RELATIVISTIC KINEMATICS WITHOUT NUMERICAL CANCELLATION

By

YUMEI ZHAO

Bachelor of Science

Henan Normal University

Xinxiang, China

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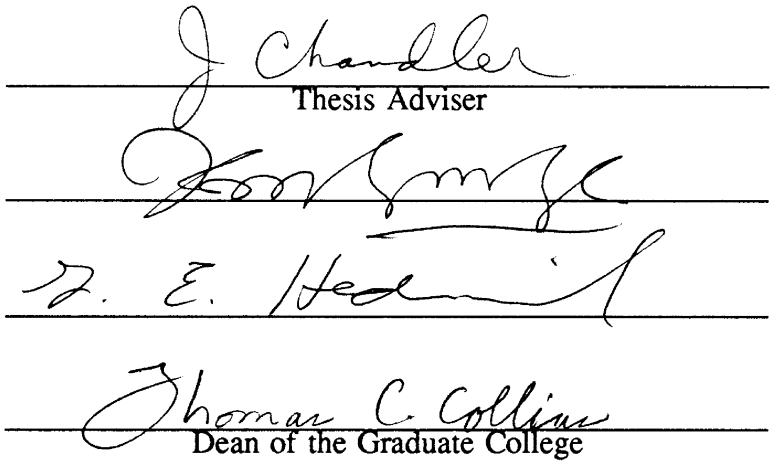

Thesis Adviser

Dean of the Graduate College

PREFACE

This thesis studies numerical cancellation of relativistic kinematics in low energy calculations. A method is developed to avoid numerical cancellation. This method can be applied to many problems in atomic physics and nuclear physics.

I would like to express my sincere gratitude to my advisor, Dr. J. P. Chandler, for his guidance, encouragement and assistance through all phases of this work. Without his help, it would be impossible for this thesis to be completed.

I would also like to thank Dr. K. M. George and Dr. G. E. Hedrick for serving on my committee. Their support was very helpful throughout the study.

Finally, I would like to thank my family for their love, support, and encouragement throughout my education. Special thanks go to my husband, Dr. X. H. Yang for his understanding and valuable suggestions.

This thesis is dedicated to my parents, Mr. and Mrs. Zhao.

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CHAPTER I

INTRODUCTION

Since the establishment of special relativity in 1904, it has found wide applications in atomic physics and nuclear physics. The essence of relativistic mechanics is that energy is intrinsically related to mass. Mass can transfer to energy and vice versa. The main difference between relativistic mechanics and classical mechanics (Newtonian mechanics) is that relativistic mechanics is valid for all energy (high or low) regimes whereas classical mechanics is an approximation that is accurate only in the low energy region. At low energy, nonrelativistic formulae are used for kinematic calculation. It is well known that these calculations will give approximate results which are very close to the exact results. The reason for not using relativistic formulas for the low energy region is that the relativistic equations as conventionally formulated cause a drastic loss of significant digits due to subtractive cancellation when used for numerical calculation. Therefore, at intermediate energies one must choose between approximation and cancellation.

Obviously it is ideal and very useful if one has a set of equations which gives accurate results in both low energy

cases and high energy cases.

The purpose of this thesis is to solve the problem of numerical cancellation of relativistic kinematics in low energy calculations so that it can be used in all energies.

Chapter II presents some background of relativistic kinematics and the problems of relativistic kinematics of particle reactions that are associated with numerical cancellation at low energy. The first problem is the kinematic limits on the momentum and production angle of one particle from a system of particles. The second problem is related to Dalitz plots.

Chapter III presents floating point computation and the analysis of the numerical cancellation of the two problems stated in Chapter II.

Chapter IV introduces the methods of avoiding numerical cancellation at low energy.

Chapter V presents the implementation of the methods developed in Chapter IV and shows results without numerical cancellation.

Finally, Chapter VI presents a summary and conclusions.

CHAPTER II

BACKGROUND ON RELATIVISTIC KINEMATICS AND NUMERICAL CANCELLATION AT LOW ENERGY

Newtonian formulas *vs.* relativistic formulas

The fundamental relativistic relationships relating energy and mass are

$$m = \frac{m_0}{(1-\beta^2)^{1/2}}$$

$$p = mv \tag{2.1}$$

$$T = mc^2 - m_0c^2$$

where m = relativistic mass of particle = total mass

m_0= rest mass of particle

β = v/c = velocity of particle in relation to c

p = momentum of particle

T = kinetic energy of particle

From the above three fundamental equations, many useful expressions can be derived, e.g.

$$T = \sqrt{p^2c^2 + m_0^2c^4} - m_0c^2 \tag{2.2}$$

The relativistic formulas simplify to the following

familiar formulas when $\beta << 1$ or $v << c$:

$$m = m_0$$

$$p = m_0 v \qquad (2.3)$$

$$T = \frac{1}{2} m_0 v^2$$

These are nonrelativistic equations, also called Newtonian equations.

The next example will illustrate that at low energy Newtonian formulas can be used for kinematic calculation to avoid the numerical cancellation. Numerical calculation is inevitable in using conventional relativistic formulas.

For a neutron, the rest energy m_0c^2=939.5731(MeV). Assume at low energy the momentum of a neutron is p=0.0100(MeV/c). Applying relativistic formulas to calculate the kinetic energy, and using rounded arithmetic with seven significant digits,

$$\begin{aligned} T &= \sqrt{p^2c^2 + m_0^2c^4} - m_0c^2 \\ &= \sqrt{0.0100_2 + 939.5731^2} - 939.5731 \\ &= \sqrt{10^{-4} + 882797.6102} - 939.5731 \\ &= 939.5731 - 939.5731 \\ &= 0.0 \end{aligned}$$

the relative error = 100%. If the arithmetic had been exact, of course, there would have been no error.

Using nonrelativistic formulas, we have

$$T = \frac{p^2}{2m_0} = \frac{0.0100}{2*939.5371} = 5.32*10^{-8}\ (Mev)$$

this result is quite accurate.

Relativistic kinematics of particle reaction

To study particle reactions, one has to have a knowledge of relativistic kinematics since the particles are completely characterized by kinematic quantities such as energy and momentum. The combination of energy and momentum gives the so-called four-momentum vector which transforms as a four-vector under Lorentz transformations. Relativistic kinematics is based on two well-established principles: the Lorentz transformation and energy-momentum conservation.

Kinematics helps us to see what is beyond the power of apparatus, trying to fill in what has not been observed by the instruments, thereby making them more sharp-sighted without altering them or taking any interest in their design.

The Lorentz transformation relates physical quantities as measured in two coordinate systems which are moving with respect to each other. For example, the lab system and center-of-momentum system are moving with respect to each other with velocity v^*. Denoting the center-of-momentum quantities with an asterisk, and the lab quantities without

an asterisk, then the Lorentz transformation of the energy-momentum four vector is

$$\begin{pmatrix} E^* \\ p_x^* \\ p_y^* \\ p_z^* \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} E \\ p_x \\ p_y \\ p_z \end{pmatrix} \tag{2.4}$$

where $\beta = |v|/c$ and v is along the z axis, $\gamma = (1-\beta^2)^{-\frac{1}{2}}$.

The inverse transformation is found just by changing the sign of β in Equation (2.4).

$$\begin{pmatrix} E \\ p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} E^* \\ p_x^* \\ p_y^* \\ p_z^* \end{pmatrix} \tag{2.5}$$

Energy and momentum conservation has the form

$$\Sigma E_i = \Sigma E_f \tag{2.6}$$

and

$$\Sigma p_{x_i} = \Sigma p_{x_f}, \quad \Sigma p_{y_i} = \Sigma p_{y_f}, \quad \Sigma p_{z_i} = \Sigma p_{z_f} \tag{2.7}$$

where i denotes the initial state and f the final state.

Now, let us look at the problems associated with numerical cancellation at low energy.

(1) First, consider the kinematic limits on the momentum and production angle of one particle from a system of particles.

It is of interest to discuss the possible angular distribution and energy distribution of the produced

particles in a particle collision. Let us restrict ourselves to the case in which the target particle is at rest. Then we know that the created particles will in general be able to travel in a distribution of angles around the forward direction. The details depend on the nature of the interaction, but some limits can be given in general.

The following example[7] best illustrates the significance of kinematic limit calculations. The maximum production angle θ_{max} is 68° for a collision of a 2.2-GeV neutron with a proton in which one pion is generated. This result can be used to establish that a particle which goes off at an angle greater than θ_{max} in an inelastic n-p (neutron-proton) collision at this beam energy cannot be a nucleon and must be a pion (or a heavy meson). Such a criterion is useful in the analysis of interactions.

Considering the particle reactions such as

$$K^- + D \rightarrow p + \pi^- + \lambda,$$
$$K^- + D \rightarrow p + \lambda + \pi^- + \pi^+ + K^- + K^0$$

We would like to compute the kinematic limits on the momentum and production angle of one particle (e.g. the proton) from a system of particles.

The procedures of solving the above problems are: (a) getting the kinematic quantities (e.g. energy, momentum, etc.) as seen from the CMS, (b) getting the above quantities in the lab system by Lorentz transformation. The reason for doing so is that we can find the answer quite obviously and

most easily in the CMS and also the Lorentz transformation is well established and easy to apply.

Figure 1 represents the change of the shape of a momentum spectrum under the Lorentz transformation qualitatively (p_1=0!). Assuming a δ-shaped isotropic distribution in the CMS, the Lorentz transformation converts the sphere into an ellipsoid by first stretching it γ-fold to the right and left and then shifting it as a whole to the right by the amount $\beta\gamma E^*$. The ellipsoid can be divided into different classes depending on the relative magnitude of β and v^* (Figure 2). In class 3, since no proton goes backward in the lab system, there must be a maximum angle $\theta < \pi/2$. At any given angle θ ($\theta < \theta_{max}$) we have two peaks in the lab spectrum: one coming from particles going forward in the CMS and one coming from particles going backward in the CMS. In class 1, particles can fly backward in the lab system. There are no two peaks and no maximum production angle exists in the lab system.

We now discuss the problem quantitatively.

By applying energy-momentum conservation, we get the momentum squared of the proton in the center-of-momentum system is[3]

$$p^{*2} = \frac{(M^2 - \Sigma m_i^2)\,(M^2 - (\Sigma m_i - 2m_k)^2)}{4M^2} \qquad (2.8)$$

where

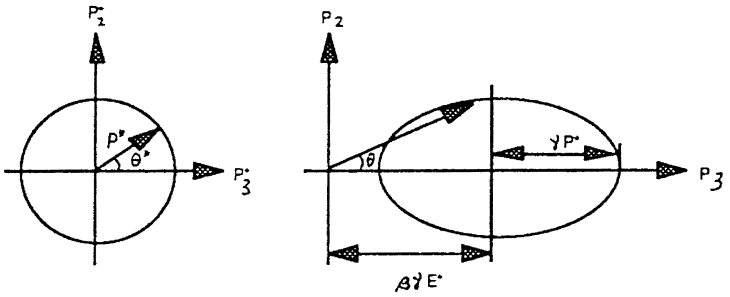


Figure 1. The momentum spectrum before and after the Lorentz transformation

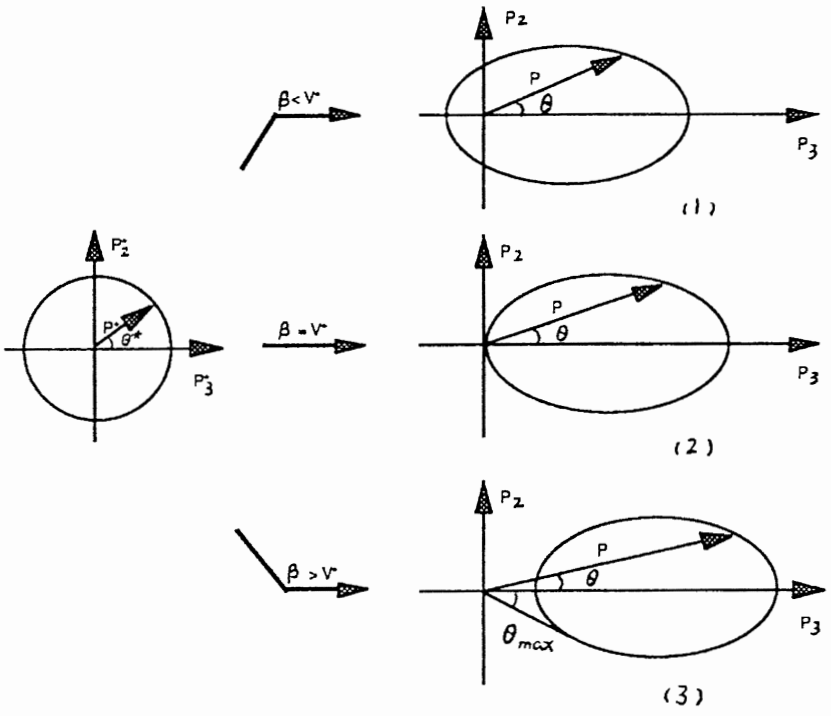


Figure 2. The classification of Lorentz transformed momentum ellipses.

$M^2 = E^2 - p^2$ is the total mass squared.

E is the total energy.

p is the total momentum. Since the target particle D is at rest, p is the beam momentum in fact.

Σm_i is the sum of the masses of particles in the system.

m_k is the mass of the proton.

The energy squared of the proton in the center-of-momentum system is

$$E^{*2} = p^{*2} + m_k^2 \qquad (2.9)$$

By using Lorentz transformation and rotating the axes such that $p_1 = 0$, we get the equation of an ellipse in the lab system.

$$\frac{p_2^2}{p^{*2}} + \frac{(p_3 - \beta\gamma E^*)^2}{\gamma^2 p^{*2}} = 1 \qquad (2.10)$$

Inserting

$$p_2 = p_3 \tan\theta \qquad (2.11)$$

into (2.10), and letting p be the momentum of the proton in the lab system, we get a quadratic equation for p:

$$(1 - \beta^2\cos^2\theta)p^2 - 2\frac{E^*\beta\cos\theta}{\gamma}p + (\beta E^*)^2 - p^{*2} = 0 \qquad (2.12)$$

where θ is the production angle of proton. Solving the above quadratic equation, we get the maximum momentum and

minimum momentum.

$$p^{(\pm)} = \frac{\beta E^* \cos\theta \pm \sqrt{\beta^2 E^{*2}\cos^2\theta - \gamma^2(1-\beta^2\cos^2\theta)(\beta^2 E^{*2} - p^{*2})}}{\gamma(1-\beta^2\cos^2\theta)} \tag{2.13}$$

If we let the radicand equal zero, we can get the maximum production angle

$$\tan^2\theta_{max} = \frac{v^{*2}}{\gamma^2(\beta^2 - v^{*2})} \tag{2.14}$$

where $v^* = p^*/E^*$.

As we discussed qualitatively, $\tan\theta_{max} = \infty$ for $\beta = v^*$ (class 2) and there is no real solution of $\tan\theta_{max}$ for $\beta < v^*$ (class 1).

Does numerical cancellation occur in low energy situations if we use the relativistic equations as conventionally formulated?

Now, we need to talk about the threshold energy (considering reaction of a general type $a + b \rightarrow c + d + \cdots$, and assuming the target b at rest in the lab system). The threshold energy for the production of a system of particles is defined as the minimum energy necessary. In CMS the minimum energy, for the reaction just to occur, is given by the condition

$$E^*_{thres} = \Sigma m_i \tag{2.15}$$

let ε_{thres} be the threshold bombarding energy of the

projectile a, then

$$2\epsilon_{thres}M + M^2 + m^2 = (\Sigma m_i)^2$$

$$\epsilon_{thres} = \frac{(\Sigma m_i)^2 - M^2 - m^2}{2M} \qquad (2.16)$$

where $m=m_a$, $M=m_b$, $\Sigma m_i=m_c+m_d+\cdots$.

Let us take as an example the reaction

$$K^- + D \rightarrow p + \lambda + \pi^- + \pi^+ + K^- + K^0$$

We calculate the maximum momentum and maximum production angle of the proton near the threshold energy.

The threshold bombarding energy for the above reaction just to occur is $\epsilon_{thres} = 1.9436407(GeV)$. The corresponding beam momentum of the K^- is

$$p_{thres} = \sqrt{\epsilon_{thres}^2 - m^2} = \sqrt{1.9436407^2 - 0.4939^2} = 1.8798409\ (GeV/c)$$

Figure 3 shows a plot of the maximum momentum of the proton as a function of beam momentum in the lab system. Figure 4 is the plot of the maximum production angle of the proton against the beam momentum in the lab system. It is clearly seen from Figure 3 that when the beam momentum is near threshold energy (1.8798409 GeV/c), the maximum momentum is getting very rough. From Figure 4 we can see that the tangents of maximum production angle are not increasing smoothly as they should. The reason we have the rough portions in Figure 3 and Figure 4 is due to numerical cancellation. The analysis will ba given in the next

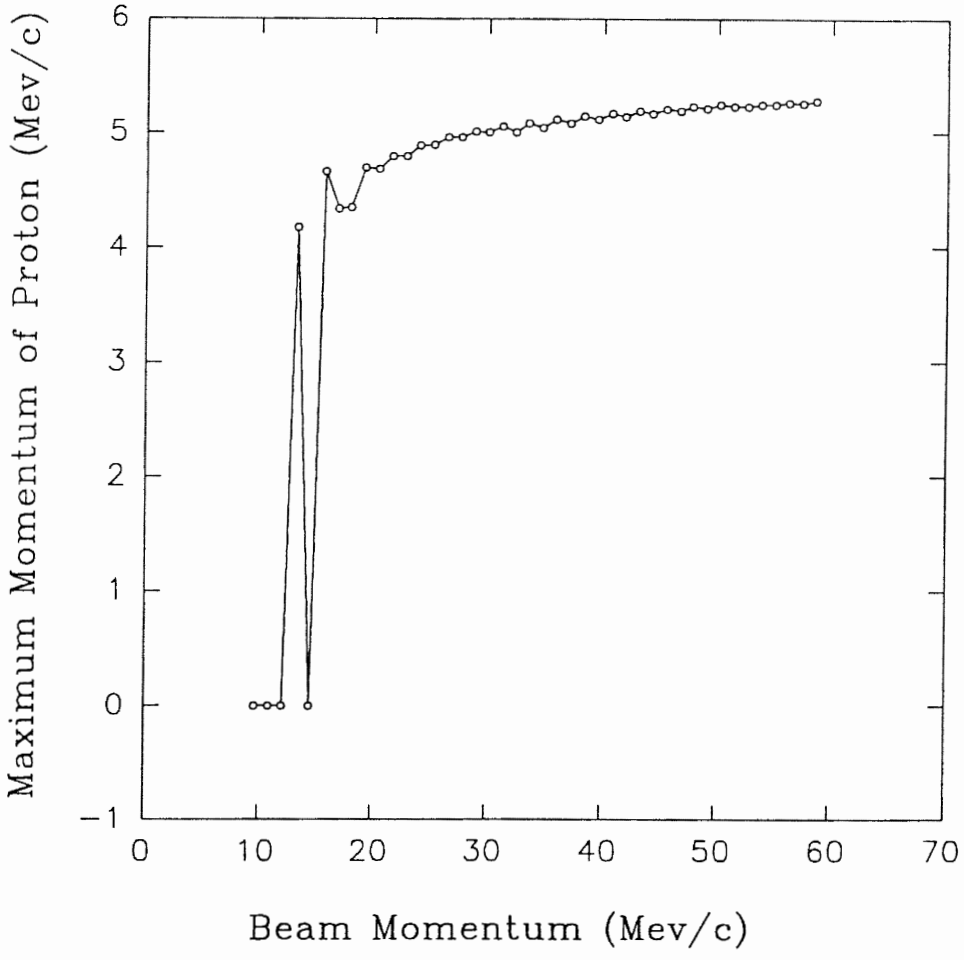
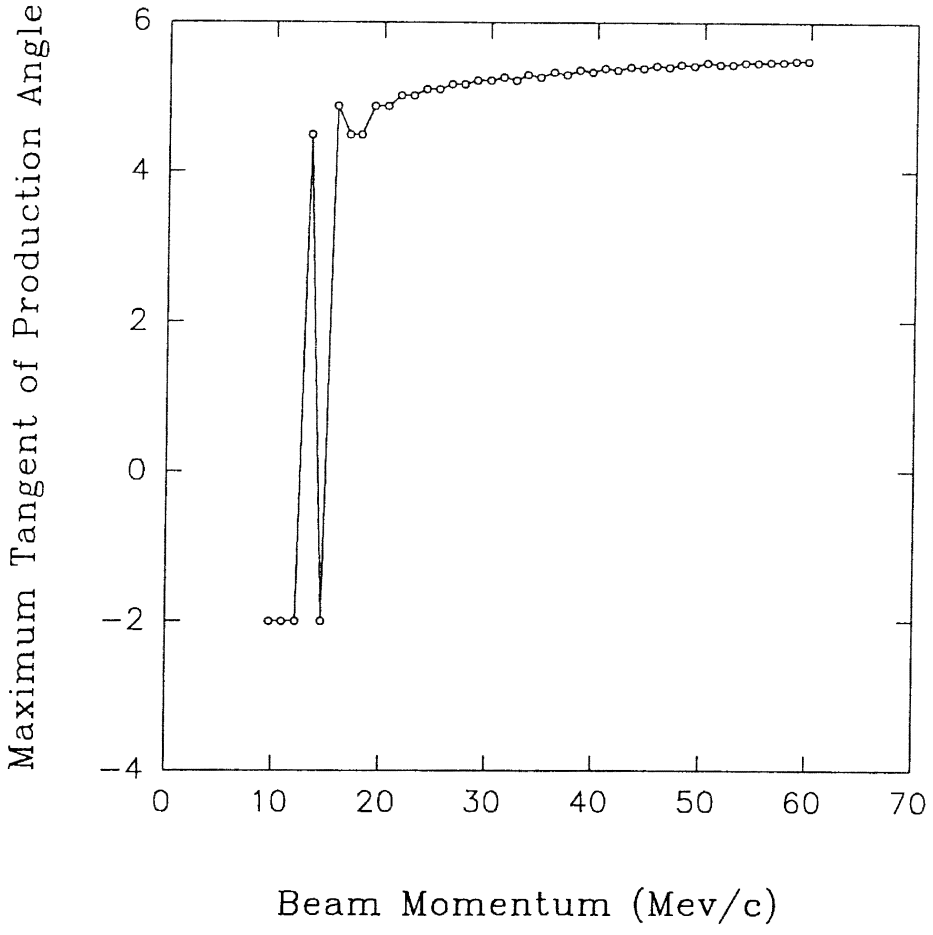


Figure 3. Maximum momentum of proton against beam momentum in the lab system for the reaction $K^- + D \rightarrow p + \lambda + \pi^- + \pi^+ + K^- + K^0$. In the plot, $x = p \cdot 10^7 - 18798400$, $y = \log(pmax \cdot 10^8 - 53066000)$ where p is the beam momentum and pmax is the maximum momentum of proton.



igure 4. Maximum tangent of production angle of proton against beam momentum in the lab system for the reaction $K^- + D \rightarrow p + \lambda + \pi^- + \pi^+ + K^- + K^0$. In the plot, $x = p \cdot 10^7 - 18798400$, $y = \log(\tan\theta \cdot 10^8 - 22900)$ where p is the beam momentum and $\tan\theta$ is the maximum tangent of production angle of proton.

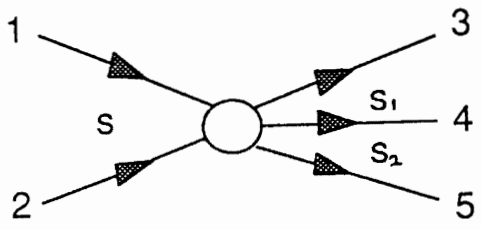


Figure 5. A reaction with three particles in the final state.

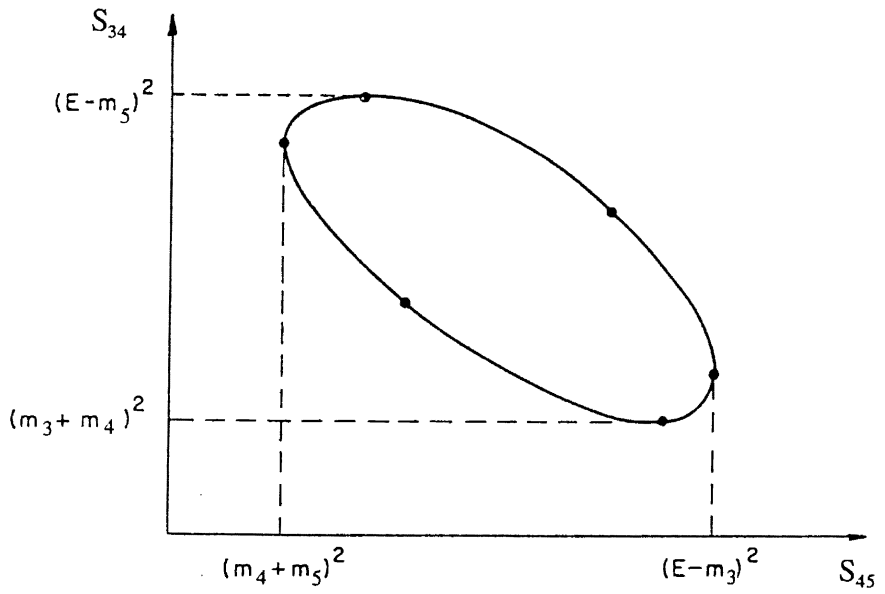


Figure 6. Dalitz plot in s_{34}, s_{45} plane. The physical region for a three-particle final state is inside the closed curve. The black dots on the plot indicate the maximum and minimum for s_{34}, s_{45} and s_{35} respectively.

chapter.

(2) Dalitz plot.

For a reaction with three particles in the final state $1 + 2 \rightarrow 3 + 4 + 5$ (Figure 5) we always have the laws of conservation of energy and momentum. Also, we have an invariant for each offspring particles. In the center-of-momentum system, the following equations hold

$$\begin{aligned} E_3 + E_4 + E_5 &= E \\ \boldsymbol{p}_3 + \boldsymbol{p}_4 + \boldsymbol{p}_5 &= 0 \\ E_3^2 - p_3^2 = m_3^2, \quad E_4^2 - p_4^2 = m_4^2, \quad E_5^2 - p_5^2 &= m_5^2 \end{aligned} \tag{2.17}$$

where E is the total energy.

It would be a good idea to represent graphically the conditions above. Assuming that one point represents a reaction and another point represents another reaction, we learn that all this kind of points fill a certain region. On the contrary, any point within this region represents a conceivable, allowable case of reaction. The region is called a Dalitz plot in nuclear physics. Figure 6 shows an example of a Dalitz plot, the physical region for a three-particle final state is inside the closed curve.

What is the significance of drawing a Dalitz plot? It is convenient to represent the results of the analysis of a photograph in which a reaction of three-particle in final state is observed as a point. One photograph, one point; another photograph, another point, so that a thousand photographs provide a thousand points. The results of the

observations of a great many reactions of the same type are all within a plot drawn beforehand. The results of a whole experiment are represented in a single picture. A vertical or horizontal band of points represents a two-particle resonant state.

Now we would look into details of drawing Dalitz plot.

A Dalitz plot can be defined as the physical region in terms of and variables related to s_1, s_2.

$$\begin{aligned} s_1 &= s_{34} = (p_3 + p_4)^2 = (p - p_5)^2 \\ s_2 &= s_{45} = (p_4 + p_5)^2 = (p - p_3)^2 \\ s_3 &= s_{53} = (p_5 + p_3)^2 = (p - p_4)^2 \end{aligned} \tag{2.18}$$

where p_3, p_4 and p_5 represent the four-momenta of particles 3,4,and 5 respectively. s_1 is the total energy squared of particles 3 and 4 in their own center-of-momentum system, s_2 is the total energy squared of particles 4 and 5 in their own center-of-momentum system, etc. $s=(p_1+p_2)^2$ is the total energy squared in the center-of-momentum system. The relation among them is

$$s_{34} + s_{45} + s_{53} = s + m_3^2 + m_4^2 + m_5^2 \tag{2.19}$$

A Dalitz plot can be defined in terms of any pair E_i, E_j, i,j=3,4,5, any pair of s_i, s_j, i,j=1,2,3. As there is a linear relation between E and s (Equation (2.20)), the (E_3, E_4)-plot and (s_1, s_2)-plot are not significantly different from each other.

$$E_3 = \frac{s - m_3^2 - s_2}{2\sqrt{s}}$$

$$E_4 = \frac{s - m_4^2 - s_3}{2\sqrt{s}} \qquad (2.20)$$

$$E_5 = \frac{s - m_5^2 - s_1}{2\sqrt{s}}$$

Conventionally, a Dalitz plot is often obtained by using a G function[7][11].

A single universal function G is given by

$$\begin{aligned} G(x,y,z,u,v,w) &= x^2y + xy^2 + z^2u + zu^2 + v^2w + vw^2 \\ &\quad + xzw + xuv + yzv + yuw - xy(z + u + v + w) \\ &\quad - zu(x + y + v + w) - vw(x + y + z + u) \end{aligned} \qquad (2.21)$$

G can also be expressed as the symmetric determinant

$$G(x,y,z,u,v,w) = -\frac{1}{2}\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & v & x & z \\ 1 & v & 0 & u & y \\ 1 & x & u & 0 & w \\ 1 & z & y & w & 0 \end{vmatrix}$$

The Dalitz plot in the s_1s_2 plane can be written as

$$G(s_1, s_2, s, m_4^2, m_3^2, m_5^2) = 0. \qquad (2.22)$$

By the symmetry properties of G, this is equivalent to

$$G(s_2, s_1, m_5^2, m_3^2, s, m_4^2) = 0. \qquad (2.23)$$

s = total energy squared - total momentum squared.

Let us consider the reaction

$$K^- + N \rightarrow K^- + \Xi + \pi$$

assuming the target particle N is at rest. The threshold

bombarding energy for this reaction just going is ε_{thres} = 1.4289446 GeV. The corresponding beam momentum of the K^- is

$$p_{thres} = \sqrt{\epsilon_{thres}^2 - m^2} = \sqrt{1.4289446^2 - 0.4939^2} = 1.3404692\,(GeV/c)$$

Let the beam momentum p=2.24(GeV/c); Figure 7 shows the Dalitz plot in the s_1s_2 plane. Let the beam momentum p=1.35(GeV/c) which is just above the threshold energy; the Dalitz plot in the s_1s_2 plane is shown in Figure 8. Comparing this two figures, we can see that the Dalitz plot at p=2.24(GeV/c) is smooth, but as shown in Figure 8, the Dalitz plot at p=1.35 (GeV/c) is very rough. The rough plot is due to the numerical calculation in deriving the data.

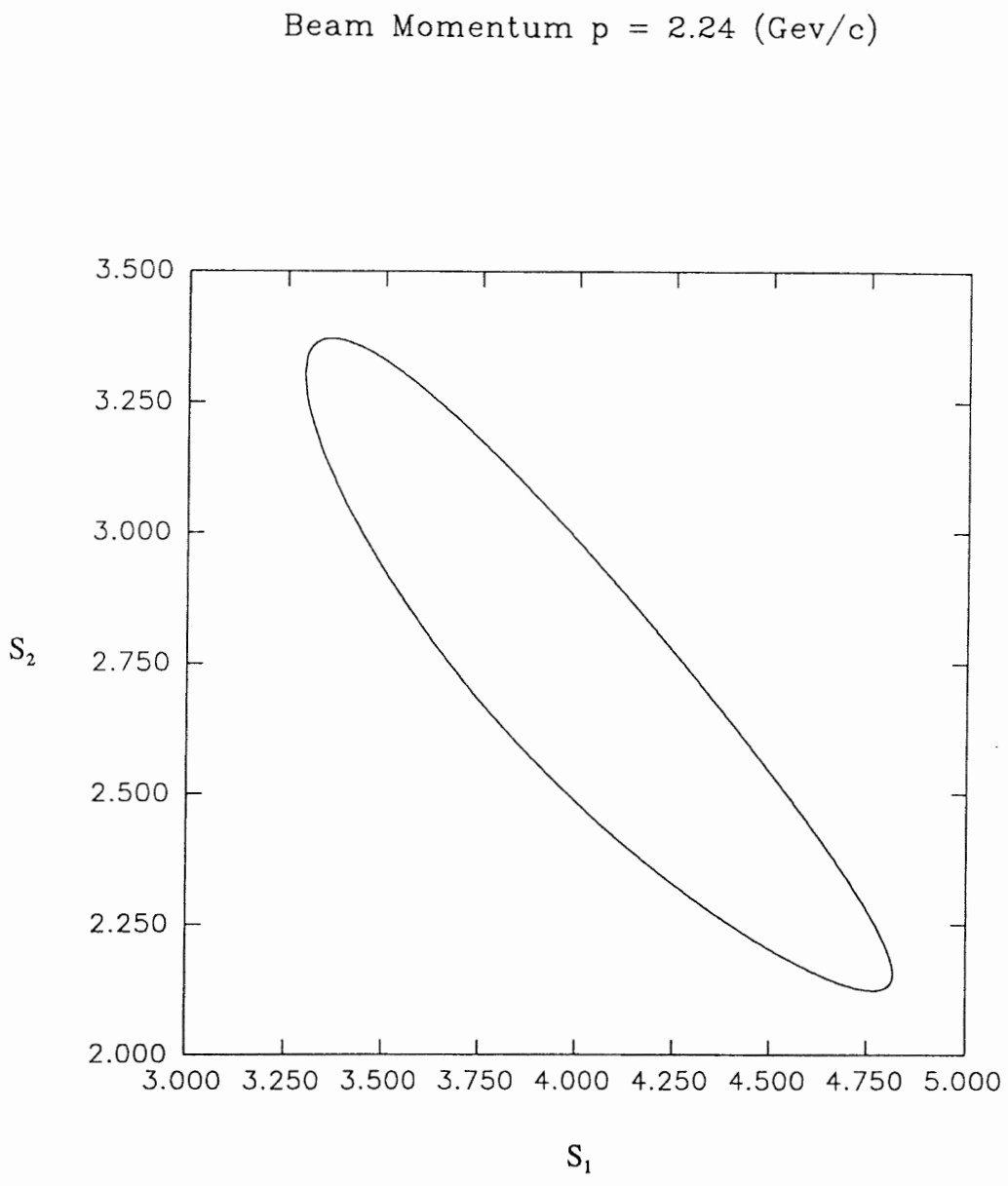


Figure 7. Dalitz plot of the physical region of $K^- + N \rightarrow K^- + \Xi + \pi$ in the s_1s_2 plane when beam momentum p equals 2.24 GeV/c.

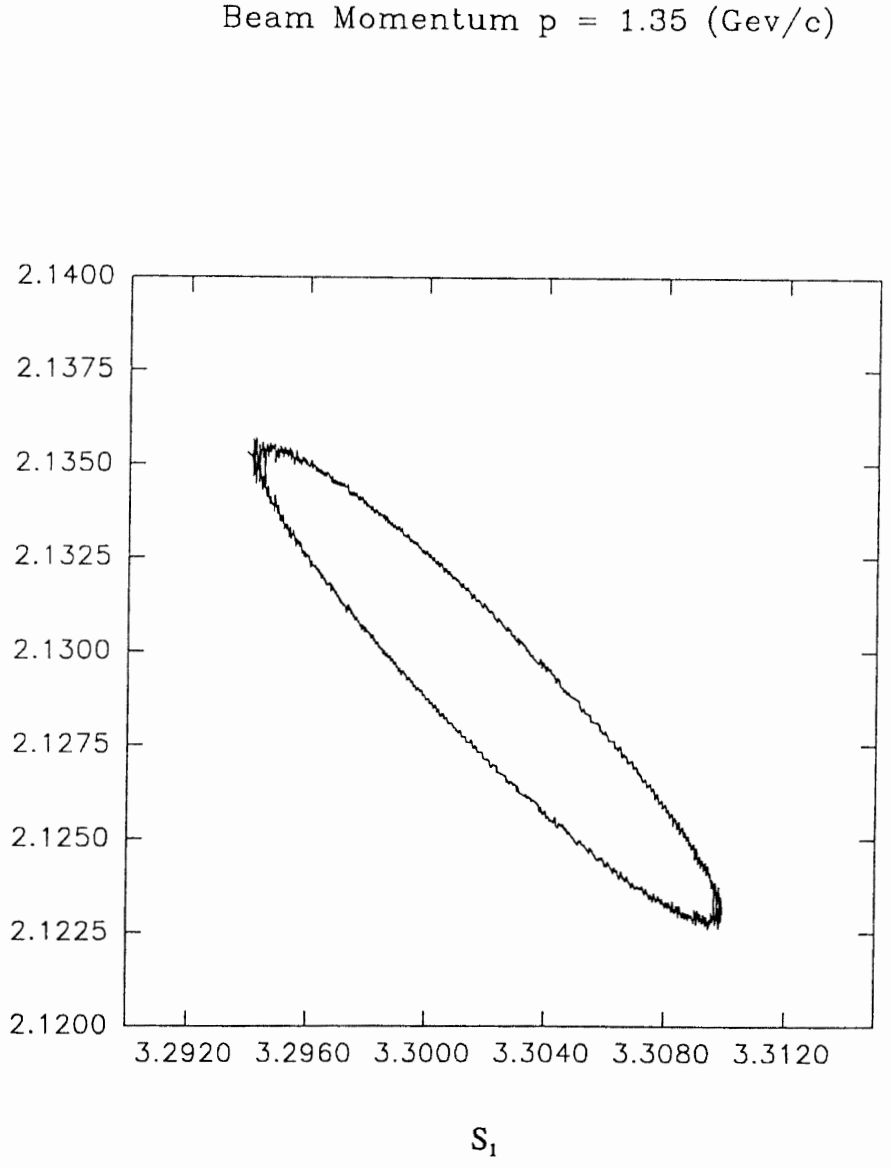


Figure 8. Dalitz plot of the physical region of $K^- + N \rightarrow K^- + \Xi + \pi$ in the $s_1 s_2$ plane when beam momentum is just above the threshold energy. p = 1.35 (GeV/c).

CHAPTER III

FLOATING POINT COMPUTATION AND ANALYSIS OF NUMERICAL CANCELLATION

Floating Point Systems

Nearly all numerical computation on a digital computer is done in floating point arithmetic. Floating point system is a number system which uses a finite number of digits to approximate the real number system which we use in exact computation. We will illustrate floating point using base=10, as used in all hand calculators.

Any non-zero real number y can be written as

$$y = \pm .d_1 d_2 \cdots d_s d_{s+1} \cdots * 10^e$$

where $1 \le d_1 \le 9$ and in general infinitely many digits are required.

A non-zero floating point number y has the form

$$y = \pm .d_1 d_2 \cdots d_s * 10^e$$

where $1 \le d_1 \le 9$, $0 \le d_k \le 9$, $k=2,3,\cdots,s$. The fractional part $d_1 d_2 \cdots d_s$ is called the mantissa, and e is called the exponent.

In the floating point system, the number zero is represented by $+.00\cdots0*10^{e}$.

In a computer, the number of digits for mantissa and exponent is limited. This means that only a finite set of numbers can be represented in a machine. For example, with three digits in the exponent, the machine's floating point variable range is 10^{-999} to 10^{999}.

Roundoff Error

In a computer, a real number a is represented by the floating point number

$$\overline{a} = \overline{m} \cdot 10^{e}$$

where m rounded to s decimals. The precision of the computer is said to be s decimal digits. If the magnitude of the error in $\overline{a}$ does not exceed $(1/2)*10^{-s}$, $\overline{a}$ is said to have s correct decimals. The digits in $\overline{a}$ which occupy positions where the units is greater than or equal to 10^{-s} are called significant digits. These s significant digits are not necessarily correct in any particular calculation.

For example, one number which is approximately equal to 2/3 is 0.66667. Another is 0.66699842593. Although the second number has many more significant digits, it is actually a worse representation of 2/3 than the first number. In the first number which represent 2/3, all five of the significant digits are correct. In the second, only

the first three digits are correct.

There are two ways of rounding off a number to s significant digits. One is called chopping in which one simply leaves off all the decimals to the right of the *s*th. The other is called rounding. In rounding, one chooses, among the numbers which can be expressed with s decimals, a number which is closest to the given number. Thus if the part of the number which stands to the right of the *s*th decimal is less than $(1/2)*10^{-s}$ in magnitude, one should leave the *s*th decimal unchanged. If it is greater than $(1/2)*10^{-s}$, then one raises the *s*th decimal by 1. In the boundary case, if that which stands to the right of the *s*th decimal is exactly $(1/2)*10^{-s}$, one should raise the *s*th decimal if it is odd or leave it unchanged if it is even. The relative error due to rounding is less than $5*10^{-s}$. The relative error in chopping is less than 10^{1-s}, which can be a whole unit in the last digit position and the maximum error will be correspondingly greater.

Computers usually have the option of doing arithmetic calculations in double precision, using two memory locations to store each number, giving twice as many bit positions for the mantissa. Thus the roundoff error is much reduced. For example, if the single precision of a machine is approximately 7 decimals, the double precision will be 14 decimals. Computers use either base=2 or , more rarely, base=16, for floating point computation, rather than

base=10.

Cancellation

When we perform a calculation for some practical purpose, we usually start with numbers that are not exact values, For example, numbers representing such quantities as length, lab momentum, and so on. These numbers are said to have inherent errors. It is possible in a complex computer program to do hundreds or thousands of additions, subtractions, multiplications, and divisions with these inexact numbers. It is also possible for the results to become succeedingly less and less exact, until the final results may be completely meaningless. Therefore, a careful check for accuracy in numerical calculations is very important.

The primary cause of loss of accuracy in numerical calculation is the subtraction of two nearly equal numbers. Let us look at the following example. In the 7 place notation

$$a = 0.9386782 * 10^3$$

$$b = 0.9386780 * 10^3$$

a and b have 7 significant digits. Upon subtraction

$$a - b = 0.0000002 * 10^3$$

When numbers are handled in floating point system, they are ordinarily adjusted so that the fractional part has no

leading zeros. Therefore

$$a - b = 0.2000000 * 10^3$$

The final answer has only one correct significant digit, assuming that a and b were not exact. Six significant digits were lost in the subtraction.

Now let us look at error propagation in the subtraction of two nearly equal numbers. An error at one point in a calculation propagates. The subtraction of two nearly equal numbers is an extreme case: even though the two numbers have small errors, the relative error in the difference may be quite large.

Suppose we have

$$z = x - y$$

and

$$\frac{e_z}{z} = \frac{x}{x-y} \cdot \left(\frac{e_x}{x}\right) - \frac{y}{x-y} \cdot \left(\frac{e_y}{y}\right)$$

where e_x, e_y and e_z are the absolute errors of x, y and z, respectively. Suppose x and y are properly rounded positive numbers

$$\left|\frac{e_x}{x}\right| \le 5*10^{-s} \quad , \quad \left|\frac{e_y}{y}\right| \le 5*10^{-s}$$

then if x and y are two nearly equal numbers, the relative error in z may be very large, even though the absolute error in z is quite small. Because it is relative error that is propagated in a floating point computation, x-y can have a drastic effect on the final results.

Assume $x = 0.5628*10^4$, $y = 0.5631*10^4$, then $z=0.0003*10^4$. Since $s = 4$,

$$\left|\frac{e_x}{x}\right| \leq 5*10^{-4} = 0.005\%$$

$$\left|\frac{e_y}{y}\right| \leq 5*10^{-4} = 0.005\%$$

0.005% is a small relative error. But for z

$$\left|\frac{e_z}{z}\right| \leq \left(\frac{0.5628}{0.0003} + \frac{0.5631}{0.0003}\right) * 0.5*10^{-4} \approx 17\%$$

which is quite large. This large relative error is propagated through all following computations. If the next operation is to multiply by $u=0.7259*10^4$, the result will be $0.2178*10^5$. In fact, there is only one digit that is correct.

The phenomenon where subtraction of two nearly equal numbers of the same sign rise to catastrophic error is known as subtractive cancellation. Most of the serious error in computer calculations are a result of subtractive cancellation. The problem of accuracy in subtraction is the most important accuracy consideration in many calculations.

Process Graphs

A process graph is a pictorial representation of the sequence in which the arithmetic operations in a calculation are carried out. It is a convenient way to handle the error propagation in a calculation.

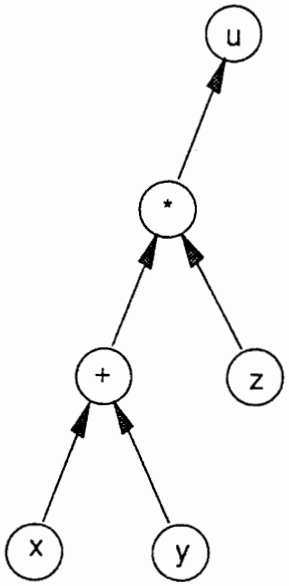


Figure 9. Process graph for the operation u = (x+y)·z.

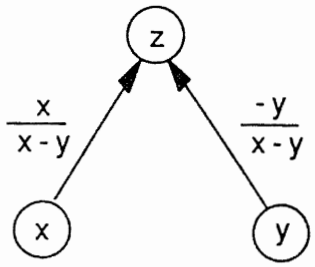


Figure 10. Process graph for the operation z = x-y.

Figure 9 is the process graph for the operation u=(x+y)*z.

Now, let us see how the process graph can show the propagation of errors. For the subtraction z=x-y, we have the process graph to show the error propagation (Figure 10).

In figure 10, one branch which is from node x to node z is labelled by x/(x-y), the other branch which is from node y to node z is labelled by -y/(x-y). If we wish to know the relative error of z, we compute the product of the relative error in x

$$(\frac{e_x}{x}) \cdot (\frac{x}{x-y})$$

and the product of the relative error in y

$$(\frac{e_y}{y}) \cdot (\frac{-y}{x-y})$$

and roundoff error r_a which created by the subtraction in node z. Thus the relative error of z is

$$(\frac{e_x}{x}) \cdot (\frac{x}{x-y}) - (\frac{e_y}{y}) \cdot (\frac{y}{x-y}) + r_a.$$

Relative Error Magnification Factor

Relative error of z=x-y is

$$\frac{e_z}{z} = (\frac{e_x}{x}) \cdot (\frac{x}{x-y}) - (\frac{e_y}{y}) \cdot (\frac{y}{x-y}) + r_a$$

by legitimate change

$$
\begin{aligned}
\left|\frac{e_z}{z}\right| &\le \left|\frac{x}{x-y}\right| \cdot \left|\frac{e_x}{x}\right| + \left|\frac{y}{x-y}\right| \cdot \left|\frac{e_y}{y}\right| + |r_a| \\
&\le \max\left(\left|\frac{x}{x-y}\right|, \left|\frac{y}{x-y}\right|\right) \cdot \left|\frac{e_x}{x}\right| + \max\left(\left|\frac{x}{x-y}\right|, \left|\frac{y}{x-y}\right|\right) \cdot \left|\frac{e_y}{y}\right| + |r_a| \\
&= \max\left(\left|\frac{x}{x-y}\right|, \left|\frac{y}{x-y}\right|\right) \cdot \left(\left|\frac{e_x}{x}\right| + \left|\frac{e_y}{y}\right|\right) + |r_a| \\
&= \frac{\max(|x|, |y|)}{|x-y|} \cdot \left(\left|\frac{e_x}{x}\right| + \left|\frac{e_y}{y}\right|\right) + |r_a|
\end{aligned}
$$

We define

$$
RERMAG = \frac{\max(|x|, |y|)}{|x-y|}
$$

which is called the relative error magnification factor. Thus

$$
\left|\frac{e_z}{z}\right| \le RERMAG \cdot \left(\left|\frac{e_x}{x}\right| + \left|\frac{e_y}{y}\right|\right) + |r_a|
$$

RERMAG can be used to better illustrate the relative error of subtraction.

Analysis of numerical cancellation

We have mentioned in last section that most of the serious errors in computer calculations are caused by subtractive cancellation. Thus careful analysis of each step of program is necessary. Especially, each step with subtraction will be examined to determine if a subtractive cancellation occurs. The relative error magnification factor can help this analysis.

Table I shows the relative error magnification factor

TABLE I

THE RERMAG OF SOME SUBTRACTIONS IN KINE1

(BEAM MOMENTUM p = 1.8798413 MeV/c)

subtraction (x-y)	x	y	RERMAG
$M^2=E^2-p^2$	14.5857935	3.5338035	0.1319744E+01
$M^2-(\Sigma m_i)^2$	11.0519905	11.0519905	--*
$A=1.0-(\beta*\cos\theta)^2$	1.0000000	0.2422771	0.1319744E+01
$ARG=B^2-4.0*A*C$	0.6463767	0.6463764	0.2711100E+07
PMIN=-ONE+TOTHER	0.0003027	0.5305208	0.1000571E+01
β^2-v^{*2}	0.2422771	0.0000001	0.1000000E+01

* -- means the RERMAG is ∞.

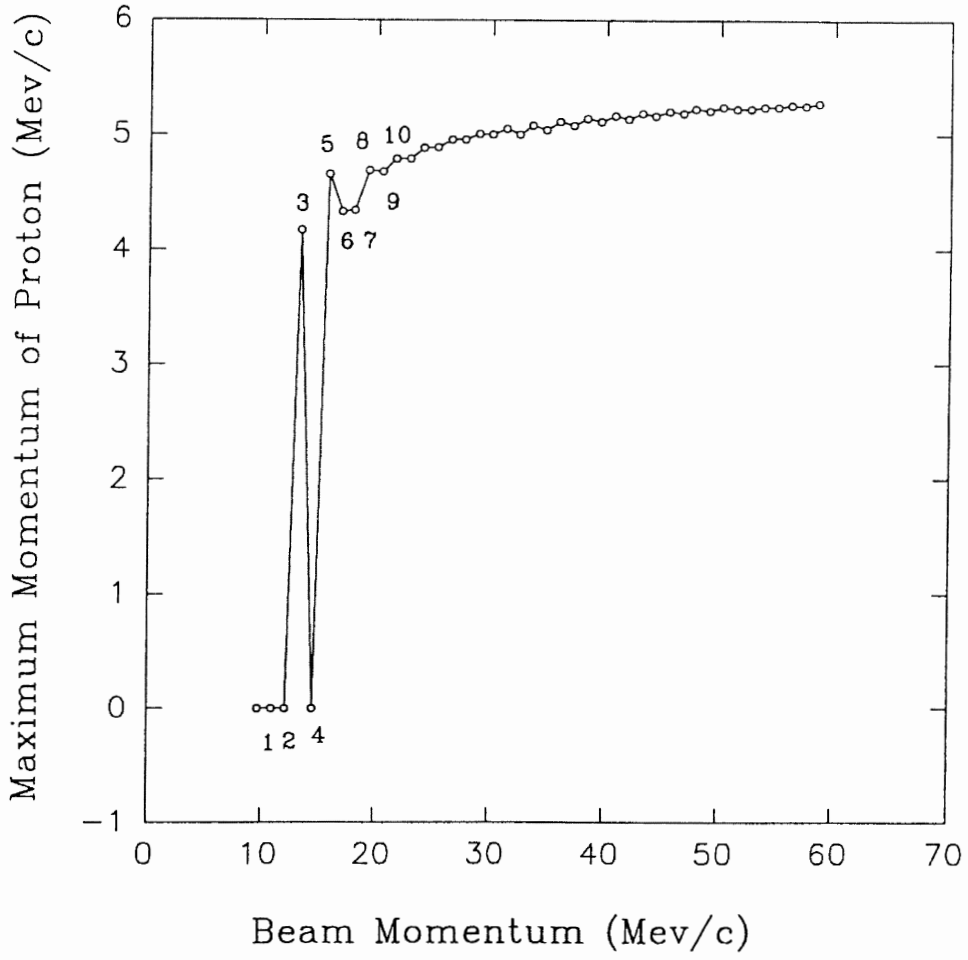


Figure 11. Maximum momentum of proton against beam momentum in the lab system for the reaction $K^- + D \rightarrow p + \lambda + \pi^- + \pi^+ + K^- + K^0$. In the plot, $x = p \cdot 10^7 - 18798400$, $y = \log(pmax \cdot 10^8 - 53066000)$ where p is the beam momentum and pmax is the maximum momentum of proton. The 10 points labeled is being studied in detail.

TABLE II

THE RERMAG OF 10 POINTS IN KINE1 PLOT

point	p	M^2	$(\Sigma m_i)^2$	RERMAG of $M^2-(\Sigma m_i)^2$	pmax
1	1.8798411	11.0519896	11.0519905	0.1158885E+08	0.0
2	1.8798412	11.0519886	11.0519905	0.5794426E+07	0.0
3	1.8798413	11.0519905	11.0519905	--	0.5308235
4	1.8798414	11.0519896	11.0519905	0.1158885E+08	0.0
5	1.8798416	11.0519915	11.0519905	0.1158885E+08	0.5311326
6	1.8798417	11.0519905	11.0519905	--	0.5308927
7	1.8798418	11.0519905	11.0519905	--	0.5308960
8	1.8798419	11.0519915	11.0519905	0.1158885E+08	0.5311697
9	1.8798420	11.0519915	11.0519905	0.1158885E+08	0.5311564
10	1.8798422	11.0519924	11.0519905	0.5794427E+07	0.5312958

of some subtraction in subroutine KINE1 which we have used to calculate the kinematic limit on the momentum and production angle of one particle from a system of particles. KINE1 is given in Appendix A. It is seen clearly from table I that the RERMAG of $M^2-(\Sigma m_i)^2$ (which is part of equation (2.8)) is very large. This means $M^2-(\Sigma m_i)^2$ is very close to zero when the energy is near the threshold. Also, we can see from table I that other subtractions with the RERMAG not large are not dangerous for the calculation. Table II shows the value of p, M^2, $(\Sigma m_i)^2$, RERMAG of $M^2-(\Sigma m_i)^2$ and pmax for 10 points in Figure 11. We can see that the RERMAG of $M^2-(\Sigma m_i)^2$ of every point is huge and the value of pmax is jumping around instead of increasing monotonically. The conclusion we get is that the main cause of numerical cancellation in KINE1 is the subtractive cancellation $M^2-(\Sigma m_i)^2$, but $B^2-4.0*A*C$ is also bad.

Table II gives the details of 10 points of Figure 11. It is obvious that the RERMAG of $M^2-(\Sigma m_i)^2$ of each point from . to 10 is very large. That causes the value of pmax to have a drastic loss of significance.

Now, we will analyze the Dalitz plot when the beam momentum is near threshold.

Letting $G(s_{34}, s_{45}, m_3^2, m_5^2, s, m_4^2)=0$, we will get the boundary of a Dalitz plot. From G=0 we get

$$y=Ax^2+Bx+C$$

here

$x=s_{34}$

$y=s_{45}$

$A=s_{34}$

$B=s_{34}*(s_{34}-m_3^2-m_4^2)+m_5^2(m_4^2-s_{34}-m_3^2)+s(m_3^2-s_{34}-m_4^2)$

$C=(m_4^2s-m_3^2m_5^2)*(m_4^2+s-m_3^2-m_5^2)+s_{34}*(m_4^2-m_5^2)*(m_3^2-s)$

then

$$s_{45} = \frac{-B \pm \sqrt{B^2 - 4*A*C}}{2*A}$$

Using above equations, a computer program DALG is written which is given in Appendix B. Let us look at part of the Dalitz plot (Figure 12 between a and b). Table III shows that the RERMAG of B^2-4AC is very large. It is infinite for some points. The huge error of B^2-4AC cause the value of s_{45} to jump around instead of decreasing monotonically when s_{34} is increasing.

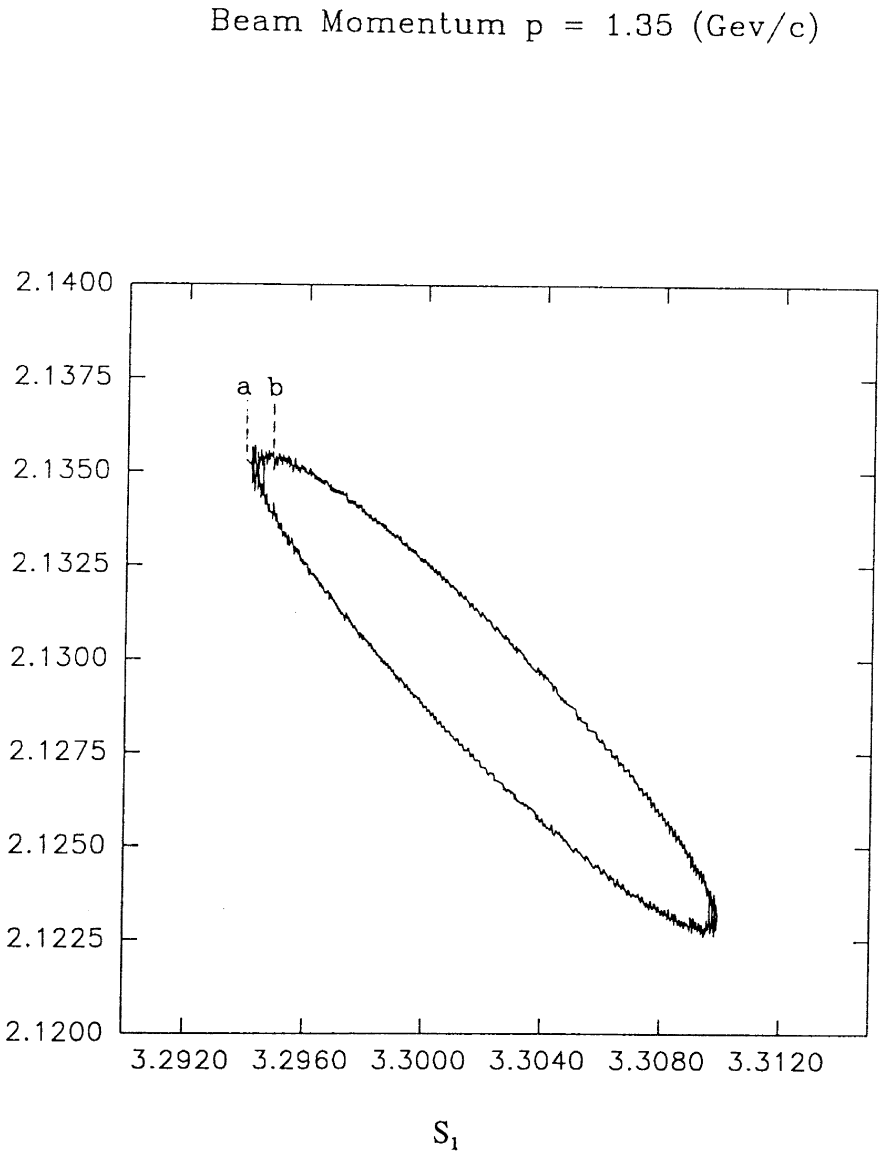


Figure 12. Dalitz plot as the physical region of $K^- + N \rightarrow K^- + \Xi + \pi$ in the s_1s_2 plane when beam momentum is just above the threshold energy; p = 1.35 (GeV/c). The part of the plot between a and b is being studied in detail.

TABLE III

PART OF THE POINTS IN DALITZ PLOT

DRAWING IN G FUNCTION

s_{34}	B^2	4.0*A*C	RERMAG OF B^2-4AC	B^2-4AC	$s_{45}^{(+)}$	$s_{45}^{(-)}$
3.2939701	197.8795319	197.8795319	--	-0.0000054	2.1352601	2.1352601
3.2940023	197.8788300	197.8788605	0.6484095E+07	-0.0000229	2.1352355	2.1352355
3.2940342	197.8781586	197.8781738	0.1296814E+08	-0.0000118	2.1352112	2.1352112
3.2940662	197.8775024	197.8775024	--	-0.0000006	2.1351869	2.1351869
3.2940981	197.8768311	197.8768158	0.1296806E+08	0.0000107	2.1356587	2.1346662
3.2941301	197.8761292	197.8761444	0.1296801E+08	-0.0000174	2.1351380	2.1351380
3.2941623	197.8754578	197.8754578	--	0.0000048	2.1354444	2.1347828
3.2941942	197.8747864	197.8747711	0.1296792E+08	0.0000162	2.1356993	2.1344793
3.2942262	197.8740845	197.8740997	0.1296788E+08	-0.0000118	2.1350648	2.1350648
3.2942581	197.8734131	197.8734131	--	-0.0000003	2.1350405	2.1350405
3.2942901	197.8727417	197.8727417	--	0.0000113	2.1355257	2.1345065
3.2943223	197.8720551	197.8720551	--	-0.0000057	2.1349914	2.1349914
3.2943542	197.8713837	197.8713684	0.1296770E+08	0.0000060	2.1353378	2.1345966
3.2943861	197.8707123	197.8706818	0.6483828E+07	0.0000177	2.1355810	2.1343048
3.2944181	197.8700104	197.8700104	--	0.0000026	2.1351635	2.1346731
3.2944503	197.8693390	197.8693237	0.1296757E+08	0.0000127	2.1354342	2.1343536
3.2944822	197.8686371	197.8686371	--	-0.0000023	2.1348693	2.1348693
3.2945142	197.8679657	197.8679657	--	0.0000096	2.1353161	2.1343741
3.2945461	197.8672943	197.8672791	0.1296743E+08	0.0000216	2.1355262	2.1341155
3.2945781	197.8666077	197.8665924	0.1296739E+08	0.0000068	2.1351919	2.1344006

CHAPTER IV

METHODS OF AVOIDING NUMERICAL CANCELLATION

Analyzing relativistic formulas carefully one could find that some conventionally formulated equations are not suitable for numerical calculations although they seem to have simpler forms. It is desirable to have formulas which are accurate at both low energy and high energy. Especially we want to use relativistic equations without loss of accuracy in numerical calculations in low energy situations. Two approaches can be used to achieve this goal:

1) Reformulating certain equations

2) Using the appropriate variables

Let's start by reformulating conventional equations. If possible, subtraction of two nearly equal numbers should be avoided since it usually give a big cancellation errors. Usually reformulating the expression containing a subtraction can achieve this. Also, a certain "square root trick" is very useful in avoiding subtractive cancellation. We now concentrate on this technique.

In the quadratic equation $ax^2 + bx + c = 0$, if we assume that all coefficients are positive, and $b^2 >> 4ac$,

then one root can be given by

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}.$$

If the formula is evaluated in finite precision, b^2-4ac may be equal to b^2 for working precision. Then x_1 will have the value 0.0 in a floating point computation which is not the right answer.

By using the "square root trick", we reformulate the equation to

$$\begin{aligned} x_1 &= \frac{-b + \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{(-b + \sqrt{b^2 - 4ac})}{2a} \frac{(-b - \sqrt{b^2 - 4ac})}{(-b - \sqrt{b^2 - 4ac})} \\ &= \frac{4ac}{2a\,(-b - \sqrt{b^2 - 4ac})} \\ &= \frac{-2c}{(b + \sqrt{b^2 - 4ac})} \end{aligned}$$

now, x_1 gives the exact root without subtractive cancellation since b and the radical are both positive and there is no subtraction.

Now we look at the application of this technique to a practical problem of reaction energy calculation. Reaction energy Q is the energy released as kinetic energy at the expense of the internal energy of the colliding systems. For the typical reaction of two-product type T(i,p)R when the target nucleus is at rest (Figure 13), reaction energy

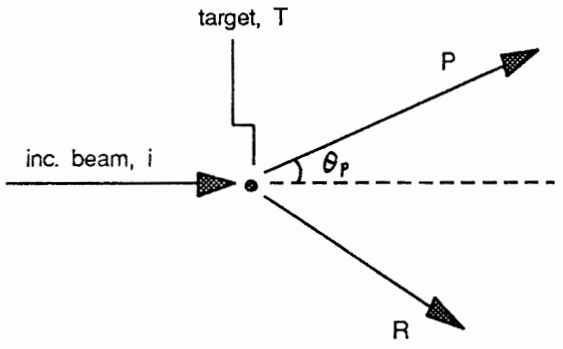


Figure 13. Reaction of the two-product type: T(i, p)R.

is

$$Q = T_p - T_i + m_R c^2 \left(\left[1 + \left(\frac{m_p}{m_R} \right)^2 \frac{T_p}{m_p c^2} \left(2 + \frac{T_p}{m_p c^2} \right) + \left(\frac{m_i}{m_R} \right)^2 \frac{T_i}{m_i c^2} \left(\frac{T_i}{m_i c^2} + 2 \right) \right.\right.$$
$$\left.\left. - 2 \frac{m_p m_i}{m_R^2} \left[\frac{T_p T_i}{m_p c^2 m_i c^2} \left(2 + \frac{T_p}{m_p c^2} \right) + \left(2 + \frac{T_i}{m_i c^2} \right) \right] \right]^{1/2} \cos\theta_p \Big]^{1/2} - 1 \Big) \qquad (4.1)$$

θ_p is the production angle[5].

Defining

$$W = \left(\frac{m_p}{m_R} \right)^2 \frac{T_p}{m_p c^2} \left(2 + \frac{T_p}{m_p c^2} \right) + \left(\frac{m_i}{m_R} \right)^2 \frac{T_i}{m_i c^2} \left(\frac{T_i}{m_i c^2} + 2 \right)$$
$$- 2 \frac{m_p m_i}{m_R^2} \left[\frac{T_p T_i}{m_p c^2 m_i c^2} \left(2 + \frac{T_p}{m_p c^2} \right) + \left(2 + \frac{T_i}{m_i c^2} \right) \right]^{1/2} \cos\theta_p \qquad (4.2)$$

Then we have

$$Q = T_p - T_i + m_R c^2 \left[(1 + W)^{1/2} - 1 \right] \qquad (4.3)$$

For the low energy situations, all the items of T/mc^2 in (4.2) can be neglected compared to unity. Therefore, at low energy, W is very close to zero. $(1+W)^{1/2} - 1$ in (4.3) will cause a drastic loss of significant digits in finite precision.

If we ignore the quantities T/mc^2 in (4.1), we will get the following approximate formula for low energy cases.

$$Q = T_p \left(1 + \frac{M_p}{m_R} \right) - T_i \left(1 - \frac{M_i}{m_R} \right) - 2 \frac{(m_p T_p m_i T_i)^{1/2}}{m_R} \cos\theta_p \qquad (4.4)$$

Equation (4.4) gives only approximate results because

the quantity T/mc^2 had been ignored. Altering Formula (4.3) using the "square root trick", one gets

$$Q = T_p - T_i + m_R c^2 \frac{W}{(1+W)^{1/2}+1} \qquad (4.5)$$

Equation (4.5) is still exact and the problem of loss of significant digits at low energy has been solved. Therefore, the formula above can be used for both high energy and low energy computation without cancellation.

Next, let us examine the usage of appropriate variables. In relativistic kinematics, the choice of the energy variable is not unique and the following alternatives are in use

a. Kinetic energy of the particle, $T = E - m_0c^2$, is mostly used in the domain where the rest energy is larger than the kinetic energy. T is the normal variable in low energy nuclear physics.

b. Total energy E is used in the high energy domain (E >= 1Gev).

c. Momentum p of the particle is normally used at all energies.

We would like to use "good" variables in formulas in order to get more accurate results. In relativistic kinematics, we expect momentum p, kinetic energy T, production angle, etc. to be good variables.

The reason of saying "good" variable instead of good variable is that the meaning of "good" or "bad" has nothing

to do with the nuclear physics itself. Here a variable is to be thought "good" or "bad" depending only on the analysis of numerical calculation.

The criterion for selecting good variables in numerical calculation with finite precision is that the variable has good limits; it has a range from zero to infinity. Finite nonzero limits cause cancellation. Therefore we try to avoid them.

The variables such as total energy E, β, γ, etc. are not good variables in numerical calculation although in nuclear physics they are widely used to describe the various particle interactions.

For example, the relativistic factor γ is defined as

$$\gamma = \frac{1}{\sqrt{1-(\frac{v}{c})^2}}$$

at low energy, $v << c$, γ is very close to 1.0. It has a nonzero limit. Therefore γ is not a good choice in calculation. But $(\gamma-1)$ would be a good variable.

The total energy E is not a good variable since for most cases, E cannot reach the limit of zero.

Let us look at an example. It is known that total energy = rest energy + kinetic energy, or $E = m_0c^2 + T$.

For proton, $m_0c^2 = 938.2796$ Mev. Assuming E = 938.31 ± 0.01 Mev; here, $e_E = 0.01$ (Mev) is the absolute error of E.

Then the relative error of E is

$$\frac{e_E}{E} = \frac{0.01}{938.31} \approx 0.0011\%.$$

Now, let us calculate kinetic energy T.

$$\begin{aligned} T &= E - m_0c^2 \\ &= 938.31 \pm 0.01 - 938.2796 \\ &= 0.03 \pm 0.01\,(Mev) \end{aligned}$$

the relative error of T is

$$\frac{0.01}{0.03} \approx 33\%.$$

The relative error of E propagates in the calculation $T = E - m_0c^2$ and the effect of error of E becomes greater. This means that the small relative error of E causes a big relative error in T as subsequent operations are carried out. The absolute error in E and T are the same, 0.01 MeV. But the relative error in T is magnified by a factor of about

$$\frac{33\%}{0.0011\%} = 30{,}000 = RERMAG.$$

It is clear that E is a "bad" variable.

In the calculation of relativistic kinematics the total energy E should be avoided. Instead kinetic energy T and momentum p are good choices for such calculations.

Though some variables do not have so-called good limits, they are still considered to be appropriate variables. look at the next example.

As stated in Chapter II, we are required to measure and use production angles which may have finite nonzero limits. However, angles are not worse variables at low energy than they are at high energy. Therefore angles can be used in relativistic kinematics calculations, and in fact this is unavoidable.

For practical computation, an accuracy checklist can be used to avoid unnecessary errors, such as roundoff error and relative error.[10][12]

a. When numbers are to be added or subtracted, work with the smallest numbers first.

b. If possible, avoid subtraction of two nearly equal numbers.

c. If there are nearly equal numbers, do the subtraction before multiplying. This will avoid compounding the problem with additional roundoff errors.

d. When none of the above applies, minimize the number of arithmetic operations.

In summary, by reformulating the relativistic equations using carefully selected variables, we can avoid the loss of accuracy in low energy calculations and also avoid the Newtonian approximation at high energy.

CHAPTER V

RESULTS

In Chapter II two relativistic kinematics problems with numerical cancellation at threshold energy have been discussed. The error and the main reason of the error have been analyzed in Chapter III. Chapter IV gave the methods of avoiding numerical cancellation. Now the methods of avoiding numerical cancellation will be applied to the problems described in Chapter II.

(1) Kinematic limits of a particle from a reaction without numerical cancellation when near threshold energy

First let us look at the problem of computing kinematic limits on the momentum and production angle of one particle from a system of particles. The reaction

$$K^- + D \rightarrow p + \lambda + \pi^- + \pi^+ + K^- + K^0$$

will be used as the example.

By analyzing the relative error magnification factor in Chapter III, it was already found that the $(M^2-\Sigma m_i^2)$ in Equation (2.8) is very close to zero when the energy is near threshold. Therefore subtractive cancellation will occur and this cancellation will cause a drastic loss of significance in the value pmax which represents the maximum

momentum of the proton. Analogous analysis applies to the maximum production angle of the proton.

To avoid the subtractive cancellation, $(M^2-\Sigma m_i^2)$ has to be eliminated from Equation (2.8). It is better to use kinetic energy T and momentum p in each equation instead of total energy E.

Let us derive an equation which has good variables and does not cause subtractive cancellation.

Starting from Equation (2.8),

$$p^{*2}=\frac{(M^2-\Sigma m_i^2)\,(M^2-(\Sigma m_i-2m_k)^2)}{4M^2}$$

the numerator can be altered.

$$\begin{aligned}
&(M^2-(\Sigma m_i)^2)\,(M^2-(\Sigma m_i-2m_k)^2)\\
&=(E^2-p^2-(\Sigma m_i)^2)\,(E^2-p^2-(\Sigma m_i)^2+4m_k(\Sigma m_i-m_k))\\
&=((E^2-(\Sigma m_i)^2)-p^2)\,((E^2-(\Sigma m_i)^2)-p^2+4m_k(\Sigma m_i-m_k))
\end{aligned}$$

since

$$E = T+m_0c^2$$

then

$$\begin{aligned}
E^2-(\Sigma m_i)^2 &= (T+\Sigma m_i)^2-(\Sigma m_i)^2\\
&= T^2+2T\Sigma m_i
\end{aligned}$$

therefore, Equation (2.8) would become

$$p^{*2} = \frac{(T^2+2T\Sigma m_i-p^2)\,(T^2+2T\Sigma m_i-p^2+4m_k(\Sigma m_i-m_k))}{4M^2}$$

(5.1)

defining

$$Q^2 = T^2 + 2T\Sigma m_i - p^2 \qquad (5.2)$$

which represents the kinetic energy of the proton in the center-of-momentum system.

Replacing (5.2) into (5.1), the momentum of the proton in the center-of-momentum system would be

$$p^{*2} = \frac{Q^2 (Q^2 + 4m_k(\Sigma m_i - m_k))}{4M^2} \qquad (5.3)$$

In computer program KINE3 (that is given in Appendix C) which calculates the kinematic limits of the proton, Equation (5.3) has been used instead of (2.8) to get p^{*2}. The new plot of maximum momentum against beam momentum is shown in Figure 14. Clearly this plot is much smoother than the plot in Figure 3 which is plotted using the data generated from subroutine KINE1. The "zigzag" is gone.

Reformulating p^{*2} is not the only change we made to improve the accuracy of KINE1 to KINE3. In fact every statement in KINE3, especially those containing subtraction, is analyzed according to RERMAG to make sure that the possibility of subtractive cancellation was eliminated if possible. Following are two examples.

Example 1. In order to get the accurate kinetic energy

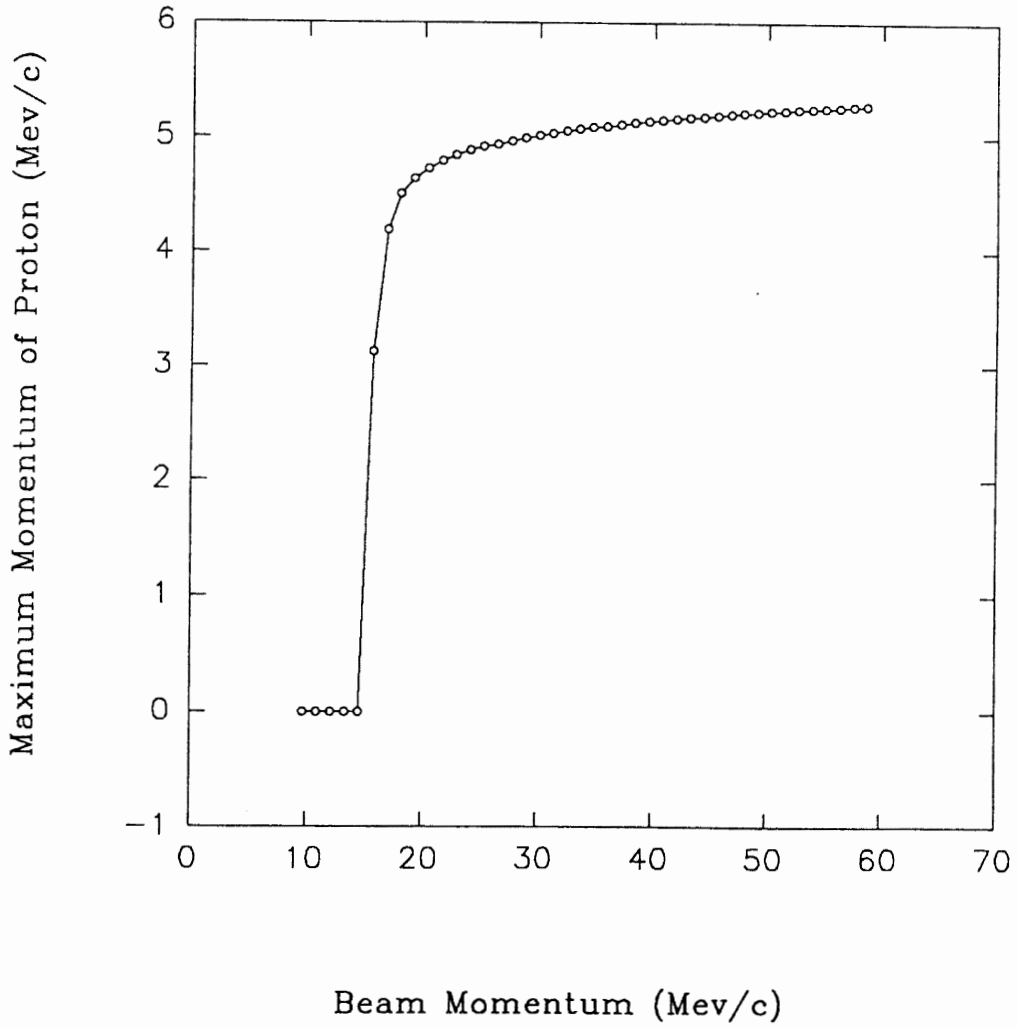


Figure 14. Maximum momentum of the proton against beam momentum in the lab system for the reaction $K^- + D \rightarrow p + \lambda + \pi^- + \pi^+ + K^- + K^0$. In the plot, $x = p \cdot 10^7 - 18798400$, $y = \log(pmax \cdot 10^8 - 53066000)$ where p is the beam momentum and pmax is the maximum momentum of the proton. The data were generated from subroutine KINE3.

T, the conventional formulated equation

$$T = E - \Sigma m_i \tag{5.4}$$

would not be used.

Let us reformulate it.

$$\begin{aligned} T &= E - \Sigma m_i \\ &= m_D + \sqrt{m_k^2 + p^2} - \Sigma m_i \\ &= \sqrt{m_k^2 + p^2} - (\Sigma m_i - m_D) \\ &= (\sqrt{m_k^2 + p^2} - (\Sigma m_i - m_D)) \frac{\sqrt{m_k^2 + p^2} + (\Sigma m_i - m_D)}{\sqrt{m_k^2 + p^2} + (\Sigma m_i - m_D)} \\ &= \frac{m_k^2 + p^2 - (\Sigma m_i - m_D)^2}{\sqrt{m_k^2 + p^2} + (\Sigma m_i - m_D)} \end{aligned}$$

Then we get

$$T = \frac{p^2 + (m_k + \Sigma m_i - m_D)\ (m_k - \Sigma m_i + m_D)}{\sqrt{m_k^2 + p^2} + (\Sigma m_i - m_D)} \tag{5.5}$$

Obviously, Equation (5.5) has no danger of subtractive cancellation.

Example 2. The minimum momentum of the proton is

$$pmin = \frac{-b - \sqrt{b^2 - 4.0 * a * c}}{2.0 * a} \tag{5.6}$$

Here, b has a negative value. The numerator of the formula above

$$-b - \sqrt{b^2 - 4.0 * a * c}$$

may cause cancellation. It need to be reformulated.

Next is the reformulated one.

$$pmin = \frac{-b - \sqrt{b^2 - 4.0*a*c}}{2.0*a}$$

$$= \frac{-b - \sqrt{b^2 - 4.0*a*c}}{2.0*a} \cdot \frac{-b + \sqrt{b^2 - 4.0*a*c}}{-b + \sqrt{b^2 - 4.0*a*c}}$$

$$= \frac{b^2 - (b^2 - 4.0*a*c)}{2.0*a*(-b + \sqrt{b^2 - 4.0*a*c})}$$

$$= \frac{4.0*a*c}{2.0*a*(-b + \sqrt{b^2 - 4.0*a*c})}$$

$$= \frac{2.0*c}{-b + \sqrt{b^2 - 4.0*a*c}}$$

$$= \frac{c}{a \cdot pmax}$$

where

$$pmax = \frac{-b + \sqrt{b^2 - 4.0*a*c}}{2.0*a} \qquad (5.7)$$

then, Equation (5.7) becomes a safe equation with respect to numerical cancellation.

Figure 15 shows the plot of maximum production angle of proton against beam momentum. The plot looks much smoother than the plot in Figure 4 which is plotted using the data generated from subroutine KINE1.

Studying Figure 14 carefully, one can find that several points in this figure are not exactly in the right spot.

Referring to Table IV, one can see that the first difference of beam momentum p and maximum momentum of proton pmax all have the same sign (all be positive). That indicates that the value of pmax is increasing as p is increasing. But the second differences of pmax are not

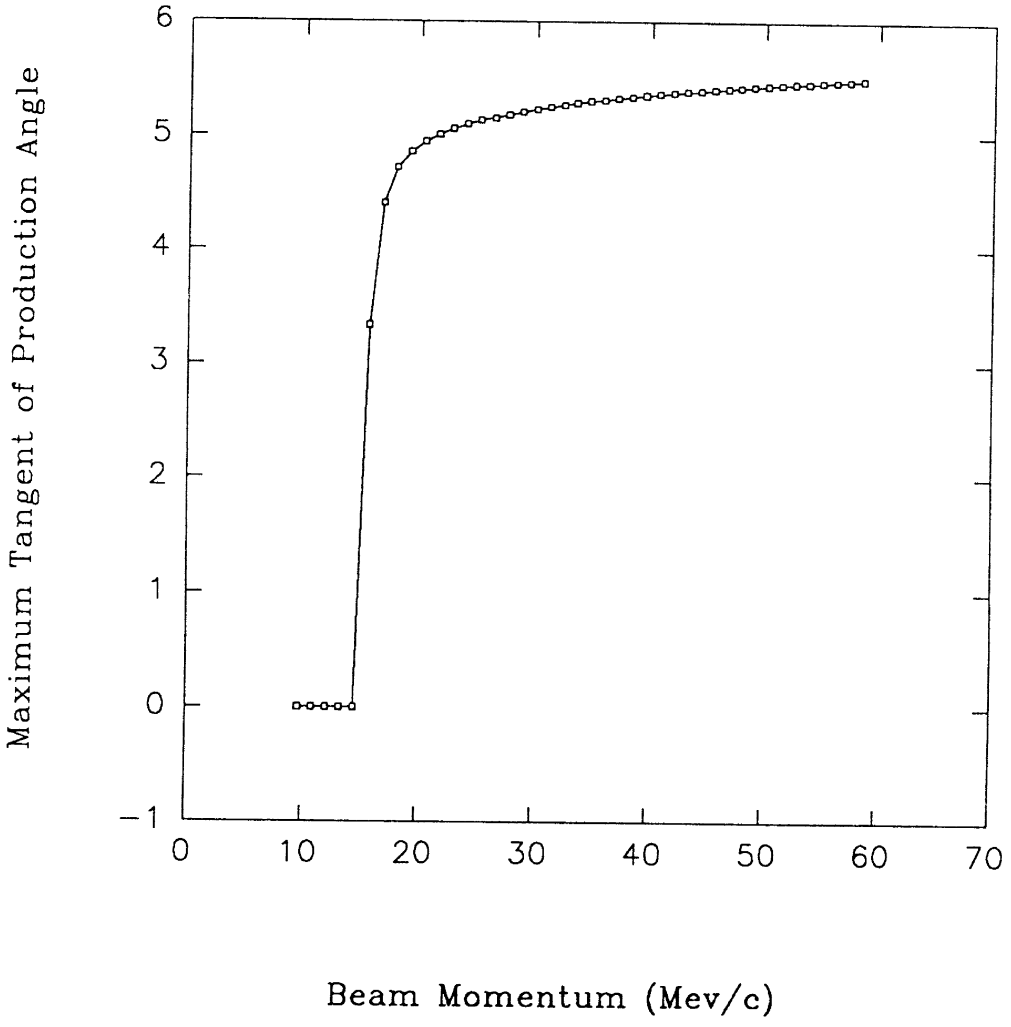


Figure 15. Maximum tangent of production angle of the proton against beam momentum in the lab system for the reaction $K^- + D \rightarrow p + \lambda + \pi^- + \pi^+ + K^- + K^0$.
In the plot, $x = p \cdot 10^7 - 18798400$, $y = \log(\tan\theta \cdot 10^8 - 22900)$ where p is the beam momentum and $\tan\theta$ is the maximum tangent of production angle of the proton. The data were generated from subroutine KINE3.

TABLE IV

The first difference and the second difference of pmax for KINE3

point	pway	pmax	Δpmax	Δ^2pmax
1	1.879840970	0.000000000	0.000000000	0.000000000
2	1.879841089	0.000000000	0.000000000	0.000000000
3	1.879841208	0.000000000	0.000000000	0.000000000
4	1.879841328	0.000000000	0.000000000	0.530673385
5	1.879841447	0.000000000	0.530673385	-0.530532539
6	1.879841566	0.530673385	0.000140846	0.000023663
7	1.879841685	0.530814230	0.000164509	-0.000044942
8	1.879841805	0.530978739	0.000119567	-0.000020921
9	1.879841924	0.531098306	0.000098646	-0.000012636
10	1.879842043	0.531196952	0.000086010	-0.000008762
11	1.879842162	0.531282961	0.000077248	-0.000006557
12	1.879842281	0.531360209	0.000070691	-0.000005007
13	1.879842401	0.531430900	0.000065684	-0.000033975
14	1.879842520	0.531496584	0.000031710	0.000027955
15	1.879842639	0.531528294	0.000059664	-0.000003099
16	1.879842758	0.531587958	0.000056565	-0.000002861
17	1.879842877	0.531644523	0.000053704	-0.000002265
18	1.879842997	0.531698227	0.000051439	-0.000002086
19	1.879843116	0.531749666	0.000049353	-0.000001729
20	1.879843235	0.531799018	0.000047624	-0.000001729

21	1.879843354	0.531846642	0.000045896	-0.000023186
22	1.879843473	0.531892538	0.000022709	0.000020981
23	1.879843593	0.531915247	0.000043690	-0.000001192
24	1.879843712	0.531958938	0.000042498	-0.000001311
25	1.879843831	0.532001436	0.000041187	-0.000000954
26	1.879843950	0.532042623	0.000040233	-0.000001132
27	1.879844069	0.532082856	0.000039101	-0.000000894
28	1.879844189	0.532121956	0.000038207	-0.000019073
29	1.879844308	0.532160163	0.000019133	0.000017703
30	1.879844427	0.532179296	0.000036836	-0.000000656
31	1.879844546	0.532216132	0.000036180	-0.000000775
32	1.879844666	0.532252312	0.000035405	-0.000000656
33	1.879844785	0.532287717	0.000034750	-0.000000715
34	1.879844904	0.532322466	0.000034034	-0.000000477
35	1.879845023	0.532356501	0.000033557	-0.000000775
36	1.879845142	0.532390058	0.000032783	-0.000016332
37	1.879845262	0.532422841	0.000016451	0.000015616
38	1.879845381	0.532439291	0.000032067	-0.000000596
39	1.879845500	0.532471359	0.000031471	-0.000000477
40	1.879845619	0.532502830	0.000030994	-0.000000417
41	1.879845738	0.532533824	0.000030577	-0.000000417
42	1.879845858	0.532564402	0.000030160	-0.000000536

TABLE V

THE RERMAG OF Q^2 FOR 17 POINTS IN THE KINE3 PLOT (FIG. 14)

point	p	$T^*(T+2\Sigma m_i)$	p^2	RERMAG of Q^2	Q^2
1	1.8798411	3.5338008	3.5338025	2117406.0	-0.0000018
2	1.8798412	3.5338016	3.5338030	2470307.3	-0.0000013
3	1.8798413	3.5338025	3.5338035	3705461.5	-0.0000008
4	1.8798414	3.5338035	3.5338039	7410924.0	-0.0000004
5	1.8798416	3.5338044	3.5338044	--	0.0000001
6	1.8798417	3.5338051	3.5338047	7410926.5	0.0000003
7	1.8798418	3.5338061	3.5338051	3705464.3	0.0000008
8	1.8798419	3.5338068	3.5338056	2964372.0	0.0000012
9	1.8798420	3.5338078	3.5338061	2117409.3	0.0000017
10	1.8798422	3.5338087	3.5338066	1646874.3	0.0000022
11	1.8798423	3.5338097	3.5338070	1347442.9	0.0000026
12	1.8798424	3.5338106	3.5338075	1140144.3	0.0000031
13	1.8798425	3.5338116	3.5338080	988125.31	0.0000036
14	1.8798426	3.5338120	3.5338082	926367.63	0.0000038
15	1.8798428	3.5338130	3.5338087	823438.13	0.0000043
16	1.8798429	3.5338140	3.5338092	741094.50	0.0000047
17	1.8798430	3.5338149	3.5338097	673722.44	0.0000052

perfect. Δ^2pmax has the sign changed irregularly. Some points have positive value, some have negative value. This phenomenon indicates that pmax is not increasing uniformly as it should be.

Analyzing the subtractions in subroutine KINE3 one would find the main reason of Δ^2pmax not having the same sign. The subtraction $Q^2=T^*(T+2\Sigma m_i)-p^2$ is a dangerous statement for numerical reason. It may cause very high cancellation.

The large RERMAG of Q^2 affects the final results of pmax, making the value of pmax not changing uniformly. Table V gives the RERMAG of Q^2 of 17 points in Figure 14 which generated from KINE3. Although the RERMAG of Q^2 is very high, the value of Q^2 increased monotonically when beam momentum p is increased and the high RERMAG of Q^2 only has an effect on the second difference of maximum momentum of proton Δ^2pmax.

Now, the reason why the plot in Figure 14 is not very smooth has been revealed. Can this problem be solved?

It is believed that there is no way to avoid the subtraction $T^*(T+2\Sigma m_i)-p^2$. The equation

$$p^{*2} = \frac{(T^2 + 2T\Sigma m_i - p^2)\,(T^2 + 2T\Sigma m_i - p^2 + 4m_k(\Sigma m_i - m_k))}{4M^2}$$

is considered to be the best formula one can get to calculate p^{*2}. Only using double precision when running the same program (KINE3) can solve the problem. Figure 16

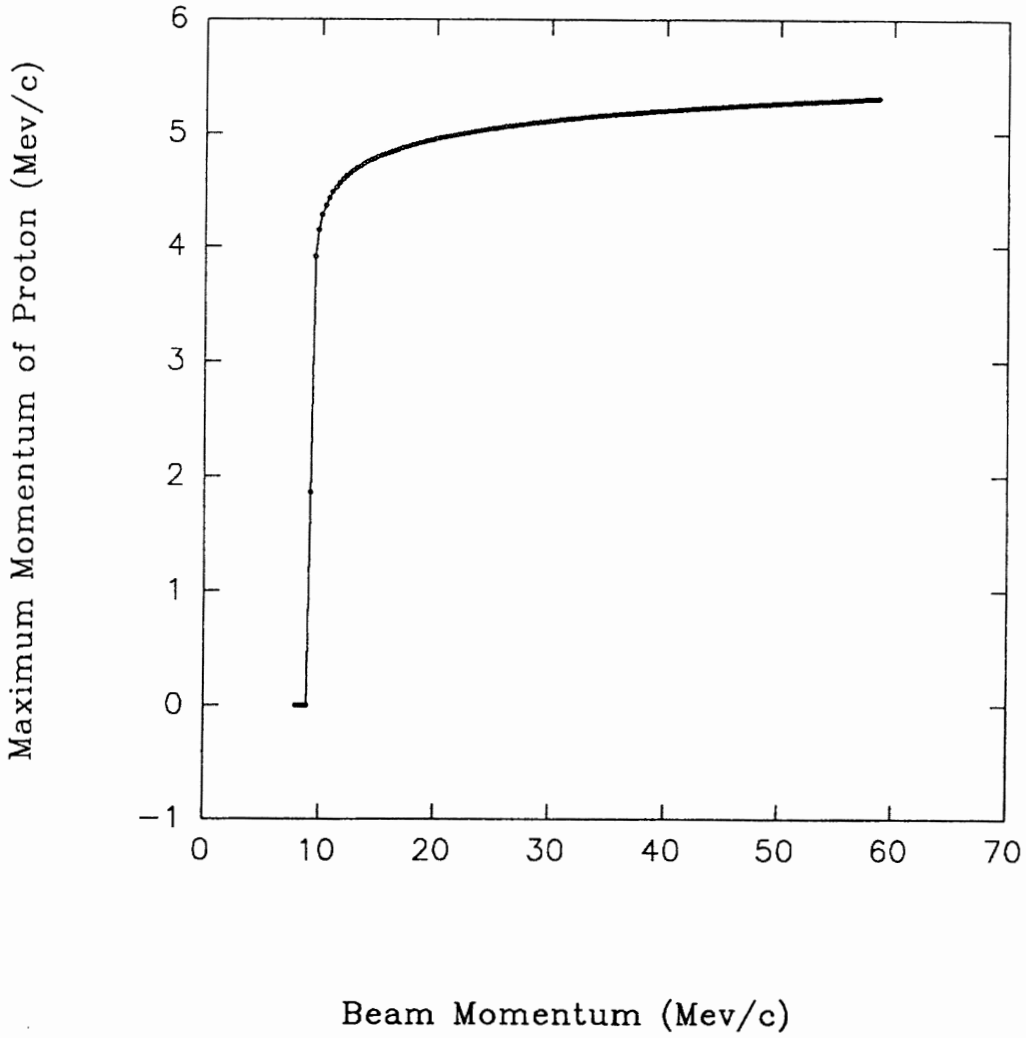


Figure 16. Maximum momentum of the proton against beam momentum in the lab system for the reaction $K^- + D \rightarrow p + \lambda + \pi^- + \pi^+ + K^- + K^0$. In the plot, $x = p \cdot 10^7 - 18798400$, $y = \log(pmax \cdot 10^8 - 53066000)$ where p is the beam momentum and pmax is the maximum momentum of the proton. The data were generated from subroutine KINE3 with double precision.

TABLE VI

The first difference and the second difference of pmax for KINE3 with double precision

point	pway	pmax	Δpmax	Δ^2pmax
1	1.879840800	0.000000000	0.000000000	0.000000000
2	1.879840830	0.000000000	0.000000000	0.000000000
3	1.879840860	0.000000000	0.000000000	0.530660725
4	1.879840890	0.000000000	0.530660725	-0.530580042
5	1.879840920	0.530660725	0.000080683	-0.000022430
6	1.879840950	0.530741408	0.000058253	-0.000010215
7	1.879840980	0.530799660	0.000048038	-0.000006202
8	1.879841010	0.530847698	0.000041835	-0.000004282
9	1.879841040	0.530889533	0.000037553	-0.000003187
10	1.879841070	0.530927086	0.000034366	-0.000002491
11	1.879841100	0.530961452	0.000031875	-0.000002017
12	1.879841130	0.530993327	0.000029858	-0.000001676
13	1.879841160	0.531023185	0.000028182	-0.000001422
14	1.879841190	0.531051367	0.000026760	-0.000001226
15	1.879841220	0.531078127	0.000025534	-0.000001072
16	1.879841250	0.531103661	0.000024462	-0.000000947
17	1.879841280	0.531128123	0.000023515	-0.000000845
18	1.879841310	0.531151638	0.000022670	-0.000000760
19	1.879841340	0.531174308	0.000021910	-0.000000688
20	1.879841370	0.531196218	0.000021222	-0.000000627
21	1.879841400	0.531217440	0.000020595	-0.000000575
22	1.879841430	0.531238036	0.000020021	-0.000000529
23	1.879841460	0.531258056	0.000019492	-0.000000489
24	1.879841490	0.531277548	0.000019003	-0.000000454
25	1.879841520	0.531296551	0.000018549	-0.000000423
26	1.879841550	0.531315100	0.000018126	-0.000000395

27	1.879841580	0.531333225	0.000017731	-0.000000370
28	1.879841610	0.531350956	0.000017360	-0.000000348
29	1.879841640	0.531368316	0.000017012	-0.000000328
30	1.879841670	0.531385328	0.000016684	-0.000000310
31	1.879841700	0.531402012	0.000016374	-0.000000293
32	1.879841730	0.531418386	0.000016081	-0.000000278
33	1.879841760	0.531434468	0.000015803	-0.000000264
34	1.879841790	0.531450271	0.000015540	-0.000000251
35	1.879841820	0.531465811	0.000015288	-0.000000239
36	1.879841850	0.531481099	0.000015049	-0.000000228
37	1.879841880	0.531496148	0.000014821	-0.000000218
38	1.879841910	0.531510969	0.000014602	-0.000000209
39	1.879841940	0.531525572	0.000014394	-0.000000200
40	1.879841970	0.531539965	0.000014193	-0.000000192
41	1.879842000	0.531554159	0.000014001	-0.000000184
42	1.879842030	0.531568160	0.000013817	-0.000000177
43	1.879842060	0.531581977	0.000013639	-0.000000171
44	1.879842090	0.531595616	0.000013469	-0.000000164
45	1.879842120	0.531609085	0.000013304	-0.000000159
46	1.879842150	0.531622389	0.000013146	-0.000000153
47	1.879842180	0.531635535	0.000012993	-0.000000148
48	1.879842210	0.531648528	0.000012845	-0.000000143
49	1.879842240	0.531661373	0.000012702	-0.000000138
50	1.879842270	0.531674075	0.000012564	-0.000000134
51	1.879842300	0.531686639	0.000012430	-0.000000130
52	1.879842330	0.531699070	0.000012301	-0.000000126
53	1.879842360	0.531711370	0.000012175	-0.000000122
54	1.879842390	0.531723546	0.000012053	-0.000000118
55	1.879842420	0.531735599	0.000011935	-0.000000115
56	1.879842450	0.531747534	0.000011820	-0.000000112
57	1.879842480	0.531759354	0.000011709	-0.000000108
58	1.879842510	0.531771063	0.000011600	-0.000000105
59	1.879842540	0.531782663	0.000011495	-0.000000103
60	1.879842570	0.531794158	0.000011392	-0.000000100

shows the plot of maximum momentum pmax against beam momentum p when applying double precision to KINE3. Table VI shows that the first differences of pmax and the second differences of pmax. It is clearly seen that the signs of Δ^2pmax do not change irregularly but keep integrity.

(2) Dalitz plot without numerical cancellation when near threshold energy.

Now, let us solve the problem of numerical cancellation at threshold energy when drawing a Dalitz plot.

First, let us try to use good variables to replace the "bad" variables in G function.

Instead of plotting the Dalitz plot in the s_1s_2 plane, we can plot the Dalitz plot on x_1x_2 plane. x_1, x_2 are defined as

$$x_1 = s_1 - (m_3 + m_4)^2$$
$$x_2 = s_2 - (m_4 + m_5)^2 \qquad (5.8)$$

x_1, x_2 are considered to be good variables, since both of them could reach the value zero.

Figure 17 is the Dalitz plot in the x_1x_2 plane. Though good variables have been used, the plot is still quite rough. This indicates that the G function contains really serious cancellation which cannot be overcome by just replacing "bad" variables by good variables.

Next, we will figure out a way of drawing the Dalitz plot which bypasses the G function.

For the reaction 1+2 $\rightarrow$ 3+4+5, the conservation law of

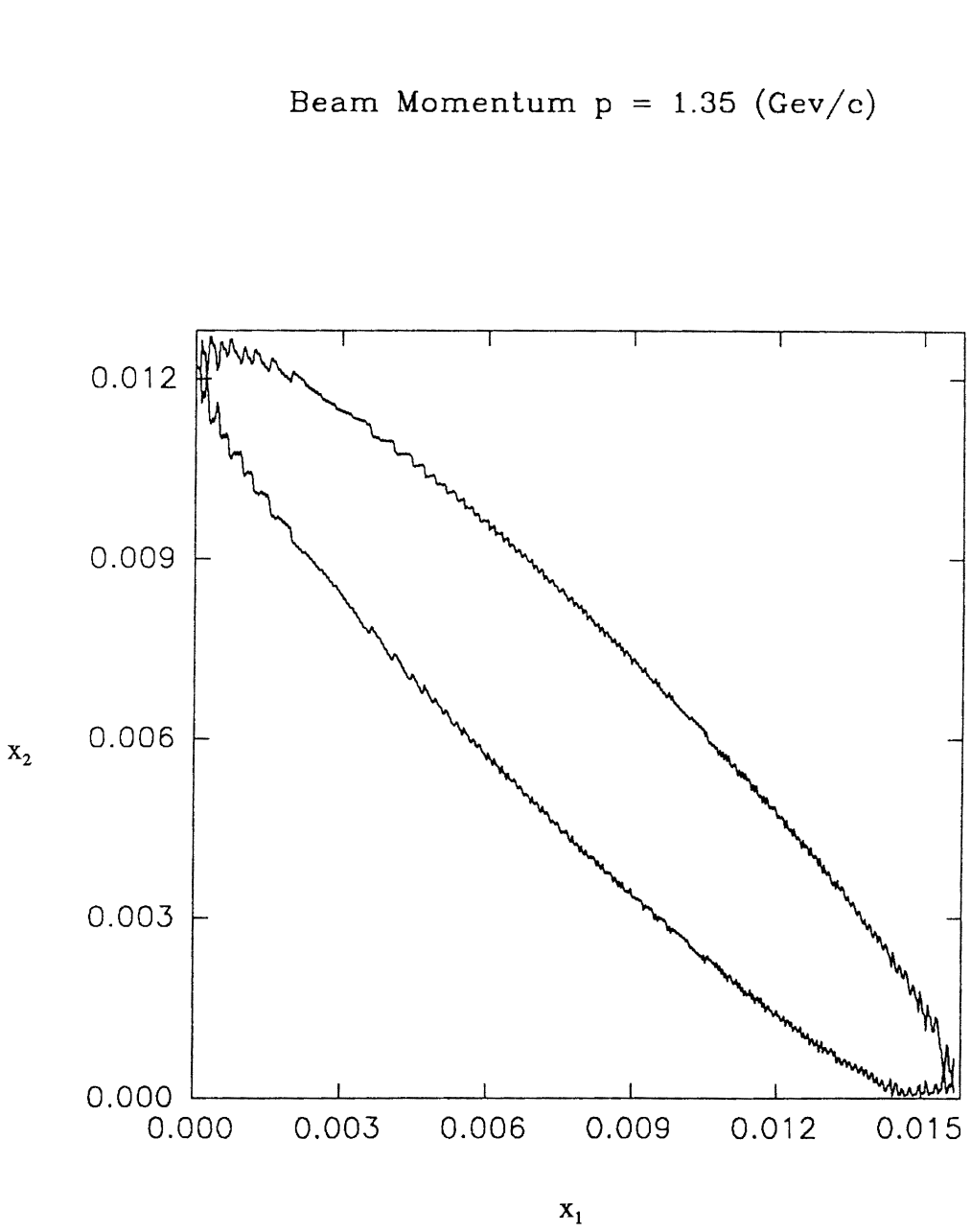


Figure 17. Dalitz plot as the physical region of $K^- + N \rightarrow K^- + \Xi + \pi$ in the x_1x_2 plane when the beam momentum is just above the threshold energy, p = 1.35 (GeV/c).

energy and momentum in the center-of-momentum system is as follows:

$$E_3 + E_4 + E_5 = \sqrt{s} \tag{5.9}$$

$$\boldsymbol{p}_3 + \boldsymbol{p}_4 + \boldsymbol{p}_5 = 0 \tag{5.10}$$

where $s=(p_1+p_2)^2$ is the total energy squared. p_1, p_2 refer to the four-momentum of particles 1 and 2.

For particle 4, it holds that

$$E_4^2 = p_4^2 + m_4^2 \tag{5.11}$$

From Equation (5.10), we have

$$\boldsymbol{p}_4 = -\boldsymbol{p}_3 - \boldsymbol{p}_5$$

Which can be written as

$$p_4^2 = p_3^2 + p_5^2 + 2p_3p_5\cos\theta_{35} \tag{5.12}$$

inserting (5.12) into (5.11), we get

$$E_4^2 = p_3^2 + p_5^2 + 2p_3p_5\cos\theta_{35} + m_4^2 \tag{5.13}$$

where the value $\cos\theta_{35}= \pm 1$ corresponds to the boundary of the physical region in the E_3E_5 plane. That is, the extreme values of E_3 and E_5 are reached when $\cos\theta_{35}= \pm 1$. This is the boundary of the Dalitz plot.

Inserting (5.13) into (5.9), we get

$$\begin{aligned}(\sqrt{s} - E_3 - E_5)^2 &= p_3^2 + p_5^2 \pm 2p_3p_5 + m_4^2 \\ &= (p_3 \pm p_5)^2 + m_4^2 \\ &= E_3^2 - m_3^2 + E_5^2 - m_5^2 \pm 2\{(E_3^2 - m_3^2)(E_5^2 - m_5^2)\}^{\frac{1}{2}} + m_4^2\end{aligned}$$

or

$$4\,(E_3^2 - m_3^2)\,(E_5^2 - m_5^2) = \{2E_3E_5 - 2\sqrt{s}\,(E_3 + E_5) + s - m_4^2 + m_3^2 + m_5^2\}^2 \tag{5.14}$$

Substituting E_3 and E_5 for the invariants s, s_1 and s_2, we get

$$s_2 = \frac{1}{2s_1}\{-s_1^2 + s_1(s + m_3^2 + m_4^2 + m_5^2) - (m_3^2 - m_4^2)(s - m_5^2)\}$$

$$\pm\frac{1}{2s_1}\{(s_1 - (m_3 - m_4)^2)(s_1 - (m_3 + m_4)^2)(s_1 - (\sqrt{s} - m_5)^2)(s_1 - (\sqrt{s} + m_4)^2)\}^{\frac{1}{2}} \tag{5.15}$$

Letting

$$A = s_1,$$

$$B = s_1^2 - s_1(s + m_3^2 + m_4^2 + m_5^2) + (m_3^2 - m_4^2)(s - m_5^2),$$

$$RAD = (s_1 - (m_3 - m_4)^2)(s_1 - (m_3 + m_4)^2)(s_1 - (\sqrt{s} - m_5)^2)(s_1 - (\sqrt{s} + m_5)^2)$$

where RAD represents the radicant, then

$$s_2^{(+)} = \frac{-B + \sqrt{RAD}}{2A}$$

$$s_2^{(-)} = \frac{-B - \sqrt{RAD}}{2A}$$

The solution of equation (5.15) exists for

(a) $s_1 \leq (m_3 - m_4)^2$

(b) $(m_3 + m_4)^2 \leq s_1 \leq (\sqrt{s} - m_5)2$

(c) $s_1 \geq (\sqrt{s} + m_5)^2$

However, for the processes under consideration, the minimum value of s_1 is $(m_3 + m_4)^2$ which occurs when particles 3 and 4 are emitted parallel with the same velocity. The maximum

value occurs when particle 5 remains at rest. In this case $(\mathbf{p}_3+\mathbf{p}_4)$ equals zero, so that s_1 is $(\sqrt{s}-m_5)^2$.

When s_1 is plotted against s_2 we get Figure 18. The Dalitz plot in Figure 18 comes out to be very smooth. Analyzing part of this plot we get Table VII which shows the data of part of the points in the Dalitz plot in Figure 18.

Comparing the column of RAD in Table VII with the same column in Table III It is clearly seen that when using G function, the value of B^2-4AC is jumping around. Whereas the value of RAD in Table VII is increasing monotonically as s_1 increases. This shows that the way of getting B^2-4AC in subroutine DALNG (see Appendix D) is a better approach. In this way the numerical cancellation could be avoided.

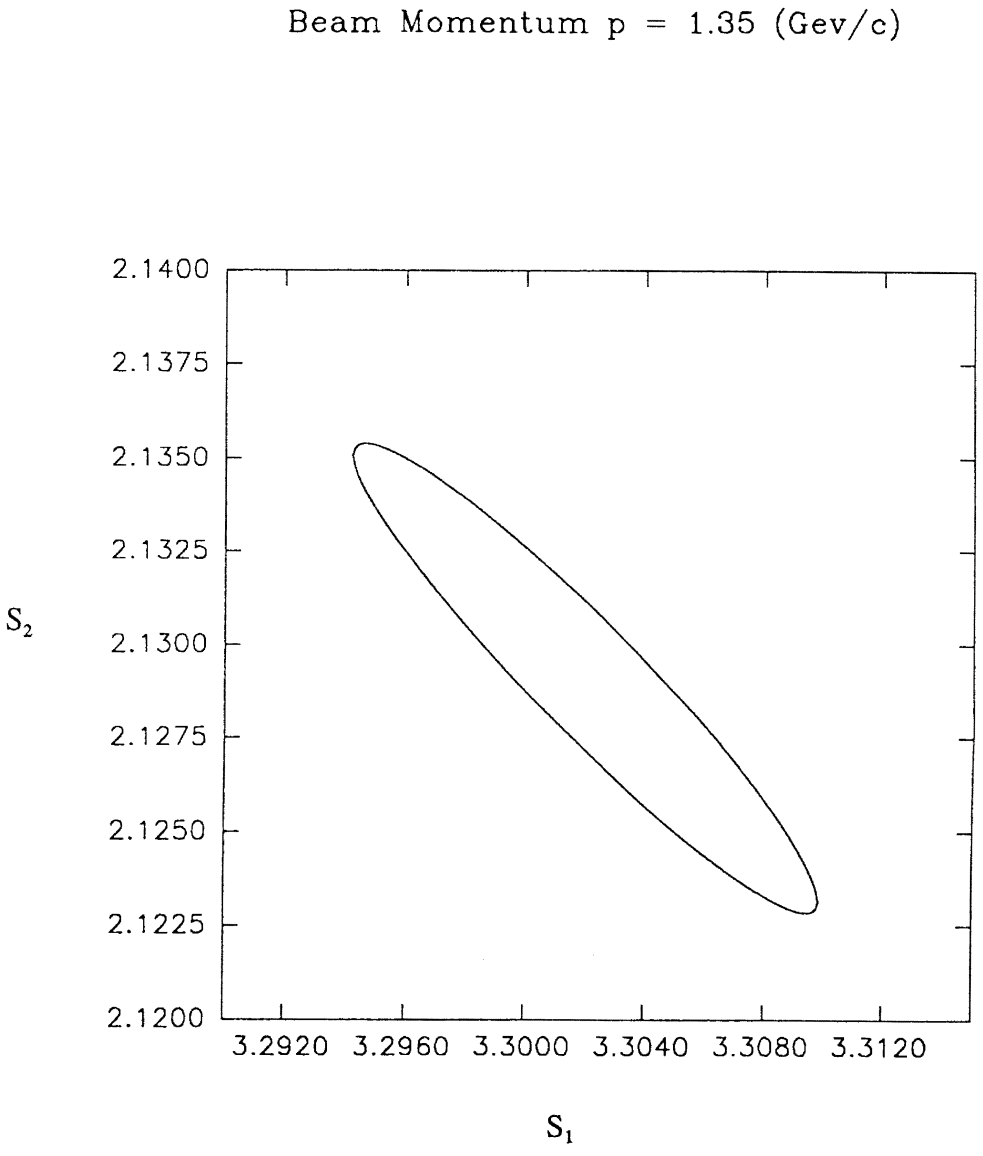


Figure 18. Dalitz plot as the physical region of $K^- + N \rightarrow K^- + \Xi + \pi$ in the $s_1 s_2$ plane when the beam momentum is just above the threshold energy, p = 1.35 (GeV/c). Function G was not used in drawing this Dalitz plot.

TABLE VII

PART OF THE POINTS IN THE DALITZ PLOT

S_{34}	RAD	$S_{45}^{(+)}$	$S_{45}^{(-)}$
3.2942221	0.0000000	2.1233864	2.1233864
3.2942262	0.0000000	2.1234066	2.1233935
3.2942302	0.0000000	2.1234121	2.1233821
3.2942343	0.0000000	2.1234143	2.1233740
3.2942381	0.0000000	2.1234150	2.1233671
3.2942421	0.0000000	2.1234155	2.1233606
3.2942462	0.0000000	2.1234152	2.1233544
3.2942502	0.0000000	2.1234148	2.1233485
3.2942543	0.0000001	2.1234143	2.1233430
3.2942581	0.0000001	2.1234136	2.1233380
3.2942622	0.0000001	2.1234126	2.1233327
3.2942662	0.0000001	2.1234114	2.1233277
3.2942703	0.0000001	2.1234102	2.1233227
3.2942741	0.0000001	2.1234090	2.1233182
3.2942781	0.0000001	2.1234078	2.1233134
3.2942822	0.0000001	2.1234064	2.1233087
3.2942863	0.0000001	2.1234047	2.1233041
3.2942901	0.0000001	2.1234031	2.1232996
3.2942941	0.0000001	2.1234016	2.1232953
3.2942982	0.0000001	2.1233997	2.1232908
3.2943022	0.0000001	2.1233981	2.1232862
3.2943063	0.0000001	2.1233962	2.1232820
3.2943101	0.0000001	2.1233945	2.1232779
3.2943141	0.0000002	2.1233926	2.1232736
3.2943182	0.0000002	2.1233904	2.1232693
3.2943223	0.0000002	2.1233885	2.1232653
3.2943261	0.0000002	2.1233866	2.1232615

CHAPTER VI

SUMMARY AND CONCLUSIONS

In this work, a method was developed for avoiding numerical cancellation at low energy in calculations in relativistic kinematics. Since most of the serious errors in computer calculation are the consequence of subtractive cancellation, this situation should be considered and avoided.

A general rule for handling subtractive cancellation could is as following.

a. Determine which subtraction in the computation is causing the drastic loss of accuracy. The analysis of RERMAG for each subtraction would help doing the job. RERMAG represents the relative error magnification factor. The RERMAG of x-y is defined as

$$RERMAG = \frac{\max(|x| , |y|)}{|x-y|} .$$

Any large value of RERMAG indicates trouble. RERMAG=10 indicates cancellation of one decimal digit; RERMAG=100 indicates loss of two digits, etc.

b. Reformulate the equations that contain the subtractive cancellation by using the appropriate variables and by applying derationalization (the "square root trick") to the

conventionally formulated equations.

c. Apply the accuracy checklist to each computation, minimizing the computing errors.

The subtractive cancellation can be avoided to a certain degree, though it is not possible to avoid cancellation entirely. The method developed was applied to several situations successfully.

The first application was the computation of kinematic limits on the momentum and production angle of one particle from a system of particles. Using the method developed in this work the problem of numerical cancellation at threshold energy has been solved to a certain degree. Cancellation could not be avoided entirely due to the inevitable use of momentum p and kinetic energy T measured in lab system.

Another application of this method was to solve the problem of numerical cancellation at threshold energy in drawing a Dalitz plot. The Dalitz plot derived after using this method was very smooth. The roughness of the plot obtained from the G function which contains big cancellation was eliminated.

Since relativistic kinematics has wide applications in atomic physics and nuclear physics, this method of avoiding numerical cancellation at low energy could be applied to many problems.

BIBLIOGRAPHY

(1) Evans, Robley D. The Atomic Nucleus. New York: McGraw-Hill Book Company, Inc. 1955

(2) Byckling, E. and Kajantie, K. Particle Kinematics. New York: John Wiley & Sons, Inc. 1973

(3) Hagedorn, R. Relativistic Kinematics. New York: W.A.Benjamin, Inc. 1963

(4) Feld, Bernard T. Models of Elementary Particles. Waltham, Massachusetts: Blaisdell Publishing Company. 1969

(5) Morrison, P. A Survey of Nuclear Reactions. Part VI in Seger, E., Editor. Experimental Nuclear Physics (Volume II). New York: John Wiley & Sons, Inc. 1953

(6) Kopylov, G. I. Elementary Kinematics of Elementary Particles. Moscow: Mir Publishers. 1983

(7) Sternheimer, R. M. Kinematic Criterion for Meson Production in Fundamental Particle Collisions. Physics Review, 93, pp. 642-643

(8) Baldin, A. M. and Gol'danskii, V. I. and Rozenthal, I. L. Kinematics of Nuclear Reactions. New York: Pergamon Press. 1961

(9) Dahlquist, G. and Bjorck, A. Numerical Methods. New Jersey: Prentice-Hall, Inc. 1974

(10) Shampine, L. F. and Allen, R. C. Jr. Numerical Computing: An Introduction. Philadelphia: W.B. Saunders Company. 1973

(11) Kajantie, K. and Lindblom, P. Physical Region on the Plane of Two Invariant Momentum Transfers for a Reaction with Three Particles in the Final State. Physical Review, Volume 175, Number 5, pp. 2203-2213

(12) Acton, F. S. Numerical Methods That Works. Harper & Row, Publishers. 1970

APPENDIX A

PROGRAM LISTING OF KINE1

```
C*****************************************************************
C
C  KINE1                  JUNE, 1994
C
C  A.N.S.I. STANDARD FORTRAN 77
C
C  J. P. CHANDLER, COMPUTER SCIENCE DEPARTMENT,
C     OKLAHOMA STATE UNIVERSITY
C  YUMEI ZHAO, COMPUTER SCIENCE DEPARTMENT,
C     OKLAHOMA STATE UNIVERSITY
C
C*****************************************************************
C
C*****************************************************************
C
C  KINE1 COMPUTES THE KINEMATIC LIMITS ON THE MOMENTUM AND
C  PRODUCTION ANGLE OF ONE PARTICLE FROM A SYSTEM OF PARTICLES.
C
C  PWAY   --  ARRAY OF BEAM MOMENTUM
C  Y      --  ARRAY OF MAXIMUM MOMENTUM OF THE PROTON
C  Z      --  ARRAY OF MINIMUM MOMENTUM OF THE PROTON
C  EM     --  ARRAY OF MASS OF OFFSPRING PARTICLES
C  NPTS   --  NUMBER OF POINTS IN THE SET
C  KP     --  NUMBER OF OFFSPRING PARTICLES
C  PMIN   --  MINIMUM VALUE OF MOMENTUM OF PROTON SET
C  PMAX   --  MAXIMUM VALUE OF MOMENTUM OF PROTON SET
C  EP     --  TOTAL ENERGY
C  TANMAX --  THE TANGENT OF MAXIMUM PRODUCTION ANGLE
C
C*****************************************************************
C
C  All VARIABLES ARE SET TO BE SINGLE PRECISION
C
      DIMENSION Y(1001)
      DIMENSION Z(1001)
      DIMENSION PWAY(1001)
      DIMENSION EM(10)
C
C  OPEN OUTPUT FILE
      OPEN(5,FILE='KINE1_OUT',STATUS='NEW')
C  OPEN A FILE FOR RELATIVE ERROR MAGNIFICATION FACTOR
      OPEN (7,FILE = 'RERMAG.K1', STATUS = 'NEW')
C
C
      NPTS=1001
      IN=5
      LP=6
      COSINE=1.0
C
C  THE REACTION IS K- + D -> P + LAMBDAO + PI- + PI+ + K- + K0
C
      KP=6
      EM(1)=.938213
      EM(2)=1.11536
      EM(3)=.13959
      EM(4)=.13959
      EM(5)=.4939
      EM(6)=.4978
C
      PMIN=1.87983
      PMAX=1.87986
C
      DO 2 I=1,NPTS
C
      KCH=1
      NGIVEN=1
C
      PWAY(I)=PMIN+(PMAX-PMIN)*(I-1)/FLOAT(NPTS-1)
      EP = 1.875494+SQRT(0.4939**2+PWAY(I)**2)
      CALL KINE1 (KP,EM,EP,PWAY(I),KCH,NGIVEN,COSINE,Y(I),
     *   Z(I),TANMAX)
    2 WRITE(5,22) PWAY(I),Y(I),Z(I),COSINE,TANMAX
   22 FORMAT(1X, 5E16.9)
C
      END
C
```

```
C
      SUBROUTINE DIAGNO(JPOINT,X,Y)
C
C  COMPUTE AND PRINT THE RELATIVE ERROR MAGNIFICATION FACTOR
C  FOR ONE FLOATING POINT SUBTRACTION.
C
      HUGE=1.0E30
      RZERO=0.0
C
      IF(X-Y.EQ.RZERO) THEN
         RERMAG=HUGE
      ELSE
         RERMAG=AMAX1(ABS(X),ABS(Y))/ABS(X-Y)
      ENDIF
C
      WRITE(7,77)JPOINT,X,Y,RERMAG
   77 FORMAT(' AT POINT',I2,' X =',F15.7,3X,'Y =',F15.7,3X,
     *       'RERMAG =',E15.7)
C
      RETURN
C
      END
C
C
      SUBROUTINE KINE1(KP,EM,EP,PWAY,KCH,NGIVEN,COSINE,PMAX,PMIN,TANMAX)
C
C  KINE1 -- COMPUTES EXTREMA OF MOMENTUM AND PRODUCTION ANGLE.
C  'KINE1' COMPUTES THE KINEMATIC LIMITS ON THE MOMENTUM AND PRODUCTION
C  ANGLE OF ONE PARTICLE FROM A SYSTEM OF PARTICLES HAVING A TOTAL
C  ENERGY EP AND TOTAL MOMENTUM OF MAGNITUDE PWAY.
C  A SYSTEM OF UNITS MUST BE USED IN WHICH C=1.0 (E.G., MEV AND MEV/C).
C  THE MOMENTUM LIMITS ARE EITHER FOR ANY ANGLE (NGIVEN=0) OR FOR AN
C  ANGLE SPECIFIED BY ITS COSINE (NGIVEN=1).
C  THERE ARE A TOTAL OF KP PARTICLES.  THE LIMITS ARE FOR PARTICLE KCH.
C  THE PARTICLE MASSES ARE EXPECTED TO BE IN THE FIRST KP LOCATIONS OF
C  THE ARRAY EM.
C  TANMAX RETURNS THE TANGENT OF THE MAXIMUM PRODUCTION ANGLE (ANGLES
C  ARE MEASURED WITH RESPECT TO THE TOTAL MOMENTUM OF THE SYSTEM).
C  THE METHOD IS TO SOLVE SIMULTANEOUSLY THE EQUATIONS FOR THE RAY AT
C  THE GIVEN ANGLE AND FOR THE ELLIPSE OUTLINING THE REGION OF ALLOWED
C  MOMENTUM.  THE REVERSE DIRECTION ALONG A RAY IS NOT CONSIDERED.
C  FOR KP=2 , ONLY POINTS LYING ON THE ELLIPSE ARE PHYSICALLY ALLOWED.
C  IF THE RAY DOES NOT INTERSECT THE ELLIPSE, PMAX IS SET EQUAL TO -1.
C  IF THE ELLIPSE ENCIRCLES THE ORIGIN, TANMAX IS SET EQUAL TO -1.0 (AN
C  OTHERWISE IMPOSSIBLE VALUE) AND, IF KP IS NOT EQUAL TO TWO, PMIN IS
C  SET EQUAL TO ZERO.
C  IF THE INPUT QUANTITIES ARE UNPHYSICAL OR ARE OTHERWISE IN ERROR,
C  EXIT OCCURS WITH PMAX=PMIN=0.0 AND TANMAX=-2.0 .
C  REFERENCES.... STERNHEIMER LETTER, PHYS. REV. 93, 642
C             .... 'RELATIVISTIC KINEMATICS' BY HAGEDORN (BENJAMIN)

      DIMENSION EM(10)
      REAL  EMSUM,EP,PWAY,ESQ,PESQY,ESQIE,PSTAR,ESTAR,BETA
C
      IF(KP)250,250,10
   10 IF(KCH)250,250,20
   20 IF(KCH-KP)30,30,250
   30 IF(EP)250,250,40
   40 IF(PWAY)250,50,50
C
C  COMPUTE THE MAXIMUM MOMENTUM SQUARED OF PARTICLE KCH IN THE CMS.
C
   50 EMSUM=0.
      DO 60 I=1,KP
      IF(EM(I))250,60,60
   60 EMSUM=EMSUM+EM(I)
C
C  COMPUTE THE RELATIVE ERROR MAGNIFICATION FACTOR OF ESQ
C
      CALL DIAGNO(1,EP**2,PWAY**2)
C
      ESQ=EP**2-PWAY**2
      IF(ESQ)250,250,70
C
C  COMPUTE THE RELATIVE ERROR MAGNIFICATION FACTOR OF ESQ-EMSUM**2
C
   70 CALL DIAGNO(2,ESQ,EMSUM**2)
C
```

```
155         PESQY=(ESQ-EMSUM**2)*(ESQ-(EMSUM-2.*EM(KCH))**2)/(4.*ESQ)
156   C
157         WRITE(7,71) PESQY
158      71 FORMAT(1X,F15.7)
159   C
160         IF(PESQY)250,80,80
161   C
162      80 ESQIE=PESQY+EM(KCH)**2
163         PSTAR=SQRT(PESQY)
164         ESTAR=SQRT(ESQIE)
165         BETA=PWAY/EP
166         GAMMA=EP/SQRT(ESQ)
167         IF(NGIVEN)90,130,90
168      90 IF(ABS(COSINE)-1.)100,100,250
169     100 BECUZ=BETA*COSINE
170   C
171   C  COMPUTE THE RELATIVE ERROR MAGNIFICATION FACTOR OF AYE
172   C
173         CALL DIAGNO(3,1.0,BECUZ**2)
174   C
175         AYE=1.0-BECUZ**2
176         BEE=-2.0*BECUZ*ESTAR/GAMMA
177         SEA=BETA**2*ESQIE-PESQY
178   C
179   C  COMPUTE THE RELATIVE ERROR MAGNIFICATION FACTOR OF ARG
180   C
181         CALL DIAGNO(4,BEE**2,4.0*AYE*SEA)
182   C
183         ARG=BEE**2-4.0*AYE*SEA
184         IF(ARG)230,110,110
185
186   C  THE RAY INTERSECTS THE ELLIPSE, BUT PERHAPS IN THE REVERSE DIRECTION.
187
188     110 TWAIN=2.0*AYE
189         IF(TWAIN)120,250,120
190     120 TOTHER=-BEE/TWAIN
191         ONE=+SQRT(ARG)/TWAIN
192         GO TO 140
193     130 ONE=+GAMMA*PSTAR
194         TOTHER=+BETA*GAMMA*ESTAR
195   C
196   C  COMPUTE THE RELATIVE ERROR MAGNIFICATION FACTOR OF PMIN
197   C
198     140 CALL DIAGNO(5,ONE,TOTHER)
199   C
200         PMAX=ONE+TOTHER
201   C
202         WRITE(7,77) PMAX
203      77 FORMAT(3X, G15.7)
204   C
205         IF(PMAX)230,150,150
206     150 PMIN=-ONE+TOTHER
207         IF(KP-2)160,160,190
208     160 IF(PMIN)170,220,240
209     170 IF(NGIVEN)180,210,180
210     180 PMIN=PMAX
211         GO TO 220
212     190 PMIN=MAX(PMIN,0.)
213         IF(PMIN)250,220,240
214     210 PMIN=ABS(PMIN)
215     220 TANMAX=-1.0
216         GO TO 270
217
218   C  THE RAY DOES NOT INTERSECT THE ELLIPSE.
219
220     230 PMAX=-1.0
221         PMIN=+1.0
222
223     240 VSTAR=PSTAR/ESTAR
224   C
225   C  COMPUTE THE RELATIVE ERROR MAGNIFICATION FACTOR OF ARG
226   C
227         CALL DIAGNO(6,BETA**2,VSTAR**2)
228   C
229         ARG=BETA**2-VSTAR**2
230         IF(ARG)250,250,260
231
232   C  ERROR EXIT.
```

```
233
234     250 PMAX=0.0
235         PMIN=0.0
236         TANMAX=-2.0
237         GO TO 270
238
239     260 TANMAX=VSTAR/(GAMMA*SQRT(ARG))
240     270 RETURN
241         END
242
```

APPENDIX B

PROGRAM LISTING OF DALG

```
C*************************************************************
C
C  DALG                    JUNE, 1994
C
C  A.N.S.I. STANDARD FORTRAN 77
C
C  J. P. CHANDLER, COMPUTER SCIENCE DEPARTMENT,
C     OKLAHOMA STATE UNIVERSITY
C  YUMEI ZHAO, COMPUTER SCIENCE DEPARTMENT,
C     OKLAHOMA STATE UNIVERSITY
C
C*************************************************************
C
C*************************************************************
C
C  DALG GENERATES DATA FOR DALITZ PLOT OF REACTION 1+2 -> 3+4+5.
C  THE DALITZ PLOT IS GOT FORM G FUNCTION.
C
C  S34     --  ARRAY OF TOTAL ENERGY SQUARED OF PARTICLES 3 AND 4
C              IN THEIR OWN CMS
C  S45     --  ARRAY OF TOTAL ENERGY SQUARED OF PARTICLES 4 AND 5
C              IN THEIR OWN CMS
C  RSTEP   --  INCREMENT OF X AXIS
C  NPTS    --  NUMBER OF POINTS IN THE SET
C  P       --  BEAM MOMENTUM
C  S       --  TOTAL ENERGY SQUARED IN CMS
C  S34N    --  MINIMUM VALUE OF S34
C  S34X    --  MAXIMUM VALUE OF S34
C
C*************************************************************
C
C  All VARIABLES ARE SET TO BE SINGLE PRECISION
C
      DIMENSION S34(501,2),S45(501,2)
C
C  OPEN A FILE FOR OUTPUT DATA
      OPEN (4,FILE = 'FILE.DAL', STATUS = 'NEW')
C  OPEN A FILE FOR RELATIVE ERROR MAGNIFICATION FACTOR
      OPEN (7,FILE = 'GRERMAG.DAL', STATUS = 'NEW')
C
      LP=6
      RZERO=0.0
      RSTEP=0.001
C
      NPT=501
      EM1=0.495
      EM2=0.939
      EM3=0.495
      EM4=1.32
      EM5=0.137
C     P=2.24
C  THE REACTION IS K- + N -> K- + CASCADE + PI
C  EM1+EM2 < EM3+EM4+EM5, THE REACTION IS ENDOENERGETIC
C  THE THRESHOLD ENERGY IS 1.4289446. WHILE N IS AT REST, THE TOTAL LAB
C  MOMENTUM IS SQRT(E**2-EM1**2) = 1.3404692.
      P=1.35
      WRITE(4,44) P
   44 FORMAT(1X, F5.3)
      S=(EM2+SQRT(EM1**2+P**2))**2-P**2
C
      SLOW=2.0
      S34N=GTXMIN(SLOW,RZERO,S,EM4,EM3,EM5)
      IF(S34N.LT.RZERO) GO TO 3
      SHIG=S34N+RSTEP
      S34X=GTXMAX(SHIG,RZERO,S,EM4,EM3,EM5)
      IF(S34X.LE.S34N) GO TO 3
C
      DO 1 J=1,NPT
      S34(J,1)=S34N+(S34X-S34N)*(J-1)/FLOAT(NPT-1)
      GG=CHANG(S34(J,1),RZERO,S,EM4**2,EM3**2,EM5**2,0,
     *   S45(J,1),S45(J,2),KOMPLX)
      FF=  CHANG(S34(J,1),(S45(J,1)+S45(J,2))/2.0 ,S,EM4**2,EM3**2,
     X   EM5**2,0,XYZX,XYZN,KK)
    1 WRITE(4, 2) S34(J,1), S45(J,1), S45(J,2), KOMPLX, FF, GG
    2 FORMAT(1X, 3E15.6, I6, 2E15.6)
C
    3 STOP
      END
```

```
C
C
      SUBROUTINE DIAGNO(JPOINT,X,Y)
C  COMPUTE AND PRINT THE RELATIVE ERROR MAGNIFICATION FACTOR
C  FOR ONE FLOATING POINT SUBTRACTION.
C
      HUGE=1.0E30
      RZERO=0.0
C
      IF(X-Y.EQ.RZERO) THEN
         RERMAG=HUGE
      ELSE
         RERMAG=AMAX1(ABS(X),ABS(Y))/ABS(X-Y)
      ENDIF
C
      WRITE(7,77)JPOINT,X,Y,RERMAG
   77 FORMAT(' AT POINT',I2,' X =',F15.7,3X,'Y =',F15.7,3X,
     *       'RERMAG =',E15.7)
C
      RETURN
C
      END
C
C
      REAL FUNCTION CHANG(X,Y,Z,U,V,W,JXYZ,
     *   XYZMAX,XYZMIN,KOMPLX)
C
C  CHANG EVALUATES THE FUNCTION G(X,Y,Z,U,V,W) AND SOLVES G=0 FOR
C  X, Y, OR Z (ACCORDINGLY AS JXYZ IS NEGATIVE, ZERO, OR POSITIVE).
C  XYZMAX AND XYZMIN RETURN THE TWO SOLUTIONS OF G=0.
C  IF THERE ARE NO REAL ROOTS OF G=0, KOMPLX IS RETURNED NONZERO.
C
C  CHAN HONG-MO, K. KAJANTIE, AND G. RANFT, NUOVO CIMENTO 49 (1967) 178
C  SEE ALSO K. KAJANTIE AND P. LINDBLOM, PHYS. REV. 175 (1968) 2203.
C
C  CONSIDER A PROCESS WITH THREE PARTICLES IN THE FINAL STATE,
C                  1+2 GOES TO 3+4+5 .
C  G(S34,S45,S,M4**2,M3**2,M5**2)=0 DEFINES THE BOUNDARY OF A
C  DALITZ PLOT (PLUS AT LEAST ONE OTHER, UNPHYSICAL, CLOSED CURVE).
C  G IS NEGATIVE INSIDE THE BOUNDARY OF THE DALITZ PLOT.
C  G(T13,S,S45,M1**2,M2**2,M3**2)=0 DEFINES THE CURVILINEAR PORTION
C  OF THE BOUNDARY OF A CHEW-LOW PLOT.
C
C  J. P. CHANDLER, DEPT. OF PHYSICS, INDIANA UNIVERSITY
C
C
C     IMPLICIT REAL*8 (A-H,O-Z)
C
C                              EVALUATE G(X,Y,Z,U,V,W) .
C
      CHANG= X*Y*(X+Y-Z-U-V-W)+
     *       Z*U*(Z+U-X-Y-V-W)+
     *       V*W*(V+W-X-Y-Z-U)+
     *       X*(Z*W+U*V)+
     *       Y*(Z*V+U*W)
C
      RZERO=0.0
C
      IF(JXYZ)10,20,30
C                              SOLVE FOR X.
   10 D=Y
      E=U
      F=W
      G=V
      H=Z
      GO TO 40
C                              SOLVE FOR Y.
   20 D=X
      E=U
      F=V
      G=W
      H=Z
      GO TO 40
C                              SOLVE FOR Z.
   30 D=U
      E=Y
      F=W
      G=V
```

```
157         H=X
158   C                              SET UP AND SOLVE THE QUADRATIC EQUATION.
159      40 A=D
160         B=D*(D-E-F)+G*(E-D-F)+H*(F-D-E)
161         C=(E*H-F*G)*(E+H-F-G)+D*(E-G)*(F-H)
162         KOMPLX=0
163   C
164         WRITE(7,71) A
165      71 FORMAT(1X, F15.7)
166   C
167         IF(A.NE.0.) GO TO 60
168         IF(B.NE.0.) GO TO 50
169         XYZMAX=RZERO
170         XYZMIN=RZERO
171         KOMPLX=-1
172   C                            NO SOLUTION
173         GO TO 80
174   C
175      50 XYZMAX=-C/B
176         XYZMIN=XYZMAX
177   C                            ONE SOLUTION
178         KOMPLX=1
179         GO TO 80
180   C
181   C  CHECK THE RELATIVE ERROR MAGNIFICATION FACTOR
182   C
183      60 CALL DIAGNO(1,B**2,4.0*A*C)
184   C
185         ARG=B**2-4.0 *A*C
186   C
187         WRITE(7,72) ARG
188      72 FORMAT(1X, F15.7)
189   C
190         IF(ARG.GT.RZERO) GO TO 70
191         KOMPLX=1
192         ARG=RZERO
193   C                            ONE SOLUTION
194         XYZMAX=-B/(2.0D0*A)
195         XYZMIN=XYZMAX
196   C
197         WRITE(7,73) XYZMAX, XYZMIN
198      73 FORMAT(1X, 2F15.7)
199   C
200         GO TO 80
201      70 ROOT=SQRT(ARG)
202   C
203   C  CHECK THE RELATIVE ERROR MAGNIFICATION FACTOR
204   C
205         CALL DIAGNO(2,B,ROOT)
206   C
207         XYZMAX=(-B+ROOT)/(2.0 *A)
208         XYZMIN=(-B-ROOT)/(2.0 *A)
209   C
210         WRITE(7,77) XYZMAX, XYZMIN
211      77 FORMAT(1X, 2F15.7)
212   C
213         KOMPLX=0
214   C                            TWO SOLUTIONS
215      80 RETURN
216   C
217   C  END CHANG
218   C
219         END
220   C
221   C
222   C
223         REAL FUNCTION GTXMIN(SLOW,RZERO,S,EM4,EM3,EM5)
224   C
225   C  GETXMIN WILL GET THE MINIMUM S34 VALUE OF THE DALITZ PLOT.
226   C
227   C  YUMEI ZHAO, COMPUTER SCIENCE DEPARTMENT,
228   C     OKLAHOMA STATE UNIVERSITY
229   C
230   C     IMPLICIT REAL*8 (A-H,O-Z)
231   C     DOUBLE PRECISION CHANG
232         RMINUS=-1.0
233         RLIMIT=20.0
234         RSTEP=0.001
```

```
C                          DUMMY IS USELESS THERE
  100 DUMMY=CHANG(SLOW,RZERO,S,EM4**2,EM3**2,EM5**2,0,Y1,Y2,KOMPLX)
      IF(KOMPLX.GT.0) GO TO 200
      IF(KOMPLX.EQ.0) GO TO 300
      PRINT *, 'NO SOLUTION'
      SMIN=RMINUS
      GO TO 500
C
C                          ONE SOLUTION
  200 SLOW=SLOW+RSTEP
      IF(SLOW.GE.RLIMIT) GO TO 400
      GO TO 100
C
C                          TWO SOLUTION
C                          FOUND THE MINIMUM S
  300 SMIN=SLOW-RSTEP
      GO TO 500
C
C                          OUT OF THE LIMIT
  400 PRINT *, 'NO CURVE'
      SMIN=RMINUS
      GO TO 500
C
  500 GTXMIN=SMIN
      RETURN
C
      END
C
C
      REAL FUNCTION GTXMAX(SHIG,RZERO,S,EM4,EM3,EM5)
C
C  GETXMAX WILL GET THE MAXIMUM S34 VALUE OF THE DALITZ PLOT.
C
C     IMPLICIT REAL*8 (A-H,O-Z)
      RMINUS=-1.0
      RLIMIT=40.0
      RSTEP=0.001
C                          DUMMY IS USELESS THERE
   11 DUMMY=CHANG(SHIG,RZERO,S,EM4**2,EM3**2,EM5**2,0,Y1,Y2,KOMPLX)
      IF(KOMPLX.EQ.0) GO TO 22
      IF(KOMPLX.GT.0) GO TO 33
      PRINT *, 'NO SOLUTION'
      SMAX=RMINUS
      GO TO 55
C
C                          TWO SOLUTION
   22 SHIG=SHIG+RSTEP
      IF(SHIG.GE.RLIMIT) GO TO 44
      GO TO 11
C
C                          ONE SOLUTION
C                          FOUND THE MAXIMUM S34
   33 SMAX=SHIG
      GO TO 55
C
C                          OUT OF THE LIMIT
   44 PRINT *, 'NO CURVE'
      SMAX=RMINUS
      GO TO 55
C
   55 GTXMAX=SMAX
      RETURN
C
      END
```

APPENDIX C

PROGRAM LISTING OF KINE3

```
C*****************************************************************
C
C  KINE3                  JUNE, 1994
C
C  A.N.S.I. STANDARD FORTRAN 77
C
C  J. P. CHANDLER, COMPUTER SCIENCE DEPARTMENT,
C     OKLAHOMA STATE UNIVERSITY
C  YUMEI ZHAO, COMPUTER SCIENCE DEPARTMENT,
C     OKLAHOMA STATE UNIVERSITY
C
C*****************************************************************
C
C*****************************************************************
C
C  KINE3 COMPUTES THE KINEMATIC LIMITS ON THE MOMENTUM AND
C  PRODUCTION ANGLE OF ONE PARTICLE FROM A SYSTEM OF PARTICLES.
C
C  PWAY    --  ARRAY OF BEAM MOMENTUM
C  TT      --  ARRAY OF KINETIC ENERGY
C  Y       --  ARRAY OF MAXIMUM MOMENTUM OF THE PROTON
C  Z       --  ARRAY OF MINIMUM MOMENTUM OF THE PROTON
C  EM      --  ARRAY OF MASS OF OFFSPRING PARTICLES
C  X1      --  ARRAY OF FIRST DIFFERENCE OF PWAY
C  X2      --  ARRAY OF SECOND DIFFERENCE OF PWAY
C  D1      --  ARRAY OF FIRST DIFFERENCE OF MAXIMUM MOMENTUM
C              OF THE PROTON
C  D2      --  ARRAY OF SECOND DIFFERENCE OF MAXIMUM MOMENTUM
C              OF THE PROTON
C  NPTS    --  NUMBER OF POINTS IN THE SET
C  KP      --  NUMBER OF OFFSPRING PARTICLES
C  PMIN    --  MINIMUM VALUE OF MOMENTUM OF PROTON SET
C  PMAX    --  MAXIMUM VALUE OF MOMENTUM OF PROTON SET
C  EP      --  TOTAL ENERGY
C  TANMAX --  THE TANGENT OF MAXIMUM PRODUCTION ANGLE
C
C*****************************************************************
C
C  ALL VARIABLES ARE SET TO BE SINGLE PRECISION
C
      DIMENSION Y(1001)
      DIMENSION Z(1001)
      DIMENSION PWAY(1001)
      DIMENSION TT(1001)
      DIMENSION EM(10)
      DIMENSION X(1000), X1(1000), X2(999)
      DIMENSION D(1000), D1(1000), D2(999)
C
      OPEN(5,FILE='KINE3_OUT',STATUS='NEW')
      OPEN(7,FILE='DIF_OUT',STATUS='NEW')
      OPEN(8,FILE='TABL_OUT',STATUS='NEW')
      OPEN(9,FILE='RER_QQ',STATUS='NEW')
C
      NPTS=1001
C     IN=5
C     LP=6
      COSINE=1.0D0
C
C  THE REACTION IS K- + D -> P + LAMBDAO + PI- + PI+ + K- + K0
C
      KP=6
      EM(1)=.938213
      EM(2)=1.11536
      EM(3)=.13959
      EM(4)=.13959
      EM(5)=.4939
      EM(6)=.4978
C
C  FOR THE SIX MASSES ABOVE, EMSUM=3.324453
C
C  EMD = MASS OF PARTICLE D
C  EMKMI = MASS OF PARTICLE K-
C
      EMD=1.875494
      EMKMI=0.4939
C
   50 EMSUM=0.0
      DO 60 I=1,KP
```

```
      IF(EM(I))270,60,60
   60 EMSUM=EMSUM+EM(I)
C
      EMK=EM(1)
      EMR=EMSUM-EM(1)
C
      PMIN=1.87983D0
      PMAX=1.87986D0
C
      DO 2 I=1,NPTS
C
      PWAY(I)=PMIN+(PMAX-PMIN)*(I-1)/FLOAT(NPTS-1)
      EP = EMD+SQRT(EMKMI**2+PWAY(I)**2)
      TT(I)=EP-EMSUM
C
C  GETTING MORE ACCURATE TT
C
      RRR=SQRT((EMSUM-EMD+EMKMI)*(EMSUM-EMD-EMKMI))
      TT(I)=(PWAY(I)+RRR)*(PWAY(I)-RRR)/
     *       (SQRT(EMKMI**2+PWAY(I)**2)+(EMSUM-EMD))
C
      Y(I)=0.0
      Z(I)=0.0
C
      CALL KINE3(KP,EMK,EMR,TT(I),PWAY(I),NCLOSE,PCMAX,TANMAX,
     *   1.0,COSLOU,1.0,INTSEC,Y(I),COSCPX,Z(I),COSCPN)
C
    2 WRITE(5,22) PWAY(I),Y(I),Z(I),COSINE,TANMAX
   22 FORMAT(1X, 5E16.9)
C
C ELIMINATE THE DUPLICATED POINTS
C
      J=1
      DO 15 I=1,NPTS
      IF(PWAY(I+1).NE.PWAY(I)) THEN
          X(J)=PWAY(I)
          D(J)=Y(I)
          J=J+1
      ENDIF
   15 CONTINUE
      KPTS=J-1
C
      WRITE(7,71)(J,X(J),D(J),J=1,KPTS)
   71 FORMAT(I4,3X, 2F16.9)
C
C GET THE FIRST DIFFERENCE OF PWAY AND MAXIMUM MOMENTUM OF THE PROTON
C
      IMAX=KPTS-1
      DO 11 I=1,IMAX
      X1(I)=X(I+1)-X(I)
      D1(I)=D(I+1)-D(I)
C
   11 WRITE(7,72) X1(I),D1(I)
   72 FORMAT(1X, 2F16.9)
C
C GET THE SECOND DIFFERENCE OF PWAY AND MAXIMUM MOMENTUM OF THE PROTON
C
      IMAX=KPTS-2
      DO 12 I=1,IMAX
      X2(I)=X1(I+1)-X1(I)
      D2(I)=D1(I+1)-D1(I)
C
   12 WRITE(7,73) X2(I),D2(I)
   73 FORMAT(2X, 2F16.9)
C
C
      WRITE(8,81)(J,X(J),D(J),X1(J),D1(J),X2(J),D2(J),J=1,KPTS)
   81 FORMAT(I4,3X, 6F16.9)
C
  270 END
C
C
```

```
151   C
152         SUBROUTINE DIAGNO(JPOINT,X,Y)
153   C
154   C  COMPUTE AND PRINT THE RELATIVE ERROR MAGNIFICATION FACTOR
155   C  FOR ONE FLOATING POINT SUBTRACTION.
156   C
157         HUGE=1.0E30
158         RZERO=0.0
159   C
160         IF(X-Y.EQ.RZERO) THEN
161            RERMAG=HUGE
162         ELSE
163            RERMAG=AMAX1(ABS(X),ABS(Y))/ABS(X-Y)
164         ENDIF
165   C
166         WRITE(9,99)JPOINT,X,Y,RERMAG
167      99 FORMAT(' AT POINT',I2,' X =',G16.8,3X,'Y =',G16.8,3X,
168        *        'RERMAG =',G16.8)
169   C
170         RETURN
171   C
172         END
173   C
174   C
175   C
176         SUBROUTINE KINE3(
177        *    NP,EM,EMOTHR,TTLAB,PTLAB,NCLOSE,PCMAX,TANMAX,
178        *              COSCIN, COSLOU,COSLIN,
179        *              INTSEC,PLMAX,COSCPX,PLMIN,COSCPN )
180   C
181   C  KINE3 COMPUTES EXTREMA OF MOMENTA AND PRODUCTION ANGLE, AND
182   C  TRANSFORMS ANGLES.  IT IS BOTH RELATIVISTICALLY CORRECT AND ACCURATE
183   C  AT LOW ENERGIES.
184   C
185   C  COPYRIGHT 1966 J. P. CHANDLER, PHYSICS DEPT., INDIANA UNIVERSITY
186   C
187   C  THE INPUT QUANTITIES ARE NP, EM, EMOTHR, TTLAB, PTLAB, COSCIN, AND
188   C  COSLIN.
189   C  THE OUTPUT QUANTITIES ARE NCLOSE, PCMAX, TANMAX, COSLOU, INTSEC,
190   C  PLMAX, COSCPX, PLMIN, AND COSCPN.
191   C  A SYSTEM OF UNITS MUST BE USED IN WHICH C=1.0 (E.G., MEV AND MEV/C).
192   C
193   C  NP IS THE TOTAL NUMBER OF PARTICLES IN THE FINAL STATE.
194   C  EM IS THE REST MASS OF THE PARTICLE FOR WHICH OUTPUT QUANTITIES ARE
195   C  TO BE CALCULATED.
196   C  EMOTHR IS THE SUM OF THE REST MASSES OF THE OTHER OUTGOING PARTICLES.
197   C  TTLAB IS THE TOTAL KINETIC ENERGY OF ALL NP PARTICLES IN THE LAB
198   C  SYSTEM.
199   C  PTLAB IS THE MAGNITUDE OF THE TOTAL MOMENTUM IN THE LAB SYSTEM.
200   C  THESE QUANTITIES DEFINE THE PHYSICAL REGIONS OF THE MOMENTUM IN THE
201   C  LAB SYSTEM AND CMS.  THESE REGIONS ARE, RESPECTIVELY, AN ELLIPSOID
202   C  OF REVOLUTION AND A SPHERE.  FOR NP=2, ONLY THE SURFACES OF THESE
203   C  REGIONS ARE PHYSICALLY ACCESSIBLE.
204   C  NCLOSE WILL BE RETURNED EQUAL TO UNITY IF THE ELLIPSOID ENCLOSES THE
205   C  ORIGIN, AND ZERO IF IT DOES NOT (SEE BELOW).
206   C  PCMAX RETURNS THE RADIUS OF THE CMS SPHERE.  THAT IS, PCMAX IS THE
207   C  MAXIMUM MOMENTUM IN THE CMS.
208   C  TANMAX RETURNS THE TANGENT OF THE MAXIMUM PRODUCTION ANGLE IN THE
209   C  LAB, IF THE ELLIPSOID DOES NOT ENCLOSE THE ORIGIN.
210   C  (ALL ANGLES ARE MEASURED WITH RESPECT TO THE DIRECTION OF PTLAB.
211   C  IN SPECIFYING ANY DIRECTION, THE REVERSE DIRECTION ALONG THAT RAY IS
212   C  NOT CONSIDERED.)
213   C
214   C  COSCIN IS THE COSINE OF A SPECIFIED DIRECTION IN THE CMS.
215   C  COSLOU RETURNS THE COSINE OF THE LAB DIRECTION CORRESPONDING TO THE
216   C  CMS MOMENTUM DEFINED BY PCMAX AND COSCIN.
217   C
218   C  COSLIN IS THE COSINE OF A SPECIFIED DIRECTION IN THE LAB SYSTEM.
219   C  INTSEC WILL BE RETURNED EQUAL TO UNITY IF THIS DIRECTION INTERSECTS
220   C  THE SURFACE OF THE ELLIPSOID, AND ZERO OTHERWISE.
221   C  PLMAX RETURNS THE MAGNITUDE OF THE MAXIMUM MOMENTUM ALONG THE
222   C  DIRECTION SPECIFIED BY COSLIN.
223   C  COSCPX RETURNS THE COSINE OF THE CMS ANGLE CORRESPONDING TO THE LAB
224   C  MOMENTUM DEFINED BY PLMAX AND COSLIN.
225   C  SIMILARLY, PLMIN RETURNS THE MAGNITUDE OF THE SMALLEST PHYSICAL
226   C  MOMENTUM IN THE LAB IN THE DIRECTION COSLIN, AND COSCPN RETURNS THE
227   C  COSINE OF THE CORRESPONDING DIRECTION IN THE CMS.
228   C
```

```
C  FOR CLARIFICATION OF THE DEFINITIONS OF ALL QUANTITIES, THE USER
C  SHOULD CONSULT HAGEDORN'S BOOK.
C
C  IF ANY INPUT QUANTITY IS IN ERROR, EXIT OCCURS WITH NCIRCL=INTSEC=-1.
C
C  ALL CODING IS ARRANGED FOR ACCURACY, AT THE EXPENSE OF SPEED.
C  FOR SOME OF THE METHODS USED, SEE THE BOOK BY MCCRACKEN AND DORN.
C
C  REFERENCES.... STERNHEIMER LETTER, PHYS. REV. 93, P.642
C              .... 'RELATIVISTIC KINEMATICS', BY HAGEDORN
C              .... 'NUMERICAL METHODS AND FORTRAN PROGRAMMING', BY
C                                          MCCRACKEN AND DORN
C
      REAL             ZERO,ONE,TWO,FOUR,EMK,EMR,T,P,COSINE,COSCM,
     *   EMSUM,EP,ESQ,QQ,PESQY,PSTAR,ESTAR,BETA,GAMMA,PROD,PMISC,
     *   SINCIN,BECUZ,A,B,C,ARG,ROOT,PMAX,PMIN,VSTAR
C
      ZERO=0.
      ONE=1.
      TWO=2.
      FOUR=4.
C
      NN=NP
      EMK=EM
      EMR=EMOTHR
      T=TTLAB
      P=PTLAB
      COSCM=COSCIN
      COSINE=COSLIN
C
      IF(NN-2)270,10,10
   10 IF(EMK)270,20,20
   20 IF(EMR)270,30,30
   30 IF(T)270,270,40
   40 IF(P)270,50,50
C                          IF(ABSF(COSCIN)-1.) OK,OK,NO
   50 IF(COSCM+ONE)270,70,60
   60 IF(COSCM-ONE)70,70,270
C                          IF(ABSF(COSLIN)-1.) OK,OK,NO
   70 IF(COSINE+ONE)270,90,80
   80 IF(COSINE-ONE)90,90,270
C
   90 NCLOSE=0
      INTSEC=1
C
C  COMPUTE THE MAXIMUM MOMENTUM SQUARED, PESQY, IN THE CMS.
C
      EMSUM=EMK+EMR
C     EP=EMSUM+T
C     ESQ=(EP+P)*(EP-P)
      ESQ=(EMSUM+(T+P))*(EMSUM+(T-P))
      IF(ESQ)270,270,100
C
  100 WRITE(9,91) P
   91 FORMAT(F15.7)
C
C  DIAGNOSE THE RELATIVE ERROR OF QQ
C
      CALL DIAGNO(1,T*(T+TWO*EMSUM),P**2)
C
      QQ=T*(T+TWO*EMSUM)-P**2
C
      WRITE(9,92) QQ
   92 FORMAT(F15.7)
C
      PESQY=QQ*(QQ+FOUR*EMK*EMR)/(FOUR*ESQ)
      IF(PESQY)270,270,110
C
  110 PSTAR=SQRT(PESQY)
      PCMAX=PSTAR
      ESTAR=SQRT(PESQY+EMK**2)
      BETA=P/(EMSUM+T)
      IF(BETA-ONE)120,270,270
  120 GAMMA=(EMSUM+T)/SQRT(ESQ)
      PROD=BETA*ESTAR
      PMISC=GAMMA*(PSTAR*COSCM+PROD)
      ARG=(ONE+COSCM)*(ONE-COSCM)
      SINCIN=SQRT(ARG)
```

```
307         ARG=PMISC**2+(PSTAR*SINCIN)**2
308         COSLOU=PMISC/SQRT(ARG)
309         BECUZ=BETA*COSINE
310         A=(ONE+BECUZ)*(ONE-BECUZ)
311         B=-TWO*BECUZ*ESTAR/GAMMA
312         C=(PROD+PSTAR)*(PROD-PSTAR)
313         ARG=ESQ*PESQY-(P*EMK)**2*(ONE+COSINE)*(ONE-COSINE)
314         IF(ARG)210,130,130
315   C
316   C  THE LINE INTERSECTS THE ELLIPSOID, BUT PERHAPS ONLY IN THE REVERSE
317   C  DIRECTION.  IN THAT CASE PMAX WILL BE NEGATIVE.
318   C
319     130 ROOT=TWO*SQRT(ARG)/(EMSUM+T)
320         PMAX=(-B+ROOT)/(TWO*A)
321         IF(PMAX)210,140,140
322     140 ROOT=SQRT(PMAX**2+EMK**2)
323         COSCPX=GAMMA*(PMAX*COSINE-BETA*ROOT)/PSTAR
324         PMIN=C/(A*PMAX)
325         IF(NN-2)150,150,170
326     150 IF(PMIN)160,190,220
327     160 PMIN=PMAX
328         COSCPN=COSCPX
329         GO TO 200
330     170 IF(PMIN)180,190,220
331     180 PMIN=ZERO
332   C
333   C  THE ELLIPSOID ENCLOSES THE ORIGIN.
334   C
335     190 COSCPN=2.0
336     200 NCLOSE=1
337         TANMAX=-1.0
338         GO TO 250
339   C
340   C  THE RAY DOES NOT INTERSECT THE ELLIPSOID.
341   C
342     210 INTSEC=0
343         PMAX=-ONE
344         PMIN=+ONE
345         GO TO 230
346   C
347     220 ROOT=SQRT(PMIN**2+EMK**2)
348         COSCPN=GAMMA*(PMIN*COSINE-BETA*ROOT)/PSTAR
349     230 VSTAR=PSTAR/ESTAR
350         ARG=(BETA+VSTAR)*(BETA-VSTAR)
351         IF(ARG)200,200,240
352     240 TANMAX=VSTAR/(GAMMA*SQRT(ARG))
353   C
354     250 PLMAX=PMAX
355         PLMIN=PMIN
356     260 RETURN
357   C
358   C  ERROR EXIT.
359   C
360     270 NCLOSE=-1
361         INTSEC=-1
362         GO TO 260
363   C
364   C  END KINE3.
365   C
366         END
```

APPENDIX D

PROGRAM LISTING OF DALNG

```
C*********************************************************************
C
C  DALNG                    JUNE, 1994
C
C  A.N.S.I. STANDARD FORTRAN 77
C
C  J. P. CHANDLER, COMPUTER SCIENCE DEPARTMENT,
C     OKLAHOMA STATE UNIVERSITY
C  YUMEI ZHAO, COMPUTER SCIENCE DEPARTMENT,
C     OKLAHOMA STATE UNIVERSITY
C
C*********************************************************************
C
C*********************************************************************
C
C  DALNG GENERATES DATA FOR DALITZ PLOT OF REACTION 1+2 -> 3+4+5.
C  THE DALITZ PLOT IS GOT WITHOUT USING G FUNCTION.
C
C  S34     --  ARRAY OF TOTAL ENERGY SQUARED OF PARTICLES 3 AND 4
C              IN THEIR OWN CMS
C  S45     --  ARRAY OF TOTAL ENERGY SQUARED OF PARTICLES 4 AND 5
C              IN THEIR OWN CMS
C  RSTEP   --  INCREMENT OF X AXIS
C  NPTS    --  NUMBER OF POINTS IN THE SET
C  P       --  BEAM MOMENTUM
C  S       --  TOTAL ENERGY SQUARED IN CMS
C  S34N    --  MINIMUM VALUE OF S34
C  S34X    --  MAXIMUM VALUE OF S34
C
C*********************************************************************
C
C  All VARIABLES ARE SET TO BE SINGLE PRECISION
C
      DIMENSION S34(501,2),S45(501,2)
C
C  OPEN A FILE FOR OUTPUT DATA
      OPEN (4,FILE = 'FILE.DAL', STATUS = 'NEW')
C  OPEN A FILE FOR RELATIVE ERROR MAGNIFICATION FACTOR
      OPEN (7,FILE = 'GNRERMAG.DAL', STATUS = 'NEW')
C
      LP=6
      RZERO=0.0
      RSTEP=0.001
C
      NPT=501
      EM1=0.495
      EM2=0.939
      EM3=0.495
      EM4=1.32
      EM5=0.137
C     P=2.24
C  THE REACTION IS K- + N -> K- + CASCADE + PI
C  EM1+EM2 < EM3+EM4+EM5, THE REACTION IS ENDOENERGETIC
C  THE THRESHOLD ENERGY IS 1.4289446. WHILE N IS AT REST, THE TOTAL LAB
C  MOMENTUM IS SQRT(E**2-EM1**2) = 1.3404692.
      P=1.3409
      WRITE(4,44) P
   44 FORMAT(1X, F5.3)
      S=(EM2+SQRT(EM1**2+P**2))**2-P**2
C
      SLOW=2.0
      S34N=GTXMIN(SLOW,RZERO,S,EM4,EM3,EM5)
      IF(S34N.LT.RZERO) GO TO 3
      SHIG=S34N+RSTEP
      S34X=GTXMAX(SHIG,RZERO,S,EM4,EM3,EM5)
      IF(S34X.LE.S34N) GO TO 3
C
      DO 1 J=1,NPT
      S34(J,1)=S34N+(S34X-S34N)*(J-1)/FLOAT(NPT-1)
      KK=KF(S34(J,1),S,EM3,EM4,EM5,S45(J,1),S45(J,2))
    1 WRITE(4, 2) S34(J,1), S45(J,1), S45(J,2), KK
    2 FORMAT(1X, 3E15.8, I6)
C
    3 STOP
      END
C
C
C
```

```
      FUNCTION KF(X,S,EM3,EM4,EM5,Y1,Y2)
C
C  FUNCTION KF BYPASSES G FUNCTION TO GET THE DATA FOR DRAWING
C  DALITZ PLOT FOR THE REACTION 1+2 -> 3+4+5.
C  IN THIS WAY, NUMERICAL CANCELLATION AT LOW ENERGY CAN BE
C  AVOID.
C
C  YUMEI ZHAO, DEPT. OF COMPUTER SCIENCE, OKLAHOMA STATE UNIV.
C
C
      RZERO=0.0
      A=X
      B=-X**2+X*(S+EM3**2+EM4**2+EM5**2)-(EM3**2-EM4**2)*(S-EM5**2)
      SS=SQRT(S)
C
      WRITE(7,71) A
   71 FORMAT(3X,F15.7)
C
C
      IF(A.NE.RZERO) GO TO 20
      IF(B.NE.RZERO) GO TO 30
      K=-1
      Y1=RZERO
      Y2=Y1
      GO TO 50
C
C  CHECK THE RELATIVE ERROR MAGNIFICATION FACTOR
C
   20 CALL DIAGNO(1,X,(EM3+EM4)**2)
C
      DET=(X-(EM3-EM4)**2)*(X-(EM3+EM4)**2)*(X-(SS-EM5)**2)*
     *    (X-(SS+EM5)**2)
C
      WRITE(7,77)DET
   77 FORMAT(3X,F15.7)
C
      IF(DET.GT.RZERO) GO TO 40
      DET=RZERO
   30 K=1
      Y1=B/(2.0*A)
      Y2=Y1
      GO TO 50
   40 K=0
      ROOT=SQRT(DET)
C
C  CHECK THE RELATIVE ERROR MAGNIFICATION FACTOR
C
      CALL DIAGNO(2,-B,ROOT)
C
      Y1=(B+ROOT)/(2.0*A)
      Y2=(B-ROOT)/(2.0*A)
C
      WRITE(7,78)Y1,Y2
   78 FORMAT(3X,2F15.7)
C
      GO TO 50
C
   50 KF=K
      RETURN
C
C  END OF KF
C
      END


      SUBROUTINE DIAGNO(JPOINT,X,Y)
C  COMPUTE AND PRINT THE RELATIVE ERROR MAGNIFICATION FACTOR
C  FOR ONE FLOATING POINT SUBTRACTION.
C
      HUGE=1.0E30
      RZERO=0.0
C
      IF(X-Y.EQ.RZERO) THEN
         RERMAG=HUGE
      ELSE
         RERMAG=AMAX1(ABS(X),ABS(Y))/ABS(X-Y)
      ENDIF
C
```

```
      WRITE(7,77)JPOINT,X,Y,RERMAG
   77 FORMAT(' AT POINT',I2,' X =',F15.7,3X,'Y =',F15.7,3X,
     *       'RERMAG =',E15.7)
C
      RETURN
C
      END
C
C
      REAL FUNCTION CHANG(X,Y,Z,U,V,W,JXYZ,
     *   XYZMAX,XYZMIN,KOMPLX)
C
C  CHANG EVALUATES THE FUNCTION G(X,Y,Z,U,V,W) AND SOLVES G=0 FOR
C  X, Y, OR Z (ACCORDINGLY AS JXYZ IS NEGATIVE, ZERO, OR POSITIVE).
C  XYZMAX AND XYZMIN RETURN THE TWO SOLUTIONS OF G=0.
C  IF THERE ARE NO REAL ROOTS OF G=0, KOMPLX IS RETURNED NONZERO.
C
C  CHAN HONG-MO, K. KAJANTIE, AND G. RANFT, NUOVO CIMENTO 49 (1967) 178
C  SEE ALSO K. KAJANTIE AND P. LINDBLOM, PHYS. REV. 175 (1968) 2203.
C
C  CONSIDER A PROCESS WITH THREE PARTICLES IN THE FINAL STATE,
C                  1+2 GOES TO 3+4+5 .
C  G(S34,S45,S,M4**2,M3**2,M5**2)=0 DEFINES THE BOUNDARY OF A
C  DALITZ PLOT (PLUS AT LEAST ONE OTHER, UNPHYSICAL, CLOSED CURVE).
C  G IS NEGATIVE INSIDE THE BOUNDARY OF THE DALITZ PLOT.
C  G(T13,S,S45,M1**2,M2**2,M3**2)=0 DEFINES THE CURVILINEAR PORTION
C  OF THE BOUNDARY OF A CHEW-LOW PLOT.
C
C  J. P. CHANDLER, DEPT. OF PHYSICS, INDIANA UNIVERSITY
C
C     IMPLICIT REAL*8 (A-H,O-Z)
C
C                              EVALUATE G(X,Y,Z,U,V,W) .
C
      CHANG= X*Y*(X+Y-Z-U-V-W)+
     *       Z*U*(Z+U-X-Y-V-W)+
     *       V*W*(V+W-X-Y-Z-U)+
     *       X*(Z*W+U*V)+
     *       Y*(Z*V+U*W)
C
      RZERO=0.0
C
      IF(JXYZ)10,20,30
C                              SOLVE FOR X.
   10 D=Y
      E=U
      F=W
      G=V
      H=Z
      GO TO 40
C                              SOLVE FOR Y.
   20 D=X
      E=U
      F=V
      G=W
      H=Z
      GO TO 40
C                              SOLVE FOR Z.
   30 D=U
      E=Y
      F=W
      G=V
      H=X
C                              SET UP AND SOLVE THE QUADRATIC EQUATION.
   40 A=D
      B=D*(D-E-F)+G*(E-D-F)+H*(F-D-E)
      C=(E*H-F*G)*(E+H-F-G)+D*(E-G)*(F-H)
      KOMPLX=0
      IF(A.NE.0.) GO TO 60
      IF(B.NE.0.) GO TO 50
      XYZMAX=RZERO
      XYZMIN=RZERO
      KOMPLX=-1
C                          NO SOLUTION
      GO TO 80
C
   50 XYZMAX=-C/B
      XYZMIN=XYZMAX
```

```
C                          ONE SOLUTION
      KOMPLX=1
      GO TO 80
C
   60 ARG=B**2-4.0 *A*C
      IF(ARG.GT.RZERO) GO TO 70
      KOMPLX=1
      ARG=RZERO
C                          ONE SOLUTION
      XYZMAX=-B/(2.0D0*A)
      XYZMIN=XYZMAX
      GO TO 80
   70 ROOT=SQRT(ARG)
      XYZMAX=(-B+ROOT)/(2.0 *A)
      XYZMIN=(-B-ROOT)/(2.0 *A)
      KOMPLX=0
C                          TWO SOLUTIONS
   80 RETURN
C
C  END CHANG
C
      END
C
C
C
      REAL FUNCTION GTXMIN(SLOW,RZERO,S,EM4,EM3,EM5)
C
C  GETXMIN WILL GET THE MINIMUM S VALUE OF THE DALITZ PLOT.
C
C  YUMEI ZHAO, DEPT. OF COMPUTER SCIENCE, OKLAHOMA STATE UNIV.
C
C     IMPLICIT REAL*8 (A-H,O-Z)
C     DOUBLE PRECISION CHANG
      RMINUS=-1.0
      RLIMIT=20.0
      RSTEP=0.001
C                          DUMMY IS USELESS THERE
  100 KOMPLX=KF(SLOW,S,EM3,EM4,EM5,Y1,Y2)
      IF(KOMPLX.GT.0) GO TO 200
      IF(KOMPLX.EQ.0) GO TO 300
      PRINT *, 'NO SOLUTION'
      SMIN=RMINUS
      GO TO 500
C
C                          ONE SOLUTION
  200 SLOW=SLOW+RSTEP
      IF(SLOW.GE.RLIMIT) GO TO 400
      GO TO 100
C
C                          TWO SOLUTION
C                          FOUND THE MINIMUM S
  300 SMIN=SLOW-RSTEP
      GO TO 500
C
C                          OUT OF THE LIMIT
  400 PRINT *, 'NO CURVE'
      SMIN=RMINUS
      GO TO 500
C
  500 GTXMIN=SMIN
      RETURN
C
      END
C
C
      REAL FUNCTION GTXMAX(SHIG,RZERO,S,EM4,EM3,EM5)
C
C  GETXMAX WILL GET THE MAXIMUM S34 VALUE OF THE DALITZ PLOT.
C
C  YUMEI ZHAO, DEPT. OF COMPUTER SCIENCE, OKLAHOMA STATE UNIV.
C
      RMINUS=-1.0
      RLIMIT=40.0
C
      RSTEP=0.001
C                          DUMMY IS USELESS THERE
   11 KOMPLX=KF(SHIG,S,EM3,EM4,EM5,Y1,Y2)
      IF(KOMPLX.EQ.0) GO TO 22
```

```
      IF(KOMPLX.GT.0) GO TO 33
      PRINT *, 'NO SOLUTION'
      SMAX=RMINUS
      GO TO 55
C
C                          TWO SOLUTION
   22 SHIG=SHIG+RSTEP
      IF(SHIG.GE.RLIMIT) GO TO 44
      GO TO 11
C
C                          ONE SOLUTION
C                          FOUND THE MAXIMUM S34
   33 SMAX=SHIG
      GO TO 55
C
C                          OUT OF THE LIMIT
   44 PRINT *, 'NO CURVE'
      SMAX=RMINUS
      GO TO 55
C
   55 GTXMAX=SMAX
      RETURN
C
      END
C
```

VITA

YUMEI ZHAO

Candidate for the Degree of

Master of Science

Thesis: RELATIVISTIC KINEMATICS WITHOUT NUMERICAL CANCELLATION

Major Field: Computer Science

Biographical:

Personal Data: Born in Zhengzhou, China, January 23, 1963, the daughter of Ruxiang Zhao and Xiuwen Deng.

Education: Graduated from First high school Zhengzhou, China, in July, 1980; received Bachelor of Science Degree in physics from Henan Normal University, Henan, China, in July, 1984; completed requirements for the Master of Science Degree in Computer Science at Oklahoma State University, in July, 1994.

Professional Experience: Editor in Henan Educational Publishing Co. Henan, China from July, 1984 to October, 1988; Programmer (part time) in Agriculture Economics Department at Oklahoma State University from October, 1993 to March, 1994.