NUMERICAL CALCULATION OF RADIATIVE TRANSFER IN ONE-DIMENSIONAL MEDIA WITH A REFLECTIVE TOP BOUNDARY AND ANISOTROPIC SCATTERING

By

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PREFACE

The purpose of this work was to obtain the exact solutions for source function, reflective intensity, and flux at the reflective boundary of one-dimensional semi-infinite and finite media with anisotropic scattering.

I would like to express my sincere appreciation and gratitude to my major adviser, Dr. Ronald L. Dougherty, for his continuous encouragement and invaluable support throughout my graduate work, and intelligent guidance and unlimited time spent with me during the work of this research. I also wish to thank the other committee members, Dr. A. J. Ghajar and Dr. F. W. Chambers, for their assistance and help during my course and paper work.

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I would like to express my sincere thanks to my great father, Ching-Yung Liu, and my dear mother, Chun-Ying Chu, who should get all of the achievements that I have earned. I also would like to thank my grand uncle Mr. Yu and his family, to whom I am very appreciative.

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NOMENCLATURE

A	function defined in Eq. (22a)
B_k^m	function defined between Eq. (4) and Eq. (5)
C_m	function defined in Eq. (4)
f_{km}	function defined in Eq. (35c)
I	intensity
I^+	intensity in the positive τ direction
I^-	intensity in the negative τ direction
I_{Ae}^+	transmitted intensity just outside the bottom boundary of the medium in the positive τ direction with the collimated boundary condition
I_{Ae}^-	reflected intensity just outside the upper boundary of the medium in the negative τ direction with the collimated boundary condition
I_{Ain}^-	reflected intensity just inside the upper boundary of the medium in the negative τ direction with the collimated boundary condition
I_e^+	transmitted intensity just outside the bottom boundary of the medium in the positive τ direction
I_e^-	reflected intensity just outside the upper boundary of the medium in the negative τ direction
I_i	magnitude of the incident intensity inside the medium
I_i^+	incident intensity inside the medium in the positive τ direction
I_o	magnitude of the incident intensity outside the medium

I_o^+ — incident intensity outside the medium in the positive τ direction

I_ν — intensity of radiation which depends on frequency

K_{1jmk} — function defined in Eq. (26b)

K_{11jmk} — function defined in Eq. (30b)

K_{2jmk} — function defined in Eq. (26c)

K_{22jmk} — function defined in Eq. (30c)

L — number of Legendre polynomials

$L(\mu,n)$ — function defined in Eq. (26d)

n — ratio of the index of refraction of the medium to that of the material bounding on the top boundary of the medium

n' — ratio of the index of refraction of the medium to that of the material bounding on the bottom boundary of the medium

n_1 — the index of refraction of the medium

n_2 — the index of refraction of the material bounding the bottom boundary of the medium

n_o — the index of refraction of the material bounding the top boundary of the medium

P — phase function defined in Eq. (3)

P_k — Legendre polynomials

P_k^m — Associated Legendre functions

P_{km} — source function defined in Eq. (19a)

PP_{km} — source function defined in Eq. (34a)

PP_{km1} — source function defined in Eq. (35a)

PP_{km2} — source function defined in Eq. (35b)

$\overline{PP_{jm}}$ — reflection function defined in Eq. (73)

$\overline{PP_{im1}}$ — reflection function defined in Eq. (51)

$\overline{PPI_{im1}}$ — transmission function defined in Eq. (115)

q	flux
q_{km}	source function defined in Eq. (19b)
q_p	fundamental flux defined in Eq. (31a)
R_{PP_m}	reflection function defined in Eq. (79)
S	source function
S_ν	source function which depends on frequency
T_{PPI_m}	transmission function defined in Eq. (166)
x_k	expansion coefficients

Greek

δ	Dirac delta function
δ_{0m}	Kronecker delta function
θ	polar angle of incident intensity just inside the top boundary of the medium in the positive τ direction
θ_e	polar angle of reflected intensity just outside the top boundary of the medium in the negative τ direction
θ_{e2}	polar angle of transmitted intensity just outside the bottom boundary of the medium in the positive τ direction
θ_{in}	polar angle of reflected intensity just inside the top boundary of the medium in the negative τ direction
θ_o	incident [polar] angle
μ	cosine of the polar angle right inside the top boundary, θ
μ_e	cosine of the exit polar angle at top boundary, θ_e
μ_{e2}	cosine of the exit polar angle at lower boundary, θ_{e2}
μ_{in}	cosine of the polar angle in the medium, θ_{in}

μ_o	cosine of the incident [polar] angle of intensity, θ_o
ρ	interface reflection coefficient
τ	optical location
τ_0	finite optical thickness
ϕ	azimuthal angle
ϕ_{in}	azimuthal angle inside the medium
ϕ_o	azimuthal angle outside the medium
ω	single scattering albedo
ω_ν	single scattering albedo which depends on frequency

Superscripts

+	in the positive τ direction
-	in the negative τ direction

Subscripts

A	collimated boundary condition
e	emission
i	incident
in	inside the medium
o	outside the medium
ν	frequency dependent

CHAPTER I

INTRODUCTION

In the early century, many significant restrictions and assumptions had to be made in order to solve radiation problems due to the mathematical complexities involved in this field. The improvement of digital computers in the last twenty years had allowed us to compute more complicated problems by numerical simulation.

Most radiative transfer studies which have been conducted are concerned with one-dimensional semi-infinite and finite media. Less attention has been paid to two-dimensional problems. However, the exact solutions for one-dimensional either semi-infinite or finite media with one reflective boundary and anisotropic scattering are non existent.

The objective of present work is to get the exact solutions for radiative transfer properties in one-dimensional semi-infinite and finite media with one reflective boundary and anisotropic scattering. The exact expressions for the source function, reflection and transmission functions, reflected and transmitted intensities, and reflected and transmitted fluxes will be obtained in later chapters.

In this paper,the problems are simplified to solve only at the top boundary for semi-infinite media and at both boundaries for finite media. The principle of superposition as well as Ambarzumian's method are used in the solution process. Note that, for these solution methods, no information inside the media will be obtained for both finite and semi-infinite media.

For the semi-infinite case, fundamental source functions will be expressed in terms of a set of unknown functions, which are similar to Chandrasekhar's H function [1], and can be solved by the successive approximation method. On the other hand, fundamental source functions of finite case will be expressed in terms of a set of dependent integro-differential functions, which are similar to Chandrasekhar's X and Y functions [2]. These equations include first order derivatives, and can be solved by a combination of the Runge-Kutta numerical calculation method and the successive approximation method.

The solutions in this research are shown to be pretty close to those of previous work. However, it is only the fundamental step for this one. Sooner or later, we should get the results which can be applied in a two-dimensional media.

Literature Review

Extensive studies for radiative transfer exist in the public literature. The exact solutions, however, do not exist for the problem presented in this research, although some papers with certain similarities had been found during the research. Some investigations considered the influence of index of refraction alone when others studied the effect of anisotropic scattering alone. Nevertheless, the combination of these two factors has not been studied, whether the condition is one-dimensional or multi-dimensional. Many researchers have either ignored the effect of refractive index or considered average directional reflectivities for multi-dimensional problem due to the complexity. Some interesting or similar studies will be mentioned as follows.

Some typical studies of radiative transfer in one-dimensional semi-infinite media were conducted by Armaly, Lam, and Crosbie [3], Dougherty [4], and Dougherty [5]. An approximate solution was presented by Armaly, Lam, and Crosbie [3] while the exact solutions were presented by Dougherty [4] and Dougherty [5].

The exponential kernal substitution, which had been used to obtained an approximate solution, was used to predict the results for a absorbing scattering medium with index of refraction by Armaly, Lam, and Crosbie [3]. They studied the effect of index of refraction and found that an increase in the refractive index would either increase or decrease the directional emittance depending the magnitude of the scattering albedo.

A variation of Ambarzumian's method was used by

Dougherty [4] to formulate the equations for the source function, flux, and intensity at the reflective boundary. Numerical results were presented for albedo less than 0.7 and refractive indices of 1.1, 1.33, 1.5, and 2.0.

Dougherty [5] has also presented the exact solutions of source function, reflection function, intensity, and flux for an isotropically scattering medium having collimated radiation incident upon its reflective boundary. His results were precise for any refractive index between one and two, along with the albedo less than 0.7. Dougherty has given the basic results for a collimated boundary condition which allows the use of these to superpose for other results with different boundary conditions.

Approximate numerical results for one-dimensional geometry have been presented by Buckius and Tseng [6] for finite media alone, and by Armaly and El-Baz [7] for both semi-infinite and finite media. In contrast, Jiang [8] has given the exact solution for one-dimensional both semi-infinite and finite media.

Buckius and Tseng [6] have offered an approximate solution for a planar medium with emission, anisotropic scattering, and directional boundaries. The numerical results showed that the anisotropic scattering effects were more significant for the diffuse incident problem than for isothermal emission because of the directional nature of the incident solution.

Armaly and El-Baz [7] have obtained the approximate solutions for nonconservative cases at one reflective boundary with emission from an isothermal, isotropically scattering medium. They have shown that an increase in refractive index caused the source function to increase and the effect was more pronounced at higher albedo and smaller optical depths.

The exact solutions for the source function, reflection and transmission functions, and reflected and transmitted intensities and fluxes have been obtained by Jiang [8] in a medium with isotropic scattering and refractive index. The H function was used to solve the semi-infinite problem while either the X and Y functions or Ambarzumian's method was utilized to solve the finite problem. Various angles were considered for the collimated incident intensity. The results for optical thickness from 0.1 to 2.0 for different step sizes with refractive index of 1.33 and 1.5 were presented, along with albedoes of 0.5, 0.9, and 1.0.

For two-dimensional semi-infinite cylindrical media, Crosbie and Dougherty [9], Crosbie and Dougherty [10], and Crosbie and Lee [11] have presented exact solutions for isotropic, Rayleigh, and an nth order Legendre representation of anisotropic scattering, respectively. Exact expressions of equations are also derived but without solving numerically by Crosbie and Dougherty [12] for anisotropic scattering.

The solutions for the source function, flux, and intensity at the boundary of an isotropically scattering medium have been found by Crosbie and Dougherty [9]. Crosbie and Dougherty demonstrated how to use separation of variables to reduce the source function integral equation to a one-dimensional form for a Bessel function boundary condition. Solutions for other boundary conditions were then shown to be superpositions of the Bessel function solution. Crosbie and Dougherty also illustrated that the source function and flux of both the cylindrical and planar medium could be expressed in terms of the same basic function.

A modified Ambarzumian's method is used by Crosbie and Dougherty [10] to develop the integro-differential equations for the source function, flux, and intensity at the boundary of a cylindrical medium scattering with second order Legendre phase function which is represented by a series of Legendre polynomials (two terms, second order). The incident radiation was collimated, normal to the top surface, and was dependent only on the radial coordinate. Crosbie and Dougherty have investigated both boundary conditions which varied as a Bessel function and as a Gaussian distribution. The numerical results for albedoes of 0.1, 0.5, 0.9, 0.99, and 1.0 were presented in graphical and tabular forms for a Rayleigh scattering medium on both the Bessel-varying and Gaussian beam problems.

Crosbie and Lee [11] have developed a system of integral equations for the source function at the boundary of a cylindrical medium with anisotropic scattering by a modified Ambarzumian's method. The scattering phase function was represented by a series of Legendre polynomials and the incident radiation, which was Bessel-varying in the radial coordinate, was collimated and normal to the surface of the medium. The scattering medium was infinite in both the r- and z-directions. Exact solutions were presented for back-scattered intensity and flux of three and five term phase functions. Crosbie and Lee have discovered that the results for anisotropic and isotropic scattering had similar trends, but the actual values of the source function, flux, and intensity would differ substantially. They also found that at large distances from the laser beam, the results for different phase functions collapsed to the isotropic scattering results.

Crosbie and Dougherty [12] have derived the exact integral equations for source function and flux in a radially infinite cylindrical medium which scatters anisotropically. Both collimated and diffused incident radiation were considered for boundary conditions while the scattering phase function was represented by a spike in the forward direction plus a series of Legendre polynomials and without limitation on the number of Legendre polynomials. Radially varying collimated radiation was incident normal to the upper surface while the bottom boundary had no radiation incident on it. Crosbie and Dougherty also showed the two-dimensional integral equations being reduced to a one-dimensional form by separating variables when the radial variation of the incident radiation was in the form of a Bessel function.

By using the S-N discrete ordinates method, Kim and Lee [13] have performed an accurate approximation for radiative transfer in two-dimensional rectangular enclosures exposed to an arbitrarily inclined, collimated incident beam with top reflective boundary. Kim and Lee have found that anisotropy of the phase functions had a strong effect on the radiative transfer, and had more significant influence for collimated incidence than for diffuse incidence. The phase function that was applied in this case had up to 13 terms in the Legendre polynomials series expansion.

Crosbie and Dougherty [14] and Crosbie and Dougherty [15] have presented exact solutions for isotropic and linearly anisotropic scattering in a two-dimensional finite thick cylindrical medium, respectively.

Graphical and tabular results were presented by Crosbie and Dougherty [14] for the back-scattered intensity from a cylindrical medium which scattered isotropically and were exposed to a Gaussian beam of radiation with a refractive index of unity. Also, results for the source function and flux at the boundaries were presented by them. Moreover, they have found the influence of optical thickness and albedo being most pronounced at large optical radii.

The exact solutions of radiative properties were obtained by Crosbie and Dougherty [15] for a cylindrical medium which scattered in a linear anisotropic fashion.

They investigated two radial distributions which were in forms of a Bessel function and of a Gaussian laser beam. The Bessel solution was used to construct the solution for the Gaussian beam.

A multi-layer study was done by Reguigui and Dougherty [16]. They have presented a system of exact linear integral equations for the source function, intensity, and flux for a two-dimensional cylindrical medium consisting of up to four layers with reflecting boundaries between the layers. The incident radiation was collimated and had a Bessel function distribution. Their computerized model had the capability of varying the refractive index, along with handling internal reflection at the boundaries and allowing the albedo to vary through layers of variable optical thickness making up the medium.

CHAPTER II

DEVELOPMENT OF GENERAL EQUATIONS

Describe Problem

Before developing the equations, it is necessary to have a simple description of the problem which we need to solve and also mention some basic definitions which need to be used in this problem.

As mentioned in Chapter I, the problem which we are interested in is one-dimensional, with a reflective top boundary and anisotropic scattering. This is an ideal case due to the reason that the medium is supposed to have both reflective top and bottom boundaries. We will show a little bit of that derivation in this chapter to demonstrate how complicated it is for a medium which has both reflective boundaries. However, we will concentrate on the case which has only a reflective top boundary as a first step in this complex radiation study.

In this problem, we assumed that collimated incident radiation only exists at the top boundary and is a sheet of laser-like beams which make our problem become one-dimensional. Also, no boundary condition at bottom is assumed for this case. In fact, there is not a real interface between two different media at the bottom according to our assumption.

Only absorption, transmission, and reflection may occur at the interface for the incident radiation coming from outside the medium. In our case, the interface is very thin, so we may assume no absorption at the interface. Therefore, the following equation can describe what happens at the interface:

$$\text{Transmission} = 1 - \text{reflection}.$$

However, the above equation only works at top boundary. At the bottom boundary, we have only transmission due to the "no reflective bottom boundary" assumption.

A schematic of the process inside the medium is shown in Fig. (1) as follows:

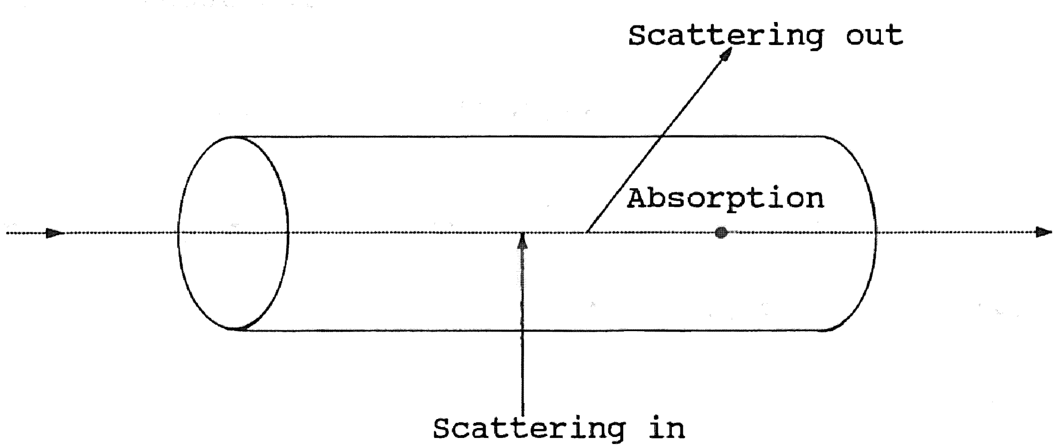

Figure 1. Geometry of Scattering and Absorption

In our case, we assumed no emission which is a good assumption only for a lower temperature medium. This assumption will fail for high temperature, for example, a combustion process. Note that anisotropic scattering is also assumed inside the medium. The definition of anisotropic scattering is that there is an unequal probability of scattering in the various directions, which is described by the phase function.

The problem which is solved in this thesis can be applied diagnostically. For example, one may determine the optical thickness for a medium of interest by measuring the intensity backscattered or transmitted by a medium and comparing the data to numerical predictions.

Fundamental Equations

The transport equation for a one-dimensional medium, see Fig. (2) on the following page, which scatters and absorbs without emitting is [4],

$$\mu_{in}\ [dI_\nu(\tau,\mu_{in},\phi_{in})/d\tau] + I_\nu(\tau,\mu_{in},\phi_{in}) = S_\nu(\tau,\mu_{in},\phi_{in}), \quad (1)$$

where
$$S_\nu(\tau,\mu_{in},\phi_{in}) = (\omega_\nu/4\pi) \int_0^{2\pi} \int_{-1}^{1} I_\nu(\tau,\mu_{in}',\phi_{in}') \times P(\mu_{in}',\phi_{in}',\mu_{in},\phi_{in})\ d\mu_{in}'\ d\phi_{in}' \quad (2)$$

is the source function, with $P(\mu_{in}',\phi_{in}',\mu_{in},\phi_{in})$ being the scattering phase function.

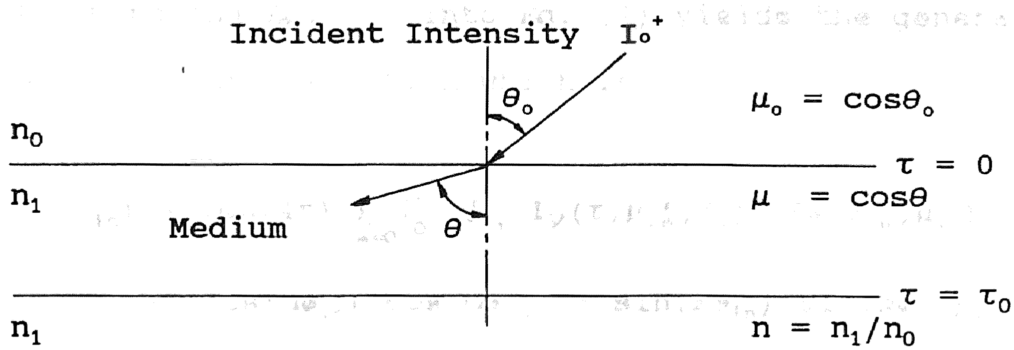

Figure 2. Geometry of a One-Dimensional Medium

Before proceeding further with the development of the transport equation, it will be useful to express the phase function in a general way.

The phase function can be written in general as a finite sum of Legendre polynomials

$$P(\mu_{1n}', \phi_{1n}', \mu_{1n}, \phi_{1n}) = \sum_{m=0}^{L} C_m(\mu_{1n}', \mu_{1n}) \cos[m(\phi_{1n} - \phi_{1n}')], \qquad (3)$$

where $$C_m(\mu_{1n}', \mu_{1n}) = (2-\delta_{0m}) \sum_{k=m}^{L} B_k^m \, P_k^m(\mu_{1n}) \, P_k^m(\mu_{1n}'), \qquad (4)$$

and $B_k^m = x_k\ (k-m)!/(k+m)!$. The x_k are the expansion coefficients. δ_{0m} is the Kronecker delta function. The P_k are Legendre polynomials [12]

$$P_k(y) = [d^k(y^2-1)^k/dy^k]/(2^k\ k!), \qquad (5)$$

and the P_k^m are Associated Legendre functions [12]

$$P_k^m(y) = (1-y^2)^{m/2}\ d^m[P_k(y)]/dy^m. \qquad (6)$$

Substituting Eq. (3) into Eq. (2) yields the general equation for source function which is

$$S_\nu(\tau,\mu_{in},\phi_{in}) = (\omega_\nu/4\pi) \sum_{m=0}^{L} \int_0^{2\pi} \int_{-1}^{1} I_\nu(\tau,\mu_{in}',\phi_{in}') \, C_m(\mu_{in}',\mu_{in}) \times [\cos(m\phi_{in}) \cos(m\phi_{in}') + \sin(m\phi_{in}) \sin(m\phi_{in}')] \times d\mu_{in}' \, d\phi_{in}'. \tag{7}$$

By using Eq. (7), Eq. (1) becomes

$$\mu_{in} \, [dI(\tau,\mu_{in},\phi_{in})/d\tau] + I(\tau,\mu_{in},\phi_{in}) = S(\tau,\mu_{in},\phi_{in})$$

$$= (\omega/4\pi) \sum_{m=0}^{L} \int_0^{2\pi} \int_{-1}^{1} I(\tau,\mu_{in}',\phi_{in}') \, C_m(\mu_{in}',\mu_{in}) \, [\cos(m\phi_{in}) \times \cos(m\phi_{in}') + \sin(m\phi_{in}) \sin(m\phi_{in}')] \, d\mu_{in}' \, d\phi_{in}'. \tag{8}$$

Notice that the frequency dependent notation has been dropped in Eq. (8) for convenience, but the frequency dependence will continue to be assumed throughout the following development.

Solving Eq. (8) by using an integrating factor yields [8]

$$I^+(\tau,\mu_{in},\phi_{in}) = I^+(0,\mu_{in},\phi_{in}) \exp(-\tau/\mu_{in}) + \int_0^{\tau} S(t,\mu_{in},\phi_{in}) \times \exp[-(\tau-t)/\mu_{in}]/\mu_{in} \, dt, \tag{9a}$$

and

$$I^-(\tau,\mu_{in},\phi_{in}) = I^-(\tau_0,\mu_{in},\phi_{in}) \exp[-(\tau_0-\tau)/\mu_{in}] + \int_{\tau}^{\tau_0} S(t,-\mu_{in},\phi_{in}) \exp[-(t-\tau)/\mu_{in}]/\mu_{in} \, dt, \tag{9b}$$

the positive τ direction, $I^-(\tau,\mu_{1n},\phi_{1n})$ is the intensity having a component in the negative τ direction, $I^+(0,\mu_{1n},\phi_{1n})$ is the incident intensity at the upper surface of the medium, $I^-(\tau_0,\mu_{1n},\phi_{1n})$ is the incident intensity at the bottom surface of the medium, and τ_0 is the finite optical thickness.

The boundary condition at the top boundary is [8]

$$I^+(0,\mu_{1n},\phi_{1n}) = I_i^+(\mu_{1n},\phi_{1n}) + I^-(0,\mu_{1n},\phi_{1n}+180°) \times \rho(\mu_{1n},n), \quad 0 \le \mu_{1n} \le 1 \tag{10a}$$

where $I^-(0,\mu_{1n},\phi_{1n}+180°)$ is the intensity component just inside the upper boundary of the medium in the negative τ direction, $I_i^+(\mu_{1n},\phi_{1n})$ is the incident intensity coming from outside of the medium, and $\rho(\mu_{1n},n)$ is the fraction of radiation incident upon the interface which is reflected back inside the medium.

Note that $\rho(\mu_{1n},n)$ is a generic reflection coefficient. The major restrictions on ρ are that it has to be a real-valued function, have a range of values between zero and unity, and is zero for unit refractive index. More information on ρ will be covered in later chapters. Also note that the refractive index n is, for example, 1.5 for glass and 1.33 for water which both compared with vacuum.

By assuming no collimated incident radiation entering from the bottom, as shown in Fig. (2), the bottom boundary condition is [8]

$$I^-(\tau_0,\mu_{1n},\phi_{1n}) = I^+(\tau_0,\mu_{1n},\phi_{1n}+180°) \times \rho(\mu_{1n},n'), \quad 0 \le \mu_{1n} \le 1 \qquad (10b)$$

where $I^+(\tau_0,\mu_{1n},\phi_{1n}+180°)$ is the intensity component just inside the bottom boundary of the medium in the positive τ direction, and $\rho(\mu_{1n},n')$ is the fraction of radiation incident upon the interface which is reflected back inside the medium.

For collimated incident intensity which is on the top boundary, the intensity just outside the boundary designated as I_o^+, and that just inside the boundary designated as I_i^+ can be represented by using the Dirac delta function, that is, [8]

$$I_o^+(\mu'',\phi) = I_o\ \delta(\mu''-\mu_o)\ \delta(\phi - \phi_o), \qquad (11)$$

and

$$I_i^+(\mu',\phi) = I_i\ \delta(\mu'-\mu)\ \delta(\phi - \phi_o), \qquad (12)$$

where δ is the Dirac delta function, I_o is the magnitude of the intensity just outside the medium, I_i is the magnitude of the intensity just inside the medium, μ_o and ϕ_o determine the direction of the collimated radiation outside the medium, μ'' refers to a given polar angle external to the medium, and with μ' and μ referring to polar angles inside the medium.

The relationship between μ_o and μ may be determined by applying Snell's Law at the interface, see Fig. (2), to yield [8]

$$\mu = [1 - (1-\mu_o^2)/n^2]^{1/2}. \qquad (13)$$

In addition, the relationship between I_o and I_i can be found by applying the conservation of energy or flux across the interface. The result is [8]

$$I_i = (\mu_o/\mu)\ [1 - \rho(\mu_o,1/n)]\ I_o. \qquad (14)$$

Finally, with the help of Eqs. (13) and (14), Eq. (12) becomes

$$I_i^+(\mu',\phi) = \mu_o I_o\ [1 - \rho(\mu_o,1/n)]\ \delta\{\mu' - [1-(1-\mu_o^2)/n^2]^{1/2}\}$$
$$\times\ \delta(\phi - \phi_o)/[1-(1-\mu_o^2)/n^2]^{1/2}. \qquad (15)$$

Source Function

Substituting Eqs. (9a), (9b), and (10a) into Eq. (2) yields the following general equation for the source function which is

$$S(\tau,\mu_{in},\phi_{in}) = (\omega/4\pi) \int_0^{2\pi} \int_0^1 \Big\{ I_i^+(\mu_{in}',\phi_{in}') \exp(-\tau/\mu_{in}')$$
$$+ I^-(\tau_0,\mu_{in}',\phi_{in}'+180°) \exp[-(\tau_0+\tau)/\mu_{in}']$$
$$\times\ \rho(\mu_{in}',n) \Big\} P(\mu_{in}',\phi_{in}',\mu_{in},\phi_{in})\ d\mu_{in}'\ d\phi_{in}'$$
$$+ (\omega/4\pi) \int_0^{2\pi} \int_0^1 \int_0^{\tau_0} \Big\{ S(t,-\mu_{in}',\phi_{in}'+180°)$$
$$\times\ \exp[-(\tau+t)/\mu_{in}']/\mu_{in}'\ \rho(\mu_{in}',n)\ P(\mu_{in}',\phi_{in}',\mu_{in},\phi_{in})$$
$$+ S[t,\mathrm{sign}(\tau-t)\mu_{in}',\phi_{in}'] \exp[-|\tau-t|/\mu_{in}']/\mu_{in}'$$

$$\times \, P[\text{sign}(\tau-t)\mu_{1n}', \phi_{1n}', \mu_{1n}, \phi_{1n}] \Big\} \, dt \, d\mu_{1n}' \, d\phi_{1n}'$$

$$+ \, (\omega/4\pi) \int_0^{2\pi} \int_0^1 I^-(\tau_0, \mu_{1n}', \phi_{1n}') \, \exp[-(\tau_0-\tau)/\mu_{1n}']$$

$$\times \, P(-\mu_{1n}', \phi_{1n}', \mu_{1n}, \phi_{1n}) \, d\mu_{1n}' \, d\phi_{1n}', \qquad (16)$$

where sign$(\tau-t)$ is 1 if $\tau \geq t$, and is -1 if $\tau < t$.

Before developing the source function further, it is very important to specify the two different situations which need to be considered. One is the medium which assumes no index of refraction change across the bottom interface, as shown in Fig. (3). The other is the medium which does have an index of refraction change across the bottom interface, as shown in Fig. (4) on the following page. Be aware that it has been assumed that no collimated incident radiation enters at the bottom interface for both of these cases.

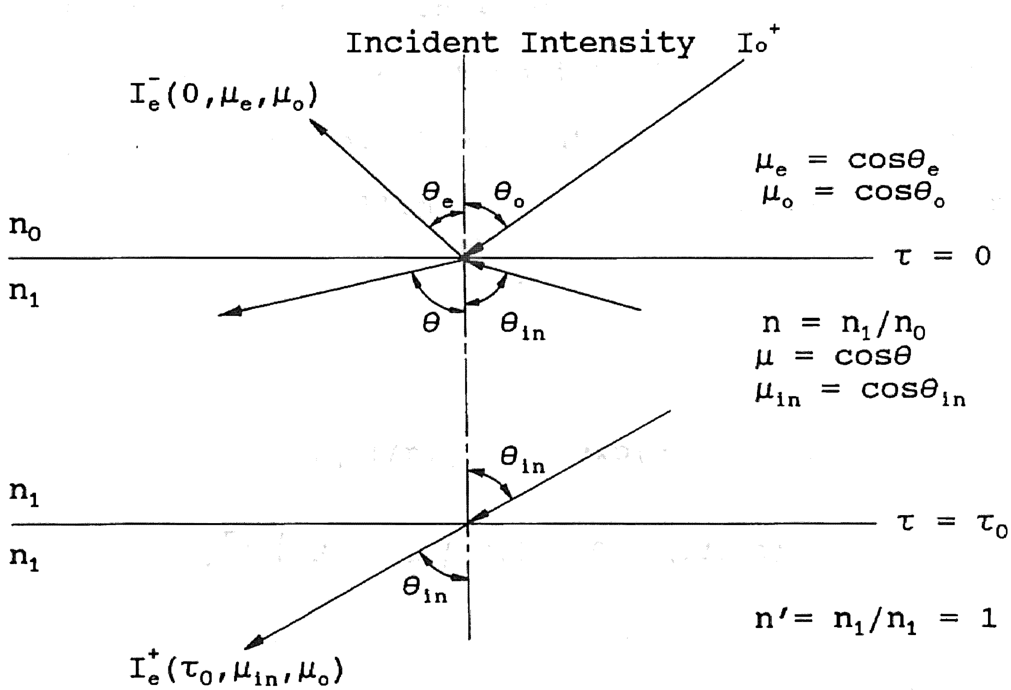

Figure 3. Geometry of a One-Dimensional Medium for Unity Refractive Index at Lower Boundary

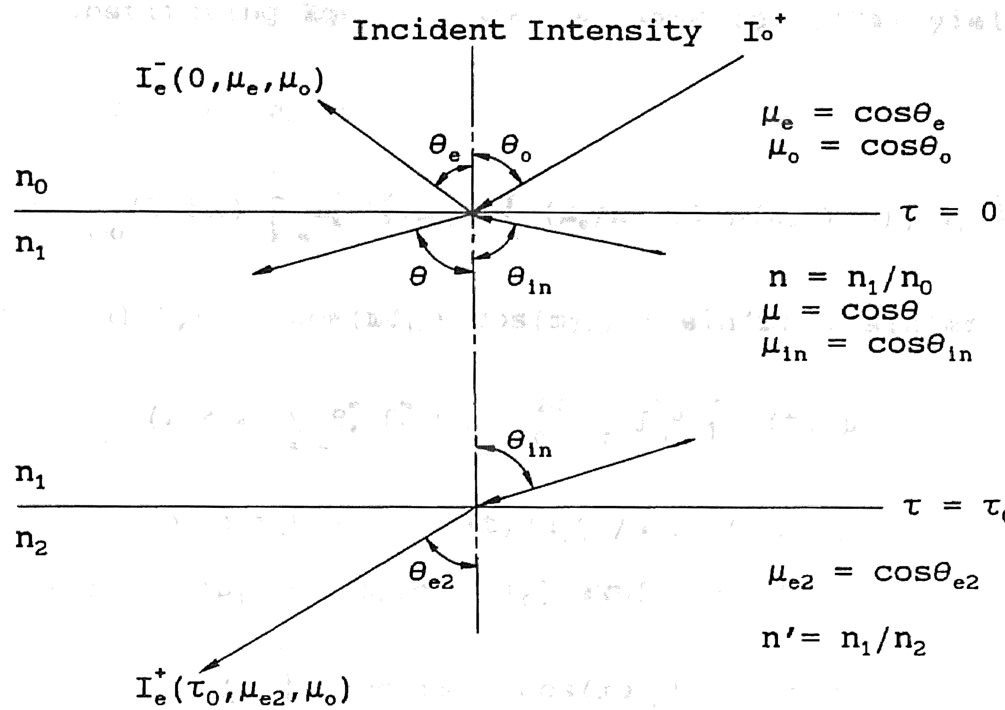

Figure 4. Geometry of a One-Dimensional Medium for Non Unity Refractive Index at Lower Boundary

At first, the situation which assumes no index of refraction effects at the bottom interface, as shown in Fig. (3), is considered. Substituting Eq. (15) into Eq. (16) and assuming no collimated incident radiation entering from the bottom (see Fig. (3)) yields

$$S(\tau,\ \mu_{1n},\ \phi_{1n},\ \mu_o,\ \phi_o,\ n;\ \tau_0)$$

$$=(\omega/4\pi)\left\{\ (\mu_o/\mu)\ \ [1-\rho(\mu_o,1/n)]\ I_o\right\}\ \exp(-\tau/\mu)\ \ P(\mu,\phi_o,\mu_{1n},\phi_{1n})$$

$$+\ (\omega/4\pi)\ \int_0^{2\pi}\int_0^1\int_0^{\tau_0}\Big\{\ S(t,-\mu_{1n}',\phi_{1n}'+180°,\mu_o,\phi_o,n;\tau_0)$$

$$\times\ \exp[-(\tau+t)/\mu_{1n}']/\mu_{1n}'\ \ \rho(\mu_{1n}',n)\ \ P(\mu_{1n}',\phi_{1n}',\mu_{1n},\phi_{1n})$$

$$+\ S[t,\mathrm{sign}(\tau-t)\mu_{1n}',\phi_{1n}',\mu_o,\phi_o,n;\tau_0]\ \ \exp[-|\tau-t|/\mu_{1n}']/\mu_{1n}'$$

$$\times\ P[\mathrm{sign}(\tau-t)\mu_{1n}',\phi_{1n}',\mu_{1n},\phi_{1n}]\ \Big\}\ dt\ d\mu_{1n}'\ d\phi_{1n}'. \qquad (17a)$$

Then, substituting Eqs. (3) and (4) into Eq. (17a) yields

$$S(\tau,\ \mu_{1n},\ \phi_{1n},\ \mu_o,\ \phi_o,\ n;\ \tau_0)$$

$$=(\omega/4\pi)\ \sum_{m=0}^{L}(2-\delta_{0m})\ \sum_{k=m}^{L} B_k^m\ P_k^m(\mu_{1n})\ \Big\{\ (\mu_o/\mu)\ [1-\rho(\mu_o,1/n)]\ I_o\ \Big\}$$

$$\times\ \exp(-\tau/\mu)\ P_k^m(\mu)\ [\cos(m\phi_{1n})\ \cos(m\phi_o)\ +\ \sin(m\phi_{1n})\ \sin(m\phi_o)]$$

$$+\ (\omega/4\pi)\ \sum_{m=0}^{L}(2-\delta_{0m})\ \sum_{k=m}^{L} B_k^m\ P_k^m(\mu_{1n})\ \int_0^{2\pi}\ \int_0^1\ \int_0^{\tau_0}\ \Big\{\ S(t,-\mu_{1n}',$$

$$\phi_{1n}'+180°,\mu_o,\phi_o,n;\tau_0)\ \exp[-(\tau+t)/\mu_{1n}']/\mu_{1n}'\ \rho(\mu_{1n}',n)\ P_k^m(\mu_{1n}')$$

$$+\ S[t,\mathrm{sign}(\tau-t)\mu_{1n}',\phi_{1n}',\mu_o,\phi_o,n;\tau_0]\ \exp[-|\tau-t|/\mu_{1n}']/\mu_{1n}'$$

$$\times\ P_k^m[\mathrm{sign}(\tau-t)\mu_{1n}']\ \Big\}\ [\cos(m\phi_{1n})\ \cos(m\phi_{1n}')\ +\ \sin(m\phi_{1n})$$

$$\times\ \sin(m\phi_{1n}')]\ dt\ d\mu_{1n}'\ d\phi_{1n}'. \qquad (17b)$$

Examination of Eq. (7) suggests that a reasonable expansion of the source function in terms of Legendre functions is

$$S(\tau,\mu_{1n},\phi_{1n};\tau_0)\ =\ (\omega/4\pi)\ \sum_{m=0}^{L}(2-\delta_{0m})\ \sum_{k=m}^{L} B_k^m\ P_k^m(\mu_{1n})\ [\cos(m\phi_{1n})$$

$$\times\ P_{km}(\tau,\tau_0)\ +\ \sin(m\phi_{1n})\ q_{km}(\tau,\tau_0)], \qquad (18)$$

where P_{km} and q_{km} are functions to be determined.

The use of Eq. (18) in Eq. (17b) results in two independent sets of equations for the P's and q's

$$P_{km}(\tau,\mu,n;\tau_0)\ =\ \Big\{\ (\mu_o/\mu)\ [1-\rho(\mu_o,1/n)]\ I_o\ \Big\}\ \exp(-\tau/\mu)\ P_k^m(\mu)$$

$$\times\ \cos(m\phi_o)\ +\ (\omega/2)\ \sum_{j=m}^{L} B_j^m\ \int_0^1\ \int_0^{\tau_0}\ \Big\{\ (-1)^m$$

$$\times\ P_{jm}(t,\mu,n;\tau_0)\ \exp[-(\tau+t)/\mu_{in}']/\mu_{in}'\ \rho(\mu_{in}',n)$$

$$\times\ P_j^m(-\mu_{in}')\ P_k^m(\mu_{in}')\ +\ P_{jm}(t,\mu,n;\tau_0)$$

$$\times\ \exp[-|\tau-t|/\mu_{in}']/\mu_{in}'\ P_j^m[\mathrm{sign}(\tau-t)\mu_{in}']$$

$$\times\ P_k^m[\mathrm{sign}(\tau-t)\mu_{in}']\Big\}\ dt\ d\mu_{in}', \qquad (19a)$$

and

$$q_{km}(\tau,\mu,n;\tau_0)\ =\ \Big\{\ (\mu_o/\mu)\ [1-\rho(\mu_o,1/n)]\ I_o\ \Big\}\ \exp(-\tau/\mu)\ P_k^m(\mu)$$

$$\times\ \sin(m\phi_o)\ +\ (\omega/2)\ \sum_{j=m}^{L} B_j^m \int_0^1 \int_0^{\tau_0} \Big\{\ (-1)^m$$

$$\times\ q_{jm}(t,\mu,n;\tau_0)\ \exp[-(\tau+t)/\mu_{in}']/\mu_{in}'\ \rho(\mu_{in}',n)$$

$$\times\ P_j^m(-\mu_{in}')\ P_k^m(\mu_{in}')\ +\ q_{jm}(t,\mu,n;\tau_0)$$

$$\times\ \exp[-|\tau-t|/\mu_{in}']/\mu_{in}'\ P_j^m[\mathrm{sign}(\tau-t)\mu_{in}']$$

$$\times\ P_k^m[\mathrm{sign}(\tau-t)\mu_{in}']\ \Big\}\ dt\ d\mu_{in}', \qquad (19b)$$

where μ and μ_o are related by Eq. (13), and n of $\rho(\mu'',n)$ is equal to n_1/n_0.

Next, the situation which has index of refraction effects at the bottom interface, as shown in Fig. (4), is considered.

Substituting Eq. (9a) into Eq. (10b) yields

$$I^-(\tau_0,\mu_{in},\phi_{in})\ =\ \Big\{\ I^+(0,\mu_{in},\phi_{in}+180°)\ \exp(-\tau_0/\mu_{in})$$

$$+\ \int_0^{\tau_0} S(t,\mu_{in},\phi_{in}+180°)\ \exp[-(\tau_0-t)/\mu_{in}]/\mu_{in}$$

$$\times\ dt\ \Big\}\ \rho(\mu_{in},n'). \qquad (20a)$$

Using Eqs. (10a) in (9b), we have

$$I^{+}(0,\mu_{in},\phi_{in}+180°) = I_{i}^{+}(\mu_{in},\phi_{in}+180°) + \Big\{ I^{-}(\tau_0,\mu_{in},\phi_{in}) \times \exp(-\tau_0/\mu_{in}) + \int_0^{\tau_0} S(t,-\mu_{in},\phi_{in}) \times \exp(-t/\mu_{in})/\mu_{in}\, dt \Big\} \rho(\mu_{in},n). \quad (20b)$$

Solving Eqs. (20a) and (20b) simultaneously, the result is

$$I^{-}(\tau_0,\mu_{in},\phi_{in}) = 1 \,/\, [1-\exp(-2\tau_0/\mu_{in})\ \rho(\mu_{in},n)\ \rho(\mu_{in},n')] \times \Big\{ I_{i}^{+}(\mu_{in},\phi_{in}+180°)\ \exp(-\tau_0/\mu_{in}) + \int_0^{\tau_0} S(t,-\mu_{in},\phi_{in})\ \exp[-(\tau_0+t)/\mu_{in}]/\mu_{in}\, dt \times \rho(\mu_{in},n) + \int_0^{\tau_0} S(t,\mu_{in},\phi_{in}+180°) \times \exp[-(\tau_0-t)/\mu_{in}]/\mu_{in}\, dt \Big\} \rho(\mu_{in},n'). \quad (20c)$$

Substituting Eq. (9b) into Eq. (10a) yields

$$I^{+}(0,\mu_{in},\phi_{in}) = I_{i}^{+}(\mu_{in},\phi_{in}) + \Big\{ I^{-}(\tau_0,\mu_{in},\phi_{in}+180°) \times \exp(-\tau_0/\mu_{in}) + \int_0^{\tau_0} S(t,-\mu_{in},\phi_{in}+180°) \times \exp(-t/\mu_{in})/\mu_{in}\, dt \Big\} \rho(\mu_{in},n). \quad (21a)$$

Also using Eqs. (9a) and (10b), we have

$$I^{-}(\tau_0,\mu_{in},\phi_{in}+180°) = \Big\{ I^{+}(0,\mu_{in},\phi_{in})\ \exp(-\tau_0/\mu_{in}) + \int_0^{\tau_0} S(t,\mu_{in},\phi_{in})\ \exp[-(\tau_0-t)/\mu_{in}]/\mu_{in} \times dt \Big\} \rho(\mu_{in},n'). \quad (21b)$$

Solving Eqs. (21a) and (21b) simultaneously, the result is

$$I^{+}(0,\mu_{1n},\phi_{1n}) = 1 \,/\, [1-\exp(-2\tau_0/\mu_{1n})\ \rho(\mu_{1n},n)\ \rho(\mu_{1n},n')]$$
$$\times \Big\{ I_1^{+}(\mu_{1n},\phi_{1n}) + \int_0^{\tau_0} S(t,\mu_{1n},\phi_{1n})$$
$$\times \exp[-(2\tau_0-t)/\mu_{1n}]/\mu_{1n}\ dt\ \rho(\mu_{1n},n)$$
$$\times \rho(\mu_{1n},n') + \int_0^{\tau_0} S(t,-\mu_{1n},\phi_{1n}+180°)$$
$$\times \exp(-t/\mu_{1n})/\mu_{1n}\ dt\ \rho(\mu_{1n},n)\Big\}. \qquad (21c)$$

Substituting Eqs. (9a), (9b), (20c), and (21c) into Eq. (2), the source function can be written as

$$S(\tau,\ \mu_{1n},\ \phi_{1n})$$
$$= (\omega/4\pi) \int_0^{2\pi} \int_0^1 \Big\{ I_1^{+}(\mu_{1n}',\phi_{1n}') \exp(-\tau/\mu_{1n}') P(\mu_{1n}',\phi_{1n}',\mu_{1n},\phi_{1n})$$
$$+ I_1^{+}(\mu_{1n}',\phi_{1n}'+180°) \exp[-(2\tau_0-\tau)/\mu_{1n}'] \rho(\mu_{1n}',n')$$
$$\times P(-\mu_{1n}',\phi_{1n}',\mu_{1n},\phi_{1n})\Big\} A(\tau_0,\mu_{1n}',n,n')\ d\mu_{1n}'\ d\phi_{1n}' + (\omega/4\pi)$$
$$\times \int_0^{2\pi} \int_0^1 \int_0^{\tau_0} \Big\{ S(t,\mu_{1n}',\phi_{1n}') \exp[-(2\tau_0+\tau-t)/\mu_{1n}'] \rho(\mu_{1n}',n')$$
$$+ S(t,-\mu_{1n}',\phi_{1n}'+180°) \exp[-(\tau+t)/\mu_{1n}'] \Big\} /\mu_{1n}'\ dt\ \rho(\mu_{1n}',n)$$
$$\times A(\tau_0,\mu_{1n}',n,n') P(\mu_{1n}',\phi_{1n}',\mu_{1n},\phi_{1n})\ d\mu_{1n}'\ d\phi_{1n}' + (\omega/4\pi)$$
$$\times \int_0^{2\pi} \int_0^1 \int_0^{\tau_0} \Big\{ S(t,-\mu_{1n}',\phi_{1n}') \exp[-(2\tau_0-\tau+t)/\mu_{1n}'] \rho(\mu_{1n}',n)$$
$$+ S(t,\mu_{1n}',\phi_{1n}'+180°) \exp[-(2\tau_0-\tau-t)/\mu_{1n}'] \Big\} /\mu_{1n}'\ dt\ \rho(\mu_{1n}',n')$$
$$\times A(\tau_0,\mu_{1n}',n,n') P(-\mu_{1n}',\phi_{1n}',\mu_{1n},\phi_{1n})\ d\mu_{1n}'\ d\phi_{1n}' + (\omega/4\pi)$$
$$\times \int_0^{2\pi} \int_0^1 \int_0^{\tau_0} S[t,\mathrm{sign}(\tau-t)\mu_{1n}',\phi_{1n}'] \exp[-|\tau-t|/\mu_{1n}']/\mu_{1n}'\ dt$$
$$\times P[\mathrm{sign}(\tau-t)\mu_{1n}',\phi_{1n}',\mu_{1n},\phi_{1n}]\ d\mu_{1n}'\ d\phi_{1n}', \qquad (22)$$

where $A(\tau_0,\mu'',n,n')$ is defined as

$$A(\tau_0,\mu'',n,n') = [1-\exp(-2\tau_0/\mu'')\ \rho(\mu'',n')\ \rho(\mu'',n)]^{-1}. \quad (22a)$$

Substituting Eqs. (15), (3), and (4) into Eq. (22) and assuming no collimated incident radiation entering from the bottom, see Fig. (4), yields the following source function

$$S(\tau,\ \mu_{1n},\ \phi_{1n},\ \mu_o,\ \phi_o,\ n,\ n';\ \tau_0)$$

$$= (\omega/4\pi) \sum_{m=0}^{L} (2-\delta_{0m}) \sum_{k=m}^{L} B_k^m P_k^m(\mu_{1n})\ (\mu_o/\mu)\ [1-\rho(\mu_o,1/n)]\ I_o$$

$$\times A(\tau_0,\mu,n,n')\ [\cos(m\phi_{1n})\ \cos(m\phi_o) + \sin(m\phi_{1n})\ \sin(m\phi_o)]$$

$$\times \Big\{ \exp(-\tau/\mu)\ P_k^m(\mu) + (-1)^m \exp[-(2\tau_0-\tau)/\mu]\ P_k^m(-\mu)\ \rho(\mu,n') \Big\}$$

$$+ (\omega/4\pi) \sum_{m=0}^{L} (2-\delta_{0m}) \sum_{k=m}^{L} B_k^m P_k^m(\mu_{1n}) \int_0^{2\pi} \int_0^1 \int_0^{\tau_0} \Big\{ S(t,\mu_{1n}',\phi_{1n}',$$

$$\mu_o,\phi_o,n,n';\tau_0)\ \exp[-(2\tau_0+\tau-t)/\mu_{1n}']\ P_k^m(\mu_{1n}')\ \rho(\mu_{1n}',n')$$

$$\times \rho(\mu_{1n}',n) + S(t,-\mu_{1n}',\phi_{1n}'+180°,\mu_o,\phi_o,n,n';\tau_0)$$

$$\times \exp[-(\tau+t)/\mu_{1n}']\ P_k^m(\mu_{1n}')\ \rho(\mu_{1n}',n) + S(t,-\mu_{1n}',\phi_{1n}',\mu_o,\phi_o,n,$$

$$n';\tau_0)\ \exp[-(2\tau_0-\tau+t)/\mu_{1n}']\ P_k^m(-\mu_{1n}')\ \rho(\mu_{1n}',n')\ \rho(\mu_{1n}',n)$$

$$+ S(t,\mu_{1n}',\phi_{1n}'+180°,\mu_o,\phi_o,n,n';\tau_0)\ \exp[-(2\tau_0-\tau-t)/\mu_{1n}']$$

$$\times P_k^m(-\mu_{1n}')\ \rho(\mu_{1n}',n') \Big\} /\mu_{1n}'\ dt\ A(\tau_0,\mu_{1n}',n,n')\ [\cos(m\phi_{1n})$$

$$\times \cos(m\phi_{1n}') + \sin(m\phi_{1n})\ \sin(m\phi_{1n}')]\ d\mu_{1n}'\ d\phi_{1n}' + (\omega/4\pi)$$

$$\times \sum_{m=0}^{L} (2-\delta_{0m}) \sum_{k=m}^{L} B_k^m P_k^m(\mu_{1n}) \int_0^{2\pi} \int_0^1 \int_0^{\tau_0} S[t,\mathrm{sign}(\tau-t)\mu_{1n}',\phi_{1n}',$$

$$\mu_o,\phi_o,n,n';\tau_0]\ \exp[-|\tau-t|/\mu_{1n}']\ /\mu_{1n}'\ dt\ P_k^m[\mathrm{sign}(\tau-t)\mu_{1n}']$$

$$\times [\cos(m\phi_{1n})\cos(m\phi_{1n}') + \sin(m\phi_{1n})\sin(m\phi_{1n}')]\ d\mu_{1n}'\ d\phi_{1n}'. \quad (23)$$

The use of Eq. (18) in Eq. (23) results in two independent sets of equations for the P's and q's

$$P_{km}(\tau,\ \mu,\ n;\ \tau_0)$$

$$= (\mu_o/\mu)\ [1-\rho(\mu_o,1/n)]\ I_o\ A(\tau_0,\mu,n,n')\ \cos(m\phi_o)\ \Big\{ \exp(-\tau/\mu)$$

$$\times\ P_k^m(\mu)\ +\ (-1)^m\ \exp[-(2\tau_0-\tau)/\mu]\ P_k^m(-\mu)\ \rho(\mu,n')\ \Big\}\ +\ (\omega/2)$$

$$\times \sum_{j=m}^{L} B_j^m \int_0^1 \int_0^{\tau_0} \Big\{ P_{jm}(t,\mu,n;\tau_0)\ \exp[-(2\tau_0+\tau-t)/\mu_{1n}']\ P_j^m(\mu_{1n}')$$

$$\times\ P_k^m(\mu_{1n}')\ \rho(\mu_{1n}',n')\ \rho(\mu_{1n}',n)\ +\ (-1)^m\ P_{jm}(t,\mu,n;\tau_0)$$

$$\times\ \exp[-(\tau+t)/\mu_{1n}']\ P_j^m(-\mu_{1n}')\ P_k^m(\mu_{1n}')\ \rho(\mu_{1n}',n)\ +\ P_{jm}(t,\mu,n;\tau_0)$$

$$\times\ \exp[-(2\tau_0-\tau+t)/\mu_{1n}']\ P_j^m(-\mu_{1n}')\ P_k^m(-\mu_{1n}')\ \rho(\mu_{1n}',n')\ \rho(\mu_{1n}',n)$$

$$+\ (-1)^m\ P_{jm}(t,\mu,n;\tau_0)\ \exp[-(2\tau_0-\tau-t)/\mu_{1n}']\ P_j^m(\mu_{1n}')\ P_k^m(-\mu_{1n}')$$

$$\times\ \rho(\mu_{1n}',n')\ \Big\}\ /\mu_{1n}'\ dt\ A(\tau_0,\mu_{1n}',n,n')\ d\mu_{1n}'\ +\ (\omega/2) \sum_{j=m}^{L} B_j^m$$

$$\times \int_0^1 \int_0^{\tau_0} P_{jm}(t,\mu,n;\tau_0)\ \exp[-|\tau-t|/\mu_{1n}']/\mu_{1n}'\ dt$$

$$\times\ P_j^m[\mathrm{sign}(\tau-t)\mu_{1n}']\ P_k^m[\mathrm{sign}(\tau-t)\mu_{1n}']\ d\mu_{1n}', \qquad (24a)$$

and

$$q_{km}(\tau,\ \mu,\ n;\ \tau_0)$$

$$= (\mu_o/\mu)\ [1-\rho(\mu_o,1/n)]\ I_o\ A(\tau_0,\mu,n,n')\ \sin(m\phi_o)\ \Big\{ \exp(-\tau/\mu)$$

$$\times\ P_k^m(\mu)\ +\ (-1)^m\ \exp[-(2\tau_0-\tau)/\mu]\ P_k^m(-\mu)\ \rho(\mu,n')\ \Big\}\ +\ (\omega/2)$$

$$\times \sum_{j=m}^{L} B_j^m \int_0^1 \int_0^{\tau_0} \Big\{ q_{jm}(t,\mu,n;\tau_0)\ \exp[-(2\tau_0+\tau-t)/\mu_{1n}']\ P_j^m(\mu_{1n}')$$

$$\times\ P_k^m(\mu_{1n}')\ \rho(\mu_{1n}',n')\ \rho(\mu_{1n}',n)\ +\ (-1)^m\ q_{jm}(t,\mu,n;\tau_0)$$

$$\times \exp[-(\tau+t)/\mu_{1n}'] \, P_j^m(-\mu_{1n}') \, P_k^m(\mu_{1n}') \, \rho(\mu_{1n}',n) + q_{jm}(t,\mu,n;\tau_0)$$

$$\times \exp[-(2\tau_0-\tau+t)/\mu_{1n}'] \, P_j^m(-\mu_{1n}') \, P_k^m(-\mu_{1n}') \, \rho(\mu_{1n}',n') \, \rho(\mu_{1n}',n)$$

$$+ (-1)^m \, q_{jm}(t,\mu,n;\tau_0) \, \exp[-(2\tau_0-\tau-t)/\mu_{1n}'] \, P_j^m(\mu_{1n}') \, P_k^m(-\mu_{1n}')$$

$$\times \rho(\mu_{1n}',n') \Big\} /\mu_{1n}' \, dt \, A(\tau_0,\mu_{1n}',n,n') \, d\mu_{1n}' + (\omega/2) \sum_{j=m}^{L} B_j^m$$

$$\times \int_0^1 \int_0^{\tau_0} q_{jm}(t,\mu,n;\tau_0) \, \exp[-|\tau-t|/\mu_{1n}']/\mu_{1n}' \, dt$$

$$\times P_j^m[\mathrm{sign}(\tau-t)\mu_{1n}'] \, P_k^m[\mathrm{sign}(\tau-t)\mu_{1n}'] \, d\mu_{1n}', \qquad (24b)$$

where μ and μ_o are related by Eq. (13), n of $\rho(\mu'',n)$ is equal to n_1/n_0, n′ of $\rho(\mu'',n')$ is equal to n_1/n_2, and $A(\tau_0,\mu'',n,n')$ is defined as Eq. (22a).

In this paper, we are not going to develop any further for the situation which has index of refraction effects at both the top and bottom interfaces as shown in Fig. (4). From this point on, we will focus our attention on obtaining the exact solution for the situation which assumes no index of refraction effect at the bottom interface as shown in Fig. (3).

Now, let us go back to the situation which assumes no index of refraction change across the bottom interface as shown in Fig. (3). Assuming that the incident azimuthal angle, ϕ_o, is equal to zero, Eqs. (19a) and (19b) become

$$P_{km}(\tau,\mu,n;\tau_0) = \Big\{ (\mu_o/\mu) \, [1-\rho(\mu_o,1/n)] \, I_o \Big\} \exp(-\tau/\mu) \, P_k^m(\mu)$$

$$+ (\omega/2) \sum_{j=m}^{L} B_j^m \int_0^1 \int_0^{\tau_0} \Big\{ (-1)^m \, P_{jm}(t,\mu,n;\tau_0)$$

$$\times \exp[-(\tau+t)/\mu_{1n}']/\mu_{1n}' \ \rho(\mu_{1n}',n) \ P_j^m(-\mu_{1n}') \ P_k^m(\mu_{1n}')$$

$$+ P_{jm}(t,\mu,n;\tau_0) \ \exp[-|\tau-t|/\mu_{1n}']/\mu_{1n}'$$

$$\times P_j^m[\mathrm{sign}(\tau-t)\mu_{1n}'] \ P_k^m[\mathrm{sign}(\tau-t)\mu_{1n}'] \Big\} \ dt$$

$$\times d\mu_{1n}', \qquad (25a)$$

and

$$q_{km}(\tau,\mu,n;\tau_0) = (\omega/2) \sum_{j=m}^{L} B_j^m \int_0^1 \int_0^{\tau_0} \Big\{ (-1)^m \ q_{jm}(t,\mu,n;\tau_0)$$

$$\times \exp[-(\tau+t)/\mu_{1n}']/\mu_{1n}' \ \rho(\mu_{1n}',n) \ P_j^m(-\mu_{1n}') \ P_k^m(\mu_{1n}')$$

$$+ q_{jm}(t,\mu,n;\tau_0) \ \exp[-|\tau-t|/\mu_{1n}']/\mu_{1n}'$$

$$\times P_j^m[\mathrm{sign}(\tau-t)\mu_{1n}'] \ P_k^m[\mathrm{sign}(\tau-t)\mu_{1n}'] \Big\} \ dt$$

$$\times d\mu_{1n}'. \qquad (25b)$$

An important comment that needs to be mentioned here is that Eq. (25b) implies that the q function is equal to zero due to the disappearance of the leading term. Therefore, in order to solve this problem, we need to only solve for the P function.

Eq. (25a) may be rewritten in compact form as

$$P_{km}(\tau,\mu,n;\tau_0) = L(\mu,n) \ P_k^m(\mu) \ \exp(-\tau/\mu) + (\omega/2) \sum_{j=m}^{L} B_j^m$$

$$\times \int_0^{\tau_0} P_{jm}(t,\mu,n;\tau_0) \Big\{ (-1)^m \ K_{1jmk}(\tau+t,n)$$

$$+ K_{2jmk}(\tau-t) \Big\} \ dt, \qquad (26a)$$

where $K_{1jmk}(\tau+t,n) = \int_0^1 \exp[-(\tau+t)/\mu_{1n}'] \ P_j^m(-\mu_{1n}') \ \rho(\mu_{1n}',n)$

$$\times\ P_k^m(\mu_{1n}')/\mu_{1n}'\ d\mu_{1n}', \tag{26b}$$

$$K_{2jmk}(\tau-t) = \int_0^1 \exp[-|\tau-t|/\mu_{1n}']\ P_j^m[\text{sign}(\tau-t)\mu_{1n}']$$

$$\times\ P_k^m[\text{sign}(\tau-t)\mu_{1n}']\ /\mu_{1n}'\ d\mu_{1n}', \tag{26c}$$

and

$$L(\mu,n) = (\mu_o/\mu)\ [1-\rho(\mu_o,1/n)]\ I_o. \tag{26d}$$

Equation (26a) is the general source function equation which will be solved later.

Intensity

Now we want to determine the intensity for a collimated beam of radiation at the top boundary. Remember that we have assumed no index of refraction effects at the bottom interface and no collimated incident radiation entering from the bottom as shown in Fig. (3).

For τ equal to zero, Eq. (9b) becomes

$$I^-(0,\mu_{1n},\phi_{1n}) = I^-(\tau_0,\mu_{1n},\phi_{1n})\ \exp(-\tau_0/\mu_{1n})$$

$$+ \int_0^{\tau_0} S(t,-\mu_{1n},\phi_{1n})\ \exp(-t/\mu_{1n})/\mu_{1n}\ dt. \tag{27a}$$

Substituting Eqs. (12), (14), and (27a) into eq. (10a) yields

$$I^+(0,\mu_{1n},\phi_{1n}) = (\mu_o/\mu)\ [1-\rho(\mu_o,1/n)]\ I_o\ \delta(\mu_{1n}-\mu)\ \delta(\phi_{1n} - \phi_o)$$

$$+ \rho(\mu_{1n},n) \int_0^{\tau_0} S(t,-\mu_{1n},\phi_{1n}+180°)\ \exp(-t/\mu_{1n})$$

$$/\ \mu_{1n}\ dt, \tag{27b}$$

where μ and μ_o are related by Eq. (13). Setting $\tau = \tau_0$ in Eq. (9a) and substituting Eqs. (27b) and (15) into the modified Eq. (9a), the following intensity function at the bottom boundary can be obtained

$$\overset{+}{I}_o(\tau_0,\mu_{in},\phi_{in}) = \overset{+}{I}_i(\mu_{in},\phi_{in}) \exp(-\tau_0/\mu_{in}) + \rho(\mu_{in},n)$$
$$\times \exp(-\tau_0/\mu_{in}) \int_0^{\tau_0} S(t,-\mu_{in},\phi_{in}+180°)$$
$$\times \exp(-t/\mu_{in})/\mu_{in}\, dt + \int_0^{\tau_0} S(t,\mu_{in},\phi_{in})$$
$$\times \exp[-(\tau_0-t)/\mu_{in}]/\mu_{in}\, dt. \qquad (27c)$$

Once the source function is determined, intensities at the top and bottom interfaces can be found.

Flux

The flux can be found by substituting Eqs. (9a), (9b), and (10a) into the general flux equation which is [8]

$$q(\tau) = \int_0^{2\pi} \int_{-1}^{1} I(\tau,\mu_{in},\phi_{in})\ \mu_{in}\, d\mu_{in}\, d\phi_{in}. \qquad (28a)$$

Note that there is no collimated incident radiation entering from the bottom boundary. Thus, we have the flux equation as

$$q(\tau) = \int_0^{2\pi} \int_0^1 \overset{+}{I}_i(\mu_{in},\phi_{in}) \exp(-\tau/\mu_{in})\ \mu_{in}\, d\mu_{in}\, d\phi_{in}$$
$$+ \int_0^{2\pi} \int_0^1 \int_0^{\tau_0} S(t,-\mu_{in},\phi_{in}+180°) \exp[-(\tau+t)/\mu_{in}]\, dt$$
$$\times \rho(\mu_{in},n)\, d\mu_{in}\, d\phi_{in} + \int_0^{2\pi} \int_0^1 \int_0^{\tau} S(t,\mu_{in},\phi_{in})$$

$$\times \exp[-(\tau-t)/\mu_{in}]dt\ d\mu_{in}\ d\phi_{in} - \int_0^{2\pi}\int_0^1\int_\tau^{\tau_0} S(t,-\mu_{in},\phi_{in})$$

$$\times \exp[-(t-\tau)/\mu_{in}]\ dt\ d\mu_{in}\ d\phi_{in}. \tag{28b}$$

Applying collimated incident boundary condition Eq. (15) to the above equation gives

$$q(\tau,\mu_o,n,\tau_0) = \mu_o I_o\ [1-\rho(\mu_o,1/n)]\ \exp\left\{ -\tau/[1-(1-\mu_o^2)/n^2]^{1/2} \right\}$$

$$+ \int_0^{2\pi}\int_0^1\int_0^{\tau_0} S(t,-\mu_{in},\phi_{in}+180°)\ \exp[-(\tau+t)/\mu_{in}]$$

$$\times dt\ \rho(\mu_{in},n)\ d\mu_{in}\ d\phi_{in} + \int_0^{2\pi}\int_0^1\int_0^{\tau} S(t,\mu_{in},\phi_{in})$$

$$\times \exp[-(\tau-t)/\mu_{in}]\ dt\ d\mu_{in}\ d\phi_{in}$$

$$- \int_0^{2\pi}\int_0^1\int_\tau^{\tau_0} S(t,-\mu_{in},\phi_{in})\ \exp[-(t-\tau)/\mu_{in}]\ dt$$

$$\times d\mu_{in}\ d\phi_{in}. \tag{28c}$$

Examination of Eqs. (25a) and (25b) suggests that a reduced expression for the source function as compared to Eq. (18) should be

$$S(\tau,\mu_{in},\phi_{in};\tau_0) = (\omega/4\pi)\sum_{m=0}^{L}(2-\delta_{0m})\sum_{k=m}^{L} B_k^m\ P_k^m(\mu_{in})\ \cos(m\phi_{in})$$

$$\times P_{km}(\tau,\mu,n;\tau_0). \tag{29}$$

Substitution of Eq. (29) into Eq. (28c) yields

$$q(\tau,\mu_o,n,\tau_0) = \mu_o I_o\ [1-\rho(\mu_o,1/n)]\ \exp\left\{ -\tau/[1-(1-\mu_o^2)/n^2]^{1/2} \right\}$$

$$+ (\omega/2)\sum_{k=0}^{L} B_k^0\ (-1)^k \int_0^1\int_0^{\tau_0} P_k^0(\mu_{in})\ P_{k0}(t,\mu,n;\tau_0)$$

$$\times \exp[-(\tau+t)/\mu_{in}]\ \rho(\mu_{in},n)\ dt\ d\mu_{in} + (\omega/2)\sum_{k=0}^{L} B_k^0$$

$$\times \int_0^1 \int_0^{\tau} P_k^0(\mu_{in})\ P_{k0}(t,\mu,n;\tau_0)\ \exp[-(\tau-t)/\mu_{in}]\ dt$$

$$\times\ d\mu_{in}\ -\ (\omega/2)\sum_{k=0}^{L} B_k^0\ (-1)^k \int_0^1 \int_{\tau}^{\tau_0} P_k^0(\mu_{in})$$

$$\times\ P_{k0}(t,\mu,n;\tau_0)\ \exp[-(t-\tau)/\mu_{in}]\ dt\ d\mu_{in}. \qquad (30a)$$

Let

$$K_{11jmk}(\tau+t,n) = \int_0^1 \exp[-(\tau+t)/\mu_{in}]\ P_j^m(-\mu_{in})\ \rho(\mu_{in},n)$$

$$\times\ P_k^m(\mu_{in})\ d\mu_{in}, \qquad (30b)$$

and

$$K_{22jmk}(\tau-t) = \int_0^1 \exp[-|\tau-t|/\mu_{in}]\ P_j^m[\mathrm{sign}(\tau-t)\mu_{in}]$$

$$\times\ P_k^m[\mathrm{sign}(\tau-t)\mu_{in}]\ d\mu_{in}. \qquad (30c)$$

Substituting Eqs. (30b) and (30c) into Eq. (30a) yields

$$q(\tau,\mu_o,n,\tau_0) = (\mu_o/\mu)\ I_o\ [1-\rho(\mu_o,1/n)]$$

$$\times\ q_p\left\{\ \tau,[1-(1-\mu_o^2)/n^2]^{1/2},n;\tau_0\ \right\}, \qquad (31)$$

where the equation for the fundamental flux q_p is defined as

$$q_p(\tau,\mu,n;\tau_0) = \mu\ \exp(-\tau/\mu)\ +\ (\omega/2)\sum_{k=0}^{L} B_k^0 \int_0^{\tau_0} P_{k0}(t,\mu,n;\tau_0)$$

$$/\ \{(\mu_o/\mu)I_o[1-\rho(\mu_o,1/n)]\}\left\{\ K_{11k00}(\tau+t,n)\right.$$

$$\left.+\ \mathrm{sign}(\tau-t)\ K_{22k00}(\tau-t)\ \right\}\ dt. \qquad (31a)$$

Setting k = 1 and m = 0 in Eq. (25a) and comparing with Eq. (30a), gives

$$q(\tau,\mu_o,n;\tau_0) = P_{10}(\tau,\mu,n;\tau_0) \qquad (32)$$

Therefore, whenever we get the solution for the P function, Eq. (32) above will automatically give us the exact solution for heat flux without doing further calculations.

So far, the equations for the source function, intensity, and heat flux are written in general forms. The assumptions made at this stage are: no index of refraction effects at the bottom interface, no collimated incident radiation entering from the bottom, and ϕ_o being equal to zero as shown in Fig. (3). In the next two chapters, these equations will be written for the cases when the optical thickness τ_0 is infinite and finite, respectively.

CHAPTER III

SOLUTION OF THE SEMI-INFINITE PROBLEM

Source Function

For infinite optical thickness, the general source function which is similar to Eq. (26a) is

$$P_{km}(\tau,\mu,n) = L(\mu,n)\ P_k^m(\mu)\ \exp(-\tau/\mu) + (\omega/2) \sum_{j=m}^{L} B_j^m \times \int_0^\infty P_{jm}(t,\mu,n) \left\{ (-1)^m K_{1jmk}(\tau+t,n) + K_{2jmk}(\tau-t) \right\} dt, \qquad (33)$$

where $K_{1jmk}(\tau+t,n)$ is defined as Eq. (26b), $K_{2jmk}(\tau-t)$ is defined as Eq. (26c), and $L(\mu,n)$ is defined as Eq. (26d).

Defining fundamental source function PP_{km} as

$$PP_{km}(\tau,\mu,n) = P_k^m(\mu)\ \exp(-\tau/\mu) + (\omega/2) \sum_{j=m}^{L} B_j^m \int_0^\infty PP_{jm}(t,\mu,n) \times \left\{ (-1)^m K_{1jmk}(\tau+t,n) + K_{2jmk}(\tau-t) \right\} dt, \qquad (34a)$$

and superposing Eq. (33) with Eq. (34a) yields

$$P_{km}(\tau,\mu,n) = L(\mu,n)\ PP_{km}(\tau,\mu,n). \qquad (34b)$$

Eq. (34a) can be rewritten as

$$PP_{km}(\tau,\mu,n) = P_k^m(\mu)\ \exp(-\tau/\mu) + (\omega/2) \sum_{j=m}^{L} B_j^m\ (-1)^m$$

$$\times \int_0^\infty PP_{jm}(t,\mu,n)\ K_{1jmk}(\tau+t,n)\ dt + (\omega/2) \sum_{j=m}^{L} B_j^m$$

$$\times \int_0^\infty PP_{jm}(t,\mu,n)\ K_{2jmk}(\tau-t)\ dt. \quad (35)$$

Following Jiang [8], let us define

$$PP_{km1}(\tau,\mu) = P_k^m(\mu)\ \exp(-\tau/\mu) + (\omega/2) \sum_{j=m}^{L} B_j^m \int_0^\infty PP_{jm1}(t,\mu)$$

$$\times K_{2jmk}(\tau-t)\ dt, \quad (35a)$$

and

$$PP_{km2}(\tau,\mu,n) = f_{km}(\tau,\mu,n) + (\omega/2) \sum_{j=m}^{L} B_j^m \int_0^\infty PP_{jm2}(t,\mu,n)$$

$$\times K_{2jmk}(\tau-t)\ dt, \quad (35b)$$

where $f_{km}(\tau,\mu,n) = (\omega/2) \sum_{j=m}^{L} B_j^m\ (-1)^m \int_0^\infty PP_{jm}(t,\mu,n)$

$$\times K_{1jmk}(\tau+t,n)\ dt, \quad (35c)$$

and $PP_{km1}(\tau,\mu)$ is not a function of refractive index n. Then, adding Eq. (35a) to Eq. (35b), and comparing with Eq. (35), we have the following equation

$$PP_{km}(\tau,\mu,n) = PP_{km1}(\tau,\mu) + PP_{km2}(\tau,\mu,n). \quad (36)$$

Ambarzumian's approach will be used to derive the fundamental source function in the following derivation.

Eq. (35a) can be written in expanded form as

$$PP_{km1}(\tau,\mu) = P_k^m(\mu)\ \exp(-\tau/\mu) + (\omega/2) \sum_{j=m}^{L} B_j^m \int_0^\tau PP_{jm1}(t,\mu)$$

$$\times K_{2jmk}(\tau-t)\ dt + (\omega/2) \sum_{j=m}^{L} B_j^m \int_\tau^\infty PP_{jm1}(t,\mu)$$

$$\times K_{2jmk}(\tau-t)\ dt. \qquad (37)$$

Using the substitution $\bar{t} = \tau-t$ in the first integral and $\bar{t} = t-\tau$ in the second integral, Eq. (37) becomes

$$PP_{km1}(\tau,\mu) = P_k^m(\mu)\ \exp(-\tau/\mu) + (\omega/2) \sum_{j=m}^{L} B_j^m \int_0^\tau PP_{jm1}(\tau-\bar{t},\mu)$$

$$\times K_{2jmk}(\bar{t})\ d\bar{t} + (\omega/2) \sum_{j=m}^{L} B_j^m \int_0^\infty PP_{jm1}(\tau+\bar{t},\mu)$$

$$\times K_{2jmk}(-\bar{t})\ d\bar{t}. \qquad (38)$$

Using Leibnitz rule to take the derivative of Eq. (38) with respect to τ yields

$$\frac{\partial\ PP_{km1}(\tau,\mu)}{\partial\ \tau} = -(1/\mu)\ P_k^m(\mu)\ \exp(-\tau/\mu) + (\omega/2) \sum_{j=m}^{L} B_j^m$$

$$\times PP_{jm1}(0,\mu)\ K_{2jmk}(\tau) + (\omega/2) \sum_{j=m}^{L} B_j^m$$

$$\times \int_0^\tau \frac{\partial\ PP_{jm1}(\tau-\bar{t},\mu)}{\partial\ \tau} K_{2jmk}(\bar{t})\ d\bar{t} + (\omega/2) \sum_{j=m}^{L} B_j^m$$

$$\times \int_0^\infty \frac{\partial\ PP_{jm1}(\tau+\bar{t},\mu)}{\partial\ \tau} K_{2jmk}(-\bar{t})\ d\bar{t}. \qquad (39)$$

Using the substitution $t = \tau-\bar{t}$ in the first integral and $t = \bar{t}+\tau$ in the second integral, Eq. (39) may be written as

$$\frac{\partial\ PP_{km1}(\tau,\mu)}{\partial\ \tau} = -(1/\mu)\ P_k^m(\mu)\ \exp(-\tau/\mu) + (\omega/2) \sum_{j=m}^{L} B_j^m$$

$$\times\ PP_{jm1}(0,\mu)\ K_{2jmk}(\tau) + (\omega/2) \sum_{j=m}^{L} B_j^m$$

$$\times \int_0^{\tau} \frac{\partial\ PP_{jm1}(t,\mu)}{\partial\ t}\ K_{2jmk}(\tau-t)\ dt + (\omega/2) \sum_{j=m}^{L} B_j^m$$

$$\times \int_{\tau}^{\infty} \frac{\partial\ PP_{jm1}(t,\mu)}{\partial\ t}\ K_{2jmk}(\tau-t)\ dt, \tag{40}$$

or

$$\frac{\partial\ PP_{km1}(\tau,\mu)}{\partial\ \tau} = -(1/\mu)\ P_k^m(\mu)\ \exp(-\tau/\mu) + (\omega/2) \sum_{j=m}^{L} B_j^m$$

$$\times\ PP_{jm1}(0,\mu)\ K_{2jmk}(\tau) + (\omega/2) \sum_{j=m}^{L} B_j^m$$

$$\times \int_0^{\infty} \frac{\partial\ PP_{jm1}(t,\mu)}{\partial\ t}\ K_{2jmk}(\tau-t)\ dt. \tag{41}$$

The solution of Eq. (41) can be found by the method of superposition. Start by replacing μ by μ_{in}' in Eq. (35a), then we get

$$PP_{km1}(\tau,\mu_{in}') = P_k^m(\mu_{in}')\ \exp(-\tau/\mu_{in}') + (\omega/2) \sum_{j=m}^{L} B_j^m$$

$$\times \int_0^{\infty} PP_{jm1}(t,\mu_{in}')\ K_{2jmk}(\tau-t)\ dt. \tag{42}$$

Then, multiplying Eq. (42) by $(\omega/2)\ B_i^m\ PP_{im1}(0,\mu)\ P_i^m(\mu_{in}')$ $/\mu_{in}'$, integrating from zero to one with respect to μ_{in}', and summing from $i = m$ to L, we obtain

$$(\omega/2)\sum_{i=m}^{L} B_i^m\, PP_{im1}(0,\mu)\int_0^1 PP_{km1}(\tau,\mu_{1n}')\, P_i^m(\mu_{1n}')/\mu_{1n}'\, d\mu_{1n}'$$

$$= (\omega/2)\sum_{i=m}^{L} B_i^m\, PP_{im1}(0,\mu)\int_0^1 \exp(-\tau/\mu_{1n}')\, P_i^m(\mu_{1n}')\, P_k^m(\mu_{1n}')/\mu_{1n}'$$

$$\times\, d\mu_{1n}' + (\omega/2)\sum_{j=m}^{L} B_j^m \int_0^\infty \Big\{ (\omega/2)\sum_{i=m}^{L} B_i^m\, PP_{im1}(0,\mu)$$

$$\times \int_0^1 PP_{jm1}(t,\mu_{1n}')\, P_i^m(\mu_{1n}')/\mu_{1n}'\, d\mu_{1n}' \Big\} K_{2jmk}(\tau-t)\, dt,$$

or using Eq. (26c), for K_{2jmk}

$$(\omega/2)\sum_{i=m}^{L} B_i^m\, PP_{im1}(0,\mu)\int_0^1 PP_{km1}(\tau,\mu_{1n}')\, P_i^m(\mu_{1n}')/\mu_{1n}'\, d\mu_{1n}'$$

$$= (\omega/2)\sum_{i=m}^{L} B_i^m\, PP_{im1}(0,\mu)\, K_{2imk}(\tau) + (\omega/2)\sum_{j=m}^{L} B_j^m \int_0^\infty \Big\{ (\omega/2)$$

$$\times \sum_{i=m}^{L} B_i^m\, PP_{im1}(0,\mu)\int_0^1 PP_{jm1}(t,\mu_{1n}')\, P_i^m(\mu_{1n}')/\mu_{1n}'\, d\mu_{1n}' \Big\}$$

$$\times\, K_{2jmk}(\tau-t)\, dt. \qquad (43)$$

Replacing i by j in the first term of the right hand side, Eq. (43) is modified as

$$(\omega/2)\sum_{i=m}^{L} B_i^m\, PP_{im1}(0,\mu)\int_0^1 PP_{km1}(\tau,\mu_{1n}')\, P_i^m(\mu_{1n}')/\mu_{1n}'\, d\mu_{1n}'$$

$$= (\omega/2)\sum_{j=m}^{L} B_j^m\, PP_{jm1}(0,\mu)\, K_{2jmk}(\tau) + (\omega/2)\sum_{j=m}^{L} B_j^m \int_0^\infty \Big\{ (\omega/2)$$

$$\times \sum_{i=m}^{L} B_i^m\, PP_{im1}(0,\mu)\int_0^1 PP_{jm1}(t,\mu_{1n}')\, P_i^m(\mu_{1n}')/\mu_{1n}'\, d\mu_{1n}' \Big\}$$

$$\times\, K_{2jmk}(\tau-t)\, dt. \qquad (44)$$

Finally, multiplying Eq. (35a) by $-(1/\mu)$ and adding to Eq. (44) give

$$-(1/\mu)\ PP_{km1}(\tau,\mu) + (\omega/2) \sum_{i=m}^{L} B_i^m\ PP_{im1}(0,\mu) \int_0^1 PP_{km1}(\tau,\mu_{in}')$$

$$\times P_i^m(\mu_{in}')/\mu_{in}'\ d\mu_{in}' = -(1/\mu)\ P_k^m(\mu)\ \exp(-\tau/\mu) + (\omega/2) \sum_{j=m}^{L} B_j^m$$

$$\times PP_{jm1}(0,\mu)\ K_{2jmk}(\tau) + (\omega/2) \sum_{j=m}^{L} B_j^m$$

$$\times \int_0^\infty \Big\{ -(1/\mu)\ PP_{jm1}(t,\mu) + (\omega/2) \sum_{i=m}^{L} B_i^m$$

$$\times PP_{im1}(0,\mu) \int_0^1 PP_{jm1}(t,\mu_{in}')\ P_i^m(\mu_{in}')/\mu_{in}'$$

$$\times d\mu_{in}' \Big\}\ K_{2jmk}(\tau-t)\ dt. \qquad (45)$$

Now, comparing Eq. (41) with Eq. (45), the solution of Eq. (41) by superposition is

$$\frac{\partial\ PP_{km1}(\tau,\mu)}{\partial\ \tau} = -(1/\mu)\ PP_{km1}(\tau,\mu) + (\omega/2) \sum_{j=m}^{L} B_j^m\ PP_{jm1}(0,\mu)$$

$$\times \int_0^1 PP_{km1}(\tau,\mu_{in}')\ P_j^m(\mu_{in}')/\mu_{in}'\ d\mu_{in}'. \qquad (46)$$

In order to solve the above integro-differential equation, we need to solve for $PP_{jm1}(0,\mu)$ first.

By setting $\tau = 0$, replacing j by i, and replacing k by j in Eq. (35a), we have the following expression for $PP_{jm1}(0,\mu)$

$$PP_{jm1}(0,\mu) = P_j^m(\mu) + (\omega/2) \sum_{i=m}^{L} B_i^m \int_0^\infty PP_{im1}(t,\mu)$$

$$\times K_{2imj}(-t)\ dt. \qquad (47)$$

Equation (26c) can be written as follows when $\tau = 0$, j is replaced by i, and k is replaced by j

$$K_{21mj}(-t) = \int_0^1 \exp(-t/\mu_{1n}') \, P_i^m(-\mu_{1n}') \, P_j^m(-\mu_{1n}')/\mu_{1n}' \, d\mu_{1n}'. \quad (48)$$

Substituting Eq. (48) into Eq. (47) yields

$$PP_{jm1}(0,\mu) = P_j^m(\mu) + (\omega/2) \sum_{i=m}^{L} B_i^m \int_0^\infty PP_{im1}(t,\mu) \int_0^1 \exp(-t/\mu_{1n}')$$

$$\times P_i^m(-\mu_{1n}') \, P_j^m(-\mu_{1n}')/\mu_{1n}' \, d\mu_{1n}' \, dt. \quad (49)$$

Interchanging the order of integration, Eq. (49) may be rewritten as

$$PP_{jm1}(0,\mu) = P_j^m(\mu) + (\omega/2) \sum_{i=m}^{L} B_i^m \int_0^1 \left\{ \int_0^\infty PP_{im1}(t,\mu) \right.$$

$$\left. \times \exp(-t/\mu_{1n}') \, dt \right\} P_i^m(-\mu_{1n}') \, P_j^m(-\mu_{1n}')/\mu_{1n}' \, d\mu_{1n}'. \quad (50)$$

Defining

$$\overline{PP_{im1}}(\mu_{1n}',\mu) = \int_0^\infty PP_{im1}(t,\mu) \, \exp(-t/\mu_{1n}') \, dt \quad (51)$$

as the transform of $PP_{im1}(t,\mu)$ with respect to t, which is also the reflection function of $PP_{im1}(t,\mu)$, Eq. (50) becomes

$$PP_{jm1}(0,\mu) = P_j^m(\mu) + (\omega/2) \sum_{i=m}^{L} B_i^m \int_0^1 \overline{PP_{im1}}(\mu_{1n}',\mu) \, P_i^m(-\mu_{1n}')$$

$$\times P_j^m(-\mu_{1n}')/\mu_{1n}' \, d\mu_{1n}'. \quad (52)$$

Therefore, we have found $PP_{jm1}(0,\mu)$ in terms of the transform of PP_{im1}. Our next objective is to find an expression for $\overline{PP_{im1}}$ which involves $PP_{im1}(0,\mu)$.

Application of the transform of Eq. (51) to Eq. (46) by multiplying Eq. (46) by $\exp(-\tau/s)$ and integrating over τ yields

$$(1/s)\ \overline{PP_{km1}}(s,\mu) - PP_{km1}(0,\mu)$$

$$= -(1/\mu)\ \overline{PP_{km1}}(s,\mu) + (\omega/2) \sum_{j=m}^{L} B_j^m\ PP_{jm1}(0,\mu) \int_0^1 \overline{PP_{km1}}(s,\mu_{1n}')$$

$$\times\ P_j^m(\mu_{1n}')/\mu_{1n}'\ d\mu_{1n}', \qquad (53)$$

Note that the transform of $\dfrac{\partial\ PP_{km1}(\tau,\mu)}{\partial\ \tau}$ is equal to $(1/s)$ $\times$ [transform of $PP_{km1}(\tau,\mu)$] $- PP_{km1}(0,\mu)$.

Equation (53) can be rearranged as

$$(1/s + 1/\mu)\ \overline{PP_{km1}}(s,\mu) = PP_{km1}(0,\mu) + (\omega/2) \sum_{j=m}^{L} B_j^m\ PP_{jm1}(0,\mu)$$

$$\times \int_0^1 \overline{PP_{km1}}(s,\mu_{1n}')\ P_j^m(\mu_{1n}')/\mu_{1n}'$$

$$\times\ d\mu_{1n}'. \qquad (54)$$

Multiplying Eq. (54) by $(\omega/2)\ B_k^m\ P_k^m(-s)$ and summing from $k = m$ to L gives

$$(1/s + 1/\mu)\ (\omega/2) \sum_{k=m}^{L} B_k^m\ P_k^m(-s)\ \overline{PP_{km1}}(s,\mu)$$

$$= (\omega/2) \sum_{k=m}^{L} B_k^m\ P_k^m(-s)\ PP_{km1}(0,\mu) + (\omega/2) \sum_{j=m}^{L} B_j^m\ PP_{jm1}(0,\mu)$$

$$\times \left\{ (\omega/2) \sum_{k=m}^{L} B_k^m \int_0^1 \overline{PP_{km1}}(s,\mu_{1n}')\ P_j^m(\mu_{1n}')\ P_k^m(-s)/\mu_{1n}' \right.$$

$$\left. \times\ d\mu_{1n}' \right\}. \qquad (55)$$

Replacing k by j and k by i in the first term and the second term of the right hand side, respectively, Eq. (55) can be written as

$$(1/s + 1/\mu)\ (\omega/2) \sum_{k=m}^{L} B_k^m\ P_k^m(-s)\ \overline{PP_{kml}}(s,\mu)$$

$$= (\omega/2) \sum_{j=m}^{L} B_j^m\ P_j^m(-s)\ PP_{jml}(0,\mu)\ +\ (\omega/2) \sum_{j=m}^{L} B_j^m\ PP_{jml}(0,\mu)$$

$$\times \Big\{\ (\omega/2) \sum_{i=m}^{L} B_i^m \int_0^1 \overline{PP_{iml}}(s,\mu_{in}')\ P_j^m(\mu_{in}')\ P_i^m(-s)/\mu_{in}'$$

$$\times\ d\mu_{in}' \Big\}. \tag{56a}$$

By knowing $P_k^m(-\mu) = (-1)^{k+m}\ P_k^m(\mu)$ [12], Eq. (56a) becomes

$$(1/s + 1/\mu)\ (\omega/2) \sum_{k=m}^{L} B_k^m\ P_k^m(-s)\ \overline{PP_{kml}}(s,\mu)$$

$$= (\omega/2) \sum_{j=m}^{L} B_j^m\ PP_{jml}(0,\mu)\ (-1)^{j+m} \Big\{\ P_j^m(s)\ +\ (\omega/2) \sum_{i=m}^{L} B_i^m$$

$$\times \int_0^1 \overline{PP_{iml}}(s,\mu_{in}')\ P_j^m(-\mu_{in}')\ P_i^m(-s)/\mu_{in}'\ d\mu_{in}' \Big\}. \tag{56b}$$

It can be shown that (App. A)

$$\sum_{i=m}^{L} B_i^m\ P_i^m(-\mu)\ \overline{PP_{iml}}(\mu,s)\ = \sum_{i=m}^{L} B_i^m\ P_i^m(-s)\ \overline{PP_{iml}}(s,\mu). \tag{57}$$

With the help of Eq. (57), we may write Eq. (52) in another form

$$PP_{jml}(0,\mu)\ = P_j^m(\mu)\ +\ (\omega/2) \sum_{i=m}^{L} B_i^m \int_0^1 \overline{PP_{iml}}(\mu,\mu_{in}')\ P_i^m(-\mu)$$

$$\times\ P_j^m(-\mu_{in}')/\mu_{in}'\ d\mu_{in}'. \tag{58}$$

Now, replacing μ with s in Eq. (58) and substituting it into Eq. (56b) yields

$$(1/s + 1/\mu)\ (\omega/2) \sum_{k=m}^{L} B_k^m\ P_k^m(-s)\ \overline{PP_{km1}}(s,\mu)$$

$$= (\omega/2) \sum_{j=m}^{L} B_j^m\ PP_{jm1}(0,\mu)\ (-1)^{j+m}\ PP_{jm1}(0,s),$$

or

$$(\omega/2) \sum_{i=m}^{L} B_i^m\ P_i^m(-s)\ \overline{PP_{im1}}(s,\mu)$$

$$= [1\ /(1/s + 1/\mu)]\ (\omega/2) \sum_{i=m}^{L} B_i^m\ PP_{im1}(0,\mu)\ (-1)^{i+m}$$

$$\times\ PP_{im1}(0,s). \tag{59}$$

Thus, we have found the transform of PP_{im1} in terms of $PP_{im1}(0,\mu)$.

Replacing s with μ_{in}' in Eq. (59) and substituting it into Eq. (52), we get

$$PP_{jm1}(0,\mu) = P_j^m(\mu) + (\omega/2) \sum_{i=m}^{L} (-1)^{i+m}\ B_i^m\ PP_{im1}(0,\mu)$$

$$\times \int_0^1 [1\ /(1/\mu_{in}' + 1/\mu)]\ PP_{im1}(0,\mu_{in}')\ P_j^m(-\mu_{in}')/\mu_{in}'$$

$$\times\ d\mu_{in}'. \tag{60}$$

The above integral equation can be solved numerically by the successive approximation method. Note that Eq. (60) is the same as that of Crosbie and Dougherty [10] if their transform variable β is set equal to 0.

In the following derivation, superposition will be used to deduce the unknown function $PP_{km2}(\tau,\mu,n)$, and also $PP_{km}(\tau,\mu,n)$ in terms of $PP_{km1}(\tau,\mu)$.

Substituting Eq. (36) into Eq. (35c) gives

$$f_{km}(\tau,\mu,n) = (\omega/2) \sum_{j=m}^{L} B_j^m (-1)^m \int_0^\infty PP_{jm1}(t,\mu)\ K_{1jmk}(\tau+t,n) \times dt + (\omega/2) \sum_{j=m}^{L} B_j^m (-1)^m \int_0^\infty PP_{jm2}(t,\mu,n) \times K_{1jmk}(\tau+t,n)\ dt. \tag{61}$$

Next, substituting Eq. (61) into Eq. (35b) yields

$$PP_{km2}(\tau,\mu,n) = (\omega/2) \sum_{j=m}^{L} B_j^m (-1)^m \int_0^\infty PP_{jm1}(t,\mu)\ K_{1jmk}(\tau+t,n) \times dt + (\omega/2) \sum_{j=m}^{L} B_j^m \int_0^\infty PP_{jm2}(t,\mu,n) \left\{ (-1)^m \times K_{1jmk}(\tau+t,n) + K_{2jmk}(\tau-t) \right\} dt. \tag{62}$$

The above function has the same kernel function as $PP_{km}(\tau,\mu,n)$ in Eq. (34a). The leading function is

$$(\omega/2) \sum_{j=m}^{L} B_j^m (-1)^m \int_0^\infty PP_{jm1}(t,\mu)\ K_{1jmk}(\tau+t,n)\ dt, \tag{63a}$$

which, using the definition of $K_{1jmk}(\tau+t,n)$ in Eq. (26b), can be represented as

$$(\omega/2) \sum_{j=m}^{L} B_j^m (-1)^m \int_0^1 \overline{PP_{jm1}}(\mu_{1n}',\mu)\ \exp(-\tau/\mu_{1n}')\ \rho(\mu_{1n}',n) \times P_k^m(\mu_{1n}')\ P_j^m(-\mu_{1n}')/\mu_{1n}'\ d\mu_{1n}'. \tag{63b}$$

$\overline{PP_{jm1}}(\mu_{in}',\mu)$ in Eq. (63b) is found from Eq. (51) and called the reflection function of $PP_{jm1}(\tau,\mu)$.

Substitution of Eq. (63b) for the first term on the right side of Eq. (62), Eq. (62) may be written in long form as

$$PP_{km2}(\tau,\mu,n) = (\omega/2) \sum_{j=m}^{L} B_j^m (-1)^m \int_0^1 \overline{PP_{jm1}}(\mu_{in}',\mu) \exp(-\tau/\mu_{in}')$$

$$\times \rho(\mu_{in}',n) P_k^m(\mu_{in}') P_j^m(-\mu_{in}')/\mu_{in}' d\mu_{in}' + (\omega/2)$$

$$\times \sum_{j=m}^{L} B_j^m \int_0^\infty PP_{jm2}(t,\mu,n) \left\{ (-1)^m K_{1jmk}(\tau+t,n) \right.$$

$$\left. + K_{2jmk}(\tau-t) \right\} dt. \qquad (64)$$

Superposition of Eq. (64) with Eq. (34a) yields the following equation

$$PP_{km2}(\tau,\mu,n) = (\omega/2) \sum_{j=m}^{L} B_j^m (-1)^m \int_0^1 \overline{PP_{jm1}}(\mu_{in}',\mu) \rho(\mu_{in}',n)$$

$$\times P_j^m(-\mu_{in}') PP_{km}(\tau,\mu_{in}',n)/\mu_{in}' d\mu_{in}'. \qquad (65)$$

Replacing j by i and μ_{in}' by s, Eq. (65) becomes

$$PP_{km2}(\tau,\mu,n) = (\omega/2) \sum_{i=m}^{L} B_i^m (-1)^m \int_0^1 \overline{PP_{im1}}(s,\mu) \rho(s,n) P_i^m(-s)$$

$$\times PP_{km}(\tau,s,n)/s\, ds. \qquad (66)$$

Then, substituting Eq. (66) into Eq. (36) gives

$$PP_{km}(\tau,\mu,n) = PP_{km1}(\tau,\mu) + (\omega/2) \sum_{i=m}^{L} B_i^m (-1)^m \int_0^1 \overline{PP_{im1}}(s,\mu)$$

$$\times\ \rho(s,n)\ P_i^m(-s)\ PP_{km}(\tau,s,n)/s\ ds. \qquad (67)$$

At the top boundary of the medium where the optical thickness τ is equal to zero, Eq. (67) can be adjusted as

$$PP_{km}(0,\mu,n) = PP_{km1}(0,\mu) + (\omega/2) \sum_{i=m}^{L} B_i^m\ (-1)^m \int_0^1 \overline{PP_{im1}}(s,\mu)$$

$$\times\ \rho(s,n)\ P_i^m(-s)\ PP_{km}(0,s,n)/s\ ds. \qquad (68)$$

Modifying Eq. (60) by replacing j by k gives

$$PP_{km1}(0,\mu) = P_k^m(\mu) + (\omega/2) \sum_{i=m}^{L} (-1)^{i+m}\ B_i^m\ PP_{im1}(0,\mu)$$

$$\times \int_0^1 [1\ /(1/\mu_{in}' + 1/\mu)]\ PP_{im1}(0,\mu_{in}')\ P_k^m(-\mu_{in}')/\mu_{in}'$$

$$\times\ d\mu_{in}'. \qquad (69)$$

Substituting Eqs. (69) and (59) into Eq. (68) yields

$$PP_{km}(0,\mu,n) = P_k^m(\mu) + (\omega/2) \sum_{i=m}^{L} (-1)^{i+m}\ B_i^m\ PP_{im1}(0,\mu)$$

$$\times \int_0^1 [1\ /(1/\mu_{in}' + 1/\mu)]\ PP_{im1}(0,\mu_{in}')\ P_k^m(-\mu_{in}')/\mu_{in}'$$

$$\times\ d\mu_{in}' + (\omega/2) \sum_{i=m}^{L} (-1)^{i}\ B_i^m\ PP_{im1}(0,\mu)$$

$$\times \int_0^1 [1\ /(1/s + 1/\mu)]\ PP_{im1}(0,s)\ \rho(s,n)$$

$$\times\ PP_{km}(0,s,n)/s\ ds. \qquad (70)$$

Replacing s by μ_{in}' and i by j, Eq. (70) can be reduced to a more compact form as follows

$$PP_{km}(0,\mu,n) = P_k^m(\mu) + (\omega/2) \sum_{j=m}^{L} (-1)^j B_j^m PP_{jm1}(0,\mu)$$

$$\times \int_0^1 [1 / (1/\mu_{in}' + 1/\mu)] PP_{jm1}(0,\mu_{in}') \{ (-1)^m$$

$$\times P_k^m(-\mu_{in}') + \rho(\mu_{in}',n) PP_{km}(0,\mu_{in}',n) \} / \mu_{in}'$$

$$\times d\mu_{in}'. \qquad (71)$$

Eq. (71) can be solved numerically by the successive approximation method, once we get the exact solution for $PP_{jm1}(0,\mu)$ from Eq. (60).

Reflection Function

The reflection function is used to determine the intensity. Now, the reflection function of $PP_{jm}(\tau,\mu,n)$ can be obtained by the superposition method. Using the definition of K_{1jmk} from Eq. (26b), the lead function $f_{km}(\tau,\mu,n)$ in Eq. (35b) can be represented as

$$f_{km}(\tau,\mu,n) = (\omega/2) \sum_{j=m}^{L} B_j^m (-1)^m \int_0^1 \overline{PP_{jm}}(\mu_{in}',\mu,n) \exp(-\tau/\mu_{in}')$$

$$\times \rho(\mu_{in}',n) P_k^m(\mu_{in}') P_j^m(-\mu_{in}')/\mu_{in}' d\mu_{in}', \qquad (72)$$

where $\overline{PP_{jm}}(\mu_{in}',\mu,n) = \int_0^\infty PP_{jm}(t,\mu,n) \exp(-t/\mu_{in}') dt \qquad (73)$

is the reflection function of $PP_{jm}(\tau,\mu,n)$.

Substituting Eq. (72) into Eq. (35b), and superposing with Eq. (35a) yields the following $PP_{km2}(\tau,\mu,n)$ function

$$PP_{km2}(\tau,\mu,n) = (\omega/2) \sum_{j=m}^{L} B_j^m (-1)^m \int_0^1 \overline{PP_{jm}}(\mu_{in}',\mu,n) \rho(\mu_{in}',n)$$

$$\times P_j^m(-\mu_{1n}') \; PP_{km1}(\tau,\mu_{1n}')/\mu_{1n}' \; d\mu_{1n}'. \tag{74}$$

Substituting Eq. (74) into Eq. (36) gives

$$PP_{km}(\tau,\mu,n) = PP_{km1}(\tau,\mu) + (\omega/2) \sum_{j=m}^{L} B_j^m \; (-1)^m$$

$$\times \int_0^1 \overline{PP_{jm}}(\mu_{1n}',\mu,n) \; \rho(\mu_{1n}',n) \; P_j^m(-\mu_{1n}') \; PP_{km1}(\tau,\mu_{1n}')$$

$$/\mu_{1n}' \; d\mu_{1n}'. \tag{75}$$

Taking the transform of Eq. (75) by multiplying $\exp(-\tau/\overline{\mu})$ on both sides and integrating from zero to infinity with respect to τ, and using Eqs. (51) and (73) to simplify yields

$$\overline{PP_{km}}(\overline{\mu},\mu,n) = \overline{PP_{km1}}(\overline{\mu},\mu) + (\omega/2) \sum_{j=m}^{L} B_j^m \; (-1)^m$$

$$\times \int_0^1 \overline{PP_{km1}}(\overline{\mu},\mu_{1n}') \; \overline{PP_{jm}}(\mu_{1n}',\mu,n) \; \rho(\mu_{1n}',n) \; P_j^m(-\mu_{1n}')$$

$$/\mu_{1n}' \; d\mu_{1n}'. \tag{76}$$

Multiplying Eq. (76) by $(\omega/2) \; B_k^m \; P_k^m(-\overline{\mu})$, and summing k from m to L gives

$$(\omega/2) \sum_{k=m}^{L} B_k^m \; P_k^m(-\overline{\mu}) \; \overline{PP_{km}}(\overline{\mu},\mu,n)$$

$$= (\omega/2) \sum_{k=m}^{L} B_k^m \; P_k^m(-\overline{\mu}) \; \overline{PP_{km1}}(\overline{\mu},\mu) + (\omega/2) \sum_{j=m}^{L} B_j^m \; (-1)^m$$

$$\times \int_0^1 \left\{ (\omega/2) \sum_{k=m}^{L} B_k^m \; P_k^m(-\overline{\mu}) \; \overline{PP_{km1}}(\overline{\mu},\mu_{1n}') \right\} \overline{PP_{jm}}(\mu_{1n}',\mu,n)$$

$$\times \rho(\mu_{1n}',n) \; P_j^m(-\mu_{1n}')/\mu_{1n}' \; d\mu_{1n}'. \tag{77}$$

Then, substituting Eq. (59) into Eq. (77), we get

$$(\omega/2) \sum_{k=m}^{L} B_k^m \, P_k^m(-\bar{\mu}) \, \overline{PP_{km}}(\bar{\mu},\mu,n)$$

$$= [1 \,/(1/\bar{\mu} + 1/\mu)] \, (\omega/2) \sum_{k=m}^{L} B_k^m \, PP_{km1}(0,\mu) \, (-1)^{k+m} \, PP_{km1}(0,\bar{\mu})$$

$$+ (\omega/2) \sum_{j=m}^{L} B_j^m \, (\omega/2) \sum_{k=m}^{L} B_k^m \, (-1)^k \, PP_{km1}(0,\bar{\mu})$$

$$\times \int_0^1 [1 \,/(1/\bar{\mu} + 1/\mu_{1n}')] \, PP_{km1}(0,\mu_{1n}') \, \overline{PP_{jm}}(\mu_{1n}',\mu,n) \, \rho(\mu_{1n}',n)$$

$$\times P_j^m(-\mu_{1n}')/\mu_{1n}' \, d\mu_{1n}'. \qquad (78)$$

Let $Rpp_m(a,\mu,n) = (\omega/2) \sum_{k=m}^{L} B_k^m \, P_k^m(-a) \, \overline{PP_{km}}(a,\mu,n)$, (79)

then Eq. (78) can be written as

$$Rpp_m(\bar{\mu},\mu,n) = [1 \,/(1/\bar{\mu} + 1/\mu)] \, (\omega/2) \sum_{k=m}^{L} B_k^m \, PP_{km1}(0,\mu)$$

$$\times (-1)^{k+m} \, PP_{km1}(0,\bar{\mu}) + (\omega/2) \sum_{k=m}^{L} B_k^m \, (-1)^k$$

$$\times PP_{km1}(0,\bar{\mu}) \int_0^1 [1 \,/(1/\bar{\mu} + 1/\mu_{1n}')] \, PP_{km1}(0,\mu_{1n}')$$

$$\times Rpp_m(\mu_{1n}',\mu,n) \, \rho(\mu_{1n}',n)/\mu_{1n}' \, d\mu_{1n}'. \qquad (80)$$

Equation (80) can be solved numerically by successive approximation also, once PP_{km1} is known. Note that if L = 0, Eq. (80) reduces to that of Jiang [8].

Reflected Intensity

The reflected intensity just inside the upper boundary of the medium is given by Eq. (27a). Substituting Eqs. (29), (34b), (73), (79), (26d), and (13) into Eq. (27a) yields the following

$$\bar{I}_{Ain}(0,\mu_{in},\mu_o,\phi_{in},n) = (I_o\ \mu_o)/(2\pi\ \mu_{in})\ [1-\rho(\mu_o,1/n)] \\ /\ [1-(1-\mu_o^2)/n^2]^{1/2} \sum_{m=0}^{L}(2-\delta_{0m})\ \cos(m\phi_{in}) \\ \times\ Rpp_m\Big\{\ \mu_{in},[1-(1-\mu_o^2)/n^2]^{1/2},n\ \Big\}, \qquad (81)$$

where the reflection function Rpp_m is defined as Eq. (79).

The subscript "in" and "A" in Eq. (81) refer to quantities inside the medium and the collimated boundary condition, respectively. An energy balance must be performed across the interface in order to determine the value of the reflected intensity just outside the upper boundary [8]. If the reflected intensity just outside the upper boundary of the medium is denoted as $\bar{I}_{Ae}$, then we have [8]

$$\bar{I}_{Ae}(0,\mu_e,\mu_o,\phi_{in},n) \\ = (I_o\ \mu_o)/2\pi\ [1-\rho(\mu_o,1/n)]\ \Big\{\ 1\ -\ \rho\{[1-(1-\mu_e^2)/n^2]^{1/2},n\}\ \Big\} \\ \times \sum_{m=0}^{L}(2-\delta_{0m})\ \cos(m\phi_{in})\ Rpp_m\Big\{\ [1-(1-\mu_e^2)/n^2]^{1/2}, \\ [1-(1-\mu_o^2)/n^2]^{1/2},n\ \Big\}\ /\ \Big\{\ n^2\ [1-(1-\mu_o^2)/n^2]^{1/2} \\ \times\ [1-(1-\mu_e^2)/n^2]^{1/2}\ \Big\}, \qquad (82)$$

where μ_e and μ_{in} are related by Snell's Law from Eq. (13).

Therefore, for collimated incident intensity, Eqs. (60) and (80) can be used to determine the intensity reflected from the medium at the upper boundary.

Flux

The flux can be easily obtained from Eq. (32) as mentioned in the previous chapter.

CHAPTER IV

SOLUTION OF THE FINITE PROBLEM

The procedure in this chapter is very similar to that in Chapter III. Instead of solving only at the top boundary as for the semi-infinite problem, both top and bottom boundaries need to be solved for finite problem.

Source Function

For finite optical thickness, the general source function equation, Eq. (26a), is

$$P_{km}(\tau,\mu,n;\tau_0) = L(\mu,n)\ P_k^m(\mu)\ \exp(-\tau/\mu) + (\omega/2) \sum_{j=m}^{L} B_j^m$$

$$\times \int_0^{\tau_0} P_{jm}(t,\mu,n;\tau_0) \left\{ (-1)^m K_{1jmk}(\tau+t,n) \right.$$

$$\left. + K_{2jmk}(\tau-t) \right\} dt, \qquad (26a)$$

where $K_{1jmk}(\tau+t,n)$ is defined as Eq. (26b), $K_{2jmk}(\tau-t)$ is defined as Eq. (26c), and $L(\mu,n)$ is defined as Eq. (26d).

Defining fundamental source function PP_{km} as

$$PP_{km}(\tau,\mu,n;\tau_0) = P_k^m(\mu)\ \exp(-\tau/\mu) + (\omega/2) \sum_{j=m}^{L} B_j^m$$

$$\times \int_0^{\tau_0} PP_{jm}(t,\mu,n;\tau_0) \left\{ (-1)^m K_{1jmk}(\tau+t,n) \right.$$

$$+ K_{2jmk}(\tau-t) \Big\} dt, \tag{83}$$

and superposing Eq. (83) with Eq. (26a) yields

$$P_{km}(\tau,\mu,n;\tau_0) = L(\mu,n)\; PP_{km}(\tau,\mu,n;\tau_0). \tag{84}$$

Equation (83) can be rewritten as

$$PP_{km}(\tau,\mu,n;\tau_0) = P_k^m(\mu)\; \exp(-\tau/\mu) + (\omega/2) \sum_{j=m}^{L} B_j^m\; (-1)^m$$
$$\times \int_0^{\tau_0} PP_{jm}(t,\mu,n;\tau_0)\; K_{1jmk}(\tau+t,n)\; dt + (\omega/2)$$
$$\times \sum_{j=m}^{L} B_j^m \int_0^{\tau_0} PP_{jm}(t,\mu,n;\tau_0)\; K_{2jmk}(\tau-t)\; dt. \tag{85}$$

Following Jiang [8], let us define

$$PP_{km1}(\tau,\mu;\tau_0) = P_k^m(\mu)\; \exp(-\tau/\mu) + (\omega/2) \sum_{j=m}^{L} B_j^m$$
$$\times \int_0^{\tau_0} PP_{jm1}(t,\mu;\tau_0)\; K_{2jmk}(\tau-t)\; dt, \tag{86}$$

and

$$PP_{km2}(\tau,\mu,n;\tau_0) = f_{km}(\tau,\mu,n;\tau_0) + (\omega/2) \sum_{j=m}^{L} B_j^m$$
$$\times \int_0^{\tau_0} PP_{jm2}(t,\mu,n;\tau_0)\; K_{2jmk}(\tau-t)\; dt, \tag{87}$$

where $f_{km}(\tau,\mu,n;\tau_0) = (\omega/2) \sum_{j=m}^{L} B_j^m\; (-1)^m \int_0^{\tau_0} PP_{jm}(t,\mu,n;\tau_0)$

$$\times K_{1jmk}(\tau+t,n)\; dt, \tag{88}$$

and $PP_{km1}(\tau,\mu;\tau_0)$ is not a function of refractive index n. Then, adding Eq. (86) to Eq. (87), and comparing with Eq.

(85), we have the following equation

$$PP_{km}(\tau,\mu,n;\tau_0) = PP_{km1}(\tau,\mu;\tau_0) + PP_{km2}(\tau,\mu,n;\tau_0), \quad (89)$$

where $PP_{km1}(\tau,\mu;\tau_0)$ is the fundamental source function in the finite medium with no reflective boundaries, i.e., unit refractive index.

Ambarzumian's approach will be used to derive the fundamental source function in the following derivation.

Equation (86) can be written in expanded form as

$$\begin{aligned} PP_{km1}(\tau,\mu;\tau_0) &= P_k^m(\mu)\ \exp(-\tau/\mu) + (\omega/2) \sum_{j=m}^{L} B_j^m \\ &\times \int_0^{\tau} PP_{jm1}(t,\mu;\tau_0)\ K_{2jmk}(\tau-t)\ dt + (\omega/2) \sum_{j=m}^{L} B_j^m \\ &\times \int_{\tau}^{\tau_0} PP_{jm1}(t,\mu;\tau_0)\ K_{2jmk}(\tau-t)\ dt. \qquad (90) \end{aligned}$$

Using the substitution $\bar{t} = \tau-t$ in the first integral and $\bar{t} = t-\tau$ in the second integral, Eq. (90) becomes

$$\begin{aligned} PP_{km1}(\tau,\mu;\tau_0) &= P_k^m(\mu)\ \exp(-\tau/\mu) + (\omega/2) \sum_{j=m}^{L} B_j^m \\ &\times \int_0^{\tau} PP_{jm1}(\tau-\bar{t},\mu;\tau_0)\ K_{2jmk}(\bar{t})\ d\bar{t} + (\omega/2) \sum_{j=m}^{L} B_j^m \\ &\times \int_0^{\tau_0-\tau} PP_{jm1}(\tau+\bar{t},\mu;\tau_0)\ K_{2jmk}(-\bar{t})\ d\bar{t}. \qquad (91) \end{aligned}$$

Using Leibnitz rule to take the derivative of Eq. (91) with respect to τ yields

$$\frac{\partial\ PP_{km1}(\tau,\mu;\tau_0)}{\partial\ \tau} = -(1/\mu)\ P_k^m(\mu)\ \exp(-\tau/\mu) + (\omega/2) \sum_{j=m}^{L} B_j^m$$

$$\times\ PP_{jm1}(0,\mu;\tau_0)\ K_{2jmk}(\tau) - (\omega/2) \sum_{j=m}^{L} B_j^m$$

$$\times\ PP_{jm1}(\tau_0,\mu;\tau_0)\ K_{2jmk}(\tau-\tau_0) + (\omega/2) \sum_{j=m}^{L} B_j^m$$

$$\times \int_0^{\tau} \frac{\partial\ PP_{jm1}(\tau-\bar{t},\mu;\tau_0)}{\partial\ \tau}\ K_{2jmk}(\bar{t})\ d\bar{t} + (\omega/2)$$

$$\times \sum_{j=m}^{L} B_j^m \int_0^{\tau_0-\tau} \frac{\partial\ PP_{jm1}(\tau+\bar{t},\mu;\tau_0)}{\partial\ \tau}\ K_{2jmk}(-\bar{t})$$

$$\times\ d\bar{t}. \qquad (92)$$

Using the substitution $t = \tau-\bar{t}$ in the first integral and $t = \bar{t}+\tau$ in the second integral, Eq. (92) may be written as

$$\frac{\partial\ PP_{km1}(\tau,\mu;\tau_0)}{\partial\ \tau} = -(1/\mu)\ P_k^m(\mu)\ \exp(-\tau/\mu) + (\omega/2) \sum_{j=m}^{L} B_j^m$$

$$\times\ PP_{jm1}(0,\mu;\tau_0)\ K_{2jmk}(\tau) - (\omega/2) \sum_{j=m}^{L} B_j^m$$

$$\times\ PP_{jm1}(\tau_0,\mu;\tau_0)\ K_{2jmk}(\tau-\tau_0) + (\omega/2) \sum_{j=m}^{L} B_j^m$$

$$\times \int_0^{\tau} \frac{\partial\ PP_{jm1}(t,\mu;\tau_0)}{\partial\ t}\ K_{2jmk}(\tau-t)\ dt + (\omega/2)$$

$$\times \sum_{j=m}^{L} B_j^m \int_{\tau}^{\tau_0} \frac{\partial\ PP_{jm1}(t,\mu;\tau_0)}{\partial\ t}\ K_{2jmk}(\tau-t)\ dt,$$

or

$$\frac{\partial\ PP_{km1}(\tau,\mu;\tau_0)}{\partial\ \tau} = -(1/\mu)\ P_k^m(\mu)\ \exp(-\tau/\mu)\ +\ (\omega/2)\ \sum_{j=m}^{L} B_j^m$$

$$\times\ PP_{jm1}(0,\mu;\tau_0)\ K_{2jmk}(\tau)\ -\ (\omega/2)\ \sum_{j=m}^{L} B_j^m$$

$$\times\ PP_{jm1}(\tau_0,\mu;\tau_0)\ K_{2jmk}(\tau-\tau_0)\ +\ (\omega/2)\ \sum_{j=m}^{L} B_j^m$$

$$\times \int_0^{\tau_0} \frac{\partial\ PP_{jm1}(t,\mu;\tau_0)}{\partial\ t}\ K_{2jmk}(\tau-t)\ dt. \qquad (93)$$

The solution of Eq. (93) can be found by the method of superposition. Start by replacing μ with μ_{in}' and j with i in Eq. (86). Then we get

$$PP_{km1}(\tau,\mu_{in}';\tau_0)\ =\ P_k^m(\mu_{in}')\ \exp(-\tau/\mu_{in}')\ +\ (\omega/2)\ \sum_{i=m}^{L} B_i^m$$

$$\times \int_0^{\tau_0} PP_{im1}(t,\mu_{in}';\tau_0)\ K_{2imk}(\tau-t)\ dt. \qquad (94)$$

Multiplying Eq. (94) by $(\omega/2)\ B_j^m\ PP_{jm1}(0,\mu;\tau_0)\ P_j^m(\mu_{in}')$ $/\mu_{in}'$, integrating from zero to one with respect to μ_{in}', and summing from j = m to L, we obtain

$$(\omega/2)\ \sum_{j=m}^{L} B_j^m\ PP_{jm1}(0,\mu;\tau_0)\ \int_0^1 PP_{km1}(\tau,\mu_{in}';\tau_0)\ P_j^m(\mu_{in}')/\mu_{in}'$$

$$\times\ d\mu_{in}'\ =\ (\omega/2)\ \sum_{j=m}^{L} B_j^m\ PP_{jm1}(0,\mu;\tau_0)\ \int_0^1 \exp(-\tau/\mu_{in}')\ P_j^m(\mu_{in}')$$

$$\times\ P_k^m(\mu_{in}')/\mu_{in}'\ d\mu_{in}'\ +\ (\omega/2)\ \sum_{i=m}^{L} B_i^m \int_0^{\tau_0} \Big\{\ (\omega/2)\ \sum_{j=m}^{L} B_j^m$$

$$\times\ PP_{jm1}(0,\mu;\tau_0)\ \int_0^1 PP_{im1}(t,\mu_{in}';\tau_0)\ P_j^m(\mu_{in}')/\mu_{in}'\ d\mu_{in}'\ \Big\}$$

$$\times\ K_{2imk}(\tau-t)\ dt,$$

or using Eq. (26c), for K_{2jmk}

$$(\omega/2) \sum_{j=m}^{L} B_j^m \, PP_{jm1}(0,\mu;\tau_0) \int_0^1 PP_{km1}(\tau,\mu_{in}';\tau_0) \; P_j^m(\mu_{in}')/\mu_{in}'$$

$$\times \, d\mu_{in}' = (\omega/2) \sum_{j=m}^{L} B_j^m \, PP_{jm1}(0,\mu;\tau_0) \; K_{2jmk}(\tau) + (\omega/2) \sum_{i=m}^{L} B_i^m$$

$$\times \int_0^{\tau_0} \Big\{ (\omega/2) \sum_{j=m}^{L} B_j^m \, PP_{jm1}(0,\mu;\tau_0) \int_0^1 PP_{im1}(t,\mu_{in}';\tau_0)$$

$$\times \, P_j^m(\mu_{in}')/\mu_{in}' \, d\mu_{in}' \Big\} K_{2imk}(\tau-t) \, dt. \qquad (95)$$

Replacing τ by $\tau_0-\tau$ and t by τ_0-t, Eq. (94) is modified as

$$PP_{km1}(\tau_0-\tau,\mu_{in}';\tau_0) = P_k^m(\mu_{in}') \exp[(\tau-\tau_0)/\mu_{in}'] + (\omega/2)$$

$$\times \sum_{i=m}^{L} (-1)^{i+k} B_i^m \int_0^{\tau_0} PP_{im1}(\tau_0-t,\mu_{in}';\tau_0)$$

$$\times \, K_{2imk}(\tau-t) \, dt. \qquad (96)$$

Multiplying Eq. (96) by $-(-1)^{j+k}(\omega/2) \, B_j^m \, PP_{jm1}(\tau_0,\mu;\tau_0)$ $P_j^m(\mu_{in}')/\mu_{in}'$, integrating from zero to one with respect to μ_{in}', and summing from j = m to L, gives

$$-(\omega/2) \sum_{j=m}^{L} (-1)^{j+k} B_j^m \, PP_{jm1}(\tau_0,\mu;\tau_0) \int_0^1 PP_{km1}(\tau_0-\tau,\mu_{in}';\tau_0)$$

$$\times \, P_j^m(\mu_{in}')/\mu_{in}' \, du_{in}' = -(\omega/2) \sum_{j=m}^{L} B_j^m \, PP_{jm1}(\tau_0,\mu;\tau_0) \; K_{2jmk}(\tau-\tau_0)$$

$$+ (\omega/2) \sum_{i=m}^{L} B_i^m \int_0^{\tau_0} \Big\{ -(\omega/2) \sum_{j=m}^{L} (-1)^{j+i} B_j^m$$

$$\times \, PP_{jm1}(\tau_0,\mu;\tau_0) \int_0^1 PP_{im1}(\tau_0-t,\mu_{in}';\tau_0)$$

$$\times\ P_j^m(\mu_{in}')/\mu_{in}'\ du_{in}' \Big\}\ K_{2imk}(\tau-t)\ dt. \qquad (97)$$

Finally, replacing i by j and j by i in the second term of the right hand sides for both Eqs. (95) and (97), multiplying Eq. (86) by $-(1/\mu)$, and adding all of these three equations together, we have

$$-(1/\mu)\ PP_{km1}(\tau,\mu;\tau_0) + (\omega/2) \sum_{j=m}^{L} B_j^m\ PP_{jm1}(0,\mu;\tau_0)$$

$$\times \int_0^1 PP_{km1}(\tau,\mu_{in}';\tau_0)\ P_j^m(\mu_{in}')/\mu_{in}'\ d\mu_{in}' - (\omega/2) \sum_{j=m}^{L} (-1)^{j+k}\ B_j^m$$

$$\times\ PP_{jm1}(\tau_0,\mu;\tau_0) \int_0^1 PP_{km1}(\tau_0-\tau,\mu_{in}';\tau_0)\ P_j^m(\mu_{in}')/\mu_{in}'\ d\mu_{in}'$$

$$= -(1/\mu)\ P_k^m(\mu)\ \exp(-\tau/\mu) + (\omega/2) \sum_{j=m}^{L} B_j^m\ PP_{jm1}(0,\mu;\tau_0)$$

$$\times\ K_{2jmk}(\tau) - (\omega/2) \sum_{j=m}^{L} B_j^m\ PP_{jm1}(\tau_0,\mu;\tau_0)\ K_{2jmk}(\tau-\tau_0) + (\omega/2)$$

$$\times \sum_{j=m}^{L} B_j^m \int_0^{\tau_0} \Big\{ -(1/\mu)\ PP_{jm1}(t,\mu;\tau_0) + (\omega/2) \sum_{i=m}^{L} B_i^m$$

$$\times\ PP_{im1}(0,\mu;\tau_0) \int_0^1 PP_{jm1}(t,\mu_{in}';\tau_0)\ P_i^m(\mu_{in}')/\mu_{in}'\ d\mu_{in}' - (\omega/2)$$

$$\times \sum_{i=m}^{L} (-1)^{j+i}\ B_i^m\ PP_{im1}(\tau_0,\mu;\tau_0) \int_0^1 PP_{jm1}(\tau_0-t,\mu_{in}';\tau_0)$$

$$\times\ P_i^m(\mu_{in}')/\mu_{in}'\ d\mu_{in}' \Big\}\ K_{2jmk}(\tau-t)\ dt. \qquad (98)$$

Now, comparing Eq. (93) with Eq. (98), the solution of Eq. (93) by superposition is

$$\frac{\partial\ PP_{km1}(\tau,\mu;\tau_0)}{\partial\ \tau} = -(1/\mu)\ PP_{km1}(\tau,\mu;\tau_0) + (\omega/2) \sum_{j=m}^{L} B_j^m$$

$$\times\ PP_{jm1}(0,\mu;\tau_0)\ \int_0^1 PP_{km1}(\tau,\mu_{in}';\tau_0)\ P_j^m(\mu_{in}')$$

$$/\ \mu_{in}'\ d\mu_{in}' - (\omega/2) \sum_{j=m}^{L} (-1)^{j+k}\ B_j^m$$

$$\times\ PP_{jm1}(\tau_0,\mu;\tau_0)\ \int_0^1 PP_{km1}(\tau_0-\tau,\mu_{in}';\tau_0)$$

$$\times\ P_j^m(\mu_{in}')/\mu_{in}'\ d\mu_{in}'. \tag{99}$$

Replacing μ_{in}' by μ and i by j in Eq. (96), and expressing it in expanded form as

$$PP_{km1}(\tau_0-\tau,\mu;\tau_0) = P_k^m(\mu)\ \exp[(\tau-\tau_0)/\mu] + (\omega/2) \sum_{j=m}^{L} (-1)^{j+k}\ B_j^m$$

$$\times \int_0^\tau PP_{jm1}(\tau_0-t,\mu;\tau_0)\ K_{2jmk}(\tau-t)\ dt + (\omega/2)$$

$$\times \sum_{j=m}^{L} (-1)^{j+k}\ B_j^m \int_\tau^{\tau_0} PP_{jm1}(\tau_0-t,\mu;\tau_0)$$

$$\times\ K_{2jmk}(\tau-t)\ dt. \tag{100}$$

Using the substitution $\bar{t} = \tau-t$ in the first integral and $\bar{t} = t-\tau$ in the second integral, Eq. (100) becomes

$$PP_{km1}(\tau_0-\tau,\mu;\tau_0) = P_k^m(\mu)\ \exp[(\tau-\tau_0)/\mu] + (\omega/2) \sum_{j=m}^{L} (-1)^{j+k}\ B_j^m$$

$$\times \int_0^\tau PP_{jm1}(\tau_0-\tau+\bar{t},\mu;\tau_0)\ K_{2jmk}(\bar{t})\ d\bar{t} + (\omega/2)$$

$$\times \sum_{j=m}^{L} (-1)^{j+k}\ B_j^m \int_0^{\tau_0-\tau} PP_{jm1}(\tau_0-\tau-\bar{t},\mu;\tau_0)$$

$$\times\ K_{2jmk}(-\bar{t})\ d\bar{t}. \tag{101}$$

Using Leibnitz rule to take the derivative of Eq. (101) with respect to τ yields

$$\frac{\partial\ PP_{km1}(\tau_0-\tau,\mu;\tau_0)}{\partial\ \tau} = (1/\mu)\ P_k^m(\mu)\ \exp[(\tau-\tau_0)/\mu] + (\omega/2)$$

$$\times \sum_{j=m}^{L} (-1)^{j+k}\ B_j^m\ PP_{jm1}(\tau_0,\mu;\tau_0)\ K_{2jmk}(\tau)$$

$$- (\omega/2) \sum_{j=m}^{L} (-1)^{j+k}\ B_j^m\ PP_{jm1}(0,\mu;\tau_0)$$

$$\times K_{2jmk}(\tau-\tau_0) + (\omega/2) \sum_{j=m}^{L} (-1)^{j+k}\ B_j^m$$

$$\times \int_0^{\tau} \frac{\partial\ PP_{jm1}(\tau_0-\tau+\bar{t},\mu;\tau_0)}{\partial\ \tau}\ K_{2jmk}(\bar{t})\ d\bar{t}$$

$$+ (\omega/2) \sum_{j=m}^{L} (-1)^{j+k}\ B_j^m$$

$$\times \int_0^{\tau_0-\tau} \frac{\partial\ PP_{jm1}(\tau_0-\tau-\bar{t},\mu;\tau_0)}{\partial\ \tau}$$

$$\times K_{2jmk}(-\bar{t})\ d\bar{t}. \qquad (102)$$

Using the substitution $t = \tau-\bar{t}$ in the first integral and $t = \bar{t}+\tau$ in the second integral, Eq. (102) may be written as

$$\frac{\partial\ PP_{km1}(\tau_0-\tau,\mu;\tau_0)}{\partial\ \tau} = (1/\mu)\ P_k^m(\mu)\ \exp[(\tau-\tau_0)/\mu] + (\omega/2)$$

$$\times \sum_{j=m}^{L} (-1)^{j+k}\ B_j^m\ PP_{jm1}(\tau_0,\mu;\tau_0)\ K_{2jmk}(\tau)$$

$$- (\omega/2) \sum_{j=m}^{L} (-1)^{j+k}\ B_j^m\ PP_{jm1}(0,\mu;\tau_0)$$

$$\times K_{2jmk}(\tau-\tau_0) + (\omega/2) \sum_{j=m}^{L} (-1)^{j+k}\ B_j^m$$

$$\times \int_0^{\tau} \frac{\partial\ PP_{jm1}(\tau_0-t,\mu;\tau_0)}{\partial\ t}\ K_{2jmk}(\tau-t)\ dt$$

$$+\ (\omega/2) \sum_{j=m}^{L} (-1)^{j+k}\ B_j^m$$

$$\times \int_{\tau}^{\tau_0} \frac{\partial\ PP_{jm1}(\tau_0-t,\mu;\tau_0)}{\partial\ t}\ K_{2jmk}(\tau-t)\ dt,$$

or

$$\frac{\partial\ PP_{km1}(\tau_0-\tau,\mu;\tau_0)}{\partial\ \tau} = (1/\mu)\ P_k^m(\mu)\ \exp[(\tau-\tau_0)/\mu] + (\omega/2)$$

$$\times \sum_{j=m}^{L} (-1)^{j+k}\ B_j^m\ PP_{jm1}(\tau_0,\mu;\tau_0)\ K_{2jmk}(\tau)$$

$$-\ (\omega/2) \sum_{j=m}^{L} (-1)^{j+k}\ B_j^m\ PP_{jm1}(0,\mu;\tau_0)$$

$$\times K_{2jmk}(\tau-\tau_0)\ +\ (\omega/2) \sum_{j=m}^{L} B_j^m$$

$$\times \int_0^{\tau_0} \frac{\partial\ PP_{jm1}(\tau_0-t,\mu;\tau_0)}{\partial\ t}\ K_{2jmk}(t-\tau)$$

$$\times\ dt. \qquad (103)$$

The solution of Eq. (103) can be found by the method of superposition. First, multiplying Eq. (94) by $(\omega/2)\ (-1)^{j+k}$ $\times\ B_j^m\ PP_{jm1}(\tau_0,\mu;\tau_0)\ P_j^m(\mu_{1n}')/\mu_{1n}'$, integrating from zero to one with respect to μ_{1n}', and summing from j = m to L, we obtain

$$(\omega/2) \sum_{j=m}^{L} (-1)^{j+k}\ B_j^m\ PP_{jm1}(\tau_0,\mu;\tau_0) \int_0^1 PP_{km1}(\tau,\mu_{1n}';\tau_0)$$

$$\times\ P_j^m(\mu_{1n}')/\mu_{1n}'\ d\mu_{1n}' = (\omega/2) \sum_{j=m}^{L} (-1)^{j+k}\ B_j^m\ PP_{jm1}(\tau_0,\mu;\tau_0)$$

$$\times\ K_{2jmk}(\tau)\ +\ (\omega/2)\ \sum_{i=m}^{L} B_i^m \int_0^{\tau_0} \Big\{ (\omega/2)$$

$$\times \sum_{j=m}^{L} (-1)^{j+i}\ B_j^m\ PP_{jm1}(\tau_0,\mu;\tau_0)$$

$$\times \int_0^1 PP_{im1}(t,\mu_{in}';\tau_0)\ P_j^m(\mu_{in}')/\mu_{in}'\ d\mu_{in}' \Big\}$$

$$\times\ K_{2imk}(t-\tau)\ dt. \qquad (104)$$

Then, multiplying Eq. (96) by $-(\omega/2)(-1)^{j+k}\ B_j^m\ PP_{jm1}(0,\mu;\tau_0)$ $(-1)^{j+k}\ P_j^m(\mu_{in}')/\mu_{in}'$, integrating from zero to one with respect to μ_{in}', and summing from j = m to L, gives

$$-(\omega/2)\ \sum_{j=m}^{L} B_j^m\ PP_{jm1}(0,\mu;\tau_0)\ \int_0^1 PP_{km1}(\tau_0-\tau,\mu_{in}';\tau_0)\ P_j^m(\mu_{in}')/\mu_{in}'$$

$$\times\ du_{in}'\ =\ -(\omega/2)\ \sum_{j=m}^{L} (-1)^{j+k}\ B_j^m\ PP_{jm1}(0,\mu;\tau_0)\ K_{2jmk}(\tau-\tau_0)$$

$$+\ (\omega/2)\ \sum_{i=m}^{L} B_i^m \int_0^{\tau_0} \Big\{ -(\omega/2)\ \sum_{j=m}^{L} B_j^m\ PP_{jm1}(0,\mu;\tau_0)$$

$$\times \int_0^1 PP_{im1}(\tau_0-t,\mu_{in}';\tau_0)\ P_j^m(\mu_{in}')/\mu_{in}'\ du_{in}' \Big\}$$

$$\times\ K_{2imk}(t-\tau)\ dt. \qquad (105)$$

Finally, replacing i by j and j by i in the second term of the right hand side for both Eqs. (104) and (105), multiplying Eq. (100) by $(1/\mu)$, and adding all three of these equations together, we have

$$(1/\mu)\ PP_{km1}(\tau_0-\tau,\mu;\tau_0)\ +\ (\omega/2)\ \sum_{j=m}^{L} (-1)^{j+k}\ B_j^m\ PP_{jm1}(\tau_0,\mu;\tau_0)$$

$$\times \int_0^1 PP_{km1}(\tau,\mu_{1n}';\tau_0)\ P_j^m(\mu_{1n}')/\mu_{1n}'\ d\mu_{1n}' - (\omega/2) \sum_{j=m}^{L} B_j^m$$

$$\times PP_{jm1}(0,\mu;\tau_0) \int_0^1 PP_{km1}(\tau_0-\tau,\mu_{1n}';\tau_0)\ P_j^m(\mu_{1n}')/\mu_{1n}'\ d\mu_{1n}'$$

$$= (1/\mu)\ P_k^m(\mu)\ \exp[(\tau-\tau_0)/\mu] + (\omega/2) \sum_{j=m}^{L} (-1)^{j+k}\ B_j^m$$

$$\times PP_{jm1}(\tau_0,\mu;\tau_0)\ K_{2jmk}(\tau) - (\omega/2) \sum_{j=m}^{L} (-1)^{j+k}\ B_j^m$$

$$\times PP_{jm1}(0,\mu;\tau_0)\ K_{2jmk}(\tau-\tau_0) + (\omega/2) \sum_{j=m}^{L} B_j^m \int_0^{\tau_0} \Big\{ (1/\mu)$$

$$\times PP_{jm1}(\tau_0-t,\mu;\tau_0) + (\omega/2) \sum_{i=m}^{L} (-1)^{j+i}\ B_i^m\ PP_{im1}(\tau_0,\mu;\tau_0)$$

$$\times \int_0^1 PP_{jm1}(t,\mu_{1n}';\tau_0)\ P_i^m(\mu_{1n}')/\mu_{1n}'\ d\mu_{1n}' - (\omega/2) \sum_{i=m}^{L} B_i^m$$

$$\times PP_{im1}(0,\mu;\tau_0) \int_0^1 PP_{jm1}(\tau_0-t,\mu_{1n}';\tau_0)\ P_i^m(\mu_{1n}')/\mu_{1n}'\ d\mu_{1n}' \Big\}$$

$$\times K_{2jmk}(t-\tau)\ dt. \qquad (106)$$

Now, comparing Eq. (103) with Eq. (106), the solution of Eq. (103) by superposition is

$$\frac{\partial\ PP_{km1}(\tau_0-\tau,\mu;\tau_0)}{\partial\ \tau} = (1/\mu)\ PP_{km1}(\tau_0-\tau,\mu;\tau_0) + (\omega/2)$$

$$\times \sum_{j=m}^{L} (-1)^{j+k}\ B_j^m\ PP_{jm1}(\tau_0,\mu;\tau_0)$$

$$\times \int_0^1 PP_{km1}(\tau,\mu_{1n}';\tau_0)\ P_j^m(\mu_{1n}')/\mu_{1n}'\ d\mu_{1n}'$$

$$- (\omega/2) \sum_{j=m}^{L} B_j^m\ PP_{jm1}(0,\mu;\tau_0)$$

$$\times \int_0^1 PP_{km1}(\tau_0-\tau,\mu_{1n}';\tau_0)\ P_j^m(\mu_{1n}')/\mu_{1n}'$$

$$\times \, d\mu_{in}' . \qquad (107)$$

In order to solve the integro-differential equations, Eqs. (99) and (107), we need to solve for $PP_{jm1}(0,\mu;\tau_0)$ and $PP_{jm1}(\tau_0,\mu;\tau_0)$ first.

By setting $\tau = 0$, replacing j by i, and replacing k by j in Eq. (86), we have the following expression for $PP_{jm1}(0,\mu;\tau_0)$

$$PP_{jm1}(0,\mu;\tau_0) = P_j^m(\mu) + (\omega/2) \sum_{i=m}^{L} B_i^m \int_0^{\tau_0} PP_{im1}(t,\mu;\tau_0) \times K_{2imj}(-t) \, dt. \qquad (108)$$

By setting $\tau = 0$, replacing j by i, and replacing k by j in Eq. (100), we have the following expression for $PP_{jm1}(\tau_0,\mu;\tau_0)$

$$PP_{jm1}(\tau_0,\mu;\tau_0) = P_j^m(\mu) \exp(-\tau_0/\mu) + (\omega/2) \sum_{i=m}^{L} (-1)^{j+i} B_i^m \times \int_0^{\tau_0} PP_{im1}(\tau_0 - t,\mu;\tau_0) \, K_{2imj}(-t) \, dt. \qquad (109)$$

Substituting Eq. (48) into Eqs. (108) and (109) yields

$$PP_{jm1}(0,\mu;\tau_0) = P_j^m(\mu) + (\omega/2) \sum_{i=m}^{L} B_i^m \int_0^{\tau_0} PP_{im1}(t,\mu;\tau_0) \times \int_0^1 \exp(-t/\mu_{in}') \, P_i^m(-\mu_{in}') \, P_j^m(-\mu_{in}')/\mu_{in}' \, d\mu_{in}' \times dt, \qquad (110)$$

and

$$PP_{jm1}(\tau_0,\mu;\tau_0) = P_j^m(\mu) \exp(-\tau_0/\mu) + (\omega/2) \sum_{i=m}^{L} (-1)^{j+i} B_i^m$$

$$\times \int_0^{\tau_0} PP_{lml}(\tau_0-t,\mu;\tau_0) \int_0^1 \exp(-t/\mu'_{in}) \, P_l^m(-\mu'_{in})$$

$$\times P_j^m(-\mu'_{in})/\mu'_{in} \, d\mu'_{in} \, dt. \tag{111}$$

Interchanging the order of integration, Eqs. (110) and (111) may be rewritten as

$$PP_{jml}(0,\mu;\tau_0) = P_j^m(\mu) + (\omega/2) \sum_{l=m}^{L} B_l^m \int_0^1 \left\{ \int_0^{\tau_0} PP_{lml}(t,\mu;\tau_0) \right.$$

$$\left. \times \exp(-t/\mu'_{in}) \, dt \right\} P_l^m(-\mu'_{in}) \, P_j^m(-\mu'_{in})/\mu'_{in}$$

$$\times d\mu'_{in}, \tag{112}$$

and

$$PP_{jml}(\tau_0,\mu;\tau_0) = P_j^m(\mu) \exp(-\tau_0/\mu) + (\omega/2) \sum_{l=m}^{L} (-1)^{j+l} B_l^m$$

$$\times \int_0^1 \left\{ \int_0^{\tau_0} PP_{lml}(\tau_0-t,\mu;\tau_0) \exp(-t/\mu'_{in}) \, dt \right\}$$

$$\times P_l^m(-\mu'_{in}) \, P_j^m(-\mu'_{in})/\mu'_{in} \, d\mu'_{in}. \tag{113}$$

Define

$$\overline{PP_{lml}}(\mu'_{in},\mu;\tau_0) = \int_0^{\tau_0} PP_{lml}(t,\mu;\tau_0) \exp(-t/\mu'_{in}) \, dt, \tag{114}$$

as the transform of $PP_{lml}(t,\mu;\tau_0)$ with respect to t, which is the reflection function of $PP_{lml}(t,\mu;\tau_0)$; and define

$$\overline{PPI_{lml}}(\mu'_{in},\mu;\tau_0) = \int_0^{\tau_0} PP_{lml}(\tau_0-t,\mu;\tau_0) \exp(-t/\mu'_{in}) \, dt, \tag{115}$$

as the transform of $PP_{lml}(\tau_0-t,\mu;\tau_0)$ with respect to t, which is the transmission function of $PP_{lml}(t,\mu;\tau_0)$.

Substituting Eqs. (114) and (115) into Eqs. (112) and

(113), respectively, Eqs.(112) and (113) become

$$PP_{jm1}(0,\mu;\tau_0) = P_j^m(\mu) + (\omega/2) \sum_{i=m}^{L} B_i^m \int_0^1 \overline{PP_{i\,m\,1}}(\mu_{in}',\mu;\tau_0)$$

$$\times P_i^m(-\mu_{in}') \; P_j^m(-\mu_{in}')/\mu_{in}' \; d\mu_{in}', \qquad (116)$$

and

$$PP_{jm1}(\tau_0,\mu;\tau_0) = P_j^m(\mu) \exp(-\tau_0/\mu) + (\omega/2) \sum_{i=m}^{L} (-1)^{j+i} B_i^m$$

$$\times \int_0^1 \overline{PPI_{i\,m\,1}}(\mu_{in}',\mu;\tau_0) \; P_i^m(-\mu_{in}') \; P_j^m(-\mu_{in}')/\mu_{in}'$$

$$\times d\mu_{in}'. \qquad (117)$$

Therefore, we have found $PP_{jm1}(0,\mu,\tau_0)$ in terms of the transform of $PP_{im1}(t,\mu,\tau_0)$ and $PP_{jm1}(\tau_0,\mu;\tau_0)$ in terms of the transform of $PP_{im1}(\tau_0-t,\mu,\tau_0)$. Our next objective is to find the expressions for $\overline{PP_{i\,m\,1}}$ and $\overline{PPI_{i\,m\,1}}$, both of which involve $PP_{im1}(0,\mu,\tau_0)$ and $PP_{im1}(\tau_0,\mu,\tau_0)$.

Application of the transform of Eqs. (114) and (115) to Eqs. (99) and (107), respectively, by multiplying both Eqs. (99) and (107) by $\exp(-\tau/s)$ and integrating over τ yields

$$(1/s) \; \overline{PP_{k\,m\,1}}(s,\mu;\tau_0) + \exp(-\tau_0/s) \; PP_{km1}(\tau_0,\mu;\tau_0)$$

$$- PP_{km1}(0,\mu;\tau_0) = -(1/\mu) \; \overline{PP_{k\,m\,1}}(s,\mu;\tau_0) + (\omega/2) \sum_{j=m}^{L} B_j^m$$

$$\times PP_{jm1}(0,\mu;\tau_0) \int_0^1 \overline{PP_{k\,m\,1}}(s,\mu_{in}';\tau_0) \; P_j^m(\mu_{in}')$$

$$/ \; \mu_{in}' \; d\mu_{in}' - (\omega/2) \sum_{j=m}^{L} (-1)^{j+k} B_j^m$$

$$\times PP_{jm1}(\tau_0,\mu;\tau_0) \int_0^1 \overline{PPI_{k\,m\,1}}(s,\mu_{in}';\tau_0) \; P_j^m(\mu_{in}')$$

$$/ \ \mu_{1n}' \ d\mu_{1n}', \tag{118}$$

and

$$(1/s) \ \overline{PPI_{km1}}(s,\mu;\tau_0) + \exp(-\tau_0/s) \ PP_{km1}(0,\mu;\tau_0)$$

$$- PP_{km1}(\tau_0,\mu;\tau_0) = (1/\mu) \ \overline{PPI_{km1}}(s,\mu;\tau_0) + (\omega/2) \sum_{j=m}^{L} (-1)^{j+k}$$

$$\times B_j^m \ PP_{jm1}(\tau_0,\mu;\tau_0) \int_0^1 \overline{PP_{km1}}(s,\mu_{1n}';\tau_0)$$

$$\times P_j^m(\mu_{1n}')/\mu_{1n}' \ d\mu_{1n}' - (\omega/2) \sum_{j=m}^{L} B_j^m$$

$$\times PP_{jm1}(0,\mu;\tau_0) \int_0^1 \overline{PPI_{km1}}(s,\mu_{1n}';\tau_0) \ P_j^m(\mu_{1n}')$$

$$/ \ \mu_{1n}' \ d\mu_{1n}', \tag{119}$$

where $\overline{PP_{km1}}(s,\mu;\tau_0)$ and $\overline{PPI_{km1}}(s,\mu;\tau_0)$ are defined as in Eqs. (114) and (115), respectively. Note that the transform of $\dfrac{\partial \ PP_{km1}(\tau,\mu;\tau_0)}{\partial \ \tau}$ is equal to $(1/s) \times$ [transform of $PP_{km1}(\tau,\mu;\tau_0)$] $+ \exp(-\tau_0/s) \ PP_{km1}(\tau_0,\mu;\tau_0) - PP_{km1}(0,\mu;\tau_0)$, and the transform of $\dfrac{\partial \ PP_{km1}(\tau_0-\tau,\mu;\tau_0)}{\partial \ \tau}$ is equal to $(1/s)$ $\times$ [transform of $PP_{km1}(\tau_0-\tau,\mu;\tau_0)$] $+ \exp(-\tau_0/s) \ PP_{km1}(0,\mu;\tau_0)$ $- PP_{km1}(\tau_0,\mu;\tau_0)$.

Equations (118) and (119) can be rarranged as

$$(1/s + 1/\mu) \ \overline{PP_{km1}}(s,\mu;\tau_0)$$

$$= PP_{km1}(0,\mu;\tau_0) - \exp(-\tau_0/s) \ PP_{km1}(\tau_0,\mu;\tau_0) + (\omega/2) \sum_{j=m}^{L} B_j^m$$

$$\times PP_{jm1}(0,\mu;\tau_0) \int_0^1 \overline{PP_{km1}}(s,\mu_{1n}';\tau_0) \ P_j^m(\mu_{1n}')/\mu_{1n}' \ d\mu_{1n}' - (\omega/2)$$

$$\times \sum_{j=m}^{L} (-1)^{j+k} B_j^m PP_{jm1}(\tau_0,\mu;\tau_0) \int_0^1 \overline{PPI_{km1}}(s,\mu_{1n}';\tau_0) P_j^m(\mu_{1n}')$$

$$/ \mu_{1n}' d\mu_{1n}', \tag{120}$$

and

$$(1/s - 1/\mu) \overline{PPI_{km1}}(s,\mu;\tau_0)$$

$$= PP_{km1}(\tau_0,\mu;\tau_0) - \exp(-\tau_0/s) PP_{km1}(0,\mu;\tau_0) + (\omega/2)$$

$$\times \sum_{j=m}^{L} (-1)^{j+k} B_j^m PP_{jm1}(\tau_0,\mu;\tau_0) \int_0^1 \overline{PP_{km1}}(s,\mu_{1n}';\tau_0) P_j^m(\mu_{1n}')$$

$$/ \mu_{1n}' d\mu_{1n}' - (\omega/2) \sum_{j=m}^{L} B_j^m PP_{jm1}(0,\mu;\tau_0) \int_0^1 \overline{PPI_{km1}}(s,\mu_{1n}';\tau_0)$$

$$\times P_j^m(\mu_{1n}')/\mu_{1n}' d\mu_{1n}'. \tag{121}$$

Multiplying Eq. (120) by $(\omega/2)$ $(-1)^k$ B_k^m $P_k^m(s)$ and summing from $k = m$ to L, and Eq. (121) by $(\omega/2)$ B_k^m $P_k^m(s)$ and summing from $k = m$ to L, gives

$$(1/s + 1/\mu) (\omega/2) \sum_{k=m}^{L} (-1)^k B_k^m P_k^m(s) \overline{PP_{km1}}(s,\mu;\tau_0)$$

$$= (\omega/2) \sum_{k=m}^{L} (-1)^k B_k^m P_k^m(s) PP_{km1}(0,\mu;\tau_0) - (\omega/2) \exp(-\tau_0/s)$$

$$\times \sum_{k=m}^{L} (-1)^k B_k^m P_k^m(s) PP_{km1}(\tau_0,\mu;\tau_0) + (\omega/2) \sum_{j=m}^{L} B_j^m$$

$$\times PP_{jm1}(0,\mu;\tau_0) \Big\{ (\omega/2) \sum_{k=m}^{L} (-1)^k B_k^m P_k^m(s) \int_0^1 \overline{PP_{km1}}(s,\mu_{1n}';\tau_0)$$

$$\times P_j^m(\mu_{1n}')/\mu_{1n}' d\mu_{1n}' \Big\} - (\omega/2) \sum_{j=m}^{L} (-1)^j B_j^m PP_{jm1}(\tau_0,\mu;\tau_0)$$

$$\times \Big\{ (\omega/2) \sum_{k=m}^{L} B_k^m P_k^m(s) \int_0^1 \overline{PPI_{km1}}(s,\mu_{1n}';\tau_0) P_j^m(\mu_{1n}')/\mu_{1n}'$$

$$\times\ d\mu_{in}' \Big\}, \tag{122}$$

and

$$(1/s - 1/\mu)\ (\omega/2) \sum_{k=m}^{L} B_k^m\ P_k^m(s)\ \overline{PPI_{km1}}(s,\mu;\tau_0)$$

$$= (\omega/2) \sum_{k=m}^{L} B_k^m\ P_k^m(s)\ PP_{km1}(\tau_0,\mu;\tau_0) - (\omega/2)\ \exp(-\tau_0/s)$$

$$\times \sum_{k=m}^{L} B_k^m\ P_k^m(s)\ PP_{km1}(0,\mu;\tau_0) + (\omega/2) \sum_{j=m}^{L} (-1)^j\ B_j^m$$

$$\times\ PP_{jm1}(\tau_0,\mu;\tau_0) \Big\{ (\omega/2) \sum_{k=m}^{L} (-1)^k\ B_k^m\ P_k^m(s)$$

$$\times \int_0^1 \overline{PP_{km1}}(s,\mu_{in}';\tau_0)\ P_j^m(\mu_{in}')/\mu_{in}'\ d\mu_{in}' \Big\} - (\omega/2) \sum_{j=m}^{L} B_j^m$$

$$\times\ PP_{jm1}(0,\mu;\tau_0) \Big\{ (\omega/2) \sum_{k=m}^{L} B_k^m\ P_k^m(s) \int_0^1 \overline{PPI_{km1}}(s,\mu_{in}';\tau_0)$$

$$\times\ P_j^m(\mu_{in}')/\mu_{in}'\ d\mu_{in}' \Big\}. \tag{123}$$

Replacing k by j in the first and second terms, and k by i in the third and fourth terms of the right hand sides, Eqs. (122) and (123) can be written as

$$(1/s + 1/\mu)\ (\omega/2) \sum_{k=m}^{L} (-1)^k\ B_k^m\ P_k^m(s)\ \overline{PP_{km1}}(s,\mu;\tau_0)$$

$$= (\omega/2) \sum_{j=m}^{L} (-1)^j\ B_j^m\ P_j^m(s)\ PP_{jm1}(0,\mu;\tau_0) - (\omega/2)\ \exp(-\tau_0/s)$$

$$\times \sum_{j=m}^{L} (-1)^j\ B_j^m\ P_j^m(s)\ PP_{jm1}(\tau_0,\mu;\tau_0) + (\omega/2) \sum_{j=m}^{L} B_j^m$$

$$\times\ PP_{jm1}(0,\mu;\tau_0) \Big\{ (\omega/2) \sum_{i=m}^{L} (-1)^i\ B_i^m\ P_i^m(s) \int_0^1 \overline{PP_{im1}}(s,\mu_{in}';\tau_0)$$

$$\times\ P_j^m(\mu_{1n}')/\mu_{1n}'\ d\mu_{1n}' \Big\} - (\omega/2) \sum_{j=m}^{L} (-1)^j\ B_j^m\ PP_{jm1}(\tau_0,\mu;\tau_0)$$

$$\times \Big\{ (\omega/2) \sum_{i=m}^{L} B_i^m\ P_i^m(s) \int_0^1 \overline{PPI_{im1}}(s,\mu_{1n}';\tau_0)\ P_j^m(\mu_{1n}')/\mu_{1n}'$$

$$\times\ d\mu_{1n}' \Big\}, \qquad (124)$$

and

$$(1/s - 1/\mu)\ (\omega/2) \sum_{k=m}^{L} B_k^m\ P_k^m(s)\ \overline{PPI_{km1}}(s,\mu;\tau_0)$$

$$= (\omega/2) \sum_{j=m}^{L} B_j^m\ P_j^m(s)\ PP_{jm1}(\tau_0,\mu;\tau_0) - (\omega/2)\ \exp(-\tau_0/s)$$

$$\times \sum_{j=m}^{L} B_j^m\ P_j^m(s)\ PP_{jm1}(0,\mu;\tau_0) + (\omega/2) \sum_{j=m}^{L} (-1)^j\ B_j^m$$

$$\times\ PP_{jm1}(\tau_0,\mu;\tau_0) \Big\{ (\omega/2) \sum_{i=m}^{L} (-1)^i\ B_i^m\ P_i^m(s)$$

$$\times \int_0^1 \overline{PP_{im1}}(s,\mu_{1n}';\tau_0)\ P_j^m(\mu_{1n}')/\mu_{1n}'\ d\mu_{1n}' \Big\} - (\omega/2) \sum_{j=m}^{L} B_j^m$$

$$\times\ PP_{jm1}(0,\mu;\tau_0) \Big\{ (\omega/2) \sum_{i=m}^{L} B_i^m\ P_i^m(s) \int_0^1 \overline{PPI_{im1}}(s,\mu_{1n}';\tau_0)$$

$$\times\ P_j^m(\mu_{1n}')/\mu_{1n}'\ d\mu_{1n}' \Big\}. \qquad (125)$$

It can be shown that (App. A)

$$\sum_{i=m}^{L} (-1)^i\ B_i^m\ P_i^m(\mu)\ \overline{PP_{im1}}(\mu,s;\tau_0) = \sum_{i=m}^{L} (-1)^i\ B_i^m\ P_i^m(s)$$

$$\times\ \overline{PP_{im1}}(s,\mu;\tau_0), \qquad (126)$$

and

$$\sum_{i=m}^{L} B_i^m\ P_i^m(\mu)\ \overline{PPI_{im1}}(\mu,s;\tau_0) = \sum_{i=m}^{L} B_i^m\ P_i^m(s)$$

$$\times \overline{PPI_{lm1}}(s,\mu;\tau_0). \quad (127)$$

With the help of Eqs. (126) and (127), we may rewrite Eqs. (116) and (117) as

$$PP_{jm1}(0,\mu;\tau_0) = P_j^m(\mu) + (\omega/2) \sum_{l=m}^{L} (-1)^{j+l} B_l^m P_l^m(\mu)$$

$$\times \int_0^1 \overline{PP_{lm1}}(\mu,\mu_{in}';\tau_0) \; P_j^m(\mu_{in}')/\mu_{in}' \; d\mu_{in}', \quad (128)$$

and

$$PP_{jm1}(\tau_0,\mu;\tau_0) = P_j^m(\mu)\exp(-\tau_0/\mu) + (\omega/2) \sum_{l=m}^{L} B_l^m P_l^m(\mu)$$

$$\times \int_0^1 \overline{PPI_{lm1}}(\mu,\mu_{in}';\tau_0) \; P_j^m(\mu_{in}')/\mu_{in}' \; d\mu_{in}'. \quad (129)$$

Now, replacing μ with s in Eqs. (128) and (129), and substituting these two equations into Eq. (124) and Eq. (125) yields

$$(1/s + 1/\mu) \; (\omega/2) \sum_{k=m}^{L} (-1)^k B_k^m P_k^m(s) \; \overline{PP_{km1}}(s,\mu;\tau_0)$$

$$= (\omega/2) \sum_{j=m}^{L} (-1)^j B_j^m \Big\{ PP_{jm1}(0,\mu;\tau_0) \; PP_{jm1}(0,s;\tau_0)$$

$$- PP_{jm1}(\tau_0,\mu;\tau_0) \; PP_{jm1}(\tau_0,s;\tau_0) \Big\},$$

and

$$(1/s - 1/\mu) \; (\omega/2) \sum_{k=m}^{L} B_k^m P_k^m(s) \; \overline{PPI_{km1}}(s,\mu;\tau_0)$$

$$= (\omega/2) \sum_{j=m}^{L} B_j^m \Big\{ PP_{jm1}(\tau_0,\mu;\tau_0) \; PP_{jm1}(0,s;\tau_0) - PP_{jm1}(0,\mu;\tau_0)$$

$$\times PP_{jm1}(\tau_0,s;\tau_0) \Big\}.$$

or changing the summation indices to i rather than k

$$(1/s + 1/\mu)\ (\omega/2) \sum_{i=m}^{L} (-1)^{i}\ B_i^m\ P_i^m(s)\ \overline{PP_{i\,m\,1}}(s,\mu;\tau_0)$$

$$= (\omega/2) \sum_{i=m}^{L} (-1)^{i}\ B_i^m \Big\{ PP_{im1}(0,\mu;\tau_0)\ PP_{im1}(0,s;\tau_0)$$

$$- PP_{im1}(\tau_0,\mu;\tau_0)\ PP_{im1}(\tau_0,s;\tau_0) \Big\}, \qquad (130)$$

and

$$(1/s - 1/\mu)\ (\omega/2) \sum_{i=m}^{L} B_i^m\ P_i^m(s)\ \overline{PPI_{i\,m\,1}}(s,\mu;\tau_0)$$

$$= (\omega/2) \sum_{i=m}^{L} B_i^m \Big\{ PP_{im1}(\tau_0,\mu;\tau_0)\ PP_{im1}(0,s;\tau_0) - PP_{im1}(0,\mu;\tau_0)$$

$$\times PP_{im1}(\tau_0,s;\tau_0) \Big\}. \qquad (131)$$

Thus, we have found the expressions which involve $PP_{im1}(0,\mu;\tau_0)$ and $PP_{im1}(\tau_0,\mu;\tau_0)$ for $\overline{PP_{i\,m\,1}}$ and $\overline{PPI_{i\,m\,1}}$.

Replacing s with μ_{in}' in Eqs. (130) and (131), and substituting these two equations into Eq. (116) and Eq. (117), respectively, we get

$$PP_{jm1}(0,\mu;\tau_0) = P_j^m(\mu) + (\omega/2) \sum_{i=m}^{L} (-1)^{i+j}\ B_i^m \int_0^1 [1\ /(1/\mu_{in}'$$

$$+ 1/\mu)] \Big\{ PP_{im1}(0,\mu;\tau_0)\ PP_{im1}(0,\mu_{in}';\tau_0)$$

$$- PP_{im1}(\tau_0,\mu;\tau_0)\ PP_{im1}(\tau_0,\mu_{in}';\tau_0) \Big\}$$

$$\times P_j^m(\mu_{in}')/\mu_{in}'\ d\mu_{in}', \qquad (132)$$

and

$$PP_{jm1}(\tau_0,\mu;\tau_0) = P_j^m(\mu)\ \exp(-\tau_0/\mu) + (\omega/2) \sum_{i=m}^{L} B_i^m$$

$$\times \int_0^1 [1 / (1/\mu_{in}' - 1/\mu)] \Big\{ PP_{im1}(\tau_0,\mu;\tau_0)$$

$$\times PP_{im1}(0,\mu_{in}';\tau_0) - PP_{im1}(0,\mu;\tau_0)$$

$$\times PP_{im1}(\tau_0,\mu_{in}';\tau_0) \Big\} P_j^m(\mu_{in}')/\mu_{in}' \, d\mu_{in}'. \quad (133)$$

Notice that Eqs. (132) and (133) are dependent equations. Therefore, these two equations should be solved simultaneously. However, Eqs. (132) and (133) can not be easily solved numerically due to the term $[1 / (1/\mu_{in}' - 1/\mu)]$ in Eq. (133). Hence, we need to find another set of equations to solve for these terms.

Replacing μ_{in}' by μ, Eq. (96) can be written as

$$PP_{km1}(\tau_0-\tau,\mu;\tau_0) = P_k^m(\mu) \exp[(\tau-\tau_0)/\mu] + (\omega/2) \sum_{i=m}^{L} (-1)^{i+k} B_i^m$$

$$\times \int_0^{\tau_0} PP_{im1}(\tau_0-t,\mu;\tau_0) \; K_{2imk}(\tau-t) \, dt. \quad (134)$$

Using Leibnitz rule to take the derivative of Eq. (86) and Eq. (134) with respect to τ_0 yields

$$\frac{\partial \; PP_{km1}(\tau,\mu;\tau_0)}{\partial \; \tau_0} = (\omega/2) \sum_{j=m}^{L} B_j^m \; PP_{jm1}(\tau_0,\mu;\tau_0) \; K_{2jmk}(\tau-\tau_0)$$

$$+ (\omega/2) \sum_{j=m}^{L} B_j^m \int_0^{\tau_0} \frac{\partial \; PP_{jm1}(t,\mu;\tau_0)}{\partial \; \tau_0} K_{2jmk}(\tau-t)$$

$$\times \, dt, \quad (135)$$

and

$$\frac{\partial \; PP_{km1}(\tau_0-\tau,\mu;\tau_0)}{\partial \; \tau_0} = -(1/\mu) \; P_k^m(\mu) \; \exp[(\tau-\tau_0)/\mu] + (\omega/2)$$

$$\times \sum_{i=m}^{L} (-1)^{i+k} B_i^m PP_{im1}(0,\mu;\tau_0) K_{2imk}(\tau-\tau_0)$$

$$+ (\omega/2) \sum_{i=m}^{L} (-1)^{i+k} B_i^m$$

$$\times \int_0^{\tau_0} \frac{\partial PP_{im1}(\tau_0-t,\mu;\tau_0)}{\partial \tau_0} K_{2imk}(\tau-t)$$

$$\times dt. \qquad (136)$$

The solutions of Eq. (135) and Eq. (136) are found by the method of superposition. We are going to solve Eq. (135) first. Starting by multiplying Eq. (97) by -1, we get

$$(\omega/2) \sum_{j=m}^{L} (-1)^{j+k} B_j^m PP_{jm1}(\tau_0,\mu;\tau_0) \int_0^1 PP_{km1}(\tau_0-\tau,\mu_{in}';\tau_0)$$

$$\times P_j^m(\mu_{in}')/\mu_{in}' du_{in}' = (\omega/2) \sum_{j=m}^{L} B_j^m PP_{jm1}(\tau_0,\mu;\tau_0) K_{2jmk}(\tau-\tau_0)$$

$$+ (\omega/2) \sum_{i=m}^{L} B_i^m \int_0^{\tau_0} \Big\{ (\omega/2) \sum_{j=m}^{L} (-1)^{j+i} B_j^m$$

$$\times PP_{jm1}(\tau_0,\mu;\tau_0) \int_0^1 PP_{im1}(\tau_0-t,\mu_{in}';\tau_0)$$

$$\times P_j^m(\mu_{in}')/\mu_{in}' du_{in}' \Big\} K_{2imk}(\tau-t) dt. \qquad (137)$$

Replacing j by i in the second term on the right hand side of Eq. (135), then comparing with Eq. (137), the solution of Eq. (135) by superposition is

$$\frac{\partial PP_{km1}(\tau,\mu;\tau_0)}{\partial \tau_0} = (\omega/2) \sum_{j=m}^{L} (-1)^{j+k} B_j^m PP_{jm1}(\tau_0,\mu;\tau_0)$$

$$\times \int_0^1 PP_{km1}(\tau_0-\tau,\mu_{in}';\tau_0) P_j^m(\mu_{in}')/\mu_{in}'$$

$$\times\ du_{in}'. \tag{138}$$

Multiplying Eq. (105) by -1 gives

$$(\omega/2)\sum_{j=m}^{L} B_j^m\, PP_{jm1}(0,\mu;\tau_0)\int_0^1 PP_{km1}(\tau_0-\tau,\mu_{in}';\tau_0)\ P_j^m(\mu_{in}')/\mu_{in}'$$

$$\times\ du_{in}' = (\omega/2)\sum_{j=m}^{L}(-1)^{j+k} B_j^m\, PP_{jm1}(0,\mu;\tau_0)\ K_{2jmk}(\tau-\tau_0)$$

$$+\ (\omega/2)\sum_{i=m}^{L} B_i^m\int_0^{\tau_0}\Big\{(\omega/2)\sum_{j=m}^{L} B_j^m\, PP_{jm1}(0,\mu;\tau_0)$$

$$\times\int_0^1 PP_{im1}(\tau_0-t,\mu_{in}';\tau_0)\ P_j^m(\mu_{in}')/\mu_{in}'\ du_{in}'\Big\}$$

$$\times\ K_{2imk}(t-\tau)\ dt. \tag{139}$$

Replacing j by i in Eq. (100) and multiplying this equation by $-(1/\mu)$, and then adding to Eq. (139), we have

$$-(1/\mu)\ PP_{km1}(\tau_0-\tau,\mu;\tau_0) + (\omega/2)\sum_{j=m}^{L} B_j^m\, PP_{jm1}(0,\mu;\tau_0)$$

$$\times\int_0^1 PP_{km1}(\tau_0-\tau,\mu_{in}';\tau_0)\ P_j^m(\mu_{in}')/\mu_{in}'\ du_{in}'$$

$$= -(1/\mu)\ P_k^m(\mu)\ \exp[(\tau-\tau_0)/\mu] + (\omega/2)\sum_{j=m}^{L}(-1)^{j+k} B_j^m$$

$$\times\ PP_{jm1}(0,\mu;\tau_0)\ K_{2jmk}(\tau-\tau_0) + (\omega/2)\sum_{i=m}^{L}(-1)^{i+k} B_i^m$$

$$\times\int_0^{\tau_0}\Big\{-(1/\mu)\ PP_{im1}(\tau_0-t,\mu;\tau_0) + (\omega/2)\sum_{j=m}^{L} B_j^m\, PP_{jm1}(0,\mu;\tau_0)$$

$$\times\int_0^1 PP_{im1}(\tau_0-t,\mu_{in}';\tau_0)\ P_j^m(\mu_{in}')/\mu_{in}'\ du_{in}'\Big\}\ K_{2imk}(\tau-t)$$

$$\times\ dt. \tag{140}$$

Now, replacing i by j in the second term on the right hand side of Eq. (136), and comparing with Eq. (140), the solution of Eq. (136) by superposition is

$$\frac{\partial\ PP_{km1}(\tau_0-\tau,\mu;\tau_0)}{\partial\ \tau_0} = -(1/\mu)\ PP_{km1}(\tau_0-\tau,\mu;\tau_0) + (\omega/2) \sum_{j=m}^{L} B_j^m$$

$$\times\ PP_{jm1}(0,\mu;\tau_0) \int_0^1 PP_{km1}(\tau_0-\tau,\mu_{in}';\tau_0)$$

$$\times\ P_j^m(\mu_{in}')/\mu_{in}'\ du_{in}'. \tag{141}$$

Finally, letting $\tau = 0$, Eqs. (138) and (141) become

$$\frac{\partial\ PP_{km1}(0,\mu;\tau_0)}{\partial\ \tau_0} = (\omega/2) \sum_{j=m}^{L} (-1)^{j+k}\ B_j^m\ PP_{jm1}(\tau_0,\mu;\tau_0)$$

$$\times \int_0^1 PP_{km1}(\tau_0,\mu_{in}';\tau_0)\ P_j^m(\mu_{in}')/\mu_{in}'\ du_{in}', \tag{142}$$

and

$$\frac{\partial\ PP_{km1}(\tau_0,\mu;\tau_0)}{\partial\ \tau_0} = -(1/\mu)\ PP_{km1}(\tau_0,\mu;\tau_0) + (\omega/2) \sum_{j=m}^{L} B_j^m$$

$$\times\ PP_{jm1}(0,\mu;\tau_0) \int_0^1 PP_{km1}(\tau_0,\mu_{in}';\tau_0)\ P_j^m(\mu_{in}')$$

$$/\ \mu_{in}'\ du_{in}'. \tag{143}$$

Equations (142) and (143) are dependent integro-differential equations. These two equations can be solved simultaneously by a combination of the Runge-Kutta numerical calculation method and the successive approximation method.

In the following derivation, superposition will be used to deduce the unknown function $PP_{km2}(\tau,\mu,n;\tau_0)$, and also $PP_{km}(\tau,\mu,n;\tau_0)$, in terms of $PP_{km1}(\tau,\mu;\tau_0)$.

By using Eqs. (89) and (88), Eq. (87) can be written as

$$PP_{km2}(\tau,\mu,n;\tau_0) = (\omega/2) \sum_{j=m}^{L} B_j^m (-1)^m \int_0^{\tau_0} PP_{jm1}(t,\mu;\tau_0)$$

$$\times K_{1jmk}(\tau+t,n)\ dt + (\omega/2) \sum_{j=m}^{L} B_j^m$$

$$\times \int_0^{\tau_0} PP_{jm2}(t,\mu,n;\tau_0) \left\{ (-1)^m K_{1jmk}(\tau+t,n) \right.$$

$$\left. + K_{2jmk}(\tau-t) \right\} dt. \qquad (144)$$

Using the definition of $K_{1jmk}(\tau+t,n)$ given in Eq. (26b), and reversing the order of integration in the first term on the right hand side of Eq. (144) gives

$$PP_{km2}(\tau,\mu,n;\tau_0) = (\omega/2) \sum_{j=m}^{L} B_j^m (-1)^m \int_0^1 \overline{PP_{jm1}}(\mu_{1n}',\mu;\tau_0)$$

$$\times \exp(-\tau/\mu_{1n}')\ \rho(\mu_{1n}',n)\ P_k^m(\mu_{1n}')\ P_j^m(-\mu_{1n}')/\mu_{1n}'$$

$$\times d\mu_{1n}' + (\omega/2) \sum_{j=m}^{L} B_j^m \int_0^{\tau_0} PP_{jm2}(t,\mu,n;\tau_0)$$

$$\times \left\{ (-1)^m K_{1jmk}(\tau+t,n) + K_{2jmk}(\tau-t) \right\}$$

$$\times dt, \qquad (145)$$

where $\overline{PP_{jm1}}(\mu_{1n}',\mu;\tau_0)$, as defined in Eq. (114), is the reflection function of $PP_{jm1}(t,\mu;\tau_0)$.

Replacing i by j and s by μ_{1n}' in Eq. (130), $\overline{PP_{jm1}}(\mu_{1n}',\mu;\tau_0)$ can be expressed in terms of $PP_{jm1}(0,\mu;\tau_0)$ and $PP_{jm1}(\tau_0,\mu;\tau_0)$ as follows

$$\sum_{j=m}^{L} (-1)^{j} B_j^m P_j^m(\mu_{1n}') \overline{PP_{jm1}}(\mu_{1n}',\mu;\tau_0)$$

$$= [1/(1/\mu_{1n}' + 1/\mu)] \sum_{j=m}^{L} (-1)^{j} B_j^m \Big\{ PP_{jm1}(0,\mu;\tau_0)$$

$$\times PP_{jm1}(0,\mu_{1n}';\tau_0) - PP_{jm1}(\tau_0,\mu;\tau_0) PP_{jm1}(\tau_0,\mu_{1n}';\tau_0) \Big\}. \quad (146)$$

Note that $PP_{km2}(\tau,\mu,n;\tau_0)$ in Eq. (145) has the same kernel function as $PP_{km}(\tau,\mu,n;\tau_0)$ in Eq. (83).

Replacing j by i and μ by μ_{1n}' in Eq. (83), and multiplying this equation by $(\omega/2)$ $(-1)^m$ B_j^m $\overline{PP_{jm1}}(\mu_{1n}',\mu;\tau_0)$ $\rho(\mu_{1n}',n)$ $P_j^m(-\mu_{1n}')/\mu_{1n}'$, integrating μ_{1n}' from zero to one, and summing from j = m to L, then replacing j by i in the second term on the right hand side of Eq. (145), an expression for $PP_{km2}(\tau,\mu,n;\tau_0)$ can be obtained by superposing the modified Eqs. (83) and (145) as shown below,

$$PP_{km2}(\tau,\mu,n;\tau_0) = (\omega/2) \sum_{j=m}^{L} B_j^m (-1)^m \int_0^1 PP_{km}(\tau,\mu_{1n}',n;\tau_0)$$

$$\times \overline{PP_{jm1}}(\mu_{1n}',\mu;\tau_0) \rho(\mu_{1n}',n) P_j^m(-\mu_{1n}')/\mu_{1n}'$$

$$\times d\mu_{1n}'. \quad (147)$$

Then, substituting Eq. (147) into Eq. (89) gives

$$PP_{km}(\tau,\mu,n;\tau_0) = PP_{km1}(\tau,\mu;\tau_0) + (\omega/2) \sum_{j=m}^{L} B_j^m (-1)^m$$

$$\times \int_0^1 PP_{km}(\tau,\mu_{1n}',n;\tau_0) \overline{PP_{jm1}}(\mu_{1n}',\mu;\tau_0) P_j^m(-\mu_{1n}')$$

$$\times \rho(\mu_{1n}',n)/\mu_{1n}' d\mu_{1n}'. \quad (148)$$

At the top boundary of the medium where the optical location τ is equal to zero, Eq. (148) could be adjusted as

$$PP_{km}(0,\mu,n;\tau_0) = PP_{km1}(0,\mu;\tau_0) + (\omega/2) \sum_{j=m}^{L} B_j^m (-1)^m$$

$$\times \int_0^1 PP_{km}(0,\mu_{in}',n;\tau_0) \overline{PP_{jm1}}(\mu_{in}',\mu;\tau_0) P_j^m(-\mu_{in}')$$

$$\times \rho(\mu_{in}',n)/\mu_{in}' \, d\mu_{in}'. \quad (149)$$

At the bottom boundary of the medium where the optical location τ is equal to τ_0, Eq. (148) could be adjusted as

$$PP_{km}(\tau_0,\mu,n;\tau_0) = PP_{km1}(\tau_0,\mu;\tau_0) + (\omega/2) \sum_{j=m}^{L} B_j^m (-1)^m$$

$$\times \int_0^1 PP_{km}(\tau_0,\mu_{in}',n;\tau_0) \overline{PP_{jm1}}(\mu_{in}',\mu;\tau_0)$$

$$\times P_j^m(-\mu_{in}') \rho(\mu_{in}',n)/\mu_{in}' \, d\mu_{in}'. \quad (150)$$

Now, substituting Eq. (146) into Eq. (149) and Eq. (150) yields

$$PP_{km}(0,\mu,n;\tau_0) = PP_{km1}(0,\mu;\tau_0) + (\omega/2) \sum_{j=m}^{L} (-1)^j B_j^m$$

$$\times \int_0^1 [1/(1/\mu_{in}' + 1/\mu)] PP_{km}(0,\mu_{in}',n;\tau_0)$$

$$\times \Big\{ PP_{jm1}(0,\mu;\tau_0) PP_{jm1}(0,\mu_{in}';\tau_0)$$

$$- PP_{jm1}(\tau_0,\mu;\tau_0) PP_{jm1}(\tau_0,\mu_{in}';\tau_0) \Big\} \rho(\mu_{in}',n)$$

$$/ \mu_{in}' \, d\mu_{in}', \quad (151)$$

and

$$PP_{km}(\tau_0,\mu,n;\tau_0) = PP_{km1}(\tau_0,\mu;\tau_0) + (\omega/2) \sum_{j=m}^{L} (-1)^j B_j^m$$

$$\times \int_0^1 [1/(1/\mu_{1n}' + 1/\mu)]\ PP_{km}(\tau_0,\mu_{1n}',n;\tau_0)$$

$$\times \Big\{ PP_{jm1}(0,\mu;\tau_0)\ PP_{jm1}(0,\mu_{1n}';\tau_0)$$

$$- PP_{jm1}(\tau_0,\mu;\tau_0)\ PP_{jm1}(\tau_0,\mu_{1n}';\tau_0) \Big\}\ \rho(\mu_{1n}',n)$$

$$/\ \mu_{1n}'\ d\mu_{1n}'. \tag{152}$$

Equations (151) and (152) can be solved numerically by the successive approximation method, once we get the exact solutions for $PP_{jm1}(0,\mu;\tau_0)$ from Eq. (142) and $PP_{jm1}(\tau_0,\mu;\tau_0)$ from Eq. (143).

Reflection and Transmission Functions

The reflection and transmission functions of source function $PP_{km}(\tau,\mu,n;\tau_0)$ can be obtained by the superposition method. Using the definitions of $\overline{PP_{jm}}$ from Eq. (154) and $K_{1jmk}(\tau+t,n)$ from Eq. (26b) and reversing the order of integration, the lead function $f_{km}(\tau,\mu,n;\tau_0)$ from Eq. (88) can be represented as

$$f_{km}(\tau,\mu,n;\tau_0) = (\omega/2) \sum_{j=m}^{L} B_j^m\ (-1)^m \int_0^1 \overline{PP_{jm}}(\mu_{1n}',\mu,n;\tau_0)$$

$$\times \exp(-\tau/\mu_{1n}')\ \rho(\mu_{1n}',n)\ P_k^m(\mu_{1n}')\ P_j^m(-\mu_{1n}')/\mu_{1n}'$$

$$\times d\mu_{1n}', \tag{153}$$

where $\overline{PP_{jm}}(\mu_{1n}',\mu,n;\tau_0) = \int_0^{\tau_0} PP_{jm}(t,\mu,n;\tau_0)\ \exp(-t/\mu_{1n}')$

$$\times dt \tag{154}$$

is the reflection function of $PP_{jm}(t,\mu,n;\tau_0)$.

Replacing μ by μ_{1n}' and j by i in Eq. (86), multiplying this equation by $(\omega/2)\ B_j^m\ (-1)^m\ \overline{PP_{jm}}(\mu_{1n}',\mu,n;\tau_0)\ \rho(\mu_{1n}',n) \times P_j^m(-\mu_{1n}')/\mu_{1n}'$, integrating μ_{1n}' from zero to one, and summing from j = m to L, then replacing j by i in the second term on the right hand side of Eq. (87), where f_{km} is from Eq. (153), an expression of $PP_{km2}(\tau,\mu,n;\tau_0)$ can be obtained by superposing the modified Eqs. (86) and (87) as follows

$$PP_{km2}(\tau,\mu,n;\tau_0) = (\omega/2) \sum_{j=m}^{L} B_j^m\ (-1)^m \int_0^1 PP_{km1}(\tau,\mu_{1n}';\tau_0) \times \overline{PP_{jm}}(\mu_{1n}',\mu,n;\tau_0)\ P_j^m(-\mu_{1n}')\ \rho(\mu_{1n}',n)/\mu_{1n}' \times d\mu_{1n}'. \tag{155}$$

Substituting Eq. (155) into Eq. (89) gives

$$PP_{km}(\tau,\mu,n;\tau_0) = PP_{km1}(\tau,\mu;\tau_0) + (\omega/2) \sum_{j=m}^{L} B_j^m\ (-1)^m \times \int_0^1 PP_{km1}(\tau,\mu_{1n}';\tau_0)\ \overline{PP_{jm}}(\mu_{1n}',\mu,n;\tau_0)\ \rho(\mu_{1n}',n) \times P_j^m(-\mu_{1n}')/\mu_{1n}'\ d\mu_{1n}'. \tag{156}$$

Taking the transform of Eq. (156) by multiplying by $\exp(-\tau/\bar{\mu})$, integrating from zero to τ_0 with respect to τ on both sides, and using Eqs. (154) and (114) to simplify yields

$$\overline{PP_{km}}(\bar{\mu},\mu,n;\tau_0) = \overline{PP_{km1}}(\bar{\mu},\mu;\tau_0) + (\omega/2) \sum_{j=m}^{L} B_j^m\ (-1)^m \times \int_0^1 \overline{PP_{km1}}(\bar{\mu},\mu_{1n}';\tau_0)\ \overline{PP_{jm}}(\mu_{1n}',\mu,n;\tau_0)\ \rho(\mu_{1n}',n)$$

$$\times\ P_j^m(-\mu_{in}')/\mu_{in}'\ d\mu_{in}'. \tag{157}$$

Multiplying Eq. (157) by $(\omega/2)\ B_k^m\ P_k^m(-\bar{\mu})$, and summing k from m to L gives

$$(\omega/2)\sum_{k=m}^{L} B_k^m\ P_k^m(-\bar{\mu})\ \overline{PP_{km}}(\bar{\mu},\mu,n;\tau_0)$$

$$= (\omega/2)\sum_{k=m}^{L} B_k^m\ P_k^m(-\bar{\mu})\ \overline{PP_{km1}}(\bar{\mu},\mu;\tau_0) + (\omega/2)\sum_{j=m}^{L} B_j^m\ (-1)^m$$

$$\times \int_0^1 \left\{ (\omega/2)\sum_{k=m}^{L} B_k^m\ P_k^m(-\bar{\mu})\ \overline{PP_{km1}}(\bar{\mu},\mu_{in}';\tau_0) \right\} \overline{PP_{jm}}(\mu_{in}',\mu,n;\tau_0)$$

$$\times\ \rho(\mu_{in}',n)\ P_j^m(-\mu_{in}')/\mu_{in}'\ d\mu_{in}'. \tag{158}$$

Then, replacing i by k and s by $\bar{\mu}$ into Eq. (130), and substituting this equation into Eq. (158), we have

$$(\omega/2)\sum_{k=m}^{L} B_k^m\ P_k^m(-\bar{\mu})\ \overline{PP_{km}}(\bar{\mu},\mu,n;\tau_0)$$

$$= [1\ /(1/\bar{\mu} + 1/\mu)]\ (\omega/2)\sum_{k=m}^{L}(-1)^{k+m}\ B_k^m \left\{ PP_{km1}(0,\mu;\tau_0)\right.$$

$$\left.\times\ PP_{km1}(0,\bar{\mu};\tau_0) - PP_{km1}(\tau_0,\mu;\tau_0)\ PP_{km1}(\tau_0,\bar{\mu};\tau_0) \right\} + (\omega/2)$$

$$\times \sum_{j=m}^{L}(-1)^m\ B_j^m \int_0^1 \left\{ [1\ /(1/\bar{\mu} + 1/\mu_{in}')]\ (\omega/2)\sum_{k=m}^{L}(-1)^{k+m}\ B_k^m\right.$$

$$\times\ [PP_{km1}(0,\mu_{in}';\tau_0)\ PP_{km1}(0,\bar{\mu};\tau_0) - PP_{km1}(\tau_0,\mu_{in}';\tau_0)$$

$$\left.\times\ PP_{km1}(\tau_0,\bar{\mu};\tau_0)] \right\} \overline{PP_{jm}}(\mu_{in}',\mu,n;\tau_0)\ \rho(\mu_{in}',n)\ P_j^m(-\mu_{in}')/\mu_{in}'$$

$$\times\ d\mu_{in}'. \tag{159}$$

Let $Rpp_m(a,\mu,n;\tau_0) = (\omega/2)\sum_{k=m}^{L} B_k^m\ P_k^m(-a)\ \overline{PP_{km}}(a,\mu,n;\tau_0)$, (160) then Eq. (159) can be written as

$$Rpp_m(\bar{\mu},\mu,n;\tau_0) = [1\ /(1/\bar{\mu} + 1/\mu)]\ (\omega/2) \sum_{k=m}^{L} (-1)^{k+m}\ B_k^m$$

$$\times \Big\{ PP_{km1}(0,\mu;\tau_0)\ PP_{km1}(0,\bar{\mu};\tau_0)$$

$$- PP_{km1}(\tau_0,\mu;\tau_0)\ PP_{km1}(\tau_0,\bar{\mu};\tau_0) \Big\} + (\omega/2)$$

$$\times \sum_{k=m}^{L} (-1)^k\ B_k^m \int_0^1 [1\ /(1/\bar{\mu} + 1/\mu_{1n}')]$$

$$\times Rpp_m(\mu_{1n}',\mu,n;\tau_0) \Big\{ PP_{km1}(0,\mu_{1n}';\tau_0)$$

$$\times PP_{km1}(0,\bar{\mu};\tau_0) - PP_{km1}(\tau_0,\mu_{1n}';\tau_0)$$

$$\times PP_{km1}(\tau_0,\bar{\mu};\tau_0) \Big\} \rho(\mu_{1n}',n)/\mu_{1n}'\ d\mu_{1n}'. \qquad (161)$$

Eq. (161) can be solved numerically by successive approximation, once $PP_{km1}(0,\mu;\tau_0)$ and $PP_{km1}(\tau_0,\mu;\tau_0)$ are available.

Next, we are going to use Eq. (148) to get the transmission function of $PP_{km}(\tau,\mu,n;\tau_0)$. Replacing τ by $\tau_0-\tau$, Eq. (148) becomes

$$PP_{km}(\tau_0-\tau,\mu,n;\tau_0) = PP_{km1}(\tau_0-\tau,\mu;\tau_0) + (\omega/2) \sum_{j=m}^{L} B_j^m\ (-1)^m$$

$$\times \int_0^1 PP_{km}(\tau_0-\tau,\mu_{1n}',n;\tau_0)\ \overline{PP_{jm1}}(\mu_{1n}',\mu;\tau_0)$$

$$\times P_j^m(-\mu_{1n}')\ \rho(\mu_{1n}',n)/\mu_{1n}'\ d\mu_{1n}'. \qquad (162)$$

Multiplying Eq. (162) by $\exp(-\tau/\bar{\mu})$ and integrating from zero to τ_0 with respect to τ on both sides, yields

$$\int_0^{\tau_0} PP_{km}(\tau_0-\tau,\mu,n;\tau_0)\ \exp(-\tau/\bar{\mu})\ d\tau$$

$$= \int_0^{\tau_0} PP_{km1}(\tau_0-\tau,\mu;\tau_0)\ \exp(-\tau/\bar{\mu})\ d\tau + (\omega/2) \sum_{j=m}^{L} B_j^m\ (-1)^m$$

$$\times \int_0^1 \left\{ \int_0^{\tau_0} PP_{km}(\tau_0-\tau,\mu_{1n}',n;\tau_0)\ \exp(-\tau/\bar{\mu})\ d\tau \right\}$$

$$\times \overline{PP_{jm1}}(\mu_{1n}',\mu;\tau_0)\ P_j^m(-\mu_{1n}')\ \rho(\mu_{1n}',n)/\mu_{1n}'\ d\mu_{1n}',$$

or

$$\overline{PPI_{km}}(\bar{\mu},\mu,n;\tau_0) = \overline{PPI_{km1}}(\bar{\mu},\mu;\tau_0) + (\omega/2) \sum_{j=m}^{L} B_j^m\ (-1)^m$$

$$\times \int_0^1 \overline{PPI_{km}}(\bar{\mu},\mu_{1n}',n;\tau_0)\ \overline{PP_{jm1}}(\mu_{1n}',\mu;\tau_0)$$

$$\times \rho(\mu_{1n}',n)\ P_j^m(-\mu_{1n}')/\mu_{1n}'\ d\mu_{1n}', \qquad (163)$$

where $\overline{PPI_{km1}}(\bar{\mu},\mu;\tau_0)$ from Eq. (115) is the transmission function of $PP_{km1}(\tau,\mu;\tau_0)$, and

$$\overline{PPI_{km}}(\bar{\mu},\mu,n;\tau_0) = \int_0^{\tau_0} PP_{km}(\tau_0-\tau,\mu,n;\tau_0)\ \exp(-\tau/\bar{\mu})\ d\tau, \qquad (164)$$

is the transmission function of $PP_{km}(\tau,\mu,n;\tau_0)$.

Multiplying Eq. (163) by $(\omega/2)\ B_k^m\ P_k^m(\bar{\mu})$, and summing k from m to L gives

$$(\omega/2) \sum_{k=m}^{L} B_k^m\ P_k^m(\bar{\mu})\ \overline{PPI_{km}}(\bar{\mu},\mu,n;\tau_0)$$

$$= (\omega/2) \sum_{k=m}^{L} B_k^m\ P_k^m(\bar{\mu})\ \overline{PPI_{km1}}(\bar{\mu},\mu;\tau_0) + (\omega/2) \sum_{k=m}^{L} B_k^m\ P_k^m(\bar{\mu})$$

$$\times \int_0^1 \overline{PPI_{km}}(\bar{\mu},\mu_{1n}',n;\tau_0) \left\{ (\omega/2) \sum_{j=m}^{L} (-1)^m\ B_j^m\ \overline{PP_{jm1}}(\mu_{1n}',\mu;\tau_0) \right.$$

$$\left. \times P_j^m(-\mu_{1n}') \right\} \rho(\mu_{1n}',n)/\mu_{1n}'\ d\mu_{1n}'. \qquad (165)$$

Let $T_{PPI_m}(\bar{\mu},a,n;\tau_0) = (\omega/2) \sum_{k=m}^{L} B_k^m P_k^m(\bar{\mu}) \overline{PPI_{km}}(\bar{\mu},a,n;\tau_0)$, (166)

then Eq. (165) can be written as

$$T_{PPI_m}(\bar{\mu},\mu,n;\tau_0) = (\omega/2) \sum_{k=m}^{L} B_k^m P_k^m(\bar{\mu}) \overline{PPI_{km1}}(\bar{\mu},\mu;\tau_0)$$
$$+ \int_0^1 T_{PPI_m}(\bar{\mu},\mu_{1n}',n;\tau_0) \left\{ (\omega/2) \sum_{j=m}^{L} (-1)^m B_j^m \right.$$
$$\left. \times \overline{PP_{jm1}}(\mu_{1n}',\mu;\tau_0) P_j^m(-\mu_{1n}') \right\} \rho(\mu_{1n}',n)/\mu_{1n}'$$
$$\times d\mu_{1n}'. \quad (167)$$

Finally, with the help of Eqs. (130) and (131), Eq. (167) becomes

$$T_{PPI_m}(\bar{\mu},\mu,n;\tau_0) = [1/(1/\bar{\mu} - 1/\mu)] (\omega/2) \sum_{k=m}^{L} B_k^m$$
$$\times \left\{ PP_{km1}(\tau_0,\mu;\tau_0) PP_{km1}(0,\bar{\mu};\tau_0) \right.$$
$$\left. - PP_{km1}(0,\mu;\tau_0) PP_{km1}(\tau_0,\bar{\mu};\tau_0) \right\} + (\omega/2)$$
$$\times \sum_{k=m}^{L} (-1)^k B_k^m \int_0^1 [1/(1/\mu_{1n}' + 1/\mu)]$$
$$\times T_{PPI_m}(\bar{\mu},\mu_{1n}',n;\tau_0) \left\{ PP_{km1}(0,\mu;\tau_0) \right.$$
$$\times PP_{km1}(0,\mu_{1n}';\tau_0) - PP_{km1}(\tau_0,\mu;\tau_0)$$
$$\left. \times PP_{km1}(\tau_0,\mu_{1n}';\tau_0) \right\} \rho(\mu_{1n}',n)/\mu_{1n}' \, d\mu_{1n}'. \quad (168)$$

Equation (168) also can be solved numerically by successive approximation when $PP_{km1}(0,\mu;\tau_0)$ and $PP_{km1}(\tau_0,\mu;\tau_0)$ are known.

Reflected and Transmitted Intensities

The reflected intensity just inside the upper boundary of the medium is given by Eq. (27a). Substituting Eqs. (29), (84), (154), (160), (26d), and (13) into Eq. (27a) yields the following

$$\bar{I}_{Ain}(0,\mu_{in},\mu_o,\phi_{in},n;\tau_0)$$

$$= (I_o\ \mu_o)/(2\pi\ \mu_{in})\ [1-\rho(\mu_o,1/n)]\ /\ [1-(1-\mu_o^2)/n^2]^{1/2}$$

$$\times \sum_{m=0}^{L}(2-\delta_{0m})\ \cos(m\phi_{in})\ Rpp_m\left\{\mu_{in},[1-(1-\mu_o^2)/n^2]^{1/2},n;\tau_0\right\}, (169)$$

where the reflection function Rpp_m is defined as Eq. (160).

The subscript "in" and "A" in Eq. (169) refer to quantities inside the medium and the collimated boundary condition, respectively. An energy balance must be performed across the interface in order to determine the value of the reflected intensity just outside the upper boundary [8]. This procedure is similar to that of the semi-infinite case in Chapter III. If the reflected intensity just outside the upper boundary of the medium is denoted as $\bar{I}_{Ae}$, then we have [8]

$$\bar{I}_{Ae}(0,\mu_e,\mu_o,\phi_{in},n;\tau_0)$$

$$= (I_o\ \mu_o)/2\pi\ [1-\rho(\mu_o,1/n)]\ \left\{1\ -\ \rho\{[1-(1-\mu_e^2)/n^2]^{1/2},n\}\right\}$$

$$\times \sum_{m=0}^{L}(2-\delta_{0m})\ \cos(m\phi_{in})\ Rpp_m\left\{[1-(1-\mu_e^2)/n^2]^{1/2},\right.$$

$$[1-(1-\mu_o^2)/n^2]^{1/2},n;\tau_0\Big\} \;/\; \Big\{ n^2\; [1-(1-\mu_o^2)/n^2]^{1/2}$$

$$\times\; [1-(1-\mu_e^2)/n^2]^{1/2} \Big\}, \qquad (170)$$

where μ_e and μ_{1n} are related by Snell's Law from Eq. (13).

Therefore, for collimated incident intensity, Eq. (161) can be used to determine the intensity reflected from the medium at the upper boundary.

Another equation which needs to be derived is the transmitted intensity at the bottom boundary of the medium. Substituting Eqs. (29), (84), and (26d) into Eq. (9a) yields

$$I^+(\tau_0,\mu_{1n},\phi_{1n}) = I^+(0,\mu_{1n},\phi_{1n})\; \exp(-\tau_0/\mu_{1n}) + (\omega\; I_o\; \mu_o)$$

$$/\; (4\pi\; \mu)\; [1-\rho(\mu_o,1/n)] \sum_{m=0}^{L} (2-\delta_{0m})\; \cos(m\phi_{1n})$$

$$\times \sum_{k=m}^{L} B_k^m\; P_k^m(\mu_{1n}) \int_0^{\tau_0} PP_{km}(t,\mu,n,\tau_0)$$

$$\times\; \exp[-(\tau_0-t)/\mu_{1n}]/\mu_{1n}\; dt, \qquad (171)$$

where μ and μ_o are related by Snell's Law from Eq. (13).

Then, substituting for $I^+(0,\mu_{1n},\phi_{1n})$ from Eq. (27b) and using the definitions of S, R_{PP_m}, and T_{PPI_m} from Eqs. (29), (160), and (166), respectively, we get

$$I_{Ae}^+(\tau_0,\mu_{1n},\mu_o,\phi_{1n},n;\tau_0)$$

$$= (I_o\; \mu_o)/\mu\; [1-\rho(\mu_o,1/n)]\; \delta(\mu_{1n}-\mu)\; \delta(\phi_{1n}-\phi_o)\; \exp(-\tau_0/\mu_{1n})$$

$$+ (I_o\; \mu_o)/(2\pi\; \mu_{1n}\; \mu)\; [1-\rho(\mu_o,1/n)] \sum_{m=0}^{L} (2-\delta_{0m}) \Big\{ \rho(\mu_{1n},n)$$

$$\times \exp(-\tau_0/\mu_{1n}) \cos[m(\phi_{1n}+180°)] R_{PP_m}(\mu_{1n},\mu,n;\tau_0) + \cos(m\phi_{1n})$$

$$\times T_{PPI_m}(\mu_{1n},\mu,n;\tau_0) \Big\}, \tag{172}$$

where μ_o and μ are related by Snell's Law from Eq. (13).

Hence, for collimated incident intensity, Eqs. (161) and (168) can be used to determine the intensity transmitted from the medium at the bottom boundary.

Reflected and Transmitted Fluxes

The reflected and transmitted fluxes can be easily obtained from Eq. (32) as mentioned in the previous chapter.

CHAPTER V

NUMERICAL CALCULATION APPROACH AND RESULTS

Numerical Calculation Approach

General Information

The equations which are derived in this paper can be applied to any interface reflection coefficient. In the numerical calculation of these equations, the following Fresnel's [4] representation has been utilized.

For $\mu \geq \mu_{cr}$,

$$\rho(\mu,n) = 0.5\ \{[(a-\mu)/(a+\mu)]^2 + [(a-\mu/n^2)/(a+\mu/n^2)]^2\},$$

for $\mu < \mu_{cr}$, $\rho(\mu,n) = 1$,

where $a = [1 - (1-\mu^2)\ n^2]^{1/2}/n$,

and $\mu_{cr} = (1 - n^{-2})^{1/2}$.

Because Fresnel's coefficient decreases rapidly as μ increases to become greater than the critical value μ_{cr}, as shown in Fig. (5) on the following page, a dense quadrature being placed around μ_{cr} is necessary [4]. Therefore, four sets of Gaussian quadrature were chosen from μ = 0 to 1 as in Reference [4]. These four intervals were zero to μ_{cr}, μ_{cr} to 1.015 μ_{cr}, 1.015 μ_{cr} to 1.085 μ_{cr}, and 1.085 μ_{cr} to one.

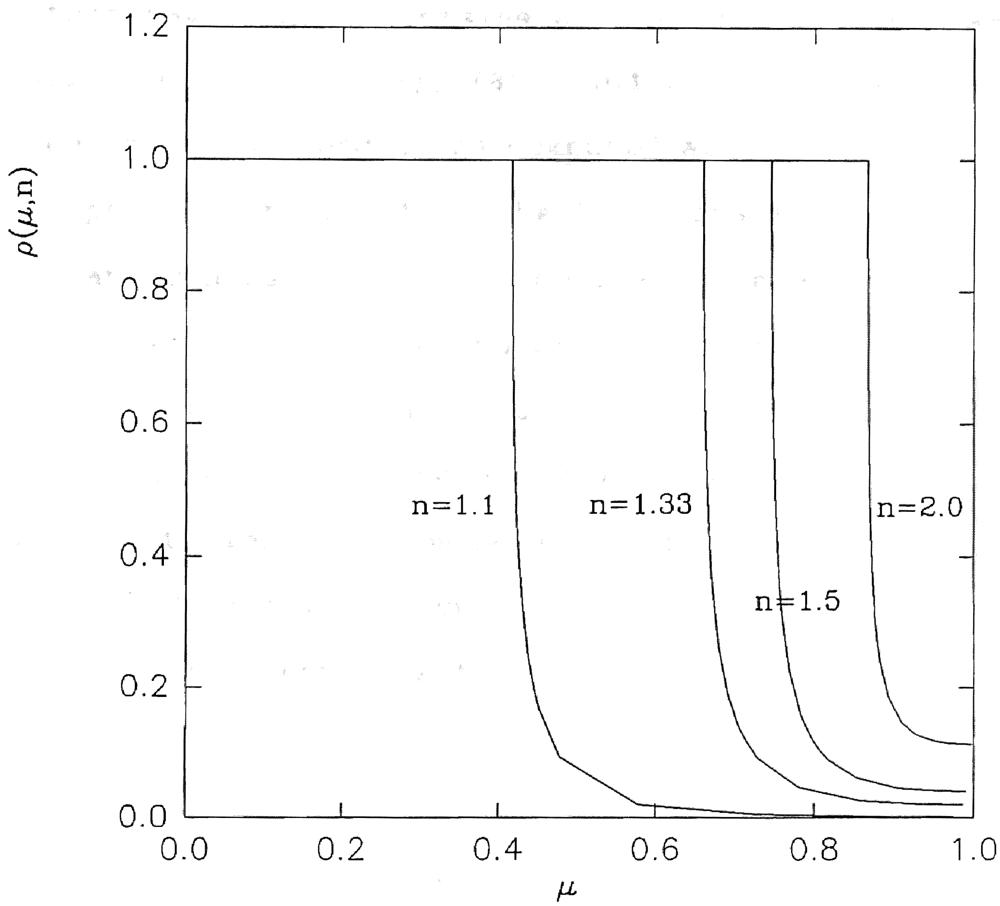

Figure 5. Effect of Refractive Index on Fresnel's Reflectivity

Figure (5) also reveals that the critical point increases as refractive index increases.

To obtain the final results, two different programs were developed. One deals with the semi-infinite case. The other handles the finite case. Both cases use Gaussian quadrature to do the integration.

Semi-Infinite Case

For this case, the successive approximation method was used to solve Eqs. (60), (71), (80), and (82) in order. The first three equations need to be computed at the μ values of quadrature points first, before the functions at the μ values we want can be solved. In general, the execution time, for the semi-infinite computer program to run a specific value of albedo, was about 15 to 20 minutes by a 486 computer while convergence error and number of quadrature points were used being 10^{-8} and 128, respectively. To get more accurate results, using a number of quadrature points not less than 128 is suggested.

Finite Case

For the finite case, at first, the combination of the Runge-Kutta numerical calculation and the successive approximation methods were used to solve Eq. (142) and Eq. (143) simultaneously. Then, the successive approximation method was used to solve Eqs. (151), (152), (161), (168), (170), and (172) in order. These equations, except the last two equations, need to be computed at the μ values of the quadrature points first, before the functions at the μ values we want can be obtained. The execution time, for this computer program to run a specific value of albedo, was about 30 to 40 minutes by a 486 computer while convergence error, number of quadrature points, step size, and optical thickness were used being 10^{-8}, 40, 0.0005, and 1.0, respectively.

Results

Some results of previous work were reproduced in order to examine the accuracy of the computer programs. As we will see later, the current results are very close to those of previous papers.

Comparsion With Previous Work

Basically, three published results were compared with the current results. Those are the results from S. Chandrasekhar [1], X. Y. Jiang [8], and Crosbie and Dougherty [15]. Note that extra cases were run for comparison by using the code of Crosbie and Dougherty [15]. To get good comparison with those results, the exact same number of quadrature points that were chosen for those results need to be used to get the current results. The fewer quadrature points we used, the worse the comparison was. A lot of hard work had to be done for comparison in order to get good accuracy for the current computer programs. Tables 1 through 8 are some examples of the comparisons. These tables yielded very similar comparisons to additional results which were computed but not included in this thesis.

Semi-Infinite Media Results. Tables 1, 2, and 3 give the comparison of the current results with those of S. Chandrasekhar [1], X. Y. Jiang [8], and Crosbie and Dougherty [15], respectively. The convergence errors for these tables are 10^{-11}, 10^{-10}, and 10^{-8}, in order, except when $\omega = 1$ in Table 1. Due to the time consuming convergence problem, the current results for $\omega = 1$ in Table 1 were generated under special conditions. For these particular results, the convergence criterion of 10^{-8} can not be achieved easily. However, these results did converge monotonically as the number of iterations increased. Thus, we decided to use these results for comparison when the number of iterations was equal to ten thousand. Tables 1 through 3 are presented as follows:

TABLE 1

COMPARISON OF SOURCE FUNCTION WITH THAT OF S. CHANDRASEKHAR [1]

When $L = 0$, $n = 1.0$, $x_0 = 1.0$			
ω	μ	$PP_{001}(0,\mu)$	% Error
0.5	0.1	1.0723687620	0.00358
0.5	1.0	1.2512595633	0.00163
0.9	0.1	1.1721430535	0.00486
0.9	1.0	1.8500985167	0.00008
1.0	0.1	1.2473072855	0.00743
1.0	1.0	2.9068047649	0.03423

TABLE 2

COMPARISON OF INTENSITY WITH THAT OF X. Y. JIANG [8]

When L = 0, x_0 = 1.0, μ_o = 1.0				
ω	n	μ_e	$I^-_{Ae}(0,\mu_e,\mu_o,\phi_{1n},n)/I_o$	% Error
0.1	1.33	0.1033816504	0.0170439243	1.0×10^{-8}
0.1	1.33	0.9982693579	0.0295523871	1.0×10^{-8}
0.3	1.33	0.1033816504	0.0617247821	1.0×10^{-8}
0.3	1.33	0.9982693579	0.1078952719	9.3×10^{-8}
0.9	2.00	0.1068921084	0.2452683010	2.4×10^{-6}
0.9	2.00	0.9991701927	0.4909618197	2.5×10^{-6}
0.99	2.00	0.1068921084	0.7759507911	1.3×10^{-5}
0.99	2.00	0.9991701927	1.5903072040	1.3×10^{-5}

TABLE 3

COMPARISON OF SOURCE FUNCTION WITH THAT OF CROSBIE AND DOUGHERTY [15]

When L = 1, n = 1.0, x_0 = 1.0, μ = 1.0					
ω	x_1	$PP_{001}(0,\mu)$	% Error	$PP_{101}(0,\mu)$	% Error
0.1	1.00	1.02104676	3.1×10^{-6}	0.99347068	1.0×10^{-5}
0.5	-1.00	1.32429145	1.0×10^{-6}	0.84063468	3.7×10^{-5}
0.9	1.00	1.69990418	1.0×10^{-6}	0.67283446	7.9×10^{-5}
0.99	-1.00	2.52644528	1.0×10^{-6}	0.21627311	1.0×10^{-5}

Finite Media Results. Table 4 shows the comparison of the current results with those of X. Y. Jiang [8], while Tables 5 through 8 demonstrate the comparison of the current results with those of Crosbie and Dougherty [15]. The convergence criterion of 10^{-8} was used for Tables 4 through 8.

The values of $\bar{I}_{Ae}(0,\mu_e,1.0,\phi_{in},n;\tau_0)/I_o$ and

$\left\{ I^{+}_{Ae}(\tau_0,\mu_{in},1.0,\phi_{in},n;\tau_0) - \text{leading term of Eq. (172)} \right\} / I_o$

from X. Y. Jiang [8] were incorrectly multiplied by μ_e and μ_{in} in reference 8, respectively. Hence, these values were divided by μ_e or μ_{in} before comparing with the current results in Table 4. Tables 4 through 8 are provided as follows:

TABLE 4

COMPARISON OF SOURCE FUNCTION, REFLECTION AND TRANSMISSION FUNCTIONS, AND REFLECTED AND TRANSMITTED INTENSITIES WITH THAT OF X. Y. JIANG [8]

When L = 0, ω = 0.5, n = 1.33, x_0 =1.0, and τ_0 =1.0		
	μ, $\bar{\mu}$, μ_e, or μ_{in} =1.0	% Error
$PP_{00}(0,\mu,n;\tau_0)$	1.4428035131	1.0×10^{-6}
$PP_{00}(\tau_0,\mu,n;\tau_0)$	0.5504674951	1.0×10^{-6}
$R_{PP_0}(\bar{\mu},1.0,n;\tau_0)$	0.6686516863	1.0×10^{-6}
$T_{PPI_0}(1.0,\mu,n,\tau_0)$	0.5775201919	5.6×10^{-6}
$\bar{I}_{Ae}(0,\mu_e,1.0,\phi_{in},n;\tau_0)/I_o$	0.1814956611	1.0×10^{-6}
$\{\overset{+}{I}_{Ae}(\tau_0,\mu_{in},1.0,\phi_{in},n;\tau_0)$ -leading term of Eq. (172)$\}$ $/I_o$	0.2912272229	5.9×10^{-6}

TABLE 5

COMPARISON OF SOURCE FUNCTION WITH THAT OF CROSBIE AND DOUGHERTY [15] FOR ω = 0.3

When L = 2, x_0 = 1.0, x_1 = 0.5, x_2 = 0.4, and τ_0 = 1.0

	μ = 0.1	% Error
$PP_{001}(0,\mu;\tau_0)$	1.040948288134	1.0×10^{-8}
$PP_{001}(\tau_0,\mu;\tau_0)$	0.005706787846	1.7×10^{-8}
$PP_{101}(0,\mu;\tau_0)$	0.087577882213	1.0×10^{-8}
$PP_{101}(\tau_0,\mu;\tau_0)$	0.003473123597	2.9×10^{-8}
$PP_{201}(0,\mu;\tau_0)$	-.495466375841	1.0×10^{-8}
$PP_{201}(\tau_0,\mu;\tau_0)$	0.000806294176	5.0×10^{-8}
$PP_{111}(0,\mu;\tau_0)$	1.002049359067	1.0×10^{-8}
$PP_{111}(\tau_0,\mu;\tau_0)$	0.000679330491	1.5×10^{-8}
$PP_{211}(0,\mu;\tau_0)$	0.293693233489	1.0×10^{-8}
$PP_{211}(\tau_0,\mu;\tau_0)$	0.001057470090	1.0×10^{-8}
$PP_{221}(0,\mu;\tau_0)$	2.981937498879	1.0×10^{-8}
$PP_{221}(\tau_0,\mu;\tau_0)$	0.000771567200	1.3×10^{-8}

TABLE 6

COMPARISON OF SOURCE FUNCTION WITH THAT OF CROSBIE AND DOUGHERTY [15] FOR $\omega = 0.7$

When $L = 2$, $x_0 = 1.0$, $x_1 = 0.5$, $x_2 = 0.4$, and $\tau_0 = 1.5$

	$\mu = 0.9$	% Error
$PP_{001}(0,\mu;\tau_0)$	1.337419849496	1.0×10^{-8}
$PP_{001}(\tau_0,\mu;\tau_0)$	0.450146720797	1.0×10^{-8}
$PP_{101}(0,\mu;\tau_0)$	0.746550220817	1.0×10^{-8}
$PP_{101}(\tau_0,\mu;\tau_0)$	0.314280975741	1.0×10^{-8}
$PP_{201}(0,\mu;\tau_0)$	0.692549389597	1.0×10^{-8}
$PP_{201}(\tau_0,\mu;\tau_0)$	0.153985047989	1.0×10^{-8}
$PP_{111}(0,\mu;\tau_0)$	0.442089960627	1.0×10^{-8}
$PP_{111}(\tau_0,\mu;\tau_0)$	0.096927745785	1.0×10^{-8}
$PP_{211}(0,\mu;\tau_0)$	1.177336725276	1.0×10^{-8}
$PP_{211}(\tau_0,\mu;\tau_0)$	0.242967144832	1.0×10^{-8}
$PP_{221}(0,\mu;\tau_0)$	0.582584439845	1.0×10^{-8}
$PP_{221}(\tau_0,\mu;\tau_0)$	0.112086794454	1.0×10^{-8}

TABLE 7

COMPARISON OF SOURCE FUNCTION WITH THAT OF CROSBIE AND DOUGHERTY [15] FOR $\omega = 0.9$

When $L = 2$, $x_0 = 1.0$, $x_1 = 0.5$, $x_2 = 0.4$, and $\tau_0 = 2.0$		
	$\mu = 1.0$	% Error
$PP_{001}(0,\mu;\tau_0)$	1.657364021291	1.0×10^{-8}
$PP_{001}(\tau_0,\mu;\tau_0)$	0.613211276143	1.0×10^{-8}
$PP_{101}(0,\mu;\tau_0)$	0.682200067598	1.0×10^{-8}
$PP_{101}(\tau_0,\mu;\tau_0)$	0.401969763043	1.0×10^{-8}
$PP_{201}(0,\mu;\tau_0)$	0.982491273638	1.0×10^{-8}
$PP_{201}(\tau_0,\mu;\tau_0)$	0.174552966775	1.0×10^{-8}
$PP_{111}(0,\mu;\tau_0)$	0.000000000000	1.0×10^{-8}
$PP_{111}(\tau_0,\mu;\tau_0)$	0.000000000000	1.0×10^{-8}
$PP_{211}(0,\mu;\tau_0)$	0.000000000000	1.0×10^{-8}
$PP_{211}(\tau_0,\mu;\tau_0)$	0.000000000000	1.0×10^{-8}
$PP_{221}(0,\mu;\tau_0)$	0.000000000000	1.0×10^{-8}
$PP_{221}(\tau_0,\mu;\tau_0)$	0.000000000000	1.0×10^{-8}

TABLE 8

COMPARISON OF SOURCE FUNCTION WITH THAT OF CROSBIE AND DOUGHERTY [15] FOR ω = 0.99

When L = 1, x_0 = 1.0, x_1 = 0.5, and τ_0 = 2.5		
	μ = 0.9	% Error
$PP_{001}(0,\mu;\tau_0)$	1.985305088418	1.0×10^{-8}
$PP_{001}(\tau_0,\mu;\tau_0)$	0.650423055928	1.0×10^{-8}
$PP_{101}(0,\mu;\tau_0)$	0.427121403977	1.0×10^{-8}
$PP_{101}(\tau_0,\mu;\tau_0)$	0.383500238705	1.0×10^{-8}
$PP_{111}(0,\mu;\tau_0)$	0.465130086848	1.0×10^{-8}
$PP_{111}(\tau_0,\mu;\tau_0)$	0.031690744258	1.0×10^{-8}

Current Semi-Infinite Media Results

The results for this part will include source function, reflection function, reflected intensity, and flux. An example of the output data is presented in Appendix D.

Current Finite Media Results

The results for the finite case will include source function, reflection and transmission functions, reflected and transmitted intensities, and reflected and transmitted fluxes. An example of the output data is presented in Appendix E.

Now, an interesting question is how to get the appropriate expansion coefficient x_k for the chosen phase function in order to get the expected results.

For demonstration, we would like to pick a phase function to show how to calculate the exact expansion coefficients x_k.

A phase function may be written as a finite sum of Legendre polynomials [12]

$$P(\mu',\phi',\mu,\phi) = P(\cos\Theta) = \sum_{k=0}^{L} x_k\, P_k(\cos\Theta),$$

where [17]

$$x_k = (2\,k + 1)/2 \int_0^{\pi} P_k(\cos\Theta)\, P(\cos\Theta)\, \sin\Theta\, d\Theta,$$

and Θ is the angle between incident and scattered radiation.

Let us assume that the phase function we want is

$$1.22903223 + 0.4860214865 \cos\Theta - 0.2916128919 \sin\Theta.$$

This phase function has been chosen to satisfy

$$1/4\pi \int P(\cos\Theta)\, d\Theta = 1.0.$$

Using above equations, we can get $x_0 = 1$, $x_1 = 0.4860214865$, and $x_2 = 0.1431451436$ for the number of Legendre polynomials equal to two. From Fig. (6), on the next page, we observe that the approximate phase functions approach the form of the original phase function as the number of Legendre polynomials increases. This is exactly what we expect. We also see that the number of Legendre polynomials being equal to two is a pretty good approximation for this particular phase function. Note that the probability of this phase function is the length P(cosΘ), as shown in Fig. (6), divided by 4π.

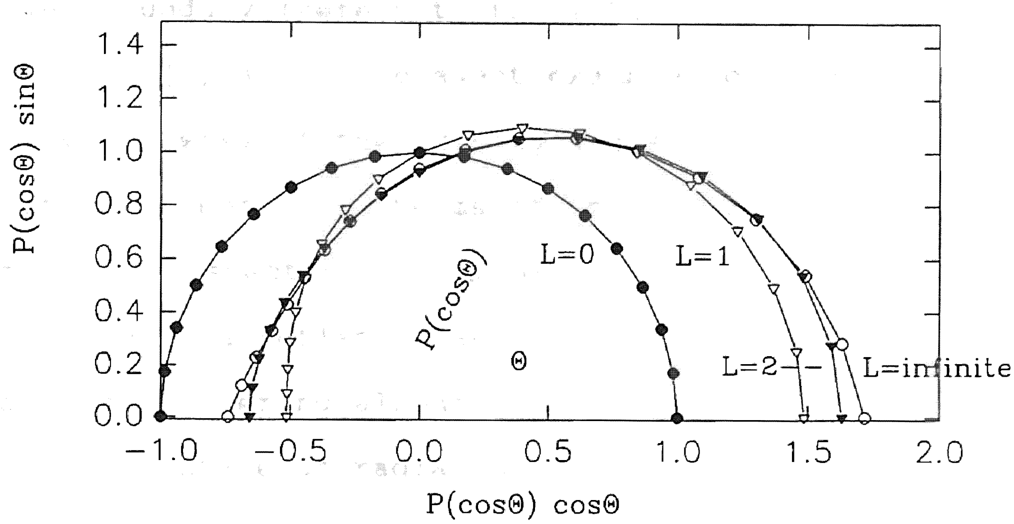

Figure 6. Plot of Phase Functions for L=0, 1, 2, and Infinite

Figures (7) to (18) are the plots of results based on the expansion coefficient x_k calculated on the previous page. The tabular results for these plots were presented in Appendix F for reference. However, nothing in literature can be compared directly to these tabular results as mentioned in Chapter I.

Note that the reflected intensity in Figs. (7), (9), (11), and (17) represents only the intensity reflected from the medium due to scattering within the medium without adding the direct interface reflection of incident radiation ($\mu_o/\mu\ \rho(\mu_o, 1/n)\ I_o$). In addition, the transmitted intensity in Figs. (8), (10), (12), and (18) represents the intensity subtracting the incident intensity reaching the bottom interface undisturbed from the total intensity penetrating the lower boundary (refers to Eq. (172)).

Figures (7) and (8) present examples of the reflected intensity in terms of the exiting angle (μ_e) and the transmitted intensity in terms of the angle inside the medium (μ_{in}), respectively. As we expected, Figs. (7) and (8) show that the reflected and transmitted intensities increase as scattering albedo increases. The reason is that a larger percentage of radiation will be scattered in the medium when scattering albedo becomes greater. This effect will contribute to the intensities on both boundaries. Figures (7) and (8) also reveal that both intensities are almost constant for all angles when albedo is equal to 0.1.

Figure (9) reveals that increasing refractive index causes reflected intensity to decrease. There are two major reasons for this tendency. One is that less incident radiation will enter the medium when the interface reflection coefficient is larger, that is, when a larger refractive index exists (refers to Eq. (14)). Hence, less intensity is reflected from the medium when less incident radiation enters as refractive index increases. The other reason is that total internal reflection occurs for all angles less than the critical angle. Thus, the energy exiting the medium decreases as the critical angle increases, that is, as refractive index increases.

Transmitted intensity increases as refractive index increases as revealed in Fig. (10). This is due to reasons similar to those mentioned in the previous paragraph. Total internal reflection occurs for all angles less than the critical angle. Therefore, the reflected energy from the top boundary of the medium increases as the critical angle increases, that is, as refractive index increases. Hence, more intensity penetrates the lower boundary of the medium when more energy is reflected from the top boundary of the medium as refractive index increases. Note that the rapid changes of the curves for $n = 1.33$ and $n = 2.0$ in Fig. (10) are caused by the critical angle of the reflectivity ρ. As mentioned before, total internal reflection occurs for all angles less than the critical angle. That is, less energy is reflected from the top boundary of the medium while the angles are larger than the critical angle. Thus, less energy penetrates the bottom boundary of the medium as the angles become larger than the critical angle.

Examples of the reflected intensity in terms of the exiting angle (μ_e) and the transmitted intensity in terms of the angle inside the medium (μ_{in}) are shown in Figs. (11) and (12), respectively. As we expected, the curve for $L = 0$ is larger than those for $L = 1$ and 2 in Fig. (11) while the curve for L = 0 is smaller than those for L = 1 and 2 in Fig. (12). This is due to the reason that we picked up a forward scattering phase function, as shown in Fig. (6). The cross-over of results for L = 1 and L = 2 in Figs. (11) and (12) is due largely to the fact that the approximate phase function does not approach smoothly the chosen phase function as the number of Legendre polynomials increases.

Examples of the top boundary source function based on the angle (μ) and the lower boundary source function based on the angle (μ) are shown in Figs. (13) and (14), in order. Both the source functions for L = 0 are the total source functions, while those for L = 1 and 2 are only one component of the total source functions (see Eq. (29)). Therefore, it is hard to tell further from Figs. (13) and (14) at this moment.

Figures (15) and (16) show that both the reflected and transmitted fluxes are almost linear for the number of Legendre polynomials equal to zero, one, or two. However, the trend in Fig. (15) seems contrary to intuition because of the reflected intensity trend in Fig. (11).

Figure (17) reveals that the reflected intensity for optical thickness approaching infinity is greater than that for optical thickness being 3 or 1. The reason is due to the fact that there is no chance to scatter outside from the bottom interface when optical thickness is infinite. Therefore, more intensity will scatter out from the top boundary when τ_0 is infinite compared with that of other optical thicknesses.

Figure (18) reveals an interesting fact. The transmitted intensity is small when optical thickness is very small, for example, $\tau_0 = 0.001$. The reason for this is that there is less chance for radiation to scatter out in a very small optical thickness. However, the transmitted intensity gets large when optical thickness increases. The explanation for this part is that intensity has a greater chance of scattering as optical thickness increases. Nevertheless, after a certain optical thickness, which is around $\tau_0 = 1.0$ in this case, the transmitted intensity decreases. This is due to the reason that the intensity not only scatters out from the bottom but also scatters out from the top. Therefore, for this case, it seems that the transmitted intensity starts to decrease for optical thickness larger than one. Note that Fig. (18) also reveals the critical angle effect, which causes a rapid change in the curve, is damped out for optical thickness equal to three.

Table 19 in Appendix F reveals the accuracy of the current results is approximately 10^{-5} for source function while it is about 10^{-8} for heat flux. There are three major reasons to require the high accuracy for the current results. One is that we need to know if the current computer codes are truly accurate from the part of comparison with previous work. Another reason is that we need to sum the source functions to get the intensity.

Thus, we need more accurate source functions in order to get the accurate intensity. The other reason is that we need to have more accurate one-dimensional results to superpose to get the accurate two-dimensional results (for future work).

Now, Figs. (7) to (18) are presented as follows:

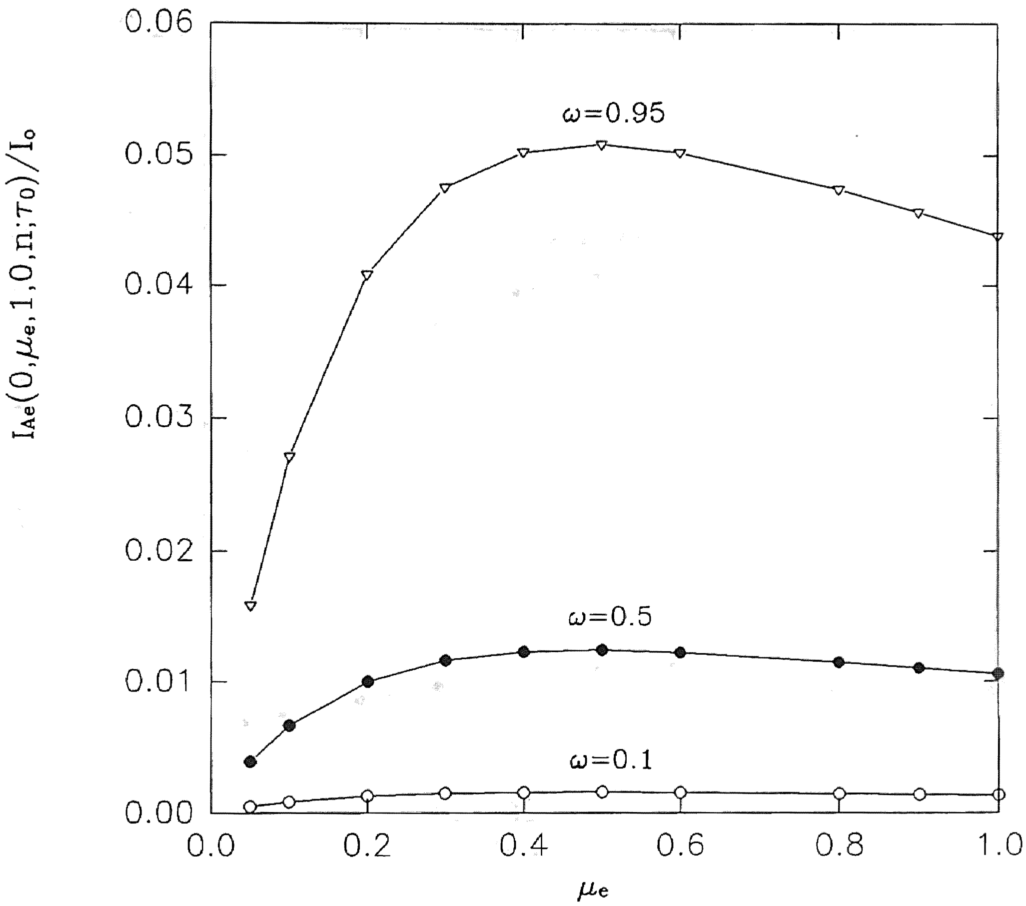

Figure 7. Effect of the Albedo on the Reflected Intensity. L=2, n=1.33, and τ_0=1.0

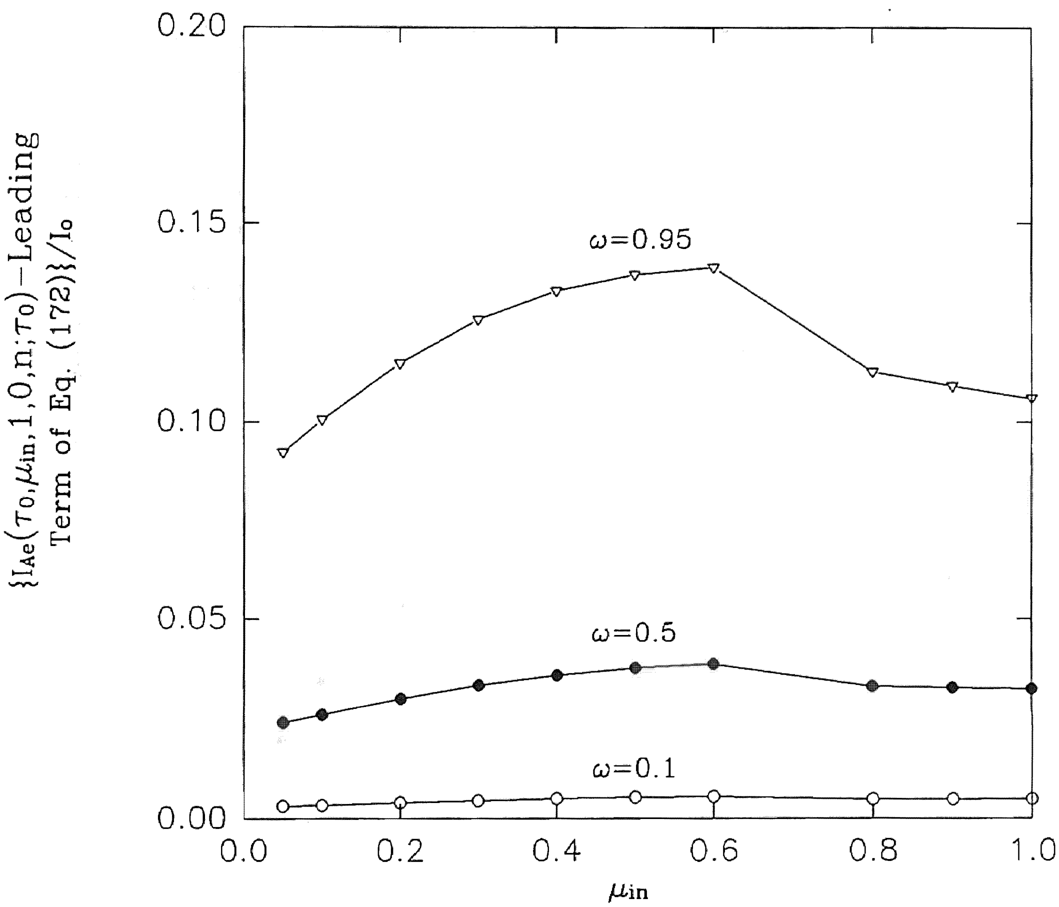

Figure 8. Effect of the Albedo on the Transmitted Intensity. L=2, n=1.33, and τ_0=1.0

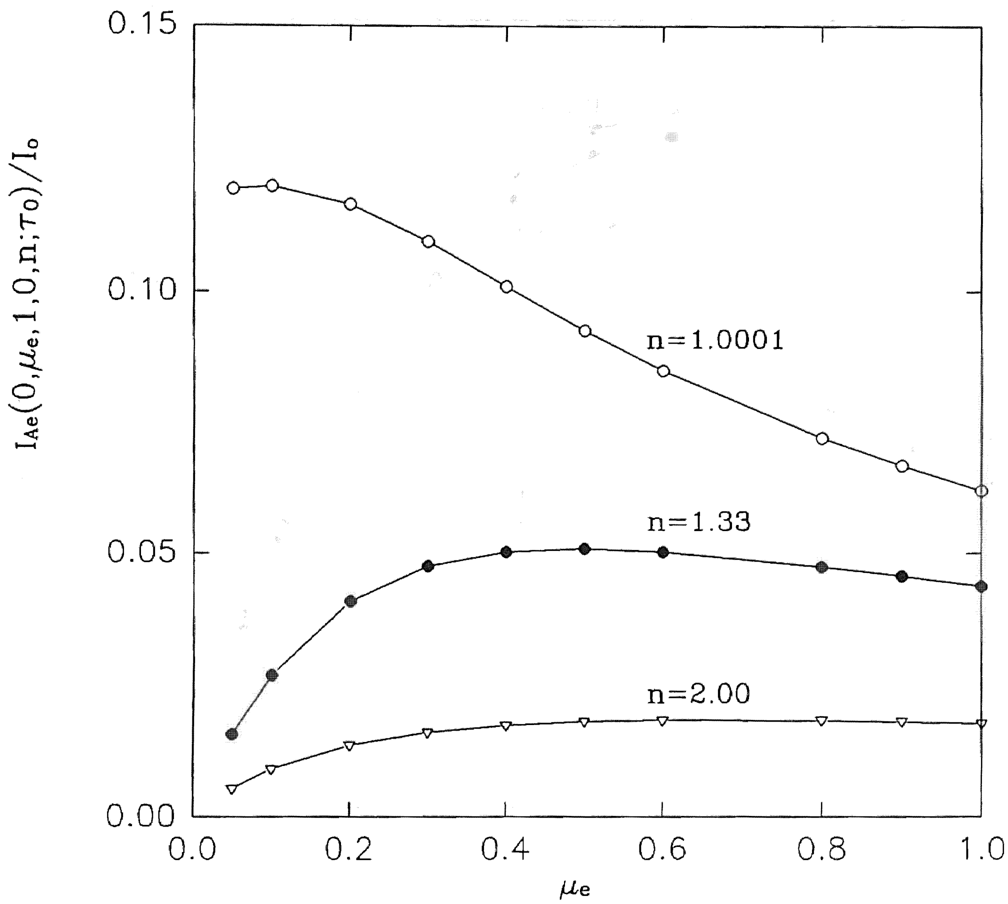

Figure 9. Effect of Refractive Index on the Reflected Intensity. L=2, ω=0.95, and τ_0=1.0

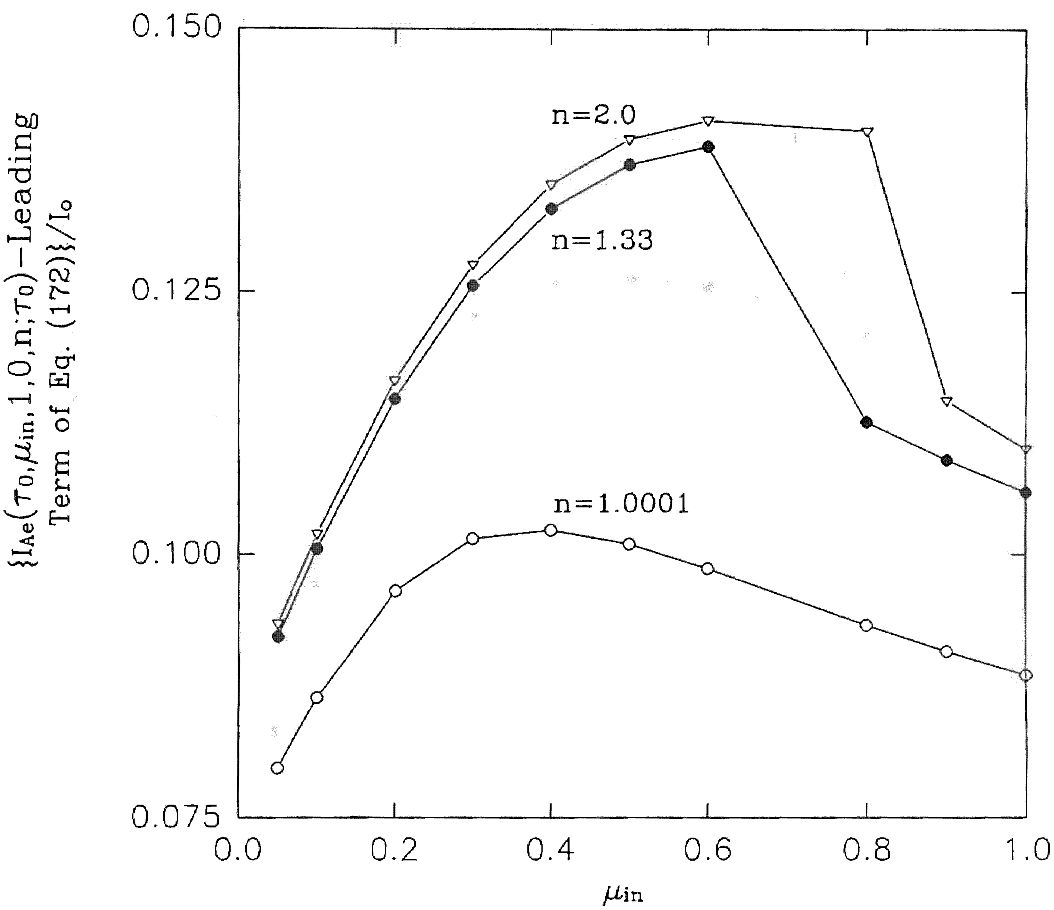

Figure 10. Effect of Refractive Index on the Transmitted Intensity. L=2, ω=0.95, and τ_0=1.0

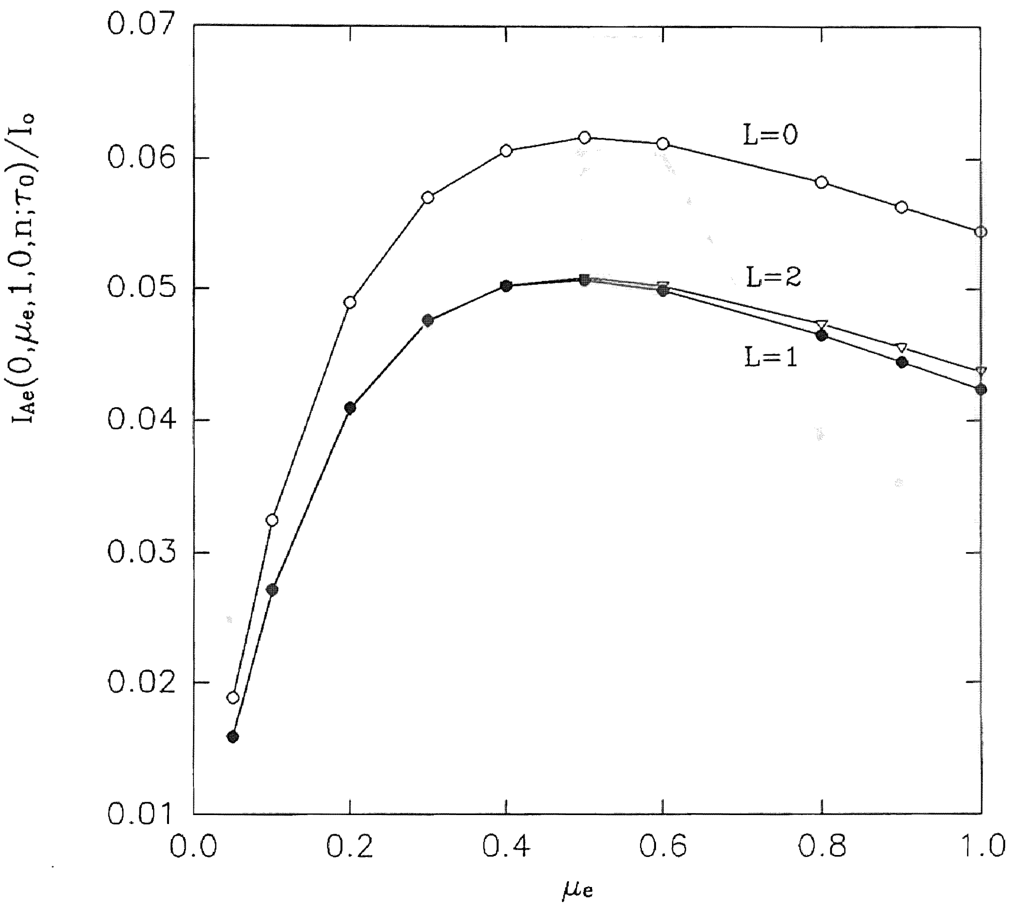

Figure 11. Effect of No. of Legendre Polynomials on the Reflected Intensity. n=1.33, ω=0.95, and τ_0=1.0

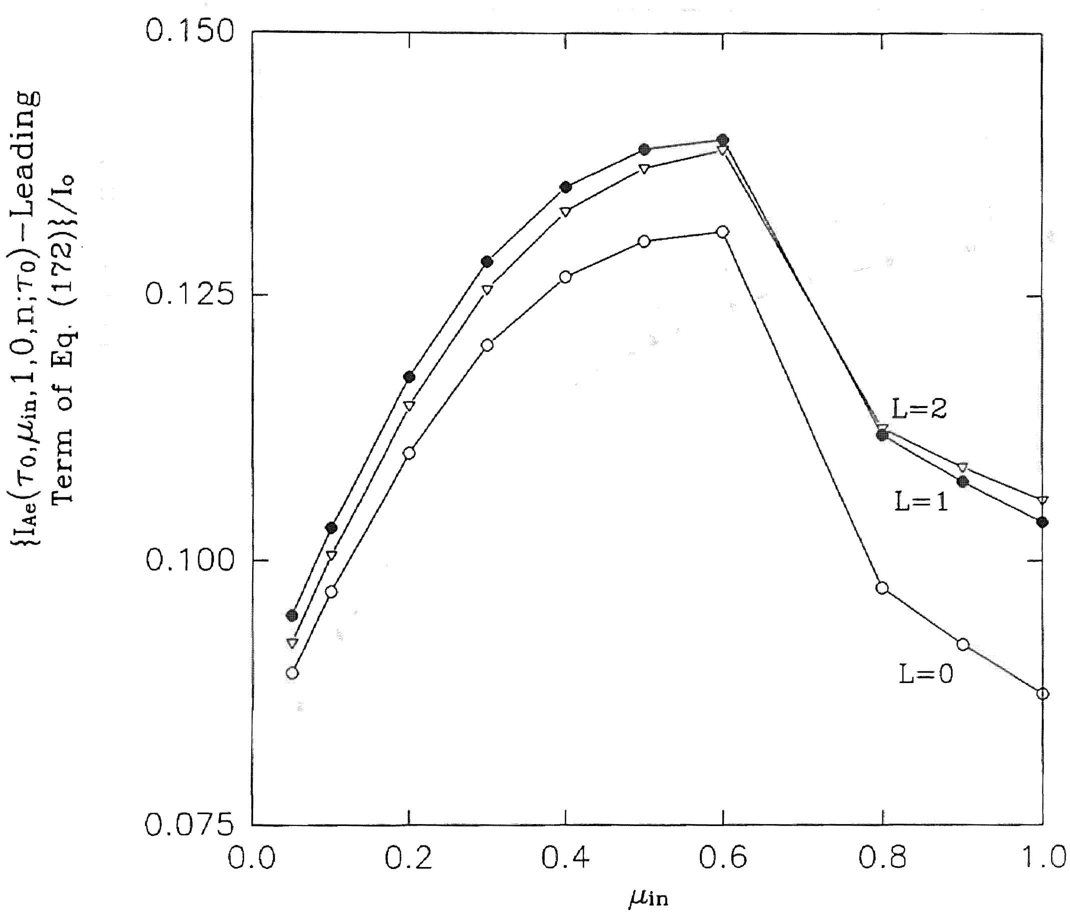

Figure 12. Effect of No. of Legendre Polynomials on the Transmitted Intensity. $n=1.33$, $\omega=0.95$, and $\tau_0=1.0$

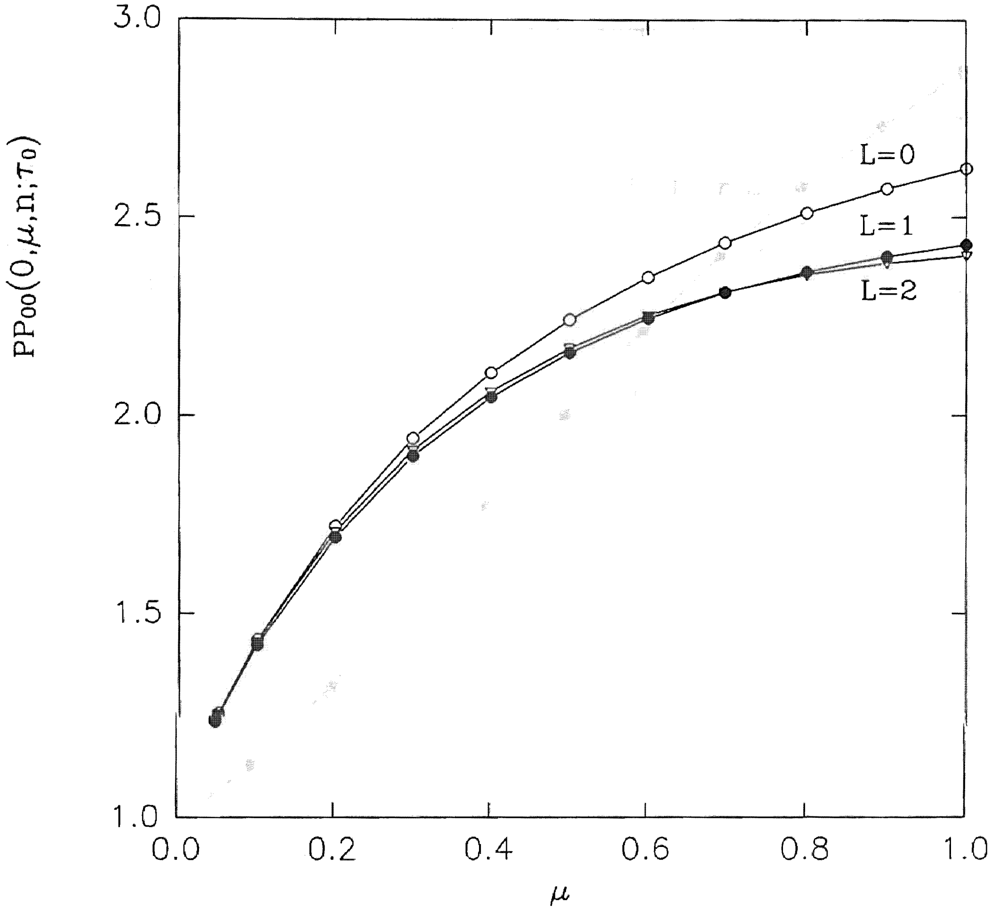

Figure 13. Effect of No. of Legendre Polynomials on the Source Function. n=1.33, ω=0.95, and τ_0=1.0

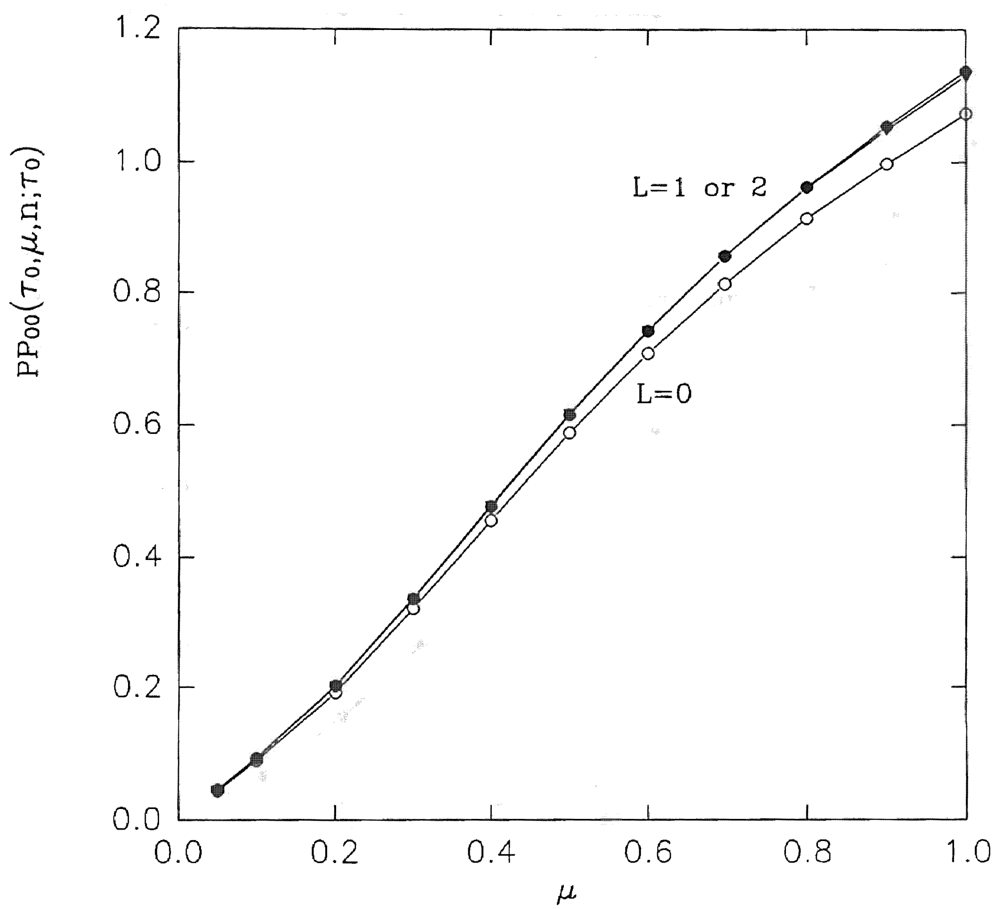

Figure 14. Effect of No. of Legendre Polynomials on the Source Function. n=1.33, ω=0.95, and τ_0=1.0

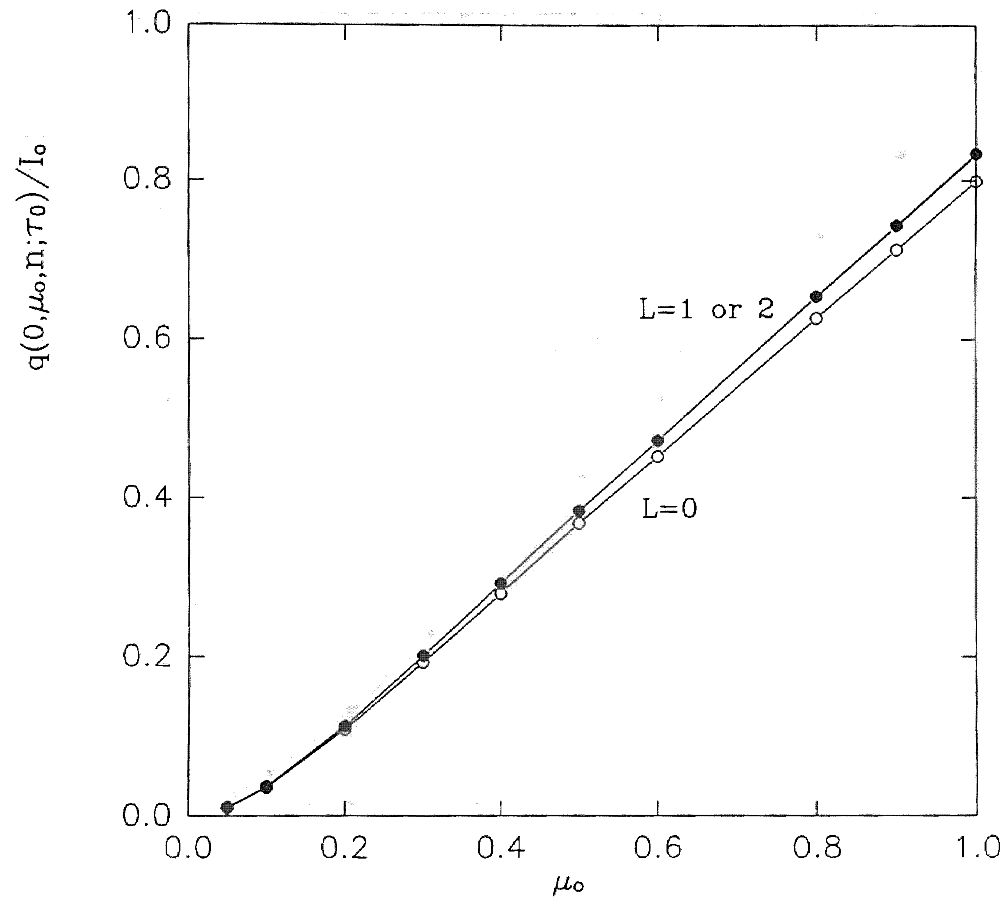

Figure 15. Effect of No. of Legendre Polynomials on the Reflected Flux. n=1.33, ω=0.95, and τ_0=1.0

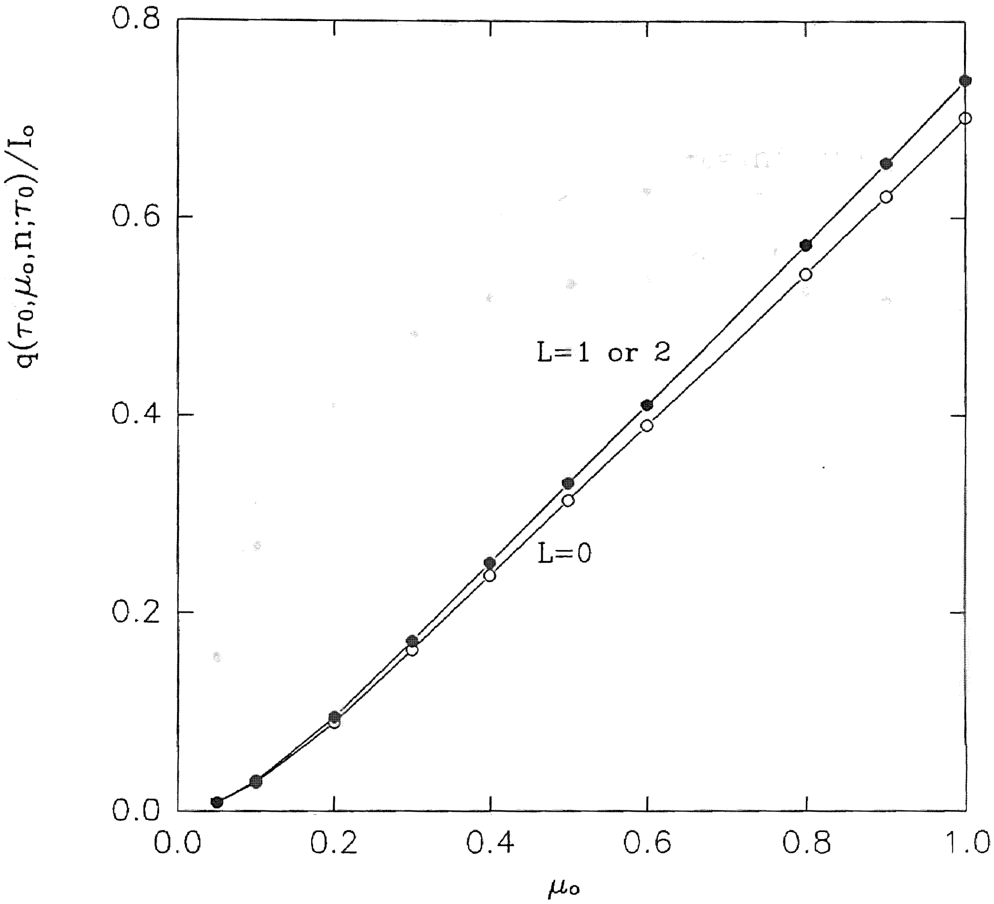

Figure 16. Effect of No. of Legendre Polynomials on the Transmitted Flux. n=1.33, ω=0.95, and τ_0=1.0

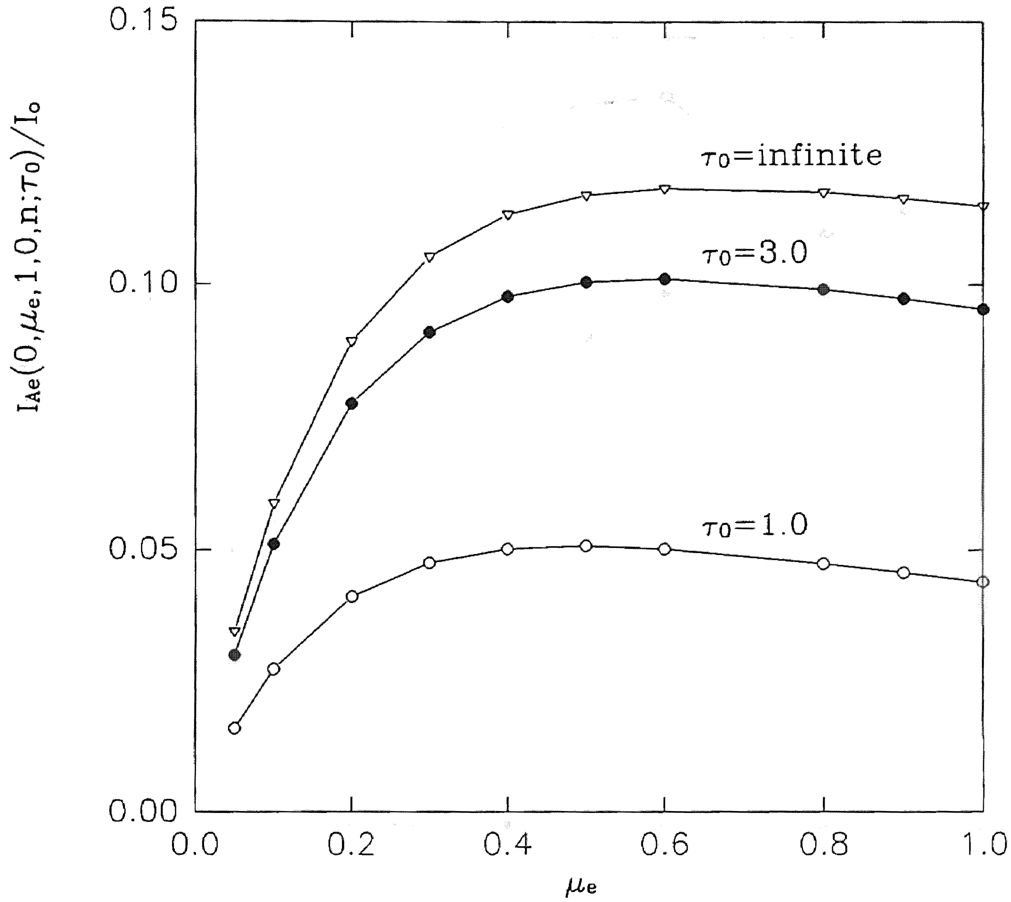

Figure17. Effect of Optical Thickness on the Reflected Intensity. L=2, n=1.33, and ω=0.95

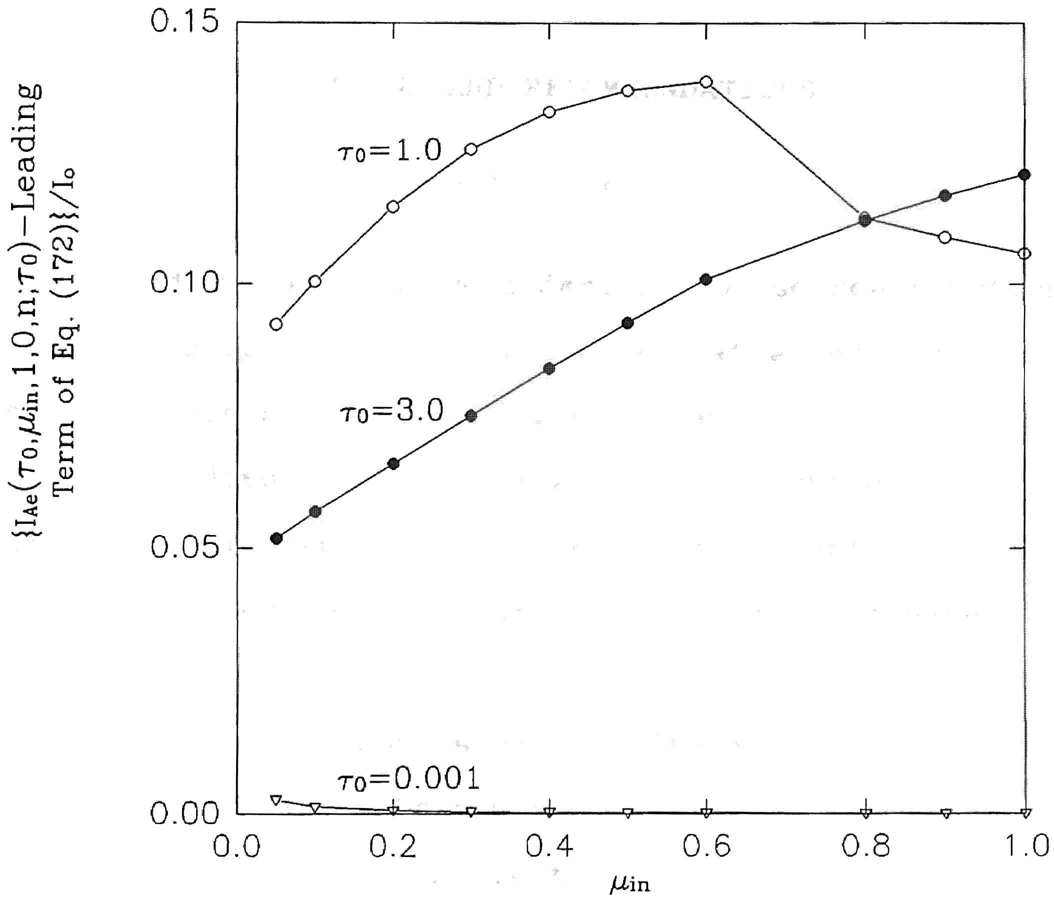

Figure 18. Effect of Optical Thickness on the Transmitted Intensity. L=2, n=1.33, and ω=0.95

CHAPTER VI

CONCLUSIONS AND RECOMMENDATIONS

Conclusions

The exact expressions were derived for source function, reflection and transmission functions, reflected and transmitted intensities, and reflected and transmitted fluxes in one-dimensional media with a reflective top boundary and anisotropic scattering. The numerical results were also obtained for these properties under the same conditions.

The programs for both semi-infinite and finite media were designed for any albedo ω from 0 to 1, cosine of incident polar angle μ from 0.0 to 1 by 0.1, and number of Legendre polynomials L from 0 to 3 for semi-infinite media while it was from 0 to 2 for finite media. The results for finite optical thickness τ_0 were computed from 0.001 to 10 by various step sizes. Both programs for semi-infinite and finite media were written on a personal computer. Therefore, the number of Legendre polynomials L can not be very large, otherwise, the execution file is too large to run on a PC. In the future, we will convert the programs to a large number of Legendre polynomials to run them on a supermini or mainframe.

Some tables for comparison of current results with those from previous work were presented in Chapter V. The results of those comparisons were good and more precise for finite media, but a little less accurate for semi-infinite media.

For the semi-infinite case, the average % errors were 8.6×10^{-3}, 3.8×10^{-6}, and 1.5×10^{-5} which compared with Chandrasekhar [1], Jiang [8], and Crosbie and Dougherty [15], respectively. The effect of albedo ω was also shown in the results. The % error was very small and time consumed for convergence was short for small albedo as compared with the error and time for higher albedo. Especially, when albedo approached to one, convergence was difficult. It took about three hours to get convergence for unit albedo compared with around ten minutes to get convergence for lower albedo ($\omega = 0.1$) in a 486 computer system.

For the finite case, the average % errors were 6.6×10^{-6} and 3×10^{-9} while compared with Jiang [8] and Crosbie and Dougherty [15], respectively. The effect of albedo ω was less sensitive in this case. It took about three minutes to get convergence for lower albedo as compared with four minutes for unit albedo on a 286 computer system while optical thickness, convergence error, and step size were used, being 0.0005, 10^{-8}, and 0.0005, respectively.

Recommendations

The personal computer is not recommended for running the semi-infinite program due to the time consuming convergence problem as mentioned in the previous section.

Future work should be focused not only on modifying the current programs to run on more powerful computers, but also expanding the current one-dimensional results to either the two-dimensional results or the one-dimensional results with reflective top and bottom boundaries. To make a two-dimensional problem, we can either change the boundary condition to a single laser beam or have the media become finite. Moreover, running more results for both cases is necessary to allow us to observe the effects of various physical phenomena.

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APPENDICES

APPENDIX A

DERIVATION OF SOME EQUATIONS FOR CHAPTER III AND CHAPTER IV

In this appendix, three important equations will be derived in detail. One is the equation (57) in Chapter III, another is the equation (126) in Chapter IV, and the other is the equation (127) also in Chapter IV.

In order to get the relationship between $\overline{PP_{lml}}(s,\mu)$ and $\overline{PP_{lml}}(\mu,s)$ of Eq. (57), we need to first consider the following set of equations. Replacing k by i in Eq. (35a), then multiplying this equation by $PP_{lml}(\tau,s)$, integrating with respect to τ from zero to infinity, and with the help of Eq. (51), we have

$$\int_0^\infty PP_{lml}(\tau,\mu)\ PP_{lml}(\tau,s)\ d\tau$$

$$= \int_0^\infty \left\{ P_l^m(\mu)\ \exp(-\tau/\mu) + (\omega/2) \sum_{k=m}^{L} B_k^m \int_0^\infty PP_{kml}(t,\mu) \right.$$

$$\left. \times\ K_{2kml}(\tau-t)\ dt \right\} PP_{lml}(\tau,s)\ d\tau$$

$$= P_l^m(\mu)\ \overline{PP_{lml}}(\mu,s) + (\omega/2) \sum_{k=m}^{L} B_k^m \int_0^\infty \left\{ \int_0^\infty PP_{kml}(t,\mu) \right.$$

$$\left. \times\ K_{2kml}(\tau-t)\ dt \right\} PP_{lml}(\tau,s)\ d\tau, \qquad (173)$$

and

$$\int_0^\infty PP_{1m1}(\tau,s)\ PP_{1m1}(\tau,\mu)\ d\tau$$

$$= \int_0^\infty \left\{ P_1^m(s)\ \exp(-\tau/s) + (\omega/2) \sum_{k=m}^{L} B_k^m \int_0^\infty PP_{km1}(t,s) \right.$$

$$\left. \times\ K_{2km1}(\tau-t)\ dt \right\} PP_{1m1}(\tau,\mu)\ d\tau$$

$$= P_1^m(s)\ \overline{PP_{1m1}}(s,\mu) + (\omega/2) \sum_{k=m}^{L} B_k^m \int_0^\infty \left\{ \int_0^\infty PP_{km1}(t,s) \right.$$

$$\left. \times\ K_{2km1}(\tau-t)\ dt \right\} PP_{1m1}(\tau,\mu)\ d\tau$$

$$= P_1^m(s)\ \overline{PP_{1m1}}(s,\mu) + (\omega/2) \sum_{k=m}^{L} B_k^m \int_0^\infty \left\{ \int_0^\infty PP_{1m1}(t,\mu) \right.$$

$$\left. \times\ K_{2km1}(t-\tau)\ dt \right\} PP_{km1}(\tau,s)\ d\tau. \qquad (174)$$

Using the definition of $K_{2jmk}(\tau-t)$ in Eq. (26c), Eq. (26c) can be written as

$$K_{2km1}(\tau-t) = \int_0^1 \exp[-|\tau-t|/\mu_{1n}']\ P_k^m[\text{sign}(\tau-t)\ \mu_{1n}']$$

$$\times\ P_1^m[\text{sign}(\tau-t)\ \mu_{1n}']/\mu_{1n}'\ d\mu_{1n}', \qquad (175)$$

and

$$K_{2km1}(t-\tau) = \int_0^1 \exp[-|t-\tau|/\mu_{1n}']\ P_k^m[\text{sign}(t-\tau)\ \mu_{1n}']$$

$$\times\ P_1^m[\text{sign}(t-\tau)\ \mu_{1n}']/\mu_{1n}'\ d\mu_{1n}',$$

or by knowing [12], $P_k^m(-\mu) = (-1)^{k+m}\ P_k^m(\mu)$, the above equation becomes

$$K_{2km1}(t-\tau) = \int_0^1 \exp[-|\tau-t|/\mu_{1n}']\ (-1)^{k+m}\ P_k^m[\text{sign}(\tau-t)\ \mu_{1n}']$$

$$\times\ (-1)^{1+m}\ P_1^m[\text{sign}(\tau-t)\ \mu_{1n}']/\mu_{1n}'\ d\mu_{1n}'$$

$$= (-1)^{i+k} \int_0^1 \exp[-|\tau-t|/\mu_{1n}'] \, P_k^m[\text{sign}(\tau-t)\ \mu_{1n}']$$

$$\times P_i^m[\text{sign}(\tau-t)\ \mu_{1n}']/\mu_{1n}' \, d\mu_{1n}'. \qquad (176)$$

Comparison of Eqs. (175) and (176), yields the following useful equation

$$K_{2kmi}(t-\tau) = (-1)^{k+i} K_{2kmi}(\tau-t). \qquad (177)$$

From Eqs. (173), (174), and (177), we realize we need another set of equations. Multiplying both Eqs. (173) and (174) by $(-1)^i B_i^m$ and summing from $i = m$ to L, and with the help of Eqs. (176) and (177), we get

$$\sum_{i=m}^{L} (-1)^i B_i^m \int_0^\infty PP_{im1}(\tau,\mu)\ PP_{im1}(\tau,s)\ d\tau$$

$$= \sum_{i=m}^{L} (-1)^i B_i^m P_i^m(\mu)\ \overline{PP_{im1}}(\mu,s) + (\omega/2) \sum_{i=m}^{L} (-1)^i B_i^m \sum_{k=m}^{L} B_k^m$$

$$\times \int_0^\infty \left\{ \int_0^\infty PP_{km1}(t,\mu)\ K_{2kmi}(\tau-t)\ dt \right\} PP_{im1}(\tau,s)\ d\tau, \qquad (178)$$

and

$$\sum_{k=m}^{L} (-1)^k B_k^m \int_0^\infty PP_{km1}(\tau,s)\ PP_{km1}(\tau,\mu)\ d\tau$$

$$= \sum_{k=m}^{L} (-1)^k B_k^m \left\{ P_k^m(s)\ \overline{PP_{km1}}(s,\mu) + (\omega/2) \sum_{i=m}^{L} B_i^m \right.$$

$$\left. \times \int_0^\infty \{ \int_0^\infty PP_{km1}(t,\mu)\ K_{2imk}(t-\tau)\ dt \} PP_{im1}(\tau,s)\ d\tau \right\}$$

$$= \sum_{k=m}^{L} (-1)^k B_k^m P_k^m(s)\ \overline{PP_{km1}}(s,\mu) + (\omega/2) \sum_{k=m}^{L} (-1)^k B_k^m \sum_{i=m}^{L} B_i^m$$

$$\times \int_0^\infty \left\{ \int_0^\infty PP_{km1}(t,\mu)\ K_{2kmi}(t-\tau)\ dt \right\} PP_{im1}(\tau,s)\ d\tau$$

$$= \sum_{k=m}^{L} (-1)^k B_k^m P_k^m(s) \overline{PP_{kml}}(s,\mu) + (\omega/2) \sum_{k=m}^{L} (-1)^k B_k^m \sum_{l=m}^{L} B_l^m$$

$$\times \int_0^\infty \left\{ \int_0^\infty PP_{kml}(t,\mu) \ (-1)^{k+l} \ K_{2kml}(\tau-t) \ dt \right\} PP_{lml}(\tau,s) \ d\tau$$

$$= \sum_{k=m}^{L} (-1)^k B_k^m P_k^m(s) \overline{PP_{kml}}(s,\mu) + (\omega/2) \sum_{l=m}^{L} (-1)^l B_l^m \sum_{k=m}^{L} B_k^m$$

$$\times \int_0^\infty \left\{ \int_0^\infty PP_{kml}(t,\mu) \ K_{2kml}(\tau-t) \ dt \right\} PP_{lml}(\tau,s) \ d\tau. \qquad (179)$$

Now, comparison of Eq. (178) and Eq. (179) yields

$$\sum_{l=m}^{L} (-1)^l B_l^m P_l^m(\mu) \overline{PP_{lml}}(\mu,s) = \sum_{k=m}^{L} (-1)^k B_k^m P_k^m(s) \overline{PP_{kml}}(s,\mu),$$

or

$$\sum_{l=m}^{L} (-1)^l B_l^m P_l^m(\mu) \overline{PP_{lml}}(\mu,s) = \sum_{l=m}^{L} (-1)^l B_l^m$$

$$\times P_l^m(s) \overline{PP_{lml}}(s,\mu). \qquad (180)$$

Then, by knowing [12], $P_k^m(-\mu) = (-1)^{k+m} P_k^m(\mu)$, Eq. (180) becomes

$$(-1)^m \sum_{l=m}^{L} B_l^m P_l^m(-\mu) \overline{PP_{lml}}(\mu,s) = (-1)^m \sum_{l=m}^{L} B_l^m P_l^m(-s)$$

$$\times \overline{PP_{lml}}(s,\mu),$$

or

$$\sum_{l=m}^{L} B_l^m P_l^m(-\mu) \overline{PP_{lml}}(\mu,s) = \sum_{l=m}^{L} B_l^m P_l^m(-s) \overline{PP_{lml}}(s,\mu). \qquad (57)$$

Our next objective is to get the relationship between $\overline{PP_{lml}}(s,\mu;\tau_0)$ and $\overline{PP_{lml}}(\mu,s;\tau_0)$ of Eq. (126). The procedure of this derivation is very similar to that of obtaining Eq. (57) in the previous case.

Replacing k by i in Eq. (86), then multiplying this equation by $PP_{im1}(\tau,s;\tau_0)$, integrating with respect to τ from zero to τ_0, and with the help of Eq. (114), we have

$$\int_0^{\tau_0} PP_{im1}(\tau,\mu;\tau_0)\ PP_{im1}(\tau,s;\tau_0)\ d\tau$$

$$= \int_0^{\tau_0} \Big\{ P_i^m(\mu)\ \exp(-\tau/\mu) + (\omega/2) \sum_{k=m}^{L} B_k^m \int_0^{\tau_0} PP_{km1}(t,\mu;\tau_0)$$

$$\times K_{2kmi}(\tau-t)\ dt \Big\}\ PP_{im1}(\tau,s;\tau_0)\ d\tau$$

$$= P_i^m(\mu)\ \overline{PP_{im1}}(\mu,s;\tau_0) + (\omega/2) \sum_{k=m}^{L} B_k^m \int_0^{\tau_0} \Big\{ \int_0^{\tau_0} PP_{km1}(t,\mu;\tau_0)$$

$$\times K_{2kmi}(\tau-t)\ dt \Big\}\ PP_{im1}(\tau,s;\tau_0)\ d\tau, \qquad (181)$$

and

$$\int_0^{\tau_0} PP_{im1}(\tau,s;\tau_0)\ PP_{im1}(\tau,\mu;\tau_0)\ d\tau$$

$$= \int_0^{\tau_0} \Big\{ P_i^m(s)\ \exp(-\tau/s) + (\omega/2) \sum_{k=m}^{L} B_k^m \int_0^{\tau_0} PP_{km1}(t,s;\tau_0)$$

$$\times K_{2kmi}(\tau-t)\ dt \Big\}\ PP_{im1}(\tau,\mu;\tau_0)\ d\tau$$

$$= P_i^m(s)\ \overline{PP_{im1}}(s,\mu;\tau_0) + (\omega/2) \sum_{k=m}^{L} B_k^m \int_0^{\tau_0} \Big\{ \int_0^{\tau_0} PP_{km1}(t,s;\tau_0)$$

$$\times K_{2kmi}(\tau-t)\ dt \Big\}\ PP_{im1}(\tau,\mu;\tau_0)\ d\tau$$

$$= P_i^m(s)\ \overline{PP_{im1}}(s,\mu;\tau_0) + (\omega/2) \sum_{k=m}^{L} B_k^m \int_0^{\tau_0} \Big\{ \int_0^{\tau_0} PP_{im1}(t,\mu;\tau_0)$$

$$\times K_{2kmi}(t-\tau)\ dt \Big\}\ PP_{km1}(\tau,s;\tau_0)\ d\tau. \qquad (182)$$

From Eqs. (181), (182), and (177), we understand that we need another set of equations. Multiplying both Eqs. (181) and (182) by $(-1)^i B_i^m$ and summing from i = m to L, and with the help of Eq. (177), we get

$$\sum_{i=m}^{L} (-1)^i B_i^m \int_0^{\tau_0} PP_{im1}(\tau,\mu;\tau_0) PP_{im1}(\tau,s;\tau_0) d\tau$$

$$= \sum_{i=m}^{L} (-1)^i B_i^m P_i^m(\mu) \overline{PP_{im1}}(\mu,s;\tau_0) + (\omega/2) \sum_{i=m}^{L} (-1)^i B_i^m \sum_{k=m}^{L} B_k^m$$

$$\times \int_0^{\tau_0} \left\{ \int_0^{\tau_0} PP_{km1}(t,\mu;\tau_0) K_{2kmi}(\tau-t) dt \right\} PP_{im1}(\tau,s;\tau_0)$$

$$\times d\tau, \tag{183}$$

and

$$\sum_{k=m}^{L} (-1)^k B_k^m \int_0^{\tau_0} PP_{km1}(\tau,s;\tau_0) PP_{km1}(\tau,\mu;\tau_0) d\tau$$

$$= \sum_{k=m}^{L} (-1)^k B_k^m P_k^m(s) \overline{PP_{km1}}(s,\mu;\tau_0) + (\omega/2) \sum_{i=m}^{L} B_i^m \sum_{k=m}^{L} (-1)^k B_k^m$$

$$\times \int_0^{\tau_0} \left\{ \int_0^{\tau_0} PP_{km1}(t,\mu;\tau_0) K_{2imk}(t-\tau) dt \right\} PP_{im1}(\tau,s;\tau_0) d\tau$$

$$= \sum_{k=m}^{L} (-1)^k B_k^m P_k^m(s) \overline{PP_{km1}}(s,\mu;\tau_0) + (\omega/2) \sum_{i=m}^{L} (-1)^i B_i^m \sum_{k=m}^{L} B_k^m$$

$$\times \int_0^{\tau_0} \left\{ \int_0^{\tau_0} PP_{km1}(t,\mu;\tau_0) K_{2kmi}(\tau-t) dt \right\} PP_{im1}(\tau,s;\tau_0)$$

$$\times d\tau. \tag{184}$$

Now, comparison of Eq. (183) and Eq. (184) yields

$$\sum_{i=m}^{L} (-1)^i B_i^m P_i^m(\mu) \overline{PP_{im1}}(\mu,s;\tau_0) = \sum_{k=m}^{L} (-1)^k B_k^m P_k^m(s)$$

$$\times \overline{PP_{km1}}(s,\mu;\tau_0),$$

or

$$\sum_{l=m}^{L} (-1)^{l} B_l^m P_l^m(\mu) \overline{PP_{lml}}(\mu,s;\tau_0) = \sum_{l=m}^{L} (-1)^{l} B_l^m P_l^m(s) \times \overline{PP_{lml}}(s,\mu;\tau_0). \tag{126}$$

Last, we want to get the relationship between $\overline{PPI_{lml}}(s,\mu;\tau_0)$ and $\overline{PPI_{lml}}(\mu,s;\tau_0)$ of Eq. (127). The procedure of this derivation is also very similar to that of obtaining Eq. (57) at the beginning of this appendix.

Replacing k by i and τ by $\tau_0-\tau$ in Eq. (86), then multiplying this equation by $PP_{iml}(\tau,s;\tau_0)$, integrating with respect to τ from zero to τ_0, and with the help of Eq. (115), we have

$$\int_0^{\tau_0} PP_{lml}(\tau_0-\tau,\mu;\tau_0)\ PP_{iml}(\tau,s;\tau_0)\ d\tau$$

$$= \int_0^{\tau_0} \left\{ P_l^m(\mu) \exp[(\tau-\tau_0)/\mu] + (\omega/2) \sum_{k=m}^{L} B_k^m \int_0^{\tau_0} PP_{kml}(t,\mu;\tau_0) \times K_{2kml}(\tau_0-\tau-t)\ dt \right\} PP_{iml}(\tau,s;\tau_0)\ d\tau$$

$$= P_l^m(\mu) \overline{PPI_{lml}}(\mu,s;\tau_0) + (\omega/2) \sum_{k=m}^{L} B_k^m \int_0^{\tau_0} \left\{ \int_0^{\tau_0} PP_{kml}(t,\mu;\tau_0) \times K_{2kml}(\tau_0-\tau-t)\ dt \right\} PP_{iml}(\tau,s;\tau_0)\ d\tau, \tag{185}$$

and

$$\int_0^{\tau_0} PP_{lml}(\tau,s;\tau_0)\ PP_{iml}(\tau_0-\tau,\mu;\tau_0)\ d\tau$$

$$= \int_0^{\tau_0} \left\{ P_l^m(s) \exp(-\tau/s) + (\omega/2) \sum_{k=m}^{L} B_k^m \int_0^{\tau_0} PP_{kml}(t,s;\tau_0) \times K_{2kml}(\tau-t)\ dt \right\} PP_{lml}(\tau_0-\tau,\mu;\tau_0)\ d\tau$$

$$= P_l^m(s)\ \overline{PPI_{lm1}}(s,\mu;\tau_0) + (\omega/2) \sum_{k=m}^{L} B_k^m \int_0^{\tau_0} \left\{ \int_0^{\tau_0} PP_{km1}(t,s;\tau_0) \right.$$

$$\left. \times\ K_{2km1}(\tau-t)\ dt \right\} PP_{lm1}(\tau_0-\tau,\mu;\tau_0)\ d\tau$$

$$= P_l^m(s)\ \overline{PPI_{lm1}}(s,\mu;\tau_0) + (\omega/2) \sum_{k=m}^{L} B_k^m \int_0^{\tau_0} \left\{ \int_0^{\tau_0} PP_{lm1}(t,\mu;\tau_0) \right.$$

$$\left. \times\ K_{2km1}(\tau_0-\tau-t)\ dt \right\} PP_{km1}(\tau,s;\tau_0)\ d\tau. \qquad (186)$$

From Eqs. (185) and (186), we know that we need another set of equations. Multiplying both Eqs. (185) and (186) by B_l^m and summing from l = m to L, and with the help of Eq. (26c), we get

$$\sum_{l=m}^{L} B_l^m \int_0^{\tau_0} PP_{lm1}(\tau_0-\tau,\mu;\tau_0)\ PP_{lm1}(\tau,s;\tau_0)\ d\tau$$

$$= \sum_{l=m}^{L} B_l^m\ P_l^m(\mu)\ \overline{PPI_{lm1}}(\mu,s;\tau_0) + (\omega/2) \sum_{l=m}^{L} B_l^m \sum_{k=m}^{L} B_k^m$$

$$\times \int_0^{\tau_0} \left\{ \int_0^{\tau_0} PP_{km1}(t,\mu;\tau_0)\ K_{2km1}(\tau_0-\tau-t)\ dt \right\} PP_{lm1}(\tau,s;\tau_0)$$

$$\times\ d\tau, \qquad (187)$$

and

$$\sum_{k=m}^{L} B_k^m \int_0^{\tau_0} PP_{km1}(\tau,s;\tau_0)\ PP_{km1}(\tau_0-\tau,\mu;\tau_0)\ d\tau$$

$$= \sum_{k=m}^{L} B_k^m \left\{ P_k^m(s)\ \overline{PPI_{km1}}(s,\mu;\tau_0) + (\omega/2) \sum_{l=m}^{L} B_l^m \right.$$

$$\times \int_0^{\tau_0} \{ \int_0^{\tau_0} PP_{km1}(t,\mu;\tau_0)\ K_{2lmk}(\tau_0-\tau-t)\ dt \}\ PP_{lm1}(\tau,s;\tau_0)$$

$$\left. \times\ d\tau \right\}$$

$$= \sum_{k=m}^{L} B_k^m\ P_k^m(s)\ \overline{PPI_{km1}}(s,\mu;\tau_0) + (\omega/2) \sum_{l=m}^{L} B_l^m \sum_{k=m}^{L} B_k^m$$

$$\times \int_0^{\tau_0} \left\{ \int_0^{\tau_0} PP_{km1}(t,\mu;\tau_0)\ K_{2km1}(\tau_0-\tau-t)\ dt \right\} PP_{1m1}(\tau,s;\tau_0)$$

$$\times\ d\tau. \tag{188}$$

Now, comparison of Eq. (187) and Eq. (188) yields

$$\sum_{l=m}^{L} B_l^m\ P_l^m(\mu)\ \overline{PPI_{lm1}}(\mu,s;\tau_0) = \sum_{k=m}^{L} B_k^m\ P_k^m(s)\ \overline{PPI_{km1}}(s,\mu;\tau_0),$$

or

$$\sum_{l=m}^{L} B_l^m\ P_l^m(\mu)\ \overline{PPI_{lm1}}(\mu,s;\tau_0) = \sum_{l=m}^{L} B_l^m\ P_l^m(s)$$

$$\times\ \overline{PPI_{lm1}}(s,\mu;\tau_0). \tag{127}$$

APPENDIX B

COMPUTER PROGRAM FOR SEMI-INFINITE CASE

The computer program semi.for which is included in this appendix deals with the semi-infinite case. The number of Legendre polynomials which can be computed is up to three at this time. For number of Legendre polynomials being larger than three, this program may not work unless all the values of dimension 4 in those arrays are changed to appropriate values. For example, use 11 instead of 4 if number of Legendre polynomials which needs to be calculated is 10. To make this program more understandable and readable, some definitions of variables are given as follows:

W	scattering albedo ω
ERROR	convergence criterion
XK	expansion coefficient x_k
NR	ratio of the index of refraction of the medium to that of the material bounding on the top boundary of the medium
THO	the results of $\rho(\mu,n)$ at quadrature points
TH1	the results of $\rho(\mu,n)$ at desired μ values
TH2	the results of $\rho(\mu,1/n)$ at desired μ values
PP1	the first guess at the quadrature points of $PP_{jm1}(0,\mu)$ in Eq. (60)
PP1N	the final results at the quadrature points of $PP_{jm1}(0,\mu)$ in Eq. (60)

PP2 the first guess at the desired μ values of $PP_{jm1}(0,\mu)$ in Eq. (60)

PP2N the final results at the desired μ values of $PP_{jm1}(0,\mu)$ in Eq. (60)

PP3 the first guess at the quadrature points of $PP_{km}(0,\mu,n)$ in Eq. (71)

PP3N the final results at the quadrature points of $PP_{km}(0,\mu,n)$ in Eq. (71)

PP4 the results at the desired μ values of $PP_{km}(0,\mu,n)$ in Eq. (71)

RP1 the first guess at the quadrature points of $R_{PP_m}(\bar{\mu},\mu,n)$ in Eq. (80)

RP1N the final results at the quadrature points of $R_{PP_m}(\bar{\mu},\mu,n)$ in Eq. (80)

RP2 the results at the desired μ values of $R_{PP_m}(\bar{\mu},\mu,n)$ in Eq. (80)

IE the results of $\bar{I}_{Ae}(0,\mu_e,\mu_o,\phi_{in},n)$ in Eq. (82)

Q the results of $q(\tau,\mu_o,n,\tau_0)$ in Eq. (32) when $\tau = 0$

The program semi.for is presented as follows:

```
C-----MAIN PROGRAM FOR SEMI-INFINITE CASE.
      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*8 NR, MUE, MU, IE
      DIMENSION XK(11), X(128), A(128), XD(32), AD(32),
     *  P(4,4,128), MU(20), RP2(4,20,20), TH2(20), Q(10),
     *  P1(4,4,20), PP2(4,4,20), PP2N(4,4,20), TH1(20),
     *  THO(128), PP4(4,4,20), IE(128,10)
      COMMON/BLK1/PP1(4,4,128),PP1N(4,4,128)
      COMMON/BLK2/PP3(4,4,128),PP3N(4,4,128)
      COMMON/BLK3/RP1(4,128,20),RP1N(4,128,20)
      OPEN(UNIT=4,FILE='SEMI.DAT')
      OPEN(UNIT=5,FILE='SEMI.OUT')
C-----READ IN THE VALUE OF N.
      WRITE(*,*) 'N= ?'
      READ(*,*) N
C-----READ IN AND PRINT OUT THE VALUE OF L.
      WRITE(*,*) 'L= ?'
      READ(*,*) L
      WRITE(5,1) L
```

```
    1 FORMAT('NUMBER OF LEGENDRE POLYNOMIALS (L)=',I3)
C-----READ IN AND PRINT OUT THE VALUE OF W.
      WRITE(*,*) 'W= ?'
      READ(*,*) W
      WRITE(5,2) W
    2 FORMAT('ALBEDO (W)=',F5.3)
C-----READ IN AND PRINT OUT THE VALUE OF ERROR.
      WRITE(*,*) 'ERROR= ?'
      READ(*,*) ERROR
      WRITE(5,3) ERROR
    3 FORMAT('ERROR=',F14.12)
C-----READ IN THE VALUES OF EXPANSION COEFFICIENTS XK'S.
      READ(4,*) (XK(I),I=1,L+1)
C-----READ IN THE VALUE OF NMUS.
      READ(4,*) NMUS
C-----READ IN THE VALUES OF MU'S.
      READ(4,*) (MU(J),J=1,NMUS)
C-----READ IN AND PRINT OUT THE VALUE OF NR(N1/N0).
      WRITE(*,*) 'NR= ?'
      READ(*,*) NR
      WRITE(5,4) NR
    4 FORMAT('REFRACTIVE INDEX (NR)=',F5.3)
C-----DECIDE TO READ IN OR CALCULATE THE VALUES OF
C-----AA, BB, CC, DD, EE.
      IF(NR .EQ. 1.0) THEN
C-----READ IN THE VALUES OF AA, BB, CC, DD, AND EE.
      WRITE(*,*) 'AA= ?'
      READ(*,*) AA
      WRITE(*,*) 'BB= ?'
      READ(*,*) BB
      WRITE(*,*) 'CC= ?'
      READ(*,*) CC
      WRITE(*,*) 'DD= ?'
      READ(*,*) DD
      WRITE(*,*) 'EE= ?'
      READ(*,*) EE
      ELSE
C-----CALCULATE THE VALUES OF AA, BB, CC, DD, AND EE.
      UCR=(1.D0-NR**(-2))**0.5
      WRITE(*,*) 'UCR=', UCR
      AA=0.0D0
      BB=UCR
      CC=1.015D0*UCR
      DD=1.085D0*UCR
      EE=1.D0
      ENDIF
C-----CALL SUBROUTINE DXA.
      CALL DXA(N,AA,BB,X,A)
      CALL DXA(N,BB,CC,XD,AD)
      DO 50 I=1, N
      X(N+I)=XD(I)
      A(N+I)=AD(I)
   50 CONTINUE
      CALL DXA(N,CC,DD,XD,AD)
```

```
      DO 70 I=1, N
      X(2*N+I)=XD(I)
      A(2*N+I)=AD(I)
   70 CONTINUE
      CALL DXA(N,DD,EE,XD,AD)
      DO 100 I=1, N
      X(3*N+I)=XD(I)
      A(3*N+I)=AD(I)
  100 CONTINUE
C-----CALCULATE AND PRINT OUT THE TOTAL VALUE OF N.
      N=4*N
      WRITE(5,5) N
    5 FORMAT('NUMBER OF QUADRATURE POINTS (N)=',I3)
      WRITE(5,*)'EXPANSION COEFFICIENTS :'
      DO 150 I=1, L+1
      WRITE(5,6) I-1, XK(I)
    6 FORMAT(3X,'XK(',I2,')=',F7.4)
  150 CONTINUE
C-----DO LOOP OF SUCCESSIVE APPROXIMATIONS FOR DXA POINTS
C-----BEGINS.(PP1)
      WRITE(*,*) 'CALCULATING PP1 FOR DXA POINTS.'
      WRITE(5,*) '----------------------------------------'
      WRITE(5,*)
      DO 200 M=0, L
      WRITE(5,7) M
    7 FORMAT('M=',I2)
C-----INITIAL GUESSING FOR PP1(II,M,I)
      DO 300 II=M, L
      CALL LEG(II,M,N,X,P)
      DO 400 I=1, N
      PP1(II,M,I)=P(II,M,I)
  400 CONTINUE
  300 CONTINUE
      ITER=0
      DO 500 JCONV=1, 20000
      ITER=ITER+1
      EMAX=0.0D0
      DO 600 J=M, L
      DO 700 JJ=1, N
      SUM1=0.0D0
      DO 800 K=M, L
      SUM2=0.0D0
      DO 900 KK=1, N
      SUM2=SUM2+(X(JJ)/(X(JJ)+X(KK))*PP1(K,M,KK)
     *      *P(J,M,KK))*A(KK)
  900 CONTINUE
      XX=XK(K+1)
      CALL BFUN(K,M,XX,B)
      SUM1=SUM1+(-1)**(K)*B*PP1(K,M,JJ)*SUM2
  800 CONTINUE
      PP1N(J,M,JJ)=P(J,M,JJ)+W/2.D0*(-1)**J*SUM1
      ERR=DABS(PP1N(J,M,JJ)-PP1(J,M,JJ))
      IF(ERR .GT. EMAX) EMAX=ERR
  700 CONTINUE
```

```
  600 CONTINUE
      WRITE(*,*) EMAX
      IF(EMAX .LT. ERROR) GO TO 1200
      DO 1000 J=M, L
      DO 1100 JJ=1, N
      PP1(J,M,JJ)=PP1N(J,M,JJ)
 1100 CONTINUE
 1000 CONTINUE
  500 CONTINUE
      WRITE(5,*) 'NEED MORE LOOPS !'
 1200 WRITE(5,8) ITER
    8 FORMAT('NUMBER OF ITERATIONS=',I5)
      WRITE(5,*)
  200 CONTINUE
      WRITE(5,*) '-----------------------------------------'
      WRITE(5,*)
C-----CALCULATE THE MU VALUES WE WANT.
      DO 1300 I=1, NMUS
      MU(NMUS+I)=(1.D0-(1.D0-MU(I)**2)/NR**2)**0.5
 1300 CONTINUE
      NMUS=2*NMUS
C-----DO LOOP OF SUCCESSIVE APPROXIMATIONS FOR MU VALUES
C-----BEGINS.(PP2)
      WRITE(*,*) 'CALCULATING PP2 FOR MU VALUES.'
      DO 1400 M=0, L
      WRITE(5,7) M
C-----INITIAL GUESSING FOR PP2(II,M,I)
      DO 1500 II=M, L
      CALL LEG(II,M,NMUS,MU,P1)
      DO 1600 I=1, NMUS
      PP2(II,M,I)=P1(II,M,I)
 1600 CONTINUE
 1500 CONTINUE
      ITER=0
      DO 1700 JCONV=1, 10000
      ITER=ITER+1
      EMAX=0.0D0
      DO 1800 J=M, L
      DO 1900 JJ=1, NMUS
      SUM1=0.0D0
      DO 2000 K=M, L
      SUM2=0.0D0
      DO 2100 KK=1, N
      SUM2=SUM2+(MU(JJ)/(MU(JJ)+X(KK))*PP1N(K,M,KK)
     *      *P(J,M,KK))*A(KK)
 2100 CONTINUE
      XX=XK(K+1)
      CALL BFUN(K,M,XX,B)
      SUM1=SUM1+(-1)**K*B*PP2(K,M,JJ)*SUM2
 2000 CONTINUE
      PP2N(J,M,JJ)=P1(J,M,JJ)+W/2.D0*(-1)**J*SUM1
      ERR=DABS(PP2N(J,M,JJ)-PP2(J,M,JJ))
      IF(ERR .GT. EMAX) EMAX=ERR
 1900 CONTINUE
```

```
 1800 CONTINUE
      IF(EMAX .LT. ERROR) GO TO 2400
      DO 2200 J=M, L
      DO 2300 JJ=1, NMUS
      PP2(J,M,JJ)=PP2N(J,M,JJ)
 2300 CONTINUE
 2200 CONTINUE
 1700 CONTINUE
      WRITE(5,*) 'NEED MORE LOOPS !'
 2400 WRITE(5,8) ITER
      DO 2500 J=M, L
      WRITE(5,9) J
    9 FORMAT('J=',I2)
      DO 2600 JJ=1, NMUS
      WRITE(5,10) MU(JJ),PP2N(J,M,JJ)
   10 FORMAT('MU=',F15.12,5X,'PP2=',F20.12)
      WRITE(5,*)
 2600 CONTINUE
 2500 CONTINUE
 1400 CONTINUE
      WRITE(5,*) '-----------------------------------------'
      WRITE(5,*)
C-----DO LOOP OF SUCCESSIVE APPROXIMATIONS FOR DXA POINTS
C-----BEGINS.(PP3)
      WRITE(*,*) 'CALCULATING PP3 FOR DXA POINTS.'
      CALL TH(N,X,NR,THO)
      DO 2700 M=0, L
      WRITE(5,7) M
C-----INITIAL GUESSING FOR PP3(II,M,I)
      DO 2800 II=M, L
      DO 2900 I=1, N
      PP3(II,M,I)=P(II,M,I)
 2900 CONTINUE
 2800 CONTINUE
      ITER=0
      DO 3000 JCONV=1, 10000
      ITER=ITER+1
      EMAX=0.0D0
      DO 3100 J=M, L
      DO 3200 JJ=1, N
      SUM1=0.0D0
      DO 3300 K=M, L
      SUM2=0.0D0
      DO 3400 KK=1, N
      SUM2=SUM2+(X(JJ)/(X(JJ)+X(KK))*PP1N(K,M,KK)*((-1)**J
     *      *P(J,M,KK)+THO(KK)*PP3(J,M,KK)))*A(KK)
 3400 CONTINUE
      XX=XK(K+1)
      CALL BFUN(K,M,XX,B)
      SUM1=SUM1+(-1)**K*B*PP1N(K,M,JJ)*SUM2
 3300 CONTINUE
      PP3N(J,M,JJ)=P(J,M,JJ)+W/2.D0*SUM1
      ERR=DABS(PP3N(J,M,JJ)-PP3(J,M,JJ))
      IF(ERR .GT. EMAX) EMAX=ERR
```

```
 3200 CONTINUE
 3100 CONTINUE
      IF(EMAX .LT. ERROR) GO TO 3700
      DO 3500 J=M, L
      DO 3600 JJ=1, N
      PP3(J,M,JJ)=PP3N(J,M,JJ)
 3600 CONTINUE
 3500 CONTINUE
 3000 CONTINUE
      WRITE(5,*) 'NEED MORE LOOPS !'
 3700 WRITE(5,8) ITER
      DO 3701 J=M, L
      WRITE(5,9) J
C     DO 3702 JJ=1, N
C     WRITE(5,17) X(JJ), PP3N(J,M,JJ)
   17 FORMAT('MU=',F15.12,5X,'PP3=',F20.12)
C3702 CONTINUE
      WRITE(5,*)
 3701 CONTINUE
 2700 CONTINUE
      WRITE(5,*) '-----------------------------------------'
      WRITE(5,*)
C-----DO LOOP FOR MU VALUES BEGINS.(PP4)
      WRITE(*,*) 'CALCULATING PP4 FOR MU VALUES.'
      DO 3800 M=0, L
      WRITE(5,7) M
      DO 3900 J=M, L
      WRITE(5,9) J
      DO 4000 JJ=1, NMUS
      SUM1=0.0D0
      DO 4100 K=M, L
      SUM2=0.0D0
      DO 4200 KK=1, N
      SUM2=SUM2+(MU(JJ)/(MU(JJ)+X(KK))*PP1N(K,M,KK)*((-1)**J
     *      *P(J,M,KK)+THO(KK)*PP3N(J,M,KK)))*A(KK)
 4200 CONTINUE
      XX=XK(K+1)
      CALL BFUN(K,M,XX,B)
      SUM1=SUM1+(-1)**K*B*PP2N(K,M,JJ)*SUM2
 4100 CONTINUE
      PP4(J,M,JJ)=P1(J,M,JJ)+W/2.D0*SUM1
      WRITE(5,11) MU(JJ),PP4(J,M,JJ)
   11 FORMAT('MU=',F15.12,5X,'PP4=',F20.12)
 4000 CONTINUE
 3900 CONTINUE
 3800 CONTINUE
      WRITE(5,*) '-----------------------------------------'
      WRITE(5,*)
C-----DO LOOP OF SUCCESSIVE APPROXIMATIONS FOR DXA POINTS
C-----BEGINS.(RP1)
      WRITE(*,*) 'CALCULATING RP1 FOR DXA POINTS.'
      DO 4300 M=0, L
      WRITE(5,7) M
      DO 4400 J=1, NMUS
```

```
C     IF(J .NE. 10) GO TO 4400
      WRITE(5,12) MU(J)
   12 FORMAT('MU=',F15.12)
C-----INITIAL GUESSING FOR RP1(M,II,J).
      DO 4500 II=1, N
      SUM=0.0D0
      DO 4600  I=M, L
      XX=XK(I+1)
      CALL BFUN(I,M,XX,B)
      SUM=SUM+(-1)**I*B*PP2N(I,M,J)*PP1N(I,M,II)
 4600 CONTINUE
      RP1(M,II,J)=X(II)*MU(J)/(MU(J)+X(II))*W/2.D0*(-1)**M
     *             *SUM
 4500 CONTINUE
      ITER=0
      DO 4700 JJCONV=1, 10000
      ITER=ITER+1
      EMAX=0.0D0
      DO 4800 JJ=1, N
      SUM2=0.0D0
      SUM3=0.0D0
      DO 4900 K=M, L
      SUM1=0.0D0
      DO 5000 KK=1, N
      SUM1=SUM1+(X(JJ)/(X(JJ)+X(KK))*PP1N(K,M,KK)*RP1(M,KK,J
     *     )*THO(KK))*A(KK)
 5000 CONTINUE
      XX=XK(K+1)
      CALL BFUN(K,M,XX,B)
      SUM2=SUM2+(-1)**K*B*PP1N(K,M,JJ)*SUM1
      SUM3=SUM3+(-1)**K*B*PP2N(K,M,J)*PP1N(K,M,JJ)
 4900 CONTINUE
      RP1N(M,JJ,J)=W/2.D0*(X(JJ)*MU(J)/(MU(J)+X(JJ))*(-1)**M
     *              *SUM3+SUM2)
      ERR=DABS(RP1N(M,JJ,J)-RP1(M,JJ,J))
      IF(ERR .GT. EMAX) EMAX=ERR
 4800 CONTINUE
      WRITE(*,*) EMAX
      IF(EMAX .LT. ERROR) GO TO 5200
      DO 5100 JJ=1, N
      RP1(M,JJ,J)=RP1N(M,JJ,J)
 5100 CONTINUE
 4700 CONTINUE
 5200 WRITE(5,8) ITER
C     DO 5201 JJ=1, N
C     WRITE(5,18) X(JJ), RP1N(M,JJ,J)
   18 FORMAT('MU=',F15.12,5X,'RP1=',F20.12)
C5201 CONTINUE
      WRITE(5,*)
 4400 CONTINUE
      WRITE(5,*)
 4300 CONTINUE
      WRITE(5,*) '-----------------------------------------'
      WRITE(5,*)
```

```
C-----DO LOOP FOR MU VALUES BEGINS.(RP2)
      WRITE(*,*) 'CALCULATING RP2 FOR MU VALUES.'
      DO 5300 M=0, L
      WRITE(5,7) M
      DO 5400 J=1, NMUS
C     IF(J .NE. 10) GO TO 5400
      WRITE(5,12) MU(J)
      DO 5500 JJ=1, NMUS
      SUM2=0.0D0
      SUM3=0.0D0
      DO 5600 K=M, L
      SUM1=0.0D0
      DO 5700 KK=1, N
      SUM1=SUM1+(MU(JJ)/(MU(JJ)+X(KK))*PP1N(K,M,KK)*
     *     RP1N(M,KK,J)*THO(KK))*A(KK)
 5700 CONTINUE
      XX=XK(K+1)
      CALL BFUN(K,M,XX,B)
      SUM2=SUM2+(-1)**K*B*PP2N(K,M,JJ)*SUM1
      SUM3=SUM3+(-1)**K*B*PP2N(K,M,J)*PP2N(K,M,JJ)
 5600 CONTINUE
      RP2(M,JJ,J)=W/2.D0*(MU(JJ)*MU(J)/(MU(JJ)+MU(J))*(-1)**
     *            M*SUM3+SUM2)
      IF(J .NE. 9) GO TO 5500
      WRITE(5,13) MU(JJ),RP2(M,JJ,J)
   13 FORMAT('MUBAR=',F15.12,5X,'RP2=',F20.12)
 5500 CONTINUE
      WRITE(5,*)
 5400 CONTINUE
      WRITE(5,*)
 5300 CONTINUE
C-----DO LOOP FOR DXA POINTS BEGINS.(IE)
      WRITE(*,*) 'CALCULATING IE FOR DXA POINTS.'
      PAI=3.141592654
      CALL TH(NMUS,MU,1.D0/NR,TH2)
      DO 5800 J=1, NMUS/2
C     IF(J .NE. 10) GO TO 5800
      WRITE(5,12) MU(J)
      DO 5900 I=1, N
      IF((1.D0-(1.D0-X(I)**2)*NR**2) .LT. 0.0D0) GO TO 5900
      MUE=(1.D0-(1.D0-X(I)**2)*NR**2)**0.5
      SUM1=0.0D0
      DO 6000 M=0, L
      DELTA=2.D0
      IF(M .EQ. 0) DELTA=1.D0
      SUM1=SUM1+DELTA*RP1N(M,I,J)
 6000 CONTINUE
      IE(I,J)=MU(J)/2.D0/PAI*(1.D0-TH2(J))*(1.D0-THO(I))
     *         *SUM1/NR**2/MU(NMUS/2+J)/X(I)
C     WRITE(5,15) MUE, IE(I,J)
C  15 FORMAT('MUE=',F15.12,5X,'IE=',F20.12)
 5900 CONTINUE
      WRITE(5,*)
 5800 CONTINUE
```

```
      WRITE(5,*) '-----------------------------------------'
      WRITE(5,*)
C-----DO LOOP FOR MU VALUES BEGINS.(IE)
      PAI=3.141592654
      CALL TH(NMUS,MU,NR,TH1)
      CALL TH(NMUS,MU,1.D0/NR,TH2)
      DO 5801 I=1, NMUS/2
      WRITE(5,14) MU(I)
   14 FORMAT('MUE=',F15.12)
      DO 5901 J=1, NMUS/2
      SUM1=0.0D0
      DO 6001 M=0, L
      DELTA=2.D0
      IF(M .EQ. 0) DELTA=1.D0
      SUM1=SUM1+DELTA*RP2(M,NMUS/2+I,NMUS/2+J)
 6001 CONTINUE
      IE(I,J)=MU(J)/2.D0/PAI*(1.D0-TH2(J))*(1.D0-TH1(NMUS/2+
     *        I))*SUM1/MU(NMUS/2+I)/MU(NMUS/2+J)/NR**2
      WRITE(5,15) MU(J),IE(I,J)
   15 FORMAT('MUO=',F15.12,5X,'Ie=',F20.12)
 5901 CONTINUE
      WRITE(5,*)
 5801 CONTINUE
      WRITE(5,*) '-----------------------------------------'
      WRITE(5,*)
C-----DO LOOP FOR MU VALUES BEGINS.(Q)
      DO 6100 JJ=1, NMUS/2
      Q(JJ)=MU(JJ)/MU(NMUS/2+JJ)*(1.D0-TH2(JJ))*
     *      PP4(1,0,NMUS/2+JJ)
      WRITE(5,16) MU(JJ),Q(JJ)
   16 FORMAT('MU=',F15.12,5X,'Q=',F20.12)
 6100 CONTINUE
      WRITE(5,*) '--------------END OF DATA---------------'
      STOP
      END


C-----SUBROUTINE TO CALCULATE B FUNCTION.
      SUBROUTINE BFUN(K,M,XX,B)
      DOUBLE PRECISION XX, B, FACT1, FACT2
      IF(K-M .EQ. 0) GO TO 200
      FACT1=1.D0
      DO 100 I=1, K-M
      FACT1=I*FACT1
  100 CONTINUE
      GO TO 300
  200 FACT1=1.D0
  300 IF(K+M .EQ. 0) GO TO 500
      FACT2=1.D0
      DO 400 I=1, K+M
      FACT2=I*FACT2
  400 CONTINUE
      GO TO 600
```

```
  500 FACT2=1.D0
  600 B=XX*FACT1/FACT2
      RETURN
      END


C-----SUBROUTINE TO CALCULATE LEGENDRE FUNCTIONS.
      SUBROUTINE LEG(K,M,NYYS,YY,P)
      DOUBLE PRECISION P(4,4,128), YY(128), Y, SUM, FACT1,
     *  FACT2, FACT3, FACT4
      IF(K .LE. 5 .AND. M .LE. 5) THEN
      GO TO 100
      ELSE
      GO TO 1000
      ENDIF
  100 GO TO (200,300,400,500,600,700) M+1
  200 GO TO (210,220,230,240,250,260) K+1
  210 DO 215 I=1, NYYS
      Y=YY(I)
      P(0,0,I)=1.D0
  215 CONTINUE
      GO TO 5000
  220 DO 225 I=1, NYYS
      Y=YY(I)
      P(1,0,I)=Y
  225 CONTINUE
      GO TO 5000
  230 DO 235 I=1, NYYS
      Y=YY(I)
      P(2,0,I)=0.5D0*(3.D0*Y**2-1.D0)
  235 CONTINUE
      GO TO 5000
  240 DO 245 I=1, NYYS
      Y=YY(I)
      P(3,0,I)=0.5D0*Y*(5.D0*Y**2-3.D0)
  245 CONTINUE
      GO TO 5000
  250 DO 255 I=1, NYYS
      Y=YY(I)
      P(4,0,I)=1.D0/8.D0*(35.D0*Y**4-30.D0*Y**2+3.D0)
  255 CONTINUE
      GO TO 5000
  260 DO 265 I=1, NYYS
      Y=YY(I)
      P(5,0,I)=1.D0/8.D0*Y*(63.D0*Y**4-70.D0*Y**2+15.D0)
  265 CONTINUE
      GO TO 5000
  300 GO TO (310,320,330,340,350) K
  310 DO 315 I=1, NYYS
      Y=YY(I)
      P(1,1,I)=(1.D0-Y**2)**0.5
  315 CONTINUE
      GO TO 5000
```

```
320 DO 325 I=1, NYYS
    Y=YY(I)
    P(2,1,I)=3.D0*Y*(1.D0-Y**2)**0.5
325 CONTINUE
    GO TO 5000
330 DO 335 I=1, NYYS
    Y=YY(I)
    P(3,1,I)=1.5D0*(5.D0*Y**2-1.D0)*(1.D0-Y**2)**0.5
335 CONTINUE
    GO TO 5000
340 DO 345 I=1, NYYS
    Y=YY(I)
    P(4,1,I)=2.5D0*Y*(7.D0*Y**2-3.D0)*(1.D0-Y**2)**0.5
345 CONTINUE
    GO TO 5000
350 DO 355 I=1, NYYS
    Y=YY(I)
    P(5,1,I)=5.D0/8.D0*(63.D0*Y**4-42.D0*Y**2+3.D0)*
   *         (1.D0-Y**2)**0.5
355 CONTINUE
    GO TO 5000
400 GO TO (410,420,430,440) K-1
410 DO 415 I=1, NYYS
    Y=YY(I)
    P(2,2,I)=3.D0*(1.D0-Y**2)
415 CONTINUE
    GO TO 5000
420 DO 425 I=1, NYYS
    Y=YY(I)
    P(3,2,I)=15.D0*Y*(1.D0-Y**2)
425 CONTINUE
    GO TO 5000
430 DO 435 I=1, NYYS
    Y=YY(I)
    P(4,2,I)=7.5D0*(1.D0-Y**2)*(7.D0*Y**2-1.D0)
435 CONTINUE
    GO TO 5000
440 DO 445 I=1, NYYS
    Y=YY(I)
    P(5,2,I)=105.D0/2.D0*Y*(1.D0-Y**2)*(3.D0*Y**2-1.D0)
445 CONTINUE
    GO TO 5000
500 GO TO (510,520,530) K-2
510 DO 515 I=1, NYYS
    Y=YY(I)
    P(3,3,I)=15.D0*(1.D0-Y**2)**1.5
515 CONTINUE
    GO TO 5000
520 DO 525 I=1, NYYS
    Y=YY(I)
    P(4,3,I)=105.D0*Y*(1.D0-Y**2)**1.5
525 CONTINUE
    GO TO 5000
530 DO 535 I=1, NYYS
```

```
      Y=YY(I)
      P(5,3,I)=105.D0/2.D0*(9.D0*Y**2-1.D0)*(1.D0-Y**2)**1.5
  535 CONTINUE
      GO TO 5000
  600 GO TO (610,620) K-3
  610 DO 615 I=1, NYYS
      Y=YY(I)
      P(4,4,I)=105.D0*(1.D0-Y**2)**2
  615 CONTINUE
      GO TO 5000
  620 DO 625 I=1, NYYS
      Y=YY(I)
      P(5,4,I)=945.D0*Y*(1.D0-Y**2)**2
  625 CONTINUE
      GO TO 5000
  700 DO 750 I=1, NYYS
      Y=YY(I)
      P(5,5,I)=945.D0*(1.D0-Y**2)**2.5
  750 CONTINUE
      GO TO 5000
 1000 K1=K/2
      K2=2*K1
      M1=M/2
      M2=2*M1
      IF(K .EQ. K2 .AND. M .EQ. M2) L=K/2-M/2
      IF(K .EQ. K2 .AND. M .NE. M2) L=K/2-(M+1)/2
      IF(K .NE. K2 .AND. M .EQ. M2) L=(K-1)/2-M/2
      IF(K .NE. K2 .AND. M .NE. M2) L=(K-1)/2-(M-1)/2
      DO 1100 I=1, NYYS
      Y=YY(I)
      SUM=0.0D0
      DO 1200 N=0, L
      IF(2*K-2*N .EQ. 0) GO TO 1400
      FACT1=1.D0
      DO 1300 J=1, 2*K-2*N
      FACT1=J*FACT1
 1300 CONTINUE
      GO TO 1500
 1400 FACT1=1.D0
 1500 IF(N .EQ. 0) GO TO 1700
      FACT2=1.D0
      DO 1600 J=1, N
      FACT2=J*FACT2
 1600 CONTINUE
      GO TO 1800
 1700 FACT2=1.D0
 1800 IF(K-N .EQ. 0) GO TO 2000
      FACT3=1.D0
      DO 1900 J=1, K-N
      FACT3=J*FACT3
 1900 CONTINUE
      GO TO 2100
 2000 FACT3=1.D0
 2100 IF(K-2*N-M .EQ. 0) GO TO 2300
```

```
      FACT4=1.D0
      DO 2200 J=1, K-2*N-M
      FACT4=J*FACT4
 2200 CONTINUE
      GO TO 2400
 2300 FACT4=1.D0
 2400 SUM=SUM+(-1)**N*FACT1/2.D0**K/FACT2/FACT3/FACT4*
     *     Y**(K-2*N-M)
 1200 CONTINUE
      P(K,M,I)=(1.D0-Y**2)**(M/2.D0)*SUM
 1100 CONTINUE
 5000 RETURN
      END


C-----SUBROUTINE TO CALCULATE THE INTERFACE REFLECTION
C-----COEFFICIENTS THO'S.
      SUBROUTINE TH(NUS,U,N,THO)
      DOUBLE PRECISION U(128), N, THO(128), UCR, A
      IF((1.D0-N**(-2)) .LT. 0.0D0) THEN
      UCR=0.0D0
      ELSE
      UCR=(1.D0-N**(-2))**0.5
      ENDIF
      DO 100 I=1, NUS
      IF(U(I) .GE. UCR) GO TO 200
      THO(I)=1.D0
  100 CONTINUE
      GO TO 500
  200 A=(1.D0/N**2-(1.D0-U(I)**2))**0.5
      THO(I)=0.5D0*(((A-U(I))/(A+U(I)))**2+((A-U(I)/N**2)/
     *       (A+U(I)/N**2))**2)
      GO TO 100
  500 RETURN
      END


      SUBROUTINE DXA(N,AA,BB,X,A)
```

DXA is a Gaussian quadrature subroutine to calculate the integration of the unknown function. It can use up to 96 quadrature points.

APPENDIX C

COMPUTER PROGRAM FOR FINITE CASE

The computer program finit.for works for the finite case and the number of Legendre polynomials being up to two. For the number of Legendre polynomials being larger than two, we need to change not only all of the values of dimension 3, but also all the values of dimension 24 to suitable values. For instance, use 5 instead of 3 and 60 instead of 24 if the number of Legendre polynomials which need to be computed is 4. Based upon same reasons as mentioned in Appendix B, some definitions of variables are given as follows:

W	scattering albedo ω
ERROR	convergence criterion
XK	expansion coefficient x_k
TH2	the results of $\rho(\mu,n)$ at quadrature points
TH3	the results of $\rho(\mu,n)$ at desired μ values
TH4	the results of $\rho(\mu,1/n)$ at desired μ values
PP1	the results at the desired μ values of $PP_{km1}(0,\mu;\tau_0)$ in Eq. (142)
PP2	the results at the desired μ values of $PP_{km1}(\tau_0,\mu;\tau_0)$ in Eq. (143)
PP3	the results at the quadrature points of $PP_{km1}(0,\mu;\tau_0)$ in Eq. (142)

PP4	the results at the quadrature points of $PP_{km1}(\tau_0,\mu;\tau_0)$ in Eq. (143)
PP5	the results at the desired μ values of $PP_{km}(0,\mu,,n;\tau_0)$ in Eq. (151)
PP6	the results at the desired μ values of $PP_{km}(\tau_0,\mu,n;\tau_0)$ in Eq. (152)
PP7	the first guess at the quadrature points of $PP_{km}(0,\mu,n;\tau_0)$ in Eq. (151)
PP7N	the results at the quadrature points of $PP_{km}(0,\mu,n;\tau_0)$ in Eq. (151)
PP8	the first guess at the quadrature points of $PP_{km}(\tau_0,\mu,n;\tau_0)$ in Eq. (152)
PP8N	the results at the quadrature points of $PP_{km}(\tau_0,\mu,n;\tau_0)$ in Eq. (152)
RP1	the first guess at the quadrature points of $Rpp_m(\bar{\mu},\mu,n,\tau_0)$ in Eq. (161)
RP1N	the final results at the quadrature points of $Rpp_m(\bar{\mu},\mu,n,\tau_0)$ in Eq. (161)
RP2	the results at the desired μ values of $Rpp_m(\bar{\mu},\mu,n,\tau_0)$ in Eq. (161)
TP1	the first guess at the quadrature points of $TppI_m(\bar{\mu},\mu,n,\tau_0)$ in Eq. (168)
TP1N	the final results at the quadrature points of $TppI_m(\bar{\mu},\mu,n,\tau_0)$ in Eq. (168)
TP2	the results at the desired μ values of $TppI_m(\bar{\mu},\mu,n,\tau_0)$ in Eq. (168)
RINTEN	the results at the desired μ values of $\overset{-}{I}_{Ae}(0,\mu_e,\mu_o,\phi_{in},n;\tau_0)$ in Eq. (170)
TINTEN	the results at the desired μ values of $\overset{+}{I}_{Ae}(\tau_0,\mu_{in},\mu_o,\phi_{in},n;\tau_0)$ in Eq. (172)
TQ	the results at the desired μ values of $q(\tau,\mu_o,n;\tau_0)$ in Eq. (32) when $\tau = 0$

BQ the results at the desired μ values of $q(\tau,\mu_o,n;\tau_0)$ in Eq. (32) when $\tau = \tau_0$

The program finit.for is written as follows:

```
C-----MAIN PROGRAM FOR FINITE CASE.
      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*8 NR, MU
      DIMENSION IPRINT(40), XK(11), HSTEP(40), A(44),
     *  P1(3,3,44), Y(24,44), XD(11), AD(11)
      COMMON/BLK1/L,N,NMUTOT
      COMMON/BLK2/LLL(3,3,4)
      COMMON/BLK3/P2(3,3,44),AA(44),BBB(3,3),BB(3,3)
      COMMON/BLK4/X(44),MU(20)
      COMMON/BLK7/TH2(44)
      COMMON/BLK8/ERROR
      COMMON/BLK9/PP1(3,3,20),PP2(3,3,20),PP3(3,3,44),
     *  PP4(3,3,44)
      COMMON/BLK10/PP5(3,3,20),PP6(3,3,20)
      COMMON/BLK12/LK(24,4),LM(24,4)
      COMMON/BLK13/RP2(3,20,20)
      COMMON/BLK15/W
      COMMON/BLK17/TP2(3,20,20)
      COMMON/BLK18/NR
      COMMON/BLK19/TH3(20),TH4(20)
      EXTERNAL DERV
      EXTERNAL FLAGR
      OPEN(UNIT=4,FILE='FINIT.DAT')
      OPEN(UNIT=5,FILE='FINIT.OUT')
C-----READ IN AND PRINT OUT THE VALUE OF L.
      WRITE(*,*) 'L= ?'
      READ(*,*) L
      WRITE(5,1) L
    1 FORMAT(1X,'NUMBER OF LEGENDRE POLYNOMIALS (L)=',I3)
C-----READ IN AND PRINT OUT THE VALUE OF W.
      WRITE(*,*) 'W= ?'
      READ(*,*) W
      WRITE(5,2) W
    2 FORMAT(1X,'ALBEDO (W)=',F5.3)
C-----READ IN AND PRINT OUT THE VALUE OF NR(N1/N0).
      WRITE(*,*) 'NR= ?'
      READ(*,*) NR
      WRITE(5,3) NR
    3 FORMAT(1X,'REFRACTIVE INDEX (NR)=',F5.3)
C-----READ IN AND PRINT OUT THE VALUE OF ERROR.
      WRITE(*,*) 'ERROR= ?'
      READ(*,*) ERROR
      WRITE(5,4) ERROR
    4 FORMAT(1X,'ERROR=',F14.12)
C-----READ IN AND PRINT OUT THE VALUES OF EXPANSION
C-----COEFFICIENTS XK'S.
      READ(4,*) (XK(I),I=1,L+1)
```

```
      WRITE(5,*) 'EXPANSION COEFFICIENTS :'
      DO 46 I=1, L+1
      WRITE(5,5) I-1, XK(I)
    5 FORMAT(6X,'XK(',I2,')=',F7.4)
   46 CONTINUE
C-----READ IN THE VALUE OF NMUS.
      READ(4,*) NMUS
C-----READ IN THE VALUES OF MU'S.
      READ(4,*) (MU(J),J=1, NMUS)
C-----READ IN THE VALUE OF NUMBER OF PRINTING NPRINT.
      READ(4,*) NPRINT
C-----READ IN THE VALUES OF PRINTING STEPS IPRINT'S.
      READ(4,6) (IPRINT(I),I=1,NPRINT)
    6 FORMAT(8I9)
C-----READ IN THE VALUES OF H STEPS HSTEP'S.
      READ(4,7) (HSTEP(I),I=1,NPRINT)
    7 FORMAT(4F19.12)
C-----READ IN THE VALUE OF MULTIPLIER OR DIVIDEND MDSTEP.
      READ(4,*) MDSTEP
C-----PRINT OUT THE VALUES OF PRINTING AND H STEPS.
      DMSTEP=1.D0*MDSTEP
      WRITE(5,*) '      IPRINT                           HSTEP'
      DO 49 I=1, NPRINT
      IPRINT(I)=MDSTEP*IPRINT(I)
      HSTEP(I)=HSTEP(I)/DMSTEP
      WRITE(5,8) IPRINT(I), HSTEP(I)
    8 FORMAT(I10,13X,F20.12)
   49 CONTINUE
C-----READ IN THE VALUES OF N1, N2, N3, AND N4.
      WRITE(*,*) 'N1= ?'
      READ(*,*) N1
      WRITE(*,*) 'N2= ?'
      READ(*,*) N2
      WRITE(*,*) 'N3= ?'
      READ(*,*) N3
      WRITE(*,*) 'N4= ?'
      READ(*,*) N4
C-----DECIDE TO READ IN OR CALCULATE THE VALUES OF
C-----A1, B1, CC, DD, AND EE.
      IF(NR .EQ. 1.0) THEN
C-----READ IN THE VALUES OF A1, B1, CC, DD, AND EE.
      WRITE(*,*) 'A1= ?'
      READ(*,*) A1
      WRITE(*,*) 'B1= ?'
      READ(*,*) B1
      WRITE(*,*) 'CC= ?'
      READ(*,*) CC
      WRITE(*,*) 'DD= ?'
      READ(*,*) DD
      WRITE(*,*) 'EE= ?'
      READ(*,*) EE
      ELSE
C-----CALCULATE THE VALUES OF A1, B1, CC, DD, AND EE.
      UCR=(1.D0-NR**(-2))**0.5
```

```
      WRITE(*,*) 'UCR=',UCR
      A1=0.0D0
      B1=UCR
      CC=1.015D0*UCR
      DD=1.085D0*UCR
      EE=1.0D0
      ENDIF
C-----CALL SUBROUTINE DXA.
      CALL DXA(N1,A1,B1,X,A)
      CALL DXA(N2,B1,CC,XD,AD)
      DO 50 I=1, N2
      X(N1+I)=XD(I)
      A(N1+I)=AD(I)
   50 CONTINUE
      CALL DXA(N3,CC,DD,XD,AD)
      DO 70 I=1, N3
      X(N1+N2+I)=XD(I)
      A(N1+N2+I)=AD(I)
   70 CONTINUE
      CALL DXA(N4,DD,EE,XD,AD)
      DO 100 I=1, N4
      X(N1+N2+N3+I)=XD(I)
      A(N1+N2+N3+I)=AD(I)
  100 CONTINUE
C-----CALCULATE AND PRINT OUT THE TOTAL VALUE OF N.
      N=N1+N2+N3+N4
      WRITE(5,9) N
    9 FORMAT(1X,'NUMBER OF QUADRATURE POINTS (N)=',I3)
C-----CALCULATING NUMBER OF FUNCTIONS WHICH NEED TO COMPUTE.
      NFUN=0
      DO 140 I=0, L
      NFUN=NFUN+(I+1)
  140 CONTINUE
      NFUN=2*NFUN
      NFUNP1=NFUN+1
      INFUN=2*NFUN
      WRITE(5,*) NFUN, NFUNP1,INFUN
C-----CALCULATING THE MU VALUES WE WANT.
      DO 150 I=1, NMUS
      MU(NMUS+I)=(1.D0-(1.D0-MU(I)**2)/NR**2)**0.5
  150 CONTINUE
      NMUTOT=NMUS*2
C-----CALL SUBROUTINE TH TO GET THE INTERFACE REFLECTION
C-----COEFFICIENTS FOR DXA POINTS.
      CALL TH(N,X,NR,TH2)
C-----CALL SUBROUTINE TH TO GET THE INTERFACE REFLECTION
C-----COEFFICIENTS FOR MU VALUES.
      CALL TH(NMUTOT,MU,NR,TH3)
      CALL TH(NMUTOT,MU,1.D0/NR,TH4)
C-----INITIAL VALUES FOR PP1, PP2, PP3, AND PP4.
      DO 170 M=0, L
      DO 180 K=M, L
      CALL LEG(K,M,NMUTOT,MU,P1)
      CALL LEG(K,M,N,X,P2)
```

```
  180 CONTINUE
  170 CONTINUE
C-----CALL ALL B FUNCTIONS WE NEED.
      DO 190 M=0, L
      DO 200 K=M, L
      XX=XK(K+1)
      CALL BFUN(K,M,XX,B)
      BB(K,M)=B
      BBB(K,M)=(-1)**K*BB(K,M)
  200 CONTINUE
  190 CONTINUE
      DO 210 KK=1, N
      AA(KK)=W/2.D0*A(KK)/X(KK)
  210 CONTINUE
C-----TRANSFER FROM THREE DIMENSIONS TO TWO DIMENSIONS.
      LL=0
      DO 220 M=0, L
      DO 230 K=M, L
      LL=LL+1
      LLL(K,M,1)=LL
      LLL(K,M,2)=LL+1
      LLL(K,M,3)=NFUN+LL
      LLL(K,M,4)=NFUN+LL+1
      WRITE(5,*) LLL(K,M,1), LLL(K,M,2), LLL(K,M,3),
     *  LLL(K,M,4)
      LK(LL,1)=K
      LM(LL,1)=M
      LK(NFUN+LL,3)=K
      LM(NFUN+LL,3)=M
      LL=LL+1
      LK(LL,2)=K
      LM(LL,2)=M
      LK(NFUN+LL,4)=K
      LM(NFUN+LL,4)=M
  230 CONTINUE
  220 CONTINUE
      DO 240 M=0, L
      DO 250 K=M, L
      DO 260 J=1, 2
      DO 270 I=1, NMUTOT
      Y(LLL(K,M,J),I)=P1(K,M,I)
  270 CONTINUE
  260 CONTINUE
      DO 280 JJ=3, 4
      DO 290 II=1, N
      Y(LLL(K,M,JJ),II)=P2(K,M,II)
  290 CONTINUE
  280 CONTINUE
  250 CONTINUE
  240 CONTINUE
      XN=0.D0
C-----PRINT INITIAL VALUES FOR PP1, PP2, PP3, AND PP4.
      DO 291 M=0, L
      DO 293 K=M, L
```

```
      DO 295 I=1, NMUTOT
      PP1(K,M,I)=Y(LLL(K,M,1),I)
      PP2(K,M,I)=Y(LLL(K,M,2),I)
  295 CONTINUE
      DO 297 II=1, N
      PP3(K,M,II)=Y(LLL(K,M,3),II)
      PP4(K,M,II)=Y(LLL(K,M,4),II)
  297 CONTINUE
  293 CONTINUE
  291 CONTINUE
      CALL OUT(XN)
C-----DO LOOP FOR RK5.
      LIU=0
      DO 300 J=1, NPRINT
      NIPRINT=IPRINT(J)
      H=HSTEP(J)
      DO 310 I=1, NIPRINT
      CALL RK5(NMUTOT,N,NFUN,NFUNP1,INFUN,H,XN,Y,DERV)
  310 CONTINUE
C-----TRANSFER FROM TWO DIMENSIONS TO THREE DIMENSIONS.
      DO 320 M=0, L
      DO 330 K=M, L
      DO 340 I=1, NMUTOT
C-----PP1= MU VALUES WHEN TAU=0.
      PP1(K,M,I)=Y(LLL(K,M,1),I)
C-----PP2= MU VALUES WHEN TAU=TA0.
      PP2(K,M,I)=Y(LLL(K,M,2),I)
  340 CONTINUE
      DO 350 II=1, N
C-----PP3= DXA POINTS WHEN TAU=0.
      PP3(K,M,II)=Y(LLL(K,M,3),II)
C-----PP4= DXA POINTS WHEN TAU=TA0.
      PP4(K,M,II)=Y(LLL(K,M,4),II)
  350 CONTINUE
  330 CONTINUE
  320 CONTINUE
      CALL OUT(XN)
C-----CALL SUBROUTINE PPFUN TO CALCULATE PP5, PP6, PP7, AND
C-----PP8.
      CALL PPFUN
C-----CALL SUBROUTINE RT TO CALCULATE RP1, RP2, TP1, AND
C-----TP2.
      CALL RT(FLAGR)
C-----CALL SUBROUTINE RTINTEN TO CALCULATE RINTEN, TINTEN.
      CALL RTINTEN(XN)
C-----CALL SUBROUTINE TBQ TO CALCULATE TQ AND BQ.
      CALL TBQ
      LIU=LIU+1
      WRITE(*,*) LIU
  300 CONTINUE
      STOP
      END
```

```
C-----SUBROUTINE TO CALCULATE B FUNCTION.
      SUBROUTINE BFUN(K,M,XX,B)
      DOUBLE PRECISION XX, B, FACT1, FACT2
      IF(K-M .EQ. 0) GO TO 200
      FACT1=1.D0
      DO 100 I=1, K-M
      FACT1=I*FACT1
  100 CONTINUE
      GO TO 300
  200 FACT1=1.D0
  300 IF(K+M .EQ. 0) GO TO 500
      FACT2=1.D0
      DO 400 I=1, K+M
      FACT2=I*FACT2
  400 CONTINUE
      GO TO 600
  500 FACT2=1.D0
  600 B=XX*FACT1/FACT2
      RETURN
      END


C-----SUBROUTINE TO CALCULATE LEGENDRE FUNCTIONS.
      SUBROUTINE LEG(K,M,NYYS,YY,P)
      DOUBLE PRECISION P(3,3,44), YY(44), Y, SUM, FACT1,
     *  FACT2, FACT3, FACT4
      IF(K .LE. 5 .AND. M .LE. 5) THEN
      GO TO 100
      ELSE
      GO TO 1000
      ENDIF
  100 GO TO (200,300,400,500,600,700) M+1
  200 GO TO (210,220,230,240,250,260) K+1
  210 DO 215 I=1, NYYS
      Y=YY(I)
      P(0,0,I)=1.D0
  215 CONTINUE
      GO TO 5000
  220 DO 225 I=1, NYYS
      Y=YY(I)
      P(1,0,I)=Y
  225 CONTINUE
      GO TO 5000
  230 DO 235 I=1, NYYS
      Y=YY(I)
      P(2,0,I)=0.5D0*(3.D0*Y**2-1.D0)
  235 CONTINUE
      GO TO 5000
  240 DO 245 I=1, NYYS
      Y=YY(I)
      P(3,0,I)=0.5D0*Y*(5.D0*Y**2-3.D0)
  245 CONTINUE
      GO TO 5000
```

```
250 DO 255 I=1, NYYS
    Y=YY(I)
    P(4,0,I)=1.D0/8.D0*(35.D0*Y**4-30.D0*Y**2+3.D0)
255 CONTINUE
    GO TO 5000
260 DO 265 I=1, NYYS
    Y=YY(I)
    P(5,0,I)=1.D0/8.D0*Y*(63.D0*Y**4-70.D0*Y**2+15.D0)
265 CONTINUE
    GO TO 5000
300 GO TO (310,320,330,340,350) K
310 DO 315 I=1, NYYS
    Y=YY(I)
    P(1,1,I)=(1.D0-Y**2)**0.5
315 CONTINUE
    GO TO 5000
320 DO 325 I=1, NYYS
    Y=YY(I)
    P(2,1,I)=3.D0*Y*(1.D0-Y**2)**0.5
325 CONTINUE
    GO TO 5000
330 DO 335 I=1, NYYS
    Y=YY(I)
    P(3,1,I)=1.5D0*(5.D0*Y**2-1.D0)*(1.D0-Y**2)**0.5
335 CONTINUE
    GO TO 5000
340 DO 345 I=1, NYYS
    Y=YY(I)
    P(4,1,I)=2.5D0*Y*(7.D0*Y**2-3.D0)*(1.D0-Y**2)**0.5
345 CONTINUE
    GO TO 5000
350 DO 355 I=1, NYYS
    Y=YY(I)
    P(5,1,I)=5.D0/8.D0*(63.D0*Y**4-42.D0*Y**2+3.D0)*
   *            (1.D0-Y**2)**0.5
355 CONTINUE
    GO TO 5000
400 GO TO (410,420,430,440) K-1
410 DO 415 I=1, NYYS
    Y=YY(I)
    P(2,2,I)=3.D0*(1.D0-Y**2)
415 CONTINUE
    GO TO 5000
420 DO 425 I=1, NYYS
    Y=YY(I)
    P(3,2,I)=15.D0*Y*(1.D0-Y**2)
425 CONTINUE
    GO TO 5000
430 DO 435 I=1, NYYS
    Y=YY(I)
    P(4,2,I)=7.5D0*(1.D0-Y**2)*(7.D0*Y**2-1.D0)
435 CONTINUE
    GO TO 5000
440 DO 445 I=1, NYYS
```

```
      Y=YY(I)
      P(5,2,I)=105.D0/2.D0*Y*(1.D0-Y**2)*(3.D0*Y**2-1.D0)
 445  CONTINUE
      GO TO 5000
 500  GO TO (510,520,530) K-2
 510  DO 515 I=1, NYYS
      Y=YY(I)
      P(3,3,I)=15.D0*(1.D0-Y**2)**1.5
 515  CONTINUE
      GO TO 5000
 520  DO 525 I=1, NYYS
      Y=YY(I)
      P(4,3,I)=105.D0*Y*(1.D0-Y**2)**1.5
 525  CONTINUE
      GO TO 5000
 530  DO 535 I=1, NYYS
      Y=YY(I)
      P(5,3,I)=105.D0/2.D0*(9.D0*Y**2-1.D0)*(1.D0-Y**2)**1.5
 535  CONTINUE
      GO TO 5000
 600  GO TO (610,620) K-3
 610  DO 615 I=1, NYYS
      Y=YY(I)
      P(4,4,I)=105.D0*(1.D0-Y**2)**2
 615  CONTINUE
      GO TO 5000
 620  DO 625 I=1, NYYS
      Y=YY(I)
      P(5,4,I)=945.D0*Y*(1.D0-Y**2)**2
 625  CONTINUE
      GO TO 5000
 700  DO 750 I=1, NYYS
      Y=YY(I)
      P(5,5,I)=945.D0*(1.D0-Y**2)**2.5
 750  CONTINUE
      GO TO 5000
1000  K1=K/2
      K2=2*K1
      M1=M/2
      M2=2*M1
      IF(K .EQ. K2 .AND. M .EQ. M2) L=K/2-M/2
      IF(K .EQ. K2 .AND. M .NE. M2) L=K/2-(M+1)/2
      IF(K .NE. K2 .AND. M .EQ. M2) L=(K-1)/2-M/2
      IF(K .NE. K2 .AND. M .NE. M2) L=(K-1)/2-(M-1)/2
      DO 1100 I=1, NYYS
      Y=YY(I)
      SUM=0.0D0
      DO 1200 N=0, L
      IF(2*K-2*N .EQ. 0) GO TO 1400
      FACT1=1.D0
      DO 1300 J=1, 2*K-2*N
      FACT1=J*FACT1
1300  CONTINUE
      GO TO 1500
```

```
1400 FACT1=1.D0
1500 IF(N .EQ. 0) GO TO 1700
     FACT2=1.D0
     DO 1600 J=1, N
     FACT2=J*FACT2
1600 CONTINUE
     GO TO 1800
1700 FACT2=1.D0
1800 IF(K-N .EQ. 0) GO TO 2000
     FACT3=1.D0
     DO 1900 J=1, K-N
     FACT3=J*FACT3
1900 CONTINUE
     GO TO 2100
2000 FACT3=1.D0
2100 IF(K-2*N-M .EQ. 0) GO TO 2300
     FACT4=1.D0
     DO 2200 J=1, K-2*N-M
     FACT4=J*FACT4
2200 CONTINUE
     GO TO 2400
2300 FACT4=1.D0
2400 SUM=SUM+(-1)**N*FACT1/2.D0**K/FACT2/FACT3/FACT4*
    *    Y**(K-2*N-M)
1200 CONTINUE
     P(K,M,I)=(1.D0-Y**2)**(M/2.D0)*SUM
1100 CONTINUE
5000 RETURN
     END



C-----SUBROUTINE TO CALCULATE THE INTERFACE REFLECTION
C-----COEFFICIENTS THO'S.
      SUBROUTINE TH(NUS,U,N,THO)
      DOUBLE PRECISION U(44), N, THO(44), UCR, A
      IF((1.D0-N**(-2)) .LT. 0.0D0) THEN
      UCR=0.0D0
      ELSE
      UCR=(1.D0-N**(-2))**0.5
      ENDIF
      DO 100 I=1, NUS
      IF(U(I) .GE. UCR) GO TO 200
      THO(I)=1.D0
  100 CONTINUE
      GO TO 500
  200 A=(1.D0/N**2-(1.D0-U(I)**2))**0.5
      THO(I)=0.5D0*(((A-U(I))/(A+U(I)))**2+((A-U(I)/N**2)/
     *       (A+U(I)/N**2))**2)
      GO TO 100
  500 RETURN
      END
```

```
C-----SUBROUTINE FOR RUNGE-KUTTA METHOD.
      SUBROUTINE RK5(NPIC,NZMU,NFUN,NFUNP1,INFUN,H,XN,YN,
     *FCT)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION  C(6), Z(6), A(6,5), YN(24,44), Y(24,44)
      COMMON/BLK5/DER(24,44)
      COMMON/BLK6/AK(24,6,44)
      DATA C(2),C(3),C(4),C(5),C(6)
     *  /.25D0,.25D0,.5D0,.75D0,1.0D0/
      DATA Z(1),Z(2),Z(3),Z(4),Z(5),Z(6)
     *  /7.0D0,0.0D0,32.0D0,12.0D0,32.0D0,7.0D0/
      DATA A(2,1),A(3,1),A(3,2),A(4,1),A(4,2),A(4,3)
     *  /.25D0,.125D0,.125D0,0.0D0,-.50D0,1.0D0/
      DATA A(5,1),A(5,2),A(5,3),A(5,4)
     *  /.1875D0,0.0D0,0.0D0,.5625D0/
      DATA A(6,1),A(6,2),A(6,3),A(6,4),A(6,5)
     *  /-.428571428571429D0,.285714285714285D0,
     *  1.71428571428571D0,-1.71428571428571D0,
     *  1.14285714285714D0/
      CALL FCT(XN,YN)
      DO 10 L=1,NFUN
      DO 10 I=1,NPIC
   10 AK(L,1,I)=DER(L,I)*H
      DO 15 L=NFUNP1,INFUN
      DO 15 I=1,NZMU
   15 AK(L,1,I)=DER(L,I)*H
      DO 90 K=2,6
      DO 30 L=1,NFUN
      DO 30 I=1,NPIC
      K1=K-1
      SUM=0.D0
      DO 20 J=1,K1
   20 SUM=A(K,J)*AK(L,J,I)+SUM
   30 Y(L,I)=YN(L,I)+SUM
      DO 40 L=NFUNP1,INFUN
      DO 40 I=1,NZMU
      K1=K-1
      SUM=0.D0
      DO 35 J=1,K1
   35 SUM=A(K,J)*AK(L,J,I)+SUM
   40 Y(L,I)=YN(L,I)+SUM
      X=XN+C(K)*H
      CALL FCT(X,Y)
      DO 80 L=1,NFUN
      DO 80 I=1,NPIC
   80 AK(L,K,I)=DER(L,I)*H
      DO 90 L=NFUNP1,INFUN
      DO 90 I=1,NZMU
   90 AK(L,K,I)=DER(L,I)*H
      DO 101 L=1,NFUN
      DO 101 I=1,NPIC
      PHI=0.D0
      DO 100 K7=1,6
  100 PHI=PHI+Z(K7)*AK(L,K7,I)
```

```
  101 YN(L,I)=YN(L,I)+PHI/90.D0
      DO 110 L=NFUNP1,INFUN
      DO 110 I=1,NZMU
      PHI=0.D0
      DO 108 K7=1,6
  108 PHI=PHI+Z(K7)*AK(L,K7,I)
  110 YN(L,I)=YN(L,I)+PHI/90.D0
      XN=XN+H
      RETURN
      END


C-----SUBROUTINE FOR DERIVATIVES.
      SUBROUTINE DERV(XN,Y)
      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*8 MU
      DIMENSION Y(24,44)
      COMMON/BLK1/L,N,NMUTOT
      COMMON/BLK2/LLL(3,3,4)
      COMMON/BLK3/P2(3,3,44),AA(44),BBB(3,3),BB(3,3)
      COMMON/BLK4/X(44),MU(20)
      COMMON/BLK5/DER(24,44)
      DO 500 M=0, L
      DO 510 K=M, L
      DO 520 JJ=1, N
      SUM1=0.D0
      SUM2=0.D0
      DO 530 J=M, L
      SUM=0.D0
      DO 550 KK=1, N
      SUM=(Y(LLL(K,M,4),KK)*P2(J,M,KK))*AA(KK)+SUM
  550 CONTINUE
      SUM1=BBB(J,M)*Y(LLL(J,M,4),JJ)*SUM+SUM1
      SUM2=BB(J,M)*Y(LLL(J,M,3),JJ)*SUM+SUM2
  530 CONTINUE
      DER(LLL(K,M,3),JJ)=(-1)**K*SUM1
      DER(LLL(K,M,4),JJ)=(-1.D0/X(JJ))*Y(LLL(K,M,4),JJ)+SUM2
  520 CONTINUE
  510 CONTINUE
  500 CONTINUE
      DO 600 M=0, L
      DO 610 K=M, L
      DO 620 JJ=1, NMUTOT
      SUM1=0.D0
      SUM2=0.D0
      DO 630 J=M, L
      SUM=0.D0
      DO 650 KK=1, N
      SUM=(Y(LLL(K,M,4),KK)*P2(J,M,KK))*AA(KK)+SUM
  650 CONTINUE
      SUM1=BBB(J,M)*Y(LLL(J,M,2),JJ)*SUM+SUM1
      SUM2=BB(J,M)*Y(LLL(J,M,1),JJ)*SUM+SUM2
  630 CONTINUE
```

```
      DER(LLL(K,M,1),JJ)=(-1)**K*SUM1
      DER(LLL(K,M,2),JJ)=(-1.D0/MU(JJ))*Y(LLL(K,M,2),JJ)+
     *                   SUM2
  620 CONTINUE
  610 CONTINUE
  600 CONTINUE
      RETURN
      END


C-----SUBROUTINE FOR OUTPUT FILE.
      SUBROUTINE OUT(XN)
      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*8 MU
      COMMON/BLK1/L,N,NMUTOT
      COMMON/BLK4/X(44),MU(20)
      COMMON/BLK9/PP1(3,3,20),PP2(3,3,20),PP3(3,3,44),
     *  PP4(3,3,44)
      WRITE(5,*)
      WRITE(5,11) XN
   11 FORMAT(1X,'OPTICAL THICKNESS =',F12.8)
      WRITE(5,*)
      WRITE(5,*) '                         TOP                 BOTTOM'
      DO 100 M=0, L
      WRITE(5,12) M
   12 FORMAT('M=',I2)
      DO 200 K=M, L
      WRITE(5,13) K
   13 FORMAT('K=',I2)
      DO 300 I=1, NMUTOT
      WRITE(5,14) MU(I), PP1(K,M,I), PP2(K,M,I)
   14 FORMAT('MU=',F10.8,1X,F20.12,3X,F20.12)
  300 CONTINUE
      WRITE(5,*)
  200 CONTINUE
  100 CONTINUE
C     WRITE(5,*)
C     WRITE(5,*)
C     WRITE(5,*)
C     DO 400 M=0, L
C     WRITE(5,12) M
C     DO 500 K=M, L
C     WRITE(5,13) K
C     DO 600 I=1, N
C     WRITE(5,14) X(I), PP3(K,M,I), PP4(K,M,I)
C 600 CONTINUE
C     WRITE(5,*)
C 500 CONTINUE
C 400 CONTINUE
      RETURN
      END
```

```
C-----SUBROUTINE TO CALCULATE PP5, PP6, PP7, AND PP8.
      SUBROUTINE PPFUN
      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*8 MU
      COMMON/BLK1/L,N,NMUTOT
      COMMON/BLK3/P2(3,3,44),AA(44),BBB(3,3),BB(3,3)
      COMMON/BLK4/X(44),MU(20)
      COMMON/BLK7/TH2(44)
      COMMON/BLK8/ERROR
      COMMON/BLK9/PP1(3,3,20),PP2(3,3,20),PP3(3,3,44),
     *  PP4(3,3,44)
      COMMON/BLK10/PP5(3,3,20),PP6(3,3,20)
      COMMON/BLK11/PP7(3,3,44),PP7N(3,3,44),PP8(3,3,44),
     *  PP8N(3,3,44)
C-----DO LOOP OF SUCCESSIVE APPROXIMATION FOR DXA POINTS
C-----BEGINS.(PP7)
      DO 100 M=0, L
      WRITE(5,16) M
   16 FORMAT('M=',I2)
C-----INITIAL GUESSING FOR PP7.
      DO 200 KK=M, L
      DO 300 II=1, N
C-----PP7= DXA POINTS WHEN TAU=0.
      PP7(KK,M,II)=PP3(KK,M,II)
  300 CONTINUE
  200 CONTINUE
      ITER=0
      DO 400 JCONV=1, 10000
      ITER=ITER+1
      EMAX=0.0
      DO 500 K=M, L
      DO 600 JJ=1, N
      SUM1=0.0
      DO 700 J=M, L
      SUM=0.0
      DO 800 I=1, N
      SUM=(X(I)*X(JJ))/(X(I)+X(JJ))*PP7(K,M,I)*(PP3(J,M,JJ)*
     *   PP3(J,M,I)-PP4(J,M,JJ)*PP4(J,M,I))*TH2(I)*AA(I)+SUM
  800 CONTINUE
      SUM1=BBB(J,M)*SUM+SUM1
  700 CONTINUE
      PP7N(K,M,JJ)=PP3(K,M,JJ)+SUM1
      ERR=DABS(PP7N(K,M,JJ)-PP7(K,M,JJ))
      IF(ERR .GT. EMAX) EMAX=ERR
  600 CONTINUE
  500 CONTINUE
      WRITE(*,*) EMAX
      IF(EMAX .LT. ERROR) GO TO 1100
      DO 900 K=M, L
      DO 1000 JJ=1, N
      PP7(K,M,JJ)=PP7N(K,M,JJ)
 1000 CONTINUE
  900 CONTINUE
  400 CONTINUE
```

```
      WRITE(5,*) 'NEED MORE LOOPS !'
 1100 WRITE(5,17) ITER
   17 FORMAT('NUMBER OF ITERATIONS=',I5)
      DO 1200 K=M, L
      WRITE(5,18) K
   18 FORMAT('K=',I2)
C     DO 1300 JJ=1, N
C     WRITE(5,19) X(JJ),PP7N(K,M,JJ)
C  19 FORMAT('MU=',F15.12,5X,'PP7=',F20.12)
C1300 CONTINUE
      WRITE(5,*)
 1200 CONTINUE
  100 CONTINUE
C-----DO LOOP FOR MU VALUES BEGINS.(PP5)
C-----PP5= MU VALUES WHEN TAU=0.
      WRITE(5,*) '                                                TOP'
      DO 1500 M=0, L
      WRITE(5,16) M
      DO 1600 K=M, L
      WRITE(5,18) K
      DO 1700 JJ=1, NMUTOT
      SUM1=0.0
      DO 1800 J=M, L
      SUM=0.0
      DO 1900 I=1, N
      SUM=(X(I)*MU(JJ))/(X(I)+MU(JJ))*PP7N(K,M,I)*(PP1(J,M,
     *    JJ)*PP3(J,M,I)-PP2(J,M,JJ)*PP4(J,M,I))*TH2(I)*
     *    AA(I)+SUM
 1900 CONTINUE
      SUM1=BBB(J,M)*SUM+SUM1
 1800 CONTINUE
      PP5(K,M,JJ)=PP1(K,M,JJ)+SUM1
      WRITE(5,20) MU(JJ),PP5(K,M,JJ)
   20 FORMAT('MU=',F15.12,5X,'PP5=',F20.12)
 1700 CONTINUE
      WRITE(5,*)
 1600 CONTINUE
 1500 CONTINUE
C-----DO LOOP OF SUCCESSIVE APPROXIMATION FOR DXA POINTS
C-----BEGINS.(PP8)
      DO 2100 M=0, L
      WRITE(5,16) M
C-----INITIAL GUESSING FOR PP8.
      DO 2200 KK=M, L
      DO 2300 II=1, N
C-----PP8= DXA POINTS WHEN TAU=TA0.
      PP8(KK,M,II)=PP4(KK,M,II)
 2300 CONTINUE
 2200 CONTINUE
      ITER=0
      DO 2400 JCONV=1, 10000
      ITER=ITER+1
      EMAX=0.0
      DO 2500 K=M, L
```

```
      DO 2600 JJ=1, N
      SUM1=0.0
      DO 2700 J=M, L
      SUM=0.0
      DO 2800 I=1, N
      SUM=(X(I)*X(JJ))/(X(I)+X(JJ))*PP8(K,M,I)*(PP3(J,M,JJ)*
     *   PP3(J,M,I)-PP4(J,M,JJ)*PP4(J,M,I))*TH2(I)*AA(I)+SUM
 2800 CONTINUE
      SUM1=BBB(J,M)*SUM+SUM1
 2700 CONTINUE
      PP8N(K,M,JJ)=PP4(K,M,JJ)+SUM1
      ERR=DABS(PP8N(K,M,JJ)-PP8(K,M,JJ))
      IF(ERR .GT. EMAX) EMAX=ERR
 2600 CONTINUE
 2500 CONTINUE
      WRITE(*,*) EMAX
      IF(EMAX .LT. ERROR) GO TO 3100
      DO 2900 K=M, L
      DO 3000 JJ=1, N
      PP8(K,M,JJ)=PP8N(K,M,JJ)
 3000 CONTINUE
 2900 CONTINUE
 2400 CONTINUE
      WRITE(5,*) 'NEED MORE LOOPS !'
 3100 WRITE(5,17) ITER
      DO 3200 K=M, L
      WRITE(5,18) K
C     DO 3300 JJ=1, N
C     WRITE(5,21) X(JJ),PP8N(K,M,JJ)
C  21 FORMAT('MU=',F15.12,5X,'PP8=',F20.12)
C3300 CONTINUE
      WRITE(5,*)
 3200 CONTINUE
 2100 CONTINUE
C-----DO LOOP FOR MU VALUES BEGINS.(PP6)
C-----PP6= MU VALUES WHEN TAU=TA0.
      WRITE(5,*) '                                          BOTTOM'
      DO 3500 M=0, L
      WRITE(5,16) M
      DO 3600 K=M, L
      WRITE(5,18) K
      DO 3700 JJ=1, NMUTOT
      SUM1=0.0
      DO 3800 J=M, L
      SUM=0.0
      DO 3900 I=1, N
      SUM=(X(I)*MU(JJ))/(X(I)+MU(JJ))*PP8N(K,M,I)*(PP1(J,M,
     *    JJ)*PP3(J,M,I)-P2(J,M,JJ)*PP4(J,M,I))*TH2(I)*
     *    AA(I)+SUM
 3900 CONTINUE
      SUM1=BBB(J,M)*SUM+SUM1
 3800 CONTINUE
      PP6(K,M,JJ)=PP2(K,M,JJ)+SUM1
      WRITE(5,22) MU(JJ),PP6(K,M,JJ)
```

```
   22 FORMAT('MU=',F15.12,5X,'PP6=',F20.12)
 3700 CONTINUE
      WRITE(5,*)
 3600 CONTINUE
 3500 CONTINUE
      RETURN
      END


C-----SUBROUTINE TO CALCULATE RP1, RP2, TP1, AND TP2.
      SUBROUTINE RT(FLAGR)
      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*8 MU
      DIMENSION FLAG(44)
      COMMON/BLK1/L,N,NMUTOT
      COMMON/BLK3/P2(3,3,44),AA(44),BBB(3,3),BB(3,3)
      COMMON/BLK4/X(44),MU(20)
      COMMON/BLK7/TH2(44)
      COMMON/BLK8/ERROR
      COMMON/BLK9/PP1(3,3,20),PP2(3,3,20),PP3(3,3,44),
     *  PP4(3,3,44)
      COMMON/BLK13/RP2(3,20,20)
      COMMON/BLK14/RP1(3,44,20),RP1N(3,44,20)
      COMMON/BLK15/W
      COMMON/BLK16/TP1(3,20,44),TP1N(3,20,44)
      COMMON/BLK17/TP2(3,20,20)
C-----DO LOOP OF SUCCESSIVE APPROXIMATIONS FOR DXA POINTS
C-----BEGINS.(RP1)
      DO 100 M=0, L
      WRITE(5,24) M
   24 FORMAT('M=',I2)
      DO 200 J=1, NMUTOT
      WRITE(5,25) MU(J)
   25 FORMAT('MU=',F15.12)
C-----INITIAL GUESSING FOR RP1(M,II,J)
      DO 300 II=1, N
      SUM=0.0
      DO 400 I=M, L
      SUM=SUM+BBB(I,M)*(PP1(I,M,J)*PP3(I,M,II)-PP2(I,M,J)
     *     *PP4(I,M,II))
  400 CONTINUE
      RP1(M,II,J)=X(II)*MU(J)/(X(II)+MU(J))*W/2.D0*(-1)**M*
     *            SUM
  300 CONTINUE
      ITER=0
      DO 500 JJCONV=1, 10000
      ITER=ITER+1
      EMAX=0.0
      DO 600 JJ=1, N
      SUM2=0.0
      SUM3=0.0
      DO 700 K=M, L
      SUM1=0.0
```

```
      DO 800 KK=1, N
      SUM1=SUM1+X(KK)*X(JJ)/(X(KK)+X(JJ))*RP1(M,KK,J)*(PP3(K
     *    ,M,KK)*PP3(K,M,JJ)-PP4(K,M,KK)*PP4(K,M,JJ))*TH2(KK)
     *    *AA(KK)
  800 CONTINUE
      SUM2=SUM2+BBB(K,M)*SUM1
      SUM3=SUM3+BBB(K,M)*(PP1(K,M,J)*PP3(K,M,JJ)-PP2(K,M,J)*
     *      PP4(K,M,JJ))
  700 CONTINUE
      RP1N(M,JJ,J)=W/2.D0*MU(J)*X(JJ)/(MU(J)+X(JJ))*(-1)**M*
     *              SUM3+SUM2
      ERR=DABS(RP1N(M,JJ,J)-RP1(M,JJ,J))
      IF(ERR .GT. EMAX) EMAX=ERR
  600 CONTINUE
      IF(EMAX .LT. ERROR) GO TO 1000
      DO 900 JJ=1, N
      RP1(M,JJ,J)=RP1N(M,JJ,J)
  900 CONTINUE
  500 CONTINUE
 1000 WRITE(5,26) ITER
   26 FORMAT('NUMBER OF ITERATIONS =',I5)
C     DO 1050 JJ=1, N
C     WRITE(5,27) X(JJ), RP1N(M,JJ,J)
C  27 FORMAT('MU=',F15.12,5X,'RP1=',F20.12)
C1050 CONTINUE
      WRITE(5,*)
  200 CONTINUE
  100 CONTINUE
C-----DO LOOP FOR MU VALUES BEGINS.(RP2)
      DO 1100 M=0, L
      WRITE(5,24) M
      DO 1200 J=1, NMUTOT
C     IF(J .NE. 10 .AND. J .NE. 20) GO TO 1200
      WRITE(5,25) MU(J)
      DO 1300 JJ=1, NMUTOT
      SUM2=0.0
      SUM3=0.0
      DO 1400 K=M, L
      SUM1=0.0
      DO 1500 KK=1, N
      SUM1=SUM1+X(KK)*MU(JJ)/(X(KK)+MU(JJ))*RP1N(M,KK,J)*(
     *  PP3(K,M,KK)*PP1(K,M,JJ)-PP4(K,M,KK)*PP2(K,M,JJ))*
     *  TH2(KK)*AA(KK)
 1500 CONTINUE
      SUM2=SUM2+BBB(K,M)*SUM1
      SUM3=SUM3+BBB(K,M)*(PP1(K,M,J)*PP1(K,M,JJ)-PP2(K,M,J)*
     *      PP2(K,M,JJ))
 1400 CONTINUE
      RP2(M,JJ,J)=W/2.D0*MU(J)*MU(JJ)/(MU(J)+MU(JJ))*(-1)**M
     *              *SUM3+SUM2
      IF(J .NE. 9) GO TO 1300
C     WRITE(5,28) MU(JJ), 2.D0/W*RP2(M,JJ,J)
      WRITE(5,28) MU(JJ), RP2(M,JJ,J)
   28 FORMAT('MUB=',F15.12,5X,'RP2=',F20.12)
```

```
 1300 CONTINUE
      WRITE(5,*)
 1200 CONTINUE
 1100 CONTINUE
C-----DO LOOP OF SUCCESSIVE APPROXIMATIONS FOR DXA POINTS
C-----BEGINS.(TP1)
      DO 2100 M=0, L
      WRITE(5,24) M
      DO 2200 JJ=1, NMUTOT
      WRITE(5,25) MU(JJ)
C-----INITIAL GUESSING FOR TP1(M,JJ,II)
      DO 2300 II=1, N
      SUM=0.0
      DO 2400 I=M, L
      SUM=SUM+BB(I,M)*(PP4(I,M,II)*PP1(I,M,JJ)-PP3(I,M,II)
     *    *PP2(I,M,JJ))
 2400 CONTINUE
      TP1(M,JJ,II)=X(II)*MU(JJ)/(X(II)-MU(JJ))*W/2.D0*SUM
 2300 CONTINUE
      ITER=0
      DO 2500 JJCONV=1, 10000
      ITER=ITER+1
      EMAX=0.0
      DO 2600 J=1, N
      SUM2=0.0
      SUM3=0.0
      DO 2700 K=M, L
      SUM1=0.0
      DO 2800 KK=1, N
      SUM1=SUM1+X(KK)*X(J)/(X(KK)+X(J))*TP1(M,JJ,KK)*(PP3(K,
     *     M,J)*PP3(K,M,KK)-PP4(K,M,J)*PP4(K,M,KK))*TH2(KK)*
     *     AA(KK)
 2800 CONTINUE
      SUM2=SUM2+BBB(K,M)*SUM1
      SUM3=SUM3+BB(K,M)*(PP4(K,M,J)*PP1(K,M,JJ)-PP3(K,M,J)*
     *     PP2(K,M,JJ))
 2700 CONTINUE
      TP1N(M,JJ,J)=W/2.D0*MU(JJ)*X(J)/(X(J)-MU(JJ))*SUM3
     *               +SUM2
      ERR=DABS(TP1N(M,JJ,J)-TP1(M,JJ,J))
      IF(ERR .GT. EMAX) EMAX=ERR
 2600 CONTINUE
      IF(EMAX .LT. ERROR) GO TO 3000
      DO 2900 J=1, N
      TP1(M,JJ,J)=TP1N(M,JJ,J)
 2900 CONTINUE
 2500 CONTINUE
 3000 WRITE(5,26) ITER
C     DO 3050 J=1, N
C     WRITE(5,29) X(J), TP1N(M,JJ,J)
C  29 FORMAT('MU=',F15.12,5X,'TP1=',F20.12)
C3050 CONTINUE
      WRITE(5,*)
 2200 CONTINUE
```

```
 2100 CONTINUE
C-----DO LOOP FOR MU VALUES BEGINS.(TP2)
      DO 3100 M=0, L
      WRITE(5,24) M
      DO 3200 JJ=1, NMUTOT
C     IF(JJ .NE. 10) GO TO 3200
      WRITE(5,35) MU(JJ)
   35 FORMAT('MUB=',F15.12)
      DO 3300 J=1, NMUTOT
      IF(MU(J) .EQ. MU(JJ)) GO TO 4000
      SUM2=0.0
      SUM3=0.0
      DO 3400 K=M, L
      SUM1=0.0
      DO 3500 KK=1, N
      SUM1=SUM1+X(KK)*MU(J)/(X(KK)+MU(J))*TP1N(M,JJ,KK)*(
     *      PP1(K,M,J)*PP3(K,M,KK)-PP2(K,M,J)*PP4(K,M,KK))*
     *      TH2(KK)*AA(KK)
 3500 CONTINUE
      SUM2=SUM2+BBB(K,M)*SUM1
      SUM3=SUM3+BB(K,M)*(PP2(K,M,J)*PP1(K,M,JJ)-PP1(K,M,J)*
     *      PP2(K,M,JJ))
 3400 CONTINUE
      TP2(M,JJ,J)=W/2.D0*MU(JJ)*MU(J)/(MU(J)-MU(JJ))*SUM3
     *              +SUM2
      IF(JJ .NE. 9) GO TO 3300
C     WRITE(5,30) MU(J), 2.D0/W*TP2(M,JJ,J)
      WRITE(5,30) MU(J), TP2(M,JJ,J)
   30 FORMAT('MU=',F15.12,5X,'TP2=',F20.12)
      GO TO 3300
 4000 DO 4100 K=1, N
      FLAG(K)=TP1(M,JJ,K)
 4100 CONTINUE
      TP2(M,JJ,J)=FLAGR(X,FLAG,1.D0,6,N-6,N)
      IF(JJ .NE. 9) GO TO 3300
C     WRITE(5,30) MU(J), 2.D0/W*TP2(M,JJ,J)
      WRITE(5,30) MU(J), TP2(M,JJ,J)
 3300 CONTINUE
      WRITE(5,*)
 3200 CONTINUE
 3100 CONTINUE
      RETURN
      END


C-----FUNCTION TO INTERPOLATE THE VALUES OF TP2 BY
C-----LAGRANGE'S POLYNOMIAL APPROXIATION METHOD WHEN
C-----MU(J) = MU(JJ) IN SUBROUNTINE RT.
      DOUBLE PRECISION FUNCTION FLAGR(X,Y,XARG,IDEG,MIN,N)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION X(N), Y(N)
      FACTOR=1.D0
      MAX=MIN+IDEG
```

```
      DO 2 J=MIN, MAX
      IF(XARG .NE. X(J)) GO TO 2
      FLAGR=Y(J)
      RETURN
    2 FACTOR=FACTOR*(XARG-X(J))
      YEST=0.D0
      DO 5 I=MIN, MAX
      TERM=Y(I)*FACTOR/(XARG-X(I))
      DO 4 J=MIN, MAX
    4 IF(I .NE. J) TERM=TERM/(X(I)-X(J))
    5 YEST=YEST+TERM
      FLAGR=YEST
      RETURN
      END


C-----SUBROUTINE TO CALCULATE RINTEN AND TINTEN.
      SUBROUTINE RTINTEN(XN)
      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*8 MU, NR
      DIMENSION RINTEN(10,10), TINTEN(10,10)
      COMMON/BLK1/L,N,NMUTOT
      COMMON/BLK4/X(44),MU(20)
      COMMON/BLK13/RP2(3,20,20)
      COMMON/BLK15/W
      COMMON/BLK17/TP2(3,20,20)
      COMMON/BLK18/NR
      COMMON/BLK19/TH3(20),TH4(20)
C-----DO LOOP FOR MU VALUES BEGINS.(RINTEN)
      PAI=3.141592654
      DO 100 J=1, NMUTOT/2
C     IF(J .NE. 10) GO TO 100
      WRITE(5,31) MU(J)
   31 FORMAT('MUO=',F15.12)
      DO 200 I=1, NMUTOT/2
      SUM1=0.0
      DO 300 M=0, L
      DELTA=2.D0
      IF(M .EQ. 0) DELTA=1.D0
      SUM1=SUM1+DELTA*RP2(M,NMUTOT/2+I,NMUTOT/2+J)
  300 CONTINUE
      RINTEN(I,J)=MU(J)/2.D0/PAI*(1.D0-TH4(J))*(1.D0-
     *            TH3(NMUTOT/2+I))*SUM1/NR**2/MU(NMUTOT/2+J)
     *            /MU(NMUTOT/2+I)
      IF(J .NE. 9) GO TO 200
C     WRITE(5,32) MU(I), 4.D0*PAI*RINTEN(I,J)
      WRITE(5,32) MU(I), RINTEN(I,J)
   32 FORMAT('MUE=',F15.12,5X,'RIe=',F20.12)
  200 CONTINUE
      WRITE(5,*)
  100 CONTINUE
C-----DO LOOP FOR MU VALUES BEGINS.(TINTEN)
      DO 600 J=1, NMUTOT/2
```

```
C     IF(J .NE. 10) GO TO 600
      WRITE(5,31) MU(J)
      DO 700 I=1, NMUTOT/2
C     IF(MU(I) .EQ. MU(NMUTOT/2+J)) GO TO 700
      SUM1=0.0
      DO 800 M=0, L
      DELTA=2.D0
      IF(M .EQ. 0) DELTA=1.D0
      SUM1=SUM1+DELTA*(TH3(I)*DEXP(-XN/MU(I))*DCOS(M*PAI)*
     *     RP2(M,I,NMUTOT/2+J)+TP2(M,I,NMUTOT/2+J))
C     IF(J .NE. 10) GO TO 800
C     WRITE(5,33) MU(I),2.D0*SUM1/MU(I)
  800 CONTINUE
      TINTEN(I,J)=MU(J)/2.D0/PAI/MU(I)/MU(NMUTOT/2+J)*
     *            (1.D0-TH4(J))*SUM1
      IF(J .NE. 9) GO TO 700
      WRITE(5,33) MU(I), TINTEN(I,J)
   33 FORMAT('MUin=',F15.12,5X,'TIe=',F20.12)
  700 CONTINUE
      WRITE(5,*)
  600 CONTINUE
      RETURN
      END


C-----SUBROUTINE TO CALCULATE TQ AND BQ.
      SUBROUTINE TBQ
      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*8 MU
      DIMENSION TQ(10), BQ(10)
      COMMON/BLK1/L,N,NMUTOT
      COMMON/BLK4/X(44),MU(20)
      COMMON/BLK10/PP5(3,3,20),PP6(3,3,20)
      COMMON/BLK19/TH3(20),TH4(20)
C-----DO LOOP FOR MU VALUES BEGINS.(TQ AND BQ)
      WRITE(5,*) '                     TQ                   BQ '
      WRITE(*,*)
      DO 100 J=1, NMUTOT/2
      A= MU(J)/MU(NMUTOT/2+J)*(1.D0-TH4(J))
      TQ(J)=A*PP5(1,0,NMUTOT/2+J)
      BQ(J)=A*PP6(1,0,NMUTOT/2+J)
      WRITE(5,34) MU(J), TQ(J), BQ(J)
   34 FORMAT('MU=',F10.8,1X,F20.12,3X,F20.12)
  100 CONTINUE
      RETURN
      END


      SUBROUTINE DXA(N,AA,BB,X,A)
```

This subroutine is the same as the one in Appendix B.

APPENDIX D

SAMPLE OF OUTPUT DATA FOR

SEMI-INFINITE PROGRAM

An example of output data for semi-infinite program is as follows (see App. B for definitions of the variables):

```
NUMBER OF LEGENDRE POLYNOMIALS (L)=  2
ALBEDO (W)= .100
ERROR= .000000010000
REFRACTIVE INDEX (NR)=1.330
NUMBER OF QUADRATURE POINTS (N)= 80
EXPANSION COEFFICIENTS :
   XK( 0)= 1.0000
   XK( 1)= 1.0000
   XK( 2)=  .4500
------------------------------------------------

M= 0
NUMBER OF ITERATIONS=    6
J= 0
MU=   .100000000000     PP2=      1.012608725657
MU=   .200000000000     PP2=      1.018025356420
MU=   .300000000000     PP2=      1.021044465073
MU=   .400000000000     PP2=      1.022675847937
MU=   .500000000000     PP2=      1.023385693836
MU=   .600000000000     PP2=      1.023437579465
MU=   .700000000000     PP2=      1.022997331755
MU=   .800000000000     PP2=      1.022176590902
MU=   .900000000000     PP2=      1.021054175340
MU= 1.000000000000     PP2=      1.019687748209

J= 1
MU=   .100000000000     PP2=       .096329948822
MU=   .200000000000     PP2=       .194171066111
MU=   .300000000000     PP2=       .292847775801
MU=   .400000000000     PP2=       .392064901620
MU=   .500000000000     PP2=       .491651261151
MU=   .600000000000     PP2=       .591496481568
MU=   .700000000000     PP2=       .691524818134
MU=   .800000000000     PP2=       .791682012618
MU=   .900000000000     PP2=       .891927930952
```

```
MU= 1.000000000000      PP2=       .992232125419

J= 2
MU=   .100000000000     PP2=      -.488415786749
MU=   .200000000000     PP2=      -.444288410000
MU=   .300000000000     PP2=      -.369634876067
MU=   .400000000000     PP2=      -.264738987543
MU=   .500000000000     PP2=      -.129690475063
MU=   .600000000000     PP2=       .035477525012
MU=   .700000000000     PP2=       .230752802440
MU=   .800000000000     PP2=       .456131790062
MU=   .900000000000     PP2=       .711614653591
MU= 1.000000000000      PP2=       .997203147023

M= 1
NUMBER OF ITERATIONS=    5
J= 1
MU=   .100000000000     PP2=       .999794090455
MU=   .200000000000     PP2=       .986342492572
MU=   .300000000000     PP2=       .961233743978
MU=   .400000000000     PP2=       .923970971009
MU=   .500000000000     PP2=       .873233791397
MU=   .600000000000     PP2=       .806645943332
MU=   .700000000000     PP2=       .719958671094
MU=   .800000000000     PP2=       .604721739926
MU=   .900000000000     PP2=       .439163417836
MU= 1.000000000000      PP2=       .000000000000

J= 2
MU=   .100000000000     PP2=       .295152733207
MU=   .200000000000     PP2=       .582853181899
MU=   .300000000000     PP2=       .852741092982
MU=   .400000000000     PP2=      1.093833074572
MU=   .500000000000     PP2=      1.293300313371
MU=   .600000000000     PP2=      1.434826770542
MU=   .700000000000     PP2=      1.495331919685
MU=   .800000000000     PP2=      1.436628926896
MU=   .900000000000     PP2=      1.174721871830
MU= 1.000000000000      PP2=       .000000000000

M= 2
NUMBER OF ITERATIONS=    4
J= 2
MU=   .100000000000     PP2=      2.974452151735
MU=   .200000000000     PP2=      2.886082964368
MU=   .300000000000     PP2=      2.736818260958
MU=   .400000000000     PP2=      2.526978678659
MU=   .500000000000     PP2=      2.256690445474
MU=   .600000000000     PP2=      1.926016332699
MU=   .700000000000     PP2=      1.534991850298
MU=   .800000000000     PP2=      1.083638856013
MU=   .900000000000     PP2=       .571971643632
MU= 1.000000000000      PP2=       .000000000000
-----------------------------------------------
```

M= 0
J= 0

MU=	.100000000000	PP4=	1.024117410411
MU=	.200000000000	PP4=	1.034163127596
MU=	.300000000000	PP4=	1.039675741639
MU=	.400000000000	PP4=	1.042598246127
MU=	.500000000000	PP4=	1.043801211406
MU=	.600000000000	PP4=	1.043764193096
MU=	.700000000000	PP4=	1.042782238696
MU=	.800000000000	PP4=	1.041050221805
MU=	.900000000000	PP4=	1.038703478634
MU=	1.000000000000	PP4=	1.035839619248

J= 1

MU=	.100000000000	PP4=	.098812462564
MU=	.200000000000	PP4=	.198026709386
MU=	.300000000000	PP4=	.297516700983
MU=	.400000000000	PP4=	.397194116834
MU=	.500000000000	PP4=	.496995762331
MU=	.600000000000	PP4=	.596875328339
MU=	.700000000000	PP4=	.696798131579
MU=	.800000000000	PP4=	.796737689663
MU=	.900000000000	PP4=	.896673434716
MU=	1.000000000000	PP4=	.996589149942

J= 2

MU=	.100000000000	PP4=	-.492669620625
MU=	.200000000000	PP4=	-.449952679376
MU=	.300000000000	PP4=	-.375990901985
MU=	.400000000000	PP4=	-.271416702341
MU=	.500000000000	PP4=	-.136454527815
MU=	.600000000000	PP4=	.028796720510
MU=	.700000000000	PP4=	.224288273829
MU=	.800000000000	PP4=	.449994423276
MU=	.900000000000	PP4=	.705901068906
MU=	1.000000000000	PP4=	.992000329382

M= 1
J= 1

MU=	.100000000000	PP4=	.995221617480
MU=	.200000000000	PP4=	.980163176901
MU=	.300000000000	PP4=	.954363042758
MU=	.400000000000	PP4=	.916937884016
MU=	.500000000000	PP4=	.866404595381
MU=	.600000000000	PP4=	.800306229330
MU=	.700000000000	PP4=	.714358807806
MU=	.800000000000	PP4=	.600120893086
MU=	.900000000000	PP4=	.435926358548
MU=	1.000000000000	PP4=	.000000000000

J= 2

MU=	.100000000000	PP4=	.292392225613
MU=	.200000000000	PP4=	.578738109260
MU=	.300000000000	PP4=	.847984376007

```
MU=   .400000000000      PP4=       1.088893072828
MU=   .500000000000      PP4=       1.288501958050
MU=   .600000000000      PP4=       1.430415698483
MU=   .700000000000      PP4=       1.491506273784
MU=   .800000000000      PP4=       1.433567912858
MU=   .900000000000      PP4=       1.172642691238
MU=  1.000000000000      PP4=        .000000000000

M= 2
J= 2
MU=   .100000000000      PP4=       2.978817348279
MU=   .200000000000      PP4=       2.892010867579
MU=   .300000000000      PP4=       2.743434126627
MU=   .400000000000      PP4=       2.533727792141
MU=   .500000000000      PP4=       2.263143625662
MU=   .600000000000      PP4=       1.931806594364
MU=   .700000000000      PP4=       1.539787267116
MU=   .800000000000      PP4=       1.087129008927
MU=   .900000000000      PP4=        .573860133815
MU=  1.000000000000      PP4=        .000000000000
---------------------------------------------------

M= 0
MU=   .90000000000
MUBAR=   .10000000       RP2=        .003621125513
MUBAR=   .20000000       RP2=        .006012935025
MUBAR=   .30000000       RP2=        .007565257298
MUBAR=   .40000000       RP2=        .008558933485
MUBAR=   .50000000       RP2=        .009199198572
MUBAR=   .60000000       RP2=        .009638977788
MUBAR=   .70000000       RP2=        .009994360241
MUBAR=   .80000000       RP2=        .010354977575
MUBAR=   .90000000       RP2=        .010791095618
MUBAR=  1.00000000       RP2=        .011358556282

M= 1
MU=   .90000000000
MUBAR=   .10000000       RP2=        .000866329156
MUBAR=   .20000000       RP2=        .001345316435
MUBAR=   .30000000       RP2=        .001521824360
MUBAR=   .40000000       RP2=        .001467077552
MUBAR=   .50000000       RP2=        .001242805670
MUBAR=   .60000000       RP2=        .000905759178
MUBAR=   .70000000       RP2=        .000512468453
MUBAR=   .80000000       RP2=        .000126500686
MUBAR=   .90000000       RP2=       -.000162516154
MUBAR=  1.00000000       RP2=        .000000000000

M= 2
MU=   .90000000000
MUBAR=   .10000000       RP2=        .000143876358
MUBAR=   .20000000       RP2=        .000253737522
MUBAR=   .30000000       RP2=        .000330781564
MUBAR=   .40000000       RP2=        .000375852883
```

MUBAR=	.50000000	RP2=	.000389559126
MUBAR=	.60000000	RP2=	.000372347782
MUBAR=	.70000000	RP2=	.000324555379
MUBAR=	.80000000	RP2=	.000246439751
MUBAR=	.90000000	RP2=	.000138201720
MUBAR=	1.00000000	RP2=	.000000000000

MUE=	.100000000000		
MUO=	.100000000000	Ie=	.000070626331
MUO=	.200000000000	Ie=	.000209024096
MUO=	.300000000000	Ie=	.000353561232
MUO=	.400000000000	Ie=	.000477657797
MUO=	.500000000000	Ie=	.000571050953
MUO=	.600000000000	Ie=	.000631301557
MUO=	.700000000000	Ie=	.000659725839
MUO=	.800000000000	Ie=	.000659375770
MUO=	.900000000000	Ie=	.000633738658
MUO=	1.000000000000	Ie=	.000572869702
MUE=	.200000000000		
MUO=	.100000000000	Ie=	.000104512048
MUO=	.200000000000	Ie=	.000309325854
MUO=	.300000000000	Ie=	.000523273280
MUO=	.400000000000	Ie=	.000707077214
MUO=	.500000000000	Ie=	.000845636734
MUO=	.600000000000	Ie=	.000935469936
MUO=	.700000000000	Ie=	.000978712872
MUO=	.800000000000	Ie=	.000980179043
MUO=	.900000000000	Ie=	.000945613526
MUO=	1.000000000000	Ie=	.000864446232
MUE=	.300000000000		
MUO=	.100000000000	Ie=	.000117853744
MUO=	.200000000000	Ie=	.000348848853
MUO=	.300000000000	Ie=	.000590258005
MUO=	.400000000000	Ie=	.000797908123
MUO=	.500000000000	Ie=	.000954943508
MUO=	.600000000000	Ie=	.001057686873
MUO=	.700000000000	Ie=	.001108908215
MUO=	.800000000000	Ie=	.001114598869
MUO=	.900000000000	Ie=	.001082336496
MUO=	1.000000000000	Ie=	.001007968151
MUE=	.400000000000		
MUO=	.100000000000	Ie=	.000119414449
MUO=	.200000000000	Ie=	.000353538607
MUO=	.300000000000	Ie=	.000598431092
MUO=	.400000000000	Ie=	.000809532249
MUO=	.500000000000	Ie=	.000970033840
MUO=	.600000000000	Ie=	.001076583145
MUO=	.700000000000	Ie=	.001132513660
MUO=	.800000000000	Ie=	.001144702736
MUO=	.900000000000	Ie=	.001122383227

MUO=	1.000000000000	Ie=	.001072274359
MUE=	.500000000000		
MUO=	.100000000000	Ie=	.000114210191
MUO=	.200000000000	Ie=	.000338254694
MUO=	.300000000000	Ie=	.000572966105
MUO=	.400000000000	Ie=	.000776027072
MUO=	.500000000000	Ie=	.000931738611
MUO=	.600000000000	Ie=	.001037385082
MUO=	.700000000000	Ie=	.001096822536
MUO=	.800000000000	Ie=	.001117625585
MUO=	.900000000000	Ie=	.001110513868
MUO=	1.000000000000	Ie=	.001095100864
MUE=	.600000000000		
MUO=	.100000000000	Ie=	.000105216926
MUO=	.200000000000	Ie=	.000311823312
MUO=	.300000000000	Ie=	.000528843436
MUO=	.400000000000	Ie=	.000717722097
MUO=	.500000000000	Ie=	.000864487568
MUO=	.600000000000	Ie=	.000967234110
MUO=	.700000000000	Ie=	.001030285589
MUO=	.800000000000	Ie=	.001061731842
MUO=	.900000000000	Ie=	.001073504907
MUO=	1.000000000000	Ie=	.001097915600
MUE=	.700000000000		
MUO=	.100000000000	Ie=	.000094246548
MUO=	.200000000000	Ie=	.000279632249
MUO=	.300000000000	Ie=	.000475246378
MUO=	.400000000000	Ie=	.000647150663
MUO=	.500000000000	Ie=	.000783444669
MUO=	.600000000000	Ie=	.000883101934
MUO=	.700000000000	Ie=	.000950815999
MUO=	.800000000000	Ie=	.000994953849
MUO=	.900000000000	Ie=	.001028200780
MUO=	1.000000000000	Ie=	.001093211820
MUE=	.800000000000		
MUO=	.100000000000	Ie=	.000082421971
MUO=	.200000000000	Ie=	.000245044761
MUO=	.300000000000	Ie=	.000417974576
MUO=	.400000000000	Ie=	.000572351368
MUO=	.500000000000	Ie=	.000698515991
MUO=	.600000000000	Ie=	.000796298882
MUO=	.700000000000	Ie=	.000870584618
MUO=	.800000000000	Ie=	.000929653004
MUO=	.900000000000	Ie=	.000986188992
MUO=	1.000000000000	Ie=	.001088342356
MUE=	.900000000000		
MUO=	.100000000000	Ie=	.000070415406
MUO=	.200000000000	Ie=	.000210136339
MUO=	.300000000000	Ie=	.000360778832

```
MUO=   .400000000000       Ie=        .000498836990
MUO=   .500000000000       Ie=        .000616952149
MUO=   .600000000000       Ie=        .000715669938
MUO=   .700000000000       Ie=        .000799711717
MUO=   .800000000000       Ie=        .000876612437
MUO=   .900000000000       Ie=        .000957702860
MUO=  1.000000000000       Ie=        .001087629450

MUE=  1.000000000000
MUO=   .100000000000       Ie=        .000057286970
MUO=   .200000000000       Ie=        .000172889246
MUO=   .300000000000       Ie=        .000302390445
MUO=   .400000000000       Ie=        .000428909744
MUO=   .500000000000       Ie=        .000547550432
MUO=   .600000000000       Ie=        .000658749360
MUO=   .700000000000       Ie=        .000765248274
MUO=   .800000000000       Ie=        .000870673885
MUO=   .900000000000       Ie=        .000978866505
MUO=  1.000000000000       Ie=        .001093566070
--------------------------------------------------

MU=   .100000000000      Q=        .045880310218
MU=   .200000000000      Q=        .139911720376
MU=   .300000000000      Q=        .248619260719
MU=   .400000000000      Q=        .359515741820
MU=   .500000000000      Q=        .468430062415
MU=   .600000000000      Q=        .574427408301
MU=   .700000000000      Q=        .677722699480
MU=   .800000000000      Q=        .778844048383
MU=   .900000000000      Q=        .878322684860
MU=  1.000000000000      Q=        .976598257049
--------------------END OF DATA-------------------
```

APPENDIX E

SAMPLE OF OUTPUT DATA FOR FINITE PROGRAM

The example of output data for finite program is as follows (see App. C for definitions of the variables):

```
NUMBER OF LEGENDRE POLYNOMIALS (L)=  2
ALBEDO (W)= .300
REFRACTIVE INDEX (NR)=2.000
ERROR= .000000010000
EXPANSION COEFFICIENTS :
      XK( 0)= 1.0000
      XK( 1)= 1.0000
      XK( 2)=  .4500
NUMBER OF QUADRATURE POINTS (N)= 20

OPTICAL THICKNESS =   .03750000
```

	TOP	BOTTOM
M= 0		
K= 0		
MU= .10000000	1.018825535582	.706422411717
MU= .20000000	1.019832040170	.850461148326
MU= .30000000	1.019692620070	.904929768121
MU= .40000000	1.019176264861	.933522917062
MU= .50000000	1.018452151192	.951109207759
MU= .60000000	1.017578974714	.962973192064
MU= .70000000	1.016582473529	.971463790967
MU= .80000000	1.015475711607	.977783798423
MU= .90000000	1.014266024703	.982612824125
MU=1.00000000	1.012957849686	.986363824615
K= 1		
MU= .10000000	.095785008832	.073355982369
MU= .20000000	.195648797579	.171126649589
MU= .30000000	.295771012558	.270497087264
MU= .40000000	.395963060388	.370307078981
MU= .50000000	.496183331936	.470300325361
MU= .60000000	.596417374031	.570387753486
MU= .70000000	.696658844745	.670530267576

MU= .80000000	.796904523700	.770708011351
MU= .90000000	.897152602018	.870909830116
MU=1.00000000	.997401985392	.971128974985

K= 2

MU= .10000000	-.491353195269	-.339398481558
MU= .20000000	-.446805380267	-.371385182644
MU= .30000000	-.371866170463	-.328805693299
MU= .40000000	-.266778206307	-.243338256135
MU= .50000000	-.131594757993	-.122379942672
MU= .60000000	.033665708167	.031439895009
MU= .70000000	.228995098576	.216952464838
MU= .80000000	.454389306429	.433559247406
MU= .90000000	.709846026162	.680922053867
MU=1.00000000	.995363863535	.958835383429

M= 1
K= 1

MU= .10000000	1.002516506717	.691502540189
MU= .20000000	.987631192110	.820796535269
MU= .30000000	.961575908723	.850624483935
MU= .40000000	.923754695470	.843269581715
MU= .50000000	.872732672636	.812013124312
MU= .60000000	.806053163785	.759671412596
MU= .70000000	.719408896105	.684352766197
MU= .80000000	.604303541781	.578948772069
MU= .90000000	.438925969491	.422879026505
MU=1.00000000	.000000000000	.000000000000

K= 2

MU= .10000000	.294447731510	.209600842376
MU= .20000000	.583775039163	.492413388315
MU= .30000000	.854676026337	.762976402249
MU= .40000000	1.096283313104	1.006819229340
MU= .50000000	1.295895046363	1.210574429276
MU= .60000000	1.437290163054	1.357985888915
MU= .70000000	1.497459749902	1.426352846874
MU= .80000000	1.438271220700	1.378330788511
MU= .90000000	1.175759758884	1.132101334971
MU=1.00000000	.000000000000	.000000000000

M= 2
K= 2

MU= .10000000	2.977161453187	2.048161844638
MU= .20000000	2.887527242860	2.394998984393
MU= .30000000	2.737334112231	2.416464556408
MU= .40000000	2.526864487960	2.301290062728
MU= .50000000	2.256180454588	2.093559655075
MU= .60000000	1.925303545309	1.808945397121
MU= .70000000	1.534243200725	1.454414276323
MU= .80000000	1.083004212063	1.033534133903
MU= .90000000	.571589269492	.548321014967
MU=1.00000000	.000000000000	.000000000000

TOP

M= 0
K= 0

MU=	.100000000000	PP5=	1.037871875168
MU=	.200000000000	PP5=	1.039927905624
MU=	.300000000000	PP5=	1.039673686972
MU=	.400000000000	PP5=	1.038652974281
MU=	.500000000000	PP5=	1.037205376305
MU=	.600000000000	PP5=	1.035448717753
MU=	.700000000000	PP5=	1.033434664337
MU=	.800000000000	PP5=	1.031189440012
MU=	.900000000000	PP5=	1.028727770127
MU=	1.000000000000	PP5=	1.026058560960

K= 1

MU=	.100000000000	PP5=	.099607137194
MU=	.200000000000	PP5=	.199614494703
MU=	.300000000000	PP5=	.299641520688
MU=	.400000000000	PP5=	.399669049149
MU=	.500000000000	PP5=	.499692848008
MU=	.600000000000	PP5=	.599711446731
MU=	.700000000000	PP5=	.699724199936
MU=	.800000000000	PP5=	.799730779863
MU=	.900000000000	PP5=	.899731002395
MU=	1.000000000000	PP5=	.999724756138

K= 2

MU=	.100000000000	PP5=	-.498267349432
MU=	.200000000000	PP5=	-.454177181952
MU=	.300000000000	PP5=	-.379273662800
MU=	.400000000000	PP5=	-.274067751738
MU=	.500000000000	PP5=	-.138671652626
MU=	.600000000000	PP5=	.026875734961
MU=	.700000000000	PP5=	.222557360819
MU=	.800000000000	PP5=	.448364573130
MU=	.900000000000	PP5=	.704292514636
MU=	1.000000000000	PP5=	.990338247962

M= 1
K= 1

MU=	.100000000000	PP5=	.995017720396
MU=	.200000000000	PP5=	.979823927008
MU=	.300000000000	PP5=	.953962752489
MU=	.400000000000	PP5=	.916533648310
MU=	.500000000000	PP5=	.866038796388
MU=	.600000000000	PP5=	.800008489683
MU=	.700000000000	PP5=	.714146905914
MU=	.800000000000	PP5=	.600000433420
MU=	.900000000000	PP5=	.435888031154
MU=	1.000000000000	PP5=	.000000000000

K= 2

MU=	.100000000000	PP5=	.290536024169
MU=	.200000000000	PP5=	.579805351534

```
MU=   .300000000000        PP5=        .850925879526
MU=   .400000000000        PP5=       1.092850898885
MU=   .500000000000        PP5=       1.292836586594
MU=   .600000000000        PP5=       1.434646802892
MU=   .700000000000        PP5=       1.495268242129
MU=   .800000000000        PP5=       1.436574325635
MU=   .900000000000        PP5=       1.174633343051
MU=  1.000000000000        PP5=        .000000000000

M= 2
K= 2
MU=   .100000000000        PP5=       2.984349538062
MU=   .200000000000        PP5=       2.895082460048
MU=   .300000000000        PP5=       2.744695475237
MU=   .400000000000        PP5=       2.533754479025
MU=   .500000000000        PP5=       2.262383869468
MU=   .600000000000        PP5=       1.930626792360
MU=   .700000000000        PP5=       1.538502163715
MU=   .800000000000        PP5=       1.086019583649
MU=   .900000000000        PP5=        .573184442419
MU=  1.000000000000        PP5=        .000000000000


                                           BOTTOM
M= 0
K= 0
MU=   .100000000000        PP6=        .718805419935
MU=   .200000000000        PP6=        .863474569390
MU=   .300000000000        PP6=        .917803007427
MU=   .400000000000        PP6=        .946006151683
MU=   .500000000000        PP6=        .963067319674
MU=   .600000000000        PP6=        .974310879338
MU=   .700000000000        PP6=        .982103215855
MU=   .800000000000        PP6=        .987655989120
MU=   .900000000000        PP6=        .991653787906
MU=  1.000000000000        PP6=        .994512580529

K= 1
MU=   .100000000000        PP6=        .076776270820
MU=   .200000000000        PP6=        .174669152945
MU=   .300000000000        PP6=        .273947826030
MU=   .400000000000        PP6=        .373604891433
MU=   .500000000000        PP6=        .473417770800
MU=   .600000000000        PP6=        .573309105120
MU=   .700000000000        PP6=        .673244938748
MU=   .800000000000        PP6=        .773208025584
MU=   .900000000000        PP6=        .873188677084
MU=  1.000000000000        PP6=        .973181031346

K= 2
MU=   .100000000000        PP6=       -.343163563982
MU=   .200000000000        PP6=       -.375405037844
MU=   .300000000000        PP6=       -.332843812532
MU=   .400000000000        PP6=       -.247307582066
```

```
MU=   .500000000000      PP6=     -.126226281598
MU=   .600000000000      PP6=      .027759336996
MU=   .700000000000      PP6=      .213475483374
MU=   .800000000000      PP6=      .430321101318
MU=   .900000000000      PP6=      .677956577570
MU=  1.000000000000      PP6=      .956175549849

M= 1
K= 1
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MU=   .200000000000      PP6=      .815970430786
MU=   .300000000000      PP6=      .845959685055
MU=   .400000000000      PP6=      .838888252379
MU=   .500000000000      PP6=      .807994674780
MU=   .600000000000      PP6=      .756084110594
MU=   .700000000000      PP6=      .681268323992
MU=   .800000000000      PP6=      .576459779730
MU=   .900000000000      PP6=      .421146921882
MU=  1.000000000000      PP6=      .000000000000

K= 2
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MU=   .200000000000      PP6=      .489063849418
MU=   .300000000000      PP6=      .759828538808
MU=   .400000000000      PP6=     1.003955517827
MU=   .500000000000      PP6=     1.208040604451
MU=   .600000000000      PP6=     1.355813827187
MU=   .700000000000      PP6=     1.424569338244
MU=   .800000000000      PP6=     1.376965565177
MU=   .900000000000      PP6=     1.131207587677
MU=  1.000000000000      PP6=      .000000000000

M= 2
K= 2
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MU=   .200000000000      PP6=     2.399571754376
MU=   .300000000000      PP6=     2.420923332518
MU=   .400000000000      PP6=     2.305464895328
MU=   .500000000000      PP6=     2.097319320164
MU=   .600000000000      PP6=     1.812172109786
MU=   .700000000000      PP6=     1.456996145515
MU=   .800000000000      PP6=     1.035362260407
MU=   .900000000000      PP6=      .549288179668
MU=  1.000000000000      PP6=      .000000000000


M= 0
MU=    .900000000000
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MUB=   .200000000000      RP2=      .003577805932
MUB=   .300000000000      RP2=      .003342098789
MUB=   .400000000000      RP2=      .003092620199
MUB=   .500000000000      RP2=      .002868094327
MUB=   .600000000000      RP2=      .002681976817
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MUB=  .800000000000     RP2=      .002445682016
MUB=  .900000000000     RP2=      .002400188504
MUB= 1.000000000000     RP2=      .002404711753

M= 1
MU=   .900000000000
MUB=  .100000000000     RP2=      .000880735213
MUB=  .200000000000     RP2=      .000817432241
MUB=  .300000000000     RP2=      .000689429474
MUB=  .400000000000     RP2=      .000544883931
MUB=  .500000000000     RP2=      .000397862176
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MUB=  .700000000000     RP2=      .000128625615
MUB=  .800000000000     RP2=      .000022892563
MUB=  .900000000000     RP2=     -.000045623518
MUB= 1.000000000000     RP2=      .000000000000

M= 2
MU=   .900000000000
MUB=  .100000000000     RP2=      .000146803136
MUB=  .200000000000     RP2=      .000155558382
MUB=  .300000000000     RP2=      .000151979810
MUB=  .400000000000     RP2=      .000142441939
MUB=  .500000000000     RP2=      .000128352772
MUB=  .600000000000     RP2=      .000110201615
MUB=  .700000000000     RP2=      .000088203227
MUB=  .800000000000     RP2=      .000062466680
MUB=  .900000000000     RP2=      .000033053248
MUB= 1.000000000000     RP2=      .000000000000


M= 0
MUB= .900000000000
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MU=  .200000000000     TP2=      .005424080205
MU=  .300000000000     TP2=      .006179498507
MU=  .400000000000     TP2=      .006921661820
MU=  .500000000000     TP2=      .007688955745
MU=  .600000000000     TP2=      .008494724904
MU=  .700000000000     TP2=      .009344827570
MU=  .800000000000     TP2=      .010242239441
MU=  .900000000000     TP2=      .012185017457
MU= 1.000000000000     TP2=      .012185017753

M= 1
MUB= .900000000000
MU=  .100000000000     TP2=      .001117253726
MU=  .200000000000     TP2=      .001334517290
MU=  .300000000000     TP2=      .001470958627
MU=  .400000000000     TP2=      .001563284274
MU=  .500000000000     TP2=      .001613088096
MU=  .600000000000     TP2=      .001612910899
MU=  .700000000000     TP2=      .001548010170
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MU=	.800000000000	TP2=	.001390738698
MU=	.900000000000	TP2=	.000106637767
MU=	1.000000000000	TP2=	.000000000000

M= 2

MUB=	.900000000000		
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MU=	.200000000000	TP2=	.000155355889
MU=	.300000000000	TP2=	.000151847461
MU=	.400000000000	TP2=	.000142348613
MU=	.500000000000	TP2=	.000128285292
MU=	.600000000000	TP2=	.000110153190
MU=	.700000000000	TP2=	.000088169907
MU=	.800000000000	TP2=	.000062445972
MU=	.900000000000	TP2=	.000000000000
MU=	1.000000000000	TP2=	.000000000000

MUO=	.900000000000		
MUE=	.100000000000	RIe=	.000035470006
MUE=	.200000000000	RIe=	.000053325686
MUE=	.300000000000	RIe=	.000062816783
MUE=	.400000000000	RIe=	.000067874117
MUE=	.500000000000	RIe=	.000070413348
MUE=	.600000000000	RIe=	.000071482881
MUE=	.700000000000	RIe=	.000071722276
MUE=	.800000000000	RIe=	.000071574857
MUE=	.900000000000	RIe=	.000071435723
MUE=	1.000000000000	RIe=	.000072729015

MUO=	.900000000000		
MUin=	.100000000000	TIe=	.009437674793
MUin=	.200000000000	TIe=	.005748718498
MUin=	.300000000000	TIe=	.004275174141
MUin=	.400000000000	TIe=	.003507079673
MUin=	.500000000000	TIe=	.003047479932
MUin=	.600000000000	TIe=	.002747381351
MUin=	.700000000000	TIe=	.002537394084
MUin=	.800000000000	TIe=	.002378816927
MUin=	.900000000000	TIe=	.001951602865
MUin=	1.000000000000	TIe=	.001746831669

	TQ	BQ
MU= .10000000	.040744117194	.039497258766
MU= .20000000	.123126823956	.119377425416
MU= .30000000	.219316253477	.212691755224
MU= .40000000	.319306193158	.309768476737
MU= .50000000	.419186549031	.406839406870
MU= .60000000	.517415642500	.502426079012
MU= .70000000	.613461237815	.596022832945
MU= .80000000	.707246371749	.687560370407
MU= .90000000	.798901313982	.777166191959

MU=1.00000000 .888644227678 .865049805641

APPENDIX F

SOME TABULAR RESULTS

Tables 9 through 18 offered in this appendix are certain tabular results for either finite or semi-infinite media while Table 19 reveals the accuracy of these tabular results. Now, these tabular results are presented as follows:

TABLE 9

RESULTS FOR L = 2, n = 1.33, τ_0 = 1.0, AND ω = 0.1

μ or μ_e	$PP_{00}(0,\mu,n;\tau_0)$	$PP_{00}(\tau_0,\mu,n;\tau_0)$	$I^-_{Ae}(0,\mu_e,1,0,n;\tau_0)/I_o$
.05	1.01511350	.00095837	.00050250
0.1	1.02344213	.00207708	.00086094
0.2	1.03404611	.01157041	.00129508
0.3	1.04056650	.04408151	.00150134
0.4	1.04476213	.09427242	.00158227
0.5	1.04745718	.15115260	.00159373
0.6	1.04911952	.20804729	.00156721
0.8	1.05039589	.31152080	.00146525
0.9	1.05032196	.35675062	.00140665
1.0	1.04990273	.39776763	.00134871

μ_{1n} or μ_o	$\{I^+_{Ae}(\tau_0,\mu_{1n},1,0,n;\tau_0)$ -leading term of Eq. (172)$\}$ $/I_o$	$q(0,\mu_o,n;\tau_0)$ $/I_o$	$q(\tau_0,\mu_o,n;\tau_0)$ $/I_o$
.05	.00314116	.01331880	.00317581
0.1	.00340502	.04579576	.01099364
0.2	.00399466	.13965641	.03440995
0.3	.00455625	.24817301	.06366880
0.4	.00499199	.35888525	.09690290
0.5	.00529575	.46763293	.13380941
0.6	.00549697	.57348567	.17450262
0.8	.00491962	.77768651	.26765218
0.9	.00492380	.87709639	.32003963
1.0	.00493834	.97533096	.37609312

TABLE 10

RESULTS FOR L = 2, n = 1.33, τ_0 = 1.0, AND ω = 0.5

μ or μ_e	$PP_{00}(0,\mu,n;\tau_0)$	$PP_{00}(\tau_0,\mu,n;\tau_0)$	$I_{Ae}^{-}(0,\mu_e,1,0,n;\tau_0)/I_o$
.05	1.09088456	.00891335	.00388053
0.1	1.14612403	.01860455	.00665033
0.2	1.22192764	.04830281	.01001451
0.3	1.27214538	.10388650	.01162844
0.4	1.30649125	.17696794	.01228025
0.5	1.33005791	.25497554	.01239682
0.6	1.34598087	.33083901	.01221744
0.8	1.36233833	.46627724	.01146284
0.9	1.36506477	.52499557	.01101494
1.0	1.36513000	.57815556	.01056359

μ_{in} or μ_o	$\{I_{Ae}^{+}(\tau_0,\mu_{in},1,0,n;\tau_0)$ -leading term of Eq. (172)$\}$ $/I_o$	$q(0,\mu_o,n;\tau_0)$ $/I_o$	$q(\tau_0,\mu_o,n;\tau_0)$ $/I_o$
.05	.02371231	.01273015	.00457809
0.1	.02572246	.04377840	.01581048
0.2	.02971651	.13358370	.04904101
0.3	.03322068	.23760500	.08949846
0.4	.03580027	.34402343	.13391823
0.5	.03747995	.44891130	.18149417
0.6	.03846780	.55139390	.23219768
0.8	.03299228	.75022242	.34342607
0.9	.03255815	.84753025	.40398704
1.0	.03223134	.94398173	.46771601

TABLE 11

RESULTS FOR L = 2, n = 1.33, τ_0 = 1.0, AND ω = 0.95

μ or μ_e	$PP_{00}(0,\mu,n;\tau_0)$	$PP_{00}(\tau_0,\mu,n;\tau_0)$	$I_{Ae}^{-}(0,\mu_e,1,0,n;\tau_0)/I_o$
.05	1.25150279	.04611037	.01579819
0.1	1.43078507	.09412283	.02708248
0.2	1.70770566	.20360504	.04082955
0.3	1.91190305	.33716348	.04749397
0.4	2.06280133	.48001275	.05026798
0.5	2.17415140	.61796184	.05087050
0.6	2.25605831	.74468101	.05025949
0.8	2.35816143	.95991132	.04735652
0.9	2.38705559	1.04998493	.04557076
1.0	2.40501401	1.13007073	.04373640

μ_{in} or μ_o	$\{I_{Ae}^{+}(\tau_0,\mu_{in},1,0,n;\tau_0)$ −leading term of Eq. (172)$\}/I_o$	$q(0,\mu_o,n;\tau_0)/I_o$	$q(\tau_0,\mu_o,n;\tau_0)/I_o$
.05	.09226191	.01069683	.00907462
0.1	.10050047	.03680691	.03124233
0.2	.11464160	.11256188	.09575131
0.3	.12558776	.20092275	.17150198
0.4	.13294177	.29225570	.25057391
0.5	.13711613	.38343116	.33046414
0.6	.13888421	.47378085	.41067389
0.8	.11253733	.65283951	.57275372
0.9	.10890433	.74222641	.65511146
1.0	.10581670	.83186213	.73855242

TABLE 12

RESULTS FOR $L = 2$, $\tau_0 = 1.0$, $\omega = 0.95$, AND $n = 1.0001$

μ or μ_e	$PP_{00}(0,\mu,n;\tau_0)$	$PP_{00}(\tau_0,\mu,n;\tau_0)$	$I^-_{Ae}(0,\mu_e,1,0,n;\tau_0)/I_o$
.05	1.11115121	.02290858	.11917185
0.1	1.18522191	.05080277	.11961244
0.2	1.30010163	.12698614	.11627136
0.3	1.38551553	.23503499	.10931581
0.4	1.44898091	.35869683	.10093744
0.5	1.49600511	.48231996	.09257319
0.6	1.53074587	.59838890	.08484600
0.8	1.57450338	.80008777	.07188033
0.9	1.58718796	.88619407	.06657931
1.0	1.59536960	.96369377	.06195669

μ_{in} or μ_o	$\{I^+_{Ae}(\tau_0,\mu_{in},1,0,n;\tau_0)$ -leading term of Eq. (172)$\}$ $/I_o$	$q(0,\mu_o,n;\tau_0)$ $/I_o$	$q(\tau_0,\mu_o,n;\tau_0)$ $/I_o$
.05	.07967393	.01801032	.01271694
0.1	.08639939	.03898423	.02793479
0.2	.09657534	.08890611	.06590876
0.3	.10145061	.14922576	.11507466
0.4	.10228695	.21879750	.17508517
0.5	.10095574	.29574116	.24408143
0.6	.09865437	.37832153	.32009587
0.8	.09337190	.55534722	.48714352
0.9	.09086046	.64806269	.57605343
1.0	.08855157	.74279544	.66757668

TABLE 13

RESULTS FOR L = 2, τ_0 = 1.0, ω = 0.95, AND n = 2.00

μ or μ_e	$PP_{00}(0,\mu,n;\tau_0)$	$PP_{00}(\tau_0,\mu,n;\tau_0)$	$I_{Ae}^{-}(0,\mu_e,1,0,n;\tau_0)/I_o$
.05	1.28896643	.06104397	.00537984
0.1	1.50345638	.12309025	.00912575
0.2	1.84286845	.25749663	.01374091
0.3	2.09760668	.41121929	.01622364
0.4	2.28766472	.56969521	.01757537
0.5	2.42888849	.71956757	.01827479
0.6	2.53350340	.85535232	.01857450
0.8	2.66590019	1.08268464	.01849558
0.9	2.70459927	1.17668055	.01826096
1.0	2.72981637	1.25967476	.01795234

μ_{in} or μ_o	$\{I_{Ae}^{+}(\tau_0,\mu_{in},1,0,n;\tau_0)$ -leading term of Eq. (172)$\}$ $/I_o$	$q(0,\mu_o,n;\tau_0)$ $/I_o$	$q(\tau_0,\mu_o,n;\tau_0)$ $/I_o$
.05	.09348613	.01116347	.00975127
0.1	.10190582	.03790854	.03311708
0.2	.11641720	.11458814	.10015322
0.3	.12768076	.20419568	.17861475
0.4	.13524480	.29746899	.26048709
0.5	.13952424	.39080943	.34269059
0.6	.14131532	.48281320	.42404973
0.8	.14031347	.66133580	.58308890
0.9	.11454589	.74792805	.66087788
1.0	.10995189	.83299065	.73774650

TABLE 14

RESULTS FOR $\tau_0 = 1.0$, $\omega = 0.95$, $n = 1.33$, AND $L = 0$

μ or μ_e	$PP_{00}(0,\mu,n;\tau_0)$	$PP_{00}(\tau_0,\mu,n;\tau_0)$	$I^-_{Ae}(0,\mu_e,1,0,n;\tau_0)/I_o$
.05	1.25158210	.04355143	.01890110
0.1	1.43389231	.08893130	.03241513
0.2	1.72127298	.19279343	.04894880
0.3	1.94070466	.32025333	.05708876
0.4	2.11086612	.45676520	.06063671
0.5	2.24485392	.58838474	.06162685
0.6	2.35225913	.70895389	.06118494
0.8	2.51255017	.91282960	.05828813
0.9	2.57367435	.99781986	.05641796
1.0	2.62574958	1.07326779	.05446653

μ_{in} or μ_o	$\{I^+_{Ae}(\tau_0,\mu_{in},1,0,n;\tau_0)$ -leading term of Eq. (172)$\}$ $/I_o$	$q(0,\mu_o,n;\tau_0)$ $/I_o$	$q(\tau_0,\mu_o,n;\tau_0)$ $/I_o$
.05	.08936286	.01025490	.00860358
0.1	.09708442	.03528680	.02962143
0.2	.11026038	.10791939	.09079343
0.3	.12029853	.19265422	.16265004
0.4	.12679091	.28026506	.23769507
0.5	.13012800	.36775963	.31356463
0.6	.13105870	.45450397	.38979162
0.8	.09743193	.62656619	.54399476
0.9	.09205618	.71254325	.62243525
1.0	.08738713	.79881896	.70196219

TABLE 15

RESULTS FOR τ_0 = 1.0, ω = 0.95, n = 1.33,
AND L = 1

μ or μ_e	$PP_{00}(0,\mu,n;\tau_0)$	$PP_{00}(\tau_0,\mu,n;\tau_0)$	$I_{Ae}^{-}(0,\mu_e,1,0,n;\tau_0)/I_o$
.05	1.24506107	.04570301	.01587853
0.1	1.42052339	.09333865	.02721306
0.2	1.69302197	.20210245	.04098326
0.3	1.89576777	.33506939	.04758884
0.4	2.04753691	.47766600	.05024339
0.5	2.16174524	.61584126	.05068192
0.6	2.24830068	.74332998	.04987405
0.8	2.36462470	.96181630	.04650967
0.9	2.40295830	1.05438656	.04446967
1.0	2.43185381	1.13754698	.04237007

μ_{in} or μ_o	$\{I_{Ae}^{+}(\tau_0,\mu_{in},1,0,n;\tau_0)$ -leading term of Eq. (172)$\}$ $/I_o$	$q(0,\mu_o,n;\tau_0)$ $/I_o$	$q(\tau_0,\mu_o,n;\tau_0)$ $/I_o$
.05	.09482924	.01069136	.00907197
0.1	.10313853	.03678913	.03123375
0.2	.11736296	.11252020	.09573145
0.3	.12822633	.20088476	.17148489
0.4	.13526840	.29227108	.25058463
0.5	.13890808	.38356351	.33053470
0.6	.13995365	.47410256	.41084069
0.8	.11194310	.65377736	.57323238
0.9	.10754557	.74359692	.65580870
1.0	.10367990	.83375061	.73951115

TABLE 16

RESULTS FOR L = 2, ω = 0.95, n = 1.33, AND τ_0 = 3.0

μ or μ_e	$PP_{00}(0,\mu,n;\tau_0)$	$PP_{00}(\tau_0,\mu,n;\tau_0)$	$I^-_{Ae}(0,\mu_e,1,0,n;\tau_0)/I_o$
.05	1.28342170	.01951993	.02971836
0.1	1.49585768	.03978917	.05102991
0.2	1.84498889	.08313877	.07743397
0.3	2.13325140	.13103201	.09101441
0.4	2.37907655	.18457810	.09766179
0.5	2.59119843	.24476356	.10046517
0.6	2.77485372	.31169995	.10109662
0.8	3.07080036	.46189352	.09914947
0.9	3.18847572	.54226003	.09740044
1.0	3.28888563	.62426080	.09541685

μ_{in} or μ_o	$\{I^+_{Ae}(\tau_0,\mu_{in},1,0,n;\tau_0)$ -leading term of Eq. (172)$\}/I_o$	$q(0,\mu_o,n;\tau_0)/I_o$	$q(\tau_0,\mu_o,n;\tau_0)/I_o$
.05	.05184348	.00860256	.00417875
0.1	.05688042	.02960139	.01439509
0.2	.06621140	.09053658	.04422232
0.3	.07518691	.16164985	.07952843
0.4	.08401200	.23524616	.11687551
0.5	.09266370	.30888486	.15533034
0.6	.10101771	.38212148	.19488723
0.8	.11209054	.52845637	.27848656
0.9	.11679941	.60228145	.32312718
1.0	.12073850	.67691966	.36995193

TABLE 17

RESULTS FOR L = 2, ω = 0.95, n = 1.33, AND τ_0 = 0.001

μ or μ_e	$PP_{00}(0,\mu,n;\tau_0)$	$PP_{00}(\tau_0,\mu,n;\tau_0)$	$I_{Ae}^{-}(0,\mu_e,1,0,n;\tau_0)/I_o$
.05	1.00574548	.98581933	.00001199
0.1	1.00575093	.99569867	.00002050
0.2	1.00571164	1.00066840	.00003056
0.3	1.00565289	1.00231624	.00003493
0.4	1.00558186	1.00312331	.00003616
0.5	1.00549998	1.00358783	.00003569
0.6	1.00540773	1.00387614	.00003433
0.8	1.00519286	1.00417392	.00003071
0.9	1.00507039	1.00423893	.00002887
1.0	1.00493795	1.00426655	.00002713

μ_{in} or μ_o	$\{I_{Ae}^{+}(\tau_0,\mu_{in},1,0,n;\tau_0)$ -leading term of Eq. (172)$\}$ $/I_o$	$q(0,\mu_o,n;\tau_0)/I_o$	$q(\tau_0,\mu_o,n;\tau_0)/I_o$
.05	.00271313	.01340563	.01340461
0.1	.00137248	.04609325	.04608976
0.2	.00069441	.14055084	.14054039
0.3	.00046898	.24972651	.24970850
0.4	.00035757	.36106442	.36103938
0.5	.00029190	.47036991	.47033877
0.6	.00024914	.57670487	.57666860
0.8	.00013744	.78166180	.78161780
0.9	.00012826	.88136237	.88131550
1.0	.00012192	.97984116	.97979194

TABLE 18

RESULTS FOR L = 2, ω = 0.95, n = 1.33, AND $\tau_0 = \infty$

μ, μ_e or μ_o	$PP_{00}(0,\mu,n)$	$I^-_{Ae}(0,\mu_e,1,0,n)/I_o$	$q(0,\mu_o,n)/I_o$
.05	1.29208864	.03417905	.00798302
0.1	1.51352511	.05871295	.02746723
0.2	1.88190196	.08923441	.08398083
0.3	2.19139107	.10515977	.14986220
0.4	2.46081402	.11324759	.21792898
0.5	2.69931522	.11702635	.28588408
0.6	2.91242244	.11839542	.35329195
0.8	3.27649370	.11762442	.48740700
0.9	3.43234200	.11640357	.55478371
1.0	3.57314384	.11493097	.62272965

Note that the data from Tables 9 through 18 were obtained by using x_0 = 1.0, x_1 = 0.4860214865, and x_2 = 0.1431451436 for the expansion coefficients while convergence error, step size, and number of quadrature points for each interval were used being 10^{-8}, 0.0005, and 10, respectively. As mentioned in Chapter V., four sets of Gaussian quadrature were chosen from μ = 0 to 1. These four intervals were zero to μ_{cr}, μ_{cr} to 1.015 μ_{cr}, 1.015 μ_{cr} to 1.085 μ_{cr}, and 1.085 μ_{cr} to one.

TABLE 19

TABLE FOR ACCURACY OF RESULTS

	PP_{00}(0, 1.0, 1.33; 0.1)
Case 1	1.260580282636
% Error of Case 2	4.28×10^{-3}
% Error of Case 3	2.86×10^{-6}
% Error of Case 4	4.28×10^{-3}
	PP_{00}(0.1, 1.0, 1.33; 0.1)
Case 1	1.134886337867
% Error of Case 2	1.37×10^{-3}
% Error of Case 3	2.64×10^{-6}
% Error of Case 4	1.37×10^{-3}
	q(0, 1.0, 1.33; 0.1) / I_o
Case 1	0.967712670814
% Error of Case 2	2.92×10^{-6}
% Error of Case 3	1.84×10^{-6}
% Error of Case 4	2.68×10^{-6}
	q(0.1, 1.0, 1.33; 0.1) / I_o
Case 1	0.961768586238
% Error of Case 2	3.72×10^{-6}
% Error of Case 3	1.83×10^{-6}
% Error of Case 4	3.47×10^{-6}

Note that cases 1 and 2 in Table 19 represent the results which have the number of quadrature points equal to 10 and 20, respectively, for each interval while the four intervals remain the same as mentioned in Chapter V. However, cases 3 and 4 in Table 19 represent the results which have the number of quadrature points equal to 10 and 20, respectively, for each interval while the intervals are different than those of cases 1 and 2. These four intervals are zero to μ_{cr}, μ_{cr} to 1.02 μ_{cr}, 1.02 μ_{cr} to 1.09 μ_{cr}, and 1.09 μ_{cr} to one.

VITA 2

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Master of Science

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