APPROXIMATE ANALYTICAL SOLUTIONS
OF THE BOUSSINESQ EQUATION

By

VICKE' A. EVANS

Bachelor of Science

Northeastern State University

Tahlequah, Oklahoma

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Thesis Advisor

Dean of the Graduate College

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Introduction

Environmental concerns have focused attention on the interaction of streams with groundwater and led to the development of both analytic and numerical ground water models. Analytic models are of particular interest because they provide qualitative solutions as well as numerical data, and avoid many of the difficulties inherent in descretization that is required in the use of numerical models. One such model, the Boussinesq equation, $K\frac{\partial}{\partial x}(h\frac{\partial h}{\partial x}) = S\frac{\partial h}{\partial t}$, is a nonlinear partial differential equation which describes one dimensional flow in a phreatic aquifer. Several solutions including a separable solution first reported by Dumm in 1954 have been presented since its introduction in 1907 [Boussinesq, 1907]. However, exact solutions are available only for specialized initial and boundary conditions. Some numerical solutions are available, and attempts have been made to find analytic solutions by connecting the equation with linear equations; specifically the heat equation, $\frac{\partial^2 h}{\partial x^2} = \beta\frac{\partial h}{\partial t}$ as explained by McWhorter in 1979. Other investigations, [Kirkham, 1964] for example, include approaches to this problem using potential theory.

The objective of this paper is to analyze existing analytic solutions of the Boussinesq equation and to develop a tool that can be used to solve this equation for arbitrary initial conditions. Methods of linearizing to the heat equation and examples using the one and two drain problem are provided.

Derivation of the Boussinesq Equation

Darcy's Law relates the ground water velocity to the hydraulic properties of the porous medium,

$$q = -K\frac{\partial h}{\partial x} \tag{1}$$

q is the Darcy velocity, K is hydraulic conductivity, and $h(x,t)$ is the head at location x and time t. The volume of water per unit width, Q, is the integral over the vertical line (see figure 1) of the velocity q. If we adopt the Dupuit-Forchheimer assumption, then this is just qh, so that

$$Q = -Kh\frac{\partial h}{\partial x} \ . \tag{2}$$

We consider water flowing through the control volume displayed in Figure 1. We can calculate the change in water in the control volume over a time period Δt in two ways. Using Darcy's law as stated in equation (2), the volume of water per unit width flowing into the box is $Kh(x+\Delta x,t)\frac{\partial h}{\partial x}(x+\Delta x,t)\Delta t$, and the volume of water per unit width flowing out of the box is $Kh(x,t)\frac{\partial h}{\partial x}(x,t)\Delta t$. Thus the total change in volume per unit width in the box is approximated by

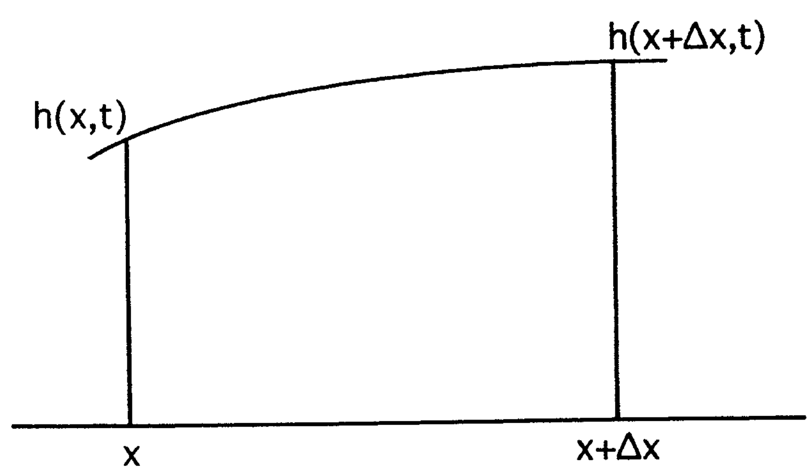

Figure 1: **Cross sectional view of water flow in an aquifer**

$$(Kh(x+\Delta x,t)\frac{\partial h}{\partial x}(x+\Delta x,t) - Kh(x,t)\frac{\partial h}{\partial x}(x,t))\Delta t \ . \tag{3}$$

Alternatively we can approximate the change in volume of water as the change in head from time t to time $t + \Delta t$, times the width, Δx, of the box all multiplied by the specific yield, S. This is gives,

$$S(h(x,t+\Delta t) - h(x,t))\Delta x. \tag{4}$$

Equating equations (3) and (4) and dividing by $\Delta x \Delta t$ yields

$$K\frac{h(x+\Delta x,t)\frac{\partial h}{\partial x}(x+\Delta x,t) - h(x,t)\frac{\partial h}{\partial x}(x,t)}{\Delta x} \approx S\frac{h(x,t+\Delta t) - h(x,t)}{\Delta t}\ . \tag{5}$$

Taking the limit as Δx and Δt go to zero yields the Boussinesq equation,

$$K\frac{\partial}{\partial x}(h\frac{\partial h}{\partial x}) = S\frac{\partial h}{\partial t}. \tag{6}$$

Or, if $\alpha = \frac{S}{K}$, the Boussinesq equation takes the form

$$\frac{\partial}{\partial x}(h\frac{\partial h}{\partial x}) = \alpha\frac{\partial h}{\partial t}\ . \tag{7}$$

Alternative derivations of this equation as it relates to water flow may be found in [McWhorter, 1977] or [Bear, 1972].

Dupuit-Forchheimer: As is noted in [Youngs, 1966], it is difficult to arrive at a usable mathematical model without the assumption that water flow is in the horizontal direction and at a constant velocity over the entire depth; however, this implies that the water table is flat. Therefore it can only be used when $\frac{\partial h}{\partial x}$ is near zero. We should be aware that whenever this assumption is applied to a water table with steep slopes in the x direction, the extent of the error built into the model may be difficult to determine.

The Two Drain Problem

One application for the Boussinesq equation that has received attention in the literature is the *Two Drain Problem* [Dumm, 1954, 1964]. This problem considers water table fluctuation in a phreatic aquifer with a horizontal base lying between parallel drains as shown in Figure 2.

We assume that the head at each of two drains remains constant, and that the water table between the two drains is initially given by a function $f(x)$. Thus $f(0) = f(L) = D$. Furthermore it is reasonable to assume that f is symmetric about the line $x = \frac{L}{2}$ so that $f'(\frac{L}{2}) = 0$. We will consider a falling water table so that the water level begins at the level of $f(x)$ and decreases over time. It follows that $\frac{\partial h}{\partial t}(x,t)$ is always negative except at the drains where it is zero. The two drain problem is thus expressed as,

$$\frac{\partial}{\partial x}(h\frac{\partial h}{\partial x}) = \alpha\frac{\partial h}{\partial t} , \tag{8}$$

$$h(x,0) = f(x) , \qquad \text{(Initial condition)} \tag{9}$$

$$h(0,t) = D = h(L,t) \text{ for all } t, \quad \text{(Boundary condition) and} \tag{10}$$

$$h(\frac{L}{2},0) = M \qquad \frac{\partial h}{\partial x}(\frac{L}{2},0) = 0 \quad \text{(Symmetry assumptions)} . \tag{11}$$

A Separable Solution

A separable solution for the case $D = 0$ was obtained by Boussinesq [Boussinesq, 1907] and by Glover as reported by Dumm [Dumm, 1954]. Jan van Schilfgaarde [van Schilfgaarde, 1963] purports to generalize this solution to the case $D > 0$, but later modified this result because the solution was not consistent with the boundary conditions [van Schilfgaarde, 1964].

Separability assumes that $h(x,t) = W(x)V(t)$. We can choose this separation so that $V(0) = 1$ which gives $h(x,t) = W(x)$ at $t = 0$. (If $h(x,t) = W^*(x)V^*(t)$ is any

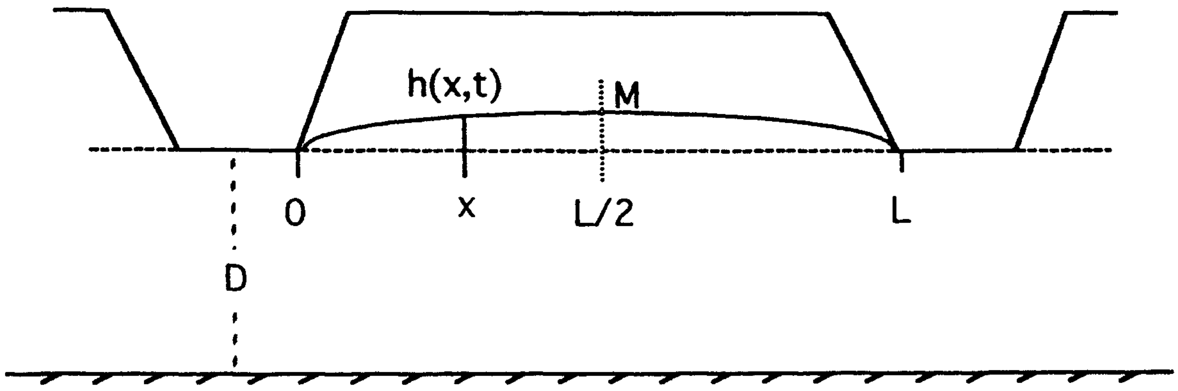

Figure 2: **The two drain problem**

separation of h, we can define $W(x) = V^*(0)W^*(x)$ and $V(t) = \dfrac{V^*(t)}{V^*(0)}$ to obtain the condition $V(0) = 1$.) The conditions on h can be summarized in terms of V and W as,

$$V(0) = 1 \,, \qquad \text{(Separation assumption)} \tag{12}$$

$$W(x) = f(x) \,, \qquad \text{(Initial condition)} \tag{13}$$

$$W(0) = 0 = W(L) \,, \qquad \text{(Boundary condition)} \tag{14}$$

$$W(\frac{L}{2}) = M \quad W'(\frac{L}{2}) = 0 \,, \qquad \text{(Symmetry conditions) and} \tag{15}$$

$$V'(t) < 0 \quad \text{(Decreasing water table assumption)} \,. \tag{16}$$

We note that the initial condition, $W(x) = f(x)$, will not be used in obtaining the solution; only the separation assumption, boundary assumption, symmetry conditions, and decreasing water table assumption are used. This is because the separability assumption is not consistent with an arbitrary initial condition; in fact these assumptions and conditions determine the initial condition.

Proceeding with the solution we replace h by VW in the Boussinesq equation,

$$\frac{d}{dx}(WV\frac{dW}{dx}V) = \alpha W\frac{dV}{dt}\ , and \tag{17}$$

$$\frac{1}{W}\frac{d}{dx}(W\frac{dW}{dx}) = \frac{\alpha}{V^2}\frac{dV}{dt}\ . \tag{18}$$

Since the left-hand side of this equation depends only on x, the right-hand side depends only on t, and they are equal for all values of x and t, both must be constant. Furthermore, since $V'(t)$ is negative, there is a positive constant A that satisfies,

$$\frac{\alpha V'}{V^2} = -A\ , and \tag{19}$$

$$\frac{1}{W}\frac{d}{dx}(WW') = -A\ . \tag{20}$$

Equation (19) can be integrated directly to yield $V = \dfrac{\alpha}{C + At}$, and the condition $V(0) = 1$ gives $C = \alpha$, so that

$$V(t) = \frac{\alpha}{\alpha + At}\ . \tag{21}$$

The ordinary differential equation (20) does not have a closed form solution, but we can evaluate the constant, A, and express W in implicit form:

$$\frac{d}{dx}WW' = -AW \tag{22}$$

$$WW'\frac{d}{dx}WW' = -AW^2W' \tag{23}$$

$$\frac{(WW')^2}{2} = C - A\frac{W^3}{3} \tag{24}$$

One's initial thought might be to determine C by evaluation of equation (24) at $x = 0$. This gives $C = 0$, and $\frac{(WW')^2}{2} = -A\frac{W^3}{3}$. The left side is nonnegative, and since the water table never dips below the level of the drains, the right side is nonpositive. It follows that $W = 0$ leading to the trivial solution, $h = 0$. The difficulty with this substitution is that it tacitly assumes that $W'(0)$ is not ∞, but if the water table begins with nonzero head, a vertical tangent at $x = 0$ is exactly what is expected. More interesting results are obtained if we evaluate at $x = \frac{L}{2}$. Since $W(\frac{L}{2}) = M$ and $W'(\frac{L}{2}) = 0$, we obtain $C = \frac{AM^3}{3}$. Substituting for C in (24) and simplifying we have

$$(WW')^2 = \frac{2A}{3}(M^3 - W^3), \text{ and} \tag{25}$$

$$\frac{WW'}{\sqrt{M^3 - W^3}} = \sqrt{\frac{2A}{3}} . \tag{26}$$

Integrating as x goes from 0 to x yields

$$\int_0^{W(x)} \frac{W}{\sqrt{M^3 - W^3}}\, dW = x\sqrt{\frac{2A}{3}} .$$

With the substitution $z = \frac{W}{M}$, this becomes

$$\sqrt{M} \int_0^{\frac{W(x)}{M}} \frac{z}{\sqrt{1 - z^3}}\, dz = x\sqrt{\frac{2A}{3}} . \tag{27}$$

We can now determine A by evaluating (27) at $x = \frac{L}{2}$ where $W = M$. Thus,

$$\sqrt{M} \int_0^1 \frac{z}{\sqrt{1 - z^3}}\, dz = \frac{L}{2}\sqrt{\frac{2A}{3}} . \tag{28}$$

Integrating equation (28) and solving for A, we obtain

$$A = 24\pi \frac{\Gamma^2(\frac{2}{3})}{\Gamma^2(\frac{1}{6})} \frac{M}{L^2} \approx 4.46209 \frac{M}{L^2} . \tag{29}$$

The gamma function is defined by

$$\Gamma(x) = \int_0^\infty t^{x-1}\epsilon^{-t}\, dt \ .$$

It can be thought of as a smoothing of the factorial function since it is a differentiable function that is defined for all positive x, and for positive integers n, $\Gamma(n+1) = n!$.

Now, using this value of A in equations (21) and (27),

$$V(t) = \frac{1}{1 + 4.46209\frac{KM}{L^2S}t} \ , \tag{30}$$

$$x = 0.579797L\int_0^{\frac{W(x)}{M}} \frac{z}{\sqrt{1-z^3}}\, dz \ , \text{ and} \tag{31}$$

$$h(x,t) = W(x)V(t) \ . \tag{32}$$

An example: For a sandy soil, we can take S to be 0.2 and K to be 2 meters per day. We will look at graphs for $M = 2$ meters and $L = 25$ meters. With this data, $V(t) = \dfrac{1}{1+0.142786t}$, and $W(\frac{L}{2}) = M = 2$. Thus at a point midway between the two drains, $h(\dfrac{L}{2},t) = \dfrac{2}{1+0.142786t}$. The graph in Figure 3 shows the height of the water table in meters at the midpoint, $\dfrac{L}{2}$, as a function of time in days.

Even though we cannot calculate $W(x)$ in closed form, we can use a trick to get the graph of $h(x,t)$ for specific values of t. Giving the right hand side of equation (31) a function name, we can write $x = P(W)$, or $W = P^{-1}(x)$. That is, the right hand side of equation (31) can be considered the inverse of the function W. In order to recover a function from its inverse, we need to reverse the roles of the independent and dependent variables. Thus if we make a parametric graph of the ordered pairs $(P(W), W)$ we have a graph of $W(x)$. Plotting the right hand side of equation (31) requires the use of Simpson's rule (or some other integral approximation technique) at many points. Finally, the nature of the function makes it difficult to get an accurate graph by evaluating at evenly spaced points; an adaptive graphing technique that samples many points where the graph changes rapidly needs to be used. Any of the most popular computer algebra systems will do all this automatically and produce

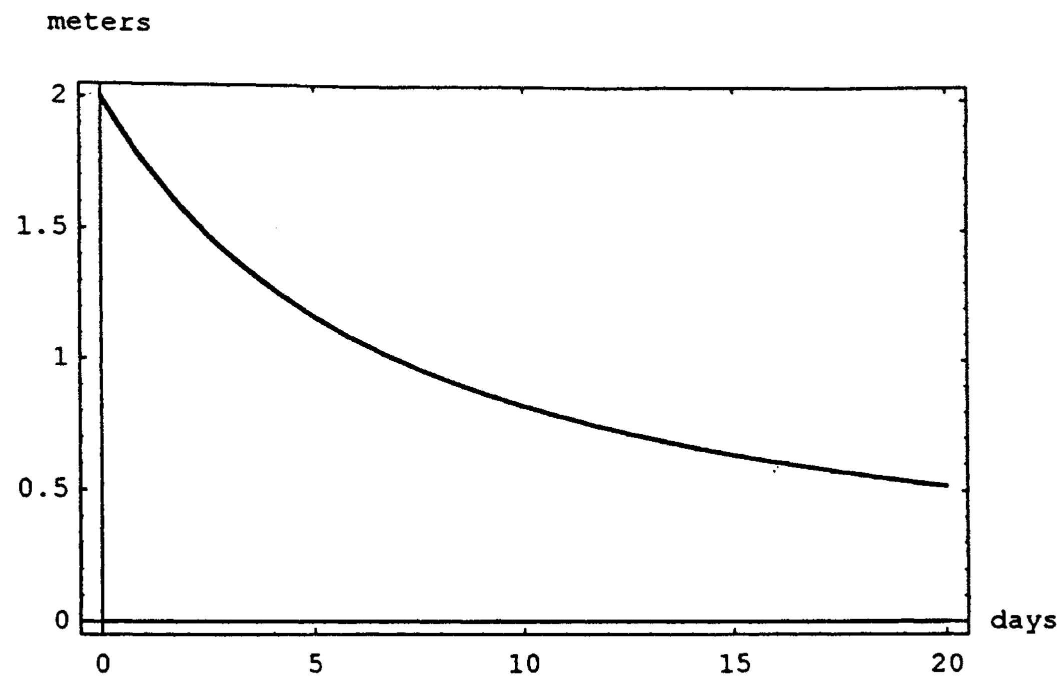

Figure 3: **Head at the midpoint given by the separable solution**

the graph. In Figure 4 we have plotted $h(x,t)$ for $t = 0, 5, 10, \ldots, 30$ days.

Limitations of the separable solution: We have already noted that a separable solution is not consistent with the selection of an arbitrary initial condition. The functions $W(x)$ and $V(t)$ are determined by the separability condition and symmetry assumptions, and the initial condition, $f(x) = h(x,0) = W(x)$, is forced after the fact by separability rather than from any consideration of natural phenomena. For most initial conditions, f, there is no separable solution.

Even the small generalization of placing the drain above the impermeable layer as proposed by van Schilfgaarde [van Schilfgaarde, 1963] has difficulties which he later noted and corrected [van Schilfgaarde, 1964]. The difficulty is that the assumption that $h(0,t) = D_0 = h(L,t)$ is not consistent with the existence of a separable solution. If $D_0 \neq 0$, then the equation $W(0)V(t) = D_0$, guarantees that $W(0) \neq 0$ so that

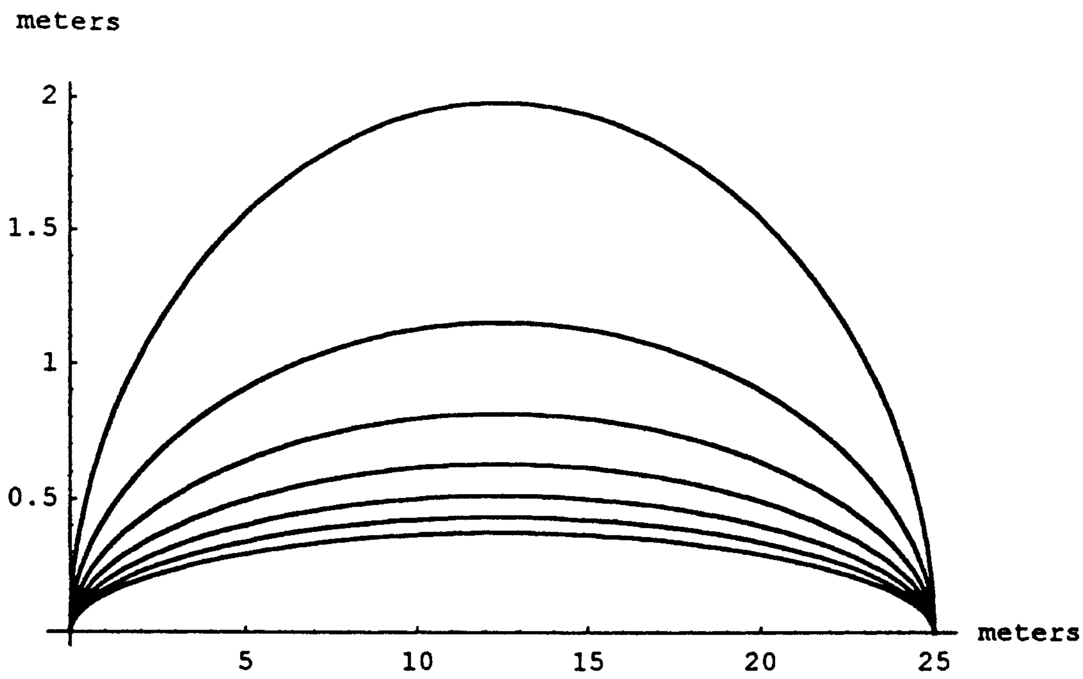

Figure 4: **Water levels predicted by the separable solution**

$V(t) = \dfrac{D_0}{W(0)}$ which is a constant. But if $V(t)$ is constant, then the water table does not change with time. This only occurs in the trivial case where the water table begins at the level of the drains.

One result of the separable solution is that it imposes a qualitative behavior on the solution. The equation $h(x,t) = W(x)V(t)$ gives the following information about the water table at various times.

Time	Water table
$t = 0$	$h = W(x)$
$t = 1$	$h = V(1)W(x)$
$t = 2$	$h = V(2)W(x)$

This predicts an equally proportional drop in the water table at any point between the two drains. Thus, if at any point the water drops to half its original level in two days, then the water table at all points between the two drains drops to half its level in two days. This phenomena can be seen in Figure 4.

Connections with the Heat Equation

Nonlinear partial differential equations are notoriously difficult to analyze, and a common method of dealing with them is to connect them with linear partial differential equations which are easier to solve. As it develops, the Boussinesq equation is closely related to the heat equation, $\frac{\partial^2 H}{\partial x^2} = \beta\frac{\partial H}{\partial t}$. This equation is linear, it is important in many areas of mathematics and engineering, and its solutions are well known. It is interesting to consider just how important the comparison of water flow with heat dispersion may be.

We consider here how to connect the Boussinesq equation, $\frac{\partial}{\partial x}(h\frac{\partial h}{\partial x}) = \alpha\frac{\partial h}{\partial t}$ with initial condition $h(x,0) = f(x)$ with the heat equation. We shall for the moment ignore boundary conditions though it is not necessary to do so. In succeeding sections we will look at the structure of solutions of the heat equation.

The Standard Connection: One way to get to the heat equation is to follow the method used in [McWhorter, 1976]. The idea is to replace h in the expression $\frac{\partial}{\partial x}(h\frac{\partial h}{\partial x})$ by its average value, A with the condition that h' is small relative to h:

$$\begin{aligned}
\frac{\partial}{\partial x}(h\frac{\partial h}{\partial x}) &= \alpha\frac{\partial h}{\partial t} \qquad (33)\\
\frac{\partial}{\partial x}(A\frac{\partial h}{\partial x}) &= \alpha\frac{\partial h}{\partial t} \qquad (34)\\
A\frac{\partial^2 h}{\partial x^2} &= \alpha\frac{\partial h}{\partial t} \qquad (35)\\
\frac{\partial^2 h}{\partial x^2} &= \frac{\alpha}{A}\frac{\partial h}{\partial t} \qquad (36)
\end{aligned}$$

Thus to obtain an approximate solution of the Boussinesq equation by this method, we should use the solution, $h(x,t)$, of the heat equation, $\frac{\partial^2 h}{\partial x^2} = \beta\frac{\partial h}{\partial t}$, with initial condition, $h(x,0) = f(x)$, where $\beta = \frac{\alpha}{A}$, and A is the average value of h.

What is the average value of h? Since we begin with no explicit formula for h, its average value is not likely to be known. We will follow the lead of [Dumm, 1954] and [Brooks, 1961] who use this equation as a model for agricultural drainage, and assign an intuitive value to A that may be derived from the initial condition. For example, in case f is a square wave with height 2, we would take $A = 1$. It should be emphasized that we have no way of knowing if this is in fact the true average value of h, but the choice will be made consistently so that our later comparison of methods remains valid.

Difficulties with the standard connection: A cardinal rule of numerical analysis has been violated here. Whenever possible, approximations should not be made before derivatives are taken. This is because functions that are point-wise very close may have vastly different derivatives as the following example illustrates. Let $f(x) = \frac{\sin(10000x)}{1000}$. Notice that the absolute value of f is never greater than 0.001, and we may feel comfortable approximating it with the zero function. But the derivative of f is $f'(x) = 10\cos(1000x)$, a function that varies between -10 and 10. In Figure 5 the graph of $f(x)$ cannot be distinguished from the x-axis, but $f'(x)$ is not close to the zero function.

This type of behavior can be expected from any function that changes rapidly. In cases such as the example described above, while the rate of change in head may be small some distance from the drain, near the drain the rate of change could be very steep.

The standard connection replaces $\frac{\partial}{\partial x}(h\frac{\partial h}{\partial x})$ by $A\frac{\partial^2 h}{\partial x^2}$. If we differentiate first we see that this is replacing $(\frac{\partial h}{\partial x})^2 + h\frac{\partial^2 h}{\partial x^2}$ by $A\frac{\partial^2 h}{\partial x^2}$ and that the true effect not only replaces h by A but also to discards the term $\left(\frac{\partial h}{\partial x}\right)^2$. Two methods for approximating

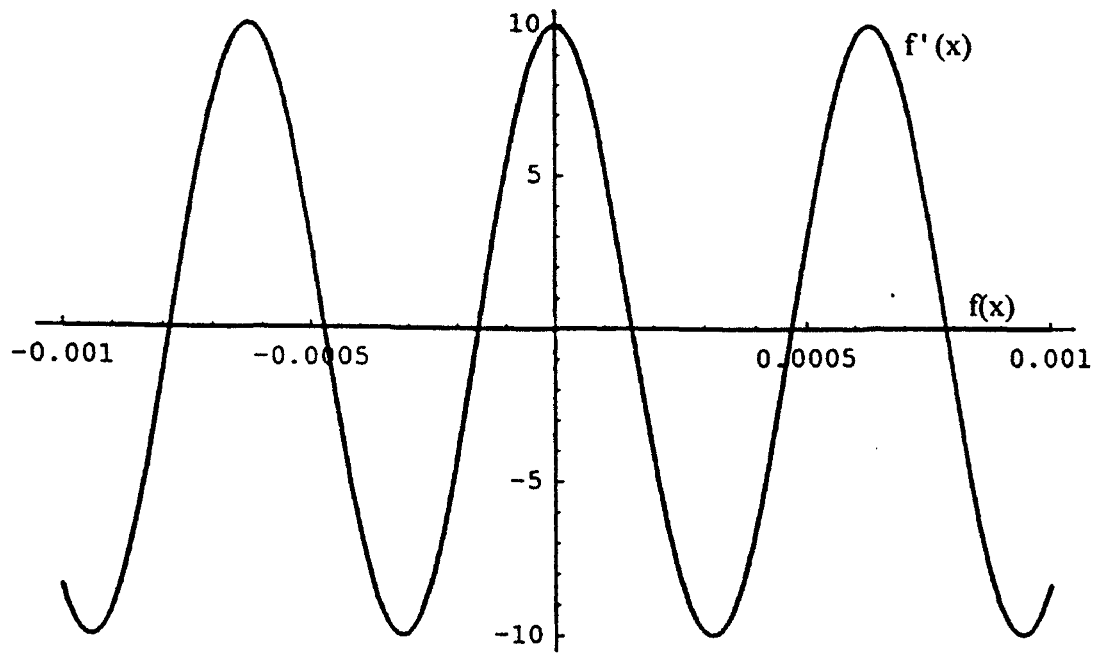

Figure 5: **Close functions whose derivatives are not close**

this expression follow; both avoid the difficulties discussed here.

The Logarithm Connection: We linearize the equation in two steps. First expand the derivative, $\frac{\partial}{\partial x}(h\frac{\partial h}{\partial x})$ to obtain

$$(\frac{\partial h}{\partial x})^2 + h\frac{\partial^2 h}{\partial x^2} = \alpha\frac{\partial h}{\partial t} , \tag{37}$$

and replace the occurrence of h in equation (37) by A so that

$$(\frac{\partial h}{\partial x})^2 + A\frac{\partial^2 h}{\partial x^2} = \alpha\frac{\partial h}{\partial t} . \tag{38}$$

For the second step we make the substitution $h = A\ln w$. Then $\frac{\partial h}{\partial x} = Aw^{-1}\frac{\partial w}{\partial x}$, and $\frac{\partial^2 h}{\partial x^2} = Aw^{-1}\frac{\partial^2 w}{\partial x^2} - Aw^{-2}(\frac{\partial w}{\partial x})^2$.

Plugging this into equation (38) and simplifying, we obtain

$$\frac{\partial^2 w}{\partial x^2} = \frac{\alpha}{A}\frac{\partial w}{\partial t} . \tag{39}$$

Thus to obtain an approximate solution of the Boussinesq equation by this method, we should use the solution, $h(x,t) = A \ln w(x,t)$, where w is the solution of the heat equation, $\frac{\partial^2 w}{\partial x^2} = \beta \frac{\partial w}{\partial t}$, with initial condition, $w(x,0) = e^{\frac{f(x)}{A}}$, where $\beta = \frac{\alpha}{A}$, and A is the average value of h.

A Square Root Connection: Notice first of all that $\frac{\partial}{\partial x}(h\frac{\partial h}{\partial x}) = \frac{\partial^2}{\partial x^2}(\frac{h^2}{2})$. Thus:

$$h\frac{\partial^2}{\partial x^2}(\frac{h^2}{2}) = \alpha h \frac{\partial h}{\partial t} \tag{40}$$

$$h\frac{\partial^2}{\partial x^2}(\frac{h^2}{2}) = \alpha \frac{\partial}{\partial t}(\frac{h^2}{2}) \tag{41}$$

$$h\frac{\partial^2}{\partial x^2}h^2 = \alpha \frac{\partial}{\partial t}h^2 \tag{42}$$

Let $z = h^2$, then

$$\sqrt{z}\frac{\partial^2 z}{\partial x^2} = \alpha \frac{\partial z}{\partial t} . \tag{43}$$

Since $z = h^2$, its average value should be near A^2. Thus we replace $\sqrt{z}$ in (43) by A:

$$A\frac{\partial^2 z}{\partial x^2} = \alpha \frac{\partial z}{\partial t} \tag{44}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\alpha}{A}\frac{\partial z}{\partial t} \tag{45}$$

Thus to obtain an approximate solution of the Boussinesq equation by this method, we should use, $h(x,t) = \sqrt{z(x,t)}$, where z is the solution of the heat equation, $\frac{\partial^2 z}{\partial x^2} = \beta \frac{\partial z}{\partial t}$, with initial condition, $z(x,0) = f^2(x)$, $\beta = \frac{\alpha}{A}$, and A is the average value of h. Werner uses a square root transformation in [Werner, 1957].

A comparison of these three methods must await our development of at least one solution of the heat equation.

One Solution of the Heat Equation: The Boltzman Transformation

The Boltzman transformation provides a way to turn some partial differential equations, including the heat equation $\frac{\partial^2 h}{\partial x^2} = \alpha\frac{\partial h}{\partial t}$, into ordinary differential equations. We look for a solution of the form $h(x,t) = y(xt^{-\frac{1}{2}})$. We first calculate the needed partial derivatives of h:

$$\frac{\partial h}{\partial x} = t^{-\frac{1}{2}}y'(xt^{-\frac{1}{2}}) \tag{46}$$

$$\frac{\partial^2 h}{\partial x^2} = t^{-1}y''(xt^{-\frac{1}{2}}) \tag{47}$$

$$\frac{\partial h}{\partial t} = -\frac{1}{2}xt^{-\frac{3}{2}}y'(xt^{-\frac{1}{2}}) \tag{48}$$

If we put $u = xt^{-\frac{1}{2}}$, substitute into the heat equation, and simplify, we obtain the ordinary differential equation,

$$y''(u) = -\frac{\alpha}{2}uy'(u)\ . \tag{49}$$

Or putting $\gamma = \frac{\alpha}{2}$, we obtain

$$y'' = -\gamma uy'\ . \tag{50}$$

We can separate the variables and solve the equation:

$$\frac{y''}{y'} = -\gamma u \tag{51}$$

$$\ln(y') = -\frac{\gamma u^2}{2} + C \tag{52}$$

$$y' = De^{-\frac{\gamma u^2}{2}} \tag{53}$$

$$y = E\mathrm{erf}\left(u\sqrt{\frac{\gamma}{2}}\right) + F \tag{54}$$

Reversing the substitution, $u = \frac{x}{\sqrt{t}}$ and $\gamma = \frac{\alpha}{2}$, we obtain the solution, $h = E\text{erf}\left(\frac{x\sqrt{\alpha}}{2\sqrt{t}}\right) + F$ to the heat equation, $\frac{\partial^2 h}{\partial x^2} = \alpha\frac{\partial h}{\partial t}$. Notice that since $\text{erf}(-\infty) = -1$, $\text{erf}(0) = 0$, and $\text{erf}(\infty) = 1$, the solution h has the property that, $h(x,0) = -E + F$ if $x < 0$, $h(x,0) = E + F$ if $x > 0$, and $h(0,t) = F$.

One Solution of the Heat Equation

$$
\begin{array}{rrcl}
 & \frac{\partial^2 h}{\partial x^2} & = & \alpha\frac{\partial h}{\partial t} \\
\text{Initial Condition} & h(x,0) & = & P, x > 0 \\
\text{Boundary Condition} & h(0,t) & = & Q \\
\text{Solution} & h(x,t) & = & (P-Q)\text{erf}\left(\frac{x\sqrt{\alpha}}{2\sqrt{t}}\right) + Q
\end{array}
$$

Applications of the solution above might seem to be limited by the nature of its initial condition. In fact its initial condition amounts to a heavyside or step function whose derivative is a Dirac Delta function. As we shall see, this will allow us to construct solutions of the heat equation for any initial condition. For the moment, it will at least allow us to explore some examples of the connections of the Boussinesq equation with the heat equation discussed in the previous section.

Comparing Routes from the Boussinesq to the Heat Equation

In order to compare the three routes that we have discussed from the Boussinesq to the Heat Equation, we need to look at a problem that has a closed form solution. One such is the *One Drain Problem*. It is the same as the two drain problem, except we consider the water table to extend across the entire positive x-axis with a single drain at the origin. We will use the same initial data as we did with the two drain problem. That is, we take $S = 0.2$, $K = 2$ meters per day, and the initial height of the water table to be 2 meters. We take the average value, A, of h to be 1. Thus we wish to solve:

$$\begin{aligned} \frac{\partial}{\partial x}(h\frac{\partial h}{\partial x}) &= 0.1\frac{\partial h}{\partial t} && (55)\\ \text{Initial Condition} \quad h(x,0) &= \begin{cases} 0 \text{ if } & x=0 \\ 2 \text{ if } & x>0 \end{cases} && (56)\\ \text{Boundary Condition} \quad h(0,t) &= 0 && (57) \end{aligned}$$

Solution given by the standard connection: We solve $\dfrac{\partial^2 h}{\partial x^2} = 0.1\dfrac{\partial h}{\partial t}$ with $h(x,0) = 2$ for $x > 0$ and $h(0,t) = 0$.

$$h(x,t) = 2\text{erf}\left(\frac{x}{2\sqrt{10t}}\right) . \tag{58}$$

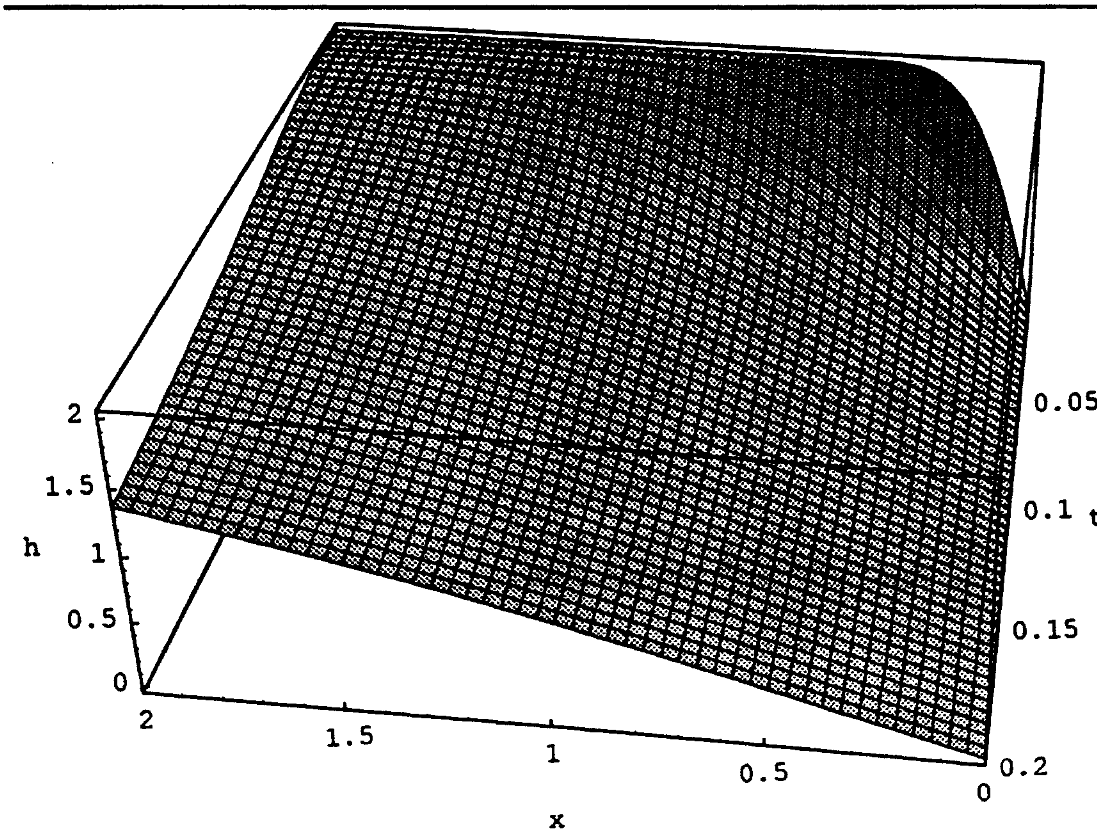

Figure 6: Solution of the one drain problem given by the standard method

Solution given by the square root connection: We solve $\dfrac{\partial^2 z}{\partial x^2} = 0.1\dfrac{\partial z}{\partial t}$ with

$z(x,0) = 4$ for $x > 0$, and $z(0,t) = 0$, and take the square root of the solution to obtain

$$h(x,t) = 2\sqrt{\operatorname{erf}\left(\frac{x}{2\sqrt{10t}}\right)}\,. \tag{59}$$

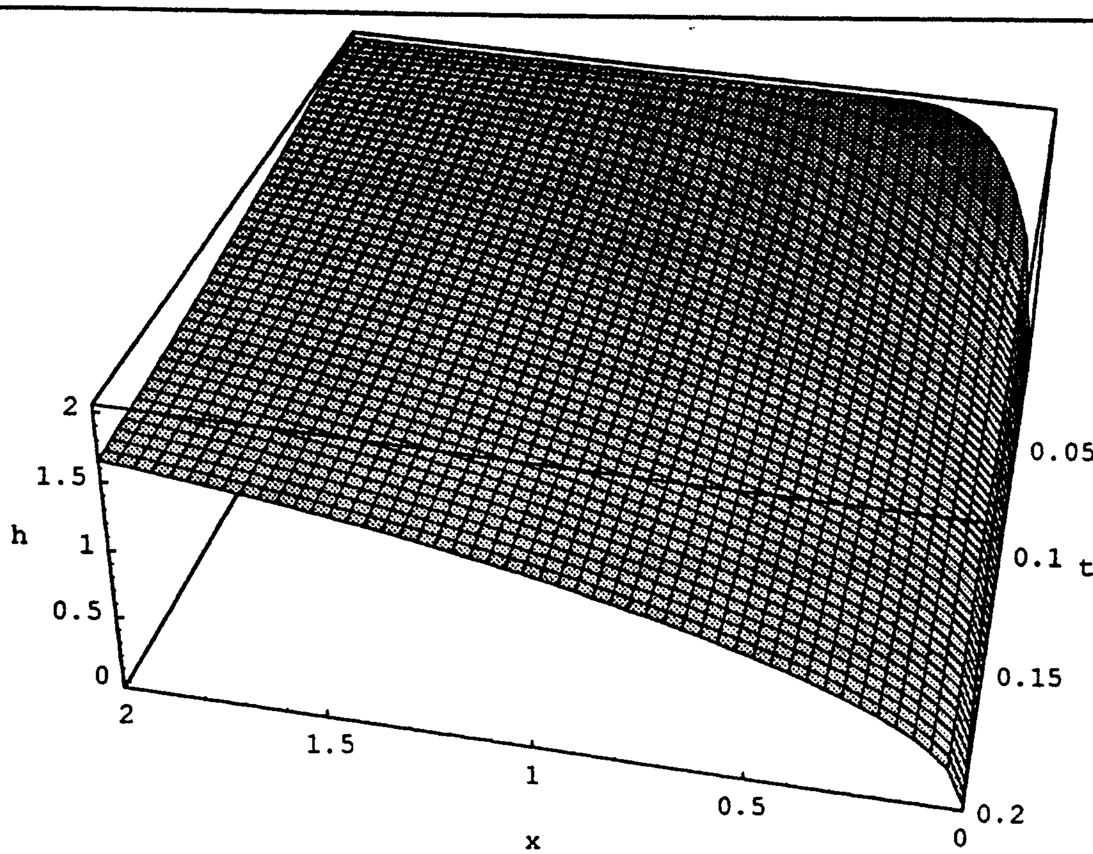

Figure 7: **Solution of the one drain problem given by the square root method**

Solution given by the logarithm connection: We solve $\dfrac{\partial^2 w}{\partial x^2} = 0.1\dfrac{\partial w}{\partial t}$, with $w(x,0) = e^2$, for $x > 0$, and $w(0,t) = 1$. The solution is $\ln w(x,t)$, or

$$h(x,t) = \ln\left((e^2 - 1)\operatorname{erf}\left(\frac{x}{2\sqrt{10t}}\right) + 1\right)\,. \tag{60}$$

Comparing three solutions: The three Figures, 6, 7, and 8, show graphs that are similar in shape, but it can be seen that the standard method predicts somewhat faster drainage than do either the square root or logarithm methods.

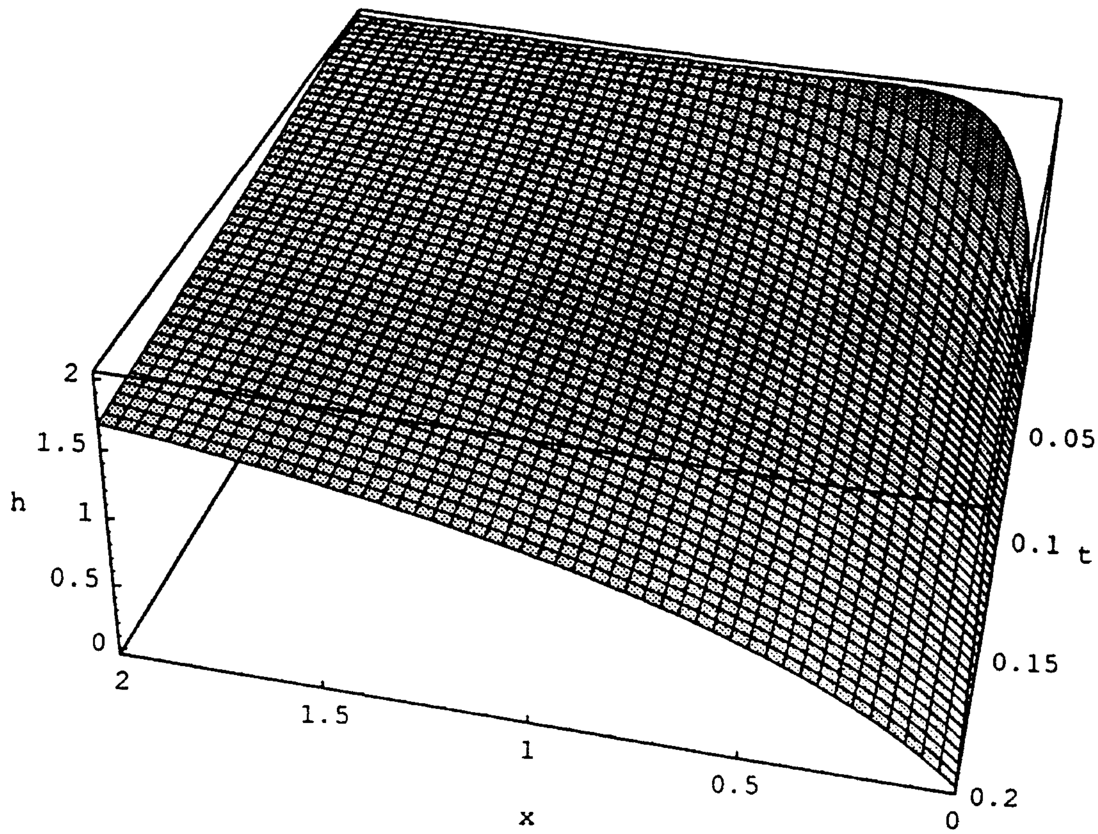

Figure 8: **Solution of the one drain problem given by the logarithm method**

In Figure 9 we fix t at 0.2 days and compare the three solutions as x varies. The bottom curve is the standard solution, the curve that is on top near the origin is the square root solution, and the logarithm solution begins between the square root and standard solutions but becomes the largest of the three for about $x > 1$. Observe that the logarithm and square root solutions are very close, but both predict a significantly slower drainage than does the standard solution.

In Figure 10 we fix x at 2 meters and compare the three solutions as t increases. The standard solution is the bottom curve, the logarithm solution is the middle curve, and the square root solution is on top. These show that the standard solution predicts much faster drainage at $x = 2$ meters than does either of the other two. The logarithm solution shows somewhat faster drainage than does the square root solution.

To make a further comparison of the three solutions, we calculate the volume per

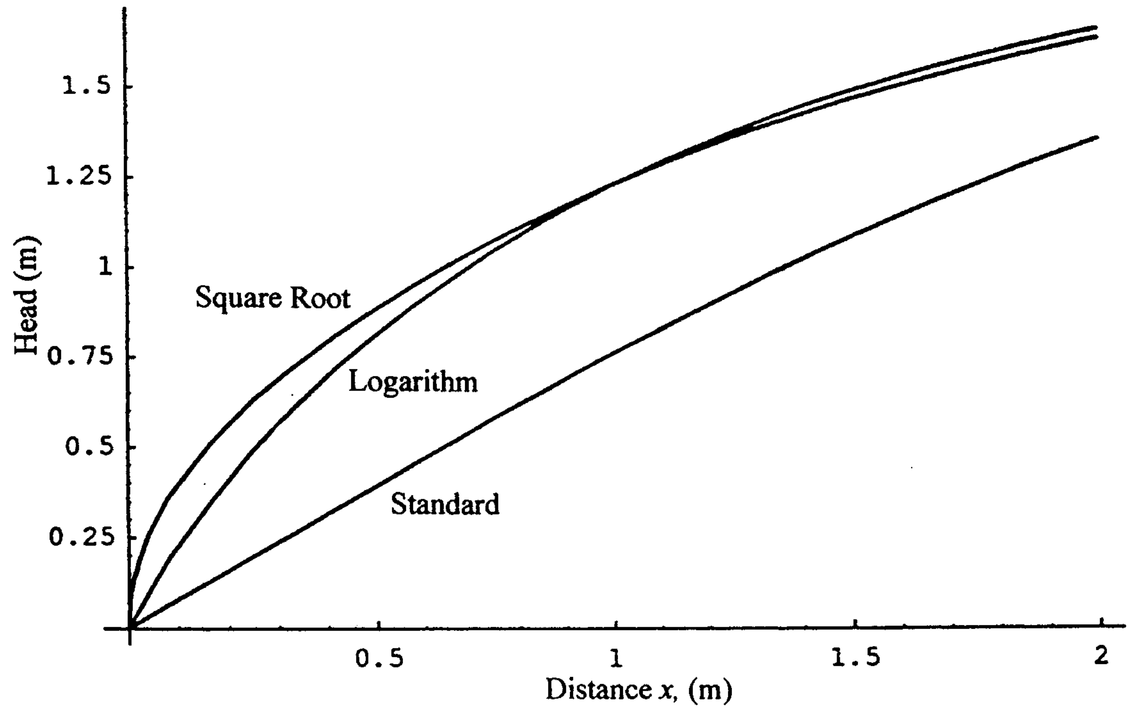

Figure 9: **Standard, square root, and log solutions for $t = 0.2$ days**

unit width of water that each of the three solutions predict will have entered the drain by time $t = 0.2$ days. This is given by

$$V(x = 0, t = 0.2) = \int_0^\infty 2 - h(x, 0.2)\; dx\ .$$

The values for the three methods are

$$\begin{aligned} \text{Standard} &= 3.19153m^3, \\ \text{Logarithm} &= 2.07031m^3, \text{and} \\ \text{Square root} &= 2.04716m^3. \end{aligned}$$

Therefore we can see that the standard solution predicts approximately 50% more drainage in the first .2 days than either of the other solutions.

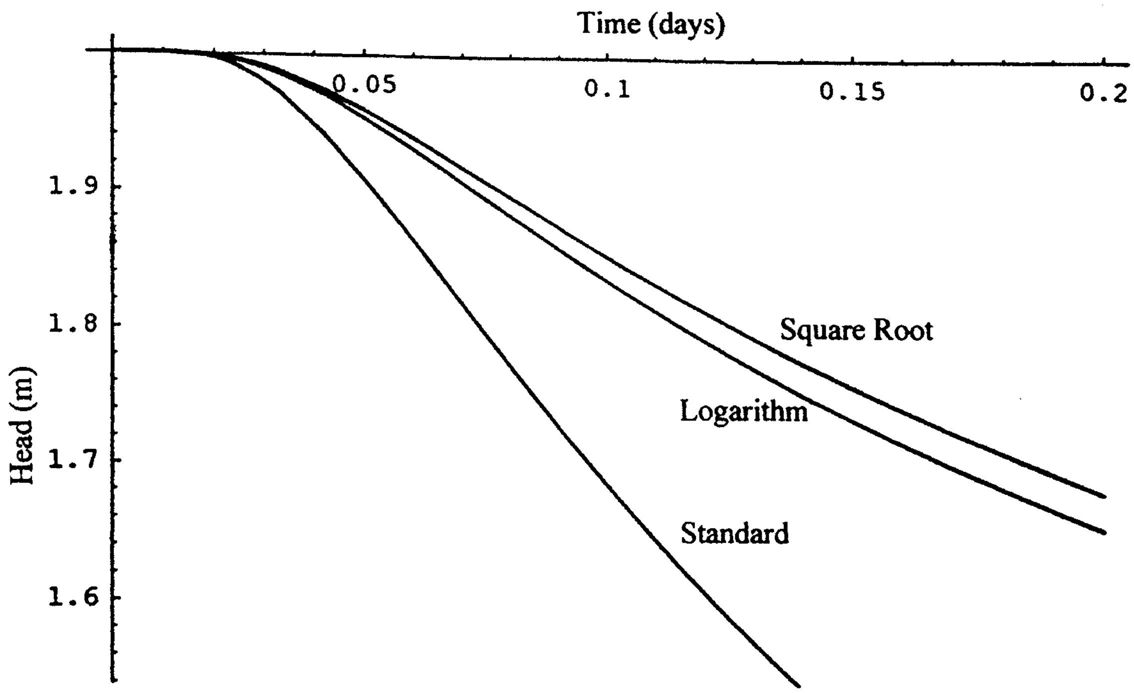

Figure 10: **Standard, square root, and log solutions for $x = 2$ meters**

Solution of the Heat Equation with Arbitrary Initial Condition

One of the features of nonlinear equations, such as the Boussinesq equation, that makes them difficult to handle is that familiar techniques for solving ordinary differential equations do not apply. In particular, any change in the boundary or initial conditions results in an entirely new problem, and solutions corresponding to different boundary conditions have no elementary connection. With ordinary differential equations, we can often obtain a general solution of the equation and then apply the initial condition to that. A simple example will illustrate this. Let's solve the differential equation $y' = 2x$ subject to the initial condition $y(0) = 2$. The general solution of the equation is $y = x^2 + C$, where C is an arbitrary constant. The initial condition tells us that $C = 2$ so that the solution of the initial value problem is $y = x^2 + 2$. If we change the initial condition to $y(0) = 5$, we can still use the general solution $y = x^2 + C$ to get the solution, $y = x^2 + 5$. For nonlinear partial differential equations, this technique is in general inapplicable; if you change the boundary condition, you have an entirely new problem. But for linear partial differential equations much of the spirit of the technique used above can be generalized in the sense that it is sometimes possible to obtain a particular solution, known as the *Green's function*, of the equation that can be used to generate solutions corresponding to any initial condition.

Consider a linear partial differential equation $L(h) = 0$. With initial condition $h(x,0) = f(x)$. The Green's function, $G(x,t)$, for $L(h) = 0$ is a solution of the equation such that $G(x,0)$ is the *Dirac Delta* function. That is $G(x,0)$ is identically 0 for $x \neq 0$, but $G(0,0) = \infty$. The following are the important properties of a Green's function [Friedman, 1956].

$$\begin{aligned} \text{Let} \;\; h(x,t) &= \int_{-\infty}^{\infty} f(u)G(x-u,t)\, du. \\ \text{Then} \;\; h(x,t) \text{ is a solution of } L(h) &= 0, \text{ and} \\ h(x,0) &= f(x) \end{aligned}$$

A Green's function for the heat equation: For linear equations such as the heat equation, if we take the partial derivative of a solution, we get another solution. We note that the solution of the heat equation that we obtained earlier, $h(x,t) = \mathrm{erf}\left(\frac{x\sqrt{\alpha}}{2\sqrt{t}}\right)$, satisfies the initial condition, $h(x,0)$ the *heavyside function*; that is, $h(x,0)$ is -1 for $x < 0$ and 1 for $x > 0$. Its derivative is the Dirac Delta function. Thus, we can take the partial derivative with respect to x of this solution to obtain a Green's function for the heat equation [Friedman, 1956].

Green's function for the heat equation

$$\begin{aligned}\frac{\partial^2 h}{\partial x^2} &= \alpha\frac{\partial h}{\partial t}\\ G(x,t) &= \sqrt{\frac{\alpha}{4\pi t}}e^{-\frac{\alpha x^2}{4t}}\end{aligned}$$

The solution of $\dfrac{\partial^2 h}{\partial x^2} = \alpha\dfrac{\partial h}{\partial t}$
with initial condition, $h(x,0) = f(x)$,
is $\int_{-\infty}^{\infty} f(u)G(x-u,t)\,du$

Initial conditions on the x-axis and symmetry: If we wish to use the Green's function to provide an alternate method for solving the one drain problem, we immediately encounter a problem; the Green's function that we have let's us specify an initial condition $f(x)$ that is valid on the entire x-axis, but it does not directly allow us to address the boundary condition on the line, $x = 0$. A simple trick using symmetry will allow us to fix the problem. If we insist that $h(x,0)$ be 2 for $x > 0$ and -2 for $x < 0$, symmetry will insure that the boundary condition, $h(0,t) = 0$ holds for all t. Thus, for this problem, we need to use the initial condition

$$f(x) = \begin{cases} 2, & x > 0 \\ -2, & x < 0 \end{cases} \quad . \tag{61}$$

The solution is

$$h(x,t) = \int_{-\infty}^{0} -2G(x-u,t)\,du \;+\; \int_{0}^{\infty} 2G(x-u,t)\,du = 2\mathrm{erf}\left(\frac{x}{2\sqrt{10t}}\right) \quad . \tag{62}$$

We note that this is identical with the standard solution we obtained earlier.

The Green's function provides real power in dealing with different initial conditions for the heat equation, but it does not directly address the boundary conditions. In practice, this difficulty can often be overcome using symmetry. Ideally we would like a Green's function for the heat equation that was zero on the boundary of the rectangular region, $-1 < x < 1,\ t > 0$ except at the origin where it is infinite. Solutions for given initial and boundary conditions could then be obtained by taking a line integral over the boundary of the region. Unfortunately the author does not know how to make such a function.

An erf Series Solution to the Two Drain Problem

With the added power we have in solving the heat equation and connecting with the Boussinesq equation, we can apply more robust methods to the two drain problem.

We are looking at the following problem:

$$\frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right) = \alpha\frac{\partial h}{\partial t} \tag{63}$$

$$\text{Initial Condition}\quad h(x,0) = M,\ \text{for } 0 < x < L \tag{64}$$

$$\text{Boundary Condition}\quad h(0,t) = h(L,t) = 0 \tag{65}$$

We select the square root connection to the heat equation. Thus we need to solve:

$$\frac{\partial^2 z}{\partial x^2} = \beta\frac{\partial z}{\partial t} \tag{66}$$

$$\text{Initial Condition}\quad z(x,0) = M^2,\ \text{for } 0 < x < L \tag{67}$$

$$\text{Boundary Condition}\quad z(0,t) = z(L,t) = 0 \tag{68}$$

We will accomplish this using the Green's function, but care is needed in selecting the initial condition, $f(x)$. We want $f(x) = M^2$, for $0 < x < L$, and we need to extend f across the entire x-axis so that symmetry will insure that the solution is zero for all t at $x = 0$ and $x = L$. Thus, f needs to be an odd function about both $x = 0$, and $x = L$. This requires that f be a square wave function that is M^2 on the

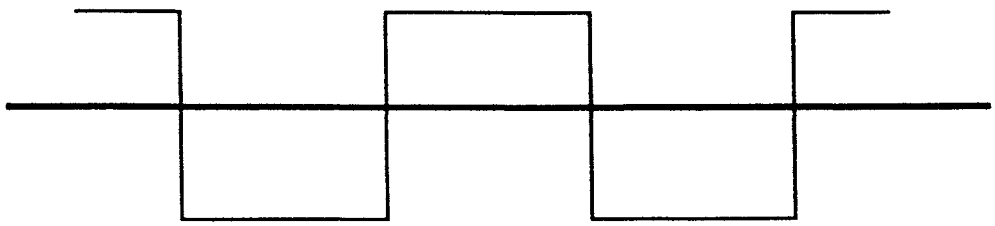

Figure 11: **A square wave function**

intervals, $[2kL,(2k+1)L]$, and $-M^2$ on the intervals, $[(2k+1)L, 2(k+1)L]$. (See Figure 11).

Now $\int_{-\infty}^{\infty} f(u)G(x-u,t)\,du$ breaks up into an infinite series of integrals of the form

$$\pm\int_{kL}^{(k+1)L} M^2\sqrt{\frac{\alpha}{4\pi t}}e^{-\frac{\alpha(x-u)^2}{4t}}\,du.$$

These integrals evaluate to

$$\frac{M^2}{2}\left(\operatorname{erf}\left(\frac{(x-kL)\sqrt{\frac{\alpha}{t}}}{2}\right)-\operatorname{erf}\left(\frac{(x-(k+1)L)\sqrt{\frac{\alpha}{t}}}{2}\right)\right).$$

If we use only the first three terms of the series and take the square root, we obtain

$$M\left(\operatorname{erf}\left(\frac{\sqrt{a}}{\sqrt{2t}}(x-4L)\right)-2\operatorname{erf}\left(\frac{\sqrt{a}}{\sqrt{2t}}(x-3L)\right)+2\operatorname{erf}\left(\frac{\sqrt{a}}{\sqrt{2t}}(x-2L)\right)-2\operatorname{erf}\left(\frac{\sqrt{a}}{\sqrt{2t}}(x-L)\right)\right.$$

$$+2\text{erf}\left(\frac{\sqrt{a}}{\sqrt{2t}}(x)\right) - 2\text{erf}\left(\frac{\sqrt{a}}{\sqrt{2t}}(x+L)\right) + 2\text{erf}\left(\frac{\sqrt{a}}{\sqrt{2t}}(x+2L)\right) - \text{erf}\left(\frac{\sqrt{a}}{\sqrt{2t}}(x+3L)\right)\Bigg)^{\frac{1}{2}}. \tag{69}$$

The graph of this approximate solution of the two drain problem appears in Figure 12. It is important to note that this technique applies without change to other initial value problems. Given any initial condition, $f(x)$, defined on the x-axis, we get an approximate solution of the Boussinesq equation immediately by calculating

$$h(x,t) = \sqrt{\int_{-\infty}^{\infty} f^2(u)G(x-u,t)\,du}. \tag{70}$$

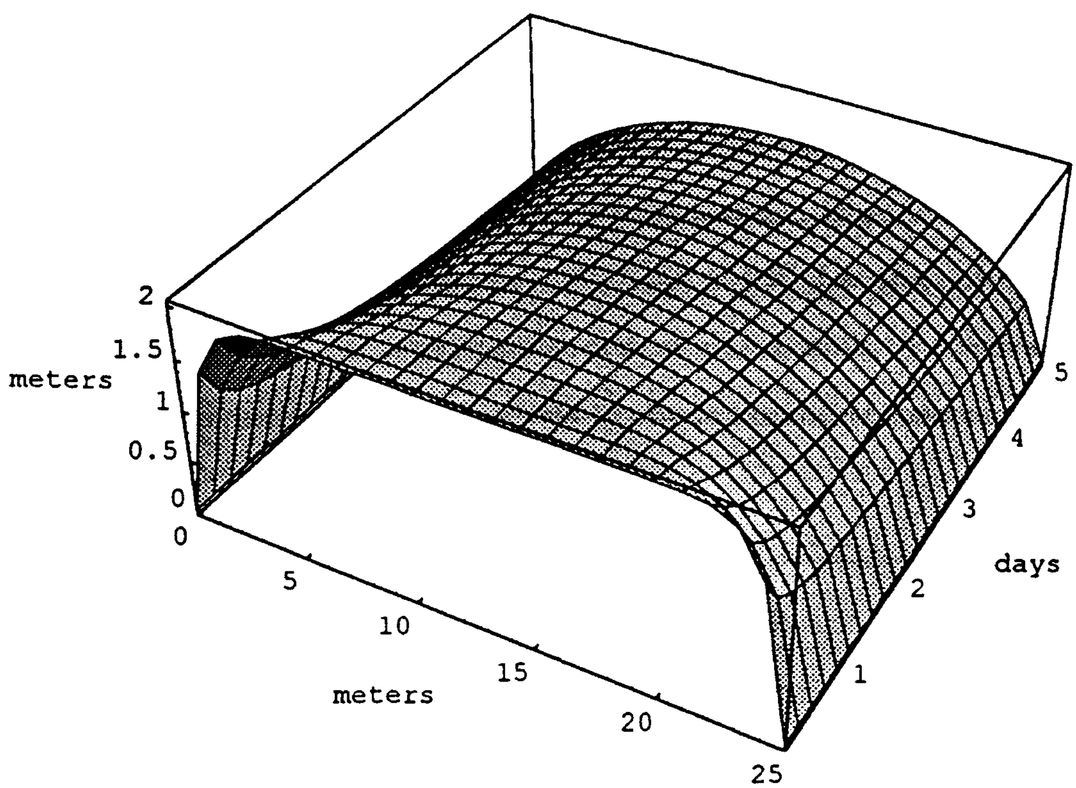

Figure 12: **An approximate solution of the two drain problem**

Conclusion

The separable approach to the Boussinesq equation provides a direct (if implicit) solution, but it is severely limited in its application due to restrictions imposed on the initial condition by the separability assumption.

One method commonly used to avoid these difficulties is to relate the Boussinesq equation with a well studied linear differential equation; the heat equation. One relation that appears in literature cited here makes this connection by replacing an occurrence of the variable head by an estimated average value prior to expanding a partial derivative of a product. Two other relations, referred to here as the square root connection and the logarithm connection, avoid this numerically unsound technique. In examples that have closed form solutions, in particular the *one drain problem*, the square root and logarithm methods provide similar predictions that are significantly different from those of the standard connection.

Once the connection with the heat equation is established, there are many well known methods for obtaining solutions. We use here the Green's function technique because it is so easily adaptable to a wide range of initial conditions. In particular, for the two drain problem, it provides a series of erf functions that converges rapidly enough that only a few terms are needed to obtain accurate predictions.

References

Bear, J. 1972. Dynamics of Fluids in Porous Media. American Elsevier Publishing Co., New York.

Boussinesq, M.J. 1904. Recherches theoriques sur l'ecoulement des nappes d'eau infiltrees dans le sol et sur le degit des sources. Journal of Mathematiques. tome X, Fasc. I, p. 5.

Brooks, R.H. 1961. Unsteady Flow of Ground Water Into Drain Tile. Journal of the Irrigation and Drainage Division, Proceedings of the American Society of Civil Engineers. v. 87, pp. 27-37.

Dumm, Lee D. 1954. Drain-Spacing Formula. Agricultural Engineering. October. pp. 726-730.

Friedman, B. 1956. Principles and Techniques of Applied Mathematics. John Wiley and Sons, New York.

Kirkham, Don. 1964. Physical artifices and formulas for approximating water table fall in tile-drained land. Soil Science Society of America, Proc. 28, pp. 585-590.

McWhorter, David B. and Harold R. Duke. 1976. Transient Drainage with Non-linearity and Capillarity. Journal of the Irrigation and Drainage Division, Proceedings of the American Society of Civil Engineers. v. 102, pp. 193-204.

McWhorter, David B. and Daniel K. Suneda. 1977. Ground-water Hydrology and Hydraulics. Water Resources Publications.

van Schilfgaarde, J. 1963. Design of Tile Drainage for Falling Water Tables. Journal of the Irrigation and Drainage Division, Proceedings of the American Society of Civil Engineers. v. 89, pp. 1-11.

van Schilfgaarde, J. 1964. Closing discussion: design of tile drainage for falling water tables. American Society of Civil Engineers, Proc. 90(IR3), pp. 71-73.

Werner, P. 1957. Some Problems in Non-Artesian Ground-Water Flow. Transactions, American Geophysical Union. v. 38, pp. 511-518.

Youngs, E.G. 1966. Exact Analysis of Certain Problems of Ground-Water Flow with Free Surface Conditions. Journal of Hydrology. v. 4, pp. 277-281.

Zauderer, E. 1983. Partial Differential Equations of Applied Mathematics. John Wiley and Sons, New York.

VITA

Vicke' A. Evans

Candidate for the Degree of

Master of Science

Thesis: APPROXIMATE ANALYTICAL SOLUTIONS OF THE BOUSSINESQ EQUATION

Major Field: Environmental Science

Biographical:

Personal Data: Born in Redwood City, California, on July 13, 1958. The daughter of Roswell and Jonell Adams.

Education: Graduated from Northeastern State University, Tahlequah, Oklahoma, in August 1984; received Bachelor of Science degree in Mathematics. Completed the requirements for the Master of Science degree with a major in Environmental Science in December 1994.

Experience: Currently employed by Frontier Engineering as a software engineer; software team leader in the Aircraft System group; employed by Frontier Engineering since 1989. Previous employment includes analyst at Stillwater Milling Co. and lecturer at Texas A&M University.