A METHOD FOR CONSTRUCTING NONDOMINATED K-COTERIES

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CHAPTER 1

INTRODUCTION

A distributed system may have shared resources which must be accessed in a mutually exclusive way. If a set of k identical resources may be simultaneously accessed by processes, we say that multiple entries to critical sections are allowed. The concept of a coterie introduced by Garcia-Molina and Barbara [GMB85] can be extended to be used in the distributed multiple mutual exclusion problems [HJK93, KFYA91]. If a shared resource allows up to k processes to enter critical sections, it is called the k-mutual exclusion problem. The distributed k-mutual exclusion problem is the problem of managing processes in a distributed system in such a way that at most k processes can enter their critical sections simultaneously. Several distributed k-mutual exclusion algorithms have been proposed [KFYA91, KFYA93, FY91, NM92, HJK93].

For example, suppose that there are k servers that contain identical license resources that are shared by nodes in a distributed system. Each license resource may only be accessed by one node at a time, and each node may access at most one license resource at a time [NM94]. In this situation, a mutual exclusion algorithm can be used to control access to the servers.

Huang, Jiang, and Kuo [HJK93] proposed the concept of a k-coterie, which is the extension of the concept of coterie introduced by Garcia-Molina and Barbara. The definition of a k-coterie proposed by Huang, Jiang, and Kuo satisfies two properties: intersection property and minimality property. Given a set of processes [nodes] S in the system, a k-coterie under S is a collection of subsets of S in which any $k+1$ subsets have a non-empty intersection. This property is called the intersection property. The intersection property guarantees that at most k processes can enter their critical sections. The other property, minimality property, says that any two distinct quorums are not a subset of each other.

Independently, Kakugawa, Fujita,Yamashita, and Ae [KFYA91] proposed the same concept of a k-coterie. The definition for a k-coterie given by Kakugawa et. al. is more restrictive. Three properties must be satisfied: intersection property, minimality property, and non-intersection property. The non-intersection property assures that up to k processes can enter their critical sections. A subset of a k-coterie is called a quorum.

Garcia-Molina and Barbara [GMB85] clasified coteries into two categories: Dominated and Nondominated. Nondominated coteries are the most resilient coteries [NM94]. Since the nondominated coteries are the most resilient, it is beneficial to find as general as possible a method for constructing them [IK93]. Some researchers have observed and analyzed the advantages of using nondominated coteries [NM94, GMB85].

1.1 Thesis

We propose a method for constructing nondominated k-coteries for any value of N, the number of nodes in a distributed system, and k, the number of processes allowed to enter their critical sections simultaneously. The proposed method is an extension of the Maj_k method introduced by Kakugawa, Fujita, Yamashita, and Ae [KFYA93]. An equivalent vote assignment is also introduced in this thesis. Using this vote assignment, we can easily construct nondominated k-coteries.

1.2 Organization

The thesis is organized into the following chapters:

- Chapter 2: Detailed basic theory of the coterie and its characteristics.

- Chapter 3: The proposed algorithm is introduced.

- Chapter 4: Analysis of the proposed algorithm is evaluated.
- Chapter 5: An equivalent vote assigment model for the proposed algorithm is introduced and the correctness of the model is also evaluated.
- Chapter 6: Summary and future work.

CHAPTER 2

BASIC THEORY

A k-coterie C is a set of subsets (also called quorums) of an underlying set of nodes, such that in any collection of $k+1$ pairwise quorums there exists at least two quorums that intersect each other. This concept, introduced in two papers, [HJK93] and [KFYA91], independently, is an extension of the concept of a coterie. Huang, Jiang, and Kuo [HJK93] defined a set C to be a k-coterie if it satisfies two properties: intersection and minimality properties. Intersection property assures that at most k processes can enter their critical sections. Minimality property says that there is no quorum in C which is a subset of the others. The second paper was written by Kakugawa, Fujita, Yamashita, and Ae [KFYA91]. In the second paper, the definition of a k-coterie is more restrictive. A set C is said to be a k-coterie if it satisfies three properties: minimality, intersection, and non-intersection properties.

First, we consider the definition of a k-coterie proposed in [HJK93].

Definition 1. k-coterie.

Let S be a set of N nodes in the system and let k be a natural number ($k \leq N$), respectively. Then a set of subsets C which satisfies the following two conditions is called a *k-coterie* under S:

1. Intersection Property.

   For any k+1-set $\{Q_1, Q_2, \ldots, Q_{k+1}\} \subseteq C$, there exists two elements Q_i and Q_j such that $Q_i \cap Q_j \neq \emptyset$.

2. Minimality Property.

   For any two distinct elements Q_i and Q_j in C, $Q_i \not\subset Q_j$.

An element Q of C is called a *quorum*.□

In this thesis, we clasifiy a new class of k-coteries: *proper k-coterie.* A set C is called a proper k-coterie when it satisfies three properties as said in [KFYA91]. The first two properties are exactly the same as in Definition 1. The third property is the non-intersection property.

Definition 2. Proper k-coterie.

A set C of subsets (quorums) is said a *proper k-coterie* if it is a k-coterie and satisfies the non-intersection property.

- Non-intersection Property.
  For any integer $h < k$, if an h-set $\{Q_1, Q_2, \ldots, Q_h\} \subseteq C$ satisfies $Q_i \cap Q_j = \emptyset$, for all $i \neq j$, $1 \leq i, j \leq h$, then there exists an element $Q \in C$, such that $Q \cap Q_i = \emptyset$ for all $1 \leq i \leq h$. $\square$

The non-intersection property guarantees that even if $h(< k)$ processes have received permission from h quorums and are in their critical sections, a process can find a quorum that does not intersect with each of the h quorums. In other words, even if $h(< k)$ processes are in their critical sections, another process can still enter its critical section.

2.1 Dominated and Nondominated Coteries

Some characteristics of a coterie also have been observed [GMB85, KFYA93, FY91, NM94]. These characteristics relate to some metrics to measure the goodness of a coterie. Garcia-Molina and Barbara classified coteries into two types: dominated and nondominated coteries [GMB85]. Nondominated coteries are the most resilient to network and site failures.[NM94].

Definition 3. Dominated k-coterie.

Let C_1 and C_2 be k-coteries under S. Then, C_1 *dominates* C_2 iff

1. $C_1 \neq C_2$

2. $\forall H \in C_2$, then $\exists Q \in C_1$, such that $Q \subseteq H$.$\square$

A k-coterie C under S is said a dominated k-coterie if there is another k-coterie under S that dominates C. If there is no such k-coterie, then C is a nondominated k-coterie[NM92]. On the basis of this definition (dominated k-coterie), Neilsen and Mizuno proposed a simple method to determine if a k-coterie is dominated.

Theorem 1. *A k-coterie C is said dominated if and only if there exists a set $H \subseteq S$ that satisfies two properties:*

1. $\forall Q \in C, Q \not\subseteq H$.

2. For any collection of k pairwise disjoint quorums $\{Q_1, Q_2, \ldots, Q_k\} \subseteq C$, $H \cap Q_i \neq \emptyset$, for some $1 \leq i \leq k$.

By this theorem, a k-coterie can be determined whether it is dominated or not by finding a subset $H \subseteq S$ satisfying the two conditions, not necessarily finding another k-coterie that dominates it. As mentioned above, nondominated k-coteries are more resilient to network and site failures; this means that for some nodes failure where a quorum cannot be constructed in a dominated k-coterie, in a nondominated coterie the quorum can be formed. If C_2 is dominated by C_1, then any quorum in C_2 contains a quorum in C_1. In other words, if a quorum from C_2 can be constructed, then the quorum in C_1 can also be constructed, but not conversely.

For an example, $S = \{1, 2, 3, 4\}$ and

$$C_2 = \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}$$

is a dominated 2-coterie under S. It can be seen that when $H = \{1\}$, then H satisfies the above two properties. But for

$$C_1 = \{\{1\}, \{2,3\}, \{2,4\}, \{3,4\}\},$$

C_1 is a nondominated 2-coterie, since there is no H satisfying two properties in Theorem 1. Moreover, C_1 dominates C_2. From the example, it can be seen that every quorum in C_2 contains a quorum in C_1.

2.2 Symmetric Coteries

Fujita and Yamashita[FY91] classified coteries into so called symmetric coteries. A coterie is said to be symmetric if it satisfies the following two properties:

1. all quorums in coterie C have the same size,

2. each element in the set of processes in a distributed system S occurs in the same number of quorums.

These conditions assure that every process has an equal right to give permission. In other words, the k mutual exclusion problem can be solved in a distributed manner by constructing symmetric k-coteries.

2.2.1 The *Cube* and The *Hypercube* Methods

Fujita and Yamashita[FY91] proposed two basic algorithms, called the *Cube* and *Hypercube* algorithms, producing coteries that satisfy two symmetric conditions for a good coterie. Even though the coteries produced by these algorithms are dominated coteries, it is beneficial to consider them as good coteries based on their message complexities. Since each quorum has the same size and each element of nodes occurs in the same number of quorums, the algorithm also produces a solution to solve distributed k-mutual exclusion problems in such a way that every process has equal right to grant resource access requests. Unfortunately, the *Cube* and *Hypercube* algorithms have limitations in producing such coteries. For some combination of N (number of nodes in the system) and k (number of processes allowed to enter critical sections), these algorithms cannot produce a symmetric coterie.

The *Cube* method can produce a symmetric k-coterie if $k = N^{(1/3)^i}$ [FY91]. This method also produces symmetric coteries that asymptotically achieve a lower bound $\sqrt{N/k}$ on the quorum size. The quorum size of k-coterie produced by the *Cube* method is

$$3^i N^{(1-(1/3)^i)/2}.$$

If $N^{(1/3)^i}$ is not an integer, the algorithm may not produce a symmetric k-coterie. The *Hypercube* method can also produce symmetric k-coteries if

$$k = N^{1/(2d-1)}.$$

The quorum size q of k-coteries constructed by the *Hypercube* is

$$(k-1)^{d-1} < q < k^{d-1}4^d.$$

The *Hypercube* method may not produce a symmetric k-coteire if $N^{1/(2d-1)}$ is not an integer. For this situation, neither the *Cube* nor *Hypercube* produce symmetric k-coterie, Fujita, Yamashita, and Ae suggested to combine with another method such as Maj_k. Although the k-coteries constructed by these two methods have, in average, smaller quorum sizes, the k-coteries are dominated coteries.

To show that the *Cube* method, for example, produces dominated k-coteries is by taking a look at an example for $N = 8$. Then, the *Cube* method can produce 2-coterie ($N^{1/3}$-coterie),

$$C = \{\{0,1,2,4\}, \{0,1,3,5\}, \{0,2,3,6\}, \{0,4,5,6\},$$

$$\{1,2,3,7\},\{1,4,5,7\},\{2,4,6,7\},\{3,5,6,7\}\}$$

It is easy to see that if $H = \{0,3,4,7\}$, then $H \subseteq S = \{0,1,2,3,4,5,6,7\}$, the subset $H \subseteq S$ satisfies the two conditions of Theorem 1; that is, (1) for all $Q \in C$, $Q \not\subseteq H$, and (2) for any 2 pairwise disjoint quorums $\{Q_1, Q_2\}$, $Q_i \cap H \neq \emptyset$, for some $1 \leq i \leq 2$.

2.2.2 The Maj_k Method

Another simple and excellent algorithm that produces symmetric coteries is Maj_k algorithm. The algorithm produces a coterie in which each quorum has the same size [KFYA93]. Even for some combinations of N and k; i.e., when $(N+1) = (k+1)w$, where w is an integer, this algorithm can produce a nondominated coterie that is more resilient to network and site failures[HJK93].

For any N nodes in the distributed system and a natural number k, the majority k-coterie C is defined as a collection of subsets of S, where each subset has the size of w and $w = \lceil (N+1)/(k+1) \rceil$, the coterie $C = \{Q \| Q| = w\}$. The algorithm is very simple, but the k-coterie produced is consider one of the good k-coteries. Although it is a good and simple algorithm, there are some drawbacks. For some combinations of N and k, the algorithm cannot produce exactly a k-coterie. As an example, for $N = 5$ and $k = 3$, then $w = 2$. The 3-coterie constructed is

$$C = \{\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{2,3\},$$

$$\{2,4\},\{2,5\},\{3,4\},\{3,5\},\{4,5\}\}.$$

It is easy to see that this coterie cannot grant 3 processes to enter critical sections simultaneously; only two processes are granted. So, for this combination $N = 5$ and $k = 3$, Maj_k cannot produce a 3-coterie. For the combination $N = 6$, $k = 2$, the 2-coterie constructed by the Maj_k algorithm is a dominated coterie. The coterie, constructed by the Maj_k algorithm,

$$C = \{\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,2,6\},\{1,3,4\},\{1,3,5\},\{1,3,6\},$$

$$\{1,4,5\},\{1,4,6\},\{1,5,6\},\{2,3,4\},\{2,3,5\},\{2,3,6\},\{2,4,5\},$$

$$\{2,4,6\},\{2,5,6\},\{3,4,5\},\{3,4,6\},\{3,5,6\},\{4,5,6\}\}$$

is a dominated 2-coterie. To show that the coterie is a dominated coterie, we can take $H = \{1\}(\subseteq S)$, where for all quorums $Q \in C$, $Q \not\subseteq H$, but for any collection of 2 pair disjoint quorums, $Q_i \cap H \neq \emptyset$, for some $1 \leq i \leq 2$. In other words, for the combination of $N = 6$ and $k = 2$, the Maj_k method cannot produce nondominated k-coteries.

CHAPTER 3

PROPOSED ALGORITHM

3.1 Motivation

In this thesis, we propose an algorithm that constructs nondominated k-coteries for any combination of N (number of nodes in the system) and k (number of processes allowed to enter critical sections simultaneously). Since the nondominated coteries are the most resilient to network and node failures, it is beneficial to consider the construction methods producing nondominated k-coterie. This algorithm is an extension of the Maj_k algorithm. The algorithm proposed works for every combination of N (number of nodes) and k (number of processes allowed to enter critical sections). By constructing coteries for any combination of N and k by this algorithm, the problems in the construction of coteries by the *Cube*, the *Hypercube*, and the Maj_k algorithms are solved for any value of N and k. However, the algorithm may not construct so-called symmetric coteries.

The idea of this algorithm is to avoid some cases for which the Maj_k method cannot produce exactly a k-coterie. Moreover, this algorithm produces nondominated k-coteries for any combination of N and k. The important thing of this method is in reducing the number of votes for some nodes. Later in Chapter 5, we introduce vote assignments and how some nodes have more votes than the others.

3.2 Algorithm

Let $S = \{a_1, a_2, \ldots, a_N\}$ is a set of N nodes in a distributed system where N is a non-negative integer. Let k be an integer, where $1 \leq k \leq N$, representing the number of processes that can enter to their critical sections simultaneously.

Algorithm.

1. Set $w = \lceil \frac{N+1}{k+1} \rceil$.

2. Set $m = (k+1)w - (N+1)$.

3. Set $C' = \{Q \subseteq S \| Q| = w\}$.

4. Let E be a set of m elements of S, $\{e_1, e_2, \ldots, e_m\}$.

5. Set $P = \{Q \in C' | E \cap Q \neq \emptyset\}$.

6. Set $P' = \emptyset$.

7. If $m \leq \frac{w-1}{2}$ then
   for $i = 1$ to m
   $P' = P' \cup \{Q \subseteq S \| Q \cap E| = i$, and $|Q| = w - i\}$.

8. else ($m > \frac{w-1}{2}$)

   (a) $min = \lfloor \frac{w-1}{2} \rfloor + 1$.
   (b) $P' = P' \cup \{Q \subseteq E| \ |Q| = min\}$.
   (c) for $i = 1$ to $min - 1$
   $P' = P' \cup \{Q \subseteq S \| Q \cap E| = i$, and $|Q| = w - i\}$.

9. Set $C = (C' - P) \cup P'$.

From the algorithm, quorums that contain exactly one element of a set

$$E = \{e_1, e_2, \ldots, e_m\}$$

constructed by the algorithm have the size $(w-1)$. The quorums that contain two elements of E have the size $(w-2)$. Generally speaking, the quorums that contain h elements of E have the size $(w-h)$, where $1 \leq h \leq min$. There is an exception when w is odd. Since $min = \lfloor \frac{w-1}{2} \rfloor + 1$, then $min = \frac{w+1}{2}$. We can see that $w - min < min$. In this case, where $Q \subseteq E$, the quorums do not follow the above rule saying that quorums containing h elements of E have the size $(w-h)$.

Since $m = (k+1)w - (N+1)$, it is easy to see that $m \leq k$. It is also easy to see that any collection of k pairwise disjoint quorums contains at least $(N - w + 1)$ nodes. Any k pairwise disjoint quorums contain exactly $N - w + 1$ elements of S (nodes) if all quorums forming it have size $(w-1)$ or $(N+1) = (k+1)n$, where n is an integer.

To show how the algorithm works, take a look at an example for $N = 6$, and $k = 2$. After step 3, we get $w = 3$, $m = 2$, and

$$C' = \{\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,2,6\},\{1,3,4\},\{1,3,5\},\{1,3,6\},$$
$$\{1,4,5\},\{1,4,6\},\{1,5,6\},\{2,3,4\},\{2,3,5\},\{2,3,6\},\{2,4,5\},$$
$$\{2,4,6\},\{2,5,6\},\{3,4,5\},\{3,4,6\},\{3,5,6\},\{4,5,6\}\}.$$

Assume $E = \{1,2\}$, then after step 8, we have

$$P = \{\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,2,6\},\{1,3,4\},\{1,3,5\},\{1,3,6\},$$
$$\{1,4,5\},\{1,4,6\},\{1,5,6\},\{2,3,4\},\{2,3,5\},\{2,3,6\},\{2,4,5\},$$
$$\{2,4,6\},\{2,5,6\}\}.$$

and

$$P' = \{\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{1,6\},\{2,3\},\{2,4\},\{2,5\},\{2,6\}\}$$

Finally, we have a 2-coterie C

$$C = \{\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{1,6\},\{2,3\},\{2,4\},\{2,5\},$$
$$\{2,6\},\{3,4,5\},\{3,4,6\},\{3,5,6\},\{4,5,6\}\}.$$

It is easy to see that C is a 2-coterie and nondominated. It can also be seen that for any collection of 2 pairwise disjoint quorums $\{Q_1, Q_2\}$, the number of elements (nodes) involved is between 4 ($\geq N - w + 1$) and 6 ($\leq N$). For $N = 5$ and $k = 3$, the algorithm produces

$$C = \{\{1\}, \{2\}, \{3,4\}, \{3,5\}, \{4,5\}\}.$$

This set C is a 3-coterie and even nondominated.

CHAPTER 4

ANALYSIS

To evaluate the correctness of the algorithm, we have to obtain properties that lead to the conclusion that C is a k-coterie. Furthermore, since the objection of the algorithm is to construct nondominated k-coterie, we also demonstrate that C is a nondominated k coterie.

Before we get to the conclusion, there are some properties (Lemma) that are obtained from the algorithm. Since the idea of the algorithm is to avoid most cases in which the Maj_k method produces dominated k-coterie and not-exactly k-coterie by selecting m special elements, we have interesting properties of m.

Lemma 0. $m \leq k$.

Proof:

Assume $m > k$. Since m and k are integers, let $m = k + 1 + \epsilon$, for $\epsilon \geq 0$. Since $m = (k+1)w - (N+1)$ we have

$$\begin{aligned} N+1 &= (k+1)w - m \\ &= (k+1)w - (k+1+\epsilon) \\ &= (k+1)(w-1) - \epsilon \\ &\leq (k+1)(w-1) \\ \lceil \frac{N+1}{k+1} \rceil &\leq w - 1. \end{aligned}$$

This contradicts that $\lceil \frac{N+1}{k+1} \rceil = w$. □

Lemma 1. $m(w-1) \leq N$

Proof:

Since $\lceil \frac{N+1}{k+1} \rceil = w$, then $w - 1 < \lceil \frac{N+1}{k+1} \rceil$ or

$$(k+1)(w-1) \;<\; N+1$$

$$\begin{aligned}(k+1)(w-1) &\leq N \\ k(w-1) &\leq N\end{aligned}$$

By Lemma 0, $m(w-1) \leq N$. This completes the proof.$\square$

Lemma 2. *Any collection of k pairwise disjoint quorums contains at least $N-w+1$ elements.*

Proof:

Let $R = \{Q_1, Q_2, \ldots, Q_k\}$ be a collection of k pairwise disjoint quorums. There are two cases to consider.

1. If $\lceil \frac{N+1}{k+1} \rceil = w$, where w is an integer. The number of elements is $kw = N-w+1$.

2. If $\frac{N+1}{k+1}$ is not an integer. There are two cases to consider.

   (a) $|E \cap (\cup_{i=1}^{k} Q_i)| = m$. There are two possibilities.

   - There are m quorums in R that have size $(w-1)$. In other words, there are m quorums in R that contain exactly one element of E. The number of elements in R is

   $$\begin{aligned}m(w-1)+(k-m)w &= kw-m \\ &= kw-(k+1)w+(N+1) \\ &= N-w+1\end{aligned}$$

   - There are some quorums in R that contain more than one element of E, or $|Q_i \cap E| > 1$, where $Q_i \in R$ and for some $1 \leq i \leq k$. Without loss of generality, assume there exists a quorums $Q_l \in R$ that $|Q_l \cap E| = j$, for $1 \leq l \leq k$, and $j \geq 2$. There are two cases:

     i. $Q_l \not\subseteq E$. By the algorithm, $|Q_l| = w-j$. Since there are k pairwise disjoint quorums in R, then there exists $j-1$ quorums in R that have size w. The number of elements included in the j quorums $= w - j + (j-1)w = j(w-1)$. The average size of these quorums is $\frac{j(w-1)}{j} = w - 1$.

ii. $Q_l \subseteq E$. If $min = w - j$ or if w is even, then this case is the same as previous one. If $min = w - j + 1$ or if w is odd , there exists j quorums in R that have size w. The number elements included in the $j+1$ quorums is $w - j + 1 + j(w) = w(j+1) - (j+1) + 2$ and the average size of the $j+1$ quorums is larger than $w - 1$.

Since any quorum $Q \in R$ such that $|Q \cap E| = j$, $\exists j - 1$ or j quorums that have size w, then the average size of m quorums in R containing m elements of E at least $w - 1$. Therefore, by previous proof, we have $| \cup_{i=1}^{k} Q_i| \geq N - w + 1$.

(b) $|E \cap (\cup_{i=1}^{k} Q_i)| = m - j$. There are two possibilities.

- $Q_i \not\subseteq E$, for all $1 \leq i \leq k$. By previous proof, the average size of the $m - j$ quorums is $w - 1$. Consequently, the number of elements is $(m-j)(w-1) + (k-m+j)w = kw - m + j = N - w + j + 1$.
- $Q_i \subseteq E$, for some $1 \leq i \leq k$. By previous proof, the average size of the $m - j$ quorums is equal to or greater than $w - 1$. Therefore, the number of elements in R is equal to or greater than $N - w + j + 1$.□

From the **Lemma 1** and **Lemma 2**, there is always a collection of k pairwise disjoint quorums which contains m disjoint quorums having the sizes of $(w-1)$. Consequently, there is a collection of k pairwise disjoint quorums with m disjoint quorums having the sizes of $(w-1)$ and $(k-m)$ disjoint quorums having the sizes of w.

Lemma 3. *There exists a collection of k pairwise disjoint quorums in C that consists* *exactly of $N - w + 1$ elements.*

Proof :

From Lemma 1, we get $m(w-1) \leq N$, or $m(w-2) \leq N - m$. This means that there are $(N - m)$ elements which are sufficient enough to form a collection of m pairwise disjoint quorums that each quorum has the size of $(w-1)$. Then by Lemma 2, $m(w-1) + (k-m)w = N - w + 1$.□

From Lemma 0, Lemma 1, and Lemma 2, we obtain the following Theorem 2.

Theorem 2. *C is a k-coterie under S.*

Proof:

To prove the Theorem, we have to show that C has two properties: intersection and minimality properties.

1. Minimality Property. From the algorithm, it is obvious that every quorum produced satisfies the minimality property, since for two distinct quorums Q_i and Q_j, $Q_i \not\subseteq Q_j$.

2. Intersection Property. Let $R = \{Q_1, Q_2, \ldots, Q_k\}$ be a collection of k pairwise disjoint quorums and let $Q \in C$ be another distinct quorum. There are two cases to consider:

   - $|Q_i| = w$, $\forall 1 \leq i \leq k$. This means that $kw \leq N - m$. Since $m = (k+1)w - (N+1)$, it implies that $kw = m + N + 1 - w \leq N - m$ or $m \leq \frac{w-1}{2}$. By the algorithm, the smallest quorums produced have size $w - m$. Then, the number of elements of $k+1$ pairwise quorums is $kw + w - m = N + 1 > N$.

   - $|Q_i| < w$ for some $1 \leq i \leq k$. This means that some elements of E are included in R. Assume $m - j$ elements of E are included in R, where $0 \leq$

$j \leq m$, then the number of elements is, by Lemma 2, at least $N-w+j+1$. In this case, there are two possibilities:

(a) $j \geq min$. Then, the smallest size of Q or $|Q| = min$ and $w-j \leq min$. Therefore, the number of elements in R is $N - w + j + 1 + min = N - (w - j) + min + 1 \geq N + 1 > N$.

(b) $j < min$. Then the smallest $|Q| = w - j(> min)$. The number of elements is $N - w + j + 1 + w - j = N + 1 > N$.

Consequently, in any $k+1$ pairwise quorums, there exists at least two quorums that intersect each other.□

Theorem 3. *C is a proper k-coterie under S if one of the following properties is satisfied:*

1. w is even.

2. w is odd and $m < 2w$.

Proof:

1. First, we will prove that if w is even, C satisfies the non-intersection property. From Lemma 2, if $|E \cap (\cup_{i=1}^{k-1} Q_i)| = m - j$, the average size of the $m - j$ quorums is $w - 1$. The number of elements in a collection of $k - 1$ pairwise disjoint quorums is $(k-1)w - m + j$. Let $f(j) = (k-1)w - m + j$. We can easily see that $f(j)$ is a monotone increasing function.

   - If $j = m$, then $f(j) = (k-1)w$ and $f(j) \leq N - m$. We can form another quorum $Q \subseteq E$ and $|Q| = min$.

   - If $j = 0$, then $f(j) = (k-1)w - m$. By Lemma 2, we can find another quorum so that the number of the collection of k pairwise disjoint quorums is $N - w + 1$.

This completes the proof for number 1.

2. Second, if w is odd, and $m < 2w$, C satisfies the non-intersection property. The only difference from the previous one is when the collection of $k-1$ pairwise disjoint quorums contains all possible quorums Q, where $Q \subseteq E$. Since w is odd, then $min = \frac{w+1}{2}$. The number of quorums $Q \subseteq E$ is $\lfloor \frac{2m}{w+1} \rfloor$. Since $m \leq 2w-1$, then we have

$$\begin{aligned} \frac{2m}{w+1} &\leq \frac{2(2w-1)}{w+1} \\ &\leq \frac{2(w+1)}{w+1} + \frac{2(w-2)}{w+1} \\ \lfloor \frac{2m}{w+1} \rfloor &\leq 3. \end{aligned}$$

The number of other elements (nodes) is $N-m$. These elements can form other quorums. The number of quorums that can be formed from these elements is $\lfloor \frac{N-m}{w} \rfloor$. Since $N = (k+1)w - (m+1)$ and $m \leq 2w-1$, then we have

$$\begin{aligned} \frac{N-m}{w} &= \frac{(k+1)w - (m+1) - m}{w} \\ &= \frac{(k+1)w - (2m+1)}{w} \\ &\geq (k+1) - \frac{2(2w-1)+1}{w} \\ &\geq (k+1) - \frac{4w-1}{w} \\ \lfloor \frac{N-m}{w} \rfloor &> k-3. \end{aligned}$$

This implies that the non-intersection property is satisfied.

By Theorem 2, C is k-coterie under S. Because C also satisfies the non-intersection property, it can be concluded that C, with above two conditions, is a proper k-coterie under S.$\square$

Theorem 4. *C is a nondominated k-coterie under S.*

Proof:

Since any collection of k pairwise disjoint quorums $\{Q_1, Q_2, \ldots, Q_k\}$ contains x elements, where $x \geq N - w + 1$, there is no subset H of S satisfying Theorem 1. Assume that C is a dominated coterie. By Theorem 1, there exists a subset $H \subseteq S$ satisfying two conditions: (1) for every $Q \in C$, $Q \not\subset H$, and (2) for any collection of k pairwise disjoint quorums $\{Q_1, Q_2, \ldots, Q_k\} \subseteq C$, $H \cap Q_i \neq \emptyset$, for some i, $1 \leq i \leq k$. The size of H must be less than or equal to $(w-1)$. There are two possibilities of the size of H or $|H|$.

1. If $min \leq |H| \leq w-1$, then H must be a subset of some quorums which do not contain any element of a set $E = \{e_1, e_2, \ldots, e_m\}$. But then, this H does not satisfy the second condition of Theorem 1, since by Lemma 2 and Lemma 3, we can always select a collection of k pairwise quorums that contains exactly $N - w + 1$ elements that do not include any element of the subset $H \subseteq S$.

2. If $|H| < min$, then, again, this H cannot satisfy the second condition of the Theorem 1. There are two cases

   (a) If $H \cap E = \emptyset$. This means that H is a subset of a collection of quorums that do not consist of any element of set E. By the previous proof, H does not satisfies the second condition of Theorem 1.

   (b) If $H \cap E \neq \emptyset$. Form a set of k pairwise disjoint quorums which consists of $(m - \lfloor\frac{w-1}{2}\rfloor)$ quorums that have sizes of $(w-1)$ and $(k - (m - \lfloor\frac{w-1}{2}\rfloor))$ quorums that have sizes of w. Let

$$f(m - \lfloor\frac{w-1}{2}\rfloor) = (m - \lfloor\frac{w-1}{2}\rfloor)(w-1) + (k - (m - \lfloor\frac{w-1}{2}\rfloor))w.$$

then we have

$$\begin{aligned}
f(m - \lfloor \frac{w-1}{2} \rfloor) &= kw - (m - \lfloor \frac{w-1}{2} \rfloor) \\
&= kw - ((k+1)w - (N+1) - \lfloor \frac{w-1}{2} \rfloor) \\
&= N - \lfloor \frac{w-1}{2} \rfloor \\
&= N - min + 1
\end{aligned}$$

Again, since we can form a collection of k pairwise disjoin quorums that contains at most $N - min + 1$, H does not satisfy the second condition of Theorem 1.

It can be concluded there is no H satisfying two conditions of Theorem 1. So C is nondominated k-coterie.□

CHAPTER 5

VOTE ASSIGNMENTS

The proposed algorithm in Chapter 3 gives an inpiration to construct k-coterie in another way. This inpiration comes from the generality of quorums produced by the algorithm; i.e., every quorum that contains h elements of E has the size of $w - h$. Although there is an exception when w is odd, we can see a consistency of the algorithm.

That inspiration motivates the author to introduce a vote assignment as a method to construct an equivalent nondominated k-coterie produced by the algorithm. This vote assignment theory was introduced by Garcia-Molina and Barbara [GMB85].

[BG87] indicated that for systems with six or more nodes, it is difficult to search exhaustively for the best assignment. Surprisingly, the assignment of votes presented here is very simple and is considered as one of the best assignments. This characteristic can be seen later when we show that this assignment produces nondominated k-coteries.

5.1 Definitions

Definition 4. Vote Assignment.

Let S be the set of N nodes that compose the system and let k be an integer $(1 \leq k \leq N)$. A *vote assignment* is a function $v : S \rightarrow Z$, (Z is the nonnegative integers), $v(a)$ is the number of votes assigned to the node a.$\square$

Defnition 5. Total and Majority.

For a vote assignment v over S, TOT and MAJ are defined by

$$TOT = (k+1)MAJ - 1$$

where

$$MAJ = \lceil \frac{N+1}{k+1} \rceil . \square$$

Now, we define the vote assignment for every node in the distributed system. From the Definition 5, we have $TOT \geq N$. If $N = TOT$, every node has the same number of votes: one vote. If $N < TOT$, let M be a subset of S such that $|M| = TOT - N$. Every node in M has two votes, the others have one vote. The definition of vote assignment function is as follows:

Definition 6. Vote Assignment Function.

Let $v(a)$ be a *vote assignment function* and $a \in S$. Let $v(a)$ be defined as follows: Let $M \subset S$ such that $|M| = TOT - N$, where TOT is as defined above.

$$v(a) = \begin{cases} 2 & \forall a \in M; \\ 1 & \forall a \in S - M. \square \end{cases}$$

Nodes in M has one more vote than the others. This tells us that these nodes are more power than the others. In selecting nodes to be members of M, it is good to consider some aspect of the realibility of communication lines and sites. However, that is not the focus of this thesis.

Definition 7. A Quorum and Coterie.

- A subset $Q \subseteq S$ is called a *quorum* if

$$v(Q) = \begin{cases} MAJ + 1 & \text{if } MAJ \text{ is odd and } Q \subseteq M; \\ MAJ & \text{otherwise.} \end{cases}$$

- *A k-coterie* C is a collection of quorums. $\square$

To illustrate how the vote assignment works, take a look at an example. Let $S = \{1, 2, 3, 4, 5, 6\}$, and $k = 2$. Then $MAJ = 3$ and $TOT = 8$. Since $TOT > N$, select a set $M = \{1, 2\}$, such that $|M| = TOT - N$. The nodes $\{1, 2\}$ have two votes, and the others have one vote. All possible quorums that can be formed are:

$$\{\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{1,6\},\{2,3\},\{2,4\},$$

$$\{2,5\},\{2,6\},\{3,4,5\},\{3,4,6\},\{4,5,6\}\}$$

A set C of these quorums is equal to the example in Chapter 3, which is constructed by the algorithm.

5.2 Correctness

From these definitions [Definition 4, 5, 6, and 7], we obtain the following properties:

Theorem 5. *A set C of quorums defined by Definition 7 is a k-coterie.*

Proof:

We need to show that a set C satisfies two properties: the minimality and intersection properties.

1. Minimality Property.

   There are two possibilities:

   - If $N = TOT$, then $v(Q) = MAJ$ for all $Q \in C$. In this case, it is obvious that there are no two distinct quorums Q_i and Q_j in C such that $Q_i \subseteq Q_j$.
   - If $N < TOT$, there may be some $Q \in C$ such that $v(Q) = MAJ + 1$. Let $G = \{Q \in C | v(Q) = MAJ + 1\}$. By definition, $Q \subseteq M$ for all $Q \in G$. For all $Q \in C - G$, $v(Q) = MAJ$. Hence, there are no two distinct quorums Q_i and Q_j in C such that $Q_i \subseteq Q_j$.

2. Intersection Property.

   Let $R = \{Q_1, Q_2, \ldots, Q_k\}$ be a collection of k pairwise disjoint quorums. Since

R is a collection of k pairwise disjoint quorums and $v(Q_i) \geq MAJ$ for all $1 \leq i \leq k$, we have $v(R) \geq k(MAJ)$. Let Q be another quorum. We will show that Q intersects some member of R. From definition of a quorum, we have $v(Q) \geq MAJ$.

$$\begin{aligned} v(R) + v(Q) &\geq k(MAJ) + MAJ. \\ v(R) + v(Q) &\geq (k+1)MAJ. \\ v(R) + v(Q) &> TOT. \end{aligned}$$

This implies that in any collection of $k+1$ quorums, there exists at least two quorums that intersect each other.□

Like the proposed algorithm, this method also mainly produces proper k-coteries. Such k-coteries are constructed whenever either MAJ is even or $TOT - N < 2MAJ$.

Theorem 6. *A set C of quorums as defined by Definition 7 is a proper k-coterie if one of the following properties holds.*

1. *MAJ is even.*

2. *MAJ is odd, and $TOT - N < 2MAJ$.*

Proof:

To show that a set C is a proper k-coterie, we need to prove that the non-intersection property holds.

1. MAJ is even. By the definition, $v(Q) = MAJ$ for all $Q \in C$. Then we can easily see that $k(MAJ) < TOT$. The non-intersection property holds.

2. MAJ is odd, then $v(Q) = MAJ + 1$ for some $Q \in C$; that is, for all Q, where $Q \subseteq M$. Since each node in M has two votes, then the number of such quorums

is $\lfloor \frac{2(TOT-N)}{MAJ+1} \rfloor$.

$$\begin{aligned}
\frac{2(TOT-N)}{MAJ+1} &\leq \frac{4MAJ-2}{MAJ+1} \\
&\leq \frac{2(MAJ+1)+2(MAJ-2)}{MAJ+1} \\
&\leq 2+\frac{2(MAJ-2)}{MAJ+1}
\end{aligned}$$

$\lfloor \frac{2(TOT-N)}{MAJ+1} \rfloor = 3$, if $MAJ > 5$. Then we can always find k pairwise disjoint quorums, because there are three possibilities:

(a) If $\lfloor \frac{2(TOT-N)}{MAJ+1} \rfloor = 3$. Then $MAJ > 5$ and $(k-3)MAJ+3(MAJ+1) \leq TOT$. This means that the non-intersection property holds.

(b) If $\lfloor \frac{2(TOT-N)}{MAJ+1} \rfloor = 2$. Then $(k-2)MAJ + 2(MAJ+1) \leq TOT$. The non-intersection property holds.

(c) If $\lfloor \frac{2(TOT-N)}{MAJ+1} \rfloor = 1$. Then $(k-1)MAJ+MAJ+1 = k(MAJ)+1 \leq TOT$. This implies that the non-intersection property is satisfied.

In other words, we can say that a set C is a proper k-coterie.$\square$

If the conditions on Theorem 6 are not satisfied, the algorithm may not produce proper k-coteries. For example, $N = 14$ and $k = 6$, then $MAJ = 3$ and $|M| = 6$. In this situation, the conditions on the Theorem 6 does not hold. There are 6 nodes having 2 votes; and the rests have 1 vote. Assume that each node in $M = \{1,2,3,4,5,6\}$ has two votes. Each node in $S-M = \{7,8,9,10,11,12,13,14\}$, has one vote. When five pairwise disjoint quorums $\{\{1,2\},\{3,4\},\{5,6\},\{7,8,9\},\{10,11,12\}\}$ have been selected, then we cannot form another quorum that does not interect to the five quorums. Thus, when this situation occurs, the algorithm cannot construct a proper 3-coterie.

Eventhough, for some combination of N and k, the algorithm may not produce proper k-coteries, still the algorithm produces nondominated k-coteries for any value of N and k.

Theorem 7. *A set C of quorums defined by Definition 7 is a nondminated k-coterie.*
Proof:

Assume that C is a dominated k-coterie, then there must be a subset $H \subseteq S$ that satisfies two conditions:

1. $\forall Q \in C$, $Q \not\subseteq H$, and
2. for any collection of k pairwise disjoint quorums $\{Q_1, Q_2, \ldots, Q_k\}$, $H \cap Q_i \neq \emptyset$ for some i, $1 \leq i \leq k$.

Since $v(Q_i) \geq MAJ$, then $v(H) < MAJ$. Let $R = \{Q_i \in C | v(Q_i) = MAJ, 1 \leq i \leq k\}$ be a collection of k pairwise disjoint quorums. This collection can be found in any C because there are two possibilities.

1. If $TOT = N$, it is obvious that $v(Q) = MAJ$.
2. If $TOT > N$, we can select $TOT - N$ quorums that contain exactly one element of M and the rests are quorums that do not contain any element of M. By this selection, we have a collection of k pairwise disjoint quorums $R = \{Q_1, Q_2, \ldots, Q_k\}$, where $v(Q_i) = MAJ$ for all i.

Now, we have

$$\begin{aligned} v(R) + v(H) &\leq k(MAJ) + MAJ - 1 \\ &\leq (k+1)MAJ - 1 = TOT. \end{aligned}$$

This concludes that H does not satisfy the second condition of Theorem 1. In other words, C is a nondominated k-coterie.□

By the properties presented above, we know that this vote assignment is equivalent to the proposed algorithm discussed in Chapter 3.

CHAPTER 6

SUMMARY AND FUTURE WORK

6.1 Summary

The method for constructing nondominated k-coteries is presented. This method works for any combination of N, the number of nodes in a distributed system, and k, the number of processes allowed to enter critical sections symultaneously. The method is the extension of Maj_k method, which produces mainly dominated k-coteries. The proposed algorithm also produces nearly symmetric k-coteries.

A vote assignment is also presented which is an equivalent method to the proposed algorithm. The vote assignment makes the constructing nondominated k-coteries easier. The correctness and equivalency of the both methods are also presented.

6.2 Future Work

Although we have proposed a method for constructing nondominated k-coteries for any value of N and k, there is an open challenging problem. The problem found here is how to find a method that can construct symmetric and proper nondominated k-coteries. What we have here is that still our method may not construct symmetric coteries especially when $N + 1 \neq (k + 1)w$. The other is that the algorithm may not produce proper coteries when w is odd and $m \geq 2w$.

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