoton. By the term "not monotonic," we mean is the following. Let $\mathfrak{D} = (\mathfrak{F}, \Delta)$ be a default theory with an extension $\mathfrak{E}$. Suppose that $\mathfrak{D}' = (\mathfrak{F}', \Delta')$ is a default theory with $\mathfrak{F} \subseteq \mathfrak{F}'$ and $\Delta \subseteq \Delta'$. It is usual that there is no extension $\mathfrak{E}'$ for $\mathfrak{D}'$ such that $\mathfrak{E} \subseteq \mathfrak{E}'$.

The following example will show these nonmonotonicity properties more precisely.

Example 2.2.6:

Let $\mathfrak{D} = (\mathfrak{F}, \Delta = \{\delta\})$ be a default theory, where

$\mathfrak{F} = \{\text{bird(Tweety)}\}$,

and

$$\delta = \frac{\text{bird}(x):\text{fly}(x)}{\text{fly}(x)}.$$

As shown previously, this default theory has only one extension $\mathfrak{E}$, where

$\mathfrak{E} = \text{Th}_{\mathfrak{L}}(\mathfrak{F} \cup \{\text{fly(Tweety)}\})$.

Now, consider a default theory $\mathfrak{D}' = (\mathfrak{F}', \Delta')$, where

$\mathfrak{F}' = \mathfrak{F} \cup \{\text{penguin(Tweety)}, \forall x\ \text{penguin}(x) \rightarrow \neg\text{fly}(x)\}$

and

$\Delta' = \Delta$.

That is, besides "Tweety is a bird," we also know that "Tweety is a penguin" and "penguins cannot fly." An extension for $\mathfrak{D}'$ is $\mathfrak{E}'$, where

$\mathfrak{E}' = \text{Th}_{\mathfrak{L}}(\mathfrak{F}')$,

since default δ is not applicable in this default theory. Furthermore, there are no other extensions for $\mathfrak{D}'$. Therefore, $\mathfrak{F} \subseteq \mathfrak{F}'$ and $\Delta \subseteq \Delta'$, but $\mathfrak{E} \not\subset \mathfrak{E}'$.

Default theories may result in an inconsistent extension. In the Reiter approach, the consistency of extensions relies on the consistency of the hard fact set. Furthermore, defaults cannot introduce inconsistency.

Theorem 2.2.2: Inconsistent extensions [39]

A default theory has an inconsistent extension if and only if its hard fact set is inconsistent and it is a unique extension.

■

That is, if a default theory has an inconsistent extension, then it is the only extension for the default theory. For this reason, we will not be concerned with an inconsistent hard fact set from now on. Therefore, we assume that a hard fact set is always consistent.

It is in question whether adding derivable facts from an extension for a default theory can enlarge the extension. This question can be answered by the following theorem, which is that adding facts that are derivable from an extension does not expand the extension.

Theorem 2.2.3: [39]

If a default theory $\mathfrak{D} = (\mathfrak{F}, \Delta)$ has an extension $\mathfrak{E}$, then it is also an extension for every default theory $\mathfrak{D}' = (\mathfrak{F}', \Delta)$, where $\mathfrak{F} \subseteq \mathfrak{F}' \subseteq \mathfrak{E}$.

■

2.2.3 Normal Default Theories

In general, default theories lack extensions. Those theories do not define the notion of consequences. The following example is a simple non-normal default theory that does not have an extension.

Example 2.2.7: A default theory not having extensions.

Let $\mathfrak{D} = (\mathfrak{F}, \Delta = \{\delta\})$ be a default theory, where

$\mathfrak{F} = \{P\}$,

and

$$\delta = \frac{P:Q}{\neg Q}.$$

That is, the fact is a proposition constant P and the default is "If P is believed and it is consistent to believe Q, then infer $\neg$Q." Suppose that this default theory has an extension. Therefore, a candidate set must contain P. There are only two candidates: $E_1 = Th(\{P\})$ and $E_2 = Th(\mathcal{F} \cup \{\neg Q\}) = Th(\{P, \neg Q\})$. E_1 cannot be an extension since the default is applicable, causing $\neg$Q in E_1, though it is not. Furthermore, E_2 cannot also be an extension since the default is not applicable, causing $\neg$Q not in E_2, though it is. Thus, $\mathcal{D}$ does not have an extension.

Reiter found a class of default theories that has some interesting properties [39]. This class consists of normal default theories. Recall that a default theory is called normal if and only if its default set consists of normal defaults.

Some interesting properties are the existence of extension, semi-monotonicity, and orthogonality of extensions. For normal default theories, the existence of extensions is guaranteed. That is, every normal default theory has at least one extension. Semi-monotonicity is another property of normal default theories that can be used to deal with incomplete knowledge.

In the rest of this section will discuss those properties of normal defaults. The most important property is the existence of extensions for normal default theories. It is given by the following theorem.

Theorem 2.2.4: Existence of extension for normal default theories [39]

For every normal default theory, there is at least one extension.

As previously mentioned, default theories are not monoton. By "not monotonic," we mean that adding facts to the hard fact set does not guarantee that an extension will be enlarged. Therefore, it could be that adding facts causes an extension not to change, to "shrink," or to expand. However, adding defaults to a default set will not shrink an extension. In other words, if $\mathfrak{D} = (\mathfrak{F}, \Delta)$ is a default theory with an extension $\mathfrak{E}$, and $\mathfrak{D} = (\mathfrak{F}', \mathfrak{D}')$ is a default theory such that $\mathfrak{F} \subseteq \mathfrak{F}'$ and $\mathfrak{D} \subseteq \mathfrak{D}'$, it could be that there is no extension $\mathfrak{E}'$ for $\mathfrak{D}'$ such that $\mathfrak{E} \subseteq \mathfrak{E}'$. These are called semi-monotonicity properties of an extension for a default theory.

As the following theorem concluded, normal default theories have semi-monotonicity properties.

Theorem 2.2.5: Semi-monotonicity [39]

Let $\mathfrak{E}$ be an extension for a normal default theory $\mathfrak{D} = (\mathfrak{F}, \Delta)$. A default theory $\mathfrak{D}' = (\mathfrak{F}, \Delta')$, where $\Delta \subseteq \Delta'$, has an extension $\mathfrak{E}'$ such that $\mathfrak{E} \subseteq \mathfrak{E}'$.

$\blacksquare$

The proof of the existence of an extension can be seen as a direct consequence of the semi-monotonicity properties. The extension for any default theory $\mathfrak{D} = (\mathfrak{F}, \emptyset)$, i.e., its default set is empty, is $Th(\mathfrak{F})$, that is, the set of all facts derivable from the hard fact set $\mathfrak{F}$. This is true because it follows from the first and second conditions for an extension plus no applicable defaults. Therefore, using the semi-monotonicity properties, it follows that an extension for normal default theories exists.

The next theorem will show the orthogonality of extensions for normal default theories.

Theorem 2.2.6: Orthogonality of extensions [39]

The union of two distinct extensions for normal default theories is inconsistent.

$\blacksquare$

Therefore, if both $\mathfrak{E}_1$ and $\mathfrak{E}_2$ are two distinct extensions for a normal default theory, then $\mathfrak{E}_1 \cup \mathfrak{E}_2$ is inconsistent.

2.3 Lukaszewicz Default Theories

In 1988, Lukaszewicz developed a way of deriving an extension so that any arbitrary default theory has an extension [24]. He defined so-called modified extensions in terms of a fixed point of two operators. He found that these theories enjoy some properties that normal default theories have.

2.3.1 The Need for Non-normal Default Theories

In 1981, the need for non-normal default theories was proposed by Reiter and Criscuolo [42]. They found naturally occurring non-normal default theories. Some of those default theories are caused by interaction among normal defaults such as blocking the transitivity. Furthermore, most of them lack some properties that normal default theories enjoy.

The following example is a normal default theory that has an unacceptable extension. It can be modified so that its extension is well behaved. Some other examples can be found Reiter and Criscuolo [42].

Example 2.3.1:

A problem with two typical facts "Typically, drop-out-students are adults" and "Typically, adults are employed" is formalized by a normal default theory $\mathcal{D} = (\mathcal{F}, \Delta = \{\delta_1, \delta_2\})$, where

$$\delta_1 = \frac{\text{drop-out-student}(x):\text{adult}(x)}{\text{adult}(x)}$$

and

$$\delta_2 = \frac{\text{adult}(x):\text{employed}(x)}{\text{employed}(x)}.$$

Suppose that we know only that "John is a drop-out student." Then, by default δ_1, "John is an adult." Furthermore, the fact "John is also employed" is derivable using default δ_2. Therefore, drop-out students are also employed unless the contrary exists.

This kind of behavior of extensions for default theories is not acceptable. There should be a mechanism to block the applicability of default δ_2 as a result of δ_1. In order to remedy this problem, Reiter and Criscuolo proposed non-normal default theories. They introduced semi-normal defaults.

In example 2.3.2 above, the source of a problem is the default δ_2. If δ_1 is applicable, it is likely that δ_2 is applicable. This is a kind of a transitivity. To block a transitivity, a semi-normal default can be used [Reiter and Criscuolo, 1981]. For example, the default δ_2 is replaced using a semi-normal default δ'_2, where

$$\delta'_2 = \frac{\text{adult}(x) : \text{employed}(x) \wedge \neg\text{drop-out-student}(x)}{\text{employed}(x)}.$$

Therefore, δ'_2 is no longer applicable for a given fact "John is a drop out student." So, we can only infer, using the default δ_1, that "John is an adult" but the fact "John is employed" is not derivable.

2.3.2 Modified Extensions for Default Theories

In order to ensure that every default theory has an extension, Lukaszewicz modified the applicability of defaults. His definition an extension is a little bit complicated. He proposed and defined an extension in terms of a fixed point of two operators. The first operator is the same as that of Reiter's. The second operator is used to control the applicability of defaults by collecting all justifications of applied defaults and checking the consistency of the consequents and justifications of all already-applied defaults. In this approach, an extension is warranted to exist.

Following is the Lukaszewicz definition for building a modified extension.

Definition: Modified extensions for default theories [24]

Let $\mathcal{D} = (\mathcal{F}, \Delta)$ *be a default theory and* (S, T) *be a pair of closed well-formed formula sets of first order language* $\mathcal{L}$. *Suppose that* $(\Gamma_1(S, T), \Gamma_2(S, T))$ *denotes a pair of the smallest sets satisfying the three following conditions*:

D1 $\mathcal{F} \subseteq \Gamma_1(S, T)$;

D2 $Th_{\mathcal{L}}(\Gamma_1(S, T)) = \Gamma_1(S, T)$; *and*

D3 if $\delta = \dfrac{A(x):B(x)}{C(x)} \in \Delta$, $A \in \Gamma_1(S, T)$, *and* $S \cup \{C(x)\} \not\vdash \neg F, \forall F \in T \cup \{B\}$,

then $C(x) \in \Gamma_1(S, T)$ *and* $B(x) \in \Gamma_2(S, T)$).

A closed well-formed formula set m-$\mathcal{E}$ *is a modified extension for* $\mathcal{D}$ with respect to $\mathcal{J}$ *if and only if* $(\Gamma_1(\text{m-}\mathcal{E}, \mathcal{J}), \Gamma_2(\text{m-}\mathcal{E}, \mathcal{J})) = (\text{m-}\mathcal{E}, \mathcal{J})$.

This definition provides criteria for a set $\mathcal{E}$ to be a modified extension for a default theory with respect to a set $\mathcal{J}$. The first two conditions are the same as those of Reiter. The third condition gives a way to infer using defaults. In contrast to the Reiter approach, the applicability of a default refers not only to the defaults under consideration, but to the other defaults as well. So, a default is applicable if and only if its prerequisite is believed, and adding its consequent to the modified extension neither leads to inconsistency nor contradicts the justification of this and other already applied defaults. Furthermore, a fact that is in a modified extension, is believed, if and only if it is either an axiom, monotonically derivable from other beliefs, or the consequent of an applicable default [24].

Example 2.3.2:

Let us consider the non-normal default theory above $\mathcal{D} = (\mathcal{F}=\{P\}, \Delta=\{\delta\})$,

where $\delta = \dfrac{P:Q}{\neg Q}$. This default theory does not have an extension but it has a modified extension m-$\mathcal{E}$ = Th($\mathcal{F}$) with respect to $\mathcal{J} = \varnothing$.

$\blacksquare$

2.3.3 General Properties of Modified Extensions for Default Theories

In this section we present some properties that belong to the Lukaszewicz formalization. We present the same properties as Reiter's above.

The first property is the characterization of a modified extension.

Theorem 3.3.1: Characterization of modified extensions [24]

Let $\mathcal{D} = (\mathcal{F}, \Delta)$ be a default theory and m-$\mathcal{E}$ be a modified extension with respect to $\mathcal{J}$. Suppose that $E_0, E_1, E_2, \ldots$ and $J_0, J_1, J_2, \ldots$ are two sequences of closed well-formed formula sets such that

$E_0 = \mathcal{F}$,

$J_0 = \varnothing$,

and, for i > 0

$$E_i = Th_{\mathcal{L}}(E_{i-1}) \cup \{C \mid \delta = \frac{A:B}{C} \in \Delta, A \in E_{i-1} \text{ and } \text{m-}\mathcal{E} \cup \{C\} \not\vdash \neg F, \forall F \in \mathcal{J} \cup \{B\}\},$$

$$J_i = J_{i-1} \cup \{B \mid \delta = \frac{A:B}{C} \in \Delta, A \in E_{i-1} \text{ and } \text{m-}\mathcal{E} \cup \{C\} \not\vdash \neg F, \forall F \in \mathcal{J} \cup \{B\}\}.$$

Then,

$$\text{m-}\mathcal{E} = \bigcup_{i=0}^{\infty} E_i \text{ and } \mathcal{J} = \bigcup_{i=0}^{\infty} J_i .$$

$\blacksquare$

The next theorem is the semi-monotonicity property, i.e., monotonicity with respect to defaults.

Theorem 3.3.2: Semi-monotonicity [24]

Let $\mathcal{D} = (\mathcal{F}, \Delta)$ be a default theory and m-$\mathcal{E}$ be a modified extension with respect to $\mathcal{G}$. A default theory $\mathcal{D}' = (\mathcal{F}', \Delta')$, where $\Delta \subseteq \Delta'$, has a modified extension m-$\mathcal{E}'$ with respect to $\mathcal{G}'$ such that m-$\mathcal{E} \subseteq$ m-$\mathcal{E}'$ and $\mathcal{G} \subseteq \mathcal{G}'$.

$\blacksquare$

The next theorem is a warrant for existence of a modified extension for a default theory. It can be proved using the semi-monotonicity properties of modified extensions for default theories.

Theorem 3.3.3: Existence of modified extensions [24]

Every default theory has a modified extension.

$\blacksquare$

CHAPTER III

A NEW APPROACH

3.1 Introduction

As previously mentioned, non-normal default theories occur in nature. However, most of them have no extensions. Furthermore, in default theories, an extension is essential. In order to remedy this problem, Lukaszewicz developed a new way of defining an extension, called a modified extension, so that every default theory does have an extension.

In using both the Reiter and the Lukaszewicz approaches, we find some disadvantages. Both fail to deal with disjunctive fact problems concerning justifications and prerequisites.

Using the Reiter approach, besides having no extension, disjunctive fact problems concerning justifications may cause default theories not to work properly, resulting in unacceptable extensions. This is due to inconsistent extensions that may arise during the application of defaults, for example, applications of defaults that are based on inconsistent justifications. In the Lukaszewicz approach, a modified extension for a default theory may result in inconsistency with its justification set. That is, a default theory may result in modified extensions that are consistent but they are inconsistent with their justification sets.

Both approaches fail to deal with disjunctive fact problems concerning prerequisites of defaults having the same justification. Disjunctive fact problems concerning prerequisites may cause a default theory to fail to derive facts even though it is a normal default theory.

In this chapter, we will see these problems more clearly and develop a new approach to overcoming the problems. First, we identify disjunctive fact problems concerning justifications, propose a new approach to solving them, and investigate the properties of the new approach. Second, we distinguish disjunctive fact problems concerning prerequisites of defaults having the same justification. To overcome these problems, we develop the first new approach by extending the applicability of defaults by including disjunctive forms of prerequisites in applying them. In doing so, we will follow conventions from the previous chapter for all terms such as a language, a default, a default theory, and an extension.

All notations are also used as in the previous chapters. A well-formed formula will be denoted using capital and italic characters like *A* and *B*. A special black-chancerry character like $\mathcal{A}$ and $\mathcal{B}$. will be used to denote a special set such as hard fact sets, extensions, and justification sets. Other sets such as a sequence of sets are denoted by a plain character like A and B.

Furthermore, we assume that we are dealing with a consistent hard fact set since an inconsistent hard fact set will result in inconsistent extensions.

3.2 Disjunctive Fact Problems Concerning Justification

There are some problems in using both the Reiter and the Lukaszewicz approaches to disjunctive fact problems concerning justifications. Disjunctive facts consist of two or more facts such that we do not have sufficient evidence to believe any of them individually but we only believe their disjunctive forms. For example, we have two birds, for example bird(Tweety) and bird(Clyde). In addition, we see that one of them has broken wings but we do not know which one.

Therefore, we believe either Tweety or Clyde has broken wings, that is, ¬broken-wing(Tweety)∨¬broken-wing(Clyde). The problems in both the Reiter and the Lukaszewicz approaches are due to inconsistent facts that may result when applying defaults in the presence of disjunctive facts.

3.2.1 Preliminary Discussion

Disjunctive fact problems concerning justifications of defaults usually arise in non-normal default theories. In using both the Reiter and the Lukaszewicz approaches, they may result in extensions that are counter-intuitive.

Consider the following example. In this example, neither approaches properly work.

Example 3.2.1:

Suppose that all we know is "Tweety and Clyde are birds" and either "Tweety has broken wings" or "Clyde has broken wings" but we do not know which of them has broken wings. Furthermore, we have a semi-normal default "A bird can fly in the absence of evidence that it cannot fly and its wings are not broken." This problem can be formalized using a semi-normal default theory $\mathfrak{D}$ = ($\mathfrak{F}$, $\Delta = \{\delta\}$), where

$\mathfrak{F}$ = {bird(Tweety), bird(Clyde), broken-wing(Tweety)∨broken-wing(Clyde)}

and

$$\delta = \frac{\mathrm{bird}(x):\mathrm{fly}(x) \wedge \neg\mathrm{broken\text{-}wing}(x)}{\mathrm{fly}(x)}.$$

Since both facts "¬(fly(Tweety)∧¬broken-wing(Tweety))" and "¬(fly(Clyde)∧¬broken-wing(Clyde))" are not inferable, then the default δ is applicable for both "bird(Tweety)" and "bird(Clyde)" resulting in both "fly(Tweety)" and "fly(Clyde)" in an extension. It can be shown that $\mathfrak{D}$ has exactly one extension $\mathfrak{E}$,

$\mathfrak{E} = \mathrm{Th}(\mathfrak{F} \cup \{\mathrm{fly(Tweety)}, \mathrm{fly(Clyde)}\})$.

Furthermore, a modified extension for $\mathfrak{D}$ is m-$\mathfrak{E}$,

m-$\mathfrak{E} = \mathrm{Th}(\mathfrak{F} \cup \{\mathrm{fly(Tweety)}, \mathrm{fly(Clyde)}\})$

with respect to

$\mathfrak{J}$={fly(Tweety)∧¬broken-wing(Tweety), fly(Clyde)∧¬broken-wing(Clyde)}).

Both extension and modified extension contain both facts "Tweety can fly" and "Clyde can fly." Since an extension contains all the facts that a person may hold, these results are counter-intuitive. A person should not be able to infer that both birds Tweety and Clyde are able to fly since the person knows that one of them has broken wings, and thus it cannot fly.

▒

Another example is presented by Poole [35]. This example illustrates a difficulty that arises in some attempts to use both Reiter and Lukaszewicz in the presence of disjunctive facts. In both the Reiter and the Lukaszewicz approaches, the default theory results in both an extension and a modified extension that are counter-intuitive.

Example 3.2.2: Disjunctive fact problems concerning justification [35]

Suppose that we know that one of John's hands is broken but we do not know which hand is broken. Suppose that by default, a person's left or right arm is usable unless the person's left or right arm, respectively, is broken. This problem is formalized using a semi-normal default theory $\mathfrak{D} = (\mathfrak{F}, \Delta = \{\delta_1, \delta_2\})$, where

$\mathfrak{F}$ = {person(John), right-arm-broken(John)∨left-arm-broken(John)},

$$\delta_1 = \frac{\mathrm{person}(x)\mathrm{:left\text{-}arm\text{-}usable}(x) \wedge \neg \mathrm{left\text{-}arm\text{-}broken}(x)}{\mathrm{left\text{-}arm\text{-}usable}(x)},$$

and

$$\delta_2 = \frac{\mathrm{person}(x)\mathrm{:right\text{-}arm\text{-}usable}(x) \wedge \neg \mathrm{right\text{-}arm\text{-}broken}(x)}{\mathrm{right\text{-}arm\text{-}usable}(x)}$$

Let us identify an extension for this default theory. Since δ_1 is applicable, the fact "John's left arm is usable" is an extension. Furthermore, since δ_2 is also applicable, the fact "John's right arm is usable" is in the extension. The extension for this default theory is

$$\mathcal{E} = \mathrm{Th}(\mathcal{F} \cup \{\text{left-arm-usable(John), right-arm-usable(John)}\}).$$

It can be shown that it is the only extension for this default theory. Now, we see a modified extension for this default theory. A modified extension for this default theory is

$$\text{m-}\mathcal{E} = \mathrm{Th}(\mathcal{F} \cup \{\text{left-arm-usable(John), right-arm-usable(John)}\})$$

with respect to

$$\mathcal{G} = \{\text{left-arm-usable(John)} \wedge \neg\text{left-arm-broken(John)},$$
$$\text{right-arm-usable(John)} \wedge \neg\text{right-arm-broken(John)}\}$$

since both defaults are applicable. Moreover, it is the only modified extension for this default theory.

However, neither approaches agrees with the intuition. That is, both approaches result in both of John's hands being usable. Since we know that at least one of John's hands is broken, at most, only one of John's hands should be usable.

$\blacksquare$

These examples show that both the Reiter and the Lukaszewicz approaches are unwell-behaved to deal with disjunctive fact problems concerning justifications.

3.2.2 New Approach to Solving Disjunctive Fact Problems Concerning Justifications

The Reiter approach cannot fully control the applicability of defaults. The restriction on applying defaults in this approach is the derivability of their prerequisites in addition to consistency. Therefore, if a prerequisite of a default is provable, it is more likely that the default is applicable. So, in default theories in which the applicability of defaults leads to counter-intuitive results such as the examples above, the approach does not work properly. It may result in default theories having extensions that do not agree with intuition.

The same reasoning is applied to the Lukaszewicz approach. The applicability of defaults in this approach is more complex. In contrast, it refers not only to the default under consideration but also to other applied defaults. Therefore, a default is applicable if its prerequisite is provable and adding its consequent does not result in inconsistency nor contradict any justification of other applied defaults including itself. However, the approach does not enforce consistency among the justifications as a basis for applying the default. For example, δ_2 is applicable in example 3.2.1 since adding the consequent of δ_2 to m-$\mathcal{E}$ does not result in inconsistency (i.e., $\neg$fly(Clyde)$\notin$ m-$\mathcal{E}$) and contradict any justification of δ_1 and δ_2 (i.e., $\neg$(fly(Tweety)$\wedge\neg$broken-wing(Tweety))$\notin \mathcal{J}$ and $\neg$(fly(Clyde)$\wedge\neg$broken-wing(Clyde))$\notin \mathcal{J}$).

In other words, in the Lukaszewicz approach, the contradiction as a result of applying defaults by adding their consequents is restricted to each individual element of the justification set. So, the applicability of a default is checked against an individual element of a justification set in addition to the provability of its prerequisite and consistency of its justification.

In order to block inferring the fact "fly(Clyde)," we need to block the applicability of δ_2. This can be done by enforcing the contradiction not only to each individual element but to the whole justification set as well. This is the main idea behind a new approach to dealing with disjunctive fact problems concerning justifications.

This idea will be given formally in the following definition. This definition extends the restriction on applying defaults concerning their justifications. It uses two operators, as in the Lukaszewicz approach, to define a pair of sets. This pair of sets is called a pair of an alternative extension and an assumption set. We also refer them as an alternative extension a-$\mathcal{E}$ based on an assumption set $\mathcal{A}$ for a default theory $\mathcal{D}$.

Definition: Alternative extensions

Let $\mathcal{D} = (\mathcal{F}, \Delta)$ *be a default theory and* (E, A) *be a pair of closed well-formed formula sets. Suppose that* $(\Gamma_1(E, A), \Gamma_2(E, A))$ *is a pair of the smallest sets satisfying the following three conditions*:

R1. $\mathcal{F} \subseteq \Gamma_i(E, A)$, *for* $i=1,2$;

R2. $\mathrm{Th}_{\mathcal{L}}(\Gamma_i(E, A)) = \Gamma_i(E, A)$, *for* $i=1,2$; *and*

R3. if $\delta = \dfrac{A:B}{C} \in \Delta$, $A \in \Gamma_1(E, A)$ *and* $A \cup \{B, C\}$ *consistent, then*

- $C \in \Gamma_1(E, A)$, *and*
- $B, C \in \Gamma_2(E, A)$.

A closed well-formed formula set a-$\mathcal{E}$ *is an alternative extension for a default theory* $\mathcal{D}$ *if and only if there is a closed well-formed formula set* $\mathcal{A}$, *called an assumption set, such that*

a-$\mathcal{E} = \Gamma_1$(a-$\mathcal{E}, \mathcal{A}$) and $\mathcal{A} = \Gamma_2$(a-$\mathcal{E}, \mathcal{A}$).

This definition is more complex than those of the Reiter and the Lukaszewicz approaches. It provides criteria for a closed well-formed formula set to be an alternative extension a-$\mathcal{E}$ for a default theory $\mathcal{D}$ based on an assumption set $\mathcal{A}$. A pair composed of an alternative extension a-$\mathcal{E}$ and an assumption set $\mathcal{A}$ is a fixed point of operators Γ_1 and Γ_2 based on three conditions; i.e., a-$\mathcal{E} = \Gamma_1$(a-$\mathcal{E}$, $\mathcal{A}$) and $\mathcal{A} = \Gamma_2$(a-$\mathcal{E}, \mathcal{A}$). The first condition insists that both alternative extension and assumption set contain the entire hard fact set. The second condition requires that both alternative extension and assumption set be inductively closed; that is, all facts derivable from an alternative extension or an assumption set in terms of first order theory must be in the alternative extension or in the assumption set, respectively.

The third condition is related to an applicability of a default δ. It extends the derivability of first order theory. It requires that a prerequisite of a default be inferable from an alternative extension a-$\mathfrak{E}$. In addition, its justification and consequent are consistent with the related assumption set $\mathcal{A}$. If these requirements are satisfied, then the justification and consequent of the default must be in the assumption set $\mathcal{A}$, and the consequent of the default must be also in the alternative extension a-$\mathfrak{E}$. In this case, the default δ is called applicable to an alternative extension a-$\mathfrak{E}$ based on an assumption set $\mathcal{A}$. In addition, when applying a default, its prerequisite, justification, and consequent are closed well-formed formulae.

In addition, the smallness of the pair assures that no more facts are added to it without any reason. In other words, a fact is in an alternative extension if it is a hard fact, a consequent of an applicable default, or a fact derivable from the set of hard facts and the set of consequents of applicable defaults. Furthermore, a fact is in an assumption set if it is a hard fact, a consequent or a justification of an applicable default, or a fact derivable from the set of hard facts and the set of consequents and justifications of applicable defaults.

There are some important notes to be identified in this definition. By combining the first two conditions, we can conclude that all facts derivable from hard facts are in both an alternative extension and an assumption set; i.e., Th($\mathfrak{F}$) $\subseteq$ a-$\mathfrak{E}$ and Th($\mathfrak{F}$) $\subseteq \mathcal{A}$. In addition, by condition R3, an assumption set contains consequents and justifications of applicable defaults. Meanwhile, an alternative extension contains all consequents of applicable defaults. Therefore, an alternative extension is a subset of an assumption set; i.e., a-$\mathfrak{E} \subseteq \mathcal{A}$. Furthermore, since an assumption set must be consistent, an alternative extension is also consistent.

To check whether a pair of sets is a pair composed of an alternative extension and an assumption set, we need to verify the three conditions above in addition to the smallness of the pair. Furthermore, a pair of closed well-formed formula sets may satisfy one, two, or three of the conditions based on a default theory $\mathcal{D} = (\mathcal{F}, \Delta)$ but not necessarily be a pair of an alternative extension and an assumption set. So, A pair of sets (E, A) satisfies condition R1 if $\mathcal{F} \subseteq \mathrm{E}$ and $\mathcal{F} \subseteq \mathrm{A}$. It satisfies condition R2 if Th(E) = E and Th(A) = A. Finally, it satisfies condition R3 if $\delta = \frac{A:B}{C}$ is a default, $A \in \mathrm{E}$ and $\mathrm{A} \cup \{B, C\}$ is consistent, then $C \in \mathrm{E}$ and $B, C \in \mathrm{A}$. (Notice the characters used. The italic characters like *A* and *B* denote well-formed formulae whereas the plain characters like A and B denote a set.)

The main reason in using an assumption set is that it assures that we apply defaults by inferring facts based on consistent assumptions, that is, a consequent and justification of an applicable default consistent with an assumption set. In this approach, we assure that no inconsistency problems arise during the application of defaults, for example in the previous examples. Therefore, the disjunctive fact problems concerning justifications will not arise, as we will see later on.

The notion of consequences in the new approach is the same as in that of the Reiter approach. A consequence of a default theory is a fact that belongs to an alternative extension. So, to find out whether a fact is a consequence of a default theory, we have to find an alternative extension containing the fact.

Example 3.2.3: The ability of birds to fly

Let $\mathcal{D} = (\mathcal{F}, \Delta = \{\delta\})$ be a default theory, where

$\mathcal{F} = \{\text{bird(Tweety)}\}$

and

$$\delta = \frac{\text{bird(x):fly(x)}}{\text{fly(x)}}.$$

It is easy to see that this default theory has only one alternative extension a-$\mathcal{E}$,

a-$\mathcal{E}$ = Th({bird(Tweety), fly(Tweety)})

based on an assumption set $\mathcal{A}$,

$\mathcal{A}$ = Th({bird(Tweety), fly(Tweety)}).

In addition, the fact "fly(Tweety)" is a consequence of $\mathcal{D}$. Therefore, in the absence of information to the contrary, "$\neg$fly(Tweety)," we can infer that "Tweety can fly."

$\blacksquare$

From this example, it can be seen that an alternative extension for a normal default theory is identical to an assumption set. This can be seen in the following theorem.

Theorem 3.2.1: Similarity of a-$\mathcal{E}$ and $\mathcal{A}$ for normal default theories

An alternative extension for normal default theories is identical to its assumption set.

$\blacksquare$

Proof: Theorem 3.2.1

Both alternative extension a-$\mathcal{E}$ and assumption set $\mathcal{A}$ contain the entire hard fact set, and the difference between the two sets occurs during the application of defaults. That is, if a default $\delta = \frac{A:B}{C}$ is applicable, then $C \in$ a-E and $B, C \in \mathcal{A}$. Hence, a-$\mathcal{E} \subseteq \mathcal{A}$ and no other facts are in both a-$\mathcal{E}$ and $\mathcal{A}$. Furthermore, since the consequent of a normal default is logically equivalent with its justification, then every justification and consequent of applicable defaults must be in both an alternative extension and an assumption set. So, the two sets are identical.

$\blacksquare$

However, it is not generally true for arbitrary default theories. Consider the following example. The following example shows a non-normal default theory having an alternative extension based on a non-identical assumption set.

Example 3.2.4: Non-identical assumption set

Let $\mathcal{D} = (\mathcal{F}, \Delta = \{\delta_1, \delta_2\})$ be a default theory, where $\mathcal{F} = \{P\}$, $\delta_1 = \frac{P:Q}{Q}$, and $\delta_2 = \frac{Q:R}{S}$. This default theory has an alternative extension a-$\mathcal{E}$,

a-$\mathcal{E} = \mathrm{Th}(\mathcal{F} \cup \{Q, S\})$

based on an assumption set $\mathcal{A}$,

$\mathcal{A} = \mathrm{Th}(\mathcal{F} \cup \{Q, R, S\})$.

That is, an alternative extension a-$\mathcal{E}$ for an arbitrary default theory is not identical to its assumption set $\mathcal{A}$.

Let us apply this new approach to the two problems in examples 3.2.1 and 3.2.2 above. The following is the first problem.

Example 3.2.5:

Let us examine the first problem in example 3.2.1, that is, the default theory $\mathcal{D} = (\mathcal{F}, \Delta)$, where

$\mathcal{F} = \{\text{bird(Tweety), bird(Clyde), broken-wing(Tweety)} \vee \text{broken-wing(Clyde)}\}$

and

$$\delta = \frac{\text{bird}(x):\text{fly}(x) \wedge \neg\text{broken-wing}(x)}{\text{fly}(x)}.$$

Let us apply the default δ to "bird(Tweety)" by inferring "fly(Tweety)," i.e., fly(Tweety) in an extension a-$\mathcal{E}_1$ based on an assumption set $\mathcal{A}_1$. It needs justification facts "fly(Tweety)" and "fly(Tweety)$\wedge\neg$broken-wing(Tweety)." Therefore, fly(Tweety)$\in \mathcal{A}_1$ and fly(Tweety)$\wedge\neg$broken-wing(Tweety)$\in \mathcal{A}_1$. So, $\neg$broken-wing(Tweety) is in $\mathcal{A}_1$. Hence, the default δ is no longer applicable to "bird(Clyde)" since "broken-wing(Clyde)" is inferable from $\mathcal{A}_1$. It can be shown that one of the extensions for this default theory is a-$\mathcal{E}_1$, where

a-$\mathcal{E}_1 = \mathrm{Th}(\mathcal{F} \cup \{\text{fly(Tweety)}\})$

based on an assumption set $\mathcal{A}_1$,

$\mathcal{A}_1 = \mathrm{Th}(\mathcal{F} \cup \{\text{fly(Tweety)}, \text{fly(Tweety)} \wedge \neg \text{broken-wing(Tweety)}\})$.

The other extension for this default theory is a-$\mathcal{E}_2$, where

a-$\mathcal{E}_2 = \mathrm{Th}(\mathcal{F} \cup \{\text{fly(Clyde)}\})$

based on an assumption set $\mathcal{A}_2$, where

$\mathcal{A}_2 = \mathrm{Th}(\mathcal{F} \cup \{\text{fly(Clyde)}, \text{fly(Clyde)} \wedge \neg \text{broken-wing(Clyde)}\})$.

These are the only alternative extensions for the default theory. Both facts "fly(Tweety)" and "fly(Clyde)" are not simultaneously in any extension. This solves the first problem. So, in the new approach, the counter-intuition statement does not arise.

▒

In the new approach, there are two distinct extensions. The first extension contains the fact "fly(Tweety)." Since a consequence for a default theory is a sentence that belongs to an extension, the fact is a consequence of default theory $\mathcal{D}$. Meanwhile, the second extension contains the fact "fly(Clyde)." So, the fact is also a consequence of $\mathcal{D}$. However, they are not simultaneously consequences of $\mathcal{D}$. In other words, one may believe either of the facts but not both.

The next example shows how the new approach solves the second problem in examples 3.2.2.

Example 3.2.6:

The problem is formalized using a semi-normal default theory $\mathcal{D} = (\mathcal{F}, \Delta = \{\delta_1, \delta_2\})$, where

$\mathcal{F} = \{\text{person(John)}, \text{right-arm-broken(John)} \vee \text{left-arm-broken(John)}\}$,

$$\delta_1 = \frac{\text{person(x)} : \text{left-arm-usable(x)} \wedge \neg \text{left-arm-broken(x)}}{\text{left-arm-usable(x)}},$$

and

$$\delta_2 = \frac{\text{person(x)} : \text{right-arm-usable(x)} \wedge \neg \text{right-arm-broken(x)}}{\text{right-arm-usable(x)}}$$

Let us see first the applicability of the default δ_1. This default is applicable to an alternative extension a-$\mathcal{E}_1$ based on an assumption set $\mathcal{A}_1$, where

a-$\mathcal{E}_1$ = Th($\mathcal{F} \cup$ {left-arm-usable(John)})

$\mathcal{A}_1$ = Th($\mathcal{F} \cup$ {left-arm-usable(John),

left-arm-usable(John)$\wedge\neg$left-arm-broken(John)}),

since is not possible to derive both facts "$\neg$left-arm-usable(John)" and "left-arm-broken(John)})." However, the default δ_2 is not applicable since "right-arm-broken(John)" is inferable from $\mathcal{A}_1$; i.e., both "right-arm-broken(John)$\vee$left-arm-broken(John)" and "$\neg$left-arm-broken(John)" in $\mathcal{A}_1$. So, both facts "left-arm-usable(John)" and "right-arm-usable(John)" are not in a-$\mathcal{E}_1$. Now, we see another alternative extension for $\mathcal{D}$. The default δ_2 is applicable in an alternative a-$\mathcal{E}_2$ based on an assumption set $\mathcal{A}_2$, where

a-$\mathcal{E}_2$ = Th($\mathcal{F} \cup$ {right-arm-usable(John)})

$\mathcal{A}_2$ = Th($\mathcal{F} \cup$ {right-arm-usable(John),

right-arm-usable(John)$\wedge\neg$right-arm-broken(John)}).

But, the default δ_1 is inapplicable since "$\neg$left-arm-broken(John)" is inferable from $\mathcal{A}_2$. Therefore, both facts "left-arm-usable(John)" and "right-arm-usable(John)" are also not in a-$\mathcal{E}_2$. It can be shown that a-$\mathcal{E}_1$ and a-$\mathcal{E}_2$ are the only alternative extensions for this default theory. Furthermore, both alternative extensions follow intuition.

▒

The following example shows a pair of sets satisfying the conditions R1, R2, and R3 but it is not necessarily a pair of an alternative extension and an assumption set.

Example 3.2.7: The necessity for the smallest sets

Let $\mathcal{D}$ = ($\mathcal{F}$, Δ = {δ_1, δ_2}) be a normal default theory, where

$$\mathfrak{F} = \{P\},\ \delta_1 = \frac{P:Q}{Q},\ \text{and}\ \delta_2 = \frac{R:S}{S}.$$

It is easy to see that a-$\mathfrak{E}$ = Th({P, Q}) is an alternative extension for $\mathfrak{D}$ based on assumption set $\mathcal{A}$ = Th({P, Q}).

Now, let us examine a pair of sets (E = Th({P, Q, S}), A = Th({P, Q, S})). Since $\mathfrak{F} \subseteq E$ and $\mathfrak{F} \subseteq A$, the pair (E, A) satisfies the condition R1. Also, the pair (E, A) satisfies the condition R2 because Th(E) = Th(Th({P, Q, S})) = Th({P, Q, S}) = E and Th(A) = Th(Th({P, Q, S})) = Th({P, Q, S}) = A. Lastly, the default δ_1 is applicable since $P \in E$ and $\neg Q \notin A$, causing $Q \in E$ and $Q \in A$. So, it satisfies R3. However, S is in E but it is neither a hard fact, a consequent of an applicable default, nor a fact derivable from the set of hard facts and the set of consequents of applicable defaults. Therefore, E is not an alternative extension for $\mathfrak{D}$.

3.2.3 Basic Properties of the New Approach

In this section we will see some properties of alternative extensions for a default theory. We will attempt to provide and prove some properties similar to the Reiter and Lukaszewicz approaches.

In the Reiter approach, a similar theorem, which more intuitively characterizes an alternative extension based on an assumption set for arbitrary default theories, can be given. It is called the characterization theorem. In the theorem, we construct a sequence of pairs, called a sequence of pairs related to a pair of sets. The following theorem sums up this property.

Theorem 3.2.2: The characterization of alternative extensions for a default theory

Suppose $\mathfrak{D} = (\mathfrak{F}, \Delta)$ is a default theory, and a-$\mathfrak{E}$ and $\mathcal{A}$ are closed well-formed sets. Let (E_0, A_0), (E_1, A_1), ... be a sequence of pairs of closed well-formed formula sets, noted as (E_i, A_i), such that

$$E_0 = \mathcal{W}$$

$$A_0 = \mathcal{W}$$

and, for $i \geq 1$

$$E_i = Th(E_{i-1}) \cup \{C \mid \delta = \frac{A:B}{C} \in \Delta, A \in E_{i-1}, \text{ and } \mathcal{A} \cup \{B, C\} \text{ is consistent}\}$$

$$A_i = Th(A_{i-1}) \cup \{B, C \mid \delta = \frac{A:B}{C} \in \Delta, A \in E_{i-1}, \text{ and } \mathcal{A} \cup \{B, C\} \text{ is consistent}\}$$

A set of closed well-formed formulae a-$\mathcal{E}$ is an alternative extension based on an assumption set $\mathcal{A}$ for $\mathcal{D}$ if and only if

$$\text{a-}\mathcal{E} = \bigcup_{i=0}^{\infty} E_i$$

and

$$\mathcal{A} = \bigcup_{i=0}^{\infty} A_i$$

This theorem provides a more intuitive way to check whether (a-$\mathcal{E}$, $\mathcal{A}$) of closed well-formed formula sets satisfies as a pair of an alternative extension and an assumption set for a default theory, that is, by constructing a sequence of pairs of sets. The pair satisfies as an alternative extension and an assumption set if and only if the pair is identical with the pair of the unions of all elements of the constructed sequence, i.e., (a-$\mathcal{E}$, $\mathcal{A}$) = ($\bigcup_{i=0}^{\infty} E_i$, $\bigcup_{i=0}^{\infty} A_i$).

In this theorem, we characterize the difference between two consecutive sets in the sequence. The next set in the sequence consists of a theory of the predecessor set and all consequences of applicable defaults with respect to the predecessor set; i.e., $E_{i+1} - Th(E_i) = \{C \mid C$s are consequents of applicable default δs using the pair (E_i, $\mathcal{A}$)$\}$ and $A_{i+1} - Th(A_i) = \{B, C \mid B$s and Cs are justifications and consequents, respectively, of applicable default δs using the pair (E_i, $\mathcal{A}$)$\}$. In addition, for normal default theories, $E_i = A_i$.

In the following, we will present a lemma that is very useful in the proofs of theorems in the entire paper. It is related to the construction of a sequence as in theorem 3.2.2.

Lemma 3.2.1:

Suppose that a sequence of sets A_0, A_1, ... such that $Th(A_{i-1}) \subseteq A_i$, for $i \geq 1$. Then, $Th(\bigcup_{i=0}^{\infty} A_i) = \bigcup_{i=0}^{\infty} A_i$.

▒

Proof: Lemma 3.2.1

Assume that $Th(A_{i-1}) \subseteq A_i$, for $i \geq 1$. By the monotonicity properties, $\bigcup_{i=0}^{\infty} A_i \subseteq Th(\bigcup_{i=0}^{\infty} A_i)$. Suppose that $X \in Th(\bigcup_{i=0}^{\infty} A_i)$. By the compactness of first order theory, there is a finite set S such that $X \in Th(S)$. By the finiteness of S, there is an A_n such that $S \subseteq A_n$. $X \in Th(A_n)$ since $Th(S) \subseteq Th(A_n)$ by the monotonicity. So, X is in $Th(A_{n+1})$ and also in $\bigcup_{i=0}^{\infty} A_i$. Hence, $Th(\bigcup_{i=0}^{\infty} A_i) \subseteq \bigcup_{i=0}^{\infty} A_i$. We conclude that $Th(\bigcup_{i=0}^{\infty} A_i) = \bigcup_{i=0}^{\infty} A_i$.

▒

In order to prove theorem 3.2.2, we need the following three lemmas. The first lemma concerns the property of the pair ($\bigcup_{i=0}^{\infty} E_i$, $\bigcup_{i=0}^{\infty} A_i$) satisfying R1, R2, and R3. The last two lemmas concern the sufficient and necessary conditions for (a-$\mathcal{E}$, $\mathcal{A}$) as a pair of an alternative extension and an assumption set for a default theory related to the sequence.

Lemma 3.2.2:

Let a sequence of pairs (E_i, A_i) be constructed as in theorem 3.2.2 using a pair (a-$\mathcal{E}$, $\mathcal{A}$). Then,

$$\Gamma_2(\text{a-}\mathcal{E}, \mathcal{A}) \subseteq \bigcup_{i=0}^{\infty} A_i \quad \text{and} \quad \Gamma_1(\text{a-}\mathcal{E}, \mathcal{A}) \subseteq \bigcup_{i=0}^{\infty} E_i.$$

▒

This lemma says that the pair of the unions of all elements of sequences E_0, E_1, ... and A_0, A_1, ... satisfies conditions R1, R2, and R3.

Proof: Lemma 3.2.2

Since (Γ_1(a-$\mathfrak{E}$, $\mathcal{A}$), Γ_2(a-$\mathfrak{E}$, $\mathcal{A}$)) is a pair of the smallest sets satisfying the three conditions R1, R2, and R3, it is enough to prove that the pair ($\bigcup_{i=0}^{\infty} E_i$, $\bigcup_{i=0}^{\infty} A_i$) satisfies R1, R2, and R3.

R1. Both E_0 and A_0 contain $\mathcal{F}$. Therefore, the pair ($\bigcup_{i=0}^{\infty} E_i$, $\bigcup_{i=0}^{\infty} A_i$) satisfies R1.

R2. By lemma 3.2.1, we conclude $Th(\bigcup_{i=0}^{\infty} E_i) = \bigcup_{i=0}^{\infty} E_i$ and $Th(\bigcup_{i=0}^{\infty} A_i) = \bigcup_{i=0}^{\infty} A_i$. So the pair ($\bigcup_{i=0}^{\infty} E_i$, $\bigcup_{i=0}^{\infty} A_i$) satisfies R2.

R3. We need to prove that if $\delta = \frac{A:B}{C} \in \Delta$, $A \in \bigcup_{i=0}^{\infty} E_i$, and $\{B, C\} \cup \bigcup_{i=0}^{\infty} A_i$ is consistent, then $C \in \bigcup_{i=0}^{\infty} E_i$ and $B, C \in \bigcup_{i=0}^{\infty} A_i$.

Suppose that $\delta = \frac{A:B}{C} \in \Delta$, $A \in \bigcup_{i=0}^{\infty} E_i$, and $\{B, C\} \cup \bigcup_{i=0}^{\infty} A_i$ is consistent. Since $E_{i-1} \subseteq E_i$, for $i \geq 1$, there is an n such that $A \in E_n$ by $A \in \bigcup_{i=0}^{\infty} E_i$. Since $\{B, C\} \cup \bigcup_{i=0}^{\infty} A_i$ is consistent, then $\{B, C\} \cup A_n$ is consistent. Hence, $C \in E_{n+1}$ and $B, C \in A_{n+1}$ by the construction of E_i and A_i. Therefore, $C \in \bigcup_{i=0}^{\infty} E_i$ and $B, C \in \bigcup_{i=0}^{\infty} A_i$.

∎

Next is the second lemma. In the construction, no fact is added to the sequence of pairs without a reason, if the construction is based on a pair of an alternative extension and an assumption set.

Lemma 3.2.3:

Let a-$\mathfrak{E}$ be an alternative extension based on an assumption set $\mathcal{A}$ for a default theory. Suppose a sequence of pairs (E_i, A_i) constructed as in theorem 3.2.2 using a pair (a-$\mathfrak{E}$, $\mathcal{A}$). Then,

$\bigcup_{i=0}^{\infty} E_i \subseteq \Gamma_1$(a-$\mathfrak{E}$, $\mathcal{A}$) and $\bigcup_{i=0}^{\infty} A_i \subseteq \Gamma_2$(a-$\mathfrak{E}$, $\mathcal{A}$).

∎

Proof: Lemma 3.2.3

Assume that a-$\mathfrak{E}$ is an alternative extension based on an assumption set $\mathcal{A}$ for a default theory $\mathcal{D}$; i.e., a-$\mathfrak{E}$ = Γ_1(a-$\mathfrak{E}$, $\mathcal{A}$) and $\mathcal{A}$ = Γ_2(a-$\mathfrak{E}$, $\mathcal{A}$). We prove that

$\bigcup_{i=0}^{\infty} E_i \subseteq \Gamma_1$(a-$\mathcal{E}$, $\mathcal{A}$) and $\bigcup_{i=0}^{\infty} A_i \subseteq \Gamma_2$(a-$\mathcal{E}$, $\mathcal{A}$) by induction on i that $E_i \subseteq \Gamma_1$(a-$\mathcal{E}$,$\mathcal{A}$) and $A_i \subseteq \Gamma_2$(a-$\mathcal{E}$, $\mathcal{A}$) for all i.

Basis: $E_0 = \mathcal{F} \subseteq \Gamma_1$(a-$\mathcal{E}$, $\mathcal{A}$) and $A_0 = \mathcal{F} \subseteq \Gamma_2$(a-$\mathcal{E}$, $\mathcal{A}$).

Hypothesis: $E_i \subseteq \Gamma_1$(a-$\mathcal{E}$, $\mathcal{A}$) implies $E_{i+1} \subseteq \Gamma_1$(a-$\mathcal{E}$, $\mathcal{A}$) and

$A_i \subseteq \Gamma_2$(a-$\mathcal{E}$, $\mathcal{A}$) implies $A_{i+1} \subseteq \Gamma_2$(a-$\mathcal{E}$, $\mathcal{A}$).

Step: Assume $E_n \subseteq \Gamma_1$(a-$\mathcal{E}$, $\mathcal{A}$) and $A_n \subseteq \Gamma_2$(a-$\mathcal{E}$, $\mathcal{A}$). By the monotonicity properties and R2, Th(E_n) $\subseteq \Gamma_1$(a-$\mathcal{E}$, $\mathcal{A}$) and Th(A_n) $\subseteq \Gamma_2$(a-$\mathcal{E}$, $\mathcal{A}$).

If E_{n+1} = Th(E_n), and so is A_{n+1} = Th(A_n), we then finish the proof. (Recall that, for $n \geq 1$,

E_{n+1} - Th(E_n) = $\{C \mid \delta = \frac{A{:}B}{C} \in \Delta, A \in E_n$ and $\{B, C\} \cup \mathcal{A}$ is consistent$\}$

A_{n+1} - Th(A_n) = $\{B, C \mid \delta = \frac{A{:}B}{C} \in \Delta, A \in E_n$ and $\{B, C\} \cup \mathcal{A}$ is consistent$\}$.)

Otherwise, suppose there are B, C such that $C \in E_{n+1}$ - Th(E_n) and B, $C \in A_{n+1}$ - Th(A_n). So, there is a default $\delta = \frac{A{:}B}{C} \in \Delta$ such that $A \in E_n$ and $\{B, C\} \cup \mathcal{A}$ is consistent. By $E_n \subseteq \Gamma_1$(a-$\mathcal{E}$, $\mathcal{A}$), $A \in E_n$ implies $A \in \Gamma_1$(a-$\mathcal{E}$, $\mathcal{A}$). Since $A \in \Gamma_1$(a-$\mathcal{E}$, $\mathcal{A}$) and $\{B, C\} \cup \mathcal{A}$ is consistent, C is in Γ_1(a-$\mathcal{E}$, $\mathcal{A}$) and B, C are in Γ_2(a-$\mathcal{E}$, $\mathcal{A}$) by R3. Since B and C are arbitrary, $E_{n+1} \subseteq \Gamma_1$(a-$\mathcal{E}$, $\mathcal{A}$) and $A_{n+1} \subseteq \Gamma_2$(a-$\mathcal{E}$, $\mathcal{A}$).

∎

The following is the third lemma. It says that if the pair of the union of all E_is and the union of all A_is in the sequence is identical with the pair that is used as a basis for the construction, then no fact is added to the sequence of pairs without any reasons.

Lemma 3.2.4:

Let a sequence of pairs (E_i, A_i) be constructed as in theorem 3.2.2 using a pair (a-$\mathcal{E}$, $\mathcal{A}$). If a-$\mathcal{E}$ = $\bigcup_{i=0}^{\infty} E_i$ and $\mathcal{A} = \bigcup_{i=0}^{\infty} A_i$, then,

$\bigcup_{i=0}^{\infty} E_i \subseteq \Gamma_1$(a-$\mathcal{E}$, $\mathcal{A}$) and $\bigcup_{i=0}^{\infty} A_i \subseteq \Gamma_2$(a-$\mathcal{E}$, $\mathcal{A}$).

∎

Proof: Lemma 3.2.4

We prove $\cup_{i=0}^{\infty} E_i \subseteq \Gamma_1(\text{a-}\mathcal{E}, \mathcal{A})$ and $\cup_{i=0}^{\infty} A_i \subseteq \Gamma_2(\text{a-}\mathcal{E}, \mathcal{A})$ by induction on i that $E_i \subseteq \Gamma_1(\text{a-}\mathcal{E}, \mathcal{A})$ and $A_i \subseteq \Gamma_2(\text{a-}\mathcal{E}, \mathcal{A})$ for all i.

Basis: $E_0 = \mathcal{F} \subseteq \Gamma_1(\text{a-}\mathcal{E}, \mathcal{A})$ and $A_0 = \mathcal{F} \subseteq \Gamma_2(\text{a-}\mathcal{E}, \mathcal{A})$.

Hypothesis: $E_i \subseteq \Gamma_1(\text{a-}\mathcal{E}, \mathcal{A})$ implies $E_{i+1} \subseteq \Gamma_1(\text{a-}\mathcal{E}, \mathcal{A})$ and

$A_i \subseteq \Gamma_2(\text{a-}\mathcal{E}, \mathcal{A})$ implies $A_{i+1} \subseteq \Gamma_2(\text{a-}\mathcal{E}, \mathcal{A})$.

Step: Assume that $E_n \subseteq \Gamma_1(\text{a-}\mathcal{E}, \mathcal{A})$ and $A_n \subseteq \Gamma_2(\text{a-}\mathcal{E}, \mathcal{A})$. By the monotonicity properties and by R2, $Th(E_n) \subseteq \Gamma_1(\text{a-}\mathcal{E}, \mathcal{A})$ and $Th(A_n) \subseteq \Gamma_2(\text{a-}\mathcal{E}, \mathcal{A})$.

If $E_{n+1} = Th(E_n)$, and so is $A_{n+1} = Th(A_n)$, we then finish the proof. Otherwise, suppose there are B, C such that $C \in E_{n+1} - Th(E_n)$ and $B, C \in A_{n+1} - Th(A_n)$. So, there is a default $\delta = \frac{A:B}{C} \in \Delta$ such that $A \in E_n$ and $\{B, C\} \cup \mathcal{A}$ is consistent. $A \in E_n$ implies $A \in \cup_{i=0}^{\infty} E_i = \text{a-}\mathcal{E}$. Since $A \in \text{a-}\mathcal{E}$ and $B, C \cup \mathcal{A}$ consistent, $C \in \Gamma_1(\text{a-}\mathcal{E}, \mathcal{A})$ and $B, C \in \Gamma_2(\text{a-}\mathcal{E}, \mathcal{A})$ by R3. Since B and C are arbitrary, $E_{n+1} \subseteq \Gamma_1(\text{a-}\mathcal{E}, \mathcal{A})$ and $A_{n+1} \subseteq \Gamma_2(\text{a-}\mathcal{E}, \mathcal{A})$.

∎

Now, we proceed to prove characterization theorem 3.2.2 above.

Proof: Theorem 3.2.2

We need to prove that: $\Gamma_1(\text{a-}\mathcal{E}, \mathcal{A}) = \cup_{i=0}^{\infty} E_i$ and $\Gamma_2(\text{a-}\mathcal{E}, \mathcal{A}) = \cup_{i=0}^{\infty} A_i$.

(if part)

Suppose that a-$\mathcal{E}$ is an alternative extension based on an assumption set $\mathcal{A}$ for a default theory $\mathcal{D}$; i.e., $\text{a-}\mathcal{E} = \Gamma_1(\text{a-}\mathcal{E}, \mathcal{A})$ and $\mathcal{A} = \Gamma_2(\text{a-}\mathcal{E}, \mathcal{A})$. So,

1. $\Gamma_1(\text{a-}\mathcal{E}, \mathcal{A}) \subseteq \cup_{i=0}^{\infty} E_i$ and $\Gamma_2(\text{a-}\mathcal{E}, \mathcal{A}) \subseteq \cup_{i=0}^{\infty} A_i$ by lemma 3.2.2; and
2. $\Gamma_1(\text{a-}\mathcal{E}, \mathcal{A}) \supseteq \cup_{i=0}^{\infty} E_i$ and $\Gamma_2(\text{a-}\mathcal{E}, \mathcal{A}) \supseteq \cup_{i=0}^{\infty} A_i$ by lemma 3.2.3.

(if part)

Suppose that $\text{a-}\mathcal{E} = \cup_{i=0}^{\infty} E_i$ and $\mathcal{A} = \cup_{i=0}^{\infty} A_i$. Therefore,

1. $\Gamma_1(\text{a-}\mathcal{E}, \mathcal{A}) \subseteq \cup_{i=0}^{\infty} E_i$ and $\Gamma_2(\text{a-}\mathcal{E}, \mathcal{A}) \subseteq \cup_{i=0}^{\infty} A_i$ by lemma 3.2.2; and
2. $\Gamma_1(\text{a-}\mathcal{E}, \mathcal{A}) \supseteq \cup_{i=0}^{\infty} E_i$ and $\Gamma_2(\text{a-}\mathcal{E}, \mathcal{A}) \supseteq \cup_{i=0}^{\infty} A_i$ by lemma 3.2.4.

Therefore, the theorem is proved.

▒

Let us find a sequence of pairs for the first problem in example 3.2.1 above.

Example 3.2.8:

Consider the default theory $\mathcal{D} = (\mathfrak{F}, \Delta)$ in example 3.2.1 above, where

$\mathfrak{F}$ = {bird(Tweety), bird(Clyde), broken-wing(Tweety)$\vee$broken-wing(Clyde)}

and

$$\delta = \frac{\text{bird}(x)\text{:fly}(x) \wedge \neg\text{broken-wing}(x)}{\text{fly}(x)}.$$

An alternative extension for $\mathcal{D}$ is a-$\mathcal{E}_1$,

a-$\mathcal{E}_1$ = Th({bird(Tweety), bird(Clyde), fly(Tweety),

broken-wing(Tweety)$\vee$broken-wing(Clyde)}),

based on an assumption set $\mathcal{A}_1$,

$\mathcal{A}_1$ = Th({bird(Tweety), bird(Clyde), fly(Tweety)$\wedge\neg$broken-wing(Tweety),

fly(Tweety), broken-wing(Tweety)$\vee$broken-wing(Clyde)}).

The sequence of pairs related to pair (a-$\mathcal{E}_1$, $\mathcal{A}_1$) is (E_i, A_i), where

(E_0, A_0) = ($\mathfrak{F}$, $\mathfrak{F}$),

(E_1, A_1) = (Th($\mathfrak{F}$)$\cup${fly(Tweety)},

Th($\mathfrak{F}$)$\cup${fly(Tweety),fly(Tweety)$\wedge\neg$broken-wing(Tweety)}),

and for $i \geq 2$,

(E_i, A_i) = (Th($\mathfrak{F}\cup${fly(Tweety)})$\cup${fly(Tweety)},

Th($\mathfrak{F}\cup${fly(Tweety),fly(Tweety)$\wedge\neg$broken-wing(Tweety)}) $\cup$

{fly(Tweety),fly(Tweety)$\wedge\neg$broken-wing(Tweety)})

= (Th($\mathfrak{F}\cup${fly(Tweety)}),

Th($\mathfrak{F}\cup${fly(Tweety),fly(Tweety)$\wedge\neg$broken-wing(Tweety)}).

▒

Let us verify the previous example 3.2.7 about the smallness of an alternative extension for a default theory using the characterization theorem.

Example 3.2.9:

Consider a default theory $\mathfrak{D} = (\mathfrak{F}, \Delta=\{\delta_1, \delta_2\})$, where $\mathfrak{F} = \{P\}$, $\delta_1 = \frac{P:Q}{Q}$, and $\delta_2 = \frac{R:S}{S}$. We prove by a contradiction that E = Th({P, Q, S}) is not an alternative extension; that is, suppose that E is an alternative extension. So, there is an assumption set A. Since $\mathfrak{D}$ is a normal default theory, then A = E by theorem 3.2.1. Let us construct a sequence of pairs (E_i, A_i) related to the pair (E, A). That is,

$$E_0 = A_0 = \mathfrak{F} = \{P\},$$

$$E_1 = A_1 = Th(\{P\}) \cup \{Q\},$$

$$E_2 = A_2 = Th(Th(\{P\} \cup \{Q\}) \cup \{Q\} = Th(\{P, Q\}),$$

and, for $i \geq 3$

$$E_i = A_i = Th(E_2) \cup \{Q\} = Th(Th(\{P, Q\}) \cup \{Q\} = Th(\{P, Q\}) = E_2.$$

Therefore, $\cup_{i=0}^{\infty} E_i = Th(\{P, Q\})$. But, $E \neq \cup_{i=0}^{\infty} E_i$ and $A \neq \cup_{i=0}^{\infty} A_i$. This is the contradiction. So, it cannot be that E is an alternative extension. This is because (E, A) is not a pair of the smallest sets; i.e., $S \in E$ but is neither a hard fact, a consequent of an applicable default, nor derivable from the set of hard facts and consequents of applicable defaults.

∎

Now, we are ready to describe the relation of an alternative extension and an extension for normal default theories; that is, every alternative extension for a normal default theory is an extension for the default theory.

Theorem 3.2.3: Identity of alternative extensions for normal default theories

Suppose that a-$\mathfrak{E}$ is an alternative extension for a normal default theory $\mathfrak{D}$ based on an assumption set $\mathcal{A}$. Therefore, a-$\mathfrak{E}$ is an extension for $\mathfrak{D}$.

∎

Proof: Theorem 3.2.3

Let a-$\mathfrak{E}$ be an alternative extension based on an assumption set $\mathcal{A}$ for a normal default theory $\mathfrak{D} = (\mathfrak{F}, \Delta)$. So, a-$\mathfrak{E} = \mathcal{A}$ by theorem 3.2.1. Now, by characterization theorem 3.2.2, we construct a sequence of pairs (E_i, A_i), where

$$E_0 = A_0 = \mathfrak{F}$$

and, for $i \geq 1$

$$E_i = A_i = Th(E_{i-1}) \cup \{B \mid \delta = \tfrac{A:B}{B} \in \Delta,\ A \in E_{i-1}, \text{ and a-}\mathfrak{E} \cup \{B\} \text{ is consistent}\}$$

such that a-$\mathfrak{E} = \bigcup_{i=0}^{\infty} E_i$ and $\mathcal{A} = \bigcup_{i=0}^{\infty} A_i$. Construct a sequence of sets $E'_0, E'_1, \ldots$ related to a-$\mathfrak{E}$ according to Reiter's characterization theorem, 2.2.1:

$$E'_0 = \mathfrak{F},$$

and, for $i \geq 1$

$$E'_i = Th(E'_{i-1}) \cup \{B \mid \delta = \tfrac{A:B}{B} \in \Delta,\ A \in E'_{i-1}, \text{ and } \neg B \notin \text{a-}\mathfrak{E}\}.$$

To prove that a-$\mathfrak{E}$ is an extension for $\mathfrak{D}$ (i.e., a-$\mathfrak{E} = \bigcup_{i=0}^{\infty} E'_i$), it is enough to show $E'_i = E_i$ for $i \geq 0$. We prove it by induction.

Basis: $E'_0 = E_0 = \mathfrak{F}$.

Hypothesis: $(E_i = E'_i) \rightarrow (E_{i+1} = E'_{i+1})$.

Step: Assume that $E'_n = E_n$. It implies that $Th(E_n) = Th(E'_n)$, and also, $Th(E_n) \subseteq E_{n+1}$ and $Th(E'_n) \subseteq E'_{n+1}$ by the monotonicity properties.

We prove that $E'_{n+1} = E_{n+1}$ by showing $E_{n+1} \subseteq E'_{n+1}$ and $E'_{n+1} \subseteq E_{n+1}$. If $E_{n+1} = Th(E_n)$, then we have proven that $E_{n+1} \subseteq E'_{n+1}$ since $Th(E_n) = Th(E'_n) \subseteq E'_{n+1}$. Otherwise, suppose there is B in $E_{n+1} - Th(E_n)$. By the construction of E_{n+1}, there is a default $\delta = \frac{A:B}{B} \in \Delta$ such that $A \in E_n$ and B is consistent with a-$\mathfrak{E}$. Since $E'_n = E_n$, $A \in E'_n$, and since $B \in E_{n+1}$, $\neg B$ is not in a-$\mathfrak{E}$. Therefore, B must be in E'_{n+1} by the construction of E'_{n+1}. Hence, $E_{n+1} \subseteq E'_{n+1}$ since $Th(E_n) = Th(E'_n) \subseteq E'_{n+1}$.

If $E'_{n+1} = Th(E'_n)$, then we have proven that $E'_{n+1} \subseteq E_{n+1}$ since $Th(E'_n) = Th(E_n) \subseteq E_{n+1}$. Otherwise, suppose there is B in $E'_{n+1} - Th(E'_n)$. By the construction of E'_{n+1}, there is a default $\delta = \frac{A:B}{B} \in \Delta$ such that $A \in E'_n$ and $\neg B \notin$ a-$\mathfrak{E}$. By assumption $E'_n = E_n$, then, $A \in E_n$. Also, B is consistent with a-$\mathfrak{E}$. Therefore, B must be in E_{n+1} by the construction of E_{n+1}. Hence, $E'_{n+1} \subseteq E_{n+1}$ since $Th(E'_n) = Th(E_n) \subseteq E_{n+1}$.

▒

In the next theorem, we will see the maximality of alternative extensions. That is, if there are two pairs (a-$\mathfrak{E}$, $\mathcal{A}$) and (a-$\mathfrak{E}'$, $\mathcal{A}'$) of alternative extensions and assumption sets, and a-$\mathfrak{E} \subseteq$ a-$\mathfrak{E}'$ and $\mathcal{A} \subseteq \mathcal{A}'$, then the pairs are identical.

Theorem 3.2.4: Maximality of alternative extensions

Suppose that a-$\mathfrak{E}$ and a-$\mathfrak{E}'$ are alternative extensions for a default theory $\mathfrak{D} = (\mathfrak{F}, \Delta)$ based on assumption sets $\mathcal{A}$ and $\mathcal{A}'$, respectively, such that a-$\mathfrak{E} \subseteq$ a-$\mathfrak{E}'$ and $\mathcal{A} \subseteq \mathcal{A}'$. Then, a-$\mathfrak{E}$ = a-$\mathfrak{E}'$ and $\mathcal{A} = \mathcal{A}'$.

▒

That is, there is no pair of an alternative extension and assumption set such that it properly "supersedes" another pair of an alternative extension and an assumption set.

Proof: Theorem 3.2.4

Assume that (a-$\mathfrak{E}$, $\mathcal{A}$) and (a-$\mathfrak{E}'$, $\mathcal{A}'$) are pairs of alternative extensions and assumption sets for a default theory $\mathfrak{D} = (\mathfrak{F}, \Delta)$ such that a-$\mathfrak{E} \subseteq$ a-$\mathfrak{E}'$ and $\mathcal{A} \subseteq \mathcal{A}'$. Suppose that the sequences of pairs (E_i, A_i) and (E'_i, A'_i) are related to the pairs (a-$\mathfrak{E}$, $\mathcal{A}$) and (a-$\mathfrak{E}'$, $\mathcal{A}'$), respectively, as in characterization theorem 3.2.2. By hypotheses a-$\mathfrak{E} \subseteq$ a-$\mathfrak{E}'$ and $\mathcal{A} \subseteq \mathcal{A}'$, we have

$$\cup_{i=0}^{\infty} E_i \subseteq \cup_{i=0}^{\infty} E'_i \text{ and } \cup_{i=0}^{\infty} A_i \subseteq \cup_{i=0}^{\infty} A'_i.$$

So, it remains to be proven that

$$\cup_{i=0}^{\infty} E'_i \subseteq \cup_{i=0}^{\infty} E_i \text{ and } \cup_{i=0}^{\infty} A'_i \subseteq \cup_{i=0}^{\infty} A_i.$$

We prove them by an induction on i that $E'_i \subseteq \cup_{i=0}^{\infty} E_i$ and $A'_i \subseteq \cup_{i=0}^{\infty} A_i$.

Basis: $E'_0 \subseteq E_0$ and $A'_0 \subseteq A_0$ since $E'_0 = E_0 = A'_0 = A_0 = \mathcal{F}$.

Hypothesis: For all i, $(E'_i \subseteq E_i) \rightarrow (E'_{i+1} \subseteq E_{i+1})$.

Step: Assume that $E'_n \subseteq E_n$ and $A'_n \subseteq A_n$. By the monotonicity property first order theory, $Th(E'_n) \subseteq Th(E_n)$ and $Th(A'_n) \subseteq Th(A_n)$.

First, we show that $E'_{n+1} \subseteq E_{n+1}$. If $E'_{n+1} = Th(E'_n)$, then we have proven that $E'_{n+1} \subseteq E_{n+1}$ since $Th(E'_n) \subseteq Th(E_n) \subseteq E_{n+1}$. Otherwise, suppose there is C such that $C \in E'_{n+1}$ but $C \notin Th(E'_n)$. So, by the construction of E'_{n+1}, there is a default $\delta = \frac{A:B}{C} \in \Delta$ such that $A \in E'_n$ and $\{B, C\} \cup \mathcal{A}'$ is consistent. Since $A \in E'_n$, A is in E_n by assumption $E'_n \subseteq E_n$. In addition, $\{B, C\} \cup \mathcal{A}$ is consistent since $\{B, C\} \cup \mathcal{A}'$ is consistent and $\mathcal{A} \subseteq \mathcal{A}'$. So, C must be in E_{n+1} by the construction of E_{n+1}. Since C is arbitrary in E'_{n+1}, therefore, $E'_{n+1} \subseteq E_{n+1}$.

Now, we prove $A'_{n+1} \subseteq A_{n+1}$. If $A'_{n+1} = Th(A'_n)$, then we have proven that $A'_{n+1} \subseteq A_{n+1}$ since $Th(A'_n) \subseteq Th(A_n) \subseteq A_{n+1}$. Otherwise, suppose there is an X such that $X \in A'_{n+1} - Th(A'_n)$. So, by the construction of A'_{n+1}, there is a default $\delta = \frac{A:B}{C} \in \Delta$, where either $X=C$ or $X=B$, such that $A \in E'_n$ and $\{B, C\} \cup \mathcal{A}'$ is consistent. So, A is in E_n, and $\{B, C\} \cup \mathcal{A}$ is consistent. Therefore, B and C must be in A_{n+1}, and so must X. Since $Th(A'_n) \subseteq Th(A_n) \subseteq A_{n+1}$, hence, $A'_{n+1} \subseteq A_{n+1}$

∎

Now, we present the semi-monotonicity properties of alternative extensions for default theories. We will show that adding new defaults to a default theory cannot shrink an alternative extension.

Theorem 3.2.5: Semi-monotonicity properties of alternative extensions

Let $\mathcal{D} = (\mathcal{F}, \Delta)$ and $\mathcal{D}' = (\mathcal{F}, \Delta')$ be default theories, where $\Delta \subseteq \Delta'$. For every alternative extension a-$\mathcal{E}$ based on assumption set $\mathcal{A}$ for $\mathcal{D}$, there is an alternative extension a-$\mathcal{E}'$ based on an assumption set $\mathcal{A}'$ for $\mathcal{D}'$ such that a-$\mathcal{E} \subseteq$

a-$\mathcal{E}'$ and $\mathcal{A} \subseteq \mathcal{A}'$.

Before proving this theorem, we will make some clarification. In the definition of a default theory in Chapter 2, we assume that the number of defaults in a default set is countable. Further, we also discussed partitioning defaults in Δ into an equivalence class of sets of defaults having the same nature. The purpose of partitioning is to abbreviate writing a lengthy default set. Therefore, it does not change the meaning of each default. In addition, we can enumerate the default set of a default theory; i.e., $\Delta = \{\delta_1, \delta_2, \delta_3, \ldots\}$. This set is possibly finite. Also, recall that a union of two countable sets is countable.

To prove the semi-monotonicity properties, we need some tools. First, we construct candidate sets for an alternative extension and its assumption set for $\mathcal{D}'$ based on construction 3.1. The construction is started with the alternative extension a-$\mathcal{E}$ and assumption set $\mathcal{A}$ for $\mathcal{D}$. Second, we prove that the candidate sets are the alternative extension and assumption set for the modified default theory $\mathcal{D}'$ using characterization theorem 3.2.1.

Construction 3.1:

Let a-$\mathcal{E}$ be an alternative extension based on an assumption set $\mathcal{A}$ for a default theory $\mathcal{D}=(\mathcal{F}, \Delta)$. Suppose $\mathcal{D}'=(\mathcal{F}, \Delta')$ is a default theory such that $\Delta \subseteq \Delta'$. Suppose, further, that $\{\delta_1 = \frac{A_1 : B_1}{C_1}, \delta_2 = \frac{A_2 : B_2}{C_2}, \delta_3 = \frac{A_3 : B_3}{C_3}, \ldots\}$ is a fixed enumeration for Δ'. We construct a sequence of pairs (E_i, A_i) as follows:

$E_0 =$ a-$\mathcal{E}$ and $A_0 = \mathcal{A}$,

and for $i \geq 1$,

$E_i = Th(E_{i-1}) \cup \bigcup_{j=0}^{\infty} E_i^j$

$A_i = Th(A_{i-1}) \cup \bigcup_{j=0}^{\infty} A_i^j$

where

$E_i^0 = E_{i-1}$,

$A_i^0 = A_{i-1}$,

and for $j \geq 1$, (we attempt to apply all default δ_js, one by one, using E_{i-1}),

$$E_i^j = E_i^{j-1} \cup \{C_j \mid \delta_j = \tfrac{A_j:B_j}{C_j} \in \Delta',\ A_j \in E_{i-1}, \text{ and } \{B_j, C_j\} \cup A_i^{j-1} \text{ is consistent}\}$$

$$A_i^j = A_i^{j-1} \cup \{C_j, B_j \mid \delta_j = \tfrac{A_j:B_j}{C_j} \in \Delta',\ A_j \in E_{i-1}, \text{ and } \{B_j, C_j\} \cup A_i^{j-1} \text{ is consistent}\}.$$

Assign $E = \bigcup_{i=0}^{\infty} E_i$ and $A = \bigcup_{i=0}^{\infty} A_i$. All sets E_is and A_is are consistent, and so are sets E and A. Since $E_0 = \text{a-}\mathcal{E}$ and $A_0 = \mathcal{A}$, we conclude that $\text{a-}\mathcal{E} \subseteq E$ and $\mathcal{A} \subseteq A$.

$\blacksquare$

From this construction, we can apply lemma 3.2.1 to the pair (E, A) resulting in Th(A) = A and Th(E) = E. In addition, we will construct a sequence of pairs related to (E, A) according to characterization theorem 3.2.2 as in construction 3.2.

Construction 3.2: A sequence of pairs using the characterization theorem.

Let $\mathcal{D}$ and $\mathcal{D}'$ be default theories and a-$\mathcal{E}$ be an alternative extension based on an assumption set $\mathcal{A}$ for $\mathcal{D}$. Assume (E,A) is a pair of sets constructed as in construction 3.1. Construct a sequence of pairs (S_i, T_i) related to the pair (E, A),

$S_0 = \mathcal{F}$ and $T_0 = \mathcal{F}$,

and for $j \geq 1$,

$$S_j = Th(S_{j-1}) \cup \{C \mid \delta = \tfrac{A:B}{C} \in \Delta',\ A \in S_{j-1} \text{ and } \{B, C\} \cup A \text{ is consistent}\},$$

$$T_j = Th(T_{j-1}) \cup \{B, C \mid \delta = \tfrac{A:B}{C} \in \Delta',\ A \in S_{j-1} \text{ and } \{B, C\} \cup A \text{ is consistent}\}.$$

Assign $S = \bigcup_{j=0}^{\infty} S_j$ and $T = \bigcup_{j=0}^{\infty} T_j$.

$\blacksquare$

To prove theorem 3.2.5, we will use one lemma. This lemma assures that S and T in construction 3.2 contain a-$\mathcal{E}$ and A, respectively. It will be used as a basis for an induction in the proof of theorem 3.2.5.

Lemma 3.2.5:

Let $\mathcal{D} = (\mathcal{F}, \Delta)$ and $\mathcal{D}' = (\mathcal{F}, \Delta')$ be default theories, where $\Delta \subseteq \Delta'$. Suppose a-$\mathcal{E}$ is an alternative extension based on assumption set $\mathcal{A}$ for $\mathcal{D}$. If the pair (S, T) is constructed as in construction 3.2, then, a-$\mathcal{E} \subseteq$ S and $\mathcal{A} \subseteq$ T.

Proof: Lemma 3.2.5

Assume that a-$\mathcal{E}$ is an alternative extension based on assumption set $\mathcal{A}$ for $\mathcal{D} = (\mathcal{F}, \Delta)$. S and T are sets assigned in construction 3.2. Using characterization theorem 3.2.3, construct a sequence of pairs (G_i, H_i) related to the pair (a-$\mathcal{E}$, $\mathcal{A}$) using the default theory $\mathcal{D} = (\mathcal{F}, \Delta)$ such that a-$\mathcal{E} = \bigcup_{i=0}^{\infty} G_i$ and $\mathcal{A} = \bigcup_{i=0}^{\infty} H_i$, where $G_0 = H_0 = \mathcal{F}$, and for $i \geq 1$,

$G_i = G_{i-1} \cup \{C \mid \delta = \frac{A:B}{C} \in \Delta, A \in G_{i-1}$ and $\{B, C\} \cup \mathcal{A}$ is consistent$\}$,

$H_i = H_{i-1} \cup \{B, C \mid \delta = \frac{A:B}{C} \in \Delta, A \in G_{i-1}$ and $\{B, C\} \cup \mathcal{A}$ is consistent$\}$.

We prove a-$\mathcal{E} \subseteq$ S and $\mathcal{A} \subseteq$ T by an induction on i that $G_i \subseteq$ S and $H_i \subseteq$ T.

Basis: $G_0 = \mathcal{F} = S_0 \subseteq S$ and $H_0 = \mathcal{F} = T_0 \subseteq T$.

Hypothesis: $G_i \subseteq S$ and $H_i \subseteq T$ imply $G_{i+1} \subseteq S$ and $H_{i+1} \subseteq T$.

Step: Assume $G_n \subseteq S$ and $H_n \subseteq T$. These imply that $Th(G_n) \subseteq S$ and $Th(H_n) \subseteq T$ since $Th(S) = S$ and $Th(T) = T$ by lemma 3.2.1.

First, we prove that $G_{n+1} \subseteq S$. If $Th(G_n) = G_{n+1}$, we have proven $G_{n+1} \subseteq S$. Otherwise, suppose there is C such that $C \in G_{n+1} - Th(G_n)$. C comes from applying a default $\delta \in \Delta$ in the construction of G_{n+1}; i.e., $\delta = \frac{A:B}{C}$, $A \in G_n$ and $\{B, C\} \cup \mathcal{A}$ is consistent. These cause B and C in H_{n+1} by the construction of H_{n+1}. $B, C \in H_{n+1}$ implies $B, C \in \bigcup_{i=0}^{\infty} H_i = \mathcal{A} = A_0$, causing B and C in $\bigcup_{i=0}^{\infty} A_i$ and also in A, since $A = \bigcup_{i=0}^{\infty} A_i$. By assumption $G_n \subseteq S$, $A \in G_n$ implies $A \in S$. Since $\bigcup_{j=0}^{\infty} S_j = S$, there is an m such that $A \in S_m$. Since $A \in S_m$ and $B, C \in A$, then, $C \in S_{m+1}$. So, $C \in S$. Hence, $G_{n+1} \subseteq S$ since $Th(G_n) \subseteq S$ and $G_{n+1} = Th(G_n) \cup \{$those Cs$\}$.

Next, we prove that $H_{n+1} \subseteq T$. If $Th(H_n) = H_{n+1}$, we have proven $H_{n+1} \subseteq T$. Otherwise, suppose there is X such that $X \in H_{n+1} - Th(H_n)$. X comes from applying a default $\delta \in \Delta$ in the construction of H_{n+1}; i.e., $\delta = \frac{A:B}{C}$, where $X=B$ or $X=C$, such that $A\in G_n$ and $\{B, C\}\cup\mathcal{A}$ is consistent. These cause B and C in H_{n+1}. B and C in H_{n+1} implies B and C in $\bigcup_{i=0}^{\infty} H_i=\mathcal{A}=A_0$, causing B and C in $\bigcup_{i=0}^{\infty} A_i$ and also in A. By assumption $G_n\subseteq S$, $A\in G_n$ implies $A\in S$. Since $\bigcup_{j=0}^{\infty} S_j=S$, there is an m such that $A\in S_m$. Since $A\in S_m$ and $B, C\in A$, $B, C\in T_{m+1}$. Therefore, $B, C\in \bigcup_{j=0}^{\infty} T_j$, causing $X\in \bigcup_{j=0}^{\infty} T_j$. $H_{n+1}\subseteq T$ since $Th(H_n)\subseteq T$ and $H_{n+1}=Th(H_n)\cup\{\text{those } X\text{s}\}$.

∎

Now, we have enough tools to prove theorem 3.2.5.

Proof: Theorem 3.2.5

Let $\mathcal{D}=(\mathcal{F}, \Delta)$ and $\mathcal{D}'=(\mathcal{F}, \Delta')$ be default theories, where $\Delta\subseteq\Delta'$. Suppose a-$\mathcal{E}$ is an alternative extension based on assumption set $\mathcal{A}$ for $\mathcal{D}$. Suppose, further, that $\{\delta_1=\frac{A_1:B_1}{C_1}, \delta_2=\frac{A_2:B_2}{C_2}, \delta_3=\frac{A_3:B_3}{C_3}, ...\}$ is a fixed enumeration for Δ'. Construct a pair (E, A) as in construction 3.1 and a pair (S, T) as in construction 3.2. Since a-$\mathcal{E}\subseteq$E and $\mathcal{A}\subseteq$A, it remains to be proven that E is an alternative extension based on A for $\mathcal{D}'$. By characterization theorem 3.2.3, we need to show E = S and A = T. That is, $\bigcup_{i=0}^{\infty} E_i=\bigcup_{j=0}^{\infty} S_j$ and $\bigcup_{i=0}^{\infty} A_i=\bigcup_{j=0}^{\infty} T_j$. First, we prove that $S\subseteq E$ and $T\subseteq A$ by induction on j that $S_j\subseteq E$ and $T_j\subseteq A$.

Basis: $S_0=\mathcal{F}\subseteq E$ and $T_0=\mathcal{F}\subseteq A$ since $E_0=\mathcal{F}$ and $A_0=\mathcal{A}$.

Hypothesis: $S_j\subseteq E$ and $T_j\subseteq A$ imply $S_{j+1}\subseteq E$ and $T_{j+1}\subseteq A$.

Step: Assume $S_n\subseteq E$ and $T_n\subseteq A$. So, $Th(S_n)\subseteq E$ and $Th(T_n)\subseteq A$ since $Th(E)=E$ and $Th(A)=A$ by lemma 3.2.1.

First, we prove $S_{n+1}\subseteq E=\bigcup_{i=0}^{\infty} E_i$. If $S_{n+1}=Th(S_n)$, we have proven that $S_{n+1}\subseteq \bigcup_{i=0}^{\infty} E_i$. Otherwise, suppose there is $C\in S_{n+1}-Th(S_n)$. C comes from applying a default δ with respect to (S_n, T_n) in the construction of S_{n+1}; i.e., a default $\delta=\frac{A:B}{C}\in\Delta'$ such that $A\in S_n$ and $\{B, C\}\cup\bigcup_{i=0}^{\infty} A_i$ is consistent. By assumption $S_n\subseteq E$, $A\in S_n$ causes A to be in E and also in $\bigcup_{i=0}^{\infty} E_i$ since $\bigcup_{i=0}^{\infty} E_i=A$. $A\in \bigcup_{i=0}^{\infty} E_i$ implies that there is an m such that $A\in$

E_m. $\delta\in\Delta'$ implies $\delta=\delta_j$. Also, $\{B, C\}\cup\bigcup_{i=0}^{\infty}A_i$ is consistent, implying that $\{B, C\}\cup A_{m+1}^{j-1}$ is consistent. Hence, C must be in E_{m+1}^{j} and also in E_{m+1}. $S_{n+1}\subseteq\bigcup_{i=0}^{\infty}E_i$ since $S_{n+1}=Th(S_n)\cup\{$those Cs$\}$.

Next, we prove $T_{n+1}\subseteq A=\bigcup_{i=0}^{\infty}A_i$. If $T_{n+1}=Th(T_n)$, we have proven that $T_{n+1}\subseteq A$. Otherwise, suppose there is $X\in T_{n+1}-Th(T_n)$. X comes from applying a default δ with respect to (S_n, T_n) in the construction of T_{n+1}; i.e., a default $\delta=\frac{A:B}{C}\in\Delta'$, where $X=B$ or $X=C$, such that $A\in S_n$ and $\{B, C\}\cup\bigcup_{i=0}^{\infty}A_i$ is consistent. By assumption $S_n\subseteq E$, $A\in S_n$ causes A to be in E and also in $\bigcup_{i=0}^{\infty}E_i$ since $\bigcup_{i=0}^{\infty}E_i=A$. $A\in\bigcup_{i=0}^{\infty}E_i$ implies that there is an m such that $A\in E_m$. $\delta\in\Delta'$ implies $\delta=\delta_j$. Also, $\{B, C\}\cup\bigcup_{i=0}^{\infty}A_i$ is consistent, implying that $\{B, C\}\cup A_{m+1}^{j-1}$ is consistent. Hence, B, C must be in A_{m+1}^{j} and also in A_{m+1}. Since $X=B$ or $X=C$, then, X must be in A_{m+1}. Therefore, $T_{n+1}\subseteq\bigcup_{i=0}^{\infty}A_i$ and also $T_{n+1}\subseteq A$.

We have finished proving $S\subseteq E$ and $T\subseteq A$. Now, we continue proving $E\subseteq S$ and $A\subseteq T$ by induction on i that $E_i\subseteq S$ and $A_i\subseteq T$ for all i.

Basis: $E_0\subseteq S$ and $A_0\subseteq T$ by lemma 3.2.5 since a-$\mathcal{E}=E_0$ and $\mathcal{A}=A_0$.

Hypothesis: $E_i\subseteq S$ and $A_i\subseteq T$ imply $E_{i+1}\subseteq S$ and $A_{i+1}\subseteq T$.

Step: Assume $E_n\subseteq S$ and $A_n\subseteq T$. So, $Th(E_n)\subseteq S$ and $Th(A_n)\subseteq T$ since $Th(S)=S$ and $Th(T)=T$ by lemma 3.2.1.

First, we will prove $E_{n+1}\subseteq S$. If $E_{n+1}=Th(E_n)$, then, we have proven $E_{n+1}\subseteq S$. Otherwise, suppose there is $C_j\in E_{n+1}-Th(E_n)$. C_j comes from applying a default δ_j using E_n in the construction of E_{n+1}^{j}; i.e., $\delta_j=\frac{A_j:B_j}{C_j}\in\Delta'$ such that $A_j\in E_n$ and $\{B_j, C_j\}\cup A_{n+1}^{j-1}$ is consistent. These cause B_j, $C_j\in A_{n+1}^{j}$ and also, B_j, $C_j\in A_{n+1}$. $A_j\in E_n$ implies $A_j\in\bigcup_{i=0}^{\infty}E_i$. Moreover, B_j, $C_j\in A_{n+1}$ implies B_j, $C_j\in\bigcup_{i=0}^{\infty}A_i$, causing $\{B_j, C_j\}\cup\bigcup_{i=0}^{\infty}A_i$ to be consistent. By assumption $E_n\subseteq S$, and $\bigcup_{j=0}^{\infty}S_j=S$, $A_j\in E_n$ implies $A_j\in\bigcup_{j=0}^{\infty}S_j$. $A_j\in\bigcup_{j=0}^{\infty}S_j$ implies there is S_k such that $A_j\in S_k$. Since $A_j\in S_k$ and $\{B_j, C_j\}\cup\bigcup_{i=0}^{\infty}A_i$ is consistent, $C_j \in S_{k+1}$. $E_{n+1} \subseteq \bigcup_{j=0}^{\infty} S_j$ since $E_{n+1} = Th(E_n) \cup \{C_j s\}$. Hence, $E_{n+1} \subseteq S$.

Next, we will prove $A_{n+1} \subseteq T$. If $A_{n+1} = Th(A_n)$, we proved that $A_{n+1} \subseteq T$. Otherwise, suppose there is $X \in A_{n+1} - Th(A_n)$. X comes from applying a default δ_j using E_n in the construction of A^j_{n+1}. That is, a default $\delta_j = \frac{A_j : B_j}{C_j} \in \Delta'$, where either $X = B_j$ or $X = C_j$, such that $A_j \in E_n$ and $\{B_j, C_j\} \cup A^{j-1}_{n+1}$ is consistent, causing $B_j, C_j \in A^j_{n+1}$ and also, $B_j, C_j \in A_{n+1}$. In addition, $B_j, C_j \in A_{n+1}$ implies $B_j, C_j \in \bigcup_{i=0}^{\infty} A_i$, causing $\{B_j, C_j\} \cup \bigcup_{i=0}^{\infty} A_i$ to be consistent. By assumption $E_n \subseteq S$, and $\bigcup_{j=0}^{\infty} S_j = S$, $A_j \in E_n$ implies $A_j \in \bigcup_{j=0}^{\infty} S_j$. $A_j \in \bigcup_{j=0}^{\infty} S_j$ implies that there is S_k such that $A_j \in S_k$. Since $A_j \in S_k$ and $\{B_j, C_j\} \cup \bigcup_{i=0}^{\infty} A_i$ is consistent, then, B_j and $C_j \in$ are in T_{k+1}, and so is X. So, $A_{n+1} \subseteq \bigcup_{j=0}^{\infty} T_j$. Hence, $A_{n+1} \subseteq T$.

We have finished proving $E \subseteq S$ and $A \subseteq T$, and so the theorem.

In the proof of theorem 3.2.5, we assume that a default set Δ' has a fixed enumeration. We apply all defaults, one by one, following the sequence of the fixed numeration to construct a sequence of pairs (E_i, A_i) as in construction 3.1. In this construction, we will have a pair of an alternative extension and an assumption for a default theory. If we have a different enumeration, we may have another similar pair. Therefore, adding some defaults into a default theory may cause the number of pairs of alternative extensions and assumption sets to increase. Each pair will "supersede" its based construction pair.

The theorem is also saying that an alternative extension for a default theory based on an assumption set is monoton when adding defaults.

However, the new approach is not monotonic.

Example 3.2.10: Not monotonicity of alternative extensions related to facts

Let $\mathcal{D} = (\mathcal{F}, \Delta = \{\delta\})$ be a default theory, where $\mathcal{F} = \{bird(Tweety)\}$ and

$\delta = \dfrac{\text{bird}(x):\text{fly}(x)}{\text{fly}(x)}$. As seen before in example 3.2.8, this default theory has only one alternative extension a-$\mathcal{E}$,

a-$\mathcal{E}$ = Th({bird(Tweety), fly(Tweety)})

based on an assumption set $\mathcal{A}$,

$\mathcal{A}$ = Th({bird(Tweety), fly(Tweety)}).

If we add facts "penguin(Tweety)" and "$\forall$x penguin(x)$\rightarrow\neg$fly(x)" to this default theory (i.e., $\mathcal{D}' = (\mathcal{F}', \Delta)$), where

$\mathcal{F}' = \mathcal{F} \cup$ {penguin(Tweety),$\forall$x penguin(x)$\rightarrow\neg$fly(x)})

the new default theory will have an alternative extension a-$\mathcal{E}'$, where

a-$\mathcal{E}'$ = Th({bird(Tweety),penguin(Tweety),$\forall$x penguin(x)$\rightarrow\neg$fly(x)})

based on an assumption set $\mathcal{A}'$ = a-$\mathcal{E}'$. The addition "shrinks" an alternative extension; i.e., a-$\mathcal{E} \not\subset$ a-$\mathcal{E}'$.

If we add facts "bird(Clyde)" to the original default theory, for example $\mathcal{D}''$ = $(\mathcal{F}'', \Delta)$, where $\mathcal{F}'' = \mathcal{F} \cup$ {bird(Clyde)}, the default theory $\mathcal{D}''$ has an extension a-$\mathcal{E}''$, where

a-$\mathcal{E}''$ = Th({bird(Tweety),bird(Clyde),fly(Tweety),fly(Clyde)})

based on an assumption set $\mathcal{A}''$ = a-$\mathcal{E}''$. The addition enlarges an alternative extension; i.e., a-$\mathcal{E} \subseteq$ a-$\mathcal{E}''$.

Furthermore, if we add a fact "fly(Tweety)" to the original default theory, the new default theory has an unchanged alternative extension based on an unchanged assumption set.

Now, the existence of an alternative extension for default theories is guaranteed by the semi--monotonicity properties.

Theorem 3.2.6: Existence of alternative extensions

Every default theory has an alternative extension a-$\mathcal{E}$ based on an assumption set $\mathcal{A}$.

$\blacksquare$

Proof: Theorem 3.2.6

Let $\mathcal{D} = (\mathcal{F}, \Delta)$ be a default theory. The default theory $\mathcal{D}' = (\mathcal{F}, \varnothing)$, where $\varnothing$ denotes the empty default set, has an alternative extension a-$\mathcal{E}' = \mathrm{Th}(\mathcal{F})$ based on itself for $\mathcal{D}'$. Since $\varnothing \subseteq \Delta$, by the semi-monotonicity properties, there is an alternative extension a-$\mathcal{E}$ based on an assumption set $\mathcal{A}$ for $\mathcal{D}$ such that a-$\mathcal{E}' \subseteq$ a-$\mathcal{E}$ and a-$\mathcal{E}' \subseteq \mathcal{A}$.

$\blacksquare$

Let us consider an example in Chapter 2 which does not have an extension.

Example 3.2.11: Non-normal default theory

Let $\mathcal{D} = (\mathcal{F}, \Delta = \{\delta\})$ be a default theory, where $\mathcal{F} = \{P\}$ and $\delta = \dfrac{P{:}Q}{\neg Q}$. This default theory has an alternative extension a-$\mathcal{E} = \mathrm{Th}(\mathcal{F})$ based on an assumption set $\mathcal{A} = \mathrm{Th}(\mathcal{F})$.

$\blacksquare$

The next property of an alternative extension for a default theory is orthogonality. In the Reiter works, the orthogonality property is relative to an extension. In the new approach, the orthogonality property is relative to an assumption set. In case of normal default theories, both approaches have the same meaning. It can be summed up as "the union of two assumption sets of two distinct alternative extensions is inconsistent." It is concluded in the following theorem.

Theorem 3.2.7: Orthogonality of alternative extensions

Let a-$\mathcal{E}$ and a-$\mathcal{E}'$ be two distinct alternative extensions based on assumption sets $\mathcal{A}$ and $\mathcal{A}'$ for a default theory $\mathcal{D} = (\mathcal{F}, \Delta)$, respectively. Then, $\mathcal{A} \cup \mathcal{A}'$ is inconsistent.

$\blacksquare$

Proof: Theorem 3.2.7

Let (E_i, A_i) and (E'_i, A'_i) be sequences of pairs related to pairs (a-$\mathcal{E}$, $\mathcal{A}$) and (a-$\mathcal{E}'$, $\mathcal{A}'$) using characterization theorem 3.2.2 such that a-$\mathcal{E} = \bigcup_{i=0}^{\infty} E_i$ and a-$\mathcal{E}' = \bigcup_{i=0}^{\infty} E'_i$ and $\mathcal{A} = \bigcup_{i=0}^{\infty} A_i$ and $\mathcal{A}' = \bigcup_{i=0}^{\infty} A'_i$. Since a-$\mathcal{E}$ and a-$\mathcal{E}'$ are two distinct alternative extensions and $E_0 = E'_0$, there is an n such that $E_n = E'_n$ but $E_{n+1} \neq E'_{n+1}$. Therefore, there is C such that $C \in E_{n+1} - E'_{n+1}$; if there is no such C, assume otherwise, $C \in E'_{n+1} - E_{n+1}$. So, there is a default $\delta = \frac{A:B}{C} \in \Delta$ such that $A \in E_n$, and $\mathcal{A} \cup \{B, C\}$ is consistent by the construction of E_{n+1}. By the construction of A_{n+1}, both B and C are in A_{n+1}, causing both B and C in $\mathcal{A}$. Since $A \in E'_n$ but $C \notin E'_{n+1}$, $\mathcal{A}' \cup \{B, C\}$ is inconsistent. Furthermore, because of B and C in $\mathcal{A}$ and $\mathcal{A}' \cup \{B, C\}$ being inconsistent, then $\mathcal{A}$ is inconsistent with $\mathcal{A}'$. Thus, $\mathcal{A} \cup \mathcal{A}'$ is inconsistent.

It is not necessary that the union of two distinct alternative extensions be inconsistent. The following is one example.

Example 3.2.12:

Let $\mathcal{D} = (\mathcal{F}, \Delta = \{\delta_1, \delta_2\})$ be a default theory, where $\mathcal{F} = \{P\}$, $\delta_1 = \frac{P:Q}{R}$, and $\delta_2 = \frac{P:\neg Q}{S}$. This default theory has two extensions a-$\mathcal{E}_1$ and a-$\mathcal{E}_2$, where

a-$\mathcal{E}_1 = Th(\{P, R\})$,

a-$\mathcal{E}_2 = Th(\{P, S\})$

based on assumption sets $\mathcal{A}_1$ and $\mathcal{A}_2$, where

$\mathcal{A}_1 = Th(\{P, Q, R\})$,

$\mathcal{A}_2 = Th(\{P, \neg Q, S\})$

respectively. It is easy to see that a-$\mathcal{E}_1 \cup$a-$\mathcal{E}_2$ is consistent but $\mathcal{A}_1 \cup \mathcal{A}_2$ is not.

3.3 Disjunctive Fact Problems Concerning Prerequisites

3.3.1 Preliminary Discussion

Another problem that may arise in using both the Reiter and the Lukaszewicz approaches is to deal with disjunctive facts concerning prerequisites. These approaches cannot deal with default theories consisting of disjunctive facts in which applicabilities of defaults are blocked due to no evidence to believe any of their prerequisites.

Consider the following example. This artificial example is observed by Lin and Shoham [22]. It shows that both the Reiter and the Lukaszewicz approaches do not work properly to derive facts.

Example 3.3.1:

Let $\mathcal{D} = (\mathfrak{F}, \Delta=\{\delta_1, \delta_2\})$ be a default theory, where $\mathfrak{F} = \{P \vee Q\}$, $\delta_1 = \frac{P:R}{R}$, and $\delta_2 = \frac{Q:R}{R}$. This normal default theory has exactly one extension $\mathfrak{E}$, where

$\mathfrak{E} = Th(\mathfrak{F})$.

It also has exactly one modified extension m-$\mathfrak{E}$, where

m-$\mathfrak{E} = Th(\mathfrak{F})$

with respect to a justification set $\mathcal{J} = \varnothing$, since there is no applicable default. Therefore, all facts that can be derived are consequences of a hard fact set $\mathfrak{F}$. This is unacceptable. We should be able to derive the fact R by δ_1 and δ_2 using $P \vee Q$. ▒

To see this more clearly, we present an example occurring naturally.

Example 3.3.2: Disjunctive fact problems concerning prerequisites

Suppose that all we know is that either "John was born in the US" or "John was naturalized by the US government." However, we do not have any evidence to believe either of them. In addition, we use two typical facts to supposedly infer the citizenship of a US person, i.e., represented using normal defaults. The typical facts are "Typically, a person born in the US is a US citizen" and "Typically, a person naturalized by the US government is a US citizen." This problem can be formalized using a normal default theory $\mathcal{D} = (\mathfrak{F}, \Delta) = (\mathfrak{F}, \{\delta_1, \delta_2\})$, where

$$\mathfrak{F} = \{\text{born-in-US(John)} \vee \text{naturalized-by-US(John)}\},$$

$$\delta_1 = \frac{\text{born-in-US(x):citizen-of-US(x)}}{\text{citizen-of-US(x)}},$$

and

$$\delta_2 = \frac{\text{naturalized-by-US(x):citizen-of-US(x)}}{\text{citizen-of-US(x)}}.$$

Since we only know the disjunctive form of the prerequisites of both defaults, that is, born-in-US(John) $\vee$ naturalized-by-US(John), both defaults are inapplicable. Hence, an extension for $\mathcal{D}$ is $\mathcal{E}$, where

$$\mathcal{E} = \text{Th}(\mathfrak{F}).$$

Furthermore, this default theory also has exactly one modified extension m-$\mathcal{E}$, where

$$\text{m-}\mathcal{E} = \text{Th}(\mathfrak{F})$$

with respect to a justification set $\mathcal{J} = \varnothing$ since there is no applicable default. So, both approaches result in the same set, that is, all facts derivable from the hard fact set $\mathfrak{F}$ in terms of first order theory. Therefore, the consequences of the default theories are all facts that can be derived from the hard fact set. This is unacceptable since we should be able to infer that "citizen-of-US(John)"; i.e., John is a US citizen.

▒

From this example, even though we know a disjunctive fact (i.e., "John was born in the US" or "John was naturalized by the US government") we are not able to use either defaults to derive any fact, even though no inconsistency has occurred. So, both the Reiter and Lukaszewicz approaches should be revised in order to deal with disjunctive fact problems concerning prerequisites.

The major problem concerning prerequisites is that we cannot apply defaults due to no evidence to believe their prerequisites. Those defaults have the same nature in the justification. They have the same justification. So, we can construct a class of equivalence sets. Each equivalence set consists of defaults having the same justification and is called a default set associated to a justification. Since the number of defaults in a default theory is countable, we should have a countable class of such associated sets. For any associated default set, we construct a set of disjunctive forms consisting of prerequisites of the defaults in the set. These forms come from defaults having the same justifications. In such a set, we may find special disjunctive forms that may be useful in applying the related defaults. Those disjunctive forms are called prime. So, by forcing consistency in applying defaults using prime disjunctive forms consisting of prerequisites of some defaults, we can infer some facts in the form of disjunctions consisting of consequents of the related defaults. This approach should solve the disjunctive fact problems concerning prerequisites as mentioned above. In addition, this new approach should also solve the disjunctive fact problems concerning justifications as mentioned in the previous section since we are forcing consistency in the assumption set when applying defaults using a prime disjunctive form. This is the idea behind the new approach.

3.3.2 New Approach to Solving Disjunctive Fact Problems Concerning Prerequisites

As previously mentioned, we need to revise both the Reiter and the

Lukaszewicz approaches in order to deal with disjunctive fact problems concerning prerequisites. The main problem in dealing with fact problems concerning prerequisites is that we cannot apply some defaults due to no evidence to believe their prerequisite (though there is evidence to believe their disjunctive form) even though consistency is guaranteed. So, a disjunctive fact can cause the application of some defaults to be blocked.

These problems can be overcome by letting a system apply defaults using not only their prerequisites but their disjunctive form consisting of their prerequisites as well. In addition, we need to force consistency not only in a set of the justification, the consequents, and the disjunctive forms consisting of prerequisites of defaults under consideration, but in the whole set of hard facts and justifications, consequents, and disjunctive forms consisting of prerequisites of all applicable defaults as well. Therefore, the applicability of defaults is checked against their prerequisites in disjunctive forms. By doing this, we extend the applicability of default in terms of prerequisites.

This can be accomplished by generating a class of sets of disjunctive forms based on prerequisites of defaults having the same justification. That is, for any distinct justification, we define a set of disjunctive forms consisting of prerequisites of defaults, which have the same justification. Therefore, in applying defaults, we need to check against their associated sets. In doing so, we extend the applicability of defaults.

Before extending the applicability of defaults, we need some definitions. First, we partition defaults into sets based on justification Bs. Each set, noted $\overline{\delta}_B$ and called a default set associated to justification B, consists of defaults having the same B. It is defined in the following.

Definition: Associated default sets

Let Δ be a set of defaults. The set of *defaults associated to justification B,* *noted* $\bar{\delta}_B$, is as *follows*:

$$\bar{\delta}_B = \{\delta = \frac{A:B}{C} \mid \delta \in \Delta\}.$$

$\bar{\delta}_B$ *is called an associated default set to a justification B.*

A set $\bar{\delta}_B$ is associated to B appearing in place of the justification of defaults. So, the number of sets $\bar{\delta}_B$s depends on the number of distinct justifications in Δ.

Example 3.3.3:

Let $\Delta = \{\delta_1 = \frac{\text{bird(Tweety):fly(Tweety)}}{\text{fly(Tweety)}}, \delta_2 = \frac{\text{pet(Tweety):fly(Tweety)}}{\text{fly(Tweety)}}, \delta_3 = \frac{\text{bird(Clyde):fly(Clyde)}}{\text{fly(Clyde)}}\}$ be a set of defaults. Since two distinct justifications appears in defaults in Δ, there are only two default sets associated to them: "fly(Tweety)" and "fly(Clyde)." They are $\bar{\delta}_{\text{fly(Tweety)}}$ and $\bar{\delta}_{\text{fly(Clyde)}}$, where

$$\bar{\delta}_{\text{fly(Tweety)}} = \{\delta_1, \delta_2\}$$

and

$$\bar{\delta}_{\text{fly(Clyde)}} = \{\delta_3\}.$$

For each set $\bar{\delta}_B$, we generate a set of disjunctive forms consisting of prerequisites of defaults in the set $\bar{\delta}_B$. Each form is called a disjunctive form consisting of prerequisites.

Definition: Set of disjunctive forms consisting of prerequisites

Let $\mathfrak{D} = (\mathfrak{F}, \Delta)$ *be a default theory and* $\bar{\delta}_B$ *denote a default set associated to a justification B. Construct a set noted* $\mathcal{H}_B$ *as follows*:

$$\mathcal{H}_B = \{\vee_i A_i \mid \delta = \tfrac{A_i:B_i}{C_i} \in \bar{\delta}_B \text{ and } A_i\text{s are distinct}\}.$$

$\mathcal{H}_B$ *is called a set of disjunctive forms consisting of prerequisites associated to a justification B. An element in the set,* $\vee_i A_i$, *is called a disjunctive form consisting of prerequisites.*

That is, a set of all disjunctive forms consist of prerequisites of defaults having the same justification. The disjunctive operator $\vee_i$ may take any number of operands as long as all prerequisites, A_is, are different. However, since a default set $\overline{\delta}_B$ is countable, the generated set should be countable.

Example 3.3.4:

Let $\Delta = \{\delta_1, \delta_2, \delta_3\}$ be a default set of a default theory $\mathcal{D}$, where $\delta_1 = \frac{P:Q}{Q}$, $\delta_2 = \frac{Q:R}{S}$, and $\delta_3 = \frac{T:R}{R}$. There are two distinct justifications. So, there are two default sets associated to justifications: $\overline{\delta}_Q = \{\delta_1\}$ and $\overline{\delta}_R = \{\delta_2, \delta_3\}$. A set of a disjunctive form consisting of a prerequisite related to $\overline{\delta}_Q$ is $\mathcal{H}_Q = \{P\}$ and a set of disjunctive forms consisting of prerequisites related to $\overline{\delta}_R$ is $\mathcal{H}_R = \{Q, T, Q \vee T\}$.

▒

The word "distinct" means no duplicate prerequisite in any disjunctive form $\vee_i A_i$. For example, $A_1 \vee A_2 \vee A_1$ is not a disjunctive form since $A_1 \vee A_2 \vee A_1 = A_1 \vee A_2$. In addition, $A \vee \tau$ is also not, since $A \vee \tau = \tau$, where τ stands for verum as opposed to the symbol $\perp$, falsum. So, τ will not appear in another disjunctive form but itself. It is related to a default without a prerequisite.

Before presenting a new approach to dealing with disjunctive facts, we need to define a prime element in a set of disjunctive forms with respect to a set. It is needed to apply defaults and is called a prime disjunctive form.

Definition: Prime disjunctive forms

Let $\mathcal{H}_B$ denote a set of disjunctive forms consisting of prerequisites based on a justification B. A disjunctive form $\vee_{i=1}^{n} A_i$ in $\mathcal{H}_B$ is prime with respect to a set $\mathcal{S}$ if and only if $\vee_{i=1}^{n} A_i \in \mathcal{S}$ but $\vee_{i=1}^{n} A_i$ is not derivable from a set $\{A^j = \vee_{\substack{i=1 \\ i \neq j}}^{n} A_i \mid A^j \in \mathcal{H}_B, A^j \in \mathcal{S}, 1 \leq j \leq n\}$ in $\mathcal{S}$. *We call it a prime disjunctive form of prerequisites.*

▒

In other words, a disjunctive form $A = \vee_{i=1}^{n} A_i$ in $\mathcal{H}_B$ is prime with respect to a set $\mathcal{S}$ if and only if it is in $\mathcal{S}$ but cannot be derived from a set of disjunctive forms A^js in $\mathcal{S}$ obtained by removing one prerequisite A_j from A; i.e., $A^j = \vee_{\substack{i=1 \\ i \neq j}}^{n} A_i$.

If A is a prerequisite of a default, i.e., $n = 1$, it is likely that it is a prime disjunctive form. We only need to check its membership in $\mathcal{S}$. Therefore, it is possible that several disjunctive forms in $\mathcal{H}_B$ are prime but they are countable.

In addition, a special treatment should be taken for τ. This form τ is always prime. It is related to a default without a prerequisite, and it is likely applicable.

Also, for every prime disjunctive form $\vee_{i=1}^{n} A_i$, there are B, C_1, ..., and C_n related to it. B is the related justification of the defaults. The related disjunctive form $\vee_{i=1}^{n} C_i$ is a disjunctive form consisting of the consequents of the defaults.

Example 3.3.5: Prime disjunctive forms

Let us see the previous example. Let $\mathcal{H}_Q$ and $\mathcal{H}_R$ be sets of disjunctive forms consisting of prerequisites, where

$$\mathcal{H}_Q = \{P\}$$

and

$$\mathcal{H}_R = \{Q, T, Q \vee T\}.$$

Suppose, further, that $\mathcal{S} = Th(\{P, Q, Q \vee T\})$. Therefore, both P and Q are prime disjunctive forms in $\mathcal{H}_Q$ and $\mathcal{H}_R$, respectively, with respect to $\mathcal{S}$. Since Q is in $\mathcal{S}$, $Q \vee T$ is not a prime disjunctive form in $\mathcal{H}_R$ with respect to $\mathcal{S}$ since it is derivable from Q in $\mathcal{S}$. Furthermore, T is not a prime disjunctive form since $T \notin \mathcal{S}$.

The following lemma is very useful in the proof of theorems in the rest of paper. It is related to the derivability of a disjunctive form.

Lemma 3.3.1:

Let $\vee_{i=1}^{n} A_i$ be a disjunctive form in a set $\mathcal{H}_B$. Suppose that a set $\mathcal{S}$ is deductively closed. If $\vee_{i=1}^{n} A_i$ in $\mathcal{S}$, then $\vee_{i=1}^{n} A_i$ is derivable from a set of prime disjunctive forms in $\mathcal{H}_B$ with respect to $\mathcal{S}$.

∎

This lemma says that if a disjunctive form in $\mathcal{H}_B$ is derivable from a deductively closed set $\mathcal{S}$, then it is derivable from a set of prime disjunctive forms in $\mathcal{S}$; they are in $\mathcal{H}_B$.

Proof: Lemma 3.3.1

Let $\vee_{i=1}^{n} A_i$ be a disjunctive form in a set $\mathcal{H}_B$ and $\mathcal{S}$ is deductively closed. If $\vee_{i=1}^{n} A_i$ is prime with respect to $\mathcal{S}$, then we proved the lemma. Otherwise, suppose that $\vee_{i=1}^{n} A_i$ is not prime. Therefore, $\vee_{i=1}^{n} A_i$ is derivable from a set of A^js in $\mathcal{S}$ and $\mathcal{H}_B$, $1 \leq j \leq n$, each A^j is resulted from removing one prerequisite from $\vee_{i=1}^{n} A_i$. If all A^js are prime with respect to $\mathcal{S}$, then, we have proven the lemma. Otherwise, suppose that there is an A^j that is not prime. By the same reason, A^j can be derived from a set of smaller disjunctive forms in $\mathcal{S}$ and $\mathcal{H}_B$ by removing one prerequisite from A^j. This continues until all smaller disjunctive forms are prime with respect to $\mathcal{S}$. This is guaranteed because the smallest disjunctive forms in $\mathcal{H}_B$ consist of one prerequisite. In addition, they must be prime with respect to $\mathcal{S}$.

∎

Example 3.3.6:

Let H_B be a set {P, Q, R, S, T, P∨Q, P∨R, P∨S, P∨T, ..., P∨Q∨R∨S∨T}. Suppose that $\mathcal{S}$=Th({P, P∨Q, Q∨R, R∨S∨T}). Disjunctive forms P, Q∨R, R∨S∨T are in $\mathcal{S}$, and they are prime with respect to $\mathcal{S}$. Disjunctive form A = P∨Q∨R∨S∨T is in $\mathcal{S}$. However, it is not prime and is derivable from {P∨Q∨R, R∨S∨T} in $\mathcal{S}$. A disjunctive form R∨S∨T is prime with respect to $\mathcal{S}$ but P∨Q∨R is not. P∨Q∨R can be derived from {P∨Q, Q∨R} in $\mathcal{S}$. Also, Q∨R is prime with respect to $\mathcal{S}$ but P∨Q is not. P∨Q is derivable from P in $\mathcal{S}$, and P is prime. So, A is derivable from a set {P, Q∨R, R∨S∨T} in $\mathcal{S}$, and P, Q∨R, R∨S∨T are all prime with respect to $\mathcal{S}$.

∎

Before presenting a new approach, we need to make some clarification in dealing with disjunctive forms. As Kautz and Selman observed, some defaults may have the same justification but they may have prerequisites in the form of complements to each other [19]. That is, their disjunctive form consists of prerequisites of defaults, implying that is always true. For example, consider the following two defaults δ_1 and δ_2, where $\delta_1=\frac{P:Q}{Q}$ and $\delta_2=\frac{\neg P:Q}{Q}$. The disjunctive form consisting of prerequisites of both defaults, $P\vee\neg P$, is always true. Therefore, we are likely able to conclude that the disjunctive form is prime. So, we are likely able to infer Q regardless of P. It seems weird. However, by looking at both defaults, we can combine the two defaults into one default which does not have a prerequisite. That is, a default $\delta=\frac{\tau:Q}{Q}$. Therefore, it is more likely that this default is applicable. So, it is reasonable to assume that the disjunctive form is replace by τ. Furthermore, τ will not appear in another disjunctive form but itself.

Example 3.3.7:

Let $\mathcal{H}_B$ be a set of disjunctive forms consisting of prerequisites, where

$\mathcal{H}_B = \{ \tau, P, Q, R, P\vee Q, P\vee R, Q\vee R, P\vee Q\vee R\}$,

and $\mathcal{S}$ be a set, where

$\mathcal{S} = Th(\{P\vee R, Q\vee R, S\})$

Therefore, the forms τ, $P\vee R$ and $Q\vee R$ are prime disjunctive forms in $\mathcal{H}_B$ with respect to $\mathcal{S}$. However, $P\vee Q\vee R$ is not since both $P\vee R$ and $Q\vee R$ are in $\mathcal{S}$, and $P\vee Q\vee R$ is derivable from $\{P\vee R, Q\vee R\}$ in $\mathcal{S}$.

Now, we are in a position to define formally a new version to deal with disjunctive fact problems concerning prerequisites. In addition, it also can deal with disjunctive fact problems concerning justifications. The main idea is to check the applicability of defaults using their related set of disjunctive forms consisting of prerequisites. A pair of sets defined in the definition is called an alternative extension and an assumption set for a default theory.

Definition: Alternative extensions

Let $\mathcal{D} = (\mathcal{F}, \Delta)$ *be a default theory and* (E, A) *be a pair of closed well-formed formula sets. Suppose that* $\mathcal{H}_B$ *denotes a set of disjunctive forms consisting of prerequisites based on a justification B of defaults in* Δ. *Suppose that* $(\Gamma_1(\mathrm{E}, \mathrm{A}), \Gamma_2(\mathrm{E}, \mathrm{A}))$ is *a pair of the smallest sets satisfying the following three conditions*:

S1. $\mathcal{F} \subseteq \Gamma_i(\mathrm{E}, \mathrm{A})$, *for* i=1,2;

S2. $\mathrm{Th}_{\mathcal{L}}(\Gamma_i(\mathrm{E}, \mathrm{A})) = \Gamma_i(\mathrm{E}, \mathrm{A})$, *for* i=1,2; *and*

S3. if $A = \vee_{i=1}^{n} A_i \in \mathcal{H}_B$ *is a prime disjunctive form with respect to* $\Gamma_1(\mathrm{E}, \mathrm{A})$ *and* $\mathrm{A} \cup \{B, C_1, \ldots, C_n\}$ *is consistent, then*

- $\vee_{i=1}^{n} C_i$ *is in* $\Gamma_1(\mathrm{E}, \mathrm{A})$; *and*
- *B and* $C_1, \ldots, C_n$ *are in* $\Gamma_2(\mathrm{E}, \mathrm{A})$.

A closed well-formed formula set a-$\mathcal{E}$ *is an alternative extension for a default theory* $\mathcal{D}$ *if and only if there is a closed well-formed formula set* $\mathcal{A}$, *called an assumption set, such that*

$$\text{a-}\mathcal{E} = \Gamma_1(\text{a-}\mathcal{E}, \mathcal{A}) \text{ and } \mathcal{A} = \Gamma_2(\text{a-}\mathcal{E}, \mathcal{A}).$$

This definition is a little more complex than that of the previous section. It provides criteria for a closed well-formed formula set a-$\mathcal{E}$ to be an alternative extension based on an assumption set $\mathcal{A}$ for a default theory $\mathcal{D}$. A pair of an alternative extension a-$\mathcal{E}$ and an assumption set $\mathcal{A}$ is a fixed point of operators Γ_1 and Γ_2, a-$\mathcal{E} = \Gamma_1(\text{a-}\mathcal{E}, \mathcal{A})$ and $\mathcal{A} = \Gamma_2(\text{a-}\mathcal{E}, \mathcal{A})$, based on three conditions. The first two conditions are identical with those in the previous section, that is, both an alternative extension and an assumption set containing the entire hard fact set and being inductively closed.

The difference is in the third condition, and it is related to the applicability of some defaults. It requires that there be a prime disjunctive form consisting of prerequisites in $\mathcal{H}_B$ with respect to alternative extension a-$\mathcal{E}$. In addition, justification B and the related consequents C_1, ..., C_n are consistent with the assumption set $\mathcal{A}$. If these requirements are satisfied, then B and the related consequents must be in assumption set $\mathcal{A}$. The related disjunctive form consisting of consequents must also be in alternative extension a-$\mathcal{E}$. In this case, the defaults are called applicable to an alternative extension a-$\mathcal{E}$ based on an assumption set $\mathcal{A}$. In addition, when some defaults are applied, a justification B and a disjunctive form consisting of prerequisite, related consequents are closed well-formed formulae.

In addition, the smallness of both alternative extension and assumption set assures that no more facts are added to them without any reason. In other words, a fact is in an alternative extension if it is a hard fact, a disjunctive form consisting of consequents of applicable defaults, or a fact derivable from the set of hard facts and the set of disjunctive forms consisting of consequents of applicable defaults. Furthermore, a fact is in an assumption set if it is a hard fact or a justification or consequents of applicable defaults, or a fact derivable from the set of hard facts, justifications, and consequents of applicable defaults.

As already noticed in the previous section, by combining the first two conditions, we can conclude that all facts derivable from a hard fact set are in both an alternative extension and an assumption set; i.e., $\mathrm{Th}(\mathcal{F}) \subseteq$ a-$\mathcal{E}$ and $\mathrm{Th}(\mathcal{F}) \subseteq \mathcal{A}$. In addition, by condition S3, an assumption set contains all justifications and consequents of applicable defaults. Meanwhile, an alternative extension contains all disjunctive forms consisting of consequents of applicable defaults. Therefore, an alternative extension is a subset of an assumption set; i.e., a-$\mathcal{E} \subseteq \mathcal{A}$. Furthermore, since an assumption set must be consistent, an alternative extension is also consistent.

To check whether a pair of sets is a pair of an alternative extension and an assumption set for a default theory, we need to verify the satisfaction of the three conditions above in addition to the smallness of the pair. In other words, a pair of closed well-formed formula sets may satisfy one, two, or three of the conditions with respect to a default theory $\mathcal{D} = (\mathcal{F}, \Delta)$ but not necessarily be a pair of an alternative extension and an assumption set. A pair of sets (E, A) may satisfy condition S1; i.e., $\mathcal{F} \subseteq E$ and $\mathcal{F} \subseteq A$. It may satisfy condition S2; i.e., Th(E) = E and Th(A) = A. Also, it may satisfy condition S3; i.e., if $\vee_{i=1}^{n} A_i$ is a prime disjunctive form consisting of prerequisites in $\mathcal{H}_B$, a set of disjunctive forms consisting of prerequisites related to justification B, with respect to E, and $A \cup \{B, C_1, ..., C_n\}$ is consistent, then $\vee_{i=1}^{n} C_i \in E$ and $B, C_1, ..., C_n \in A$. However, (E, A) may not be a pair of the smallest sets.

The main reason for using prime disjunctive forms in applying defaults is that we assure that we derive all disjunctive forms whenever possible and reasonable. By doing so, we do not have disjunctive fact problems concerning prerequisites as mentioned early in this chapter. So, the new approach will handle those problems adequately. Furthermore, we still force the consistency of the assumption set in applying defaults. Therefore, the disjunctive fact problems concerning justifications will not arise in the new approach. So, the new approach will handle both problems adequately.

Example 3.3.8:

As mentioned early in example 3.3.1, default theory $\mathcal{D} = (\mathcal{F}, \Delta = \{\delta_1, \delta_2\})$, where $\mathcal{F} = \{P \vee Q\}$, $\delta_1 = \frac{P:R}{R}$, and $\delta_2 = \frac{Q:R}{R}$, has exactly one extension $\mathcal{E}$ and also exactly one modified extension m-$\mathcal{E}$ with respect to $\mathcal{J} = \emptyset$, where $\mathcal{E}$ = m-$\mathcal{E}$ = Th($\mathcal{F}$). Now, we see an alternative extension a-$\mathcal{E}$ for this theory. That is,

a-$\mathcal{E}$ = Th($\mathcal{F} \cup \{R\}$)

based on an assumption set $\mathcal{A}$,

$$\mathcal{A} = \mathrm{Th}(\mathfrak{F} \cup \{R\}).$$

It is easy to see that the pair (a-$\mathfrak{E}$, $\mathcal{A}$) satisfies conditions S1 and S2. Now, we need to check condition S3. This default theory has only one default set $\bar{\delta}_R$ associated to justification R. So, the set of disjunctive forms consisting of prerequisites related to $\bar{\delta}_R$ is $\mathcal{H}_R = \{P, Q, P \vee Q\}$. Therefore, a prime disjunctive form in $\mathcal{H}_R$ with respect to a-$\mathfrak{E}$ is $P \vee Q$. It is the only prime disjunctive form in $\mathcal{H}_R$ with respect to a-$\mathfrak{E}$. Since R is consistent with $\mathcal{A}$, the related justification R and the related disjunctive form consisting of consequents $R \vee R = R$, must be in a-$\mathfrak{E}$. Also, justification R and the consequents R must be in $\mathcal{A}$. So, the pair satisfies S3. In addition, (a-$\mathfrak{E}$, $\mathcal{A}$) is a pair of the smallest sets satisfying S1, S2, and S3. Hence, the pair satisfies as an alternative extension based on an assumption set for $\mathcal{D}$.

∎

The concept of a consequence in a default theory is different from other theories, for example, NMLs [26] and Disjunctive Default Theories [18]. A consequence of NMLs is a closed well-formed formula that belongs to the intersection of all extensions. For example, if A is a closed well-formed formula in all extensions for an NML, then A is a consequence of the NML. A different concept is also used in disjunctive default theories, DDTs for short. A consequence of DDTs is a closed well-formed formula that belongs to all extensions. For example, if A_i is a closed well-formed formula in a set E_i, where E_i is an extension for a DDT, $1 \leq i \leq n$, $\vee_{i=1}^{n} A_i$ is a consequence of the DDT.

In contrast, a consequence of a default theory is a sentence that belongs to an alternative extension. Therefore, we need to check whether a closed well-formed formula is in any extension before we can say that it is a consequence of the default theory. For example, if A is a closed well-formed formula in an extension for a default theory, then it is a consequence of the default theory.

Let us see how the new approach solves example 3.3.1 occurring naturally.

Example 3.3.9: Disjunctive fact problems concerning prerequisite

Consider the default theory $\mathcal{D} = (\mathcal{F}, \Delta = \{\delta_1, \delta_2\})$, where

$\mathcal{F} = \{\text{born-in-US(John)} \vee \text{naturalized-by-US(John)}\}$,

$$\delta_1 = \frac{\text{born-in-US(x):citizen-of-US(x)}}{\text{citizen-of-US(x)}},$$

and

$$\delta_2 = \frac{\text{naturalized-by-US(x):citizen-of-US(x)}}{\text{citizen-of-US(x)}}.$$

It is easy to see that a-$\mathcal{E}$ is an alternative extension based on an assumption set $\mathcal{A}$ for $\mathcal{D}$, where

a-$\mathcal{E} = \mathcal{A} = \text{Th}(\mathcal{F} \vee \{\text{citizen-of-US(John)}\})$.

In the new approach, we can conclude that "John is a US citizen" as intended. ∎

The following example is a default theory involving a default without a prerequisite. So, the symbol τ is more likely to be a prime disjunctive form.

Example 3.3.10:

Let $\mathcal{D} = (\mathcal{F}, \Delta=\{\delta_i \mid 1 \le i \le 4\})$ be a default theory, where $\mathcal{F}=\{P \vee Q\}$, $\delta_1=\frac{P:R}{R_1}$, $\delta_2=\frac{Q:R}{R_2}$, $\delta_3=\frac{S:T}{T_1}$, and $\delta_4=\frac{\neg S:T}{T_2}$. The prerequisites of δ_3 and δ_4 are complements. So, there will be τ in a set of disjunctive forms related to a justification $B = T$. Hence, combining those defaults will result in a default without prerequisite. In addition, there are two distinct justifications. So, there are two associated default sets: $\bar{\delta}_R$ and $\bar{\delta}_T$. The sets of disjunctive forms related to $\bar{\delta}_R$ and $\bar{\delta}_T$ are $\mathcal{H}_R = \{P, Q, P \vee Q\}$ and $\mathcal{H}_T = \{\tau\}$, respectively. The candidate prime disjunctive forms are P, Q, $P \vee Q$ and τ. P and Q are not prime since we do not have evidence to believe P and Q, respectively. Therefore, an alternative extension for $\mathcal{D}$ is a-$\mathcal{E}$ based on $\mathcal{A}$, where a-$\mathcal{E} = \text{Th}(\{P \vee Q, R_1 \vee R_2, T_1 \vee T_2\})$ and $\mathcal{A} = \text{Th}(\{P \vee Q, R_1, R_2, T_1, T_2\})$. It is the only alternative extension for $\mathcal{D}$.

$\blacksquare$

Gelfond et al. developed a totally new theory, called Disjunctive Default Theories, to deal with disjunctive facts [18]. Their approach is not only to change the representation of a default theory and the derivation of extensions, but also to change the representation of defaults. Their theory can deal adequately with disjunctive problems. Furthermore, their approach is more general than the new approach.

However, deriving facts using a disjunctive default theory is similar to that of the Reiter approach. In their approach, the applicability of a default is checked against the derivability of its prerequisite and the consistency of its justification. So, it may lack extensions, as the Reiter approach may. Furthermore, this approach may fail to derive facts, as we will see in the following example.

Consider the following example. This is an artificial example.

Example 3.3.11:

Let $\mathcal{G} = (\{P \mid \neg Q,\ R \leftrightarrow S, \delta_1 = \frac{P:R}{R}, \delta_2 = \frac{\neg Q:R}{R}, \delta_3 = \frac{P:\neg S}{\neg S}, \delta_4 = \frac{\neg Q:\neg S}{\neg S}\})$ be a disjunctive default theory. ($P \mid \neg Q$ is to be understood as of a disjunctive default according to the Gelfond et al. formulation.) This disjunctive default theory $\mathcal{G}$ has exactly four extensions: $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$, and $\mathcal{E}_4$, where

$\mathcal{E}_1 = Th(\{P, R \leftrightarrow S, R\})$,

$\mathcal{E}_2 = Th(\{\neg Q, R \leftrightarrow S, R\})$,

$\mathcal{E}_3 = Th(\{P, R \leftrightarrow S, \neg S\})$,

and

$\mathcal{E}_4 = Th(\{\neg Q, R \leftrightarrow S, \neg S\})$.

Both $\mathcal{E}_1$ and $\mathcal{E}_2$ contain both R and S whereas both $\mathcal{E}_3$ and $\mathcal{E}_4$ contain both $\neg R$ and $\neg S$. Therefore, $R \vee \neg R$, $R \vee \neg S$, $S \vee \neg R$, and $S \vee \neg S$ are consequences of the disjunctive default theory $\mathcal{G}$. (A consequence for a disjunctive default theory is a sentence that belongs to all extensions [18].) That is, nothing is additionally derivable using defaults. So, R is not a consequence of $\mathcal{G}$; neither is S. This is not acceptable since we should be able to derive R by applying defaults δ_1 and δ_2 using the disjunctive fact $P \vee \neg Q$.

Let us consider a default theory $\mathcal{D} = (\mathcal{F}, \Delta)$, where $\mathcal{F} = \{P \vee \neg Q, R \leftrightarrow S\}$, and $\Delta = \{\delta_1 = \frac{P:R}{R}, \delta_2 = \frac{\neg Q:R}{R}, \delta_3 = \frac{P:\neg S}{\neg S}, \delta_4 = \frac{\neg Q:\neg S}{\neg S}\}$. This default theory has two alternative extensions: a-$\mathcal{E}_1$ and a-$\mathcal{E}_2$, where

$$\text{a-}\mathcal{E}_1 = Th(\{P \vee \neg Q, R \leftrightarrow S, R\})$$

and

$$\text{a-}\mathcal{E}_2 = Th(\{P \vee \neg Q, R \leftrightarrow S, \neg S\}),$$

based on assumption sets $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively, where

$$\mathcal{A}_1 = Th(\{P \vee \neg Q, R \leftrightarrow S, R\})$$

and

$$\mathcal{A}_2 = Th(\{P \vee \neg Q, R \leftrightarrow S, \neg S\}).$$

Therefore, R is a consequence of the default theory $\mathcal{D}$ as intended.

∎

The new approach should not destroy the ability to derive facts using a default in the absence of information as Reiter intended. In the new approach, the applicability of defaults is related to prime disjunctive forms consisting of prerequisites. Furthermore, a prime disjunctive form is obtained from defaults by making a disjunctive form of their prerequisite. So, a default could be applicable since its prerequisite could be a prime disjunctive form. Hence, the new approach does not destroy the capability of defaults in deriving facts as Reiter intended.

Example 3.3.12: The ability of birds to fly

Let $\mathcal{D} = (\mathcal{F}, \Delta = \{\delta\})$ be a default theory, where $\mathcal{F} = \{bird(Tweety)\}$ and

$$\delta = \frac{\text{bird}(x):\text{fly}(x)}{\text{fly}(x)}.$$

Since there is only one distinct justification, there is only one default set associated to justification "fly(Tweety)." The set is $\bar{\delta}_{\text{fly(Tweety)}} = \{\delta\}$. Furthermore, the set of disjunctive forms consisting of a prerequisite related to $\bar{\delta}_{\text{fly(Tweety)}}$ is $\mathcal{H}_{\text{fly(Tweety)}}$, where $\mathcal{H}_{\text{fly(Tweety)}} = \{\text{bird(Tweety)}\}$. So, "bird(Tweety)" is the only candidate to be a prime disjunctive form in $\mathcal{H}_{\text{fly}}$. It is easy to see that a-$\mathcal{E}$ is the only alternative extension for $\mathcal{D}$ based on an assumption set A, where

a-$\mathcal{E}$ = Th({bird(Tweety), fly(Tweety)}),

$\mathcal{A}$ = Th({bird(Tweety), fly(Tweety)}).

▒

3.3.3 Basic Properties of the New Approach

In this section we will see some properties of the new approach for building an alternative extension for a default theory. We will attempt to provide and prove some properties similar to those of the previous approaches.

The characterization theorem for an alternative extension for arbitrary default theories based on an assumption set will be given in the following. This theorem provides a more intuitive way to identify a pair (E, A) satisfying as an alternative extension and an assumption set for a default theory. According to the theorem, we can construct a sequence of pairs (E_i, A_i) that can be used to identify a pair (E, A) satisfying the conditions for an alternative extension E and an assumption set A. The sufficient and necessary condition for a pair (E, A) satisfying as an alternative extension E and an assumption set A for a default theory is that the pair is identical with the pair of the unions of all elements of the constructed sequence; i.e., (E, A) = ($\bigcup_{j=0}^{\infty} E_j$, $\bigcup_{j=0}^{\infty} A_j$). The following theorem sums up this property.

Theorem 3.3.1: The characterization of an alternative extension

Suppose $\mathfrak{D} = (\mathfrak{F}, \Delta)$ is a default theory, and a-$\mathfrak{E}$ and $\mathcal{A}$ are closed well-formed formula sets. Suppose that $\mathfrak{H}_B$ denotes a set of disjunctive forms consisting of prerequisites associated to a justification B of defaults in Δ. Let $(E_0, A_0), (E_1, A_1), \ldots$ be a sequence of pairs of closed well-formed formula sets, noted as (E_j, A_j), such that

$$E_0 = \mathfrak{F} \text{ and } A_0 = \mathfrak{F}$$

and, for $j \geq 1$

$$E_j = Th(E_{j-1}) \cup \{\vee_{i=1}^{n} C_i \mid \vee_{i=1}^{n} A_i \in \mathfrak{H}_B \text{ is a prime disjunctive form with respect to } E_{j-1}, \mathfrak{H}_B \text{ is associated to a justification } B \text{ of defaults in } \Delta, \text{ and } \mathcal{A} \cup \{B, C_1, \ldots, C_n\} \text{ is consistent}\},$$

$$A_j = Th(A_{j-1}) \cup \{B, C_1, \ldots, C_n \mid \vee_{i=1}^{n} A_i \in \mathfrak{H}_B \text{ is a prime disjunctive form with respect to } E_{j-1}, \mathfrak{H}_B \text{ is associated to a justification } B \text{ of defaults in } \Delta, \text{ and } \mathcal{A} \cup \{B, C_1, \ldots, C_n\} \text{ is consistent}\}.$$

A set of closed well-formed formulae a-$\mathfrak{E}$ is an alternative extension based on an assumption set $\mathcal{A}$ for $\mathfrak{D}$ if and only if

$$\text{a-}\mathfrak{E} = \bigcup_{j=0}^{\infty} E_j \text{ and } \mathcal{A} = \bigcup_{j=0}^{\infty} A_j .$$

$\blacksquare$

To prove that a pair (E, A) of sets is an alternative extension and an assumption set, we first construct a sequence of pairs (E_i, A_i) using a pair (E, A). Then, we need to check whether the pair is identical with the pair of the union of all members of the sequence. If so, the pair (E, A) is an alternative extension and an assumption set.

To prove the theorem, we use three lemmas. The following is the first. It says that the pair $(\cup_{j=0}^{\infty} E_j, \cup_{j=0}^{\infty} A_j)$ satisfies S1, S2, and S3.

Lemma 3.3.2:

Let a-$\mathcal{E}$ be an alternative extension based on an assumption set $\mathcal{A}$ for a default theory $\mathcal{D}=(\mathcal{F}, \Delta)$; i.e., a-$\mathcal{E}=\Gamma_1$(a-$\mathcal{E}$, $\mathcal{A}$) and $\mathcal{A}=\Gamma_2$(a-$\mathcal{E}$, $\mathcal{A}$). Then,

Γ_1(a-$\mathcal{E}$, $\mathcal{A}$) $\subseteq \bigcup_{j=0}^{\infty} E_j$ and Γ_2(a-$\mathcal{E}$, $\mathcal{A}$) $\subseteq \bigcup_{j=0}^{\infty} A_j$

Proof: Lemma 3.3.2

Since (Γ_1(a-$\mathcal{E}$, $\mathcal{A}$), Γ_2(a-$\mathcal{E}$, $\mathcal{A}$) is a pair of the smallest set satisfying S1, S2, and S3, it is enough to show that the pair $(\bigcup_{j=0}^{\infty} E_j, \bigcup_{j=0}^{\infty} A_j)$ satisfies three conditions S1, S2, and S3.

S1. Both E_0 and A_0 contain $\mathcal{F}$. Therefore, the pair $(\bigcup_{j=0}^{\infty} E_j, \bigcup_{j=0}^{\infty} A_j)$ satisfies S1.

S2. Since $Th(E_{i-1}) \subseteq E_i$ and $Th(A_{i-1}) \subseteq A_i$ for $i \geq 1$, then, $Th(\bigcup_{j=0}^{\infty} A_j) = \bigcup_{j=0}^{\infty} A_j$ and $Th(\bigcup_{j=0}^{\infty} E_j) = \bigcup_{j=0}^{\infty} E_j$ by lemma 3.2.1. So, the pair satisfies S2.

S3. We need to prove that if $\vee_{i=1}^{n} A_i$ is a prime disjunctive form in $\mathcal{H}_B$ with respect to $\bigcup_{j=0}^{\infty} E_j$, and $\{B, C_1, ..., C_n\} \cup \bigcup_{j=0}^{\infty} A_j$ is consistent, then, $\vee_{i=1}^{n} C_i \in \bigcup_{j=0}^{\infty} E_j$ and $B, C_1, ..., C_n \in \bigcup_{j=0}^{\infty} A_j$.

Suppose $\vee_{i=1}^{n} A_i$ is a prime disjunctive form in $\mathcal{H}_B$ with respect to $\bigcup_{j=0}^{\infty} E_j$, and $\{B, C_1, ..., C_n\} \cup \bigcup_{j=0}^{\infty} A_j$ is consistent. By the construction of E_j, there is an m such that $\vee_{i=1}^{n} A_i$ is a prime disjunctive form in $\mathcal{H}_B$ with respect to E_m. Since $\{B, C_1, ..., C_n\} \cup \bigcup_{j=0}^{\infty} A_j$ is consistent, so is $\{B, C_1, ..., C_n\} \cup A_m$. Hence, $\vee_{i=1}^{n} C_i \in E_{m+1}$ and $B, C_1, ..., C_n \in A_{m+1}$ by the construction of E_{m+1} and A_{m+1}. So, $\vee_{i=1}^{n} C_i \in \bigcup_{j=0}^{\infty} E_j$ and $B, C_1, ..., C_n \in \bigcup_{j=0}^{\infty} A_j$. Hence, the pair $(\bigcup_{j=0}^{\infty} E_j, \bigcup_{j=0}^{\infty} A_j)$ satisfies R3.

The following is the second lemma. It says that the necessary condition for the construction not to add unnecessary facts to a sequence of pairs (E_i, A_i) is that we use a pair of an alternative extension and an assumption set as a basis.

Lemma 3.3.3:

Suppose that a-$\mathcal{E}$ is an alternative extension based on an assumption set $\mathcal{A}$ for a default theory $\mathcal{D}$; i.e., a-$\mathcal{E} = \Gamma_1$(a-$\mathcal{E}$, $\mathcal{A}$) and $\mathcal{A} = \Gamma_2$(a-$\mathcal{E}$, $\mathcal{A}$). If (E_i, A_i) is a sequence of pairs constructed in theorem 3.4.1, then

$\cup_{j=0}^{\infty} E_j \subseteq \Gamma_1$(a-$\mathcal{E}$, $\mathcal{A}$) and $\cup_{j=0}^{\infty} A_j \subseteq \Gamma_2$(a-$\mathcal{E}$, $\mathcal{A}$).

Proof: Lemma 3.3.3

We need to prove that $\cup_{j=0}^{\infty} E_j \subseteq$ a-$\mathcal{E}$ and $\cup_{j=0}^{\infty} A_j \subseteq \mathcal{A}$. We will first prove $\cup_{j=0}^{\infty} E_j \subseteq$ a-$\mathcal{E}$ by a contradiction; i.e., suppose $\cup_{j=0}^{\infty} E_j \not\subset$ a-$\mathcal{E}$. Since $E_0 \subseteq$ a-$\mathcal{E}$ and $E_{j-1} \subseteq E_j$ for $j \geq 1$, there is an m such that $E_m \subseteq$ a-$\mathcal{E}$ but $E_{m+1} \not\subset$ a-$\mathcal{E}$. By the monotonicity property of first order theory, Th(E_m) $\subseteq$ Th(a-$\mathcal{E}$), and Th(E_m) $\subseteq$ a-$\mathcal{E}$ by S2 since a-$\mathcal{E} = \Gamma_1$(a-$\mathcal{E}$, $\mathcal{A}$). Since $E_{m+1} \not\subset$ a-$\mathcal{E}$, there is $\vee_{i=1}^{n} C_i \in E_{m+1}$ - a-$\mathcal{E}$. So, $\vee_{i=1}^{n} C_i \notin$ Th(E_m), and $\vee_{i=1}^{n} C_i$ is a disjunctive form of consequents of applicable defaults by the construction of E_{m+1}. Therefore, there is $\vee_{i=1}^{n} A_i$ being a prime disjunctive form in $\mathcal{H}_B$ with respect to E_m, and $\{B, C_1, ..., C_n\} \cup \mathcal{A}$ is consistent. So, $\vee_{i=1}^{n} A_i \in E_m$. By the assumption $E_m \subseteq$ a-$\mathcal{E}$, $\mathcal{A} \in E_m$ implies $\vee_{i=1}^{n} A_i \in$ a-$\mathcal{E}$. We need to find a contradiction; i.e., $\vee_{i=1}^{n} C_i \in$ a-$\mathcal{E}$. If $\vee_{i=1}^{n} A_i$ is a prime disjunctive form with respect to a-$\mathcal{E}$, by S3, $\vee_{i=1}^{n} C_i \in$ a-$\mathcal{E}$ since $\{B, C_1, ..., C_n\} \cup \mathcal{A}$ is consistent. Otherwise, suppose $\vee_{i=1}^{n} A_i$ is not a prime disjunctive form with respect to a-$\mathcal{E}$. Therefore, $\vee_{i=1}^{n} A_i$ is derivable from a set of prime disjunctive forms consisting of prerequisites in a-$\mathcal{E}$ by lemma 3.3.1. Since $\{B, C_1, ..., C_n\} \cup \mathcal{A}$ is consistent, $\{B, C_{k_1}, ..., C_{k_n}\} \cup \mathcal{A}$ is consistent, where $C_{k_i} = C_j$, $1 \leq j \leq n$, causing that those related $\vee_i C_{k_i}$ s are derivable and are in Γ_1(a-$\mathcal{E}$, $\mathcal{A}$) by S3. In addition, $\vee_{i=1}^{n} C_i$ is derivable from a set of the related $\vee_i C_{k_i}$ s. By monotonicity, $\vee_{i=1}^{n} C_i$ is in Γ_1(a-$\mathcal{E}$, $\mathcal{A}$). The contradiction is found. It must be $\cup_{j=0}^{\infty} E_j \subseteq \Gamma_1$(a-$\mathcal{E}$, $\mathcal{A}$).

We finish proving by a contradiction the lemma that $\cup_{j=0}^{\infty} A_j \subseteq \mathcal{A}$; i.e., suppose $\cup_{j=0}^{\infty} A_j \not\subset \mathcal{A}$. Since $A_0 \subseteq \mathcal{A}$ and $A_{j-1} \subseteq A_j$ for $j \geq 1$, there is an m such that A_m

$\subseteq \mathcal{A}$ but $A_{m+1} \not\subset \mathcal{A}$. So, $Th(A_m) \subseteq Th(\mathcal{A})$, implying $Th(A_m) \subseteq \mathcal{A}$ by S2 since $\mathcal{A} = \Gamma_2(\text{a-}\mathcal{E}, \mathcal{A})$. Since $A_{m+1} \not\subset \mathcal{A}$, there is $X \in A_{m+1} - \mathcal{A}$ by the construction of A_{m+1}, implying $X \notin Th(E_m)$. So, X is either a justification or a consequent of applicable defaults. We need to find a contradiction; i.e., $X \in \mathcal{A}$.

By the construction of A_{m+1}, there is a prime disjunctive form $\vee_{i=1}^{n} A_i \in \Psi_B$ with respect to E_m and $\{B, C_1, ..., C_n\} \cup \mathcal{A}$ is consistent, where $X = B$ or $X = C_j$. Since $\vee_{i=1}^{n} A_i \in E_m$, $\vee_{i=1}^{n} A_i$ in a-$\mathcal{E}$ by the assumption $E_m \subseteq$ a-$\mathcal{E}$. If $\vee_{i=1}^{n} A_i$ is a prime disjunctive form with respect to a-$\mathcal{E}$, then X must be in $\mathcal{A}$ by S3. Otherwise, suppose $\vee_{i=1}^{n} A_i$ is not a prime disjunctive form with respect to a-$\mathcal{E}$. By lemma 3.3.1, therefore, $\vee_{i=1}^{n} A_i$ is derivable from a set of prime disjunctive forms consisting of prerequisites in a-$\mathcal{E}$. Since $\{B, C_1, ..., C_n\} \cup \mathcal{A}$ is consistent, $\{B, C_{k_1}, ..., C_{k_n}\} \cup \mathcal{A}$ is consistent, where $C_{k_i} = C_j$, $1 \leq j \leq n$, causing that those related B, C_1, ..., C_n are in $\Gamma_2(\text{a-}\mathcal{E}, \mathcal{A})$ by S3. So, X is in $\Gamma_2(\text{a-}\mathcal{E}, \mathcal{A})$. This is the contradiction. It must be $\bigcup_{j=0}^{\infty} A_j \subseteq \Gamma_2(\text{a-}\mathcal{E}, \mathcal{A})$.

∎

The following is the last lemma used to prove theorem 3.3.1. It says that the sufficient condition for the construction not to add unnecessary facts to a sequence of pairs (E_i, A_i) is that we use the pair $(\bigcup_{j=0}^{\infty} E_j, \bigcup_{j=0}^{\infty} A_j)$ as a basis for the construction of the sequence of pairs.

Lemma 3.3.4:

Let $(\text{a-}\mathcal{E}, \mathcal{A})$ be a pair of sets, and a sequence of pairs (E_i, A_i) be constructed as in theorem 3.3.1 using default theory $\mathcal{D}=(\mathcal{F}, \Delta)$. If a-$\mathcal{E} = \bigcup_{j=0}^{\infty} E_j$ and $\mathcal{A} = \bigcup_{j=0}^{\infty} A_j$, then

$$\bigcup_{j=0}^{\infty} E_j \subseteq \Gamma_1(\text{a-}\mathcal{E}, \mathcal{A}) \text{ and } \bigcup_{j=0}^{\infty} A_j \subseteq \Gamma_2(\text{a-}\mathcal{E}, \mathcal{A}).$$

∎

Proof:

Assume that a-$\mathcal{E} = \bigcup_{j=0}^{\infty} E_j$ and $\mathcal{A} = \bigcup_{j=0}^{\infty} A_j$. We will prove by induction on j

that $E_j \subseteq \Gamma_1(\text{a-}\mathfrak{E}, \mathcal{A})$ and $A_j \subseteq \Gamma_2(\text{a-}\mathfrak{E}, \mathcal{A})$ for all j.

Basis: $E_0 = \mathfrak{F} \subseteq \Gamma_1(\text{a-}\mathfrak{E}, \mathcal{A})$ and $A_0 = \mathfrak{F} \subseteq \Gamma_2(\text{a-}\mathfrak{E}, \mathcal{A})$.

Hypothesis: $E_j \subseteq \Gamma_1(\text{a-}\mathfrak{E}, \mathcal{A})$ and $A_j \subseteq \Gamma_2(\text{a-}\mathfrak{E}, \mathcal{A})$ imply $E_{j+1} \subseteq \Gamma_1(\text{a-}\mathfrak{E}, \mathcal{A})$ and $A_{j+1} \subseteq \Gamma_2(\text{a-}\mathfrak{E}, \mathcal{A})$

Step: Assume that $E_m \subseteq \Gamma_1(\text{a-}\mathfrak{E}, \mathcal{A})$ and $A_m \subseteq \Gamma_2(\text{a-}\mathfrak{E}, \mathcal{A})$. Then, by S2 and monotonicity, $Th(E_m) \subseteq \Gamma_1(\text{a-}\mathfrak{E}, \mathcal{A})$ and $Th(A_m) \subseteq \Gamma_2(\text{a-}\mathfrak{E}, \mathcal{A})$.

First, we will prove that $E_{m+1} \subseteq \Gamma_1(\text{a-}\mathfrak{E}, \mathcal{A})$. If $E_{m+1} = Th(E_m)$, we have proven it. Otherwise, suppose there is a $\vee_{i=1}^{n} C_i$ such that $\vee_{i=1}^{n} C_i \in E_{m+1} - Th(E_m)$. By the construction of E_{m+1} and A_{m+1}, there is a prime disjunctive form $\vee_{i=1}^{n} A_i$ consisting of prerequisites of applicable defaults with respect to E_m and $\{B, C_1, ..., C_n\} \cup \mathcal{A}$ is consistent. $\vee_{i=1}^{n} A_i \in E_m$ implies $\vee_{i=1}^{n} A_i \in \Gamma_1(\text{a-}\mathfrak{E}, \mathcal{A})$ by the assumption $E_m \subseteq \Gamma_1(\text{a-}\mathfrak{E}, \mathcal{A})$. If $\vee_{i=1}^{n} A_i$ is also prime with respect to a-$\mathfrak{E}$, then $\vee_{i=1}^{n} C_i \in \Gamma_1(\text{a-}\mathfrak{E}, \mathcal{A})$ and $B, C_1, ..., C_n \in \Gamma_2(\text{a-}\mathfrak{E}, \mathcal{A})$ by S3. Otherwise, suppose that it is not. By lemma 3.3.1, $\vee_{i=1}^{n} A_i$ is derivable from a set of prime disjunctive forms in $\mathfrak{H}_B$ and $\Gamma_1(\text{a-}\mathfrak{E}, \mathcal{A})$. Since $\{B, C_1, ..., C_n\} \cup \mathcal{A}$ is consistent, $\{B, C_{k_1}, ..., C_{k_n}\} \cup \mathcal{A}$ is consistent, where $C_{k_i} = C_j$, $1 \leq j \leq n$, causing that those related $\vee_i C_{k_i}$ s are derivable and are in $\Gamma_1(\text{a-}\mathfrak{E}, \mathcal{A})$ by S3. In addition, $\vee_{i=1}^{n} C_i$ is derivable from a set of the those related $\vee_i C_{k_i}$ s. By monotonicity, $\vee_{i=1}^{n} C_i$ is in $\Gamma_1(\text{a-}\mathfrak{E}, \mathcal{A})$. Since $E_{m+1} = Th(E_m) \cup \{\text{those } \vee_{i=1}^{n} C_i\text{s}\}$, $E_{m+1} \subseteq \Gamma_1(\text{a-}\mathfrak{E}, \mathcal{A})$.

Next, we prove that $A_{m+1} \subseteq \Gamma_2(\text{a-}\mathfrak{E}, \mathcal{A})$. If $A_{m+1} = Th(A_m)$, we have proven it. Otherwise, suppose there is X such that $X \in E_{m+1} - Th(E_m)$. By the construction of A_{m+1}, there is a prime disjunctive form $\vee_{i=1}^{n} A_i$ consisting of prerequisites of applicable defaults with respect to E_m and $\{B, C_1, ..., C_n\} \cup \mathcal{A}$ is consistent. $\vee_{i=1}^{n} A_i \in E_m$ implies $\vee_{i=1}^{n} A_i \in \Gamma_1(\text{a-}\mathfrak{E}, \mathcal{A})$ by the assumption $E_m \subseteq \Gamma_1(\text{a-}\mathfrak{E}, \mathcal{A})$. If $\vee_{i=1}^{n} A_i$ is also prime with respect to a-$\mathfrak{E}$, then $\vee_{i=1}^{n} C_i \in \Gamma_1(\text{a-}\mathfrak{E}, \mathcal{A})$ and $B, C_1, ..., C_n \in \Gamma_2(\text{a-}\mathfrak{E}, \mathcal{A})$ by S3. Otherwise, suppose that it is not. By lemma 3.3.1, $\vee_{i=1}^{n} A_i$ is derivable from a set of prime disjunctive forms in $\mathcal{H}_B$ and Γ_1(a-$\mathcal{E}$, $\mathcal{A}$). Since $\{B, C_1, ..., C_n\} \cup \mathcal{A}$ is consistent, $\{B, C_{k_1}, ..., C_{k_n}\} \cup \mathcal{A}$ is consistent, where $C_{k_i} = C_j$, $1 \le j \le n$, causing that those related B, C_1, ..., C_n are in Γ_2(a-$\mathcal{E}$, $\mathcal{A}$) by S3. So, X is in Γ_2(a-$\mathcal{E}$, $\mathcal{A}$). Since $A_{m+1} = Th(A_m) \cup \{$those Xs$\}$, $A_{m+1} \subseteq \Gamma_2$(a-$\mathcal{E}$, $\mathcal{A}$).

∎

Using these three lemmas, it is easy to prove theorem 3.4.1. This is the proof of the theorem.

Proof: Theorem 3.4.1

(if part)

Suppose that a-$\mathcal{E}$ is an alternative extension based on an assumption set $\mathcal{A}$ for a default theory $\mathcal{D}$; i.e., a-$\mathcal{E} = \Gamma_1$(a-$\mathcal{E}$, $\mathcal{A}$) and $\mathcal{A} = \Gamma_2$(a-$\mathcal{E}$, $\mathcal{A}$). Then,

- Γ_1(a-$\mathcal{E}$, $\mathcal{A}$) $\subseteq \cup_{j=0}^{\infty} E_j$ and Γ_2(a-$\mathcal{E}$, $\mathcal{A}$) $\subseteq \cup_{j=0}^{\infty} A_j$ by lemma 3.3.2; and
- Γ_1(a-$\mathcal{E}$, $\mathcal{A}$) $\supseteq \cup_{j=0}^{\infty} E_j$ and Γ_2(a-$\mathcal{E}$, $\mathcal{A}$) $\supseteq \cup_{j=0}^{\infty} A_j$ by lemma 3.3.3.

(only if part)

Suppose that a-$\mathcal{E} = \cup_{j=0}^{\infty} E_j$ and $\mathcal{A} = \cup_{j=0}^{\infty} A_j$. Then,

- Γ_1(a-$\mathcal{E}$, $\mathcal{A}$) $\subseteq \cup_{j=0}^{\infty} E_j$ and Γ_2(a-$\mathcal{E}$, $\mathcal{A}$) $\subseteq \cup_{j=0}^{\infty} A_j$ by lemma 3.3.2; and
- Γ_1(a-$\mathcal{E}$, $\mathcal{A}$) $\supseteq \cup_{j=0}^{\infty} E_j$ and Γ_2(a-$\mathcal{E}$, $\mathcal{A}$) $\supseteq \cup_{j=0}^{\infty} A_j$ by lemma 3.3.4.

Hence, we prove the theorem.

∎

Let us verify the pair of an alternative extension and an assumption set in the example 3.3.8 using the characterization theorem.

Example 3.3.12:

We will prove using the characterization that a pair (a-$\mathcal{E} = Th(\mathcal{F} \cup \{R\})$, $\mathcal{A} = Th(\mathcal{F} \cup \{R\})$) is a pair of an alternative extension and an assumption set for the default theory $\mathcal{D} = (\mathcal{F}, \Delta = \{\delta_1, \delta_2\})$, where $\mathcal{F} = \{P \vee Q\}$, $\delta_1 = \frac{P:R}{R}$, and $\delta_2 = \frac{Q:R}{R}$.

Since there is only one distinct justification, there is only one default set associated to the justification: $\bar{\delta}_R$. Hence, the set of disjunctive forms consisting of prerequisites related to $\bar{\delta}_R$ is $\mathcal{H}_R = \{P, Q, P \vee Q\}$. So, a prime disjunctive form consisting of prerequisites in $\mathcal{H}_R$ with respect to a-$\mathcal{E}$ is $P \vee Q$. It is the only prime disjunctive form consisting of prerequisites in $\mathcal{H}_R$. Now, let us calculate the sequence of pairs (E_i, A_i) related to the pair (a-$\mathcal{E}$, $\mathcal{A}$).

$$E_0 = A_0 = \mathcal{F},$$
$$E_1 = Th(E_0) \cup \{R \vee R\} = Th(\mathcal{F}) \cup \{R\},$$
$$A_1 = Th(A_0) \cup \{R, R, R\} = Th(\mathcal{F}) \cup \{R\},$$
$$E_2 = Th(E_1) \cup \{R \vee R\} = Th(Th(\mathcal{F}) \cup \{R\}) \cup \{R\} = Th(\mathcal{F} \cup \{R\}),$$
$$A_2 = Th(A_1) \cup \{R, R, R\} = Th(Th(\mathcal{F}) \cup \{R\}) \cup \{R\} = Th(\mathcal{F} \cup \{R\}),$$

and, for $j \geq 3$

$$E_j = Th(E_2) \cup \{R \vee R\} = Th(E_2),$$
$$A_j = Th(A_2) \cup \{R, R, R\} = Th(A_2).$$

It is easy to verify that $(\bigcup_{j=0}^{\infty} E_j, \bigcup_{j=0}^{\infty} A_j) = (Th(E_2), Th(A_2)) =$ (a-$\mathcal{E}$, $\mathcal{A}$).

From the examples above we can see that an alternative extension is identical to its assumption set for a normal default theory. This property is summed up by the following theorem. We will use characterization theorem 3.4.1 to prove the theorem.

Theorem 3.3.2:

If a-$\mathcal{E}$ is an alternative extension based on an assumption set $\mathcal{A}$ for a normal default theory $\mathcal{D}$, then a-$\mathcal{E} = \mathcal{A}$.

Proof:

Let (a-$\mathcal{E}$, $\mathcal{A}$) be a pair of an alternative extension and an assumption set for a normal default theory $\mathcal{D} = (\mathcal{F}, \Delta)$. Construct a sequence of pairs (E_i, A_i) related to the pair. Recall that

$$E_0 = \mathcal{F} \text{ and } A_0 = \mathcal{F},$$

and, for $j \geq 1$

$$E_j = Th(E_{j-1}) \cup \{\vee_{i=1}^{n} C_i \mid \vee_{i=1}^{n} A_i \in \mathcal{H}_B \text{ is a prime disjunctive form with respect to } E_{j-1}, \mathcal{H}_B \text{ is associated to a justification } B \text{ and } \mathcal{A} \cup \{B, C_1, \ldots, C_n\} \text{ is consistent}\},$$

$$A_j = Th(A_{j-1}) \cup \{B, C_1, \ldots, C_n \mid \vee_{i=1}^{n} A_i \in \mathcal{H}_B \text{ is a prime disjunctive form with respect to } E_{j-1}, \mathcal{H}_B \text{ is associated to a justification } B \text{ and } \mathcal{A} \cup \{B, C_1, \ldots, C_n\} \text{ is consistent}\}.$$

Let $\bar{\delta}_B$ be a set of defaults having the same justification B; i.e., $\bar{\delta}_B = \{ \delta = \frac{A_i : B}{C_i} \mid \delta \in \Delta\}$. Since $\mathcal{D}$ is normal, $C_i = B$ for all δ's in $\bar{\delta}_B$. So, $E_j = A_j$.

∎

3.4 Properties of the New Approach

In this section, we will investigate further the properties of an alternative extension for a default theory. We will also look for the relation between an extension and an alternative extension, especially for normal default theories. We expect to see that an alternative extension for a default theory is a natural extension of that of the Reiter approach.

First, we will see the behavior of alternative extensions when adding some facts to a default theory. It is expected that, as seen in section 3.2, the new approach will not have monotonicity properties. However, adding facts derivable from an alternative extension does not cause any changes to the alternative extension. In addition, it may affect other alternative extensions. In other words, *for every alternative extension a-$\mathcal{E}$ for a default theory* $\mathcal{D} = (\mathcal{F}, \Delta)$, then a-$\mathcal{E}$ is also an alternative extension for $\mathcal{D}' = (\mathcal{F} \cup \mathcal{S}, \Delta)$, where $\mathcal{S} \subseteq$ a-$\mathcal{E}$. For example, if we add "fly(Tweety)" to $\mathcal{F}$ in example 3.2.5, it does not change a-$\mathcal{E}_1$ but does cause no other alternative extension.

Theorem 3.4.1:

Let a-$\mathfrak{E}$ be an alternative extension for a default theory $\mathfrak{D} = (\mathfrak{F}, \Delta)$ based on an assumption set $\mathcal{A}$. Then, a-$\mathfrak{E}$ is also an alternative extension for a default theory $\mathfrak{D}' = (\mathfrak{F} \cup \mathfrak{S}, \Delta)$, where $\mathfrak{S} \subseteq$ a-$\mathfrak{E}$.

$\blacksquare$

Proof:

Assume that a-$\mathfrak{E}$ is an alternative extension based on an assumption set $\mathcal{A}$ for a default $\mathfrak{D} = (\mathfrak{F}, \Delta)$. Now, consider a default theory $\mathfrak{D}' = (\mathfrak{F} \cup \mathfrak{S}, \Delta)$, where $\mathfrak{S} \subseteq$ a-$\mathfrak{E}$. We need to check the three conditions S1, S2, and S3. The pair (a-$\mathfrak{E}$, $\mathcal{A}$) satisfies S1 and S2; i.e., $\mathfrak{F} \cup \mathfrak{S} \subseteq$ a-$\mathfrak{E}$ and $\mathfrak{F} \cup \mathfrak{S} \subseteq \mathcal{A}$ and Th(a-$\mathfrak{E}$) = a-$\mathfrak{E}$ and Th($\mathcal{A}$) = $\mathcal{A}$. In addition, suppose that $\vee_{i=1}^{n} A_i$ is a prime disjunctive form in $\mathfrak{H}_B$ with respect to a-$\mathfrak{E}$ and $\{B, C_1, \ldots, C_n\} \cup \mathcal{A}$ is consistent. Then, $\vee_{i=1}^{n} C_i \in$ a-$\mathfrak{E}$ and $B, C_1, \ldots, C_n \in \mathcal{A}$ by S3. Γ_1(a-$\mathfrak{E}$, $\mathcal{A}$) $\subseteq$ a-$\mathfrak{E}$ and Γ_2(a-$\mathfrak{E}$, $\mathcal{A}$) $\subseteq$ a-$\mathfrak{E}$ by the smallness of Γ_1(a-$\mathfrak{E}$, $\mathcal{A}$) and Γ_2(a-$\mathfrak{E}$, $\mathcal{A}$). We need to check the smallness of the pair (a-$\mathfrak{E}$, $\mathcal{A}$). (Γ_1(a-$\mathfrak{E}$, $\mathcal{A}$), Γ_2(a-$\mathfrak{E}$, $\mathcal{A}$)) is a pair of the smallest sets satisfying S1, S2, and S3 with respect to $\mathfrak{D}$ and $\mathfrak{D}'$. Recall that (a-$\mathfrak{E}$, $\mathcal{A}$) is a pair of the smallest sets satisfying S1, S2, and S3 with respect to $\mathfrak{D}$; i.e., a-$\mathfrak{E}$=Γ_1(a-$\mathfrak{E}$, $\mathcal{A}$) and $\mathcal{A}$=Γ_2(a-$\mathfrak{E}$, $\mathcal{A}$) with respect $\mathfrak{D}$. So, it must be a-$\mathfrak{E}$=Γ_1(a-$\mathfrak{E}$, $\mathcal{A}$) and $\mathcal{A}$=Γ_2(a-$\mathfrak{E}$, $\mathcal{A}$) with respect to $\mathfrak{D}'$. Hence, the pair is an alternative extension for $\mathfrak{D}'$.

$\blacksquare$

Next, we will investigate the behavior of alternative extensions when adding some defaults. Adding some defaults to a default theory will not shrink any extension. This means that the new approach is monoton in terms of adding defaults.

The following theorem shows the semi-monotonicity properties of an alternative extension for a default theory.

Theorem 3.4.2: Semi-monotonicity properties of alternative extensions

Let $\mathcal{D} = (\mathcal{F}, \Delta)$ be a default theory. For every alternative extension a-$\mathcal{E}$ based on an assumption set $\mathcal{A}$ for $\mathcal{D}$, there is an alternative extension a-$\mathcal{E}'$ and an assumption $\mathcal{A}'$ for $\mathcal{D}' = (\mathcal{F}, \Delta')$, where $\Delta \subseteq \Delta'$, such that a-$\mathcal{E} \subseteq$ a-$\mathcal{E}'$ and $\mathcal{A} \subseteq \mathcal{A}'$.

This theorem guarantees that when adding defaults to a default theory $\mathcal{D}$, for any alternative extension a-$\mathcal{E}$ for $\mathcal{D}$, there is an alternative extension for the newly added default theory such that it supersedes a-$\mathcal{E}$. In other words, adding defaults will not shrink any extension. Therefore, default theories are monoton in terms of defaults.

To prove the theorem, we first construct a pair (a-$\mathcal{E}'$, $\mathcal{A}'$) of sets based on an alternative extension a-$\mathcal{E}$ and an assumption set $\mathcal{A}$ for $\mathcal{D}$. See the construction 3.3. Since the pair (a-$\mathcal{E}'$, $\mathcal{A}'$) is based on the pair (a-$\mathcal{E}$, $\mathcal{A}$), then, a-$\mathcal{E} \subseteq$ a-$\mathcal{E}'$ and $\mathcal{A} \subseteq \mathcal{A}'$. So, we only need to prove that (a-$\mathcal{E}'$, $\mathcal{A}'$) is a pair of an alternative extension and an assumption for $\mathcal{D}'$. To prove it, we use characterization theorem 3.4.1 by constructing a sequence of pairs (S_i, T_i) as in construction 3.4.

Construction 3.3. is similar to construction 3.1. However, we apply defaults using their disjunctive forms consisting of prerequisites. Therefore, we need to check the consistency of their justification and consequents, in addition to the derivability of the disjunctive forms.

Construction 3.3:

Suppose a-$\mathcal{E}$ is an alternative extension based on an assumption set $\mathcal{A}$ for a default theory $\mathcal{D} = (\mathcal{F}, \Delta)$. Suppose Δ' is countable, possibly finite, where $\Delta \subseteq \Delta'$. So, a set of distinct justifications of defaults in Δ' is countable, possibly finite: $\{B_1, B_2, B_3, \dots\}$. Let $\bar{\delta}_j$ denote a set of defaults having the same justification B_k,

$$\bar{\delta}_j = \{\delta_{i,j} = \frac{A_{i,j} : B_j}{C_{i,j}} \mid \delta_{i,j} \in \Delta\}.$$

Therefore, we have a countable, possibly finite, class of such sets. Suppose that $\bar{\delta}$

$= \{\overline{\delta}_1, \overline{\delta}_2, ...\}$ is a fixed enumeration of the class. $\mathcal{H}_j$ is a set of disjunctive forms consisting of prerequisites of defaults in $\overline{\delta}_j$. This set is countable. Suppose that $\mathcal{H}_j=\{\overline{A}_h^j \mid \overline{A}_h^j$ is a disjunctive form consisting of prerequisites of defaults in $\overline{\delta}_j\}$. So, $\overline{A}_h^j=\vee_i^n A_{i,j}$, where $A_{i,j}$s are prerequisites of defaults in $\overline{\delta}_j$, and the set of the related consequents is $\overline{C}_h^j=\{C_{1,j},...,C_{n,j}\}$, and the related disjunctive form consisting of consequents is $\vee_{C\in\overline{C}_h^j} C$. Let us construct a sequence of pairs (E_i, A_i) as follows:

$$E_0 = \text{a-}\mathcal{E} \text{ and } A_0 = \mathcal{A},$$

and for $k \geq 1$,

$$E_k = Th(E_{k-1}) \cup \bigcup_{j=0}^{\infty} E_k^j, \text{ and}$$

$$A_k = Th(A_{k-1}) \cup \bigcup_{j=0}^{\infty} A_k^j,$$

where

$$E_k^0 = \varnothing, \text{ and}$$

$$A_k^0 = A_{k-1},$$

and for $j \geq 1$, (we attempt to apply defaults in $\overline{\delta}_j$)

$$E_k^j = \bigcup_{\overline{A}_h^j\in\mathcal{H}_j} E_k^{jh} \text{ and } A_k^j = A_k^{j\text{-}1} \cup \bigcup_{\overline{A}_h^j\in\mathcal{H}_j} A_k^{jh},$$

where

$$A_k^{j0} = A_k^{j\text{-}1},$$

and for $\overline{A}_h^j\in\mathcal{H}_j$, (we attempt to check all disjunctive forms in $\mathcal{H}_j$)

$$E_k^{jh} = \{\vee_{C\in\overline{C}_h^j} C \mid \overline{A}_h^j \text{ is a prime disjunctive form with respect to } E_{k-1} \text{ and } \{B_j\} \cup \overline{C}_h^j \cup A_k^{jh\text{-}1} \text{ is consistent}\}$$

$$A_k^{jh} = A_k^{jh\text{-}1} \cup \{B_j, C\in \overline{C}_h^j \mid \overline{A}_h^j \text{ is a prime disjunctive form with respect to } E_{k-1} \text{ and } \{B_j\} \cup \overline{C}_h^j \cup A_k^{jh\text{-}1} \text{ is consistent}\}.$$

Assign $E = \bigcup_{k=0}^{\infty} E_k$ and $A = \bigcup_{k=0}^{\infty} A_k$. All A_ks are consistent causing A to be consistent, and so is E. Since $E_0=\text{a-}\mathcal{E}$ and $A_0=\mathcal{A}$, $\text{a-}\mathcal{E}\subseteq E$ and $\mathcal{A}\subseteq A$. ▒

Let us attempt to apply construction 3.4.3.

Example 3.4.1:

Let $\mathcal{D}' = (\mathcal{F}, \Delta'=\{\delta_i \mid 1\le i\le 9\})$ be a default theory, where $\mathcal{F} = \{P\vee Q\}$, $\delta_1=\frac{P:R}{R}$, $\delta_2=\frac{Q:R}{R}$, $\delta_3=\frac{P:S}{S}$, $\delta_4=\frac{Q:S}{S}$, $\delta_5=\frac{R:T}{T}$, $\delta_6=\frac{S:U}{U}$, $\delta_7=\frac{S:V}{V}$, $\delta_8=\frac{U:W}{W}$, and $\delta_9=\frac{V:W}{W}$. There are five distinct justifications of defaults in Δ': B_1=R, B_2=S, B_3=T, B_4=U, B_5=V, and B_6=W. So, we have $\bar{\delta}_1=\bar{\delta}_R=\{\delta_1, \delta_2\}$, $\bar{\delta}_2=\bar{\delta}_S=\{\delta_3, \delta_4\}$, $\bar{\delta}_3=\bar{\delta}_T=\{\delta_5\}$, $\bar{\delta}_4=\bar{\delta}_U=\{\delta_6\}$, $\bar{\delta}_5=\bar{\delta}_V=\{\delta_7\}$, and $\bar{\delta}_6=\bar{\delta}_W=\{\delta_8, \delta_9\}$. Also, $\mathcal{H}_1=\mathcal{H}_R=\{\bar{A}_1^1=P, \bar{A}_2^1=Q, \bar{A}_3^1=P\vee Q\}$, $\mathcal{H}_2=\mathcal{H}_S=\{\bar{A}_1^2=P, \bar{A}_2^2=Q, \bar{A}_3^2=P\vee Q\}$, $\mathcal{H}_3=\mathcal{H}_T=\{\bar{A}_1^3=T\}$, $\mathcal{H}_4=\mathcal{H}_U=\{\bar{A}_1^4=S\}$, $\mathcal{H}_5=H_V=\{\bar{A}_1^5=S\}$, and $\mathcal{H}_6=\mathcal{H}_W=\{\bar{A}_1^6=U, \bar{A}_2^6=V, \bar{A}_3^6=U\vee V\}$. Suppose a-$\mathcal{E}=\mathcal{F}$ and $\mathcal{A}=\mathcal{F}$. Construct a sequence of pairs (E_i, A_i) as in construction 3.3. Initially, we have

$$E_0 = A_0 = \mathcal{F} = \{P\vee Q\}.$$

For $k = 1$, we have

$$E_1^0 = \emptyset \text{ and } A_1^0 = A_0 = \mathcal{F}.$$

For $j = 1$, we attempt to apply defaults in $\bar{\delta}_1$ and to check the three disjunctive forms in $\mathcal{H}_1=\{\bar{A}_1^1, \bar{A}_2^1, \bar{A}_3^1\}$. Set $A_1^{1,0}=A_1^0$. For $h = 1$ and $h = 2$, since $\bar{A}_h^1$ is not in E_0, $E_1^{1,h}=\emptyset$ and $A_1^{1,h}=A_1^{1,0}$. For $h = 3$, $E_1^{1,3}=\{\vee_{C\in\bar{C}_3^1} C\}=\{R\vee R\}=\{R\}$ and $A_1^{1,3} = A_1^{1,2}\cup\{B_1\}\cup\bar{C}_3^1=\mathcal{F}\cup\{R\}$ since $\bar{A}_3^1=P\vee Q$ is prime in E_0. So,

$$E_1^1 = \bigcup_{\bar{A}_h^1\in\mathcal{H}_1} E_1^{1,h} = \{R\}$$

$$A_1^1 = A_1^0 \cup \bigcup_{\bar{A}_h^1\in\mathcal{H}_1} A_1^{1,h} = \mathcal{F}\cup\{R\}.$$

Next, for $j = 2$, we attempt to apply defaults in $\bar{\delta}_2$ by checking the three disjunctive forms in $\mathcal{H}_2=\{\bar{A}_1^2, \bar{A}_2^2, \bar{A}_3^2\}$. Set $A_1^{2,0}=A_1^1$. For $h = 1$ and $h = 2$, then $E_1^{2,h}=\emptyset$ and $A_1^{2,h}=A_1^{2,0}$ since $\bar{A}_h^2$ is not in E_0. For $h = 3$, since $\bar{A}_3^2=P\vee Q$ is prime in E_0, then $E_1^{2,3}=\vee_{C\in\bar{C}_3^2} C=\{S\vee S\}=\{S\}$ and $A_1^{2,3}=A_1^{2,2}\cup\{B_2\}\cup\bar{C}_3^2=\mathcal{F}\cup\{S\}$. So,

$$E_1^2 = \bigcup_{\bar{A}_h^2\in\mathcal{H}_2} E_1^{2,h} = \{S\}$$

$$A_1^2 = A_1^1 \cup \bigcup_{\bar{A}_h^2\in\mathcal{H}_2} A_1^{2,h} = \mathcal{F}\cup\{R, S\}.$$

For $j > 2$, we will have $E_1^j=\emptyset$ and $A_1^j=A_1^2$. Thus,

$$E_1 = Th(E_0)\cup\bigcup_{j=0}^{\infty} E_1^j = Th(\mathcal{F})\cup\{R, S\} = Th(\mathcal{F})\cup\{R, S\}, \text{ and}$$

$$A_1 = Th(A_0) \cup \bigcup_{j=0}^{\infty} A_1^j = Th(\mathfrak{F}) \cup \mathfrak{F} \cup \{R, S\} = Th(\mathfrak{F}) \cup \{R, S\}.$$

For $k = 2$, we have

$$E_2^0 = \varnothing \text{ and } A_2^0 = A_1 = Th(\mathfrak{F}) \cup \{R, S\}$$

and get the same result for $j = 1$ and $j = 2$: $E_2^j = E_1^j$ and $A_2^j = A_2^0$. For $j = 3$, we attempt to apply defaults in $\overline{\delta}_3=\{\delta_5\}$ by checking the only one disjunctive form in $\mathcal{H}_3 = \{\overline{A}_1^3\}$. Set $A_2^{3,0}= A_2^2$. For $h = 1$, since $\overline{A}_1^3 = T$ is prime in E_1, then $E_2^{3,1} = \{\vee_{C\in\overline{C}_1^3} C\} = \{T\}$ and $A_2^{3,1} = A_2^{3,0}\cup\{B_3\}\cup\overline{C}_1^3 = Th(\mathfrak{F})\cup\{R, S, T\}$. So,

$$E_2^3 = \bigcup_{\overline{A}_h^3\in\mathcal{H}_3} E_2^{1,h} = \{T\}$$

$$A_2^3 = A_2^2 \cup \bigcup_{\overline{A}_h^3\in\mathcal{H}_3} A_2^{3,h} = Th(\mathfrak{F}) \cup \{R, S, T\}.$$

Next, for $j = 4$, we apply a default in $\overline{\delta}_4= \{\delta_6\}$ by checking the only one disjunctive form in $\mathcal{H}_4 = \{\overline{A}_1^4\}$. Set $A_2^{4,0} = A_2^3$. For $h = 1$, since $\overline{A}_1^4 = S$ is prime in E_1, $E_2^{4,h} = \{\vee_{C\in\overline{C}_1^4} C\} = \{U\}$ and $A_2^{4,h} = A_2^{4,0}\cup\{B_4\}\cup\overline{C}_3^4 = Th(\mathfrak{F})\cup\{R, S, T, U\}$. So,

$$E_2^4 = \bigcup_{\overline{A}_h^4\in\mathcal{H}_4} E_2^{4,h} = \{T, U\}$$

$$A_2^4 = A_2^3 \cup \bigcup_{\overline{A}_h^4\in\mathcal{H}_4} A_2^{4,h} = Th(\mathfrak{F}) \cup \{R, S, T, U\}.$$

For $j = 5$, we attempt to apply a default in $\overline{\delta}_5= \{\delta_7\}$ by checking the only one disjunctive form in $\mathcal{H}_5 = \{\overline{A}_1^5\}$. Set $A_2^{5,0}= A_2^4$. For $h = 1$, since $\overline{A}_1^5 = S$ is in E_1, $E_2^{5,1} = \{\vee_{C\in\overline{C}_1^5} C\}= \{V\}$ and $A_2^{5,1} = A_2^{5,0}\cup\{B_5\}\cup\overline{C}_1^5 = Th(\mathfrak{F})\cup\{R, S, T, U, V\}$. So,

$$E_2^5 = \bigcup_{\overline{A}_h^5\in\mathcal{H}_5} E_2^{5,h} = \{R, S, T, U, V\} \text{ and}$$

$$A_2^5 = A_2^4 \cup \bigcup_{\overline{A}_h^5\in\mathcal{H}_5} A_2^{5,h} = Th(\mathfrak{F}) \cup \{R, S, T, U, V\}.$$

Continuing calculation for $j > 5$, we will get the same result: $E_2^5 = \varnothing$ and $A_2^j = A_2^5$.

So, we get, for k = 2,

$$E_2 = Th(E_1)\cup\bigcup_{j=0}^{\infty} E_1^j = Th(E_1)\cup\mathfrak{F}\cup\{R, S, T, U\} = Th(E_1)\cup\{T, U, V\} \text{ and}$$

$$A_2 = Th(A_1)\cup\bigcup_{j=0}^{\infty} A_1^j = Th(A_1)\cup\mathfrak{F}\cup\{R, S, T, U\} = Th(A_1\})\cup\{T, U, V\}.$$

For $k = 3$, we have

$$E_3^0 = \varnothing \text{ and } A_3^0 = A_2 = Th(A_1) \cup \{T, U, V\}.$$

For $j = 1, 2, 3, 4$, and 5, we will have the same result: $E_3^j = E_2^j$ and $A_3^j = A_3^0$. For $j = 6$, we attempt to apply defaults in $\overline{\delta}_6 = \{\delta_8, \delta_9\}$ by checking the three disjunctive form in $\mathcal{H}_6 = \{\overline{A}_1^6, \overline{A}_2^6, \overline{A}_3^6\}$. Set $A_3^{6,0} = A_3^5$. For $h = 1$ and $h = 2$, we get $E_3^{6,h} = \varnothing$ and $A_3^{6,h} = A_3^{6,0}$. For $h = 3$, $E_3^{6,3} = \{\vee_{C \in \overline{C}_1^6} C\} = \{W \vee W\} = \{W\}$ and $A_3^{6,3} = A_3^{6,2} \cup \{B_6\} \cup \overline{C}_1^6 = Th(A_2) \cup \{T, U, V, W\}$ since $\overline{A}_3^6 = U \vee V$ is prime in E_2. So,

$$E_3^6 = \cup_{\overline{A}_h^6 \in \mathcal{H}_6} E_2^{6,h} = \{W\} \text{ and}$$

$$A_3^6 = A_3^5 \cup \cup_{\overline{A}_h^6 \in \mathcal{H}_6} A_2^{6,h} = Th(A_2) \cup \{W\}.$$

Continuing calculation for $j > 6$, we will get the same result: $E_3^j = \varnothing$ and $A_4^j = A_3^6$. Consequently, we get, for $k = 3$,

$$E_3 = Th(E_2) \cup \cup_{j=0}^{\infty} E_3^j = Th(E_2) \cup \{W\} \text{ and}$$

$$A_3 = Th(A_2) \cup \cup_{j=0}^{\infty} A_3^j = Th(A_2) \cup \{W\}.$$

For $k = 4$, we will get the same result: $E_4^j = E_3^j$ and $A_4^j = A_3^j$. Therefore,

$$E_4 = Th(E_3) \cup \cup_{j=0}^{\infty} E_4^j = Th(E_3) \text{ and}$$

$$A_4 = Th(A_3) \cup \cup_{j=0}^{\infty} A_4^j = Th(A_3).$$

Continuing calculation for $k > 4$, we will get the same thing,

$$E_k = Th(E_4) = E_4 \text{ and } A_k = Th(A_4) = A_4.$$

Hence,

$$E = \cup_{k=0}^{\infty} E_k = E_4 = Th(\mathfrak{F} \cup \{R, S, T, U, W\}) \text{ and}$$

$$A = \cup_{k=0}^{\infty} A_k = A_4 = Th(\mathfrak{F} \cup \{R, S, T, U, W\}).$$

It is easy to verify that (E, A) is a pair of an alternative extension and an assumption set for the default theory $\mathcal{D}'$.

∎

The following construction is similar to construction 3.2 but uses disjunctive forms consisting of prerequisites of defaults.

Construction 3.4:

Suppose $\mathcal{D}' = (\mathfrak{F}, \Delta')$ is a default theory. Let $\overline{\delta}_{B'}$ denote a set of defaults having justification B'. Since Δ' is countable, there is at most a countable class of such set $\overline{\delta}_{B'}$s. Now, construct a sequence of pairs (S_i, T_i) based on characterization theorem 3.3.1 related to a pair (E, A). (In the proof of theorem 3.4.1, this pair is related to the pair constructed in construction 3.3.) That is, $S_0 = \mathcal{F}$ and $T_0 = \mathcal{F}$, and for $k' \geq 1$,

$S_{k'} = Th(S_{k'-1}) \cup \{ \vee_{i=1}^{n} C_i' \mid \vee_{i=1}^{n} A_i' \in \mathcal{H}_{B'}$ is a prime disjunctive form with respect to $S_{k'-1}$, $\mathcal{H}_{B'}$ is associated to a justification B' of defaults in Δ', and $\{B', C_1', ..., C_n'\} \cup A$ is consistent$\}$,

$T_{k'} = Th(T_{k'-1}) \cup \{B', C_1', ..., C_n' \mid \vee_{i=1}^{n} A_i' \in \mathcal{H}_{B'}$ is a prime disjunctive form with respect to $S_{k'-1}$, $\mathcal{H}_{B'}$ is associated to a justification B' of defaults in Δ', and $\{B', C_1', ..., C_n'\} \cup A$ is consistent$\}$.

Assign $S = \cup_{k'=0}^{\infty} S_{k'}$ and $T = \cup_{k'=0}^{\infty} T_{k'}$.

To prove theorem 3.4.1, we will use the following lemma. Construction 3.4 based on the pair (E, A) constructed in construction 3.3 will result in pair (S, T) superseding the pair used to construct the pair (a-$\mathcal{E}$, $\mathcal{A}$) in construction 3.3.

Lemma 3.4.1:

Let $\mathcal{D} = (\mathcal{F}, \Delta)$ be a default theory, and let a-$\mathcal{E}$ be an alternative extension based on an assumption set $\mathcal{A}$ for $\mathcal{D}$. Suppose that (S, T) is a pair of sets constructed as in construction 3.4 related to a default theory $\mathcal{D}' = (\mathcal{F}, \Delta')$ and to a pair (E, A), where $\Delta \subseteq \Delta'$, a-$\mathcal{E} \subseteq E$ and $\mathcal{A} \subseteq A$. Then, a-$\mathcal{E} \subseteq S$ and $\mathcal{A} \subseteq T$.

Proof: Lemma 3.4.1

Assume that a-$\mathcal{E}$ is an alternative extension based on an assumption set $\mathcal{A}$ for $\mathcal{D}$. Using characterization theorem 3.3.1, we construct a sequence of pairs (G_k, H_k) related to (a-$\mathcal{E}$, $\mathcal{A}$) and $\mathcal{D}$. So, a-$\mathcal{E} = \cup_{k=0}^{\infty} G_k$ and $\mathcal{A} = \cup_{k=0}^{\infty} H_k$, where

$G_0 = H_0 = \mathcal{F}$,

and for $k \geq 1$,

$G_k = Th(G_{k-1}) \cup \{ \vee_{i=1}^{n} C_i \mid \vee_{i=1}^{n} A_i \in \mathcal{H}_B$ is a prime disjunctive form with respect to G_{k-1}, $\mathcal{H}_B$ is associated to a justification B of defaults in

Δ, and $\{B, C_1, ..., C_n\}\cup\mathcal{A}$ is consistent},

$H_k = Th(H_{k-1})\cup\{B, C_1, ..., C_n \mid \vee_{i=1}^{n} A_i \in \Psi_B$ is a prime disjunctive form with respect to G_{k-1}, Ψ_B is associated to a justification B of defaults in Δ, and $\{B, C_1, ..., C_n\}\cup\mathcal{A}$ is consistent}.

Suppose that $\mathcal{D}' = (\mathfrak{F}, \Delta')$ is a default theory, where $\Delta\subseteq\Delta'$. Construct a pair of sets (S, T) as in construction 3.4 related to pair of consistent sets (E, A) and to $\mathcal{D}'$, where a-$\mathcal{E}\subseteq$E and A$\subseteq$A. We will prove a-$\mathcal{E}\subseteq$ S and $\mathcal{A}\subseteq$T by induction on k that $G_k\subseteq S$ and $H_k\subseteq T$ for all k.

Basis: $G_0 = \mathfrak{F} = S_0 \subseteq S$ and $H_0 = \mathfrak{F} = T_0 \subseteq T$.

Hypothesis: $G_k \subseteq S$ and $H_k \subseteq T$ imply $G_{k+1} \subseteq S$ and $H_{k+1} \subseteq T$.

Step: Assume that $G_m\subseteq S$ and $H_m\subseteq T$. By monotonicity and lemma 3.2.1, $Th(G_m) \subseteq S$ and $Th(H_m) \subseteq T$.

First, we prove $G_{m+1} \subseteq S$. If $G_{m+1} = Th(G_m)$, we have proven it. Otherwise, suppose there is $\vee_{i=1}^{n} C_i$ such that $\vee_{i=1}^{n} C_i \in G_{m+1} - Th(G_m)$. By the construction of G_{m+1}, there is a prime disjunctive form $\vee_{i=1}^{n} A_i \in \Psi_B$ with respect to G_m and $\{B, C_1, ..., C_n\}\cup\mathcal{A}$ is consistent, causing $B, C_1, ..., C_n$ to be in $\cup_{k=0}^{\infty} H_k$. $B, C_1, ..., C_n$ in $\cup_{k=0}^{\infty} H_k = \mathcal{A}$ implies that $\{B, C_1, ..., C_n\}\cup A$ is consistent. By the assumption $G_m \subseteq S$, $\vee_{i=1}^{n} A_i \in G_m$ implies $\vee_{i=1}^{n} A_i \in S$. If $\vee_{i=1}^{n} A_i$ is a prime disjunctive form with respect to $S=\cup_{k'=0}^{\infty} S_{k'}$, then there is k' such that $\vee_{i=1}^{n} A_i$ is a prime disjunctive form with respect to $S_{k'}$. So, $\vee_{i=1}^{n} C_i \in S_{k'+1}$ and $B, C_1, ..., C_n \in T_{k'+1}$ since $\{B, C_1, ..., C_n\}\cup A$ is consistent. These imply that $\vee_{i=1}^{n} C_i \in S$ and B, $\vee_{i=1}^{n} C_i \in T$. Otherwise, suppose $\vee_{i=1}^{n} A_i$ is not a prime disjunctive form with respect to S. By lemma 3.4.1, $\vee_{i=1}^{n} A_i$ is derivable from a set of smaller prime disjunctive forms in Ψ_B and in S. Since $\{B, C_1, ..., C_n\}\cup A$ is consistent, $B, C_{k_1}, ..., C_{k_n}\}\cup A$ is consistent, where $C_{k_i} = C_j$, $1 \leq j \leq n$, causing those related $\vee_i C_{k_i}$s to be derivable and in S by S3. In addition, $\vee_{i=1}^{n} C_i$ is derivable from a set of the related $\vee_i C_{k_i}$s.

By monotonicity, $\vee_{i=1}^{n} C_i$ is in S. Since $G_{m+1} = Th(G_m) \cup \{$those $\vee_{i=1}^{n} C_i$s$\}$, consequently, $G_{m+1} \subseteq S$.

Next we prove $H_{m+1} \subseteq T$. If $H_{m+1} = Th(H_m)$, we have proven it. Otherwise, suppose there is $X \in H_{m+1} - Th(H_m)$. By the construction of H_{m+1}, there is a prime disjunctive form $\vee_{i=1}^{n} A_i \in \mathcal{H}_B$ with respect to G_m and $\{B, C_1, ..., C_n\} \cup \mathcal{A}$ is consistent, causing $B, C_1, ..., C_n$ to be in $\cup_{k=0}^{\infty} H_k$. $B, C_1, ..., C_n$ in $\cup_{k=0}^{\infty} H_k = \mathcal{A}$ implies that $\{B, C_1, ..., C_n\} \cup A$ is consistent. By the assumption $G_m \subseteq S$, $\vee_{i=1}^{n} A_i \in G_m$ implies $\vee_{i=1}^{n} A_i \in S$. If $\vee_{i=1}^{n} A_i$ is a prime disjunctive form with respect to $S = \cup_{k'=0}^{\infty} S_{k'}$, then there is k' such that $\vee_{i=1}^{n} A_i$ is a prime disjunctive form with respect to $S_{k'}$. So, $\vee_{i=1}^{n} C_i \in S_{k'+1}$ and $B, C_1, ..., C_n \in T_{k'+1}$ by S3 since $\{B, C_1, ..., C_n\} \cup A$ is consistent. These imply that $B, C_1, ..., C_n \in T$. Otherwise, suppose that $\vee_{i=1}^{n} A_i$ is not a prime disjunctive form with respect to S. By lemma 3.4.1, $\vee_{i=1}^{n} A_i$ is derivable from a set of prime disjunctive forms in $\mathcal{H}_B$ and S. Since $\{B, C_1, ..., C_n\} \cup \mathcal{A}$ is consistent, $\{B, C_{k_1}, ..., C_{k_n}\} \cup \mathcal{A}$ is consistent, where $C_{k_i} = C_j$, $1 \leq j \leq n$, causing those related $B, C_1, ..., C_n$ to be in T by S3. So, X is in T. Since $H_{m+1} = Th(H_m) \cup \{B$, those Xs$\}$, $H_{m+1} \subseteq T$.

∎

Next, we will prove the semi-monotonicity theorem 3.4.2.

Proof: Theorem 3.4.2

Let a-$\mathcal{E}$ be an alternative extension for a default theory $\mathcal{D} = (\mathcal{F}, \Delta)$ based on an assumption set $\mathcal{A}$. Since Δ' is countable, we have a countable, possibly finite, set of justifications of defaults in Δ': $\{B_1, B_2, ...\}$. Let $\bar{\delta}_j$ denote a set of defaults having the same predicate symbol B_k in place of justifications; i.e.,

$$\bar{\delta}_j = \{\delta_{i,j} = \tfrac{A_{i,j} : B_j}{C_{i,j}} \mid \delta_{i,j} \in \Delta\}.$$

Therefore, we have a countable, possibly finite class of such sets, and let $\bar{\delta} = \{\bar{\delta}_1, \bar{\delta}_2, \bar{\delta}_3, ...\}$ be a fixed enumeration of the class. $\mathcal{H}_j$ is a set of disjunctive forms consisting of prerequisites of defaults having the justification B_j. Using construction 3.3, construct a pair (E, A). We know a-$\mathcal{E} \subseteq$ E and $\mathcal{A} \subseteq$ A. Therefore, we only need to prove that (E, A) is a pair of an alternative extension and an assumption set for a default theory $\mathcal{D}'$. We will use characterization theorem 3.3.1 to prove it. Using construction 3.4, construct a pair (S, T) based on the characterization theorem related to the pair (E, A). (Note that in constructions 3.3 and 3.4, there is a j such that $B' = B_j$ and $\mathcal{H}_{B'} = \mathcal{H}_{B_j}$). To prove E is an alternative extension based on A for $\mathcal{D}'$, we show

$$E = S \text{ and } A = T$$

or

$$\cup_{k=0}^{\infty} E_k = \cup_{k'=0}^{\infty} S_{k'} \text{ and } \cup_{k=0}^{\infty} A_k = \cup_{k'=0}^{\infty} T_{k'}.$$

First, we prove S $\subseteq$ E and T $\subseteq$ A by induction on k' that $S_{k'} \subseteq$ E and $T_{k'} \subseteq$ A.

Basis: $S_0 = \mathcal{F} \subseteq E_0 \subseteq E$ and $T_0 = \mathcal{F} \subseteq A_0 \subseteq A$.

Hypothesis: $S_{k'} \subseteq E$ and $T_{k'} \subseteq A$ imply $S_{k'+1} \subseteq E$ and $T_{k'+1} \subseteq A$.

Step: Assume $S_{m'} \subseteq E$ and $T_{m'} \subseteq A$. So, $Th(S_{m'}) \subseteq E$ and $Th(T_{m'}) \subseteq A$ by monotonicity and lemma 3.2.1.

First, we prove that $S_{m'+1} \subseteq E$. If $S_{m'+1} = Th(S_{m'})$, we have proven it. Otherwise, suppose that there is $\vee_{i=1}^{n} C_i' \in S_{m'+1} - Th(S_{m'})$. By construction of $S_{m'+1}$, there is a prime disjunctive form $\vee_{i=1}^{n} A_i'$ in $\mathcal{H}_{B'}$ with respect to $S_{m'}$, and $\{B', C_1', ..., C_n'\} \cup A$ is consistent. By assumption $S_{m'} \subseteq E$, $\vee_{i=1}^{n} A_i' \in S_{m'}$ implies $\vee_{i=1}^{n} A_i'$ is in E. There are two possibilities for $\vee_{i=1}^{n} A_i'$:

- If $\vee_{i=1}^{n} A_i'$ is prime with respect to E, $\vee_{i=1}^{n} A_i' \in E$ implies that there are E_{k-1} and $\mathcal{H}_j$ containing $\overline{A}_h^j$, where $\mathcal{H}_{B'} = \mathcal{H}_j$, $\overline{A}_h^j = \vee_{i=1}^{n} A_i'$, $B_j = B'$, and $\overline{C}_h^j = \{C'_1, ..., C'_n\}$, such that $\overline{A}_h^j$ is prime with respect to E_{k-1}, and $\{B'\} \cup \overline{C}_h^j \cup A_k^{j,h-1}$ is consistent in the construction of $E_k^{j,l}$. So, $\vee_{C \in \overline{C}_h^j} C = \vee_{i=1}^{n} C_i'$ must be in $E_k^{j,h}$, E_k^j, and E_k. So, $\vee_{i=1}^{n} C_i' \in E$.
- Otherwise, suppose $\vee_{i=1}^{n} A_i'$ is not prime with respect to E. By lemma

3.4.1, $\vee_{i=1}^{n} A_i' \in$ E implies that $\vee_{i=1}^{n} A_i'$ is derivable from a set of smaller prime disjunctive forms, $\overline{A}_h^j$s, in $\mathcal{H}_j = \mathcal{H}_{B'}$ and E. Since $\overline{A}_h^j$ is prime with respect to E, there is E_k such that the related $\overline{C}_h^j$ is $\{C'_{m_1}, ..., C'_{m_g}\}$, and $\vee_{C\in\overline{C}_h^j} C$ must be in $E_k^{j,h}$, E_k^j, E_k, and E. In addition, $\vee_{i=1}^{n} C_i'$ is derivable from the set of those $\vee_{C\in\overline{C}_h^j} C$s. By monotonicity, $\vee_{i=1}^{n} C_i'$ is in E since $\vee_{C\in\overline{C}_h^j} C$s are in E.

Next, we prove that $T_{m'+1} \subseteq$ A. If $T_{m'+1} = Th(T_{m'})$, we have proven it. Otherwise, suppose that there is $X \in T_{m'+1} - Th(T_{m'})$. By construction of $T_{m'+1}$, there is a prime disjunctive form $\vee_{i=1}^{n} A_i'$ in $\mathcal{H}_{B'}$ with respect to $S_{m'}$, and $\{B', C'_1, ..., C'_n\} \cup$A is consistent, where $X = B'$ or $X = C'_i$. By assumption $S_{m'} \subseteq$ E, $\vee_{i=1}^{n} A_i' \in S_{m'}$ implies $\vee_{i=1}^{n} A_i'$ is in E. There are two possibilities for $\vee_{i=1}^{n} A_i'$:

- If $\vee_{i=1}^{n} A_i'$ is prime with respect to E, $\vee_{i=1}^{n} A_i' \in$ E implies that there are E_{k-1} and $\mathcal{H}_j$ containing $\overline{A}_h^j$, where $\mathcal{H}_{B'} = \mathcal{H}_j$, $\overline{A}_h^j = \vee_{i=1}^{n} A_i'$, $B_j = B'$, and $\overline{C}_h^j = \{C'_1, ..., C'_n\}$, such that $\overline{A}_h^j$ is prime with respect to E_{k-1}, and $\{B'\} \cup \overline{C}_h^j \cup A_k^{j,h-1}$ is consistent in the construction of E_k. So, B', $C'_1, ..., C'_n$ are in $A_k^{j,h}$, A_k^j, A_k, and A.
- Otherwise, suppose $\vee_{i=1}^{n} A_i'$ is not prime with respect to E. By lemma 3.4.1, $\vee_{i=1}^{n} A_i' \in$ E implies $\vee_{i=1}^{n} A_i'$ is derivable from a set of smaller prime disjunctive forms, $\overline{A}_h^j$s, in $\mathcal{H}_j = \mathcal{H}_{B'}$ and E. Since $\overline{A}_h^j$ is prime with respect to E, there is E_k such that the related $\overline{C}_h^j$ is $\{C'_{m_1}, ..., C'_{m_g}\}$, and B', $C'_{m_1}, ..., C'_{m_g}$ are in $A_k^{j,h}$, A_k^j, A_k, and A. In addition, the union of those $\overline{C}_h^j$s is $\{C_1', ..., C_n'\}$. So, B', $C'_1, ..., C'_n$ are in A.

We have proven that S $\subseteq$ E and T $\subseteq$ A. Now, we will prove that E $\subseteq$ S and A $\subseteq$ T by induction on k that $E_k \subseteq$ S and $A_k \subseteq$ T.

Basis: $E_0 = a\text{-}\mathcal{E} \subseteq \cup_{k'=0}^{\infty} S_{k'}$ and $A_0 = \mathcal{A} \subseteq \cup_{k'=0}^{\infty} T_{k'}$ by lemma 3.4.2.

Hypothesis: $E_{k-1} \subseteq$ S and $A_{k-1} \subseteq$ T imply $E_k \subseteq$ S and $A_k \subseteq$ T.

Step: Assume that $E_{k-1} \subseteq S$ and $A_{k-1} \subseteq T$ causing $Th(E_{k-1}) \subseteq S$ and $Th(A_{k-1}) \subseteq T$ by lemma 3.2.1 on sequences E_k and A_k and by monotonicity.

First, we prove that $E_k \subseteq S$. If $E_k = Th(E_{k-1})$, we have proven it. Otherwise, suppose there is $\vee_{C \in \overline{C}_h^j} C \in E_k - Th(E_{k-1})$ by the construction of E_k. So, there is a prime disjunctive form $\overline{A}_h^j$ in Ψ_j with respect to E_{k-1}, $\{B_j\} \cup \overline{C}_h^j \cup A_k^{j,h-1}$ is consistent, and B_j, $C \in \overline{C}_h^j$ are in $A_k^{j,h}$, A_k^j, A_k, and A. By assumption $E_{k-1} \subseteq S$, $\overline{A}_h^j \in E_{k-1}$ implies $\overline{A}_h^j \in S$, and B_j, $C \in \overline{C}_h^j$ are in A implies $\{B_j\} \cup \overline{C}_h^j \cup A$ is consistent. If $\overline{A}_h^j$ is a prime disjunctive form with respect to S, $\vee_{C \in \overline{C}_h^j} C \in S$. Otherwise, suppose it is not prime. By lemma 3.4.1, $\overline{A}_h^j$ is derivable in S from a set of smaller prime disjunctive forms, noted $\overline{\overline{A}}_h^j$, in Ψ_j and S. $\overline{\overline{A}}_h^j$ in S implies there is an $S_{k'-1}$ such that $\overline{\overline{A}}_h^j \in S_{k'-1}$ and $\overline{\overline{A}}_h^j$ is prime with respect to $S_{k'-1}$. Since $\{B_j\} \cup \overline{C}_h^j \cup A$ is consistent, the related B_j, $(C \in \overline{\overline{C}}_h^j)$ are in $T_{k'}$, and $\vee_{C \in \overline{\overline{C}}_h^j} C$ in $S_{k'}$, where $\overline{\overline{C}}_h^j \subseteq \overline{C}_h^j$. So, $\vee_{C \in \overline{\overline{C}}_h^j} C$ in S. In addition, $\vee_{C \in \overline{C}_h^j} C$ is derivable from {those $C \in \overline{\overline{C}}_h^j$s}. By monotonicity, $\vee_{C \in \overline{C}_h^j} C$ is in S. Since $E_k = Th(E_{k-1}) \cup$ {those $\vee_{C \in \overline{C}_h^j} C$s}, $E_k \subseteq S$.

Next, we prove that $A_{k'} \subseteq T$. If $A_{k'} = Th(A_{k-1})$, we have proven it. Otherwise, suppose there is an $X \in A_{k'} - Th(A_{k-1})$. By the construction of A_k, there is a prime disjunctive form $\overline{A}_h^j$ in Ψ_j with respect to E_{k-1}, $\{B_j\} \cup \overline{C}_h^j \cup A_k^{j,h-1}$ is consistent, and B_j, $C \in \overline{C}_h^j$ are in $A_k^{j,h}$, A_k^j, A_k, and A, where $X = B_j$ or $X \in \overline{C}_h^j$. By the assumption $E_{k-1} \subseteq S$, $\overline{A}_h^j \in E_{k-1}$ implies $\overline{A}_h^j \in S$, and B_j, $(C \in \overline{C}_h^j)$ are in A implying that $\{B_j\} \cup \overline{C}_h^j \cup A$ is consistent. If $\overline{A}_h^j$ is a prime disjunctive form with respect to S, then, B_j, $(C \in \overline{C}_h^j)$ are in T. Otherwise, suppose $\overline{A}_h^j$ is not prime. By lemma 3.4.1, $\overline{A}_h^j$ is derivable in S from a set of smaller prime disjunctive forms, noted $\overline{\overline{A}}_h^j$, in Ψ_j and in S. $\overline{\overline{A}}_h^j$ in S implies there is an $S_{k'-1}$ such that $\overline{\overline{A}}_h^j \in S_{k'-1}$ and $\overline{\overline{A}}_h^j$ is prime with respect to $S_{k'-1}$. $\{B_j\} \cup \overline{C}_h^j \cup A$ is consistent, and so is $\{B_j\} \cup \overline{\overline{C}}_h^j \cup A$, where $\overline{\overline{C}}_h^j \subseteq \overline{C}_h^j$.

Then, the related B_j, $(C \in \overline{\overline{C}}{}_h^j)$ are in $T_{k'}$. In addition, the union of those $\overline{\overline{C}}{}_h^j$s is $\overline{C}_h^j$. So, B_j, $(C \in \overline{C}_h^j)$ are in T. Since $A_k = Th(A_{k-1}) \cup \{\text{those } X\text{s}\}$, $A_k \subseteq S$. We have now proven that $E \subseteq S$ and $A \subseteq T$, and so the theorem.

$\blacksquare$

Using the semi-monotonicity properties above, we can easily prove that every default theory has at least one extension. The idea is to construct an alternative extension from a hard fact set. A set of all facts derivable from a hard fact set in terms of first order theory is an alternative extension for a default theory without defaults. So, by adding defaults to every default theory having no extension, we will always have an alternative extension for the newly added default theory. These properties are summed in the following theorem. The proof is similar to 3.2.6 and will not be given here.

Theorem 3.4.3: Existence of alternative extensions

Every default theory has at least one alternative extension a-$\mathcal{E}$ based on an assumption set $\mathcal{A}$.

$\blacksquare$

Now, we are ready to describe the relation between an extension and an alternative extension for normal default theories. That is, for every extension $\mathcal{E}$ for a normal default theory there is an alternative extension a-$\mathcal{E}$ for the default theory such that $\mathcal{E} \subseteq$a-$\mathcal{E}$

Theorem 3.4.4:

For every extension $\mathcal{E}$ for a normal default theory $\mathcal{D}$, there is an alternative extension a-$\mathcal{E}$ based on an assumption set $\mathcal{A}$ for $\mathcal{D}$ such that $\mathcal{E} \subseteq$ a-$\mathcal{E}$.

$\blacksquare$

This theorem assures that the new approach is a natural extension of the Reiter approach for normal default theories. The idea is that the result of the new approach will be that, all consequences of a normal default theory using the Reiter approach are preserved in the new approach. In other words, all consequences of the Reiter normal default theories are also consequences in the new approach. In addition, there are some additional consequences derived due to the applicability of defaults using disjunctive facts. So, the applicability of defaults is extended using disjunctive facts.

The proof of this theorem is based on the existence of an alternative extension for a default theory.

Proof: Theorem 3.4.4

Let $\mathfrak{E}$ be an extension for a normal default theory $\mathfrak{D} = (\mathfrak{F}, \Delta)$. Consider a default theory $\mathfrak{D}' = (\mathfrak{E}, \Delta)$. According to theorem 3.4.2, there is an alternative extension a-$\mathfrak{E}$ based on assumption set $\mathcal{A}$ for $\mathfrak{D}'$. By theorem 3.3.2, a-$\mathfrak{E} = \mathcal{A}$. Furthermore, $\mathfrak{E} \subseteq$a-$\mathfrak{E}$ by S1. It is easy to see that the pair (a-$\mathfrak{E}$, a-$\mathfrak{E}$) satisfies conditions S1, S2, and S3 with respect to $\mathfrak{D}$. In addition, it is a pair of the smallest sets satisfying Γ_1 and Γ_2 with respect to $\mathfrak{D}'$, Since no other facts are involved in S3, it is also pair of the smallest sets satisfying Γ_1 and Γ_2 with respect to $\mathfrak{D}$. So, we have proven the theorem.

∎

However, theorem 3.4.4 cannot be extended into an arbitrary default theory. In other words, there is a default theory that has an extension but does not have an alternative extension. For example, we have an extension $\mathfrak{E}$ for $\mathfrak{D}$, where $\mathfrak{E}$ = Th($\mathfrak{F} \cup$ {fly(Tweety), fly(Clyde)}), in example 3.2.1, and alternative extensions a-$\mathfrak{E}_1$ and a-$\mathfrak{E}_2$ based on assumption sets $\mathcal{A}_1$ and $\mathcal{A}_2$ for $\mathfrak{D}$, respectively, where a-$\mathfrak{E}_1$ = Th($\mathfrak{F} \cup$ {fly(Tweety)}), a-$\mathfrak{E}_2$ = Th($\mathfrak{F} \cup$ {fly(Clyde)}), $\mathcal{A}_1$ = Th($\mathfrak{F} \cup$ {fly(Tweety), fly(Tweety)$\wedge\neg$broken-wing(Tweety)}), and $\mathcal{A}_2$ = Th($\mathfrak{F} \cup$ {fly(Clyde), fly(Clyde)$\wedge\neg$broken-wing(Clyde)}). However, there is no alternative extensions for $\mathfrak{D}$ such that $\mathfrak{E} \subseteq$ a-$\mathfrak{E}$.

In addition, the new approach may have some more alternative extensions for a default theory that do not have the related extensions. It is easy to find a counter-example that shows that an alternative extension for a default theory may not be always an extension. That is, we may have an alternative extension that is not an extension. One is in example 3.3.8. This is another artificial example.

Example 3.4.1:

Let $\mathcal{D} = (\mathcal{F} = \{P \vee Q\}, \Delta = \{\delta_i, 1 \leq i \leq 5\})$ be a default theory, where $\delta_1 = \frac{P:R}{R}$, $\delta_2 = \frac{Q:R}{R}$, $\delta_3 = \frac{P \vee Q:S}{S}$, $\delta_4 = \frac{R:\neg S}{\neg S}$, and $\delta_5 = \frac{S:\neg R}{\neg R}$. This default theory has exactly one extension $\mathcal{E}$, where $\mathcal{E} = Th(\{P \vee Q, S, \neg R\})$. There are four distinct justifications of defaults in Δ. So, there are four sets of disjunctive forms consisting of prerequisites based on those justifications. They are $\mathcal{H}_R = \{P, Q, P \vee Q\}$, $\mathcal{H}_{\neg R} = \{S\}$, $\mathcal{H}_S = \{P \vee Q\}$ and $\mathcal{H}_{\neg S} = \{R\}$. The default theory $\mathcal{D}$ has exactly three alternative extensions a-$\mathcal{E}_i$ based on a-$\mathcal{E}_i$:

a-$\mathcal{E}_1$ = Th({P∨Q, S, R}),

a-$\mathcal{E}_2$ = Th({P∨Q, S, ¬R}),

and

a-$\mathcal{E}_3$ = Th({P∨Q, R, ¬S}).

We can see that $\mathcal{E}$=a-$\mathcal{E}_2$. However, there is no extension for D such that it supersedes either a-$\mathcal{E}_1$ or a-$\mathcal{E}_3$.

▒

In the next theorem, we will see the maximality of alternative extensions. The maximality of an alternative extension means that if there are two pairs (a-$\mathcal{E}$, $\mathcal{A}$) and (a-$\mathcal{E}'$, $\mathcal{A}'$) of alternative extensions and assumption sets, and a-$\mathcal{E} \subseteq$ a-$\mathcal{E}'$ and $\mathcal{A} \subseteq \mathcal{A}'$, then the pairs are identical.

Theorem 3.4.6: Maximality of alternative extensions

Suppose that a-$\mathcal{E}$ and a-$\mathcal{E}'$ are alternative extensions based on assumption sets $\mathcal{A}$ and $\mathcal{A}'$, respectively, for a default theory $\mathcal{D} = (\mathcal{F}, \Delta)$ such that a-$\mathcal{E} \subseteq$ a-$\mathcal{E}'$

and $\mathcal{A} \subseteq \mathcal{A}'$. Then, a-$\mathcal{E}$=a-$\mathcal{E}'$ and $\mathcal{A}=\mathcal{A}'$.

The theorem says that we cannot have a pair of an alternative extension and an assumption set such that it properly supersedes another pair of an alternative extension and an assumption set.

Proof:

Assume that a-$\mathcal{E}$ and a-$\mathcal{E}'$ are alternative extensions based on assumption sets $\mathcal{A}$ and $\mathcal{A}'$, respectively, for a default theory $\mathcal{D}=(\mathcal{F}, \Delta)$ such that a-$\mathcal{E} \subseteq$ a-$\mathcal{E}'$ and $\mathcal{A} \subseteq \mathcal{A}'$. Let sequences of pairs (E_i, A_i) and (E'_i, A'_i) be related to the pairs (a-$\mathcal{E}$, $\mathcal{A}$) and (a-$\mathcal{E}'$, $\mathcal{A}'$), respectively, as in the characterization theorem. It is enough to prove that $\cup_{j=0}^{\infty} E'_j \subseteq \cup_{j=0}^{\infty} E_j$ and $\cup_{j=0}^{\infty} A'_j \subseteq A = \cup_{j=0}^{\infty} A_j$. We prove them by induction on i that $E'_i \subseteq E_i$ and $A'_i \subseteq A_i$.

Basis: For i=0, $E'_0 \subseteq E_0$ and $A'_0 \subseteq A_0$ since $E'_0 = E_0 = A'_0 = A_0 = \mathcal{F}$.

Hypothesis: $E'_i \subseteq E_i$ and $A'_i \subseteq A_i$ imply $E'_{i+1} \subseteq E_{i+1}$ and $A'_{i+1} \subseteq A_{i+1}$

Step: Assume that $E'_n \subseteq E_n$ and $A'_n \subseteq A_n$. So, $Th(E'_n) \subseteq E_{i+1}$ and $Th(A'_n) \subseteq A_{i+1}$.

First, we prove that $E'_n \subseteq E_n$. If $E'_{n+1} = Th(E'_n)$, then we have proven it. Otherwise, suppose that there is $\vee_{i=1}^{n} C_i$ such that $\vee_{i=1}^{n} C_i \in E'_{n+1}$ - $Th(E'_n)$. By the construction of E'_{n+1}, there is a prime disjunctive form $\vee_{i=1}^{n} A_i$ in $\mathcal{H}_B$ with respect to A'_n, and $\{B, C_1, ..., C_n\} \cup \mathcal{A}'$ is consistent. So, $\vee_{i=1}^{n} A_i$ is in E_n by assumption $E'_n \subseteq E_n$, and $\{B, C_1, ..., C_n\} \cup \mathcal{A}$ is consistent since $\mathcal{A} \subseteq \mathcal{A}'$. Hence, $\vee_{i=1}^{n} C_i$ is in E_{n+1}. Since $E'_{n+1} = Th(E'_n) \cup \{$those $\vee_{i=1}^{n} C_i$ s$\}$, then $E'_{n+1} \subseteq E_{n+1}$.

Next we prove that $A'_{n+1} \subseteq A_{n+1}$. If $E'_{n+1} = Th(E'_n)$, we have proven it. Otherwise, suppose there is an X such that $X \in A'_{n+1}$ - $Th(A'_n)$. By the construction of A'_{n+1}, there is a prime disjunctive form $\vee_{i=1}^{n} A_i$ in $\mathcal{H}_B$ with respect to A'_n, and $\{B, C_1, ..., C_n\} \cup \mathcal{A}'$ is consistent, where $X = B$, or $X = C_i$. So, $\vee_{i=1}^{n} A_i$ is in E_n by assumption $A'_n \subseteq A_n$, and $\{B, C_1, ..., C_n\} \cup \mathcal{A}$ is consistent since $\mathcal{A} \subseteq \mathcal{A}'$. Therefore, B, C_1, ..., C_n are in A_{n+1}, and so is X. Since $A'_{n+1} = Th(A'_n) \cup$ {those Xs}, then $A'_{n+1} \subseteq A_{n+1}$.

∎

Next, we will present the last properties of an alternative extension for a default theory. The last properties are about the orthogonality. The orthogonality properties for alternative extensions mean that the union of two assumption sets is inconsistent. The orthogonality properties of an alternative extension for a default theory are summed up in the following theorem.

Theorem 3.4.7: Orthogonality of alternative extensions

Let a-$\mathcal{E}$ and a-$\mathcal{E}'$ be two distinct alternative extensions for a default theory $D = (\mathcal{F}, \Delta)$ based on assumption sets $\mathcal{A}$ and $\mathcal{A}'$, respectively. Then, $\mathcal{A} \cup \mathcal{A}'$ is inconsistent.

∎

Proof:

By characterization theorem 3.3.1, let (E_i, A_i) and (E'_i, A'_i) be sequences of pairs related to alternative extensions a-$\mathcal{E}$ and a-$\mathcal{E}'$ based on $\mathcal{A}$ and $\mathcal{A}'$ for $\mathcal{D}$, respectively, such that a-$\mathcal{E} = \bigcup_{i=0}^{\infty} E_i$, $\mathcal{A} = \bigcup_{i=0}^{\infty} A_i$, a-$\mathcal{E}' = \bigcup_{i=0}^{\infty} E'_i$ and $\mathcal{A}' = \bigcup_{i=0}^{\infty} A'_i$. Since a-$\mathcal{E}$ and a-$\mathcal{E}'$ are two distinct alternative extensions and $E_0 = E'_0$, there is an n such that $E_n = E'_n$ but $E_{n+1} \neq E'_{n+1}$; i.e., there is $\vee_{i=1}^{n} C_i \in E_{n+1} - E'_{n+1}$. By construction of E_{n+1}, $\vee_{i=1}^{n} C_i$ is a disjunctive form consisting of consequents of defaults, and there is a prime disjunctive form $\vee_{i=1}^{n} A_i$ in $\mathcal{H}_B$ with respect to E_n such that $\{B, C_1, ..., C_n\} \cup \mathcal{A}$ is consistent. So, B, C_1, ..., and C_n are in A_{n+1}, causing both B, C_1, ..., and C_n in $\mathcal{A}$. Since $E_n = E'_n$, $\vee_{i=1}^{n} A_i$ is prime with respect to E_n and to E'_n. Since $\vee_{i=1}^{n} C_i \in E_{n+1} - E'_{n+1}$, then $\vee_{i=1}^{n} C_i \notin E'_{n+1}$. So, $\mathcal{A}' \cup \{B, C_1, ..., C_n\}$ is inconsistent by S3. Furthermore, because B, C_1, ..., and C_n are in $\mathcal{A}$ and $\mathcal{A}' \cup \{B, C_1, ..., C_n\}$ is inconsistent, then $\mathcal{A}$ is inconsistent with $\mathcal{A}'$.

∎

As previously mentioned in section 3.2, the union of two distinct alternative extensions may be consistent. It can be seen in example 3.2.12.

3.5 Survey of Complexity Analysis of the New Approach

This section completes the discussion of the new approach for a default theory dealing with disjunctive fact problems. In this section, we will conduct a survey of a complexity analysis of the new approach. Therefore, we do not expect to have a detailed analysis.

In a default theory, we deal with first order theory. As known, the computability of first order theory is semi-decidable. So, we do not expect the computability of the new approach to be more than that. In most cases, as Kautz and Selman observed, checking the membership of extensions may be intractable [19]. However, constructing an extension for a default theory could be tractable for some areas of interests.

There are two sources of complexity in default theories. First is the inherent complexity of first order theory. Second is the checking of consistency in applying defaults. The inherent complexity of first order theory cannot be avoided. However, by restricting the set to deal with, we may find some decidable problems, for example by using propositional logic. Since the derivability of a propositional logic is decidable, we can expect a better result if we use a default theory consisting of propositional logic.

To simplify the analysis, we will not consider the disjunctive fact problems concerning justifications. As Kautz and Selman noticed, these problems are more complex [19]. They restricted their analysis to disjunctive free default theories. Furthermore, the analysis is based on the Reiter approach.

As Kautz and Selman observed, there are three problems concerning default theories [19]. The first is to find an extension. Finding an extension is "easier" than the other problems. The second is to determine whether a proposition is derivable from an extension. The last is to determine whether a proposition is derivable from all extensions. The last two problems seem to be intractable even for a very simple default theory. Therefore, in the rest of this section, we will conduct analysis for finding an extension.

The basis of our survey is construction 3.1. The idea is that construction 3.1 will result in an alternative extension for a default theory as theorem 3.2.1 concluded. However, as the theorem required, we need to calculate an infinite sequence of pairs of sets to complete the construction.

To conduct a survey, we will make some assumptions to deal with. First is that the number of defaults in Δ of a default theory is finite, say *n*. We enumerate them and suppose that $\{\delta_1, \delta_2, ..., \delta_n\}$ is a fixed enumeration for Δ. Second is that the set of hard facts is consistent. Because if a given inconsistent hard fact set is inconsistency, the result is an inconsistent alternative extension. In addition, during the survey, we will attempt to identify some areas of interest.

The following procedure is a modification of construction 3.1. Since this procedure is based on the construction, we should have a proof that it will converge to an alternative extension, especially dealing with constant propositional well-formed formulae. However, we have not succeeded in proving this procedure. Also, we have not found a counter-example to this procedure. So, we conjecture that this procedure will converge to an alternative extension a-$\mathfrak{E}$ based on an assumption set $\mathcal{A}$ for a default theory.

This procedure will terminate if two consecutive approximations give the same result. The input for this procedure is a default theory $\mathfrak{D} = (\mathfrak{F}, \Delta)$. Set $\mathfrak{F}$ is hard facts consisting of finite sentences known to be believed. Δ is the set of defaults representing potential beliefs. The main step is step 3. For one approximation, it is done in n iterations, each is for a default in Δ. So, if the procedure 1 converges in m approximations, the procedure will terminate in $n \times m$ $\times$ the amount of time to check the applicability of a default δ.

Procedure 1: Finding-alternative-extension($\mathcal{F}$, Δ).

Input : A set of hard facts $\mathcal{F}$ and a set of defaults Δ.

Output : A pair of sets (E, A) such that Th(E) and Th(A) are an alternative extension and an assumption set for a default theory $\mathcal{D} = (\mathcal{F}, \Delta)$.

Method : 1. Assign $E_0 = \mathcal{F}$, $A_0 = \mathcal{F}$, and k = 1.

2. $E_k = E_{k-1}$ and $A_k = A_{k-1}$.

3. For i from 1 to n, do the following.

If applicable(δ_i, E_{k-1}, A_k), then,

$E_k = E_k \cup \{con(\delta_i)\}$ and $A_k = A_k \cup \{jus(\delta_i), con(\delta_i)\}$.

Otherwise,

$E_k = E_k$ and $A_k = A_k$.

4. If $E_k = E_{k-1}$ and $A_k = A_{k-1}$, then stop and return $E = E_k$ and $A = A_k$.

Otherwise, assign k=k+1.

5. Repeat steps 2, 3, and 4.

∎

The following procedure is central to construction 3.1 It appears as a true-false value function in procedure 1. It is related to the applicability of a default with respect to two sets: a set to check the derivability of a prerequisite of a given default and a set to check the consistency of a justification and a consequent of the default.

Procedure 2: applicable(δ, E, A)

Input : A default δ, a set E, and a set A.

Output : Return yes, if the default δ is applicable; otherwise, no.

Method : 1. Check whether the pre(δ) of δ is derivable from E. If not, return no.

2. Check whether the jus(δ) and the con(δ) are consistent with A.
   If not, return no. Otherwise, return yes.

Note: pre(δ) = the prerequisite of default δ;
jus(δ) = the justification of default δ; and
con(δ) = the consequent of default δ.

As previously mentioned, this procedure 2 is central to the construction to check the applicability of a default. It consists of two steps: checking the derivability of the prerequisite and checking the consistency of the justification and the consequent. These steps are the sources of problems in dealing with a default theory in order to find an alternative extension. The first step is referred to as the inherent complexity of first order theory. Generally, this step is semi-decidable. However, by restricting the area of interest, we may find a better result, for example, dealing with constant propositional well-formed formulae. The derivability in propositional logic should be obvious.

Another area of interest is observed by Kautz and Selman [19]. The area uses Horn clauses. As Dowling and Gallier found, the satisfiability of propositional Horn clauses can be decided in linear time [10]. So, in this area, we may find default theories shining.

The next step is to check the consistency. For an arbitrary first order default theory, checking the consistency is undecidable [19]. By restricting the area of the problem, we may find some interesting results. However, we are not going to conduct a further survey.

We close this section by presenting some examples to find an alternative extension and an assumption set for a default theory using procedure 1.

Example 3.5.1: This default theory is taken from Chen [5].

Let $\mathcal{D} = (\mathcal{F}, \Delta = \{\delta'_1, \delta'_2, \delta'_3, \delta'_4\})$ be a default theory, where $\mathcal{F} = \{A \vee G, J,$

$\neg B \vee \neg C \vee \neg P \vee \neg Q\}$, $\delta'_1 = \frac{A \vee P : (\neg C \vee \neg J) \wedge \neg A}{\neg A}$, $\delta'_2 = \frac{\tau : A \wedge B}{B}$ $\delta'_3 = \frac{B \vee S : P \wedge Q}{Q}$, and $\delta'_4 = \frac{J : \neg A \wedge C}{C}$.
Let us calculate an alternative extension and an assumption set for $\mathcal{D}$ using procedure 1 with an enumeration: $\{\delta_1 = \delta'_1, \delta_2 = \delta'_2, \delta_3 = \delta'_3, \delta_4 = \delta'_4\}$.

$E_0 = A_0 = \mathcal{F} = \{ A \vee G, J, \neg B \vee \neg C \vee \neg P \vee \neg Q\}$

$E_1 = \{ A \vee G, J, \neg B \vee \neg C \vee \neg P \vee \neg Q, \neg A\}$

$A_1 = \{ A \vee G, J, \neg B \vee \neg C \vee \neg P \vee \neg Q, \neg A, (\neg C \vee \neg J) \wedge \neg A\}$

$E_2 = \{ A \vee G, J, \neg B \vee \neg C \vee \neg P \vee \neg Q, \neg A\} = E_1$

$A_2 = \{ A \vee G, J, \neg B \vee \neg C \vee \neg P \vee \neg Q, \neg A, (\neg C \vee \neg J) \wedge \neg A\} = A_1$

(E_3, A_3) is a pair of an alternative extension and an assumption set for $\mathcal{D}$. Let us calculate for another enumeration: $\{\delta_1 = \delta'_2, \delta_2 = \delta'_1, \delta_3 = \delta'_3, \delta_4 = \delta'_4\}$.

$E_0 = A_0 = \mathcal{F} = \{ A \vee G, J, \neg B \vee \neg C \vee \neg P \vee \neg Q\}$

$E_1 = \{ A \vee G, J, \neg B \vee \neg C \vee \neg P \vee \neg Q, B\}$

$A_1 = \{ A \vee G, J, \neg B \vee \neg C \vee \neg P \vee \neg Q, B, A \wedge B, Q, P \wedge Q\}$

$E_2 = \{ A \vee G, J, \neg B \vee \neg C \vee \neg P \vee \neg Q, B, Q\}$

$A_2 = \{ A \vee G, J, \neg B \vee \neg C \vee \neg P \vee \neg Q, B, A \wedge B, Q, P \wedge Q\}$

$E_3 = \{ A \vee G, J, \neg B \vee \neg C \vee \neg P \vee \neg Q, B, Q\} = E_2$

$A_3 = \{ A \vee G, J, \neg B \vee \neg C \vee \neg P \vee \neg Q, B, A \wedge B, Q, P \wedge Q\} = A_2$

(E_3, A_3) is another pair of an alternative extension and assumption set for $\mathcal{D}$. ▒

Let us calculate another example taken from Etherington [12].

Example 3.5.2:

Suppose $\mathcal{D} = (\mathcal{F}, \Delta = \{\delta'_1, \delta'_2, \delta'_3, \delta'_4\})$ is a default theory, where $\mathcal{F} = \{P\}$ and $\delta'_1 = \frac{P : Q}{Q}$, $\delta'_2 = \frac{P : R}{R}$, $\delta'_3 = \frac{Q : S}{S}$, and $\delta'_4 = \frac{Q : \neg S \wedge \neg R}{\neg R}$. Let us calculate using procedure 1 with an enumeration: $\{\delta_1 = \delta'_1, \delta_2 = \delta'_2, \delta_3 = \delta'_3, \delta_4 = \delta'_4\}$.

$E_0 = A_0 = \mathcal{F} = \{P\}$

$E_1 = A_1 = \{P, Q, R\}$

$E_2 = A_2 = \{P, Q, R, S\}$

$E_3 = A_3 = \{P, Q, R, S\} = E_2 = A_2$

(E_3, A_3) is a pair of an alternative extension and an assumption set for $\mathcal{D}$. Also, we get the same thing with an enumeration $\{\delta_1 = \delta'_1, \delta_2 = \delta'_2, \delta_3 = \delta'_4, \delta_4 = \delta'_3\}$.

Next, we calculate with an enumeration: $\{\delta_1 = \delta'_1, \delta_2 = \delta'_4, \delta_3 = \delta'_2, \delta_4 = \delta'_3\}$

$E_0 = A_0 = F = \{P\}$

$E_1 = \{P, Q, \neg R\}$

$A_1 = \{P, Q, \neg R, \neg S \wedge \neg R\}$

$E_2 = \{P, Q, \neg R\} = E_1$

$A_2 = \{P, Q, \neg R, \neg S \wedge \neg R\} = A_1$

(E_2, A_2) is also a pair an alternative extension and an assumption set for $\mathcal{D}$. We get the same thing with an enumeration $\{\delta_1 = \delta'_1, \delta_2 = \delta'_4, \delta_3 = \delta'_3, \delta_4 = \delta'_2\}$.

Lastly, we will calculate a default theory similar to example 3.4.13.

Example 3.5.3:

Let $\mathcal{D} = (\mathcal{F}, \Delta=\{\delta_i \mid 1 \leq i \leq 7\})$ be a default theory, where $\mathcal{F} = \{P\}$, $\delta_1 = \frac{P:Q}{Q}$, $\delta_2 = \frac{P:R}{R}$, $\delta_3 = \frac{Q:S}{S}$, $\delta_4 = \frac{R:T}{T}$, $\delta_5 = \frac{R \wedge T:U}{U}$, $\delta_6 = \frac{U:V}{V}$, and $\delta_7 = \frac{S \wedge V:W}{W}$. Let us calculate an alternative extension and an assumption set for $\mathcal{D}$.

$E_0 = A_0 = F = \{P\}$

$E_1 = A_1 = \{P, Q, R\}$

$E_2 = A_2 = \{P, Q, R, S, T\}$

$E_3 = A_3 = \{P, Q, R, S, T, U\}$

$E_4 = A_4 = \{P, Q, R, S, T, U, V\}$

$E_5 = A_5 = \{P, Q, R, S, T, U, V, W\}$

$E_6 = A_6 = \{P, Q, R, S, T, U, V, W\} = E_5 = A_5$

It is easy to see that the pair (E_6, A_6) is a pair of an alternative extension and an assumption set for $\mathcal{D}$.

CHAPTER IV

SUMMARY AND SUGGESTED FUTURE RESEARCH

A default theory is a formal system logic that deals with incomplete knowledge. A system employing a default theory permits us to infer facts using the so-called defaults in the absence of information to the contrary. There is much research to improve the capability of default theories. The Reiter and Lukaszewicz approaches have some problems in dealing with disjunctive fact problems concerning both prerequisites and justifications. In this paper, we propose another approach to dealing with those problems.

Disjunctive fact problems concerning justifications may result in an extension and a modified extension that are counter-intuitive. The Reiter and the Lukaszewicz approaches let the system employing default theories infer some facts that are unacceptable. The system may derive some facts using defaults based on inconsistent justifications. It may result in an extension and a modified extension that are unwell-behaved.

Disjunctive fact problems concerning prerequisites may cause the applicability of some default to be blocked. These defaults have the same justifications. The applicability of these defaults is blocked due to no evidence to believe their justification but their disjunctive form. It may result in an extension and a modified extension that are unacceptable.

To deal with disjunctive fact problems, we propose a new approach. To deal with disjunctive fact problems concerning justifications, we make the new approach enforce consistency not only with an individual justification of a default under consideration but also with a whole set of justifications of already applied defaults. By doing this we will not have those problems concerning justifications. To deal with disjunctive fact problems concerning prerequisites, we develop the new approach to extend the applicability of some defaults using their disjunctive forms. These defaults have the same justification. These disjunctive forms consist of prerequisites of defaults having the same justification. A disjunctive form may be applicable. If so, it is called prime and we may apply it by inferring the related disjunctive form consisting of their consequents. In addition, we also enforce consistency with their justification and consequents as mentioned before. In other words, we apply defaults by inferring the related disjunctive forms consisting of consequents and check their justification and any individual consequents. We do not check the consistency with the related disjunctive form. This is to guarantee that disjunctive fact problems concerning justifications will not arise. The new approach will result in an alternative extension and a consistent assumption set for a default theory.

In doing so, we do not lose the properties of the Reiter and the Lukaszewicz approaches. The new approach will always have an alternative extension and an assumption set for an arbitrary default theory. Also, we still have the characterization theorem that characterizes an alternative extension and an assumption set more intuitively. Furthermore, the new approach also has the semi-monotonicity properties and maximality. In addition, the new approach has the properties of orthogonality with respect to the assumption set. Lastly, we have an important result: for every extension for normal default theory, we can find an alternative extension for the default theory such that it supersedes the extension. It means that the new approach is a natural extension of the Reiter approach.

We also conducted a survey of complexity analysis for the new approach.

However, it is not a complete and detailed analysis.

There are two benefits of the new approach:

1. It can deal with disjunctive fact problems concerning justifications; and
2. It can deal with disjunctive fact problems concerning prerequisites of defaults having the same justification.

Further research to improve the new approach is possible. One improvement would be to find a proof of or a counter-example to the conjecture about the convergence of procedure 1. Second, we need to find another approach to dealing with disjunctive fact problems concerning justifications in which the applicability of defaults that may have the same symbol in the place of justification, but perhaps different terms, is blocked.

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APPENDIX

FIRST ORDER THEORIES

Introduction

First order theory is by far the most important and commonly used logical system. It is known that first order theory is adequately powerful for complete knowledge classical mathematics. It extends propositional calculus in two ways: providing an inner structure for well-formed formulae that are viewed as expressing relations between things and giving the means to express and to reason with. The first concerns the syntactic part, whereas the second concerns with the semantic part.

The concept of a formal theory should be clear. A formal theory consists of a set of countable symbols from which are built expressions called well-formed formulae, a set of axioms, and a set of rules of inference. For example, rules of inferences in first order logics are modus ponens and generalization.

The notion of the syntax of a formal theory is the concept which describes the formal theory as a symbol system. It starts with defining the symbols used and then defines the language consisting of well-formed formulae. The well-formed formulae denote an assertion in which their truth or falsehood depends on the interpretation used.

The semantic part of a formal theory deals with a model of the theory. It provides the meaning of well-formed formulae that emerges from various interpretations that may be given. The interpretation supplies a meaning for each of the symbols in a formal theory such that any well-formed formula can be understood as a statement that is true or false in the interpretation. Such interpretation is a model for a set of well-formed formulae if every well-formed formula in the set is true under the interpretation and we say that the interpretation provides a model for the formal theory.

The study of first order theory will be given in the next several sections. The sequence is first order language, first order interpretation/model, and first order properties. Most materials of first order theories are taken from and proofs of theorems and corollary can be found in, Mendelson [28].

First Order Language

The first requirement for description of a formal theory is to describe the syntax of the language. Therefore, for first order language, we need to describe the symbols of the language and the rules for generating well-formed formulae of the language. The set of well-formed formulae is the first order language.

First, we need to define an alphabet used for the language. It consists of several symbol sets. Then, we define rules of creating well-formed formulae based on those symbols. In order to do that, we need some formal definitions.

An alphabet consists of four symbol sets. The first symbol set is usually the same for a formal theory. It is called a logical symbol set and is usually finite. The second set is a denumerable number of variable symbols. In first order theory, a quantification is allowed only over these variables. The third is a countable set of constant symbols. The next is a countable set of function symbols. The constant and function symbol sets are possibly empty. The last is a countable set of non-empty predicate symbols. An alphabet of a first order theory will be precisely defined in the following definition.

Definition: Alphabet

An alphabet consists of four symbols sets:

- *a logical symbol set:*

  $\{\tau, \perp, \neg, \wedge, \vee, \rightarrow, \leftrightarrow, \forall, \exists\}$;

- *a denumerable individual variable set:*
- $\{x_1, x_2, x_3, \ldots\}$;
- *a countable, possibly empty, function symbol set:*

  $\{f_0^0, f_0^1, f_1^1, f_0^2, f_1^2, \ldots\}$; and

- *a countable, non-empty, predicate symbol set:*

  $\{P_0^0, P_0^1, P_1^1, P_0^2, P_1^2, \ldots\}$.

The symbol τ reads verum, as opposed to symbol $\perp$, called falsum. The logical symbols $\forall$ and $\exists$ are called universal and existential quantifiers, respectively. The subscript on individual variables, function symbols, and predicate symbols is just an indexing number to distinguish different symbols. The superscript on function and predicate symbols indicates the arity of the symbol being superscripted, that is, the number of arguments. As this notation is very cumbersome, the superscript will be dropped if there is no danger of confusion. Function symbols of arity zero are also called individual constants, and they will also be noted as c_i. Predicate symbols of arity zero are called propositional symbols or propositional constant symbols to be precise. Sometimes, the words constant and variable mean an individual constant and an individual variable unless otherwise specified.

To avoid getting symbols cluttered with parentheses, the precedence hierarchies among the logic symbols are taken on. The highest to the lowest precedence hierarchy is listed as the following: τ, $\perp$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $\forall$, and $\exists$. Parentheses are used to change the grouping of logical symbols in terms of the highest precedence. Square brackets are sometimes used in order to improve readability instead of parentheses.

The informal semantics of the connectives and quantifiers is as usual. That is, the connective symbols $\perp$, $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$ mean negation, conjunction (and), disjunction (or), implication, and equivalence, respectively. Also, the quantifier symbols $\forall$ and $\exists$ mean "for all x," and "there exists an x," respectively. Furthermore, it is unnecessary to define the existential symbol $\exists$ as a primitive logical symbol since it can be defined as an abbreviation as follows:

$\exists x A$ stands for $\neg(\forall x \neg A)$.

Next, some definitions are presented in order to define the first order language over a given alphabet. First, a term is defined, followed by an atomic formula, a well-formed formula, and so on.

A term is generated by applying individual variables and individual constants to function symbols. It is defined inductively as the following definition:

Definition: Terms

Terms in first order language $\mathcal{L}$ *are defined as follows*:

- x_i *is a term for all individual variable* x_i; *and*
- $f_i^n(t_1, t_2, ..., t_n)$ is *a term*, if f_i^n *is an n-ary function symbol and* $t_1, t_2, ..., t_n$ *are terms*.

Terms include all individual constant symbols since they are zero-ary forms.

An atomic formula is generated by applying terms to predicate symbols applied. It is defined inductively as follows.

Definition: Atomic formulae

Atomic formulae are defined as follows:

- τ *and* $\perp$ *are atomic formulae*; *and*

- $P_i^n(t_1, t_2, ..., t_n)$ is *an atomic formula if* P_i^n is *an n-ary predicate symbol and* $t_1, t_2,..., t_n$ *are terms.*

Atomic formulae include all propositional constant symbols. Well-formed formulae are of forms using logical symbols as defined by the following.

Definition: Well-formed formulae

Well-formed formulae are defined as follows:

- *Atomic formulae are well-formed formulae;*
- *If A is a well-formed formula, so is* $\neg A$; *that is, the denial of a well-formed formula is also a well-formed formula;*
- *If A and B are well-formed formulae, then so are* $A \wedge B$, $A \vee B$, $A \rightarrow B$, *and* $A \leftrightarrow B$; *and*
- *If A* is *a well-formed formula for each individual variable* x, *then so are* $\forall x A$ and $\exists x A$.

Example:

The following forms are well-formed formulae: $\forall x\ \exists y\ (P(x, y) \rightarrow Q(x))$ and $\neg \exists x P(x, c) \wedge Q(f(x))$.

Definition: First Order Language $\mathfrak{L}$

The first order language $\mathfrak{L}$ *for a given alphabet consists of a set of all well-formed formulae that can be constructed from the symbols in the alphabet.*

The scope of a quantifier is a well-formed formula to which it applies. For example, in $\forall x\ \neg A \vee B$, the scope of the quantifier $\forall\forall x$ is $\neg A \vee B$. Furthermore, consecutive quantifying occurrences of variables of the same nature can be attached to a single quantifier. So, for example, $\forall x_1 ... \forall x_n A$ can be abbreviated by $\forall x_1 ... x_n A$.

Definition: Subformulae

A subformula is defined as follows:

- *A is a subformula of a well-formed formula* A, $\neg A$, $\forall xA$, and $\exists xA$; *and*
- *A and B are subformulae of well-formed formulae* $A \wedge B$, $A \vee B$, $A \rightarrow B$ *and* $A \leftrightarrow B$.

A subformula is also a well-formed formula.

The occurrence of a variable in a well-formed formula can be classified into two: bound and free. The scope of quantifiers determines the classes of variables.

Definition: Bound and free occurrence

An occurrence of a variable x is bound in a well-formed formula A if and only if it is either the variable of a quantifier, $\forall xA$ *or* $\exists xA$*, or it is the same variable that is quantified and is within the scope of the quantifier. Otherwise, the occurrence is said to be free in A.*

Definition: Free variables

A variable x *is said to be free or bound in a well-formed formula A if and only if it has a free or bound occurrence in the well-formed formula, respectively.*

It is possible that a variable may have both free and bound occurrences in a given well-formed formula. Therefore, a variable may be both free and bound in a well-formed formula. For example, in a well-formed formula $A(x, y) \rightarrow \forall xB(x)$, the first occurrence of x is free, and the second and third are bound and the occurrence of y in the well-formed formula is free.

Definition: Terms free for variables

A term t is free for a variable x_i *in a well-formed formula A if and only if no free occurrences of* x_i *in A lie within the scope of quantifier,* $\forall x_j A$ *or* $\exists x_j A$.

The definition of "a term free for a variable" is quite complicated. The idea is that a term t is free for a variable x unless there is a free occurrence of x within the scope of a quantifier binding a variable in t. For example, in a well-formed formula

$$\forall x_3\,(P_1^3(x_1, x_2, x_3) \wedge \forall x_2\, P_1^3(f(x_1), x_2, x_3) \wedge P_1^1(x_3))$$

a term $f(g(c, x_2))$ is free for x_2 and z but not for x_1 since it is within the scope of binding variable x_2 of the term f.

The definition of free and bound variables makes clear that a bound variable of well-formed formula *A* is one that has only bound occurrences in *A* meanwhile a free variable of *A* is one that has free occurrences in *A*. That definition results in a set of free variables of a well-formed formula. Furthermore, well-formed formulae without a free variable have an important role in first order theory. They are called closed well-formed formulae.

Definition: Set of free variables

A set of free variables of well-formed formula A is defined as follows:

- FV(*A*) *is the set of all free variables occurring in A*, *if it is an atomic formula*;
- FV($\neg A$) = FV(*A*);
- FV(*A*) = FV(*B*)$\cup$FV(*C*), *where A is either* $B\wedge C$, $B\vee C$, $B\rightarrow C$, or $B\leftrightarrow C$; *and*
- FV(*A*) = FV(*B*) - {x}, *where A is either* $\forall$x*B* or $\exists$x*B*.

The notion of free variables results in a well-formed formula called a closed well-formed formula.

Definition: Closed well-formed formulae

A closed well-formed formula A is a well-formed formula that has no free variable; i.e., FV(*A*) = $\emptyset$, *the empty set. A closed well-formed formula is also* *called a sentence.*

Example:

The well-formed formula $\forall x \exists y (P(x, y) \wedge Q(x))$ is closed but $\exists y (P(x, y) \wedge Q(x))$ is not since the variable x is not a free occurrence.

A closed well-formed formula can be obtained by adding a universal quantifier for every free variable occurring in a well-formed formula. This closed well-formed formula is called the universal closure of the well-formed formula.

Definition: Universal closure

The universal closure of a well-formed formula A whose $FV(A) = \{ x_1, ..., x_n\}$ *is of the form*: $\forall x_1 ... \forall x_n A$.

Other important well-formed formulae are ground formulae. They have no occurrence of variables and can be obtained by substituting all variables occurring in well-formed formulae.

Definition: Ground formula

A well-formed formula A is ground if and only if it has no occurrence of variables.

All propositional constant symbols are ground formulae.

Example:

$P(f(c_1, c_2)) \rightarrow P(c_1)$ is a ground formula, where c_i is an individual constant symbol, for each *i*.

First Order Interpretation/Model

Well-formed formulae have a meaning when an interpretation is applied to the symbols; that is the semantic part. First order theory is concerned with the formal description of the constructs that provide an interpretation for symbols of the language. Rules are given to check whether a particular construct correctly reflects the meaning of a given well-formed formula of the language.

Definition: First order interpretation

An interpretation of a well-formed formula or a set of well-formed formulae of first order language consists of:

- *A non-empty set* D, *called a domain of the interpretation*;
- *An assignment to each n-ary function symbol* f_i^n *of an n-place operation closed over the domain* D; i.e., $\bar{f}_i^n : D^n \rightarrow D$; *and.*
- *An assignment to each n-ary predicate symbol* P_i^n *of an n-place relation on the domain* D.

Intuitively, the notion of interpretation is as follows. Given an interpretation, variables are thought of as ranging over its domain, and logical and quantifier symbols are given as their usual meaning. Furthermore, an n-ary relation in D can be thought of as a subset of D^n; i.e., the set of n-tuples of elements of D. For example, if domain D is a set of integers, then the relation "less than" can be identified by the set of all ordered pairs (x, y) such that $x \leq y$; i.e.,

$$R = \{(x, y) \mid x \leq y, \text{ where } x, y \in D\}$$

For a given interpretation, a closed well-formed formula stands for a proposition which is true or false whereas a well-formed formula with free variables represents a relation on the domain which may be true for some values in the domain of the free variables and false for the others. For example, let a domain of an interpretation for a well-formed formula be the set of positive integers and $P_i^n(y,z)$ be interpreted as $y \leq z$. Then the well-formed formula $P_i^n(x_1,x_2)$ represents the relation $x_1 \leq x_2$ which is satisfied by all the ordered pairs (c_1,c_2) of positive integers such that $c_1 \leq c_2$, and well-formed formula $\exists x_2 \forall x_1 P_i^n(x_1,x_2)$ is a true sentence; i.e., there is a smallest positive integer. Of course, if the domain is changed to the set of all integers, the well-formed formula will be false.

For a given interpretation, the notions of satisfiability and truth of well-formed formulae are in question. For that reason, the definition of satisfiability and truth will be given precisely. Doing that, we first construct a set Σ of denumerable sequences of elements of the domain D; i.e.,

$$\Sigma = \{\sigma = (d_1, d_2, \ldots) \mid d_i \in D\}$$

and then define the meaning of a sequence σ satisfying a well-formed formula *A* under the given interpretation. To do that, a function σ^* will be defined. It is a function of one argument, with terms as arguments and values in the domain D, associated to σ.

Since a term could be a variable symbol, an individual constant symbol, or a function symbol, σ^* will be defined inductively. The function σ^* is defined as the following.

Definition: Interpretation for terms

Let σ *be a sequence in* Σ *and* σ^* *be an associated function of one argument applied to terms such that*:

- $\sigma^*(t) = b_i$, *if t is a variable* x_i. *So, a sequence* σ *can be considered to be a symbol table, and* σ^* *applied to a variable simply assigns the associated value in the table*;
- $\sigma^*(t) = b$, *if t is an individual constant, where b is a fixed element of* D *assigned by the interpretation*; *and*

- $\sigma^*(f_i^n(t_1,\ldots,t_n)) = f_i^n(\sigma^*(t_1), \ldots, \sigma^*(t_n))$, *if* f_i^n *is an n-ary function symbol, where* $\bar{f}_i^n$ *is the corresponding operation in D assigned by the interpretation and* $t_1, \ldots, t_n$ *are terms.*

The function σ^* is associated to the sequence σ. It maps a syntactic expression onto its intended value. In other words, for a sequence $\sigma = (d_1, d_2, \ldots)$ and a term t, $\sigma^*(t)$ is a value in D by substituting d_i for all occurrences of variable x_i in t for each *i*, and then performing the corresponding operation of the interpretation to the function symbol f. So, if a sequence σ is considered as a symbol table containing values for variables in the language, then σ^* is an interpreter that evaluates a term t in the context of the symbol table σ.

Example:

Let term t be $f_2(x_3, f_1(x_1, c_1))$, the domain D of an interpretation be the set of integers, and c_1 be the constant 10. If f_1 and f_2 are assigned to integer addition and integer multiplication, respectively, then $\sigma^*(t) = d_3\ (d_1 + 10)$ for any sequence of integers $\sigma = (d_1, d_2, \ldots)$.

The previous definition provides a way of interpreting terms for a given interpretation. The next definition will provide a way of interpreting well-formed formulae for a given interpretation.

Definition: Interpretation for well-formed formulae

Let σ *be a sequence of elements of a domain of an interpretation. A well-formed formula A is interpreted as the following:*

- *if A is an atomic formula and* $\overline{A}$ *is the corresponding relation in the interpretation, then* σ *satisfies A if and only if the n-tuple* $(\sigma^*(t_1), \sigma^*(t_2), \ldots, \sigma^*(t_n))$ *is in the relation* $\overline{A}$;
- *if* $A = \neg B$, *where B is a well-formed formula, then* σ *satisfies A if and only if*

σ does not satisfy B;

- *if A = B$\wedge$C, where B and C are well-formed formulae, then σ satisfies* A *if and only if σ satisfies both B and C*;
- *if A = B$\vee$C, where B and C are well-formed formulae, then σ satisfies A if and only if σ satisfies either B or C*;
- *if A = B$\rightarrow$C, where B and C are well-formed formulae, then σ satisfies A if and only if either σ does not satisfy B or σ satisfies C*;
- *if A = B$\leftrightarrow$C, where B and C are well-formed formulae, then σ satisfies A if and only if σ satisfies both B$\rightarrow$C and* C$\rightarrow$*B*;
- *if A is a well-formed formula, then σ satisfies $\forall x_i$ A if and only if every sequence in Σ that differs from σ in at most the i-th component satisfies A*; *and*
- *if A is a well-formed formula, then σ satisfies $\exists x_i$ A if and only if there is a sequence in Σ that differs from σ in at most the i-th component satisfies A*.

Intuitively, a sequence $\sigma = (d_1, d_2, ...)$ satisfies a well-formed formula *A* if and only if, when d_i is substituted for all free occurrences of x_i in *A* for every *i*, the resulting proposition is true under the given interpretation.

Example:

Let *A* be a well-formed formula $P(x_1, x_2) \rightarrow Q(x_1)$, the domain of an interpretation D be the set of integers and P(x, y) and Q(x) be interpreted as "x = y" and "x > 1," respectively. Therefore, the sequence $\sigma = (2,3,...)$ satisfies *A*; i.e., "2 = 3" $\rightarrow$ "2 > 1" is true. Furthermore, for any sequence in which the first element is greater than 1, the sequence will satisfy. However, the sequence σ = (1,1, ...) does not.

The definition above also applies to an individual propositional constant symbol. For example, let P and Q be individual propositional constants interpreted as "Snow is white" and "Sugar is a hydrocarbon", respectively. Then, any sequence of elements of a domain of an interpretation satisfies the well-formed formula $P \wedge Q$.

Now, we define the truthness of a well-formed formula.

Definition:

A well-formed formula A is true for a given interpretation if and only if for every sequence σ in Σ, σ satisfies A. A is false if and only there is no sequence in Σ satisfying A.

For a given interpretation, a well-formed formula is true if and only if it is true for all possible assignments of values for its free variables and it is false if and only if it is false for all possible assignments of values to its free variables. Thus, it is possible for a well-formed formula to be neither true nor false for a given interpretation.

The following properties of well-formed formulae are direct consequences of the previous definition. All these properties can be easily proved.

Corollary:

The followings are some properties of well-formed formulae.

- A is false for a given interpretation if and only if $\neg A$ is true for that interpretation. Furthermore, A is true if only if $\neg A$ is false.
- A well-formed formula A is true for a given interpretation if and only if its closure is true for that interpretation;
- If A is a closed well-formed formula, for a given interpretation either A is true or $\neg A$ is true.
- If A and $A \rightarrow B$ are true for a given interpretation, so is B; $A \rightarrow$ B is false if

and only if *A* is true and *B* is false.

The first property says that for a given interpretation, both a well-formed formula and its denial cannot be true at the same time. The second ensures that an arbitrary well-formed formula is true if and only if $\forall x_i$ *A* is true. Since a well-formed formula may be true for some interpretations and false for others, the next property insists that either a closed well-formed formula *A* is true or its denial is true, that is *A* is false. The last property is one of the most important results in first order theories. It is the modus ponens inference rule.

Now we will present the notion of the logical validity of well-formed formulae and then define the relationship between two well-formed formulae called logical implication.

Definition: Logically valid

A well-formed formula A is said to be logically valid if and only if it is true in every interpretation.

A well-formed formula *A* is logically valid if and only if it is true for every interpretation. Note that this definition requires *A* to be true in all interpretations. Therefore, it is possible that well-formed formulae may be satisfied but not true for any interpretation. These well-formed formulae are said to be satisfiable as defined by the following.

Definition: Satisfiable

A well-formed formula A is said to be satisfiable if and only if there is an interpretation such that A is satisfied by at least one sequence of elements of the domain of that interpretation. Otherwise, it is called unsatisfiable or contradictory.

Combining the last two definitions, we have 1) A is logically valid if and only if $\neg A$ is unsatisfiable; and 2) A is satisfiable if and only if $\neg A$ is not logically valid. Therefore, for a closed well-formed formula A, A is either true or false for any given interpretation. In other words, A is satisfied by all sequences in Σ or by none. Furthermore, A is satisfiable if and only if A is true for some interpretation.

The next definition will give the relationship of two well-formed formulae. This relationship is called to "logically imply".

Definition: Logical implication

Let A and B be well-formed formulae. A is said to logically imply B if and only if any sequence in Σ satisfying A also satisfies B. Furthermore, A and B are logically equivalent if and only if they logically imply each other.

The definition of "logical implication" can be easily extended to a set of well-formed formulae.

Definition:

Let S *be a set of well-formed formulae and A be a well-formed formula.* S *is said to logically imply A if and only if any sequence satisfying any well-formed formula in S also satisfies A for every interpretation.*

The following properties of well-formed formulae are simple consequences of the definitions above.

Corollary:

Let A and B be well-formed formulae.

- A logically implies B if and only if $A \rightarrow$ B is logically valid.
- A and B are logically equivalent if and only if $A \leftrightarrow B$ is logically valid.
- If A is a logical consequence of a set S of well-formed formulae, and any

well-formed formula in S is true in a given interpretation, so is A.

First Order Proof

In the previous section, the symbols and the well-formed formulae of a first order language are presented. Then, to those well-formed formulae, we define an interpretation. To get a complete formal theory, we need to furnish the theory with the axioms and rules of inferences. Since there are numerous theories, in this paper we will follow the Church framework.

In the Church framework, there are five axioms and two rules of inferences. Those axioms and rules are given below.

Definition: First order theory

First order theory T *consists of a set of well-formed formulae of first order language* $\mathfrak{L}$ *over an alphabet, a set of axioms, and a set of rules of the following*:

- *A set of axioms is schemas:* (*A, B, and C are well-formed formulae* of T)

  (A1) $A \rightarrow (B \rightarrow A)$

  (A2) $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B)\ \ (A \rightarrow C))$

  (A3) $(\neg B \rightarrow \neg A) \rightarrow ((\neg \mathrm{B} \rightarrow A) \rightarrow B)$

  (A4) $(\forall x_i\, A(x_i)) \rightarrow A(t)$, if $A(x_i)$ *is a well-formed formula and t is a term free for* x_i *in* $A(x_i)$. *Note that if* t *is* x_i, *we have the axioms* $\forall x_i\, A(x_i) \rightarrow A(x_i)$.

  (A5) $(\forall x_i\, A \rightarrow B) \rightarrow (A \rightarrow \forall x_i B)$, *if A is well-formed formula such that* x_i FV(A).

- *A set of rules of inferences*: (*A and B are well-formed formulae of* T):

  (R1) *Modus ponens*: $A, A \rightarrow B \vdash B$

  That is, B follows from A and $A \rightarrow B$.

  (R2) *Generalization*: $A \vdash \forall x\, A$

That is, $\forall x\, A$ *follows from* A.

The Church axiomatization for first order theory T does not have a set of proper (or non-logical) axioms.

An instance of a schema is a well-formed formula obtained from the schema by substitution. For example, an instance of the axiom schema A1 is

$$(\forall x_1\ P_1^1(x_1)) \rightarrow (\forall x_1\ \exists x_2\ P_1^2(x_1, x_2) \rightarrow \forall x_1\ P_1^1(x_1))$$

with A and B as well-formed formulae $\forall x_1\ P_1^1(x_1)$ and $\forall x_1\ \exists x_2\ P_1^2(x_1, x_2)$, respectively.

A model for first order theory T is an interpretation in which all the axioms of T are true. Therefore, the rules of inferences such as modus ponens and generalization can be applied to well-formed formulae. Such rules are truth preserving. So, if the application of these rules to well-formed formulae in a given interpretation is true, the results are also true. In other words, every theorem of a first order theory T is true in any model of T. This can be guaranteed due to properties of well-formed formulae for a given interpretation.

The purpose of the logical axioms of a first order theory S is to ensure that all logical consequences of T are precisely the theorems of T. Especially, all theorems of a first order theory T are those well-formed formulae of T that are logically valid.

In applying of A4 and A5, we need extra care. That is, if term t is not free for x_i in a well-formed formula A, applying A4 would get an unexpected result. For example, let A be a well-formed formula $\neg\forall x_2\ P_1^2(x_1, x_2)$, and let t be x_2. An instance of A4 with A will result in

$$(\forall x_1\ \neg\forall x_2\ P_1^2(x_1,x_2)) \rightarrow \neg\forall x_2\ P_1^2(x_2,x_2).$$

Consider an interpretation whose domain has at least two distinct elements and the predicate P_1^2 is interpreted as the identity relation. Therefore, $\forall x_1(\neg\forall x_2\ P(x_1,x_2))$ is true but $\neg\forall x_2\ P(x_2,x_2)$ is not. So, the instance does not result in a logically valid well-formed formula because the term t is not free for x_1.

This is the right place to make some important notes. A schema is a statement form. It provides a template to form a well-formed formula while leaving some parts unspecified. The unspecified parts use "meta-variables" which do not belong to first order language. In this paper, the distinctions between meta-variables and variable are kept in the notation. The symbols P with superscript and subscript belong to the first order language. Sometimes, symbols like P, Q, and R are used instead. They are well-formed formulae. On the other hand, symbols like *A*, *B*, and *C* belong to the meta-language of first order logic denoting well-formed formulae. Notice that symbols used for meta-language are italic. Moreover, symbols like x, y, and z are sometimes used for individual variables instead of unwieldy supercripted and subscripted symbols whereas symbols a, b, and c are sometimes used for individual constants for the same reason.

When seeking suitable premises of an inference rule, the axiom schemas can be used since they define some laws. Therefore, an instance of an axiom schema is a consequence of the theory.

The notion of theorems and consequences in first order theory is defined in terms of a proof, that is a finite sequence of well-formed formulae. The last element of the sequence is the consequence. The other elements of the sequence are either a sentence, an instance of an axiom schema, or the results from application of one rule of inference: modus ponens and generalization.

Definition: Proof

A proof in a first order theory T *is a finite sequence* $A_0, A_1, \ldots, A_n$ *of well-formed formulae such that, for each i, either* A_i *is an instance of an axiom schema of* T *or it is a result from an application of either generalization to* A_j *where* $j < i$ *or modus ponens to* A_j *and* A_k *where* $j, k < i$.

Definition: Theorems

A well-formed formula A is a theorem of a first order theory S *if and only if there is a proof such that the last element of the proof is A; that is,* $A = A_n$. *Such a proof is called a proof of A.*

Definition: Consequences

A well-formed formula A is said to be a consequence of a set S *in a first order theory* T *if and only if there is a finite sequence* $A_0, A_1, ..., A_n$ *of well-formed formulae such that* $A = A_n$ *and for each i, either* A_i *in* S, *it is an instance of an axiom schema of* T *or it is a direct consequence by rule of inference of some of the preceding well-formed formulae in the sequence. Such a sequence is called a proof or a deduction of A from* S, *denoted* $A \in \mathrm{Th}_T(S)$, *and* S *is called the hypothesis or premises of the proof.*

$\mathrm{Th}_T(S)$ is the set of well-formed formulae in T that are consequences in T from S. It includes all well-formed formulae in S. If a well-formed formula *A* is in $\mathrm{Th}_T(S)$, it is said to be derived or deduced in T from S, denoted $S \vdash A$ and $\mathrm{Th}_T(S) = \{A \mid S \vdash A\}$. It is also called the deductive closure in T of S.

Of course, a set S can be a first order theory itself. That is, S = T. In this case, the subscript T in the Th can be omitted. Furthermore, if S is a set of closed well-formed formulae, then $\mathrm{Th}_T(S)$ is the set of closed well-formed formulae in T that can be derived from S.

The notion of the deductive closure of a first order theory results in the concept of consistency. It characterizes whether all sentences are in a theory.

Definition: Consistency

A first order theory S *is inconsistent if and only if* Th(S) *contains all well-formed formulae of the language. Otherwise,* S *is consistent. A set* S' *of well-*

formed formulae is consistent with S *if and only if their union is consistent.*

Properties of First Order Theory

In this section, we will present some fundamental properties of first order theory. Proofs of these properties will not be given and can be found in numerous references such as Mendelson, 1964.

The following properties are useful in the study of nonmonotonic theories.

Theorem: Idempotence for first order theory

Let T be a first order theory. Then, $Th_T(Th_T(S)) = Th_T(S)$.

Theorem: Deduction theorem for first order theory

Let T be a first order theory. If A and B *are closed well-formed formulae, then* $S \cup \{A\} \vdash B$ if and only if $S \vdash A \rightarrow B$.

This theorem says that if there is a proof of B from $S \cup \{A\}$ if and only if there is a proof of $A \rightarrow B$ from S. As previously noted, an instance of A4 and A5 for arbitrary well-formed formulae may bring an unpleasant result. This can happen if the term t is not free for the variable being substituted. Therefore, it is important that both A and B be closed well-formed formulae.

Theorem: Soundness of first order theory

If a closed well-formed formula A is a consequence of a first order theory S, then A is true in all models of S.

Theorem: Completeness of first order theory

If a closed well-formed formula A is true in all models of a first order theory S, then A is a consequence of S.

Theorem: Undecidability of first order theory

There is no decision procedure for a first order theory.

Theorem: Semi-decidability of first order theory

There is a proof procedure for a first order theory that provides a proof of a sentence entailed by the theory in a finite number of steps and a finite amount of time.

Theorem: Compactness of first order theory

A first order theory S has a model if and only if every finite subset of S has a model.

Theorem:

A closed well-formed formula A can be deduced from a first order theory S if and only if there is a finite subset T of S such that T $\vdash A$.

The notion of a proof of a well-formed formula from a first order theory S lets the system employing first order theory deduce using axiom schemas and rules of inferences.

To close the discussion of first order theory, an example of a proof will be given.

Example:

Prove that $\{\forall x\, A \rightarrow B\} \vdash \forall x\, A \rightarrow \forall x\, B$, for x is not a free occurrence.

1.	$(\forall x\, A \rightarrow B) \rightarrow (A \rightarrow B)$	A4
2.	$\forall x(A \rightarrow B)$	hypothesis
3.	$A \rightarrow B$	modus ponens 1 and 2
4.	$\forall x A$	hypothesis
5.	$\forall x\, A \rightarrow A$	A4

6.	A	modus ponens 4 and 5
7.	B	modus ponens 3 and 6
8.	$\forall x B$	generalization 7
9.	$\forall x\, A \rightarrow \forall x B$	deduction theorem 4 and 8

▒

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