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ORBITAL INTEGRAL CORRESPONDENCE FOR THE PAIR $(G_2, Sp(1, \mathbf{R}))$ VIA THE CAUCHY HARISH-CHANDRA INTEGRAL

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A DISSERTATION APPROVED FOR THE
DEPARTMENT OF MATHEMATICS

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Introduction

In [14] Jacobson classified the composition algebras $(A, \mathcal{Q})$ over any field $\mathbf{F}$, char$\mathbf{F} \neq 2$, and their automorphism groups $G = \mathrm{aut}(A, \mathcal{Q})$. In particular given $x \in A$, he describes the stabilizer G_x of x in G. If $\mathbf{F} = \mathbf{R}$ and $A = O$ is the algebra of split octonions, then G is the non-compact real connected Lie group of type G_2.

Rallis and Shiffmann in [18] used Jacobson's work to show that the action of G on O is doubly transitive, see Theorem 4.3 below. By definition $G_2 \subset SO(3,4)$. The groups $(Sp(1,\mathbf{R}), O(3,4))$ form a dual pair in the sense of Howe. This leads to a correspondence of the irreducible unitary representations between the double connected cover of $Sp(1,\mathbf{R})$ and some of the irreducible unitary representations of $O(3,4)$. By using the property of double transitivity, Rallis and Schiffmann showed that the restriction of the resulting representation to G_2 remains irreducible. However they didn't compute the characters of these representations. Neither did they compute the lowest term of the expansion of such a character, which should be the Fourier transform of an orbital integral corresponding to a nilpotent orbit, see [3]. The goal of this work is to make some progress in this direction.

The theory of the Cauchy Harish-Chandra integral developed by Przebinda in [17] is useful for attacking this type of questions in the case of dual pair of classical groups. First we showed that this theory can be extended to include the case of G_2. Then we interpret the Jacobson-Rallis-Schiffmann double transitivity theorem as a statement that there is an injection from the regular semisimple orbits of $\mathfrak{sp}(1, \mathbf{R})$ to those of $\mathfrak{g}_2$, via the unnormalized maps used in Classical Invariant Theory. See Corollary 4.2. We attempt to extend this statement to nilpotent orbits and arrive at Conjecture 5.1, parts (b) and (c). Next, we compute the Cauchy Harish-Chandra integral for any non-zero orbit in $\mathfrak{sp}(1, \mathbf{R})$. See Theorem 5.1. The resulting distributions are in fact invariant real analytic functions which look like the Fourier transforms of orbital integrals of $\mathfrak{g}_2$. We were unable to prove that they coincide, however we formulated a precise conjecture, see Conjecture 5.1, parts (a) and (d).

This work is organized as follows. In chapter 1 we review the general theory of composition algebras and located the split octonions and our group G_2 in it. In chapter 2 we recall the Shale-Weil Oscillator representation, Howe's correspondence and the theory of Cauchy Harish-Chandra integrals. Chapter 3 deals with the basic structure theory of $\mathfrak{g}_2$, classification of Cartan subalgebras and root systems. Moreover, section 3.5 contains the main technical lemma which implies the existence of the Cauchy Harish-Chandra integral for $\mathfrak{g}_2$. In chapter 4 we described the Fourier Transform of a non-zero nilpotent integral in $\mathfrak{sp}(1, \mathbf{R})$. We do it via the description of eigen-distributions of the indefinite Laplacian on $\mathbf{R}^3$ available in [12]. Finally, Chapter 5 contains the conjecture and the main theorem sketched above.

Chapter 1

Composition Algebras and Their Automorphisms

1.1 Composition Algebras

In this section and the next section we recall from [14] some basic facts on composition algebras. Let $\mathbf{F}$ be a field ($\mathrm{char}(\mathbf{F}) \neq 2$) and A a $\mathbf{F}$-algebra. An arrow $\mathcal{Q} \in \mathrm{map}(A, \mathbf{F})$ is called a **non-degenerate quadratic form** if it is homogeneous of degree 2, (i.e., if for $\alpha \in \mathbf{F}$, $a \in A$, $\mathcal{Q}(\alpha a) = \alpha^2 \mathcal{Q}(a)$) and for $a_1, a_2 \in A$,

$$\langle a_1, a_2 \rangle \;=\; \frac{1}{2} \left(\mathcal{Q}(a_1 + a_2) - \mathcal{Q}(a_1) - \mathcal{Q}(a_2) \right), \tag{1.1}$$

defines a non-degenerate bilinear form on A. Such a form is said to be **multiplicative** if $\mathcal{Q} \in \hom_{groups}(A^\times, \mathbf{F}^\times)$.

Definition 1.1 • *A* ***composition algebra*** *is a pair* $(A, \mathcal{Q})$ *where* A *is an* $\mathbf{F}$*-algebra with unit* 1*, and* $\mathcal{Q} \in \mathrm{map}(A, \mathbf{F})$ *is a multiplicative non-degenerate quadratic form.*

- $(A, \mathcal{Q})$, $(A', \mathcal{Q}')$ *are* ***isomorphic*** *if there is an algebra isomorphism that preserves the quadratic forms.*

If A is a composition algebra, it admits a decomposition of the form $A = \mathbf{F} \oplus \mathbf{F}^{\perp}$, where $\mathbf{F}^{\perp}$ is the orthogonal complement of $\mathbf{F}$ with respect to (1.1). Relative to this decomposition the involution $\overline{a} = \alpha - x$ (for $a = \alpha + x \in \mathbf{F} \oplus \mathbf{F}^{\perp}$) satisfies $a \cdot \overline{a} = \mathcal{Q}(a)1$. We write $Im(A) = \mathbf{F}^{\perp}$.

Definition 1.2 *A composition algebra is called* ***split*** *if it has zero divisors. A non-split composition algebra is called a* ***normed division algebra****.*

From now on, unless stated otherwise, our composition algebras shall be over $\mathbf{R}$.

CAYLEY-DICKSON CONSTRUCTION

Example 1.1 *$A_0 = \mathbf{R}$ with $\mathcal{Q}_0(a) = a^2$, is a composition algebra.*

Example 1.2 *For $k = 1, 2, 3$, let A_k be the vector space*

$$A_k = A_{k-1} \oplus A_{k-1}.$$

Given $\mu_1, \mu_2, \mu_3 \in \mathbf{R}^{\times}$, we define inductively a multiplication and a quadratic form in A_k by,

$$q_1 \cdot q_2 = (a_1 a_2 + \mu_k \overline{b_2} b_1, b_2 a_1 + b_1 \overline{a_2}), \qquad q_i = (a_i, b_i) \in A_k,$$

and

$$\mathcal{Q}_k((a,b)) = \mathcal{Q}_{k-1}(a) - \mu_k \mathcal{Q}_{k-1}(b) \qquad (a,b) \in A_k.$$

Then $(A_k, \mathcal{Q}_k)$ is a composition algebra denoted by $A_k[(\mu_1, .., \mu_k)]$.

Classification

Theorem 1.1 *Up to an isomorphism, the only finite dimensional composition algebras over* $\mathbf{R}$ *are: the real numbers* $\mathbf{R} = A_0$*, the complex numbers* $\mathbf{C} = A_1[(-1)]$*, the split complex numbers* $\mathbf{C}_{split} = A_1[(1)]$*, the quaternions* $\mathbf{H} = A_2[(-1,-1)]$*, the split quaternions (or co-quaternions)* $\mathbf{H}_{split} = A_2[(-1,1)]$*, the octonions* $\mathbf{O} = A_3[(-1,-1,-1)]$ *and* $\mathbf{O}_{split} = A_3[(-1,-1,1)]$ *the split octonions.*

We shall be mostly interested in this last example and we shall freely use the notation $\mathbf{O} = \mathbf{O}_{split} = A_3[(-1,-1,1)]$.

1.2 Automorphisms and Derivations of Real Composition Algebras

Let $\mathrm{aut}(A, \mathcal{Q})$ be the group of composition algebra automorphisms of $(A, \mathcal{Q})$. Also let $\mathrm{isom}(A, \mathcal{Q}) = \mathrm{aut}_{forms}(\mathcal{Q})$ the group of linear isometries of the bilinear form $\mathcal{Q}$. Then $\mathrm{aut}(A, \mathcal{Q}) \subset \mathrm{isom}(A, \mathcal{Q})$.

Also, if $\varphi \in \mathrm{aut}(A, \mathcal{Q})$, $\mathbf{R}$-linearity implies that $\varphi\mid_{\mathbf{R}} = id_{\mathbf{R}}$ and hence,

$$\mathrm{aut}(A, \mathcal{Q}) \subset \mathrm{isom}(Im(A), \mathcal{Q}).$$

Correspondingly, for derivations we have:

$$\mathfrak{Der}(A, \mathcal{Q}) \subset \mathcal{L}ie(\mathrm{isom}(Im(A), \mathcal{Q})).$$

Historically, the representation theory of all these automorphism groups, listed in table 1.2, was fully understood at an early stage, except that of $G_2 := \mathrm{aut}(\mathbf{O}_{split}, Q_3)$. In his massive work [20], D. Vogan described the unitary dual of G_2.

A	$\mathrm{aut}(A, \mathcal{Q})$	$\mathfrak{der}(A, \mathcal{Q})$
$\mathbf{R}$	$\{1\}$	$\{0\}$
$\mathbf{C}$	$\mathbf{Z}/2\mathbf{Z} = \{1, \text{ complex conjugation }\}$	$\{0\}$
$\mathbf{C}_{split}$	$\mathbf{Z}/2\mathbf{Z}$	$\{0\}$
$\mathbf{H}$	$SO(3)$	$\mathfrak{so}(3) \simeq \mathfrak{u}(\mathbf{H})$
$\mathbf{H}_{split}$	$SO(1,2)$	$\mathfrak{so}(1,2)$
$\mathbf{O}$	$(G_2)_{compact}$	$(\mathfrak{g}_2)_{compact}$
$\mathbf{O}_{split}$	$G_2 = (G_2)_{split}$	$\mathfrak{g}_2$

Table 1.1: Automorphism Groups and Derivation Algebras of Composition Algebras

Here we shall concentrate on some aspects of the theory of G_2 important within the framework of the work of Przebinda and Bernon-Przebinda on dual pairs, as explained in the next chapter.

1.3 $\mathfrak{g}_2$ as an Algebra of Derivations

THE LIE ALGEBRA $\mathfrak{so}(3,4)$

The matrix of the form $\tau_{3,4}$ with respect to the standard basis of $\mathbf{R}^7$ is $\mathcal{T} = \mathrm{diag}(I_3, -I_4)$. Then the Lie algebra $\mathfrak{so}(3,4)$ is:

$$\mathfrak{so}(3,4) = \{X \in \mathfrak{sl}(7,\mathbf{R}) : X\mathcal{T} + \mathcal{T}X^T = 0\} = \left\{ \begin{pmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{pmatrix} : \right.$$
$$\left. X_1 = \Theta(X_1) \in M_{3\times3}(\mathbf{R}), X_3 = \Theta(X_3) \in M_{4\times4}(\mathbf{R}), X_2 \in M_{3\times4}(\mathbf{R}) \right\},$$

where $\Theta(X) = -X^T$.

OCTONION MULTIPLICATION

From section 1, we have:

$$\mathbf{O} = \mathbf{O}_{split} = A_3[(-1,-1,1)] = \mathbf{H} \oplus \mathbf{H},$$

where $\mathbf{H} = e_0\mathbf{R} \oplus e_1\mathbf{R} \oplus e_2\mathbf{R} \oplus e_3\mathbf{R}$ denotes the usual non-split quaternions:

$$e_0 = 1, \qquad e_l^2 = -1, \quad l = 1,2,3, \qquad e_1e_2 = e_3 = -e_2e_1, \quad \circlearrowleft$$

where $\circlearrowleft$ indicates that this identities remain valid under cyclic permutation of the indices. Notice that, when endowed with the bracket $[a,b] = ab - ba$, the space of imaginary quaternions becomes a Lie algebra, here denoted by $\mathfrak{u}(\mathbf{H})$. We have $\mathfrak{u}(\mathbf{H}) = e_1\mathbf{R} \oplus e_2\mathbf{R} \oplus e_3\mathbf{R}$.

Remark 1.1 *Since there is no extra effort involved in considering simultaneously the compact and split cases, in this section we keep the factor $\mu = \pm 1$ coming from the Cayley-Dickson construction.*

Put $\xi_j = (e_j, 0)$, and $\xi_{j+4} = (0, e_j)$ for $j = 0,1,2,3$. We have then

$$Im(\mathbf{O}) = \bigoplus_{j=1}^{7} \mathbf{R}\xi_j, \qquad \mathcal{Q}(\sum_{j=1}^{7} a_j\xi_j) = \sum_{j=1}^{7} (-\mu)^{\lfloor \frac{j}{4} \rfloor} a_j^2,$$

and the multiplication table is:

Equivalently:

$$\xi_j \cdot \xi_k = -\mu\langle \xi_j, \xi_k \rangle \xi_0 + \sum_{l=1}^{7} (-\mu)^{\lfloor \frac{l}{4} \rfloor} C_{jkl}\xi_l, \tag{1.2}$$

Table 1.2: Multiplication Table of the Octonions

$*$	ξ_1	ξ_2	ξ_3	ξ_4	ξ_5	ξ_6	ξ_7
ξ_1	$-\xi_0$	ξ_3	$-\xi_2$	ξ_5	$-\xi_4$	$-\xi_7$	ξ_6
ξ_2	$-\xi_3$	$-\xi_0$	ξ_1	ξ_6	ξ_7	$-\xi_4$	$-\xi_5$
ξ_3	ξ_2	$-\xi_1$	$-\xi_0$	ξ_7	$-\xi_6$	ξ_5	$-\xi_4$
ξ_4	$-\xi_5$	$-\xi_6$	$-\xi_7$	$\mu\xi_0$	$-\mu\xi_1$	$-\mu\xi_2$	$-\mu\xi_3$
ξ_5	ξ_4	$-\xi_7$	ξ_6	$\mu\xi_1$	$\mu\xi_0$	$\mu\xi_3$	$-\mu\xi_2$
ξ_6	ξ_7	ξ_4	$-\xi_5$	$\mu\xi_2$	$-\mu\xi_3$	$\mu\xi_0$	$\mu\xi_1$
ξ_7	$-\xi_6$	ξ_5	ξ_4	$\mu\xi_3$	$\mu\xi_2$	$-\mu\xi_1$	$\mu\xi_0$

with

$$C_{123} = \mu C_{1(2+4)(3+4)} = \mu C_{(1+4)2(3+4)} = \mu C_{(1+4)(2+4)3} = 1 \qquad \circlearrowleft$$

$$\mu C_{415} = \mu C_{4(1+1)(5+1)} = \mu C_{4(1+2)(5+2)} = 1, \qquad \circlearrowleft$$

where $\circlearrowleft$ means that for every $\sigma \in \mathrm{Symm}_7$ that permutes the elements j, k, l, for

$$\{j,k,l\} \in \{\{1,2,3\},\{1,6,7\},\{5,6,3\},\{5,7,2\},\{4,1,5\},\{4,2,6\},\{4,3,7\}\}, \quad (1.3)$$

while leaving $\{1,2,\ldots,7\} \setminus \{j,k,l\}$ fixed, we have $C_{\sigma(j)\sigma(k)\sigma(l)} = \mathrm{sgn}(\sigma) C_{jkl}$.

Also $C_{jkl} = 0$, for any $\{j,k,l\}$ not in (1.3).

Remark 1.2 *There is a well known triple identification of Lie algebras*

$$\mathfrak{u}(\mathbf{H}) \simeq \mathfrak{so}(3) \simeq \mathfrak{su}(2).$$

For $v = v_1e_1 + v_2e_2 + v_3e_3 \in \mathfrak{u}(\mathbf{H})$, these identifications are given by:

$$2v \leftrightarrow \underline{v} = \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix} \leftrightarrow \frac{v_1}{2}\mathcal{J} + \frac{-v_2}{2}i\mathcal{H} + \frac{v_3}{2}i\mathcal{I}. \tag{1.4}$$

We may also think $v = (v_1, v_2, v_3) \in \mathbf{R}^3$.

An Explicit description of $\mathfrak{g}_2$

Let $v^k \in \mathbf{R}^3$, $k = 1, 2, 3$, and $\rho \in \mathfrak{gl}(3, \mathbf{R})$. Put

$$\begin{aligned}
V_\rho &= V_0(\rho) := \begin{pmatrix} 0 & 0 & \mathrm{diag}(1,1,\mu)\rho \\ 0 & 0 & 0 \\ \mu(\mathrm{diag}(1,1,\mu)\rho)^T & 0 & 0 \end{pmatrix}, \\
V_{v^1} &= V_1(v^1) := \begin{pmatrix} 2\underline{v^1} & 0 & 0 \\ 0 & 0 & v^1 \\ 0 & -(v^1)^T & \underline{v^1} \end{pmatrix}, \\
V_{v^2} &= V_2(v^2) := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -v^2 \\ 0 & (v^2)^T & \underline{v^2} \end{pmatrix}, \\
V_{v^3} &= V_3(v^3) := \begin{pmatrix} 0 & 2\mathrm{diag}(1,1,\mu)(v^3)^T & -\mathrm{diag}(1,1,\mu)\underline{v^3} \\ 2\mu v^3\mathrm{diag}(1,1,\mu) & 0 & 0 \\ \mu\underline{v^3}\mathrm{diag}(1,1,\mu) & 0 & 0 \end{pmatrix}.
\end{aligned}$$

Proposition 1.1 *In terms of the ordered basis $\{\xi_1, ..., \xi_7\}$,*

$$\mathfrak{g}_2 = \{V(\rho, v^1, v^2, v^3) : v^k \in \mathbf{R}^3, k = 1, 2, 3,\ \rho = \rho^T \in \mathfrak{sl}(3, \mathbf{R})\} \subset \mathfrak{so}(3, 4),$$

where

$$V(\rho, \upsilon^1, \upsilon^2, \upsilon^3) = V_\rho + V_{\upsilon^1} + V_{\upsilon^2} + V_{\upsilon^3}.$$

Proof:

It is clear that this sum is direct. On the other hand, it is relatively easy to compute $\dim G_2$. For example one could argue, mutatis mutandis, as Baez does in [2] for the compact case with his *basic triples*, that:

$$\dim G_2 = \dim S^6 + \dim S^5 + \dim S^3 = 14.$$

It is enough to exhibit independent relations satisfied by our matrices as elements of $\mathfrak{g}_2$.

Let $V \in \mathfrak{Der}(\mathbf{O}) = \mathfrak{g}_2$, and write $V(\xi_k) = \sum_{m\geq 1} V_{km}\xi_m$. Then $\langle \xi_j, V(\xi_k)\rangle = V_{kj}(-\mu)^{\lfloor \frac{j}{4} \rfloor}$, while (1.2) implies: $V(\xi_j \cdot \xi_k) = \sum_{s,l\geq 1}(-\mu)^{\lfloor \frac{s}{4} \rfloor} V_{sl} C_{jks}\xi_l$,

$$\xi_j \cdot V(\xi_k) = V_{kj}(-\mu)^{\lfloor \frac{j}{4} \rfloor + 1}\xi_0 + \sum_{l,m\geq 1} (-\mu)^{\lfloor \frac{l}{4} \rfloor} V_{km} C_{jml}\xi_l,$$

and similarly for $V(\xi_j) \cdot \xi_k$. Since V is a derivation, i.e., it satisfies the Leibniz rule:

$$V(\xi_j \cdot \xi_k) = \xi_j \cdot V(\xi_k) + \xi_k \cdot V(\xi_j),$$

then

$$\sum_{m,l\geq 1} ((-\mu)^{\lfloor \frac{l}{4} \rfloor}(V_{km}C_{jml} + V_{jm}C_{mkl}) - (-\mu)^{\lfloor \frac{m}{4} \rfloor} V_{ml} C_{jkm})\xi_l$$

$$= (V_{kj}(-\mu)^{\lfloor \frac{j}{4} \rfloor} + V_{jk}(-\mu)^{\lfloor \frac{k}{4} \rfloor})\xi_0.$$

It follows from linear independence of the ξ_j that both sides of the previous equation vanish.

Since the right hand side is zero

$$V \in \mathfrak{so}(3 + (1-\mu)2, (1+\mu)2).$$

Since $\dim \mathfrak{so}(3 + (1-\mu)2, (1+\mu)2) = 21$, and $\dim \mathfrak{g}_2 = 14$, we only need to find 7 more independent relations. The fact that the left hand side is equal to zero means that for $l \geq 1$:

$$\sum_{m \geq 1} (V_{km} C_{ljm} + \circlearrowleft) = 0. \tag{1.5}$$

where $\circlearrowleft$ indicates cyclic permutation over the indices k, l, j in the expression $V_{km} C_{ljm}$.

By an appropriate choice of indices, equation (1.5) gives the relations collected in table 1.3. **QED.**

Table 1.3: Set of Independent Relations ($\mu = 1$)

l	j	k	Relation	Type
5	1	7	$-V_{12} + V_{74} - V_{65} = 0$	compact
5	1	6	$V_{13} + V_{64} + V_{75} = 0$	compact
3	5	7	$V_{23} - V_{67} + V_{45} = 0$	compact
1	2	4	$\mu V_{43} - V_{61} + V_{25} = 0$	non-compact
1	2	5	$\mu V_{53} - V_{71} - V_{24} = 0$	non-compact
1	2	6	$\mu V_{63} + V_{41} - V_{27} = 0$	non-compact
1	2	7	$\mu V_{73} + V_{51} + V_{26} = 0$	non-compact

Chapter 2

The Oscillator Representation and the Cauchy Harish-Chandra Integral

Harmonic analysis can be interpreted broadly as a general principle that relates geometric objects and spectral objects. The two kind of objects are sometimes related by explicit formulas, and sometimes simply by parallel theories. **J. Arthur.** [1]

2.1 Reductive Dual Pairs

DUAL PAIRS OF TYPE I

Let $(\mathbf{D}, \iota)$ be a pair consisting of normed (real finite dimensional) division algebra $\mathbf{D}$ and an involution $\iota \in \{id, \text{-}\}$. Also, let (V_j, τ_j) $(j = 0, 1)$ be an ordered pair consisting of a (finite dimensional) $\mathbf{D}$-vector space and ι-hermitian (ι-skew-hermitian) non degenerate form on V_j.

By applying the $\hom_{\mathbf{R}}$ functor, we get real vector spaces $W = W(V_0, V_1) = \hom_{\mathbf{D}}(V_0, V_1)$ and $W^* = W(V_1, V_0) = \hom_{\mathbf{D}}(V_1, V_0)$, together with a notion of adjointness between these two spaces

$$^* \in \hom_{\mathbf{R}}(W, W^*), \qquad \tau_1(w(v_0), v_1) = \tau_0(v_0, w^*(v_1)), \tag{2.1}$$

for all $v_j \in V_j, j = 0, 1$. At the same time, this defines a non-degenerate symplectic form τ on W, given by the formula:

$$\tau(w, w') = \operatorname{tr}_{\mathbf{D}/\mathbf{R}}(w'^* w) \qquad w, w' \in W. \tag{2.2}$$

We now take the point of view of groups. The classical real Lie groups:

$$G_j = \operatorname{isom}(V_j, \tau_j), \qquad Sp(W) = \operatorname{isom}(W, \tau),$$

arise naturally. Further, we have a maps $G_j \hookrightarrow Sp(W)$ given by

$$g_0(w) = w g_0^{-1} \qquad g_1(w) = g_1 w, \qquad g_j \in G_j, w \in W.$$

An easy calculation shows that $(w g_0^{-1})^* = g_0 w^*$, and $(g_1 w)^* = w^* g_1^{-1}$ so that these actions indeed preserve the symplectic form. Also, by taking the functor $\mathcal{L}ie$ we get an infinitesimal version: $\mathfrak{g}_j \hookrightarrow \mathfrak{sp}(W)$,

$$x_0(w) = -w x_0, \qquad x_1(w) = x_1 w, \qquad x_j \in \mathfrak{g}_j, w \in W. \tag{2.3}$$

Then (G_0, G_1) form what is called a **irreducible dual pair of type I**. More generally we have the following definition.

Definition 2.1 *Let (W, τ) be a real symplectic vector space. A pair of subgroups (G, G') of $Sp(W)$ are said to be a* ***reductive dual pair*** *if G and G' are mutual centralizers and they act absolutely reductively on W. Such a pair is said to be* ***irreducible*** *if W cannot be expressed as a direct sum of orthogonal subspaces invariant under the combined action of G and G'.*

Table 2.1: Irreducible Dual Pairs [11]

$(\mathbf{D}, \iota)$	Pair	Type
$(\mathbf{R}, id)$	$(O(p,q), Sp(m,\mathbf{R})) \subset Sp(m(p+q),\mathbf{R})$	I
$(\mathbf{C}, id)$	$(O(p,\mathbf{C}), Sp(m,\mathbf{C})) \subset Sp(2mp,\mathbf{R})$	I
$(\mathbf{C}, -)$	$(U(p,q), U(r,s)) \subset Sp((p+q)(r+s),\mathbf{R})$	I
$(\mathbf{H}, -)$	$(Sp(p,q), O^*(2m)) \subset Sp(2m(p+q),\mathbf{R})$	I
$(\mathbf{R}, \emptyset)$	$(GL(m,\mathbf{R}), GL(n,\mathbf{R})) \subset Sp(mn,\mathbf{R})$	II
$(\mathbf{C}, \emptyset)$	$(GL(m,\mathbf{C}), GL(n,\mathbf{C})) \subset Sp(2mn,\mathbf{R})$	II
$(\mathbf{H}, \emptyset)$	$(GL(m,\mathbf{H}), GL(n,\mathbf{H})) \subset Sp(4mn,\mathbf{R})$	II

An Unnormalized Moment Map

Definition 2.2 *We let $Sp(W)$ act on $\mathfrak{sp}(W)$ and $\mathfrak{sp}(W)^*$ by the adjoint and coadjoint action, respectively. Let $G \subset Sp(W)$ be a subgroup and let $V \subset \mathfrak{sp}(W)$ be a subspace, such that G acts by restriction on V. Then, the* ***unnormalized moment map*** *τ_V, is given by:*

$$\tau_V(w)(x) = \tau(x(w), w), \qquad x \in V, w \in W.$$

This map is G-equivariant, i.e. $\tau_V \in \mathrm{map}_G(W, V^*)$.

2.2 Homogeneity, the Canonical Commutation Relations and the Heisenberg Group

We briefly sketch some of the ideas behind Mackey's beautiful perspective of quantum mechanics since they provide an ontologically simple motivation for the metaplectic representation which, in turn, is at the heart of all what follows. For details we refer to [15].

In classical mechanics the ability to express the localization of an elementary particle in $\mathbf{R}$ is achieved through the proposition system given by the boolean algebra of Borel sets $\mathcal{B}orel\ (\mathbf{R})$, seen as a complete ortho-complemented lattice. In turn, in quantum mechanics, the same purpose is served by means of an adequate complete ortho-complemented lattice representation

$$\Lambda \in \mathrm{hom}_{c.o.c.lattice}(\mathcal{B}orel\ (\mathbf{R}), \mathrm{end}(\mathcal{H}))$$

of this algebra in some Hilbert space $\mathcal{H}$. Adequate here means that Λ defines a **spectral measure**.

The framework for the prototypical example is as follows. Let $\mathcal{H} = L^2(\mathbf{R})$, and 1_S the characteristic function of $S \subset \mathbf{R}$. Then we have:

$$\Lambda : \mathcal{B}orel\ (\mathbf{R}) \to \mathrm{Pr}(\mathcal{H}) \qquad\qquad S \mapsto \Lambda(S),$$

$$\Lambda(S)\varphi = 1_S \cdot \varphi,$$

where $\mathrm{Pr}(\mathcal{H})$ denotes the set of self adjoint projections in $\mathcal{H}$.

The proposition

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"The particle is in the (Borel) set S"
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is then represented by $\Lambda(S)$.

The proposition stating that the system is located in the intersection of two sets $S_1 \cap S_2$ is represented by $\Lambda(S_1 \cap S_2)$ but also by $\Lambda(S_1)\Lambda(S_2)$, so that:

$$\begin{aligned} \Lambda(S_1 \cap S_2) &= \Lambda(S_1)\Lambda(S_2) \quad \text{and similarly:} \\ \Lambda(\bigcup_{\mathbf{N}} S_i) &= \sum_{\mathbf{N}} \Lambda(S_i) \quad \text{for } \{S_i\}_{\mathbf{N}} \quad \text{pairwise disjoint.} \\ \Lambda(S^c) &= 1_{\mathbf{R}} - \Lambda(S). \end{aligned}$$

The self adjoint operator defined by the spectral measure

$$Q = \int_{\mathbf{R}} q \mathrm{d}\Lambda(q),$$

is the **position operator** of the physical system. For $\vartheta \in \mathbf{R}$, we associate to it, via spectral calculus, the operator:

$$V(\vartheta) := \exp(i\vartheta Q) = \int_{\mathbf{R}} \exp(i\vartheta q) \mathrm{d}\Lambda(q).$$

Space homogeneity means that translations induce a symmetry of the proposition system. Namely, for $S \in \mathcal{B}orel\ (\mathbf{R})$, $\xi \in \mathbf{R}$, let $S - \xi$ denote the translate of S by $-\xi$, then

$$\Lambda(S - \xi) = U(\xi)\Lambda(S)U(\xi)^{-1},$$

where $\{U(\xi)\}_{\xi\in\mathbf{R}}$ is a family of unitary operators. It may be chosen so that

$$U : \mathbf{R} \to U(\mathcal{H}), \quad \xi \mapsto U(\xi)$$

is a representation (i.e. Λ forms a **transitive system of imprimitivities** for U).

According Stone's theorem there is an infinitesimal generator P, the **momentum operator** $U(\xi) = \exp(i\xi P)$.

Furthermore,

$$\begin{aligned} U(\xi)V(\vartheta)U(\xi)^{-1} &= \int_{\mathbf{R}} \exp(i\vartheta q)\mathrm{d}((U(\xi))\Lambda(q)U(\xi)^{-1}) \\ &= \int_{\mathbf{R}} \exp(i\vartheta q)\mathrm{d}\Lambda(q-\xi) \\ &= \int_{\mathbf{R}} \exp(i\vartheta(q+\xi)\mathrm{d}\Lambda(q) = \exp(i\vartheta\xi)V(\theta). \end{aligned} \tag{2.4}$$

This is known as the **Weyl canonical commutation relation**.

These operators act by

$$\begin{aligned} V(\vartheta)(\varphi)(x) &= \exp(i\vartheta x)\varphi(x), \qquad U(\xi)(\varphi)(x) = \varphi(x-\xi), \qquad \text{for } \varphi \in L^2(\mathbf{R}), \\ Q(\varphi)(x) &= x\varphi(x), \qquad P(\varphi)(x) = i\frac{d\varphi}{dx}, \qquad \text{for } \varphi \in \mathcal{S}(\mathbf{R}). \end{aligned}$$

Consider the space $(W,\ \tau)$, where $W = \mathbf{R}^2$, and $\tau((\vartheta,\xi),(\vartheta',\xi')) = \xi'\vartheta - \vartheta'\xi$. Define a group law on $W \times S^1$ by:

$$((w,\phi)\cdot(w',\phi') = (w+w', \phi\phi' \exp i\frac{1}{2}\tau(w,w'))),$$

and denote this group by $\mathcal{H}eis(W)$.

The relation (2.4) shows that $\rho \in \mathrm{map}(\mathcal{H}eis(W), \mathcal{U}(L^2(\mathbf{R}))$, defined by

$$\rho((\vartheta, 0), 1) = V(\vartheta),$$

$$\rho((0, \xi), 1) = U(\xi),$$

$$\rho((0, 0), \phi) = \phi \; id,$$

is in fact a unitary representation. This is the Shale-Weil (or oscillator) representation of the Heisenberg group. In the next section we shall recall an extension of this representation to the metaplectic group $\widetilde{Sp(1, \mathbf{R})}$ (the connected double cover of the symplectic group $Sp(1, \mathbf{R})$ preserving the form τ).

In fact, given a symplectic space (W, τ) we define the Heisenberg group $\mathcal{H}eis(W) = W \times S^1$ with the same multiplication law as above. For a maximal isotropic subspace $X \subset W$, we obtain a unitary representation $\rho : \mathcal{H}eis(W) \to \mathcal{U}(L^2(X))$ defined by the formulas analogous to the ones above. Furthermore, given that $Sp(W)$ preserves the relation 2.4, Schur's Lemma implies the existence of a projective unitary representation $\omega : Sp(W) \to \mathcal{U}(L^2(X))$, so that

$$\omega(g)\rho(h)\omega(g)^{-1} = \rho(g \; h \; g^{-1}),$$

and the Shale-Weil theorem says that this representation lifts to a genuine representation of $\widetilde{Sp(W)}$.

2.3 The Oscillator Representation in a Nutshell

This short section follows closely [12, Chapter III].

Consider the following basis of $\mathfrak{sp}(1,\mathbf{R})$:

$$\mathcal{H} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \mathcal{E}^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \mathcal{E}^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Define $\omega \in \hom(\mathfrak{sp}(1,\mathbf{R}), \mathrm{end}(\mathcal{S}(\mathbf{R})))$ by:

$$\begin{aligned} \omega(\mathcal{H}) &= \frac{i}{2}\{P,Q\}, \\ \omega(\mathcal{E}^+) &= \frac{i}{2}Q^2, \qquad \omega(\mathcal{E}^-) = -\frac{i}{2}P^2. \end{aligned}$$

Theorem 2.1 *(Shale-Weil) The representation ω exponentiates to a unitary*

$$\omega : \widetilde{Sp(1,\mathbf{R})} \to \mathcal{U}(L^2(\mathbf{R})),$$

called the ***oscillator*** *or* ***metaplectic representation*** *of $\widetilde{Sp(1,\mathbf{R})}$.*

Remark 2.1 *Consider $-\mathcal{J} = \mathcal{E}^- - \mathcal{E}^+$, the infinitesimal generator of the maximal compact subgroup K' of $\widetilde{Sp(1,\mathbf{R})}$. Let $\mathcal{K} = -i\mathcal{J}$, then*

- *the operator*

  $$\omega(\mathcal{K}) = \frac{1}{2}(Q^2 + P^2)$$

  is the Hamiltonian of the quantum harmonic oscillator.

- $$i^{1/2}\omega(\exp -\frac{\pi i}{2}\mathcal{K}) = \mathcal{F}_1, \qquad (2.5)$$

  is a Fourier transform in $\mathcal{S}(\mathbf{R})$, given by

  $$\mathcal{F}_1(\varphi)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} \exp(-i\xi\, x)\varphi(x)dx.$$

The representation ω is not irreducible. It is the sum of two irreducible representations consisting of even and odd functions:

$$L^2(\mathbf{R}) = L^2(\mathbf{R})_{even} \oplus L^2(\mathbf{R})_{odd}.$$

Next, we consider the tensor of the p-fold tensor product of ω with the q-fold tensor product of its dual representation. Abusing notation, we shall also denote it by ω. Explicitly, as a representation of $\mathfrak{sp}(1,\mathbf{R})$, it acts on the space $\mathcal{S}(\mathbf{R}^{p+q})$ by

$$\begin{aligned}
\omega(\mathcal{H}) &= \frac{i}{2}\{\Delta_{p,q}, r^2_{p,q}\} \\
\omega(\mathcal{E}^+) &= \frac{i}{2}r^2_{p,q}, \qquad \omega(\mathcal{E}^-) = \frac{i}{2}\Delta_{p,q}.
\end{aligned}$$

This representation exponentiates to a representation of $\widetilde{Sp(1,\mathbf{R})}$.

The group $O(p,q) = \text{isom}(\mathbf{R}^{p+q}, \tau_{p,q})$ acts on $\mathcal{S}(\mathbf{R}^{p+q})$ via the permutation representation. A direct computation shows that the $\widetilde{Sp(1,\mathbf{R})}$ and $O(p,q)$ actions $\mathcal{S}(\mathbf{R}^{p+q})$ on commute with each other. In fact, the groups $(O(p,q), Sp(1,\mathbf{R}))$ form a dual pair in $Sp(p+q,\mathbf{R}^{p+q})$. Moreover the restriction of oscillator representation of $\widetilde{Sp(p+q,\mathbf{R}^{p+q})}$ to $\widetilde{Sp(1,\mathbf{R})}$ and $O(p,q)$ coincide with the representation just described.

2.4 Geometric Objects, Spectral Objects and $\mathcal{CHC}$

Arthur [1], referring to the fundamental work of Harish-Chandra, singles out two central objects in the representation theory of real reductive groups. Given a Lie group G, its *geometric objects* are the *orbital integrals* of G, while its *spectral objects* are the *characters* of elements in $\widehat{G}$.

HOWE'S CORRESPONDENCE FOR REPRESENTATIONS

Let $(G, G') \subset Sp(n, \mathbf{R})$ be a dual pair, and $\tilde{H}$ the preimage of $H \subset Sp(n, \mathbf{R})$ in $\widetilde{Sp(n, \mathbf{R})}$.

Theorem 2.2 *[13] As a group homomorphism, ω is "almost a morphism of dual pairs", in the sense that the von Neumann algebra generated by $\omega(\tilde{G})$ is the commutant of the von Neumann algebra generated by $\omega(\tilde{G}')$, and viceversa. The following decomposition holds*

$$\omega|_{\widetilde{G}\times\widetilde{G'}} \simeq \int_{\widehat{\widetilde{G'}}}^{\oplus} \theta(\pi') \otimes \pi' d\Omega(\pi'),$$

where $\theta : \widehat{\widetilde{G'}} \rightsquigarrow \widehat{\widetilde{G}}$ is a partial function, that is injective a.e. (with respect to $d\Omega(\pi')$). Furthermore, the function θ is injective in the stable range (see below).

This representation has characters which can be expressed in terms of orbital integrals as shown by Harish-Chandra. We shall look at these notions in more detail below.

THE ORBITAL INTEGRAL

Given G reductive, let H be a Cartan subgroup. We say that an element $x \in \mathfrak{g}$ is **regular**, and write $x \in \mathfrak{g}^r$, if $\dim \mathfrak{g}^x$ is maximal.

For $x \in \mathfrak{h}^r$ and $\varphi \in \mathcal{S}(\mathfrak{g})$, define the **orbital integral** on the orbit $\mathcal{O}_x$,

$$\delta_{\mathcal{O}_x}(\varphi) = \int_{\mathcal{O}_x} \varphi \; d\mu = \int_{\mathfrak{g}} \varphi(y)\delta_{\mathcal{O}_x}(y)dy := \int_{G/H} \varphi(g \cdot x)d\dot{g},$$

where $\mu = d\dot{g}$ is a G-invariant measure on the quotient. (Note that G, H are unimodular and hence the quotient carries a unique such measure, up to scalar multiple.) The two expressions in the middle are formal and are intended for the physicists. These orbital integrals patch together to yield a G-invariant function $\mu_{\mathfrak{g}}(\varphi)$ on $\mathfrak{g}^{rs}$ (the subset of regular semisimple elements in $\mathfrak{g}$). The value of this function on the orbit passing through $x \in \mathfrak{h}^r$ is given by:

$$\mu_{\mathfrak{g}}(\varphi)(x) = \delta_{\mathcal{O}_x}(\varphi).$$

THE HARISH-CHANDRA - WEYL INTEGRATION FORMULA FOR THE LIE ALGEBRA

Let $\mathcal{C} = \{H_1, ..., H_l\}$ be a complete set of representatives of mutually non-conjugate Cartan subgroups of G. Also, for $x \in \mathfrak{h}$ put

$$D(x) = |\det(\text{ad } x)_{\mathfrak{g}/\mathfrak{h}}|^{1/2}.$$

Define the following **Harish-Chandra - Weyl pairing for $\mathfrak{g}$**:

$$\mathfrak{W}_{\mathfrak{g}}(\tau, \vartheta) = \sum_{\mathcal{C}} \frac{1}{|W(H,G)|} \int_{\mathfrak{h}} \overline{\tau(x)} D(x)^2 \vartheta(x) dx,$$

where τ, ϑ, are conjugation invariant functions on $\mathfrak{g}$ such that this is integral is absolutely convergent. The **Harish-Chandra - Weyl integration formula** states that for $\varphi \in \mathcal{S}(\mathfrak{g})$:

$$\int_{\mathfrak{g}} \varphi(x)\, dx = \sum_{\mathcal{C}} \frac{1}{|W(H,G)|} \int_{\mathfrak{h}^r} D(x)^2 \delta_{\mathcal{O}_x}(\varphi) dx = \mathfrak{W}_{\mathfrak{g}}(1_{\mathfrak{g}}, \mu_{\mathfrak{g}}(\varphi)). \tag{2.6}$$

In [17] Przebinda proposed what may be called an analytic version of Howe's correspondence, dealing with the geometric and spectral objects for a dual pair (G, G') with rank $G' \leq$ rank G. Now we proceed to sketch his idea.

$\mathcal{CHC}$ FOR THE LIE ALGEBRA

Let $\chi(x) = \exp(2\pi i x) \in \widehat{\mathbf{R}}$, and let (W, τ) be a real finite dimensional symplectic space. The moment map $\tau_{\mathfrak{sp}(W)}$ induces an injection

$$\chi_{-} : \mathfrak{sp}(W) \hookrightarrow \{ \text{ Gaussians on } W \} : \qquad \chi_x(w) = \chi\left(\frac{1}{4}\tau_{\mathfrak{sp}(W)}(w)(x)\right).$$

Let $(G, G') \subset Sp(W)$ be an irreducible dual pair. For a Cartan subgroup $H' \subset G'$, let $\mathcal{K}_{H'}$ be the compact part (consisting of elements with eigenvalues in S^1) and $\mathcal{V}_{H'}$ be the vector part (consisting of elements with eigenvalues in $\mathbf{R}^+$), so that $H' = \mathcal{K}_{H'}\mathcal{V}_{H'}$. Let $\mathcal{V} = Sp(W)^{\mathcal{V}_{H'}}$ and $\mathcal{V}' = Sp(W)^{\mathcal{V}} \subset G'$. There is a $\mathcal{V}'$-invariant open dense set $W_{\mathcal{V}'} \subset W$ such that $\mathcal{M} = \mathcal{V}' \setminus W_{\mathcal{V}'}$ is a manifold with invariant measure $d\dot{w}$ such that for $\varphi \in C_c(W_{\mathcal{V}'})$ we have

$$\int_{W_{\mathcal{V}'}} \varphi(w)dw = \int_{\mathcal{M}} \int_{\mathcal{V}'} \varphi(vw)dvd\dot{w}.$$

As shown in [17], there is a temperered distribution $\widetilde{\mathrm{chc}} \in \mathcal{S}^*(\mathcal{L}ie(\mathcal{V}))$ given by

$$\widetilde{\mathrm{chc}}(\varphi) = \int_{\mathcal{M}} \int_{\mathcal{L}ie(\mathcal{V})} \varphi(v)\chi_v(w)dvd\dot{w},$$

where every consecutive integral is absolutely convergent. Its wavefront set [13, Def 8.1.2] is given by:

$$WF(\widetilde{\mathrm{chc}}) \subseteq \{(v, \tau_{\mathcal{V}}(w)) : v(w) = 0, v \in \mathcal{V}, w \in W\}. \qquad (2.7)$$

Furthermore, any $x' \in \mathfrak{h}'^{r}$, defines an embedding:

$$\iota_{x'} : \mathfrak{g} \hookrightarrow \mathcal{L}ie(\mathcal{V}) \qquad\qquad x \mapsto x + x'.$$

The containment (2.7) and [13, 8.2.4] implies that

$$\widetilde{\mathrm{chc}}_{x'}(\varphi) = \iota^{*}_{x'}(\widetilde{\mathrm{chc}})(\varphi)$$

is well defined.

Formally, given $\varphi \in \mathcal{S}(\mathfrak{g})$, we have:

$$\widetilde{\mathrm{chc}}_{x'}(\varphi) = \int_{\mathcal{M}} \int_{\mathfrak{g}} \varphi(x)\chi_{x+x'}(w) dx d\dot{w} = \int_{\mathfrak{g}} \varphi(x)\mathrm{chc}(x + x')dx,$$

where:

$$\widetilde{\mathrm{chc}}(x + x') = \int_{\mathcal{M}} \chi_{x+x'}(w) d\dot{w}.$$

Also,

$$WF(\widetilde{\mathrm{chc}}_{x'}) \subseteq \{(x, \tau_{\mathfrak{g}}(w)) : (x + x')(w) = 0, x \in \mathfrak{g}, w \in W\}.$$

Finally, define the **Cauchy Harish-Chandra integral** $\mathcal{CHC}_{\mathfrak{g}'}(\varphi)$, as a function on $\mathfrak{h}^{r}$ for every $\mathfrak{h} \in \mathcal{C}$, by:

$$\mathcal{CHC}_{\mathfrak{g}'}(\varphi)(x') = \mathrm{chc}_{x'}(\varphi). \tag{2.8}$$

Later will introduce a normalized version of this distribution denoted by the same symbol.

FOURIER TRANSFORM

Define a Fourier transform on a finite dimensional vector space V with Lebesgue measure dx, as follows. For $\varphi \in \mathcal{S}(V),\ \xi \in V^*$, put:

$$\mathcal{F}(\varphi)(\xi) = \widehat{\varphi}(\xi) = \int_V \varphi(x)\chi(\xi(x))dx, \tag{2.9}$$

while for a distribution $u \in \mathcal{S}^*(V)$ let:

$$\mathcal{F}(u)(\varphi) = \widehat{u}(\varphi) = u(\widehat{\varphi}). \tag{2.10}$$

THE CORRESPONDENCE FOR ORBITAL INTEGRALS

If (G, G') is a dual pair of type I in stable range (so that defining module for G contains an isotropic subspace of dimension grater than or equal to the dimension of the defining module for G') then for each nilpotent coadjoint orbit $\mathcal{O}' \subset \mathfrak{g}'^*$, there exists a unique nilpotent coadjoint orbit $\mathcal{O}$ dense in $\tau_{\mathfrak{g}} \circ \tau_{\mathfrak{g}'}^{-1}(\mathcal{O}') \subset \mathfrak{g}^*$. Then, according to [17, (1.19)] , there is a constant C such that:

$$\mathcal{F}(\delta_{\mathcal{O}})(\varphi) \quad = \quad C\ \mathfrak{W}_{\mathfrak{g}'}(\mathcal{F}(\delta_{\mathcal{O}'}), \mathcal{CHC}_{\mathfrak{g}'}(\varphi)).$$

This is analogous to the Formula (2.6) and suggests that $\mathcal{CHC}_{\mathfrak{g}'}$ behaves like an orbital integral (see [3]). In fact (see [5]) under adequate hypothesis $\mathcal{CHC}_{\mathfrak{g}'} \in \hom(\mathcal{D}(\mathfrak{g}), \mathcal{I}(\mathfrak{g}'))$, where $\mathcal{I}(\mathfrak{g}')$ is the space of orbital integrals, as defined in [5].

Using the language of pairings, the correspondence reads,

$$\mathfrak{W}_{\mathfrak{g}}(\mathcal{F}(\delta_{\mathcal{O}}), \mu_{\mathfrak{g}}(\varphi)) \quad = \quad C\ \mathfrak{W}_{\mathfrak{g}'}(\mathcal{F}(\delta_{\mathcal{O}'}), \mathcal{CHC}_{\mathfrak{g}'}(\varphi)),$$

which can be cast in a more abstract form as,

$$\mathcal{T}_{\mathfrak{g}}(\mu_{\mathfrak{g}})[\mathcal{O}] = \mathcal{T}_{\mathfrak{g}'}(\mathcal{CHC}_{\mathfrak{g}'})[\mathcal{O}'] \in \mathcal{S}(\mathfrak{g})^*,$$

where the notation is self explanatory.

THE CORRESPONDENCE FOR CHARACTERS

There is an analogous theory at the group level. In [17, (2.18)], under adequate hypothesis, Przebinda conjectures a correspondence of the form:

$$\Theta_{\theta(\pi')}(\varphi) \quad = \quad \chi_{\pi'}((\tilde{-1}))\Theta_{\omega}((\tilde{-1}))\sum_{C}\frac{1}{|W(H')|}\int_{\widetilde{H'}^{r}}\overline{\Theta_{\pi'}(h')}D_G^2(h')\mathcal{CHC}_{G'}(\varphi)(h')dh',$$

where $\mathcal{CHC}_G$ is an analogue at the group level of $\mathcal{CHC}_{\mathfrak{g}}$, Θ_{π} is the character of π, $\theta(\pi')$ is the representation associated to π' via Howe correspondence, as in Theorem 2.2, while $\chi_{\pi'}((\tilde{-1}))$ is the number by which the central element $(\tilde{-1})$ acts on representation space of π'.

Using the Harish-Chandra - Weyl pairing introduced implicitly above, we can write this correspondence as:

$$\mathfrak{W}_G(\Theta_{\theta(\pi')}, \mu_{\mathfrak{g}}(\varphi)) = \mathfrak{W}_{G'}(A(\pi')\Theta_{\pi'}, \mathcal{CHC}_{G'}(\varphi)),$$

or

$$\mathcal{T}_G(\mu(\varphi))[\Theta(\theta(\pi'))] = \mathcal{T}_{G'}(\mathcal{CHC}_{G'})[\Theta(\pi')] \in \mathcal{S}(\mathfrak{g})^*.$$

Chapter 3

Structure of the Lie Algebras

3.1 Roots, Cartan Subalgebras, Weyl Groups

General Definitions

Let G be a semisimple Lie group with $\mathcal{L}ie(G) = \mathfrak{g}$ and let Θ be a Cartan involution on $\mathfrak{g}$. The spectral decomposition of Θ,

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

where $\mathfrak{k}$ is the $+1$-eigenspace of Θ and $\mathfrak{p}$ is the -1-eigenspace of Θ, is called the **Cartan decomposition** associated to Θ. Assume there is a compact Cartan subgroup $H \subseteq G$ and that $\mathfrak{h} \subset \mathfrak{k}$. The joint spectral decomposition for the adjoint action of $\mathfrak{h}_{\mathbf{C}}$ on $\mathfrak{g}_{\mathbf{C}}$ is

$$\mathfrak{g}_{\mathbf{C}} = \mathfrak{h}_{\mathbf{C}} \oplus \sum_{\Delta} (\mathfrak{g}_{\mathbf{C}})_{\lambda},$$

where $\Delta = \Delta(\mathfrak{h}, \mathfrak{g}) \subset \mathfrak{h}_{\mathbf{C}}^{*}$ is the set of **roots** of $\mathfrak{h}_{\mathbf{C}}$ in $\mathfrak{g}_{\mathbf{C}}$. Given that $\mathfrak{h}$ is Θ-stable, so is each $(\mathfrak{g}_{\mathbf{C}})_{\lambda}$.

Definition 3.1 • *Since* $\dim\,(\mathfrak{g}_{\mathbf{C}})_\lambda = 1$*, we have that either* $(\mathfrak{g}_{\mathbf{C}})_\lambda \subseteq \mathfrak{k}_{\mathbf{C}}$ *or* $(\mathfrak{g}_{\mathbf{C}})_\lambda \subseteq \mathfrak{p}_{\mathbf{C}}$*. If the former condition holds we say that* λ *is* ***compact*** *and write* $\lambda \in \Delta^c$*, if the latter we say that* λ *is* ***non-compact****, and write* $\lambda \in \Delta^n$*.*

- *For* $x \in \mathfrak{g}_{\mathbf{C}}$*, let* $\overline{x}$ *denote the conjugate of* x *with respect to the real form* $\mathfrak{g} \subseteq \mathfrak{g}_{\mathbf{C}}$*. As in [19], for each root* $\lambda \in \Delta$ *we fix elements* $H_\lambda \in i\mathfrak{h}$*,* $X_\lambda \in (\mathfrak{g}_{\mathbf{C}})_\lambda$*,* $X_{-\lambda} \in (\mathfrak{g}_{\mathbf{C}})_{-\lambda}$ *such that:*

$$[X_\lambda, X_{-\lambda}] = H_\lambda,\ [H_\lambda,\ X_\lambda] = 2X_\lambda,\ [H_\lambda,\ X_{-\lambda}] = -2X_{-\lambda},$$

$\overline{H}_\lambda = -H_\lambda = H_{-\lambda},\ \lambda \in \Delta;\quad \overline{X}_\lambda = -X_{-\lambda},\quad \lambda \in \Delta^c;$ *and* $\overline{X}_\lambda = X_{-\lambda},\quad \lambda \in \Delta^n$*. We call* $(X_\lambda,\ X_{-\lambda},\ H_\lambda)$*, an* $\mathfrak{sl}(2)$***-triple*** *associated to* λ *and* H_λ *the* ***co-root*** *corresponding to* λ*.*

- *We define the* ***Weyl Groups****:*

$$W(H_{\mathbf{C}}) = N_{G_{\mathbf{C}}}(\mathfrak{h}_{\mathbf{C}})/H_{\mathbf{C}}, \qquad W(H) = N_G(\mathfrak{h})/H.$$

Clearly, $W(H_{\mathbf{C}})$ acts on $\mathfrak{h}_{\mathbf{C}}$, and on the dual $\mathfrak{h}^*_{\mathbf{C}}$.

Parametrization By Strongly Orthogonal Sets

Definition 3.2 *Two roots are called* ***strongly orthogonal*** *if they are not proportional and neither their sum nor their difference is a root. A set of roots is called strongly orthogonal if its members are pairwise strongly orthogonal.*

For $\lambda \in \Delta$ *we denote by* s_λ *the reflection that maps* λ *to* $-\lambda$ *and that leaves the orthogonal complement of* λ *with respect to the killing form fixed.*

Fix a positive root system

$$\Psi = \Psi(\mathfrak{h}, \mathfrak{g}) \subseteq \Delta.$$

Let $\Psi^n = \Psi \cap \Delta^n$. The group $W(H)$ acts on Δ^n and therefore also on the subsets of Δ^n. Let Ψ^n_{st} denote the family of strongly orthogonal subsets of Ψ^n.For $\mathcal{S} \in \Psi^n_{st}$ let

$$[\mathcal{S}] = (W(H)(\mathcal{S} \cup (-\mathcal{S}))) \cap \Psi. \tag{3.1}$$

Clearly, $[\mathcal{S}] \subseteq \Psi^n_{st}$ and Ψ^n_{st} is a disjoint union of the sets of the form (3.1). This defines an equivalence relation on Ψ^n_{st}, where $[\mathcal{S}]$ is the equivalence class of $\mathcal{S}$. Put $[\Psi^n_{st}] = \{[\mathcal{S}] : \mathcal{S} \in \Psi^n_{st}\}$.

Definition 3.3 *We define the following* ***Cayley transforms****:*

$$c : \Psi \longrightarrow \mathrm{aut}(\mathfrak{g}_{\mathbf{C}}) \quad : \qquad c(\lambda) = \exp(-\frac{\pi i}{4}\, ad(X_\lambda + X_{-\lambda})),$$

$$c : \Psi^n_{st} \longrightarrow \mathrm{aut}(\mathfrak{g}_{\mathbf{C}}) \quad : \qquad c(\mathcal{S}) = \prod_{\mathcal{S}} c(\lambda).$$

Note that this last product does not depend on the order of the factors because, for α, β strongly orthogonal, we have $[c(\alpha), c(\beta)] = 0$. Also, let

$$\mathfrak{h}(\mathcal{S}) = \mathfrak{g} \cap c(\mathcal{S})(\mathfrak{h}_{\mathbf{C}}).$$

This is a Cartan subalgebra of $\mathfrak{g}$. Put $H(\mathcal{S}) = G^{\mathfrak{h}(\mathcal{S})}$.

Proposition 3.1 *[19] Every Cartan subalgebra of $\mathfrak{g}$ is conjugate to one of the form $\mathfrak{h}(\mathcal{S})$. Two Cartan subalgebras $\mathfrak{h}(\mathcal{S})$, $\mathfrak{h}(\mathcal{S}')$ are conjugate if and only if $[\mathcal{S}] = [\mathcal{S}']$. Thus the conjugacy classes of the Cartan subalgebras in $\mathfrak{g}$ are parametrized by $[\Psi^n]$.*

By definition, given a root $\lambda \in \mathfrak{h}^*_{\mathbf{C}}$, we have:

$$\mathfrak{h}_{\mathbf{C}} = \ker(\lambda) \oplus \mathbf{C}H_\lambda. \tag{3.2}$$

Dualizing, any co-root H_λ can be seen as an element of $(\mathfrak{h}^*_{\mathbf{C}})^*$ and hence:

$$\mathfrak{h}^*_{\mathbf{C}} = \ker(H_\lambda) \oplus \mathbf{C}\lambda.$$

Let

$$\mathfrak{h}_{\mathcal{S}} = c(\mathcal{S})^{-1}(\mathfrak{h}(\mathcal{S})) \subseteq \mathfrak{h}_{\mathbf{C}}$$

and $\mathcal{L}ie(H_{\mathcal{S}}) = \mathfrak{h}_{\mathcal{S}}$.

Proposition 3.2 *The following decomposition holds:*

$$\mathfrak{h}_{\mathcal{S}} = \mathfrak{h} \cap \mathcal{V}_{\mathcal{S}} + \sum_{\mathcal{S}} \mathbf{R}H_\lambda, \tag{3.3}$$

where $\mathcal{V}_{\mathcal{S}}$ stands for the variety of all the common zeros of the elements of $\mathcal{S}$.

Let

$$\begin{aligned}
\Delta_{\mathcal{S},\mathbf{R}} &= \{\lambda \in \Delta;\ \lambda(\mathfrak{h}_{\mathcal{S}}) \subseteq \mathbf{R}\}, \\
\Delta_{\mathcal{S},i\mathbf{R}} &= \{\lambda \in \Delta;\ \lambda(\mathfrak{h}_{\mathcal{S}}) \subseteq i\mathbf{R}\}, \\
\Delta_{\mathcal{S},\mathbf{C}} &= \Delta \setminus (\Delta_{\mathcal{S},\mathbf{R}} \cup \Delta_{\mathcal{S},i\mathbf{R}}),
\end{aligned}$$

and let the corresponding symbols with Ψ be the intersections with Ψ, for example: $\Psi_{\mathcal{S},\mathbf{R}} = \Psi \cap \Delta_{\mathcal{S},\mathbf{R}}$.

Also, put

$$W(H(\mathcal{S})) = N_G(\mathfrak{h}(\mathcal{S}))/H(\mathcal{S}).$$

This is the Weyl group corresponding to the Cartan subalgebra $\mathfrak{h}(\mathcal{S})$. Let

$$W(H_{\mathcal{S}}) = c(\mathcal{S})^{-1} W(H(\mathcal{S})) c(\mathcal{S}). \tag{3.4}$$

This is a subgroup of $W(H_{\mathbf{C}})$, isomorphic to $W(H(\mathcal{S}))$. Let

$$W(\Delta_{\mathcal{S},\mathbf{R}}) = \langle \{ s_{\Delta_{\mathcal{S},\mathbf{R}}} \} \rangle \subseteq W(H_{\mathbf{C}}).$$

According to [19] we have

$$W(H_{\mathcal{S}}) = W(H)_{\mathcal{S} \cup (-\mathcal{S})} \, W(\Delta_{\mathcal{S},\mathbf{R}}), \tag{3.5}$$

where $W(H)_{\mathcal{S} \cup (-\mathcal{S})}$ denotes the stabilizer of $\mathcal{S} \cup (-\mathcal{S})$ in $W(H)$.

3.2 Harish-Chandra Orbital Integral

For $A \subseteq \Delta$, let

$$\mathcal{A}(A) = \prod_A \frac{\lambda}{|\lambda|}, \qquad \mathcal{D}(A) = \prod_A \lambda, \qquad \text{and} \qquad \mathcal{D} = \mathcal{D}(\Psi).$$

For $\varphi \in \mathcal{S}(\mathfrak{g})$ and for $x \in \mathfrak{h}_{\mathcal{S}}^r$, define

$$\begin{aligned} \varphi(\mathcal{S})(x) &= \mathcal{D}(x) \int_{G/H(\mathcal{S})} \varphi(g \cdot c(\mathcal{S})(x))\, d\dot{g} \\ &= \mathcal{D}(x) \delta_{\mathcal{O}_{c(\mathcal{S})x}}(\varphi), \end{aligned}$$

where $g \cdot c(\mathcal{S})(x) = g\ c(\mathcal{S})(x)\ g^{-1}$.

Definition 3.4 *Let*

$$\mathcal{H}_{\mathcal{S}}\varphi = \mathcal{A}(\Psi_{\mathcal{S},\mathbf{R}})\varphi(\mathcal{S}).$$

This is the pull-back of the **Harish-Chandra orbital integral** of φ, from $\mathfrak{h}(\mathcal{S})$ to $\mathfrak{h}_{\mathcal{S}}$ via $c(\mathcal{S})$.

We now rewrite the Harish-Chandra - Weyl integration formula in the language of Cayley transforms and Harish-Chandra orbital integrals.

The formula becomes,

$$\int_{\mathfrak{g}} \varphi(x)\, dx = \sum_{[\Psi_{st}^n]} \frac{1}{|W(H_{\mathcal{S}})|} \int_{\mathfrak{h}_{\mathcal{S}}} \overline{\mathcal{D}}(x) \mathcal{A}(\Psi_{\mathcal{S},\mathbf{R}})(x) \mathcal{H}_{\mathcal{S}}\varphi(x)\, dx, \tag{3.6}$$

where in the summation, for $[\mathcal{S}] \in [\Psi_{st}^n]$, we chose exactly one representative $\mathcal{S} \in [\mathcal{S}]$.

3.3 Structure of the Lie Algebra $\mathfrak{sp}(1, \mathbf{R})$

Let τ' be a non-degenerate skew-symmetric bilinear form on $\mathbf{R}^2$. Let $Sp(1,\mathbf{R}) = \mathrm{isom}(\mathbf{R}^2, \tau')$. There is a basis $\{e_1, e_2\}$ of $V' \cong \mathbf{R}^2$ such that $\tau'(e_1, e_2) = 1$. We identify $\mathfrak{sp}(1, \mathbf{R})$ with the Lie algebra of trace zero matrices with respect to this basis. Hence, the formula $\Theta(X) = -X^t$ defines a Cartan involution. Let

$$\mathfrak{sp}(1, \mathbf{R}) = \mathfrak{k}' \oplus \mathfrak{p}',$$

be the corresponding Cartan decomposition. Then $\mathfrak{k}' = \mathbf{R}\mathcal{J}$ and $\mathfrak{p}' = \mathbf{R}\mathcal{I} \oplus \mathbf{R}\mathcal{H}$. The fundamental Cartan subalgebra $\mathfrak{h}' = \mathfrak{k}'$ has only one positive root in $\mathfrak{sp}(1, \mathbf{R})_{\mathbf{C}}$, namely λ', given by

$$\lambda'(m\mathcal{J}) = 2im.$$

Then ,

$$X_{\lambda'} = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \quad X_{-\lambda'} = \overline{X_{\lambda'}} \quad H_{\lambda'} = -i\mathcal{J}, \tag{3.7}$$

form an $\mathfrak{sl}(2)$-triple, and $\lambda' \in \Delta^n$.

Also, by direct computation,

$$c(\lambda')\begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} a & ib \\ -ic & -a \end{pmatrix},$$

so that

$$\mathfrak{h}'(\lambda') = \mathbf{R}c(\lambda')(H_{\lambda'}) = \mathbf{R}\mathcal{I} \text{ and } \mathfrak{h}'_{\{\lambda'\}} = \mathbf{R}H_{\lambda'} \subseteq \mathfrak{h}'_{\mathbf{C}}. \tag{3.8}$$

Up to a conjugation by an element of $Sp(1, \mathbf{R})$, $\mathfrak{h}'$ and $\mathfrak{h}(\lambda')$ are the only Cartan subalgebras of $\mathfrak{sp}(1, \mathbf{R})$. Furthermore, their Weyl groups are $W(H') = 1$ and $W(H_{\lambda'}) = \{1, s_{\lambda'}\}$.

3.4 A Cartan Decomposition of $\mathfrak{g}_2$

In the case of $\mathfrak{g} = \mathfrak{g}_2$, the map $\Theta(X) = -X^T$ is also a Cartan involution, so that:

$$\mathfrak{g}_2 = \mathfrak{k}_2 \oplus \mathfrak{p}_2, \tag{3.9}$$

where:

$$\mathfrak{k}_2 = \{V_{v^1} + V_{v^2} : v^k \in \mathbf{R}^3\},$$

$$\mathfrak{k}_2 = \left\{ \begin{pmatrix} 0 & -2v_3^1 & 2v_2^1 & 0 & 0 & 0 & 0 \\ 2v_3^1 & 0 & -2v_1^1 & 0 & 0 & 0 & 0 \\ -2v_2^1 & 2v_1^1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & v_1^1 - v_1^2 & v_2^1 - v_2^2 & v_3^1 - v_3^2 \\ 0 & 0 & 0 & -v_1^1 + v_1^2 & 0 & -v_3^1 - v_3^2 & v_2^1 + v_2^2 \\ 0 & 0 & 0 & -v_2^1 + v_2^2 & v_3^1 + v_3^2 & 0 & -v_1^1 - v_1^2 \\ 0 & 0 & 0 & -v_3^1 + v_3^2 & -v_2^1 - v_2^2 & v_1^1 + v_1^2 & 0 \end{pmatrix} \right\},$$

and

$$\mathfrak{p}_2 = \{V_\rho + V_{v^3} : v^3 \in \mathbf{R}^3, \rho \in \mathfrak{sl}(3, \mathbf{R})\},$$

for

$$v^3 = (v_1^3, v_1^3, v_3^3) \qquad \text{and} \qquad \rho = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & a-d \end{pmatrix}.$$

Put

$$\Gamma(A) = \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix},$$

then:

$$\mathfrak{p}_2 = \left\{ \Gamma(A) \quad : \quad A = \begin{pmatrix} 2v_1^3 & a & b+v_3^3 & c-v_2^3 \\ 2v_2^3 & b-v_3^3 & d & e+v_1^3 \\ 2v_3^3 & c+v_2^3 & e-v_1^3 & -a-d \end{pmatrix} \right\}.$$

MAXIMAL COMPACT SUBGROUP OF G_2

Let $U(\mathbf{H})$ be the group of quaternions of norm 1. Recall that this group is isomorphic to $SU(2)$. The group $K = U(\mathbf{H}) \times U(\mathbf{H})$ acts faithfully on $Im(\mathbf{O}) = \mathfrak{u}(\mathbf{H}) \oplus \mathbf{H}$ by

$$(u_1, u_2)\cdot(a, b) = (u_1 a {u_1}^{-1}, u_2 b {u_1}^{-1}), \tag{3.10}$$

preserving the multiplicative structure, as can be easily seen. That is

$$K \hookrightarrow G_2.$$

We compute the derivative:

$$(-,-)_* : \ \mathfrak{u}(\mathbf{H}) \oplus \mathfrak{u}(\mathbf{H}) \hookrightarrow \mathfrak{g}_2,$$

$$[(-,-)_*]\,(v^1, v^2)(a, b) \quad = \quad ([v^1, a], v^2 \cdot b - b \cdot v^1)). \tag{3.11}$$

The matrix the map 3.11 (with respect to the basis $\xi_j = (e_j, 0)$, $\xi_{j+4} = (0, e_j)$), can be read from the following equations:

$$([(v_1^1 e_1 + v_2^1 e_2 + v_3^1 e_3), e_1], 0) = (2(-v_2^1 e_3 + v_3^1 e_2), 0) = -2v_2^1 \xi_3 + 2v_3^1 \xi_2,$$

$$\vdots$$

$$(0, -e_0 \cdot (v_1^1 e_1 + v_2^1 e_2 + v_3^1 e_3)) = (0, -v_1^1 e_1 - v_2^1 e_2 - v_3^1 e_3) = -v_1^1 \xi_5 - v_2^1 \xi_6 - v_3^1 \xi_7,$$

$$(0, -e_1 \cdot (v_1^1 e_1 + v_1^2 e_2 + v_1^3 e_3)) = -v_1^1 \xi_4 - v_2^1 \xi_7 + v_3^1 \xi_6,$$

$$\vdots$$

Thus, in fact:

$$(-,-)_* : \mathfrak{u}(\mathbf{H}) \oplus \mathfrak{u}(\mathbf{H}) \to \mathfrak{k}_2 \qquad (v^1, v^2) \mapsto V_{v^1} + V_{v^2}. \tag{3.12}$$

The map (3.12) is a Lie algebra isomorphism and K is a maximal compact subgroup $K_2 \subseteq G_2$, as follows from the fact that they both K and K_2 are connected.

3.5 Description of $\mathfrak{g}_2^\perp \subset \mathfrak{so}(3,4)$

Here $\perp$ denotes orthogonality w.r.t. the Killing form $\langle X, Y \rangle = k\mathrm{tr}(XY)$, where k is a suitable constant. We consider the space

$$\mathfrak{g}_2^\perp = \{X \in \mathfrak{so}(3,4) : \langle X, \mathfrak{g}_2 \rangle = 0\}.$$

Since each of the relations in table 1.3 involves only either elements of $\mathfrak{k}_2$ or elements of $\mathfrak{p}_2$, and when restricted to one of these subspaces the Killing form is a multiple of the Euclidean inner product, we have the following:

$$\mathfrak{g}_2^\perp = \left\{ \begin{pmatrix} \underline{\varsigma_2} & \varsigma_1^T & \underline{\varsigma_1} + \alpha I_3 \\ \varsigma_1 & 0 & -\varsigma_2 \\ -\underline{\varsigma_1} + \alpha I_3 & \varsigma_2^T & -\underline{\varsigma_2} \end{pmatrix} \right\}, \tag{3.13}$$

where $\varsigma_j \in \mathbf{R}^3$, $\alpha \in \mathbf{R}$.

Lemma 3.1 *In terms of Proposition 1.1, the projection*

$$\pi : \mathfrak{so}(3,4) \longrightarrow \mathfrak{g}_2$$

relative to $\mathfrak{so}(3,4) = \mathfrak{g}_2 \oplus \mathfrak{g}_2^\perp$ *is given by :*

$$X \mapsto V(\frac{1}{2}(X_3+X_3^T)-\frac{\mathrm{tr}(X_3)}{3}I_3, \frac{1}{6}(2X_1+X_5+X_4), \frac{1}{2}(X_5-X_4), -\frac{1}{6}(X_3-X_3^T-2X_2)),$$

where

$$X = \begin{pmatrix} \underline{X_1} & X_2^T & X_3 \\ X_2 & 0 & X_4 \\ X_3^T & -X_4^T & \underline{X_5} \end{pmatrix}.$$

Proof:

This follows by direct computation using that $\mathfrak{so}(3,4) = \mathfrak{g}_2 \oplus \mathfrak{g}_2^\perp$. **QED.**

Here is our main technical lemma:

Lemma 3.2

$$(\mathfrak{so}(3,4))_{rk\leq 2} \cap \mathfrak{g}_2^\perp = 0.$$

Proof:

We shall prove that if the the matrix (3.13) in $\mathfrak{g}_2^\perp$ has rank ≤ 2, all 3 by 3 minors vanish and hence it must be zero. Any $N \in \mathfrak{g}_2^{\perp}$ has the form

$$N = \begin{pmatrix} 0 & \alpha & \beta & -\eta & r & \delta & \varepsilon \\ -\alpha & 0 & \gamma & \varepsilon & -\delta & r & \eta \\ -\beta & -\gamma & 0 & -\delta & -\varepsilon & -\eta & r \\ -\eta & \varepsilon & -\delta & 0 & \gamma & -\beta & \alpha \\ r & -\delta & -\varepsilon & -\gamma & 0 & \alpha & \beta \\ \delta & r & -\eta & \beta & -\alpha & 0 & \gamma \\ \varepsilon & \eta & r & -\alpha & -\beta & -\gamma & 0 \end{pmatrix}.$$

Let $N\frac{(a_1,a_2,..,a_m)}{[j,k,l;j',k',l']} \in M_{3\times 3}(\mathbf{R})$ be the matrix consisting of intersection of the rows j,k,l and the columns j',k',l', as a function of $a_1,..,a_m \in \mathbf{R}$, $(m=3,4)$. Consider

$$N\frac{(\alpha,\beta,\gamma)}{[4,6,7;5,6,7]} = \begin{pmatrix} \gamma & -\beta & \alpha \\ 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \end{pmatrix}, \quad N\frac{(\alpha,\beta,\gamma)}{[4,5,6;4,5,7]} = \begin{pmatrix} 0 & \gamma & \alpha \\ -\gamma & 0 & \beta \\ \beta & -\alpha & \gamma \end{pmatrix},$$

$$N\frac{(\alpha,\beta,\gamma)}{[4,5,6;4,6,7]} = \begin{pmatrix} 0 & -\beta & \alpha \\ -\gamma & \alpha & \beta \\ \beta & 0 & \gamma \end{pmatrix}, \quad N\frac{(\delta,\varepsilon,\gamma)}{[3,4,5;2,3,4]} = \begin{pmatrix} -\gamma & 0 & -\delta \\ \varepsilon & -\delta & 0 \\ -\delta & -\varepsilon & -\gamma \end{pmatrix},$$

$$N\frac{(r,\delta,\eta,\varepsilon)}{[5,6,7;1,2,3]} = \begin{pmatrix} r & -\delta & -\varepsilon \\ \delta & r & -\eta \\ \varepsilon & \eta & r \end{pmatrix}, \quad N\frac{(\gamma,\delta,\varepsilon)}{[2,3,5;3,4,5]} = \begin{pmatrix} \gamma & \varepsilon & -\delta \\ 0 & -\delta & -\varepsilon \\ -\varepsilon & -\gamma & 0 \end{pmatrix},$$

$$N\frac{(\delta,\varepsilon,\eta,r)}{[1,2,3;4,6,7]} = \begin{pmatrix} -\eta & \delta & \varepsilon \\ \varepsilon & r & \eta \\ -\delta & -\eta & r \end{pmatrix}.$$

Then the condition $\det(N\frac{(\alpha,\beta,\gamma)}{[4,6,7;5,6,7]}) = 0$ implies $\alpha(\alpha^2+\beta^2+\gamma^2) = 0$ and hence $\alpha = 0$. Now, $\det(N\frac{(0,\beta,\gamma)}{[4,5,6;4,5,7]}) = 0$, only if $\gamma = 0$. Also, $\det(N\frac{(0,\beta,0)}{[4,5,6;4,6,7]}) = 0$ means $\beta = 0$. In a similar fashion, $\det(N\frac{(\delta,\varepsilon,0)}{[3,4,5;2,3,4]}) = 0$ shows that $\delta = 0$, while $\det(N\frac{(r,0,\eta,\varepsilon)}{[5,6,7;1,2,3]}) = 0$ implies $r = 0$. Additionally, $\det(N\frac{(0,0,\varepsilon)}{[2,3,5;3,4,5]}) = 0$ if and only if $\varepsilon = 0$. Finally, $\det(N\frac{(0,0,\eta,0)}{[1,2,3;4,6,7]}) = 0$ expresses the fact that $\eta = 0$. **QED**.

3.6 Root Decomposition of $(\mathfrak{g}_2)_{\mathbf{C}}$

Note that the vectors

$$\Lambda_1 := V_{(1,0,0)^1}, \qquad \Lambda_2 := V_{(1,0,0)^2},$$

correspond to the images of $(e_1, 0)$ and $(0, e_1)$ under the map

$$(-,-)_* : \mathfrak{u}(\mathbf{H}) \oplus \mathfrak{u}(\mathbf{H}) \to \mathfrak{k}_2,$$

and thus they span an elliptic Cartan subalgebra of $\mathfrak{g}_2$:

$$\mathfrak{h} = \left\{ t\Lambda_1 + s\Lambda_2 : \ s,\ t \in \mathbf{R} \right\} \subset \mathfrak{k}_2. \tag{3.14}$$

This is the fundamental Cartan subalgebra.

Now we proceed to describe a root decomposition of $(\mathfrak{g}_2)_{\mathbf{C}}$ w.r.t. its fundamental Cartan subalgebra:

$$\mathfrak{g}_{\mathbf{C}} = \mathfrak{h}_{\mathbf{C}} \oplus \sum_{\Psi} \left((\mathfrak{g}_{\mathbf{C}})_{\lambda} \oplus (\mathfrak{g}_{\mathbf{C}})_{-\lambda} \right). \tag{3.15}$$

where $\mathfrak{g} = \mathfrak{g}_2$ and Ψ is as in table 3.1. Let $H = \mathrm{diag}(0, D_1, D_2, D_3) = t\Lambda_1 + s\Lambda_2 \in \mathfrak{h}$ and $D_j = x_j \mathcal{J}$, so that $x_1 = -2t, x_2 = -s + t, x_3 = -(s+t)$. Then the equation $\mathrm{ad}_H(A) = \lambda(H)A$ can be written in terms of $(1,2,2,2) \times (1,2,2,2)$ block matrices as:

$$\begin{pmatrix} 0 & -A_{12}D_1 & -A_{13}D_2 & -A_{14}D_3 \\ \star & [D_1, A_{22}] & D_1A_{23} - A_{23}D_2 & D_1A_{24} - A_{24}D_3 \\ \star & \star & [D_2, A_{33}] & D_2A_{34} - A_{34}D_3 \\ \star & \star & \star & [D_3, A_{44}] \end{pmatrix} = \lambda(H)A, \tag{3.16}$$

where

$$A = \begin{pmatrix} 0 & A_{12} & A_{13} & A_{14} \\ -A_{12}^T & A_{22} & A_{23} & A_{24} \\ A_{13}^T & A_{23}^T & A_{33} & A_{34} \\ A_{14}^T & A_{24}^T & -A_{34}^T & A_{44} \end{pmatrix}$$

and each A_{ii} is antisymmetric.

2×2 NON-DIAGONAL BLOCKS:

Any equation of 2×2 non diagonal blocks induced from (3.16) looks as follows:

$$x_j \mathcal{J} \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix} x_k \mathcal{J} = \lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{3.17}$$

This implies that $\lambda = \pm i(x_j \pm x_k)$ are all the possible eigenvalues. Explicitly:

$$\lambda = \pm 2it, \pm 2is, \pm i(s+t), \pm i(3t-s), \pm i(3t+s), \pm i(s-t).$$

Indeed, for $A \in M = M_{2\times 2}(\mathbf{C})$ put $A^{\mathcal{J}} = -\{A, \mathcal{J}\}/2$, and ${}^{\mathcal{J}}A = [\mathcal{J}, A]/2$. Thus $[\mathcal{J}, A^{\mathcal{J}}] = 0$, while $\{\mathcal{J}, {}^{\mathcal{J}}A\} = 0$. This gives rise to a decomposition:

$$M = (M)^{\mathcal{J}} \oplus^{\mathcal{J}} (M). \tag{3.18}$$

Equation (3.17) above, can be rewritten as:

$$(x_j - x_k)\mathcal{J}A^{\mathcal{J}} + (x_j + x_k)\mathcal{J}^{\mathcal{J}}A = \lambda(A^{\mathcal{J}} +^{\mathcal{J}} A).$$

If $\lambda = \pm i(x_j + x_k)$, thus $A^{\mathcal{J}} = 0$ and $\mathcal{J}^{\mathcal{J}}A = \pm i^{\mathcal{J}}A$.

If $\lambda = \pm i(x_j - x_k)$, thus ${}^{\mathcal{J}}A = 0$ and $\mathcal{J}A^{\mathcal{J}} = \pm iA^{\mathcal{J}}$.

1×2 BLOCKS:

$$-(\alpha, \beta)x_k\mathcal{J} = (\beta x_k, -\alpha x_k) = (\lambda\alpha, \lambda\beta),$$

which gives $\lambda = \pm ix_k$, $\beta = \pm i\alpha$. The relation $x_3 = x_2 + x_1$ gives their interaction with the square blocks which ensures existence of eigenvectors.

DIAGONAL BLOCKS:

Since for all $a \in \mathbf{R}$ we have $x_k\mathcal{J}a\mathcal{J} - a\mathcal{J}x_k\mathcal{J} = 0$, indeed $(\mathfrak{g}_{\mathbf{C}})_0 = \mathfrak{h}_{\mathbf{C}}$.

THE POSITIVE ROOTS

Let $\widetilde{\lambda}_1$, $\widetilde{\lambda}_2$ be the basis of $\mathfrak{h}^*$ dual to Λ_1, Λ_2 viewed as a basis of $\mathfrak{h}^*_{\mathbf{C}}$. Put $\lambda_j = -2i\widetilde{\lambda}_j$, for $j = 1, 2$. As we just showed, the elements in the second column of table 3.1 are the roots of $\mathfrak{g}_2$.

It can be easily checked that

$$\begin{aligned}\alpha_2 &= \alpha_3+\lambda_1, \quad \alpha_1=\alpha_3+2\lambda_1,\\ \alpha_4 &= \alpha_3+3\lambda_1, \quad \lambda_2=2\alpha_3+3\lambda_1.\end{aligned}$$

Hence Ψ forms a positive system, with $\{\lambda_1, \alpha_3\}$ as basis of simple roots.

Table 3.1: Set of Positive Roots Ψ

Eigenvalue	root
$-2it = i(x_3 - x_2) = ix_1$	λ_1
$-2is = i(x_3 + x_2)$	λ_2
$-i(t+s) = i(x_1 + x_2) = ix_3$	$\alpha_1 = \frac{1}{2}(\lambda_1 + \lambda_2)$
$-i(-t+s) = i(x_3 - x_1) = ix_2$	$\alpha_2 = \frac{1}{2}(-\lambda_1 + \lambda_2)$
$-i(-3t+s) = -i(x_1 - x_2)$	$\alpha_3 = \frac{1}{2}(-3\lambda_1 + \lambda_2)$
$-i(3t+s) = i(x_3 + x_1)$	$\alpha_4 = \frac{1}{2}(3\lambda_1 + \lambda_2)$

Lemma 3.3 *The $\mathfrak{sl}(2)$-triples $(X_\alpha,\ X_{-\alpha},\ H_\alpha)$ associated to these positive roots in $(\mathfrak{g}_2)_{\mathbf{C}}$ are given in table 3.2 (in column 3 only non-zero parameters are listed).*

Proof:

- By extending to the complexification, (3.12) gives rise to the isomorphism

$$(\mathfrak{u}(\mathbf{H})\otimes\mathbf{C})\oplus(\mathfrak{u}(\mathbf{H})\otimes\mathbf{C})\to\mathfrak{h}_{\mathbf{C}}\oplus\sum_{\Delta^c}((\mathfrak{g}_2)_{\mathbf{C}})_\lambda.$$

- In each summand $\mathfrak{u}(\mathbf{H})\otimes\mathbf{C}$, the vectors $H_\lambda = ie_1,\ X_\lambda = \frac{1}{2}(e_2+ie_3),\ X_{-\lambda} =$

$-\overline{X_\lambda}$ form an $\mathfrak{sl}(2)$-triple, as can be easily verified:

$$\begin{aligned}
\left[\frac{1}{2}(e_2 + ie_3), -\frac{1}{2}(e_2 - ie_3)\right] &= ie_1, \\
\left[ie_1, \frac{1}{2}(e_2 + ie_3)\right] &= 2\frac{1}{2}(e_2 + ie_3), \\
\left[ie_1, -\frac{1}{2}(e_2 - ie_3)\right] &= 2\frac{1}{2}(e_2 - ie_3).
\end{aligned}$$

By the isomorphism (3.12), we have:

$$\xi_k \mapsto V_{(e_k)^1} \quad \xi_{k+4} \mapsto V_{(e_k)^2},$$

and hence the the first two lines in the table follow.

Also, if ρ_1, ρ_2 are symmetric, then $[V_{\rho_1}, V_{\rho_2}] = \mathrm{diag}([\rho_1, \rho_2], 0, [\rho_1, \rho_2])$. By looking at equations (3.7) we get the $\mathfrak{sl}(2)$-triple for α_4. We compute the rest using Mathematica© and the remarks above. **QED**.

3.7 Structure of the Root System of G_2

For $Y_1,\ Y_2 \in \mathfrak{h}_{\mathbf{C}}$, the Killing form is:

$$\langle Y_1,\ Y_2\rangle := \mathrm{tr}(\mathrm{ad}(Y_1)\mathrm{ad}(Y_2)) = 2\sum_{\Psi} \alpha(Y_1)\alpha(Y_2). \tag{3.19}$$

We have:

$$\alpha(H_{\lambda_j}) = \frac{2\langle H_\alpha, H_{\lambda_j}\rangle}{|H_\alpha|^2},$$

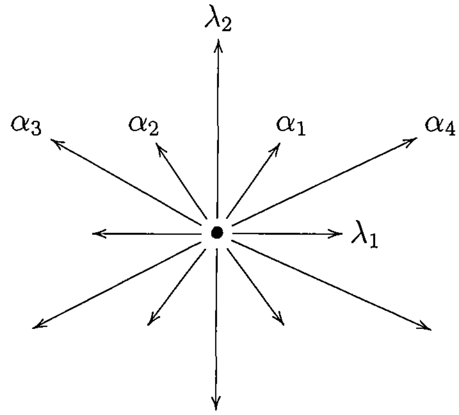
where $|H_\alpha|^2 = \langle H_\alpha, H_\alpha\rangle$. Hence we get the identification $\mathfrak{h}_{\mathbf{C}} \simeq \mathfrak{h}_{\mathbf{C}}^*$ that takes the following form for elements in Ψ given by:

$$\alpha \leftrightarrow \frac{2}{|H_\alpha|^2} H_\alpha. \tag{3.20}$$

Put

$$(\alpha, \beta) = \frac{4}{|H_\alpha|^2 |H_\beta|^2} \langle H_\alpha, H_\beta \rangle.$$

Roots System G_2



In our context strong orthogonality is equivalent to orthogonality with respect to (,).

The Weyl group $W(H) = \langle s_\alpha \rangle_{\Psi^c} \subset W(H_{\mathbf{C}})$ has order four. The $W(H)$-equivalence classes of strongly orthogonal sets of non-compact roots are:

$$\begin{aligned}
\mathcal{S}_0 &= \varnothing \\
\mathcal{S}_1 &= [\{\alpha_1\}\ ] = \{\{\alpha_1\}, \{\alpha_2\}\} \\
\mathcal{S}_2 &= [\{\alpha_3\}\ ] = \{\{\alpha_3\}, \{\alpha_4\}\} \\
\mathcal{S}_3 &= [\{\alpha_1, \alpha_3\}\ ] = \{\{\alpha_1, \alpha_3\}, \{\alpha_2, \alpha_4\}\}.
\end{aligned}$$

The complex Weyl group $W(H_{\mathbf{C}})$, generated by the reflections with respect to all the roots $\lambda \in \Psi$, contains a subgroup

$$W(H_{\mathbf{C}})^{+} = \langle s_\alpha s_\beta \rangle_{\alpha,\beta\in\Psi} \subset W(H_{\mathbf{C}}).$$

This is the subgroup of rotations of $W(H_{\mathbf{C}})$. It acts on orthogonal pair of roots as follows:

$$\begin{aligned} &\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \mapsto \begin{pmatrix} \alpha_1 \\ \alpha_3 \end{pmatrix} \mapsto \begin{pmatrix} \alpha_2 \\ -\alpha_4 \end{pmatrix} \mapsto \begin{pmatrix} -\lambda_1 \\ -\lambda_2 \end{pmatrix} \\ &\mapsto \begin{pmatrix} -\alpha_1 \\ -\alpha_3 \end{pmatrix} \mapsto \begin{pmatrix} -\alpha_2 \\ \alpha_4 \end{pmatrix} \mapsto \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \end{aligned} \tag{3.21}$$

Table 3.2: $\mathfrak{sl}(2)$-triples in $(\mathfrak{g}_2)_{\mathbf{C}}$

$\alpha = \frac{1}{2}(a\lambda_1 + b\lambda_2)$ $\updownarrow$ $i(\alpha(\Lambda_1), \alpha(\Lambda_2))$	$H_\alpha = \frac{i}{2}(a\Lambda_1 + b\Lambda_2)$ $\updownarrow$ $(\lambda_1(H_\alpha), \lambda_2(H_\alpha))$	$X_\alpha = V(v^1, v^2, v^3, \rho)$	$X_{-\alpha}$
$\lambda_1 = (2,0)$	$(2,0)$	$v^1 = \frac{1}{2}(0,1,i)$	$-\overline{X_\alpha}$
$\lambda_2 = (0,2)$	$(0,2)$	$v^2 = \frac{1}{2}(0,1,i)$	$-\overline{X_\alpha}$
$\alpha_1 = (1,1)$	$(1,3)$	$v^3 = \frac{1}{4}(0,i,1), \rho = \frac{3}{4}\begin{pmatrix} 0 & 1 & -i \\ 1 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}$	$\overline{X_\alpha}$
$\alpha_2 = (-1,1)$	$(-1,3)$	$v^3 = \frac{1}{2}(1,0,0), \rho = \frac{1}{2}\mathrm{diag}(2i, -i, -i)$	$\overline{X_\alpha}$
$\alpha_3 = (-3,1)$	$(-1,1)$	$v^3 = \frac{1}{4}(0,1,-i), \rho = \frac{1}{4}\begin{pmatrix} 0 & i & 1 \\ i & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\overline{X_\alpha}$
$\alpha_4 = (3,1)$	$(1,1)$	$\rho = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & i \\ 0 & i & -1 \end{pmatrix}$	$\overline{X_\alpha}$

Table 3.3: $\alpha(H_\beta)$

	H_{λ_1}	H_{λ_2}	H_{α_1}	H_{α_2}	H_{α_3}	H_{α_4}
λ_1	2	0	1	-1	-1	1
λ_2	0	2	3	3	1	1
α_1	1	1	2	1	0	1
α_2	-1	1	1	2	1	0
α_3	-3	1	0	3	2	-1
α_4	3	1	3	0	-1	2

Table 3.4: $\langle\ H_\alpha, H_{\alpha'}\ \rangle/8$

$\langle\ ,\ \rangle/8$	H_{λ_1}	H_{λ_2}	H_{α_1}	H_{α_2}	H_{α_3}	H_{α_4}
H_{λ_1}	6	0	3	-3	-3	3
H_{λ_2}	$\star$	2	3	3	1	1
H_{α_1}	$\star$	$\star$	6	3	0	3
H_{α_2}	$\star$	$\star$	$\star$	6	3	0
H_{α_3}	$\star$	$\star$	$\star$	$\star$	2	-1
H_{α_4}	$\star$	$\star$	$\star$	$\star$	$\star$	2

Table 3.5: (α, β)

$4(\ ,\ )$	λ_1	λ_2	α_1	α_2	α_3	α_4
λ_1	$\frac{1}{3}$	0	$\frac{1}{6}$	-$\frac{1}{6}$	-$\frac{1}{2}$	$\frac{1}{2}$
λ_2	$\star$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
α_1	$\star$	$\star$	$\frac{1}{3}$	$\frac{1}{6}$	0	$\frac{1}{2}$
α_2	$\star$	$\star$	$\star$	$\frac{1}{3}$	$\frac{1}{2}$	0
α_3	$\star$	$\star$	$\star$	$\star$	1	$-\frac{1}{2}$
α_4	$\star$	$\star$	$\star$	$\star$	$\star$	1

Table 3.6: Objects Attached to Strongly Orthogonal Sets 1

$\mathcal{S}$	$\mathfrak{h}_{\mathcal{S}}$	$W(H_{\mathcal{S}})$	$\Psi_{\mathcal{S},\mathbf{R}}$
$\varnothing$	$\mathfrak{h}$	$\langle\{s_{\lambda_1}, s_{\lambda_2}\}\rangle = \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$	$\varnothing$
$\{\alpha_1\}$	$\mathbf{R}H_{\alpha_1} + \mathbf{R}iH_{\alpha_3}$	$\langle\{s_{\alpha_1}, s_{\alpha_3}\}\rangle = \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$	$\{\alpha_3\}$
$\{\alpha_3\}$	$\mathbf{R}H_{\alpha_3} + \mathbf{R}iH_{\alpha_1}$	$\langle\{s_{\alpha_3}, s_{\alpha_1}\}\rangle = \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$	$\{\alpha_1\}$
$\{\alpha_1, \alpha_3\}$	$\mathbf{R}H_{\alpha_1} + \mathbf{R}H_{\alpha_3}$	$\langle s_\Psi \rangle = \mathrm{Dih}_6$	Ψ

Table 3.7: Objects Attached to Strongly Orthogonal Sets 2

$\mathcal{S}$	$\mathcal{V}_{\mathcal{S}}$	$\mathfrak{h}_{\mathcal{S}}$	$\Psi_{\mathcal{S},\mathbf{R}}$	$\Psi_{\mathcal{S},i\mathbf{R}}$	$\Psi_{\mathcal{S},\mathbf{C}}$
$\varnothing$	$\mathfrak{h}$	$\mathfrak{h}$	$\varnothing$	Ψ	$\varnothing$
$\{\alpha_1\}$	$\mathbf{C}H_{\alpha_3}$	$\mathbf{R}(-\Lambda_1 + \Lambda_2) + \mathbf{R}i(\Lambda_1 + 3\Lambda_2)$	$\{\alpha_1\}$	$\{\alpha_3\}$	$\{\lambda_1, \lambda_2, \alpha_2, \alpha_4\}$
$\{\alpha_3\}$	$\mathbf{C}H_{\alpha_1}$	$\mathbf{R}(\Lambda_1 + 3\Lambda_2) + \mathbf{R}i(-\Lambda_1 + \Lambda_2)$	$\{\alpha_3\}$	$\{\alpha_1\}$	$\{\alpha_2, \alpha_4, \lambda_1, \lambda_2\}$
$\{\alpha_1, \alpha_3\}$	0	$(\mathbf{R}i\Lambda_1 + \mathbf{R}i\Lambda_2)$	Ψ	$\varnothing$	$\varnothing$

Chapter 4

The Orbits

4.1 Tempered Eigen-distributions of $\Delta_{2,1}$

The group $SO(2,1)_1$ is the identity component of $\mathrm{isom}(\mathbf{R}^3, \tau_{2,1})$. We identify $[0, 2\pi]$ with S^1 by $\theta \mapsto \exp(i\theta)$. For $\alpha \in \mathbf{R}$, put

$$\mathcal{B}(\alpha) = \{x \in \mathbf{R}^3 : \tau_{2,1}(x,x) = \alpha\}.$$

Let $\epsilon = \pm 1$ (or sometimes just $\pm$, according to the context). For $S \subset \mathbf{R}$, put

$$\begin{aligned} \mathcal{B}(S) &= \bigcup_S \mathcal{B}(\alpha), \\ \mathcal{B}(S)^\epsilon &= \mathcal{B}(S) \cap \{x \in \mathbf{R}^3 : \epsilon x_3 > 0\}. \end{aligned}$$

Then $\mathcal{B}(S) = \mathcal{B}(S)^+ \cup \mathcal{B}(S)^-$. Also $\mathbf{R}^3 = \mathcal{B}(\mathbf{R})$ and, by restriction, we obtain a foliation of $\mathbf{R}^3 \setminus \{0\}$ into $SO(2,1)_1$-orbits: $\mathcal{B}(\alpha)$, $\mathcal{B}(-\alpha)^+, \mathcal{B}(-\alpha)^-$ $\alpha > 0$.

The following maps:

$$\begin{aligned}\Upsilon_0 &: \mathbf{R}^+ \times \mathbf{R} \times S^1 \to \mathcal{B}(\mathbf{R}^+)\\ (\rho, a, \theta) &\mapsto \rho(\cosh a \cos\theta, \cosh a \sin\theta, \sinh a),\end{aligned}$$

$$\begin{aligned}\Upsilon_\epsilon &: \mathbf{R}^+ \times \mathbf{R}^+ \times S^1 \to \mathcal{B}(\mathbf{R}^-)^\epsilon\\ (\rho, a, \theta) &\mapsto \rho(\sinh a \cos\theta, \sinh a \sin\theta, \epsilon\cosh a),\end{aligned}$$

define diffeomorphisms onto open dense subsets with Jacobian equal to $\rho^2 \cosh a$ ($\rho^2 \sinh a$ resp.). Moreover $\Upsilon_0(\{\rho\} \times \mathbf{R} \times S^1) = \mathcal{B}(\rho^2)$ and $\Upsilon_\epsilon(\{\rho\} \times \mathbf{R}^+ \times S^1) = \mathcal{B}(-\rho^2)^\epsilon$. Hence, for a test function $\varphi \in \mathcal{S}(\mathbf{R}^3)$,

$$\begin{aligned}\int_{\mathcal{B}(\mathbf{R}^+)} \varphi\, dx &= \int_{\mathbf{R}^+} \nu^1_\rho(\varphi)\rho^2\, d\rho,\\ \int_{\mathcal{B}(\mathbf{R}^-)^\epsilon} \varphi\, dx &= \int_{\mathbf{R}^+} \nu^2_{\epsilon\rho}(\varphi)\rho^2\, d\rho,\end{aligned} \tag{4.1}$$

where,

$$\begin{aligned}\nu^1_\rho(\varphi) &= \int_{S^1\times\mathbf{R}} \varphi(\Upsilon_0(\rho, a, \theta)) \cosh a\, d\theta\, da,\\ \nu^2_{\epsilon\rho}(\varphi) &= \int_{S^1\times\mathbf{R}^+} \varphi(\Upsilon_\epsilon(\rho, a, \theta)) \sinh a\, d\theta\, da.\end{aligned} \tag{4.2}$$

Since,

$$\int_{S^1\times\mathbf{R}} (1 + (\rho \cosh a \cos\theta)^2 + (\rho \cosh a \sin\theta)^2 + (\rho \sinh a)^2)^{-p}$$
$$\cosh a\, d\theta\, da = 2\pi \int_{\mathbf{R}} (1 + \rho^2(1 + 2\sinh^2 a))^{-p}(\cosh a)\, da < \infty,$$

for $2p > 1$, ν^1_ρ is a tempered distribution and analogously for $\nu^-_{\epsilon\rho}(\varphi)$. Furthermore, given that

$$\begin{aligned}
\nu^1_\rho(\varphi) &= \frac{1}{\rho^2}\frac{d}{d\rho}\int_{(0,\rho)} r^2\,\nu^1_r(\varphi)\,dr = \frac{1}{\rho^2}\frac{d}{d\rho}\int_{\mathcal{B}(0,\rho^2)}\varphi(x)\,dx, \quad \text{and} \\
\nu^2_{\epsilon\rho}(\varphi) &= \frac{1}{\rho^2}\frac{d}{d\rho}\int_{(0,\rho)} r^2\,\nu^2_{\epsilon r}(\varphi)\,dr = \frac{1}{\rho^2}\frac{d}{d\rho}\int_{\mathcal{B}(-\rho^2,0)^\epsilon}\varphi(x)\,dx,
\end{aligned} \tag{4.3}$$

and since the Lebesgue measure dx is $SL(3,\mathbf{R})$-invariant, we conclude that

$$\nu^1_\rho,\ \nu^2_{\epsilon\rho} \in \mathcal{S}^*(\mathbf{R}^3)^{SO(2,1)_1}.$$

Changing variables we get,

$$\rho\nu^2_{\epsilon\rho}(\varphi) = \int_{S^1\times(\rho,\infty)} \varphi(\sqrt{r^2-\rho^2}\cos\theta, \sqrt{r^2-\rho^2}\sin\theta, \epsilon r)\,d\theta\,dr,$$

and hence 4.3 implies that the following limit exists in $\mathcal{S}^*(\mathbf{R}^3)$

$$\nu^\epsilon_0(\varphi) = \lim_{\rho\to 0}\rho\nu^2_{\epsilon\rho}(\varphi) = \int_{S^1\times\mathbf{R}^+}\varphi(r(\cos\theta,\sin\theta,\epsilon))\,d\theta\,dr, \tag{4.4}$$

so it defines an element in $\mathcal{S}^*(\mathbf{R}^3)^{SO(2,1)_1}$ supported on the cone $\mathcal{B}(0)^\epsilon$.

If $\psi \in L^1_{loc}(\mathbf{R}^3)^{SO(2,1)_1}$, then

$$\begin{aligned}
\int_{\mathcal{B}(\mathbf{R}^+)}\psi(x)\,\varphi(x)\,dx &= \int_{\mathbf{R}^+}(\psi(x_1e_1)\,x_1)\nu^1_{x_1}(\varphi)\,dx_1, \\
\int_{\mathcal{B}(\mathbf{R}^-)}\psi(x)\,\varphi(x)\,dx &= \int_{\mathbf{R}}(\psi(x_3e_3)x_3)\,x_3\nu^2_{x_3}(\varphi)\,dx_3.
\end{aligned} \tag{4.5}$$

We shall use the formulas (4.5) in order to compute Fourier transforms of the measures (4.2).

Let us define the following Fourier transforms, $p, q \in \mathbf{N}$:

$$\begin{aligned}\mathcal{F}_{p,q}\varphi(y) &= \left(\frac{1}{\sqrt{2\pi}}\right)^{p+q} \int_{\mathbf{R}^{p+q}} \varphi(x) e^{-i\tau_{p,q}(x,y)}\, dx, \\ \mathcal{F}_1 &= \mathcal{F}_{1,0}, \qquad \mathcal{F}_{-1} = \mathcal{F}_{0,1},\end{aligned}$$

for $\varphi \in \mathcal{S}(\mathbf{R}^{p+q})$, while for $u \in \mathcal{S}^*(X)$ we define by (2.10).

It is elementary that, for $u \in \mathcal{S}^*(\mathbf{R}^3)$,

$$\Delta_{2,1}\mathcal{F}_{2,1}u = \mathcal{F}_{2,1}(-r_{2,1}^2 u).$$

Hence:

$$\begin{aligned}\Delta_{2,1}\mathcal{F}_{2,1}\nu_\rho^1 &= -\rho^2\mathcal{F}_{2,1}(\nu_\rho^1), \\ \Delta_{2,1}\mathcal{F}_{2,1}\nu_{\epsilon\rho}^2 &= \rho^2\mathcal{F}_{2,1}(\nu_{\epsilon\rho}^2).\end{aligned}$$

Thus the Fourier transforms $\mathcal{F}_{2,1}\nu_\rho^1$ and $\mathcal{F}_{2,1}\nu_{\epsilon\rho}^-$ are the $SO(2,1)_1$ - invariant eigen-distributions for the indefinite Laplacian $\Delta_{2,1}$ with eigenvalue $\alpha = -\rho^2$ and $\alpha = \rho^2$ respectively.

As explained in [12, (VI 5.1.22)] any such eigen-distribution, when restricted to the open sets $\mathcal{B}(\mathbf{R}^+)$ and $\mathcal{B}(\mathbf{R}^-)^\epsilon$ is of a very concrete type of smooth function. however, in order to extend them to the whole of $\mathbf{R}^3$ one has to impose a few more conditions, see [12, Theorem VI 5.1.23].

Theorem 4.1 *[12, Corollary VI 5.2.1] The space of tempered $SO(2,1)_1$ - invariant eigen-distributions for the indefinite Laplacian $\Delta_{2,1}$ is a finite dimensional subspace of $L^1_{loc}(\mathbf{R}^3)^{SO(2,1)_1}$. More precisely, the subspace corresponding to the eigenvalue $\alpha > 0$ (resp. $\alpha < 0$) is the 2 (resp 1) dimensional subspace with* *basis:*

$\alpha > 0$

$$u_1(x_1e_1) = 0, \quad u_1(x_3e_3) = \frac{\cos(\rho x_3)}{x_3}, \tag{4.6}$$

$$u_2(x_1e_1) = -\frac{e^{-\rho|x_1|}}{|x_1|}, \quad u_2(x_3e_3) = \frac{\sin(\rho x_3)}{x_3}, \tag{4.7}$$

$\alpha < 0$

$$u_3(x_1e_1) = \frac{\cos(\rho x_1)}{|x_1|}, \quad u_3(x_3e_3) = 0. \tag{4.8}$$

(The minus sign in (4.7) is missing in the original statement [12, Corollary VI 5.2.1], but it is not difficult to check that [12, Theorem VI 5.1.23] implies that the minus is necessary.)

Hence, up to scalars, we can identify $\mathcal{F}_{2,1}\nu^1_\rho$ and $\mathcal{F}_{2,1}\nu^-_{\epsilon\rho}$ with the distributions $u_1,\ u_2,\ u_3$ above.

Fourier Transform of ν^1_ρ

Given the commutativity of the $O(2,1)$ and $\widetilde{Sp(1,\mathbf{R})}$ in the metaplectic group, the space $\mathcal{S}^*(\mathbf{R}^3)^{O(2,1)}$ carries also a metaplectic representation.

Let $\Phi_1 \in \hom(\mathcal{S}(\mathbf{R})_{even}, \mathcal{S}^*(\mathbf{R}^3)^{O(2,1)})$ be given by:

$$\varphi \mapsto \int_{\mathbf{R}^+} \varphi(r) r \nu^1_r \, dr.$$

As explained in [12, Theorem VI 5.2.5 (a)],

$$\Phi_1 \in \hom_{\widetilde{Sp(1,\mathbf{R})}}(\mathcal{S}(\mathbf{R})_{even}, \mathcal{S}^*(\mathbf{R}^3)^{O(2,1)}).$$

Since the different Fourier transforms live in the same metaplectic group (see [12, (VI 1.1.5)]), we have $\mathcal{F}_{2,1}(\Phi_1(\varphi)) = \Phi_1(\mathcal{F}_1(\varphi))$, i.e.,

$$\mathcal{F}_{2,1}\left(\int_{\mathbf{R}^+} \varphi(r) r \nu_r^1 \, dr\right) = \int_{\mathbf{R}^+} (\mathcal{F}_1\varphi)(r) r \nu_r^1 \, dr. \tag{4.9}$$

By a standard approximation argument we may use the formula (4.9) with $\varphi = \delta_\rho + \delta_{-\rho}$, where δ_a is the Dirac delta at a. This leads to

$$\mathcal{F}_{2,1}(\rho\nu_\rho^1) = \int_{\mathbf{R}^+} \frac{1}{\sqrt{2\pi}} \left(e^{i\rho r} + e^{-i\rho r}\right) r \nu_r^1 \, dr. \tag{4.10}$$

By combining (4.5) with (4.10) we see that

$$\begin{aligned} \mathcal{F}_{2,1}\nu_\rho^1(x_1e_1) &= \frac{1}{\sqrt{2\pi}} \frac{e^{i\rho x_1} + e^{-i\rho x_1}}{\rho x_1}, \\ \mathcal{F}_{2,1}\nu_\rho^1(x_3e_3) &= 0. \end{aligned}$$

Fourier Transform of $\nu_{\epsilon\rho}^2$

Let $\mathcal{S}^*(\mathbf{R}^3)_+^{SO(2,1)_1} \subset \mathcal{S}^*(\mathbf{R}^3)^{SO(2,1)_1}$ be the subspace of distributions supported in $\mathcal{B}(\{0\} \cup \mathbf{R}^+)$. This space is invariant under the $\mathfrak{sp}(1,\mathbf{R})$ action. Hence $\mathcal{S}^*(\mathbf{R}^3)^\diamondsuit = \mathcal{S}^*(\mathbf{R}^3)^{SO(2,1)_1}/\mathcal{S}^*(\mathbf{R}^3)_+^{SO(2,1)_1}$ is a $\mathfrak{sp}(1,\mathbf{R})$ module. Furthermore the map

$$\Phi_2 \in \hom(\mathcal{S}(\mathbf{R}), \mathcal{S}^*(\mathbf{R}^3)^\diamondsuit),$$

defined by

$$\varphi \mapsto \int_{\mathbf{R}} \varphi(r) r \nu_r^2,$$

is intertwining

$$\Phi_2 \in \hom_{\widetilde{Sp(1,\mathbf{R})}}(\mathcal{S}(\mathbf{R}), \mathcal{S}^*(\mathbf{R}^3)^\diamondsuit),$$

as explained in [12, Theorem VI 5.2.5 (b)]. As before, one gets

$$i^{\frac{-2+1}{2}}\mathcal{F}_{2,1}\left(\int_{\mathbf{R}}\varphi(r)r\nu_r^1\,dr\right)\Big]_{\mathcal{B}(\mathbf{R}^-)}=\int_{\mathbf{R}^+}i^{\frac{1}{2}}(\mathcal{F}_{-1}\varphi)(r)r\nu_r^1\,dr\Big]_{\mathcal{B}(\mathbf{R}^-)}.$$

where $]_{\mathcal{B}(\mathbf{R}^-)}$ denotes the restriction of the distribution to the open set $\mathcal{B}(\mathbf{R}^-)$.

Now, put $f=\delta_{\epsilon\rho}$, to get

$$\mathcal{F}_{2,1}(\epsilon\rho\nu_{\epsilon\rho}^2)\Big]_{\mathcal{B}(\mathbf{R}^-)}=\int_{\mathbf{R}}\frac{i}{\sqrt{2\pi}}e^{i\epsilon\rho r}r\nu_r^2\,dr. \tag{4.11}$$

By combining (4.5) with (4.11) we see that

$$\mathcal{F}_{2,1}\nu_{\epsilon\rho}^2(x_3e_3)=\frac{i}{\sqrt{2\pi}}\frac{e^{i\epsilon\rho x_3}}{\epsilon\rho x_3}. \tag{4.12}$$

Theorem 4.1 and (4.12) imply the value of $\mathcal{F}_{2,1}\nu_{\epsilon\rho}^2$ on the other axis:

$$\mathcal{F}_{2,1}\nu_{\epsilon\rho}^2(x_1e_1)=\frac{1}{\sqrt{2\pi}}\frac{e^{-\rho|x_1|}}{\rho|x_1|}. \tag{4.13}$$

Also, by taking limits with $\rho\to 0$ (see 4.4) we are able to compute $\mathcal{F}_{2,1}\nu_0^\epsilon$. We summarize the part of discussion that we shall use later in the following theorem.

Theorem 4.2 *The Fourier transforms $\mathcal{F}_{2,1}\nu_\rho^1$, $\mathcal{F}_{2,1}\nu_{\epsilon\rho}^2$, $\mathcal{F}_{2,1}\nu_0^\epsilon$ belong to $L^1_{loc}(\mathbf{R}^3)^{SO(2,1)_1}$. Explicitly*

$$\mathcal{F}_{2,1}\nu_\rho^1(x_1e_1) = \frac{1}{\sqrt{2\pi}}\frac{e^{i\rho x_1}+e^{-i\rho x_1}}{\rho|x_1|}, \tag{4.14}$$

$$\mathcal{F}_{2,1}\nu_\rho^1(x_3e_3) = 0; \tag{4.15}$$

$$\mathcal{F}_{2,1}\nu_{\epsilon\rho}^2(x_1e_1) = \frac{1}{\sqrt{2\pi}}\frac{e^{-\rho|x_1|}}{\rho|x_1|}, \tag{4.16}$$

$$\tag{4.17}$$

$$\mathcal{F}_{2,1}\nu_{\epsilon\rho}^2(x_3e_3) = \frac{i}{\sqrt{2\pi}}\frac{e^{i\epsilon\rho x_3}}{\epsilon\rho x_3}; \tag{4.18}$$

$$\mathcal{F}_{2,1}\nu_0^\epsilon(x_1e_1) = \frac{1}{\sqrt{2\pi}}\frac{1}{|x_1|}, \tag{4.19}$$

$$\mathcal{F}_{2,1}\nu_0^\epsilon(x_3e_3) = \frac{i}{\sqrt{2\pi}}\frac{1}{\epsilon x_3}. \tag{4.20}$$

4.2 Co-Adjoint Orbits for $\mathfrak{sp}(1,\mathbf{R})$

There is an intertwining isomorphism ψ

$$(SO(2,1)_1, \mathbf{R}^3) \xrightarrow{\psi} (Sp(1,\mathbf{R})/\pm I, \mathbf{R}\mathcal{H}\oplus\mathbf{R}\mathcal{I}\oplus\mathbf{R}\mathcal{J})$$

between the standard and the adjoint representations. Explicitly

$$\psi(x_1,x_2,x_3) = x_1\mathcal{H} + x_2\mathcal{I} - x_3\mathcal{J}. \tag{4.21}$$

Using this identification, the co-adjoint orbits in $\mathfrak{sp}(1,\mathbf{R})$ are listed in table 4.1.

Table 4.1: Co-adjoint Orbits in $\mathfrak{sp}(1, \mathbf{R})$ $(p > 0)$

$\mathcal{O}$	Type	Name	Description
$\{0\}$	nilp	TRIVIAL	
$\mathcal{O}_0^+$	nilp	UPPER LIGHT-CONE	$SO(2,1)_1(1,0,1) = \mathcal{B}(0)^+$
$\mathcal{O}_0^-$	nilp	LOWER LIGHT-CONE	$SO(2,1)_1(1,0,-1) = \mathcal{B}(0)^-$
$\mathcal{O}_q^1$	ss	1-SHEETED HYPERBOLOID	$SO(2,1)_1(q,0,0) = \mathcal{B}(q^2)$
$\mathcal{O}_{+p}^2$	ss	2-SHEETED HYPERBOLOID (UPPER S)	$SO(2,1)_1(0,0,p) = \mathcal{B}(-p^2)^+$
$\mathcal{O}_{-p}^2$	ss	2-SHEETED HYPERBOLOID (LOWER S)	$SO(2,1)_1(0,0,-p) = \mathcal{B}(-p^2)^-$

4.3 Nilpotent Orbits in $\mathfrak{g}_2$

Let $\mathcal{O} \subseteq \mathfrak{g}_2$ be a non-zero nilpotent orbit. Then, as explained in [7, 9.4], there are elements,

$$e \in \mathcal{O},\ h \in \mathfrak{p}_2,$$

such that, if we denote by e' the element $-\Theta(e)$, then

$$[e, e'] = h,\ [h, e] = 2e,\ [h, e'] = -2e'.$$

Then (e, e', h) is a **Cayley triple** corresponding to the orbit $\mathcal{O}$. Let

$$c_e = \exp(-i\frac{\pi}{4}\mathrm{ad}(e + e'))$$

Table 4.2: Nilpotent Orbits in $\mathfrak{g}_2$ [7, 9.6]

$\mathcal{O}$	$\dim \mathcal{O}$	Extended Weighted Dynkin diagram for $\mathcal{O}_{\mathbf{R}}$ $\overset{\lambda_1(H_{\mathcal{O}})}{\lambda_1} \Lleftarrow \circ \longrightarrow \overset{\lambda_2(H_{\mathcal{O}})}{\lambda_2}$
$\mathcal{O}_0$	0 trivial	$\overset{0}{\circ} \Lleftarrow \circ \longrightarrow \overset{0}{\circ}$
$\mathcal{O}_6$	6 minimal	$\overset{1}{\circ} \Lleftarrow \circ \longrightarrow \overset{1}{\circ}$
$\mathcal{O}_8$	8	$\overset{1}{\circ} \Lleftarrow \circ \longrightarrow \overset{3}{\circ}$
$\mathcal{O}_{10}^1$	10 sub-regular	$\overset{2}{\circ} \Lleftarrow \circ \longrightarrow \overset{2}{\circ}$
$\mathcal{O}_{10}^2$	10 sub-regular	$\overset{0}{\circ} \Lleftarrow \circ \longrightarrow \overset{4}{\circ}$
$\mathcal{O}_{12}$	12 principal	$\overset{4}{\circ} \Lleftarrow \circ \longrightarrow \overset{8}{\circ}$

be the corresponding Cayley transform, and define the following elements in $(\mathfrak{g}_2)_{\mathbf{C}}$:

$$X = c_e(e),\ X' = c_e(e'),\ H = c_e(h).$$

Then, as it is well known, [8, Prop. 6.1],

$$X = \frac{1}{2}(e + e' - ih),\ X' = \overline{X},\ H = -i(e - e'). \tag{4.22}$$

In particular,

$$[X, X'] = H,\ [H, X] = 2X,\ [H, X'] = -2X',\ H = -\overline{H}.$$

The equation

$$e = Re(X) + \frac{i}{2}H,$$

which follows from (4.22) is used to calculate the last column of table 4.3.

Since $H \in (\mathfrak{k}_2)_{\mathbf{C}}$, $X,\ X' \in (\mathfrak{p}_2)_{\mathbf{C}}$, the elements $(X,\ X',\ H)$ form what is called a **normal triple** associated to the orbit $\mathcal{O}$. Conjugating the above elements by the same element of the maximal compact subgroup $K_2 \subseteq G_2$ if necessary, we may assume that $H \in \mathfrak{h}_{\mathbf{C}}$ and that $\lambda_1(H) > 0$ and $\lambda_2(H) > 0$.

Then the pair of numbers $(\lambda_1(H), \lambda_2(H))$ determines the orbit $\mathcal{O}$ completely. In these terms, the **Kostant-Sekiguchi correspondence**

$$\text{nilpotent } G_2 \text{ orbits in } \mathfrak{g}_2 \leftrightarrow \text{ nilpotent } (K_2)_{\mathbf{C}} \text{ orbits in } (\mathfrak{p}_2)_{\mathbf{C}},$$

is given by

$$\mathcal{O}_e = G_2 \cdot e \leftrightarrow K_{\mathbf{C}} \cdot X. \tag{4.23}$$

Also

$$\dim_{\mathbf{R}}(\mathcal{O}_e) = \dim_{\mathbf{C}}(G_{\mathbf{C}}X).$$

Dimension of Orbits

The dimension of the $G_{\mathbf{C}}$ adjoint orbit through X is

$$\dim_{\mathbf{C}}(\mathfrak{g}^X)_{\mathbf{C}} = \dim_{\mathbf{C}} \mathfrak{g}_{\mathbf{C}} - \dim_{\mathbf{C}}(\mathfrak{g}^X)_{\mathbf{C}}$$

$$\dim(\mathcal{O}_X) = \dim_{\mathbf{C}} \mathfrak{g}_{\mathbf{C}} - \#\{\alpha \in \Delta : [X, X_\alpha] = 0\} - \dim(\mathfrak{h}^X).$$

Hence the dimensions appearing in table 4.3 can computed as follows:

Notice that $\alpha_4+\gamma$ is not a root for $\gamma = \lambda_1, \lambda_2, \alpha_1, \alpha_2, -\alpha_3, -\alpha_2$.

$$\dim(\mathcal{O}_{X_{\alpha_4}}) = 14 - 7 - 1 = 6.$$

Since $\alpha_1+\gamma$ is not a root for $\gamma = \lambda_2, \alpha_4, \alpha_3, -\alpha_3$, then

$$\dim(\mathcal{O}_{X_{\alpha_1}}) = 14 - 5 - 1 = 8.$$

4.4 An Explicit Correspondence

EXPLICIT FORMULAS FOR THE MOMENT MAPS We make the identification $W \simeq M_{2\times 7}(\mathbf{R}) = \{\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\}$, $w_j \in \mathbf{R}^7$. Define adjunction in W (see (2.1)) by:

$$w^* = \mathcal{T} w^t \mathcal{J}, \tag{4.24}$$

where $\mathcal{J}$ is as before, while $\mathcal{T} = \mathrm{diag}(I_3, -I_4)$.

Hence, if we identify $\mathfrak{sp}(1,\mathbf{R})$ with its dual $\mathfrak{sp}(1,\mathbf{R})^*$ using the trace form, then, for $w \in W$, we have

$$\tau_{\mathfrak{sp}(1,\mathbf{R})}(w) = ww^* = \begin{pmatrix} -\tau_{3,4}(w_1,w_2) & \tau_{3,4}(w_1,w_1) \\ -\tau_{3,4}(w_2,w_2) & \tau_{3,4}(w_2,w_1) \end{pmatrix}. \tag{4.25}$$

If we identify $\mathfrak{so}(3,4) = \mathfrak{so}(3,4)^*$ in the same way, then

$$\tau_{\mathfrak{g}_2} = \pi \circ \tau_{\mathfrak{so}(3,4)},$$

where $\pi : \mathfrak{so}(3,4) \to \mathfrak{g}_2$ is as in an in Lemma 3.1.

Explicitly, if we split $w \in W$ into 3 blocks of sizes 2×3, 2×1, 2×3, namely $w = (\omega_1, \omega_2, \omega_3)$, then

$$\tau_{\mathfrak{g}_2}(w) = V(\rho, v^1, v^2, v^3), \tag{4.26}$$

where

$$\begin{aligned}
\underline{v^1} &= -\frac{1}{3}\omega_1^t \mathcal{J}\omega_1 + \frac{1}{6}\omega_3^t \mathcal{J}\omega_3 + \underline{\frac{1}{6}\omega_2^t \mathcal{J}\omega_3}, \quad \underline{v^2} = \frac{1}{2}(\omega_3^t \mathcal{J}\omega_3 - \underline{\omega_2^t \mathcal{J}\omega_3}), \\
\underline{v^3} &= \frac{1}{6}(\omega_3^t \mathcal{J}\omega_1 - (\omega_3^t \mathcal{J}\omega_1)^t + 2\underline{\omega_2^t \mathcal{J}\omega_1}), \\
\rho &= \frac{1}{2}(\omega_3^t \mathcal{J}\omega_1 + (\omega_3^t \mathcal{J}\omega_1)^t) + \frac{1}{3}\mathrm{tr}(\omega_1^t \mathcal{J}\omega_3) I.
\end{aligned}$$

In order to see this, note that:

$$\tau_{\mathfrak{so}(3,4)} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 & \omega_{21} & \omega_{31} & \omega_{41} & \omega_{51} & \omega_{61} & \omega_{71} \\ -\omega_{21} & 0 & \omega_{32} & \omega_{42} & \omega_{52} & \omega_{62} & \omega_{72} \\ -\omega_{31} & -\omega_{32} & 0 & \omega_{43} & \omega_{53} & \omega_{63} & \omega_{73} \\ \omega_{41} & \omega_{42} & \omega_{43} & 0 & \omega_{45} & \omega_{46} & \omega_{47} \\ \omega_{51} & \omega_{52} & \omega_{53} & -\omega_{45} & 0 & \omega_{56} & \omega_{57} \\ \omega_{61} & \omega_{62} & \omega_{63} & -\omega_{46} & -\omega_{56} & 0 & \omega_{67} \\ \omega_{71} & \omega_{72} & \omega_{73} & -\omega_{47} & -\omega_{57} & -\omega_{67} & 0 \end{pmatrix},$$

where $w_1 = (x_1, ..., x_7)$, $w_2 = (y_1, ..., y_7)$, $\omega_{ij} = x_i y_j - x_j y_i$, and apply the formula in Lemma 3.1 to this element.

Double Transitivity Let G_2 act on $\mathbf{R}^7$, by matrix multiplication.

For $\alpha \in \mathbf{R}^\times$, let

$$O(\alpha) = \{x \in \mathbf{R}^7 : \tau_{3,4}(x, x) = \alpha\}.$$

Theorem 4.3 *[18, Thm 1, pg. 807] The set $O(\alpha)$ is a G_2-orbit. Also, if $x \in O(\alpha)$, for any $\beta \in \mathbf{R}^{\times}$, $(G_2)_x$ acts transitively on the intersection of $O(\beta)$ with $x^{\perp}$, the hyperplane $\tau_{3,4}$-orthogonal to x.*

Corollary 4.1 *Let $\xi \in \mathfrak{sp}(1,\mathbf{R})^{rs}$. Then*

$$\tau^{-1}_{\mathfrak{sp}(1,\mathbf{R})}(\xi) \subseteq W,$$

is a single G_2-orbit.

Proof:

Each conjugacy class in $\mathfrak{sp}(1,\mathbf{R})^{rs}$ intersects $\mathbf{R}^{\times}\mathcal{J} \cup \mathbf{R}^{+}\mathcal{I}$ at a unique point. If $\xi \in \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix}$, then

$$\tau^{-1}_{\mathfrak{sp}(1,\mathbf{R})}(\xi) = \{w \in W;\ w_1 \in O(\alpha),\ w_2 \in O(-\alpha),\ \tau_{3,4}(w_1, w_2) = 0\ \},$$

while if $\xi \in \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}$, then

$$\tau^{-1}_{\mathfrak{sp}(1,\mathbf{R})}(\xi) = \{w \in W;\ w_1 \in O(\alpha),\ w_2 \in O(\alpha),\ \tau_{3,4}(w_1, w_2) = 0\ \}.$$

Let $w, w' \in \tau^{-1}_{\mathfrak{sp}(1,\mathbf{R})}(\xi)$. Theorem 4.3 implies the existence of $g \in G_2$ such that $g \cdot w_1 = w_1'$.

Thus

$$\tau_{3,4}(g \cdot w_2, w_1') = \tau_{3,4}(g \cdot w_2, g \cdot w_1) = \tau_{3,4}(w_2, w_1) = 0.$$

Hence, again by Theorem 4.3, there is $h \in (G_2)_{w_1'}$ such that $h \cdot (g \cdot w_2) = w_2'$.

Clearly hg maps w to w'. **QED**.

Corollary 4.2 *Let $\mathcal{O}' \subseteq \mathfrak{sp}(1,\mathbf{R})$ be a regular semisimple orbit. Then*

$$\tau^{-1}_{\mathfrak{sp}(1,\mathbf{R})}(\mathcal{O}') \subset W$$

is a single $G_2 \cdot Sp(1,\mathbf{R})$-orbit. Moreover,

$$\mathcal{O} = \tau_{\mathfrak{g}_2}(\tau^{-1}_{\mathfrak{sp}(1,\mathbf{R})}(\mathcal{O}')) \subseteq \mathfrak{g}_2$$

is a single G_2 orbit.

Furthermore, under our identifications, for $p > 0$,

$$\mathcal{O}' = \mathcal{O}^2_p \text{ if and only if } \mathcal{O} = G_2 \cdot (\frac{p}{3}\Lambda_1),$$

$$\mathcal{O}' = \mathcal{O}^2_{-p} \text{ if and only if } \mathcal{O} = G_2 \cdot \frac{p}{6}(\Lambda_1 + 3\Lambda_2),$$

and

$$\mathcal{O}' = \mathcal{O}^1_p \text{ if and only if } \mathcal{O} = G_2 \cdot \frac{-p}{3}c(H_{\alpha_1}),$$

where $c = c(\alpha_1)$, as in Definition 3.3.

Proof:

The first two statements are clear from 4.1. If

$$w = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

then, by (4.27),

$$\tau_{\mathfrak{sp}(1,\mathbf{R})}(w) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } \tau_{\mathfrak{g}_2}(w) = \frac{1}{3}\Lambda_1.$$

Similarly, if

$$w = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

then, by (4.27),

$$\tau_{\mathfrak{sp}(1,\mathbf{R})}(w) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } \tau_{\mathfrak{g}_2}(w) = \frac{1}{6}\Lambda_1 + \frac{1}{2}\Lambda_2.$$

We see from Lemma 3.15 that, in terms of Proposition 3.3, $c(H_{\alpha_1}) = i(X_{\alpha_1} - X_{-\alpha_1})$ corresponds to $v^1 = 0$, $v^2 = 0$, $v^3 = (0, \frac{1}{2}, 0)$ and $\rho = \begin{pmatrix} 0 & 0 & -\frac{3}{2} \\ 0 & 0 & 0 \\ -\frac{3}{2} & 0 & 0 \end{pmatrix}$.

Therefore, if

$$w = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

then, by (4.27),

$$\tau_{\mathfrak{sp}(1,\mathbf{R})}(w) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \tau_{\mathfrak{g}_2}(w) = \frac{-1}{3}c(H_{\alpha_1}).$$

Hence, the other statements follow. **QED.**

4.5 Normalization of Measures

Define, for $\mathfrak{J} \in \mathrm{end}(W)$,

$$\mathfrak{J}(w) = -\mathcal{J} w \mathcal{T}. \tag{4.27}$$

Then

$$\begin{aligned} \langle \mathfrak{J}(w), w \rangle &= \mathrm{tr}(w^*(-\mathcal{J} w \mathcal{T})) = \mathrm{tr}(w^t w), \\ \mathfrak{J}^2 &= -id \quad \text{and} \quad \mathfrak{J} \in Sp(W). \end{aligned}$$

Thus $\mathfrak{J}$ is a compatible positive definite complex structure on W and therefore conjugation by $\mathfrak{J}$ defines a Cartan involution Θ on $\mathfrak{sp}(7, \mathbf{R})$. It is easy to check that Θ preserves the subspaces $\mathfrak{so}(3,4)$, $\mathfrak{sp}(1, \mathbf{R})$ and that it restricts to the previously defined Cartan involutions on these Lie subalgebras. The formula

$$\kappa(x, y) = -\mathrm{tr}_{\mathrm{end}(W)}(\Theta(x)y), \tag{4.28}$$

for $x, y \in \mathfrak{sp}(7, \mathbf{R})$ defines an inner product on $\mathfrak{sp}(7, \mathbf{R})$ which determines a normalization $d\mu$ of the Lebesgue measure dx on each subspace of $\mathfrak{sp}(7, \mathbf{R})$, so that the volume of the unit cube is 1, namely $d\mu = \sqrt{|\det(\kappa(e_i, e_j))|}dx$.

For any unimodular closed Lie subgroups $F \subseteq E \subseteq Sp(7, \mathbf{R})$, the measure μ induces Haar measures on E and E/F. We shall denote these induced measures also by μ. Specifically if $\varphi \in C_c(E)$ support near the identity, then

$$\int_E \varphi(g) d\mu(g) = \int_{\mathfrak{e}} \varphi(\exp(X)) |Jac(X)| d\mu(X)$$

and

$$\int_F \int_{E/F} \varphi(gh) d\mu(gF) d\mu(h) = \int_E \varphi(g) d\mu(g).$$

In particular, if we view the matrix $\mathcal{J}$ as an element of $\mathfrak{sp}(1, \mathbf{R})$, then

$$\begin{aligned} \kappa(\mathcal{J}, \mathcal{J}) &= -\mathrm{tr}_{\mathrm{end}(W)}(\Theta(\mathcal{J})\mathcal{J}) = -\mathrm{tr}_{\mathrm{end}(W)}(\mathcal{J}^2) = -\mathrm{tr}_{\mathrm{end}(W)}(-I) \quad (4.29) \\ &= \mathrm{tr}_{\mathrm{end}(W)}(I) = \dim W = 14, \quad (4.30) \end{aligned}$$

and therefore,

$$\mu(H') = \int_{S^1} d\mu(r\mathcal{J}) = \int_{S^1} \sqrt{\kappa(\mathcal{J}, \mathcal{J})}\, dr = \sqrt{14}\, 2\pi. \quad (4.31)$$

4.6 The Fourier Transform of an Adjoint Orbit in $\mathfrak{sp}(1, \mathbf{R})$

We can see from (4.22) and (3.7) that:

$$c(\lambda')H_{\lambda'} = i(X_{\lambda'} - X_{-\lambda'}) = -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$c(\lambda')X_{\lambda'} = Re(X_{\lambda'}) - \frac{i}{2}H_{\lambda'} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix},$$

$$c(\lambda')X_{-\lambda'} = Re(X_{\lambda'}) + \frac{i}{2}H_{\lambda'} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}.$$

Moreover, $(c(\lambda')X_{\lambda'},\ c(\lambda')X_{-\lambda'},\ c(\lambda')H_{\lambda'})$ form a standard Cayley triple.

In particular, the Kostant-Sekiguchi correspondence of nilpotent orbits associated to our Cayley transform looks as follows

$$\mathcal{O}_0^\epsilon = Sp(1,\mathbf{R})\cdot c(\lambda')X_{\epsilon\lambda'} \leftrightarrow K'_{\mathbf{C}}\ X_{\epsilon\lambda'}$$

where $K'_{\mathbf{C}}\cdot X_{\epsilon\lambda'} = H'_{\mathbf{C}}\cdot X_{\epsilon\lambda'} = \mathbf{C}^\times\ X_{\epsilon\lambda'}$. (Here, $g\cdot x = gxg^{-1}$.)

Lemma 4.1 *Let $\rho\in\mathbf{R}^+$, $\epsilon=\pm$ and let N'_ϵ be the centralizer of $c(\lambda')X_{\epsilon\lambda'}$ in $Sp(1,\mathbf{R})$. Then, using the normalization and identification (4.21), for any $\varphi\in\mathcal{S}(\mathfrak{sp}(1,\mathbf{R}))$,*

$$\int_{Sp(1,\mathbf{R})/H'(\lambda')}\varphi(g\cdot\rho c(\lambda')H_{\lambda'})\,d\mu(gH(\lambda')) = \delta_{\mathcal{O}^1_\rho}(\varphi) = 7\nu^1_\rho(\varphi), \tag{4.32}$$

$$\int_{Sp(1,\mathbf{R})/H'}\varphi(g\cdot(-\epsilon i\rho H_{\lambda'}))\,d\mu(gH') = \delta_{\mathcal{O}^2_{\epsilon\rho}}(\varphi) = 7\nu^2_{\epsilon\rho}(\varphi), \tag{4.33}$$

$$\int_{Sp(1,\mathbf{R})/N'_\epsilon}\varphi(g\cdot c(\lambda')X_{\epsilon\lambda'})\,d\mu(gN'_\epsilon) = \delta_{\mathcal{O}^\epsilon_0}(\varphi) = \frac{7}{\sqrt{2}}\nu^\epsilon_0(\varphi). \tag{4.34}$$

Proof:

In order to shorten the notation let

$$\widehat{\sigma}_1 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\ \widehat{\sigma}_2 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\ \widehat{\sigma}_3 = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}.$$

Then, in particular, for $1\le j,k\le 3$

$$-\tau_{2,1}(\Theta\widehat{\sigma}_j,\widehat{\sigma}_k) = \delta_{j,k}. \tag{4.35}$$

Thus, for $x, y \in \mathfrak{sp}(1, \mathbf{R})$,

$$\tilde{\kappa}(x, y) = -\mathrm{tr}(\Theta(x)y)_{\mathrm{end}(W)} = -7\mathrm{tr}(\Theta(x)y) = -14\tau_{2,1}(\Theta(x), y). \tag{4.36}$$

A straightforward computation shows that for $\theta, a \in \mathbf{R}$,

$$(\exp(\theta\frac{1}{2}\widehat{\sigma}_3)\exp(-a\frac{1}{2}\widehat{\sigma}_2)) \cdot \rho\widehat{\sigma}_1 = \Upsilon_0(\rho, a, \theta).$$

Hence, by (4.2),

$$\nu_\rho^1(\varphi) = \int_{\mathbf{R}^+ \times S^1} \varphi((\exp(\theta\frac{1}{2}\widehat{\sigma}_3)\exp(-a\frac{1}{2}\widehat{\sigma}_2)) \cdot \rho\widehat{\sigma}_1) \cosh a \, d\theta \, da.$$

We have,

$$\frac{\left|\det\begin{pmatrix} \tilde{\kappa}(\frac{1}{2}\widehat{\sigma}_3, \frac{1}{2}\widehat{\sigma}_3) & \tilde{\kappa}(\frac{1}{2}\widehat{\sigma}_3, -\frac{1}{2}\widehat{\sigma}_2) & \tilde{\kappa}(\frac{1}{2}\widehat{\sigma}_3, \widehat{\sigma}_1) \\ \tilde{\kappa}(-\frac{1}{2}\widehat{\sigma}_2, \frac{1}{2}\widehat{\sigma}_3) & \tilde{\kappa}(-\frac{1}{2}\widehat{\sigma}_2, -\frac{1}{2}\widehat{\sigma}_2) & \tilde{\kappa}(-\frac{1}{2}\widehat{\sigma}_2, \widehat{\sigma}_1) \\ \tilde{\kappa}(\widehat{\sigma}_1, \frac{1}{2}\widehat{\sigma}_3) & \tilde{\kappa}(\widehat{\sigma}_1, -\frac{1}{2}\widehat{\sigma}_2) & \tilde{\kappa}(\widehat{\sigma}_1, \widehat{\sigma}_1) \end{pmatrix}\right|}{|\tilde{\kappa}(\widehat{\sigma}_1, \widehat{\sigma}_1)|}$$

$$= \frac{\left|\det\begin{pmatrix} \frac{7}{2} & 0 & 0 \\ 0 & \frac{7}{2} & 0 \\ 0 & 0 & 14 \end{pmatrix}\right|}{14} = \left(\frac{7}{2}\right)^2.$$

Since the group $H(\lambda')$ has two connected components, equation (4.32) follows.

Similarly,

$$(\exp(\theta\frac{1}{2}\widehat{\sigma}_3)\exp(a\frac{1}{2}\widehat{\sigma}_2))\cdot(\widehat{\sigma}_1+\epsilon\widehat{\sigma}_3)=e^a(\cos\theta\,\widehat{\sigma}_1+\sin\theta\,\widehat{\sigma}_2+\epsilon\widehat{\sigma}_3).$$

Also, $c(\lambda')X_{\epsilon\lambda'}=\frac{1}{2}(\widehat{\sigma}_1+\widehat{\sigma}_3)$. Hence, by (4.2),

$$\begin{aligned}
\nu_0^\epsilon(\varphi) &= \int_{\mathbf{R}^+\times S^1}\varphi(r(\cos\theta\,\widehat{\sigma}_1+\sin\theta\,\widehat{\sigma}_2+\epsilon\,\widehat{\sigma}_3))\,d\theta\,dr \\
&= \frac{1}{2}\int_{\mathbf{R}\times S^1}\varphi\;\frac{r}{2}(\cos\theta\,\widehat{\sigma}_1+\sin\theta\,\widehat{\sigma}_2+\epsilon\,\widehat{\sigma}_3)\,d\theta\,dr \\
&= \frac{1}{2}\int_{\mathbf{R}\times S^1}\varphi\;(\frac{e^a}{2}(\cos\theta\,\widehat{\sigma}_1+\sin\theta\,\widehat{\sigma}_2+\epsilon\,\widehat{\sigma}_3))\,e^a\,d\theta\,da. \\
&= \frac{1}{2}\int_{\mathbf{R}\times S^1}\varphi((\exp(\theta\frac{1}{2}\widehat{\sigma}_3)\exp(a\frac{1}{2}\widehat{\sigma}_2))\cdot c(\lambda')X_{\epsilon\lambda'})\,e^a\,d\theta\,da.
\end{aligned}\tag{4.37}$$

Furthermore, by (4.35) and (4.36),

$$\frac{\left|\det\begin{pmatrix}\tilde{\kappa}(\frac{1}{2}\widehat{\sigma}_3,\frac{1}{2}\widehat{\sigma}_3) & \tilde{\kappa}(\frac{1}{2}\widehat{\sigma}_3,\epsilon\frac{1}{2}\widehat{\sigma}_2) & \tilde{\kappa}(\frac{1}{2}\widehat{\sigma}_3,c(\lambda')X_{\epsilon\lambda'}) \\ \tilde{\kappa}(\epsilon\frac{1}{2}\widehat{\sigma}_2,\frac{1}{2}\widehat{\sigma}_3) & \tilde{\kappa}(\epsilon\frac{1}{2}\widehat{\sigma}_2,\epsilon\frac{1}{2}\widehat{\sigma}_2) & \tilde{\kappa}(\epsilon\frac{1}{2}\widehat{\sigma}_2,c(\lambda')X_{\epsilon\lambda'}) \\ \tilde{\kappa}(c(\lambda')X_{\epsilon\lambda'},\frac{1}{2}\widehat{\sigma}_3) & \tilde{\kappa}(c(\lambda')X_{\epsilon\lambda'},\epsilon\frac{1}{2}\widehat{\sigma}_2) & \tilde{\kappa}(c(\lambda')X_{\epsilon\lambda'},c(\lambda')X_{\epsilon\lambda'})\end{pmatrix}\right|}{|\tilde{\kappa}(c(\lambda')X_{\epsilon\lambda'},c(\lambda')X_{\epsilon\lambda'})|}$$

$$=\frac{\left|\det\begin{pmatrix}\frac{7}{2} & 0 & \frac{7}{2} \\ 0 & \frac{7}{2} & 0 \\ \frac{7}{2} & 0 & 7\end{pmatrix}\right|}{7}=\frac{7^2}{2^3}.$$

Thus $\delta_{Sp(1,\mathbf{R})\cdot c(\lambda')X_{\epsilon\lambda'}}=\sqrt{\frac{7^2}{2^3}}\cdot 2\cdot\nu_0^\epsilon$, and (4.34) follows.

In order to verify (4.33) we shall rewrite the formula (4.2) defining the measure

$\nu_{\epsilon\rho}^2$ in a different way. Since $(\widehat{\sigma}_1 + \widehat{\sigma}_2)^2 = 0$,

$$\exp(t(\widehat{\sigma}_1 + \widehat{\sigma}_2)) = I + t(\widehat{\sigma}_1 + \widehat{\sigma}_2).$$

Thus, for $x \in \mathfrak{sp}(1, \mathbf{R})$,

$$\begin{aligned} & \exp(t(\widehat{\sigma}_1 + \widehat{\sigma}_2)) \cdot x \\ = \;& (I + t(\widehat{\sigma}_1 + \widehat{\sigma}_2))x(I - t(\widehat{\sigma}_1 + \widehat{\sigma}_2)) = x + t[\widehat{\sigma}_1 + \widehat{\sigma}_2, x] - t^2(\widehat{\sigma}_1 + \widehat{\sigma}_2)x(\widehat{\sigma}_1 + \widehat{\sigma}_2). \end{aligned}$$

Explicitly,

$$\begin{aligned} & \exp(t(\widehat{\sigma}_1 + \widehat{\sigma}_2)) \cdot (x_1 \widehat{\sigma}_1 + x_2 \widehat{\sigma}_2 + x_3 \widehat{\sigma}_3) \\ = \;& (x_1 - 2tx_2 - 2t^2(x_1 - x_3))\widehat{\sigma}_1 + (x_2 + 2t(x_1 - x_3))\widehat{\sigma}_2 \\ + \;& (x_3 - 2tx_2 - 2t^2(x_1 - x_3))\widehat{\sigma}_3. \end{aligned}$$

Furthermore, for $b \in \mathbf{R}$,

$$\exp(b\widehat{\sigma}_2) \cdot \epsilon\rho\, \widehat{\sigma}_3 = \epsilon\rho \sinh b\, \widehat{\sigma}_1 + \epsilon\rho \cosh b\, \widehat{\sigma}_3.$$

Therefore,

$$\begin{aligned} & (\exp(t(\widehat{\sigma}_1 + \widehat{\sigma}_2)) \exp(b\widehat{\sigma}_2)) \cdot \epsilon\rho\, \widehat{\sigma}_3 \qquad (4.38) \\ = \;& (\epsilon\rho \sinh b + \epsilon 2t^2 \rho e^{-\epsilon b})\widehat{\sigma}_1 - \epsilon 2t\rho e^{-\epsilon b}\widehat{\sigma}_2 + (\epsilon\rho \cosh b + \epsilon\rho 2t^2 e^{-\epsilon b})\widehat{\sigma}_3. \end{aligned}$$

A straightforward computation shows that the equality

$$\begin{aligned} & \sinh a\, \cos\theta\, \widehat{\sigma}_1 + \sinh a\, \sin\theta\, \widehat{\sigma}_2 + \epsilon\, \cosh a\, \widehat{\sigma}_3 \qquad (4.39) \\ = \;& (\epsilon \sinh b + \epsilon 2t^2 \rho e^{-\epsilon b})\widehat{\sigma}_1 - \epsilon 2te^{-\epsilon b}\widehat{\sigma}_2 + (\epsilon\, \cosh b + \epsilon 2t^2 e^{-\epsilon b})\widehat{\sigma}_3 \end{aligned}$$

holds if and only if

$$t = -\epsilon \frac{1}{2} \sinh a\, \sin\theta (\cosh a - \epsilon\, \sinh a\, \cos\theta)^{-1}, \qquad (4.40)$$

$$b = -\epsilon\, \ln(\cosh a - \epsilon\, \sinh a\, \cos\theta). \qquad (4.41)$$

Furthermore,

$$\begin{aligned} \frac{\partial t}{\partial a} &= -\epsilon \frac{1}{2} \sin\theta (\cosh a - \epsilon\, \sinh a\, \cos\theta)^{-2} \\ \frac{\partial t}{\partial \theta} &= -\epsilon \frac{1}{2} \sinh a (\cosh a\, \cos\theta - \epsilon\, \sinh a)(\cosh(a) - \epsilon\, \sinh a\, \cos\theta)^{-2} \\ \frac{\partial b}{\partial a} &= -\epsilon (\sinh a - \epsilon\, \cosh a\, \cos\theta)(\cosh a - \epsilon\, \sinh a\, \cos\theta)^{-1} \\ \frac{\partial b}{\partial \theta} &= -\sinh a\, \sin\theta (\cosh a - \epsilon\, \sinh a\, \cos\theta)^{-1}, \end{aligned}$$

so that

$$\det \begin{pmatrix} \frac{\partial t}{\partial a} & \frac{\partial t}{\partial \theta} \\ \frac{\partial b}{\partial a} & \frac{\partial b}{\partial \theta} \end{pmatrix} = \epsilon \frac{1}{2} \sinh a. \qquad (4.42)$$

By combining (4.38), (4.39), and (4.42) we deduce

$$\int_{\mathbf{R}\times\mathbf{R}} \varphi((\exp(t(\widehat{\sigma}_1 + \widehat{\sigma}_3) \exp(b\widehat{\sigma}_2)) \cdot \epsilon\rho\widehat{\sigma}_3)\, db\, dt = \frac{1}{2} \nu^2_{\epsilon\rho}(\varphi). \qquad (4.43)$$

Moreover,

$$\frac{\left|\det\begin{pmatrix} \tilde{\kappa}(\widehat{\sigma}_1+\widehat{\sigma}_3,\widehat{\sigma}_1+\widehat{\sigma}_3) & \tilde{\kappa}(\widehat{\sigma}_1+\widehat{\sigma}_3,\widehat{\sigma}_2) & \tilde{\kappa}(\widehat{\sigma}_1+\widehat{\sigma}_3,\widehat{\sigma}_3) \\ \tilde{\kappa}(\widehat{\sigma}_2,\widehat{\sigma}_1+\widehat{\sigma}_3) & \tilde{\kappa}(\widehat{\sigma}_2,\widehat{\sigma}_2) & \tilde{\kappa}(\widehat{\sigma}_2,\widehat{\sigma}_3) \\ \tilde{\kappa}(\widehat{\sigma}_3,\widehat{\sigma}_1+\widehat{\sigma}_3) & \tilde{\kappa}(\widehat{\sigma}_3,\widehat{\sigma}_2) & \tilde{\kappa}(\widehat{\sigma}_3,\widehat{\sigma}_3) \end{pmatrix}\right|}{|\tilde{\kappa}(\widehat{\sigma}_3,\widehat{\sigma}_3)|}$$

$$= \frac{\left|\det\begin{pmatrix} 28 & 0 & 14 \\ 0 & 14 & 0 \\ 14 & 0 & 14 \end{pmatrix}\right|}{14} = (14)^2.$$

Hence,

$$\delta_{Sp(1,\mathbf{R})\cdot(-\epsilon i\rho H_{\lambda'})} = 14\cdot\frac{1}{2}\nu_{\epsilon\rho}^2,$$

which verifies (4.33).

QED.

Define a Fourier transform on $\mathfrak{sp}(1,\mathbf{R})$ as follows,

$$\mathcal{F}\varphi(y) = \int_{\mathfrak{sp}(1,\mathbf{R})} \varphi(x)\, e^{-2\pi i\,\mathrm{tr}(xy)}\, d\mu(x). \tag{4.44}$$

We will make use of the identifications,

$$\begin{aligned} \mathcal{O}_0^{\epsilon} &= Sp(1,\mathbf{R})\cdot c(\lambda')X_{\epsilon\lambda'}, \\ \mathcal{O}_\rho^{1} &= Sp(1,\mathbf{R})\cdot \rho c(\lambda')H_{\lambda'}, \\ \mathcal{O}_{\epsilon\rho}^{2} &= Sp(1,\mathbf{R})\cdot(-\epsilon i\rho H_{\lambda'}). \end{aligned}$$

Theorem 4.4 *Let $\rho \in \mathbf{R}^+$ and let $\epsilon = \pm$. The Fourier transforms of the orbital integrals in $\mathfrak{sp}(1,\mathbf{R})$ coincide with $L^1_{loc}(\mathfrak{sp}(1,\mathbf{R}))^{Sp(1,\mathbf{R})}$ functions determined by their restrictions to $\mathfrak{h}'$ and $\mathfrak{h}'(\lambda')$. Explicitly*

$$\begin{aligned}
\mathcal{F}\delta_{\mathcal{O}^1_\rho}(lc(\lambda')H_{\lambda'}) &= 7\pi\frac{e^{i\pi\rho l}+e^{-i\pi\rho l}}{\pi\rho|l|}, && (4.45)\\
\mathcal{F}\delta_{\mathcal{O}^1_\rho}(riH_{\lambda'}) &= 0;\\
\mathcal{F}\delta_{\mathcal{O}^2_{\epsilon\rho}}(lc(\lambda')H_{\lambda'}) &= 7\pi\frac{e^{-\pi\rho|l|}}{\pi\rho|l|}, && (4.46)\\
\mathcal{F}\delta_{\mathcal{O}^2_{\epsilon\rho}}(riH_{\lambda'}) &= 7\pi i\frac{e^{i\epsilon\pi\rho r}}{\epsilon\pi\rho r};\\
\mathcal{F}\delta_{\mathcal{O}^\epsilon_0}(lc(\lambda')H_{\lambda'}) &= \frac{7\pi}{\sqrt{2}}\frac{1}{\pi|l|}, && (4.47)\\
\mathcal{F}\delta_{\mathcal{O}^\epsilon_0}(riH_{\lambda'}) &= \frac{7\pi}{\sqrt{2}}\frac{1}{\epsilon i\pi r}.
\end{aligned}$$

Proof:

We shall verify (4.46). The proof of the remaining identities is very similar.

Since

$$\det\begin{pmatrix} \tilde{\kappa}(\widehat{\sigma_1},\widehat{\sigma_1}) & \tilde{\kappa}(\widehat{\sigma_1},\widehat{\sigma_2}) & \tilde{\kappa}(\widehat{\sigma_1},\widehat{\sigma_3}) \\ \tilde{\kappa}(\widehat{\sigma_2},\widehat{\sigma_1}) & \tilde{\kappa}(\widehat{\sigma_2},\widehat{\sigma_2}) & \tilde{\kappa}(\widehat{\sigma_2},\widehat{\sigma_3}) \\ \tilde{\kappa}(\widehat{\sigma_3},\widehat{\sigma_1}) & \tilde{\kappa}(\widehat{\sigma_3},\widehat{\sigma_2}) & \tilde{\kappa}(\widehat{\sigma_3},\widehat{\sigma_3}) \end{pmatrix} = \det\begin{pmatrix} 14 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 14 \end{pmatrix} = 14^3,$$

we have

$$\mathcal{F}\psi(y) = 14^{3/2}\int_{\mathbf{R}^3}\psi(x)\,e^{-i\pi\tau_{2,1}(x,y)}\,dx = 14^{3/2}\cdot\pi^{-3}\cdot(2\pi)^{3/2}\mathcal{F}_{2,1}(\psi\circ m_{\frac{1}{\pi}})(y),$$

where$\psi\circ m_{\frac{1}{\pi}}(x) = \psi(\frac{1}{\pi}x)$. Therefore,

$$
\begin{aligned}
&\mathcal{F}\delta_{Sp(1,\mathbf{R})\cdot(-\epsilon i\rho H_{\lambda'})}(\psi) = \delta_{Sp(1,\mathbf{R})\cdot(-\epsilon i\rho H_{\lambda'})}(\mathcal{F}\psi)\\
= \;& 14^{3/2}\cdot\pi^{-3}\cdot(2\pi)^{3/2}\delta_{Sp(1,\mathbf{R})\cdot(-\epsilon i\rho H_{\lambda'})}(\mathcal{F}_{2,1}(\psi\circ m_{\frac{1}{\pi}}))\\
= \;& 14^{3/2}\cdot\pi^{-3}\cdot(2\pi)^{3/2}\cdot 7\cdot\nu^2_{\epsilon\rho}(\mathcal{F}_{2,1}(\psi\circ m_{\frac{1}{\pi}}))\\
= \;& 14^{3/2}\cdot\pi^{-3}\cdot(2\pi)^{3/2}\cdot 7\cdot(\mathcal{F}_{2,1}\nu^2_{\epsilon\rho})(\psi\circ m_{\frac{1}{\pi}})\\
= \;& 14^{3/2}\cdot\pi^{-3}\cdot(2\pi)^{3/2}\cdot 7\cdot\int_{\mathbf{R}^3}(\mathcal{F}_{2,1}\nu^2_{\epsilon\rho})(x)\psi(\frac{1}{\pi}x)\,dx\\
= \;& 14^{3/2}\cdot(2\pi)^{3/2}\cdot 7\cdot\left(\int_{\mathbf{R}^+}\nu^1_r(((\mathcal{F}_{2,1}\nu^2_{\epsilon\rho})\circ m_\pi)\psi)\,r^2\,dr\right.\\
+ \;& \left.\sum_{\epsilon'}\int_{\mathbf{R}^+}\nu^-_{\epsilon' r}(((\mathcal{F}_{2,1}\nu^2_{\epsilon\rho})\circ m_\pi)\psi)\,r^2\,dr\right)\\
= \;& 14^{3/2}\cdot(2\pi)^{3/2}\cdot 7\cdot\left(\int_{\mathbf{R}^+}(\mathcal{F}_{2,1}\nu^2_{\epsilon\rho})(\pi re_2)\nu^1_r(\psi)\,r^2\,dr\right.\\
+ \;& \left.\sum_{\epsilon'}\int_{\mathbf{R}^+}(\mathcal{F}_{2,1}\nu^2_{\epsilon\rho})(\epsilon'\pi re_3)\nu^-_{\epsilon' r}(\psi)\,r^2\,dr\right)\\
= \;& 14^{3/2}\cdot(2\pi)^{3/2}\cdot\left(\int_{\mathbf{R}^+}(\mathcal{F}_{2,1}\nu^2_{\epsilon\rho})(\pi re_2)\delta_{Sp(1,\mathbf{R})\cdot re_2}(\psi)\,r^2\,dr\right.\\
+ \;& \left.\sum_{\epsilon'}\int_{\mathbf{R}^+}(\mathcal{F}_{2,1}\nu^2_{\epsilon\rho})(\epsilon'\pi re_3)\delta_{Sp(1,\mathbf{R})\cdot(-\epsilon' ire_3)}(\psi)\,r^2\,dr\right)\\
= \;& \frac{7}{2}\cdot(2\pi)^{3/2}\cdot\left(\int_{\mathbf{R}^+}(\mathcal{F}_{2,1}\nu^2_{\epsilon\rho})(\pi re_2)\delta_{Sp(1,\mathbf{R})\cdot re_2}(\psi)\,|\lambda'(c(\lambda')^{-1}re_2)|^2\,d\mu(re_2)\right.\\
+ \;& \left.\sum_{\epsilon'}\int_{\mathbf{R}^+}(\mathcal{F}_{2,1}\nu^2_{\epsilon\rho})(\epsilon'\pi re_3)\delta_{Sp(1,\mathbf{R})\cdot(-\epsilon' ire_3)}(\psi)\,|\lambda'(\epsilon' re_3)|^2\,d\mu(\epsilon' re_3)\right).
\end{aligned}
$$

Hence,

$$\mathcal{F}\delta_{Sp(1,\mathbf{R})\cdot(-\epsilon i\rho H_{\lambda'})}(re_2) = \frac{7}{2}\cdot(2\pi)^{3/2}(\mathcal{F}_{2,1}\nu^2_{\epsilon\rho})(\pi re_2) = 7\pi\frac{e^{-\rho\pi r}}{\rho\pi r},$$

and

$$\mathcal{F}\delta_{Sp(1,\mathbf{R})\cdot(-\epsilon i\rho H_{\lambda'})}(\epsilon' re_3) = \frac{7}{2}\cdot(2\pi)^{3/2}(\mathcal{F}_{2,1}\nu^2_{\epsilon\rho})(\epsilon'\pi re_3) = 7\pi i\frac{e^{i\pi\epsilon\rho\epsilon' r}}{\pi\epsilon\rho\epsilon' r}.$$

QED.

Table 4.3: Parameters of Non-trivial Non-maximal G_2 Nilpotent Orbits

H	X	$(\lambda_1(H), \lambda_2(H))$	$\mathcal{O}_e$	$e = a\Lambda_1 + b\Lambda_2 + V_{v^3} + V_\rho$ $\updownarrow$ (a, b, v^3) ρ
H_{α_4}	X_{α_4}	$(1,1)$	$\mathcal{O}_6$	$(-\frac{1}{4},\ -\frac{1}{4},\ 0)$ $\frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$
H_{α_1}	X_{α_1}	$(1,3)$	$\mathcal{O}_8$	$(-\frac{1}{4},\ -\frac{3}{4},\ \frac{1}{4}(0,1,0))$ $-\frac{3}{4}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$
$H_{\alpha_1} + H_{-\alpha_3}$	$X_{\alpha_1} + X_{-\alpha_3}$	(2,2)	$\mathcal{O}_{10}^1$	$(-\frac{1}{2},\ -\frac{1}{2},\ \frac{1}{4}(0,1,1))$ $\frac{1}{4}\begin{pmatrix} 0 & 3 & 1 \\ 3 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$
$H_{\alpha_1} + \overline{H}_{-\alpha_3}$	$X_{\alpha_1} + \overline{X}_{-\alpha_3}$	(0,4)	$\mathcal{O}_{10}^2$	$(0,\ -1,\ (0, \frac{3}{4}, 0)),$ $\rho = \begin{pmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 \end{pmatrix}$

Chapter 5

$\mathcal{CHC}$ and Orbit Correspondence

5.1 Cauchy Harish-Chandra Integral for the Pair $(\mathfrak{g}_2, \mathfrak{sp}(1, \mathbf{R}))$

By [17, Lemma 1.10] and the identification $\mathfrak{so}(3,4) = \mathfrak{so}(3,4)^*$ via the trace form, we have

$$\begin{aligned} WF(\widetilde{\mathrm{chc}}_{x'}) \subseteq &\quad \{(x, -w^*w);\ (x'+x)(w) = 0,\ x \in \mathfrak{so}(3,4),\ w \in W\} \\ \subseteq &\quad \mathfrak{so}(3,4) \times \mathfrak{so}(3,4)_{rk\leq 2}. \end{aligned} \tag{5.1}$$

The co-normal bundle to the embedding

$$\mathfrak{g}_2 \hookrightarrow \mathfrak{so}(3,4) \tag{5.2}$$

coincides with $\mathfrak{g}_2 \times \mathfrak{g}_2^{\perp}$. Since $\mathfrak{g}_2^{\perp} \cap \mathfrak{so}(3,4)_{rk\leq 2} = 0$, by Lemma 3.2, we see that the intersection of the wave front set of $\widetilde{\mathrm{chc}}_{x'}$ with the co-normal bundle to the embedding (5.2) is empty. Therefore we may define

$$\widetilde{\mathrm{chc}}_{x'} \in \mathcal{S}^*(\mathfrak{g}_2) \tag{5.3}$$

as the pull-back of $\widetilde{\mathrm{chc}}_{x'} \in \mathcal{S}^*(\mathfrak{so}(3,4))$ to $\mathfrak{g}_2$ via (5.2), see [13, 8.2.4]. The resulting distribution $\widetilde{\mathrm{chc}}_{x'}$ is G_2-invariant. Since for any Cartan subalgebra $\mathfrak{h}_j \subseteq \mathfrak{g}_2$, the G_2-orbits through a regular element of $\mathfrak{h}_j$ are transversal to $\mathfrak{h}$, we may further restrict the distribution $\widetilde{\mathrm{chc}}_{x'}$ to $\mathfrak{h}_j^r$.

By Harish-Chandra - Weyl integration formula for $\varphi \in C_c^\infty(\mathfrak{g}_2^{rs})$,

$$\begin{aligned} \widetilde{\mathrm{chc}}_{x'}(\varphi) &= \int_{\mathfrak{g}_2} \widetilde{\mathrm{chc}}(x'+x)\varphi(x)\, d\mu(x) \\ &= \sum_{[\mathcal{S}]} \frac{1}{|W(H_{\mathcal{S}})|} \int_{\mathfrak{h}_{\mathcal{S}}} \widetilde{\mathrm{chc}}(x' + c(\mathcal{S})x)\, \overline{\mathcal{D}(\Psi)(x)}\, \mathcal{A}(\Psi_{\mathcal{S},\mathbf{R}})(x)\, \mathcal{H}_{\mathcal{S}}\varphi(x)\, d\mu(x). \end{aligned} \tag{5.4}$$

The defining module for the algebra $\mathfrak{so}(3,4)$ is the space $V = \mathbf{R}^7$. Let us decompose it into $\mathfrak{h}$-irreducible subspaces, namely,

$$V_0 \oplus V_1 \oplus V_2 \oplus V_3, \tag{5.5}$$

where

$$V_0 = \left\{ \begin{pmatrix} * \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\},\ V_1 = \left\{ \begin{pmatrix} 0 \\ * \\ * \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\},\ V_2 = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ * \\ * \\ 0 \\ 0 \end{pmatrix} \right\},\ V_3 = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ * \\ * \end{pmatrix} \right\}.$$

The defining module for $\mathfrak{sp}(1, \mathbf{R})$, $V' = \mathbf{R}^2$, is $\mathfrak{h}'$-irreducible. The corresponding space $W = M_{2\times 7}(\mathbf{R})$ has symplectic form

$$\langle w', w \rangle = -\mathrm{tr}(w' \mathcal{T} w^T \mathcal{J}) \qquad \text{where } w', w \in W. \tag{5.6}$$

Also, define a complex structure on W by $\mathcal{J}'(w) = \mathcal{J}w$. We view W as a complex vector space by $iw := \mathcal{J}'(w)$. Let

$$\det : W_{\mathbf{C}} \to \mathbf{C}$$

denote the corresponding determinant.

Note that

$$\langle \mathcal{J}' \ , \ \rangle_{\mathrm{hom}(V_0 \oplus V_1, V')} < 0 \text{ and } \langle \mathcal{J}' \ , \ \rangle_{\mathrm{hom}(V_2 \oplus V_3, V')} > 0. \tag{5.7}$$

Let $2p_+$ be the maximal dimension of a subspace of W such that the restriction of the symmetric form $\langle \mathcal{J}' \ , \ \rangle$ to that subspace is positive definite. By (5.7),

$p_+ = 4$.

Let $x' \in \mathfrak{h}'^r$, and put

$$\mathfrak{sp}(W)^+_{\mathbb{C}} = \{x + iy \in \mathfrak{sp}(W)_{\mathbb{C}} : \langle y\ ,\ \rangle|_{\ker(x)} > 0\}.$$

Then Proposition 5.1 in [4] states that, as a distribution on $\mathfrak{so}(3,4)$, the unnormalized Cauchy Harish-Chandra integral is given by

$$\widetilde{\mathrm{chc}}(x' + x) = \lim_{y \to 0} \frac{(-1)^{p_+} \sqrt{2}^{\dim_{\mathbb{R}}(W)}}{\det(x' + x + iy)}, \tag{5.8}$$

where $x, y \in \mathfrak{so}(3,4)$, $x + x' + iy \ \in \mathfrak{sp}(W)^+_{\mathbb{C}}$, and by the double restriction process explained and justified earlier, we have

$$\widetilde{\mathrm{chc}}(x' + c(\mathcal{S})x) = \lim_{y \to 0} \ -2^7 \frac{1}{\det(x' + x + iy)}, \tag{5.9}$$

where $x, \in \mathfrak{h}^{rs}_{\mathcal{S}}$, $y \in \mathfrak{h} \cap \mathfrak{h}_{\mathcal{S}}$, $x + x' + iy \in \mathfrak{sp}(W)^+_{\mathbb{C}}$.

Let $x' = l\mathcal{J}$ and let $x = t\Lambda_1 + s\Lambda_2$. Then $(x' + x)(w) = x'w - wx$. Hence $x + x'$ preserves the following induced decomposition of W:

$$W = W_1 \oplus W_2 \oplus W_3 \oplus W_4,$$

where

$$W_i = \hom(V_i, V'),$$

where V' is as in Section 3.3 and V_i is as in decomposition (5.5).

Therefore

$$\det(x' + x) = \prod_{j=1}^{4} \det(x' + x)_{W_j}.$$

Furthermore, if $w \in W_1$ then $(x'+x)(w) = x'w = l\mathcal{J}w = ilw$. Thus $\det(x'+x)_{W_1} = il$. For $j = 2, 3, 4$,

$$W_j = W_j^{\mathcal{J}} \oplus {}^{\mathcal{J}}W_j,$$

analogous to (3.18), i.e $[W_j^{\mathcal{J}}, \mathcal{J}] = \{{}^{\mathcal{J}}W_j, \mathcal{J}\} = 0$. This decomposition is preserved by the action of $x' + x$ so that $\det(x' + x)_{W_j} = \det(x' + x)_{W_j^{\mathcal{J}}} \det(x' + x)_{{}^{\mathcal{J}}W_j}$. Thus altogether,

$$\begin{aligned} \det(x' + x) = \quad & i^7 l\,(l - 2t)\,(l + 2t)\,(l - t + s)\,(l + t - s) \\ & (l - t - s)\,(l + t + s). \end{aligned} \tag{5.10}$$

Moreover, $\lambda'(x') = 2il$ and by (3.2),

$$\prod_{\Psi} \lambda(x) = i^6 (2t)\,(2s)\,(t + s)\,(t - s)\,(3t - s)\,(3t + s). \tag{5.11}$$

Therefore, for any polynomial $P(l)$, of degree less or equal 5,

$$\begin{aligned} & P(l) \frac{\lambda'(x') \prod_{\Psi} \lambda(x)}{\det(x' + x)} \qquad (5.12) \\ = \quad & 2P(l) \frac{2t\, 2s\,(t + s)\,(t - s)\,(3t - s)\,(3t + s)}{(l - 2t)\,(l + 2t)\,(l - t + s)\,(l + t - s)\,(l - t - s)\,(l + t + s)}. \end{aligned}$$

is a rational function of l with the degree of the nominator less than the degree of the denominator. Hence, we may decompose it into partial fractions. The result is

$$P(l)\frac{\lambda'(x')\prod_{\Psi}\lambda(x)}{\det(x'+x)} = \begin{aligned}[t] & \frac{2s\,P(2t)}{l-2t} - \frac{2s\,P(-2t)}{l+2t} \\ & + \frac{(3t+s)\,P(t-s)}{l-t+s} - \frac{(3t+s)\,P(-t+s)}{l+t-s} \\ & + \frac{(3t-s)\,P(-t-s)}{l+t+s} - \frac{(3t-s)\,P(t+s)}{l-t-s}. \end{aligned} \tag{5.13}$$

Notice that, in terms of the group $W(H_{\mathbf{C}})^{+}$ (3.7), (5.13) may be rewritten as

$$P(l)\frac{\lambda'(x')\prod_{\Psi}\lambda(x)}{\det(x'+x)} = \sum_{\eta\in W(H_{\mathbf{C}})^{+}} \frac{i(\eta\lambda_2)(x)}{l-i(\eta\lambda_1)(x)} P(i(\eta\lambda_1)(x)). \tag{5.14}$$

Let $W_j^1 = W_j^{\mathcal{J}}$ and let $W_j^{-1} = {}^{\mathcal{J}}W_j$, $j = 2, 3, 4$. Set

$$\zeta_2 = -i\lambda_1,\ \zeta_3 = -i\alpha_2,\ \zeta_4 = i\alpha_1.$$

Then

$$y(w) = -\epsilon\,\zeta_j(y)\,\mathfrak{J}w \qquad w \in W_j^{\epsilon},\ \epsilon = \pm 1, \tag{5.15}$$

so that

$$\langle y\ ,\ \rangle_{W_j^{\epsilon}} > 0 \text{ if and only if } \langle -\epsilon\zeta_j(y)\,\mathfrak{J}w, w\rangle > 0 \text{ for } w \in W_j^{\epsilon}\setminus 0).$$

Also, for $x \in \mathfrak{h}$,

$$\det(x'+x+iy)_{W_j^{\epsilon}} = i(l-\epsilon\zeta_j(x) - i\epsilon\zeta_j(y)). \tag{5.16}$$

Moreover, by (5.7),

$$\langle \mathcal{J}w, w\rangle_{W_2^\epsilon} < 0 \text{ if } w \in W_2^\epsilon \setminus 0,$$

$$\langle \mathcal{J}w, w\rangle_{W_j^\epsilon} > 0 \text{ if } w \in W_j^\epsilon \setminus 0,\ j = 3, 4.$$

Therefore, for $x \in \mathfrak{h}^r$,

$$\begin{aligned}
& \lim_{y\to 0} P(l)\frac{\lambda'(x')\prod_\Psi \lambda(x+iy)}{\det(x'+x+iy)} \\
= \quad & \frac{2s\,P(2t)}{l-2t+i0} - \frac{2s\,P(-2t)}{l+2t+i0} \\
& + \frac{(3t+s)\,P(t-s)}{l-t+s-i0} - \frac{(3t+s)\,P(-t+s)}{l+t-s-i0} \\
& + \frac{(3t-s)\,P(-t-s)}{l+t+s-i0} - \frac{(3t-s)\,P(t+s)}{l-t-s-i0},
\end{aligned}$$

where, as in (5.9), the limit is taken over $y \in \mathfrak{h}$, with $\langle y\ ,\ \rangle|_{\ker(x'+x)} > 0$.

Suppose $x \in \mathfrak{h}^r_{\{\alpha_1\}}$. The only root which has imaginary values on x is α_3. Therefore, 5.15 and 5.16 implies $\ker(x'+x) = 0$, so that

$$\begin{aligned}
& \lim_{y\to 0} P(l)\frac{\lambda'(x')\prod_\Psi \lambda(x+iy)}{\det(x'+x+iy)} \\
= \quad & \frac{2s\,P(2t)}{l-2t} - \frac{2s\,P(-2t)}{l+2t} \\
& + \frac{(3t+s)\,P(t-s)}{l-t+s} - \frac{(3t+s)\,P(-t+s)}{l+t-s} \\
& + \frac{(3t-s)\,P(-t-s)}{l+t+s} - \frac{(3t-s)\,P(t+s)}{l-t-s},
\end{aligned}$$

Similarly, for $x \in \mathfrak{h}^r_{\alpha_2}$,

$$\begin{aligned}
&\lim_{y\to 0} P(l)\frac{\lambda'(x')\prod_{\Psi}\lambda(x+iy)}{\det(x'+x+iy)} \\
= \quad & \frac{2s\,P(2t)}{l-2t} - \frac{2s\,P(-2t)}{l+2t} \\
&+ \frac{(3t+s)\,P(t-s)}{l-t+s} - \frac{(3t+s)\,P(-t+s)}{l+t-s} \\
&+ \frac{(3t-s)\,P(-t-s)}{l+t+s} - \frac{(3t-s)\,P(t+s)}{l-t-s},
\end{aligned}$$

For $x \in \mathfrak{h}^r_{\{\alpha_1,\alpha_2\}}$ each root takes real values, so that $\ker(x'+x) = 0$. Hence,

$$\begin{aligned}
&\lim_{y\to 0} P(l)\frac{\lambda'(x')\prod_{\Psi}\lambda(x+iy)}{\det(x'+x+iy)} \\
= \quad & \frac{2s\,P(2t)}{l-2t} - \frac{2s\,P(-2t)}{l+2t} \\
&+ \frac{(3t+s)\,P(t-s)}{l-t+s} - \frac{(3t+s)\,P(-t+s)}{l+t-s} \\
&+ \frac{(3t-s)\,P(-t-s)}{l+t+s} - \frac{(3t-s)\,P(t+s)}{l-t-s},
\end{aligned}$$

Since the normalized Cauchy Harish-Chandra integral (see [4]) is given by

$$\mathrm{chc}_{x'} = \frac{1}{\mu(H')}\,\widetilde{\mathrm{chc}}_{x'}, \tag{5.17}$$

we deduce the following lemma.

Lemma 5.1 *Let $x' = il\mathcal{J} \in \mathfrak{h}'^r$. If $x \in \mathfrak{h}^r$ then*

$$
\begin{aligned}
& P(l)\lambda'(x')\,\mathrm{chc}(x'+x)\prod_{\Psi}\lambda(x) \\
= \frac{2^7}{\sqrt{14}\,2\pi} & \left(\frac{2s\,P(2t)}{l-2t+i0} - \frac{2s\,P(-2t)}{l+2t+i0}\right. \\
& + \frac{(3t+s)\,P(t-s)}{l-t+s-i0} - \frac{(3t+s)\,P(-t+s)}{l+t-s-i0} \\
& \left. + \frac{(3t-s)\,P(-t-s)}{l+t+s-i0} - \frac{(3t-s)\,P(t+s)}{l-t-s-i0}\right);
\end{aligned}
$$

If $x \in \mathfrak{h}^r_{\{\alpha_1\}}$ then

$$
\begin{aligned}
& P(l)\lambda'(x')\,\mathrm{chc}(x'+x)\prod_{\Psi}\lambda(x) \\
= \frac{2^7}{\sqrt{14}\,2\pi} & \left(\frac{2s\,P(2t)}{l-2t} - \frac{2s\,P(-2t)}{l+2t}\right. \\
& + \frac{(3t+s)\,P(t-s)}{l-t+s} - \frac{(3t+s)\,P(-t+s)}{l+t-s} \\
& \left. + \frac{(3t-s)\,P(-t-s)}{l+t+s} - \frac{(3t-s)\,P(t+s)}{l-t-s-}\right);
\end{aligned}
$$

If $x \in \mathfrak{h}^r_{\{\alpha_2\}}$ then

$$
\begin{aligned}
& P(l)\lambda'(x')\,\mathrm{chc}(x'+x)\prod_{\Psi}\lambda(x) \\
= \frac{2^7}{\sqrt{14}\,2\pi} & \left(\frac{2s\,P(2t)}{l-2t} - \frac{2s\,P(-2t)}{l+2t}\right. \\
& + \frac{(3t+s)\,P(t-s)}{l-t+s} - \frac{(3t+s)\,P(-t+s)}{l+t-s} \\
& \left. + \frac{(3t-s)\,P(-t-s)}{l+t+s} - \frac{(3t-s)\,P(t+s)}{l-t-s}\right);
\end{aligned}
$$

If $x \in \mathfrak{h}^r_{\{\alpha_1,\alpha_2\}}$ *then*

$$\begin{aligned}
&P(l)\lambda'(x')\,\mathrm{chc}(x'+x)\prod_{\Psi}\lambda(x) \\
= \frac{2^7}{\sqrt{14}\,2\pi} \quad &\Bigg(\frac{2s\,P(2t)}{l-2t} - \frac{2s\,P(-2t)}{l+2t} \\
&+ \frac{(3t+s)\,P(t-s)}{l-t+s} - \frac{(3t+s)\,P(-t+s)}{l+t-s} \\
&+ \frac{(3t-s)\,P(-t-s)}{l+t+s} - \frac{(3t-s)\,P(t+s)}{l-t-s}\Bigg).
\end{aligned}$$

In terms of the Weyl group $W(H_{\mathbf{C}})^+$, Lemma 5.1 may be rewritten as

Lemma 5.2 *Let* $x' = il\mathcal{J} \in \mathfrak{h}'^r$ *and let*

$$\epsilon(\eta,\mathcal{S}) = \begin{cases} 1 & \text{if } \eta\lambda_1 = \pm\lambda_1, \quad \mathcal{S}=\emptyset, \\ -1 & \text{if } \eta\lambda_1 \neq \pm\lambda_1, \quad \mathcal{S}=\emptyset, \\ 0 & \qquad\qquad\qquad \mathcal{S}\neq\emptyset. \end{cases}$$

Then for $x \in \mathfrak{h}^r_{\mathcal{S}}$,

$$\begin{aligned}
&P(l)\lambda'(x')\,\mathrm{chc}(x'+c(\mathcal{S})x)\prod_{\Psi}\lambda(x) \\
= \quad &\frac{2^7}{\sqrt{14}\,2\pi}\sum_{\eta\in W(H_{\mathbf{C}})^+}\frac{i(\eta\lambda_2)(x)}{l-i(\eta\lambda_1)(x)+\epsilon(\eta,\mathcal{S})i0}P(i(\eta\lambda_1)(x)).
\end{aligned}$$

Let $X \subseteq W$ be the subspace consisting of matrices

$$w = \begin{pmatrix} w_{11} & w_{12} & w_{13} & w_{14} & w_{15} & w_{16} & w_{17} \\ w_{11} & w_{12} & w_{13} & w_{14} & w_{15} & w_{16} & w_{17} \end{pmatrix}$$

and let $Y \subseteq W$ be the the subspace consisting of matrices

$$w = \begin{pmatrix} w_{11} & w_{12} & w_{13} & w_{14} & w_{15} & w_{16} & w_{17} \\ -w_{11} & -w_{12} & -w_{13} & -w_{14} & -w_{15} & -w_{16} & -w_{17} \end{pmatrix}.$$

Then

$$W = X \oplus Y \tag{5.18}$$

is a complete polarization. Let $x' = im\mathcal{J} \in \mathfrak{h}'_{\lambda'}$. Then,

$$c(\lambda')x' = m \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Hence, $c(\lambda')x'$ preserves X and Y and acts in X as mI, where I is the identity. The elements of $\mathfrak{g}_2$ also preserve X and Y. Furthermore, if $\mathcal{S} \in \Psi^n_{st}$ and $x \in \mathfrak{h}_{\mathcal{S}}$, then

$$\begin{aligned} & \det(c(\lambda')x' + c(\mathcal{S})x) = \det(mI - c(\mathcal{S})x)_{\mathrm{end}(V)} = \det(mI - c(\mathcal{S})x)_{\mathrm{end}(V_{\mathbb{C}})} \\ = \quad & \det(mI - x)_{\mathrm{end}(V_{\mathbb{C}})} = m(m^2 + 4t^2)(m^2 + (t-s)^2)(m^2 + (t+s)^2) \\ = \quad & m(m^2 - \lambda_1(x)^2)(m^2 - \alpha_2(x)^2)(m^2 - \alpha_1(x)^2). \end{aligned} \tag{5.19}$$

In particular we see that (5.19) is non-zero unless λ_1, α_1, α_2 take real values on $\mathfrak{h}_{\mathcal{S}}$. Hence

$$\det(c(\lambda')x' + x) \neq 0 \quad \text{if} \quad x \in \mathfrak{h}^r.$$

Moreover, if $x \in \mathfrak{h}^r_{\{\alpha_1\}}$, then

$$\det(c(\lambda')x' + c(\alpha_1)x) = 0 \quad \text{if and only if} \quad m^2 = \alpha_1(x)^2,$$

if $x \in \mathfrak{h}^r_{\alpha_2}$, then

$$\det(c(\lambda')x' + c(\alpha_2)x) = 0 \text{ if and only if } m^2 = \alpha_2(x)^2,$$

and for $x \in \mathfrak{h}^r_{\{\alpha_1,\alpha_2\}}$,

$$\begin{gathered}\det(c(\lambda')x' + c(\alpha_1,\alpha_2)x) = 0 \text{ if and only if} \\ m^2 = \alpha_1(x)^2 \text{ or } m^2 = \alpha_2(x)^2 \text{ or } m^2 = \lambda_1(x)^2.\end{gathered}$$

Let $\delta = \delta_0 \in S^*(\mathbf{R})$ denote the Dirac delta at the origin. We conclude that for $x \in \mathfrak{h}^r$,

$$\delta(\det(c(\lambda')x' + x)) = 0,$$

for $x \in \mathfrak{h}^r_{\{\alpha_1\}}$,

$$\begin{aligned} & \delta(\det(c(\lambda')x' + c(\alpha_1)x)) \\ = \quad & |m(\alpha_1(x)^2 - \lambda_1(x)^2)(\alpha_1(x)^2 - \alpha_2(x)^2)2\alpha_1(x)|^{-1} \\ & (\delta(m - \alpha_1(x)) + \delta(m + \alpha_1(x))),\end{aligned}$$

for $x \in \mathfrak{h}^r_{\{\alpha_2\}}$,

$$\begin{aligned} & \delta(\det(c(\lambda')x' + c(\alpha_2)x)) \\ = \quad & |m(\alpha_2(x)^2 - \lambda_1(x)^2)2\alpha_2(x)(\alpha_2(x)^2 - \alpha_1(x)^2)|^{-1} \\ & (\delta(m - \alpha_2(x)) + \delta(m + \alpha_2(x))),\end{aligned}$$

and for $x \in \mathfrak{h}^r_{\{\alpha_1,\alpha_2\}}$,

$$\begin{aligned}
& \delta(\det(c(\lambda')x' + c(\alpha_1,\alpha_2)x)) \\
= \quad & |m2\lambda_1(x)(\lambda_1(x)^2 - \alpha_2(x)^2)(\lambda_1(x)^2 - \alpha_1(x)^2)|^{-1} \\
& (\delta(m - \lambda_1(x)) + \delta(m + \lambda_1(x))) \\
+ \quad & |m(\alpha_2(x)^2 - \lambda_1(x)^2)2\alpha_2(x)(\alpha_2(x)^2 - \alpha_1(x)^2)|^{-1} \\
& (\delta(m - \alpha_2(x)) + \delta(m + \alpha_2(x))) \\
+ \quad & |m(\alpha_2(x)^2 - \lambda_1(x)^2)(\alpha_1(x)^2 - \alpha_2(x)^2)2\alpha_1(x)|^{-1} \\
& (\delta(m - \alpha_1(x)) + \delta(m + \alpha_1(x))).
\end{aligned}$$

Notice that, by (5.11),

$$\begin{aligned}
|\lambda_1(\lambda_1^2 - \alpha_2^2)(\lambda_1^2 - \alpha_1^2)|^{-1} &= \left|\prod_\Psi \lambda\right|^{-1} |\lambda_2|, \\
|(\alpha_2^2 - \lambda_1^2)\alpha_2(\alpha_2^2 - \alpha_1^2)|^{-1} &= \left|\prod_\Psi \lambda\right|^{-1} |\alpha_4|, \\
|(\alpha_1^2 - \lambda_1^2)(\alpha_1^2 - \alpha_2^2)\alpha_1|^{-1} &= \left|\prod_\Psi \lambda\right|^{-1} |\alpha_3|.
\end{aligned}$$

Thus, for $x \in \mathfrak{h}^r$,

$$|\lambda'(x')|\delta(\det(c(\lambda')x' + x))\left|\prod_\Psi \lambda(x)\right| = 0,$$

for $x \in \mathfrak{h}^r_{\{\alpha_1\}}$,

$$\begin{aligned}
& |\lambda'(x')|\delta(\det(c(\lambda')x' + c(\alpha_1)x))\left|\prod_\Psi \lambda(x)\right| \\
= \quad & |\alpha_3(x)|(\delta(m - \alpha_1(x)) + \delta(m + \alpha_1(x))),
\end{aligned}$$

for $x \in \mathfrak{h}^r_{\{\alpha_2\}}$,

$$\begin{aligned} & |\lambda'(x')|\delta(\det(c(\lambda')x' + c(\alpha_2)x)) \left|\prod_\Psi \lambda(x)\right| \\ = \quad & |\alpha_4(x)|(\delta(m - \alpha_2(x)) + \delta(m + \alpha_2(x))), \end{aligned}$$

and for $x \in \mathfrak{h}^r_{\{\alpha_1,\alpha_2\}}$,

$$\begin{aligned} & |\lambda'(x')|\delta(\det(c(\lambda')x' + c(\{\alpha_1, \alpha_2\})x)) \left|\prod_\Psi \lambda(x)\right| \\ = \quad & |\lambda_2(x)|(\delta(m - \lambda_1(x)) + \delta(m + \lambda_1(x))) \\ + \quad & |\alpha_4(x)|(\delta(m - \alpha_2(x)) + \delta(m + \alpha_2(x))) \\ + \quad & |\alpha_3(x)|(\delta(m - \alpha_1(x)) + \delta(m + \alpha_1(x))). \end{aligned}$$

Recall that, for $x' \in \mathfrak{h}'^{\,r}_{\lambda'}$, $x \in \mathfrak{h}^r_{\mathcal{S}}$, by [5, (1.1)],

$$\mathrm{chc}(c(\lambda')x' + c(\mathcal{S})x) = \frac{2^7}{\sqrt{14}}\,\delta(\det(c(\lambda')x' + c(\mathcal{S})x)).$$

Thus we have verified the following lemma.

Lemma 5.3 *Let $x' = im\mathcal{J} \in \mathfrak{h}'^{\,r}_{\lambda'}$. For $x \in \mathfrak{h}^r$,*

$$|\lambda'(x')|\mathrm{chc}(c(\lambda')x' + x) \left|\prod_\Psi \lambda(x)\right| = 0$$

For $x \in \mathfrak{h}^r_{\{\alpha_1\}}$,

$$
\begin{aligned}
&|\lambda'(x')|\mathrm{chc}(c(\lambda')x' + c(\alpha_1)x)\left|\prod_{\Psi}\lambda(x)\right| \\
= \quad & \frac{2^7}{\sqrt{14}}|\alpha_3(x)|(\delta(m-\alpha_1(x)) + \delta(m+\alpha_1(x))).
\end{aligned}
$$

For $x \in \mathfrak{h}^{\tau}_{\{\alpha_2\}}$,

$$
\begin{aligned}
&|\lambda'(x')|\mathrm{chc}(c(\lambda')x' + c(\alpha_2)x)\left|\prod_{\Psi}\lambda(x)\right| \\
= \quad & \frac{2^7}{\sqrt{14}}|\alpha_4(x)|(\delta(m-\alpha_2(x)) + \delta(m+\alpha_2(x))).
\end{aligned}
$$

For $x \in \mathfrak{h}^{\tau}_{\{\alpha_1,\alpha_2\}}$,

$$
\begin{aligned}
&|\lambda'(x')|\mathrm{chc}(c(\lambda')x' + c(\alpha_1,\alpha_2)x)\left|\prod_{\Psi}\lambda(x)\right| \\
= \frac{2^7}{\sqrt{14}} \quad & (|\lambda_2(x)|(\delta(m-\lambda_1(x)) + \delta(m+\lambda_1(x))) \\
+ \quad & |\alpha_4(x)|(\delta(m-\alpha_2(x)) + \delta(m+\alpha_2(x))) \\
+ \quad & |\alpha_3(x)|(\delta(m-\alpha_1(x)) + \delta(m+\alpha_1(x)))).
\end{aligned}
$$

In terms of the group (3.7), we may rewrite Lemma 5.3 as follows.

Lemma 5.4 *Set*

$$
\varsigma_{\mathcal{S}}(\lambda) = \begin{cases} 1 \text{ if } \lambda \in \mathcal{S} \cap (-\mathcal{S}) \\ 0 \text{ otherwise.} \end{cases}
$$

Let $x' = im\mathcal{J} \in \mathfrak{h}'^{\,r}_{\lambda'}$. *Then for* $x \in \mathfrak{h}^r_{\mathcal{S}}$,

$$\begin{aligned}
& |\lambda'(x')|\mathrm{chc}(c(\lambda')x' + c(\mathcal{S})x)\left|\prod_{\Psi}\lambda(x)\right| \\
= \quad & \frac{2^7}{\sqrt{14}}\sum_{\eta\in W(H_{\mathbf{C}})^+} \varsigma_{\mathcal{S}}(\eta\lambda_1)|(\eta\lambda_2)(x)|\delta(m + (\eta\lambda_1)(x)).
\end{aligned}$$

Proposition 5.1 *Let* $\varphi \in C^\infty_c(\mathfrak{g}^{rs})$. *Then for* $x' = l\mathcal{J} \in \mathfrak{h}'^{r}$,

$$\begin{aligned}
& \int_{\mathfrak{g}_2} P(l)\lambda'(x')\,\mathrm{chc}(x'+x)\,\varphi(x)\,d\mu(x) \\
= \quad & \sum_{[\mathcal{S}]}\frac{(-1)^{|\mathcal{S}|}}{|W(H_{\mathcal{S}})|}\frac{2^7}{\sqrt{14}2\pi}\int_{\mathfrak{h}_{\mathcal{S}}}\sum_{\eta\in W(H_{\mathbf{C}})^+}\frac{i(\eta\lambda_2)(x)}{l - i(\eta\lambda_1)(x) + \epsilon(\eta,\mathcal{S})i0}\mathcal{A}(\Psi_{\mathcal{S},\mathbf{R}})(x) \\
& P(i(\eta\lambda_1)(x))\,\mathcal{H}_{\mathcal{S}}\varphi(x)\,d\mu(x).
\end{aligned}$$

Proof:

We see from the Harish-Chandra - Weyl integration formula (2.6), that

$$\begin{aligned}
& \int_{\mathfrak{g}_2} P(l)\lambda'(x')\,\mathrm{chc}(x'+x)\,\varphi(x)\,d\mu(x) \\
= \quad & \sum_{[\mathcal{S}]}\frac{1}{|W(H_{\mathcal{S}})|}\int_{\mathfrak{h}_{\mathcal{S}}} P(l)\lambda'(x')\,\mathrm{chc}(x' + c(\mathcal{S})x)\,\overline{\prod_{\Psi}\lambda(x)}\mathcal{A}(\Psi_{\mathcal{S},\mathbf{R}})(x) \\
& \mathcal{H}_{\mathcal{S}}\varphi(x)\,d\mu(x).
\end{aligned}$$

However, it is very easy to see that, for $x \in \mathfrak{h}_{\mathcal{S}}$,

$$\overline{\prod_{\Psi}\lambda(x)} = (-1)^{|\mathcal{S}|}\prod_{\Psi}\lambda(x).$$

Hence Lemma 5.1 follows from Lemma 5.2. **QED**.

Proposition 5.2 *Set*

$$\tilde{\varsigma}_{\mathcal{S}}(\eta\lambda_1) = \begin{cases} -\varsigma_{\mathcal{S}}(\eta\lambda_1) \text{ if } \mathcal{S} = \{\alpha_2\}, \\ \varsigma_{\mathcal{S}}(\eta\lambda_1) \text{ if } \mathcal{S} \neq \{\alpha_2\}. \end{cases}$$

Let $\varphi \in C_c^\infty(\mathfrak{g}^{rs})$. *Then for* $x' = i\, m\mathcal{J} \in \mathfrak{h}'^{\,r}_{\lambda'}$,

$$\begin{aligned} &\int_{\mathfrak{g}_2} |\lambda'(x')|\, \mathrm{chc}(c(\lambda')x' + x)\, \varphi(x)\, d\mu(x) \\ = \quad & \sum_{[\mathcal{S}]} \frac{1}{|W(H_{\mathcal{S}})|} \frac{2^7}{\sqrt{14}} \int_{\mathfrak{h}_{\mathcal{S}}} \sum_{\eta \in W(H_{\mathbf{C}})^+} \tilde{\varsigma}_{\mathcal{S}}(\eta\lambda_1)\, |(\eta\lambda_2)(x)|\, \delta(m + (\eta\lambda_1)(x)) \\ & \mathcal{A}(\Psi_{\mathcal{S},i\mathbf{R}})(x)\, \mathcal{H}_{\mathcal{S}}\varphi(x)\, d\mu(x). \end{aligned}$$

Proof:

We see from the Harish-Chandra - Weyl integration formula (2.6) and Lemma 5.4, that

$$\begin{aligned} &\int_{\mathfrak{g}_2} |\lambda'(x')|\, \mathrm{chc}(c(\lambda')x' + x)\, \varphi(x)\, d\mu(x) \\ = \quad & \sum_{[\mathcal{S}]} \frac{1}{|W(H_{\mathcal{S}})|} \int_{\mathfrak{h}_{\mathcal{S}}} |\lambda'(x')|\, \mathrm{chc}(c(\lambda')x' + c(\mathcal{S})x)\, \overline{\prod_{\Psi} \lambda(x)} \mathcal{A}(\Psi_{\mathcal{S},\mathbf{R}})(x)\, \mathcal{H}_{\mathcal{S}}\varphi(x)\, d\mu(x) \\ = \quad & \sum_{[\mathcal{S}]} \frac{1}{|W(H_{\mathcal{S}})|} \int_{\mathfrak{h}_{\mathcal{S}}} \frac{2^7}{\sqrt{14}} \sum_{\eta \in W(H_{\mathbf{C}})^+} \eta_{\mathcal{S}}(\eta\alpha_1)\, |(\eta\alpha_2)(x)|\, \delta(m + (\eta\lambda_1)(x)) \\ & \left(\left| \prod_{\Psi} \lambda(x) \right|^{-1} \overline{\prod_{\Psi} \lambda(x)} \mathcal{A}(\Psi_{\mathcal{S},\mathbf{R}})(x) \right) \mathcal{H}_{\mathcal{S}}\varphi(x)\, d\mu(x). \end{aligned}$$

It remains to compute the term in the parenthesis.

Notice that for $\mathcal{S} = \{\alpha_1, \alpha_2\}$,

$$\left|\prod_{\Psi} \lambda(x)\right|^{-1} \overline{\prod_{\Psi} \lambda(x)} \mathcal{A}(\Psi_{\mathcal{S},\mathbf{R}})(x) = 1.$$

Suppose $\mathcal{S} = \{\alpha_1\}$. Then any element $x \in \mathfrak{h}_{\mathcal{S}}$ may be written as

$$x = i2aH_{\alpha_3} - 2bH_{\alpha_1} = (a + ib)\Lambda_1 + (-a + i3b)\Lambda_2,$$

where a and b are some real numbers. Hence,

$$\begin{aligned}
\prod_{\Psi} \lambda(x) = & \quad -4ts(t+s)(t-s)(3t-s)(3t+s) \\
= & \quad -4 \cdot 2 \cdot (a^2 + b^2) 2(a^2 + 9b^2)(-4b)(4ia) \\
= & \quad -\alpha_1(x)\,\alpha_3(x)\, 4 \cdot 2 \cdot (a^2 + b^2) 2(a^2 + 9b^2) \\
= & \quad \overline{\alpha_1(x)\,\alpha_3(x)} \cdot \prod_{\lambda \in \Psi \setminus \{\alpha_1, \alpha_3\}} |\lambda(x)|.
\end{aligned}$$

Therefore,

$$\left|\prod_{\Psi} \lambda(x)\right|^{-1} \overline{\prod_{\Psi} \lambda(x)} \mathcal{A}(\Psi_{\mathcal{S},\mathbf{R}})(x) = \frac{\alpha_3(x)}{|\alpha_3(x)|}.$$

Suppose $\mathcal{S} = \{\alpha_2\}$. Then any element $x \in \mathfrak{h}_{\mathcal{S}}$ may be written as

$$x = -i2aH_{\alpha_4} + 2bH_{\alpha_2} = (a + ib)\Lambda_1 + (a - i3b)\Lambda_2,$$

where a and b are some real numbers. Hence,

$$\begin{aligned}
\prod_{\Psi} \lambda(x) &= -4ts(t+s)(t-s)(3t-s)(3t+s) \\
&= 4 \cdot 2 \cdot (a^2+b^2) 2(a^2+9b^2)(-4b)(4ia) \\
&= -\alpha_2(x)\,\alpha_4(x)\,4 \cdot 2 \cdot (a^2+b^2)2(a^2+9b^2) \\
&= -\alpha_2(x)\,\alpha_4(x) \cdot \prod_{\lambda \in \Psi \setminus \{\alpha_2, \alpha_4\}} |\lambda(x)|.
\end{aligned}$$

Therefore,

$$\left|\prod_{\Psi} \lambda(x)\right|^{-1} \overline{\prod_{\Psi} \lambda(x)} \mathcal{A}(\Psi_{S,\mathbf{R}}) = \frac{\alpha_4(x)}{|\alpha_4(x)|}.$$

QED.

Proposition 5.3 *Let $\varphi \in C_c^\infty(\mathfrak{g}_2^{rs})$. Then there is a continuous seminorm q on $C_c^\infty(\mathfrak{g}_2^{rs})$ such that, for $x' = l\mathcal{J} \in \mathfrak{h}'^r$,*

$$\left|\int_{\mathfrak{g}_2} \lambda'(x')\,\mathrm{chc}(x'+x)\,\varphi(x)\,d\mu(x)\right| \leq q(\varphi)\,(1+l^2)^{-1}. \tag{5.20}$$

Moreover, as a function of $x' \in \mathfrak{h}'^{\,r}_{\lambda'}$,

$$\int_{\mathfrak{g}} \mathrm{chc}(x'+x)\,\varphi(x)\,d\mu(x) \tag{5.21}$$

is smooth and compactly supported.

Proof:

Let $P(l) = l^2$ and consider one term in the double sum of Proposition 5.1,

$$\int_{\mathfrak{h}_\mathcal{S}} \frac{i(\eta\lambda_2)(x)}{l - i(\eta\lambda_1)(x) + \epsilon(\eta, \mathcal{S})i0} \mathcal{A}(\Psi_{\mathcal{S},\mathbf{R}})(x) \\ P(i(\eta\lambda_1)(x))\, \mathcal{H}_\mathcal{S}\varphi(x)\, d\mu(x). \tag{5.22}$$

The function

$$f(x) = i(\eta\lambda_2)(x)\, \mathcal{A}(\Psi_{\mathcal{S},\mathbf{R}})(x)\, P(i(\eta\lambda_1)(x))\, \mathcal{H}_\mathcal{S}\varphi(x)$$

is smooth and compactly supported in $\mathfrak{h}_\mathcal{S}^r$. Let $\beta = -\eta\lambda_1$ and let $H_\beta \in i\mathfrak{h}$ be the corresponding element such that $\beta(H_\beta) = 2$, as in Lemma 3.15. Set $\epsilon = \epsilon(\eta, \mathcal{S})$. Then (5.22) may be rewritten as

$$\int_{\mathfrak{h}_\mathcal{S}} \frac{1}{l + i\beta(x) + \epsilon i0} f(x)\, d\mu(x). \tag{5.23}$$

Suppose β is an imaginary root for $\mathfrak{h}_\mathcal{S}$. Then

$$\mathfrak{h}_\mathcal{S} = \mathbf{R}iH_\beta \oplus \ker(\beta).$$

For $t \in \mathbf{R}$, let

$$f_\beta(tiH_\beta) = \int_{\ker(\beta)} f(t\frac{i}{2}H_\beta + y)\, d\mu(y).$$

Since, $\beta(t\frac{i}{2}H_\beta) = -t$, (5.23) is equal to

$$\int_{\mathbf{R}iH_\beta} \frac{1}{l - t + \epsilon i0} f_\beta(tiH_\beta)\, d\mu(t\frac{i}{2}H_\beta). \tag{5.24}$$

Set $\phi(t) = -f_\beta(-tiH_\beta)$. Then, up to a constant multiple coming from the normalization of the measure μ, see 4.5, (5.24) coincides with

$$\int_{\mathbf{R}} \frac{1}{t-l-\epsilon i0}\, \phi(t)\, dt = \int_{\mathbf{R}} \frac{1}{t-\epsilon i0}\, \phi(t+l)\, dt, \tag{5.25}$$

where $\phi \in C_c^\infty(\mathbf{R})$.

Fix a function $\xi \in C_c^\infty(\mathbf{R})$ such that $0 \le \xi \le 1$, $supp(\xi) \subseteq [-2,2]$ and $\xi(x) = 1$ for $x \in [-1,1]$. Then (5.25) is equal to

$$\int_{\mathbf{R}} \frac{1}{t}(1-\xi(x))\phi_l(t)\, dt + \int_{\mathbf{R}} \frac{1}{t-\epsilon i0}\, \xi(t)\phi_l(t)\, dt,$$

where $\phi_l(t) = \phi(t+l)$. Notice that

$$\left| \int_{\mathbf{R}} \frac{1}{t}(1-\xi(x))\phi_l(t)\, dt \right| \le \int_{\mathbf{R}} |\phi_l(t)|\, dt = \| \phi \|_1 .$$

Furthermore,

$$\begin{aligned} & \left| \int_{\mathbf{R}} \frac{1}{t-\epsilon i0}\, \xi(t)\phi_l(t)\, dt \right| \\ = \quad & \left| \int_{\mathbf{R}} -\ln(t-\epsilon i0)\, \partial_t(\xi\phi_l)(t)\, dt \right| \\ \le \quad & \int_{|t|\le 2} |\ln(t-\epsilon i0)| (|\xi'(t)\phi_l(t)| + |\xi(t)\phi_l'(t)|)\, dt \\ \le \quad & \left(\int_{|t|\le 2} |\ln(t-\epsilon i0)|\, dt \right) (\| \xi' \|_\infty \| \phi \|_\infty + \| \phi' \|_\infty). \end{aligned}$$

Hence, (5.25) may be dominated by

$$\left(\int_{|t|\le 2} |\ln(t-\epsilon i0)|\, dt \right) (\| \xi' \|_\infty \| \phi \|_\infty + \| \phi' \|_\infty) + \| \phi \|_1). \tag{5.26}$$

Suppose β is a real root for $\mathfrak{h}_S$. Then $\epsilon = 0$ and, for $x \in \mathfrak{h}_S$,

$$|l + i\beta(x)| \geq |\beta(x)|$$

has a strictly positive minimum on the support of ϕ, because β is non-zero there. Hence, (5.23), and therefore also (5.22) is dominated by

$$\int_{\mathfrak{h}_S} |f(x)|\, d\mu(x). \tag{5.27}$$

Suppose β is a complex root for $\mathfrak{h}_S$. Then $\epsilon = 0$ and

$$\beta : \mathfrak{h}_S \to \mathbf{C}$$

is a bijective linear isomorphism over the reals. Let $\beta^{-1} : \mathbf{C} \to \mathfrak{h}_S$ denote the inverse. The (5.23) may be rewritten as a constant multiple of

$$\int_{\mathbf{C}} \frac{1}{l + iz}\, f(\beta^{-1}(z))\, dx\, dy, \tag{5.28}$$

where $z = x + iy$ as usual. Notice that the function $\frac{1}{z}$ is locally integrable. Therefore, (5.28) may be dominated by

$$\left(max_D \int_D \frac{1}{|z|}\, dx\, dy \right) \parallel f \parallel_\infty, \tag{5.29}$$

where the maximum is taken over all the discs of radius one in the complex plane. By combining (5.26), (5.27), (5.29) with Proposition 5.1, we see that there is a continuous seminorm q on $C_c^\infty(\mathfrak{g}_2^{rs})$ such that

$$\left|\int_{\mathfrak{g}} P(l)\lambda'(x')\,\mathrm{chc}(x'+x)\,\varphi(x)\,d\mu(x)\right| \leq q(\varphi).$$

Hence, (5.20) follows.

The statement (5.21) is an easy consequence of Proposition 5.2.

QED.

5.2 A Conjecture and the Main Theorem

For $\varphi \in C_c^\infty(\mathfrak{g}_2^{rs})$, $x' \in \mathfrak{h}'^r \cup \mathfrak{h}'(\lambda')^r$, define

$$\mathcal{CHC}(\varphi)(x') = \int_{\mathfrak{g}_2} \mathrm{chc}(x'+x)\,\varphi(x)\,d\mu(x),$$

and note that this is the restriction to $\mathfrak{g}_2$ of the of the normalized version of the distribution (2.8). We know from Proposition 5.3 that

$$\lambda(x')\,\mathcal{CHC}(\varphi)(x')$$

is a smooth function of $x' \in \mathfrak{h}'^r$, which decays at infinity at least as fast as $|\lambda'(x')|^{-2}$. Also, as a function of $x' \in \mathfrak{h}'(\lambda')^r$,

$$\lambda(c(\lambda')^{-1}x')\,\mathcal{CHC}(\varphi)(x')$$

is smooth and compactly supported.

Let $\mathcal{O}' \subseteq \mathfrak{sp}(1,\mathbf{R})$ be a $Sp(1,\mathbf{R})$-orbit, and let $\mathcal{F}\delta_{\mathcal{O}'}$ be the Fourier transform of the invariant measure $\delta_{\mathcal{O}'}$, as in Theorem 4.4. Then the functions

$$\begin{aligned} &\lambda(x')\,\mathcal{F}\delta_{\mathcal{O}'}(x') && \text{for } x' \in \mathfrak{h}'^r, \\ &\lambda(c(\lambda')^{-1}x')\,\mathcal{F}\delta_{\mathcal{O}'}(x') && \text{for } x' \in \mathfrak{h}'(\lambda')^r, \end{aligned}$$

are bounded. Hence,

$$\int_{\mathfrak{h}'^r} |\mathcal{F}\delta_{\mathcal{O}'}(x')||\lambda(x')|^2|\mathcal{CHC}(\varphi)(x')|\,d\mu(x') < \infty, \tag{5.30}$$

$$\int_{\mathfrak{h}'(\lambda')^r} |\mathcal{F}\delta_{\mathcal{O}'}(x')||\lambda(c(\lambda')^{-1}x')|^2|\mathcal{CHC}(\varphi)(x')|\,d\mu(x') < \infty. \tag{5.31}$$

Therefore, following the Harish-Chandra Weyl integration formula for $\mathfrak{sp}(1,\mathbf{R})$, we may define

$$\begin{aligned} \mathcal{CHC}(\overline{\mathcal{F}\delta_{\mathcal{O}'}})(\varphi) &= \int_{\mathfrak{h}'^r} \overline{\mathcal{F}\delta_{\mathcal{O}'}(x')}|\lambda(x')|^2\mathcal{CHC}(\varphi)(x')\,d\mu(x') \\ &+ \frac{1}{2}\int_{\mathfrak{h}'(\lambda')^r} \overline{\mathcal{F}\delta_{\mathcal{O}'}(x')}|\lambda(c(\lambda')^{-1}x')|^2\mathcal{CHC}(\varphi)(x')\,d\mu(x'). \end{aligned} \tag{5.32}$$

Then $\mathcal{CHC}(\overline{\mathcal{F}\delta_{\mathcal{O}'}})$ is a G_2-invariant distribution on $\mathfrak{g}_2^{rs}$.

For an orbit $\mathcal{O} \subseteq \mathfrak{g}_2$ let $\mu_{\mathcal{O}}$ be the positive G_2-invariant measure, normalized as in section 4.5. Define the Fourier transform $\mathcal{F}\mu_{\mathcal{O}}$ as in (4.44) with the $sp(1,\mathbf{R})$ replaced by $\mathfrak{g}_2$.

Conjecture 5.1 *(a) Let $\mathcal{O}' \subseteq \mathfrak{sp}(1,\mathbf{R})$ be a non-zero semisimple orbit and let $\mathcal{O} \subseteq \mathfrak{g}_2$ be the corresponding semisimple orbit as in Corollary 4.2. Then there is a constant C independent of $\mathcal{O}'$ such that*

$$\mathcal{CHC}(\overline{\mathcal{F}\delta_{\mathcal{O}'}}) = C\,\mathcal{F}\delta_{\mathcal{O}}.$$

(b) Let $\mathcal{O}' \subseteq \mathfrak{sp}(1, \mathbf{R})$ be a non-zero nilpotent orbit. Then there is a unique nilpotent orbit $\mathcal{O} \subseteq \mathfrak{g}_2$ such that

$$\tau_{\mathfrak{g}_2}(\tau_{\mathfrak{sp}(1,\mathbf{R})}{}^{-1}(closure(\mathcal{O}'))) = closure(\mathcal{O}).$$

(c) For the two different non-zero nilpotent orbits $\mathcal{O}'_1, \mathcal{O}'_2 \subseteq \mathfrak{sp}(1, \mathbf{R})$, the corresponding orbits $\mathcal{O}_1, \mathcal{O}_2 \subseteq \mathfrak{g}_2$ are different. In terms of table 4.3, one of them corresponds to the parameter $(2, 2)$ and the other one to $(0, 4)$.

(d) The formula of part (a) holds for a non-zero nilpotent orbit $\mathcal{O}' \subseteq \mathfrak{sp}(1, \mathbf{R})$, with the $\mathcal{O} \subseteq \mathfrak{g}_2$ corresponding to $\mathcal{O}'$ as in (b).

We don't have any complete proof of this conjecture yet, but the result below provides some evidence for its validity. In particular, the distribution in question is a function on the subset of the regular semisimple elements, as is the Fourier transform of the invariant measure on the corresponding orbit (a result of Harish-Chandra).

Theorem 5.1 *Let $\mathcal{O}' \subseteq \mathfrak{sp}(1, \mathbf{R})$ be a non-zero orbit. Then the restriction of $\mathcal{CHC}(\overline{\mathcal{F}\delta_{\mathcal{O}'}})$ to the dense open set $\mathfrak{g}_2^{rs} \subseteq \mathfrak{g}_2$ is a smooth locally integrable function. Explicitly, for $\varphi \in C_c^\infty(\mathfrak{g}_2^{rs})$, if $\mathcal{O}' = \mathcal{O}_\rho^1$ with $\rho \in \mathbf{R}^+$, then*

$$\begin{aligned} \mathcal{CHC}(\overline{\mathcal{F}\delta_{\mathcal{O}'}}) &= \sum_{[S]} \frac{1}{|W(H_S)|} \int_{\mathfrak{h}_S} \frac{2^7 7}{\rho} \sum_{\eta \in W(H_\mathbf{C})^+} \tilde{\varsigma}_S(\eta\lambda_1)|(\eta\lambda_2)(x)|\mathcal{A}(\Psi_{S,i\mathbf{R}})(x) \\ &\quad (\exp(-i\pi\rho(\eta\lambda_1)(x)) + \exp(i\pi\rho(\eta\lambda_1)(x)))\mathcal{H}_S\varphi(x)\, d\mu(x); \quad (5.33) \end{aligned}$$

if $\mathcal{O}' = \mathcal{O}^2_{\epsilon\rho}$ *with* $\rho > 0$ *and* $\epsilon = \pm 1$, *then*

$$\begin{aligned} \mathcal{CHC}(\overline{\mathcal{F}\delta_{\mathcal{O}'}}) &= \sum_{[S]} \frac{1}{|W(H_S)|} \int_{\mathfrak{h}_S} \frac{2^7 7}{\rho} \sum_{\eta \in W(H_{\mathbf{C}})^+} ((-1)^{|S|} 2(-\eta\lambda_2)(x) \qquad (5.34) \\ & \mathcal{A}(\Psi_{S,\mathbf{R}})(x) \exp(\pi\rho(\eta\lambda_1)(x)) 1_{\mathbf{R}^+}(\epsilon\rho(Re(-\eta\lambda_1)(x) + \epsilon(\eta, \mathcal{S}))) \\ &+ \tilde{\varsigma}_S(\eta\lambda_1)|(\eta\lambda_2)(x)| \mathcal{A}(\Psi_{S,i\mathbf{R}})(x) \exp(-\pi\rho|(\eta\lambda_1)(x)|)) \mathcal{H}_S\varphi(x)\, d\mu(x); \end{aligned}$$

if $\mathcal{O}' = \mathcal{O}^{\epsilon}_0$ *with* $\epsilon = \pm 1$, *then*

$$\begin{aligned} \mathcal{CHC}(\overline{\mathcal{F}\delta_{\mathcal{O}'}})(\varphi) &= \sum_{[S]} \frac{1}{|W(H_S)|} \int_{\mathfrak{h}_S} \frac{2^7 7}{\sqrt{2}} \sum_{\eta \in W(H_{\mathbf{C}})^+} ((-1)^{|S|} 2(-\eta\lambda_2)(x) \mathcal{A}(\Psi_{S,\mathbf{R}}) \\ & (1_{\mathbf{R}^+}(\epsilon(-Re(\eta\lambda_1)(x) + \epsilon(\eta, \mathcal{S}))) - 1_{\mathbf{R}^+}(-\epsilon(Re(\eta\lambda_1)(x) + \epsilon(\eta, \mathcal{S})))) \\ &+ \tilde{\varsigma}_S(\eta\lambda_1)|(\eta\lambda_2)(x)| \mathcal{A}(\Psi_{S,i\mathbf{R}})) \mathcal{H}_S\varphi(x)\, d\mu(x). \qquad (5.35) \end{aligned}$$

Proof:

Suppose $v \in \mathbf{R}$ and $\epsilon \in \{-1, 0, 1\}$ are such that $v = 0$ if and only if $\epsilon \neq 0$. Let $u \in \mathbf{R}$ and let $w = u + iv \in \mathbf{C}$. Then

$$\int_{\mathbf{R}} \frac{e^{-i\xi x}}{x + w + \epsilon i0}\, dx = -2\pi i \operatorname{sgn}(v + \epsilon)\, e^{i\xi w}\, 1_{\mathbf{R}^+}(\xi(v + \epsilon)). \qquad (5.36)$$

This is to be understood as an equation of tempered distributions. By the definition (5.32), Propositions 5.1 and 5.2,

$$\begin{aligned} \mathcal{CHC}(\overline{\mathcal{F}\delta_{\mathcal{O}'}})(\varphi) = \sum_{[S]} \int_{\mathfrak{h}_S} \frac{1}{|W(H_{\mathbf{C}})|} \frac{2^7}{\sqrt{14}} \sum_{\eta \in W(H_{\mathbf{C}})^+} \Big(\frac{(-1)^{|S|}}{2\pi} \int_{\mathfrak{h}'} \qquad (5.37) \\ \frac{i(\eta\lambda_2)(x)}{l - i(\eta\lambda_1)(x) + \epsilon(\eta, \mathcal{S}) i0} \overline{\mathcal{F}\delta_{\mathcal{O}'}(x')\lambda'(x')}\, d\mu(x') \mathcal{A}(\Psi_{S,\mathbf{R}}) \\ + \frac{\tilde{\varsigma}_S(\eta\lambda_1)}{2} \int_{\mathfrak{h}'_{\lambda'}} |(\eta\lambda_2)(x)| \delta(m + (\eta\lambda_1)(x)) \overline{\mathcal{F}\delta_{\mathcal{O}'}(c(\lambda')x')\lambda'(x')}\, d\mu(x') \\ \mathcal{A}(\Psi_{S,i\mathbf{R}})) \mathcal{H}_S\varphi(x)\, d\mu(x), \end{aligned}$$

where for $x' \in \mathfrak{h}'_{\lambda'}$, $x' = miJ$, i.e. $m = -\frac{1}{2}\lambda'(x')$. Also, by (4.29),

$$\begin{aligned} & \int_{\mathfrak{h}'_{\lambda'}} \delta(m + (\eta\lambda_1)(x))\overline{\mathcal{F}\delta_{\mathcal{O}'}(c(\lambda')x')\lambda'(x')}\, d\mu(x') \\ = \ & 2\sqrt{14}\,\overline{\mathcal{F}\delta_{\mathcal{O}'}((\eta\lambda_1)(x)c(\lambda')H_{\lambda'})|(\eta\lambda_1)(x)|}. \end{aligned} \tag{5.38}$$

Suppose $\mathcal{O}'$ is as in (5.33). Then $\mathcal{F}\delta_{\mathcal{O}'}(x') = 0$ for $x' \in \mathfrak{h}'$. Hence, by (5.37), (5.38) and (4.45),

$$\begin{aligned} \mathcal{CHC}(\overline{\mathcal{F}\delta_{\mathcal{O}'}})(\varphi) = & \sum_{[\mathcal{S}]} \int_{\mathfrak{h}_\mathcal{S}} \frac{1}{|W(H_\mathbf{C})|} \frac{2^7}{\sqrt{14}} \sum_{\eta \in W(H_\mathbf{C})^+} \frac{\tilde{\varsigma}_\mathcal{S}(\eta\lambda_1)}{2} \\ & |(\eta\lambda_2)(x)| \mathcal{A}(\Psi_{\mathcal{S},i\mathbf{R}})\, 2\sqrt{14}\, 7\pi \frac{e^{i\pi\rho(\eta\lambda_1)(x)} + e^{-i\pi\rho(\eta\lambda_1)(x)}}{\pi\rho} \mathcal{H}_\mathcal{S}\varphi(x)\, d\mu(x). \end{aligned} \tag{5.39}$$

This verifies (5.33).

Suppose $\mathcal{O}'$ is as in (5.34). Then, by (4.46),

$$\mathcal{F}(\delta_{\mathcal{O}'})(liH_{\lambda'}) = 7\pi i \frac{e^{i\pi\epsilon\rho l}}{\pi\epsilon\rho}.$$

Therefore, by (5.36),

$$\begin{aligned} & \int_{\mathfrak{h}'} \frac{1}{l - i(\eta\lambda_1)(x) + \epsilon(\eta, \mathcal{S})i0} \overline{\mathcal{F}\delta_{\mathcal{O}'}(x')\lambda'(x')}\, d\mu(x') \\ = \ & -\sqrt{14}\, 14\epsilon \frac{1}{\rho} \int_\mathbf{R} \frac{1}{l - i(\eta\lambda_1)(x) + \epsilon(\eta, \mathcal{S})i0} e^{-i\pi\epsilon\rho l}\, dl \\ = \ & -\sqrt{14}\, 14\epsilon \frac{1}{\rho} (-2\pi i) \operatorname{sgn}(Im(-i(\eta\lambda_1)(x)) + \epsilon(\eta, \mathcal{S})) e^{i\pi\epsilon\rho i(\eta\lambda_1)(x)} \\ & 1_{\mathbf{R}^+}(\epsilon\rho(Im(-i(\eta\lambda_1)(x)) + \epsilon(\eta, \mathcal{S})) \\ = \ & \sqrt{14}\, 14\epsilon \frac{1}{\rho}\, 2\pi i\, \epsilon\, e^{i\pi\epsilon\rho i(\eta\lambda_1)(x)}\, 1_{\mathbf{R}^+}(-\epsilon\rho(Im(i(\eta\lambda_1)(x) + \epsilon(\eta, \mathcal{S}))). \end{aligned} \tag{5.40}$$

Also, by by (4.45),

$$\mathcal{F}\delta_{\mathcal{O}'}((\eta\lambda_1)(x)c(\lambda')H_{\lambda'})\,|(\eta\lambda_1)(x)| = \frac{7}{\rho}e^{-\pi\rho|(\eta\lambda_1)(x)|}.$$

Thus,

$$\begin{aligned} \mathcal{CHC}(\overline{\mathcal{F}\delta_{\mathcal{O}'}})(\varphi) &= \sum_{[\mathcal{S}]}\int_{\mathfrak{h}_\mathcal{S}} \frac{1}{|W(H_\mathbf{C})|}\frac{2^7}{\sqrt{14}} \sum_{\eta\in W(H_\mathbf{C})^+} \\ &\left(\frac{(-1)^{|\mathcal{S}|}}{2\pi}(i)(\eta\lambda_2)(x)\sqrt{14}\,14\,2\pi i\frac{1}{\rho}e^{\pi\rho(\eta\lambda_1)(x)}1_{\mathbf{R}^+}(\epsilon\rho(Im(-i(\eta\lambda_1)(x)) + \epsilon(\eta,\mathcal{S}))) \right.\\ &\left. \mathcal{A}(\Psi_{\mathcal{S},\mathbf{R}}) + \frac{\tilde{\varsigma}_\mathcal{S}(\eta\lambda_1)}{2}|(\eta\lambda_2)(x)|\,2\sqrt{14}\,\frac{7}{\rho}e^{-\pi\rho|(\eta\lambda_1)(x)|}\mathcal{A}(\Psi_{\mathcal{S},i\mathbf{R}})\right)\mathcal{H}_\mathcal{S}\varphi(x)\,d\mu(x). \end{aligned} \tag{5.41}$$

This verifies (5.34). Part (5.35) follows from (5.34) and (4.13) by taking the limit if $\rho \to 0$ and dividing by $\sqrt{2}$, because of our normalization (see (4.47)). Also, one has to keep in mind that

$$\lim_{\rho\to 0^+} 1_{\mathbf{R}^+}(\rho t) = 1_{\mathbf{R}^+}(t) - 1_{\mathbf{R}^+}(-t).$$

QED.

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Appendix A

Notation

$\widetilde{G}$	connected double cover of G.
G_1	identity component of G.
$\widehat{G}$	unitary dual of G.
G^H	centralizer of H in G.
$N_G(H)$	normalizer of H in G.
G_A	stabilizer of the elements of A in G.
$\langle A \rangle$	subgroup generated by the elements of A.
$\mathfrak{g} = \mathcal{L}ie(G)$	Lie algebra of G.
$\mathrm{isom}(V, \mathcal{Q}) = \mathrm{aut}_{forms}(\mathcal{Q})$	linear isometries of the bilinear form $\mathcal{Q}$.
$\mathrm{map} = \mathrm{hom}_{set}$	maps.
$\mathrm{map}_G = \mathrm{hom}_{G-set}$	G-equivariant maps.
$\mathrm{hom} = \mathrm{hom}_{lin}$	linear maps.
$Sp(n, \mathbf{R}) = \mathrm{isom}(\mathbf{R}^{2n}, St)$	real standard symplectic group.

$\{a,b\} = ab + ba$	the anticommutator of a and b.
$[a,b] = ab - ba$	the commutator of a and b.
V^G	G-invariant vectors on the G-vector space V.
$V_{\mathbf{C}}$	complexification of the vector space V.
$M_{k\times l}$	matrices with k rows and l columns.
I_n	identity matrix in $M_{n\times n}$.
$\mathcal{S}(X)$	space of Schwartz functions on X.
$\mathcal{S}^*(X)$	space of tempered distributions on X.
$C_c^\infty(X)$	smooth compactly supported functions on X.
$L^1_{loc}(X)$	locally integrable functions on X.
$\prod\limits_S F(s)$	product over the elements s of the set S.
$\sum\limits_S F(s)$	summation over the elements s of the set S.
$\mathcal{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	
$\mathcal{H} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\sigma_1 = \mathcal{I},\ \sigma_2 = -i\mathcal{J},\ \sigma_3 = \mathcal{H}$ Pauli matrices.
$\mathcal{I} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	
$\underline{v} = \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix}$	identification of the Lie algebras $\mathfrak{so}(2)$ and $\mathbf{R}^3$.
$WF(u)$	wave front set of the distribution $u \in \mathcal{D}'(X)$.
$\mathcal{F}(\varphi) = \widehat{\varphi}$	a Fourier transform of φ.

$D(x_1, x_2, x_3)$	shorthand for $\mathrm{diag}(0, x_1\mathcal{J}, x_2\mathcal{J}, x_3\mathcal{J})$.
f^*	pullback by f.
x^τ	adjoint w.r.t. the form τ.
X^T	transpose of the matrix X.
id_X	identity map on X.
1_X	characteristic function of X.
Θ	the Cartan involution $\Theta(X) = -X^T$.
$(W, \langle\ ,\ \rangle)$	a symplectic space over the reals.
$\mathfrak{sp}(W)^+_{\mathbf{C}}$	$\{x + iy \in \mathfrak{sp}(W)_{\mathbf{C}} : \langle y\ ,\ \rangle\vert_{\ker(x)} > 0\}$.
$(\mathfrak{g})_{rk\leq j}$	elements of $\mathfrak{g}$ of rank at most j.
$\mathfrak{g}^r$	set of regular elements of $\mathfrak{g}$.
Ψ	set of positive roots.
$\mathcal{A}$	$\mathcal{A}(A) = \prod_A \frac{\lambda}{\lvert\lambda\rvert}$.
$\mathcal{D}$	$\mathcal{D}(A) = \prod_A \lambda$.
Dih_n	dihedral group of order $2n$.
Symm_n	symmetric group on n letters.
S^n	n dimensional real sphere.
$\lfloor a \rfloor$	greatest integer $\leq a$.
$\#S$	cardinality of the set S.
$\tau_{p,q}$	the bilinear form $\tau_{p,q}(x, y) = \Sigma_{j=1}^p x_j y_j - \Sigma_{l=1}^q x_l y_l$.
$r_{p,q}$	short hand for $\tau_{p,q}(x, x)$.
$\Delta_{p,q}$	the Laplacian operator $\Sigma_{j=1}^p \partial_{x_j}^2 - \Sigma_{l=1}^q \partial_{x_l}^2$.