ORTHOGONAL DESIGNS FOR SCREENING IN INDUSTRIAL EXPERIMENTATION

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1996

Submitted to the Faculty of the
Graduate College of the
Oklahoma State University
in partial fulfillment of
the requirements for
the Degree of
MASTER OF SCIENCE
May, 1998

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PREFACE

This research is about the *design* of experiments, and the combinatorial problems that are intrinsic to the subject. The possibility of being able to make judgmental decisions to control or modify a process, select (or deselect) significant (or trivial) factors based on information gathered from very few runs, in what is commonly referred to as *screening experimentation* is very optimistic, both in its scope and importance. However, the design plans applicable for such experiments, while being simple in concept and creation for symmetrical experiments, pose interesting challenges for the case of asymmetrical experiments, both in attempting a generic tractable solution and remaining true to their intent, i.e., easy to comprehend and use by the not-so-naïve experimenter. To this end, this research attempts to establish and propound a new method for generating orthogonal plans, orthogonality being a necessary attribute of the design plans to maximize confidence in the screening experiment's outcome(s) and subsequent decisions therefrom.

The *method of symmetric constructions* is the outcome of this research effort and contends as a generic solution methodology for the construction of $2^k \bullet s^p$ ($s \geq 2$) orthogonal plans in $s(1+k)$ runs, uses for which are numerous, inasmuch as industrial experimentation is concerned. Also, rules for modifying these design plans, to incorporate any user-specified combination are explored and elaborated. It would only seem fair to admit that this research is incomplete, either in fully exploiting this technique or in achieving the desired objective of completely orthogonal estimates of all higher order factor main effects. The design plans constructed using the method of symmetric constructions allow for orthogonal estimates of all linear effects while the higher order factor effects (for quantitative factors) are slightly correlated with each other. However, the pros and cons of near-orthogonal arrays are not elaborated in this report. Also, generic design templates for use in the constructions of asymmetrical experiments are derived and presented herein.

It is the fond hope of the author that the *method of symmetric constructions* will find applications in other allied fields as well and in this context, the ideas initiated and shared in this report shall be found useful by interested researchers in the years to come.

ACKNOWLEDGMENTS

A lump of clay is transformed from a featureless and shapeless mass to a beautiful figurine, due in large part to the skill of the potter and the many hands that touch it along the way from the wheel to the firing kiln. Though I was not exactly a shapeless mass, the past two years of study have tempered me from a notorious brat to a slightly cynical, but rational and realistic individual by the many giants on whose shoulders I have taken the liberty of leaning on and for which I am forever thankful. And in these two years, I have met, influenced, been influenced, and betrayed by many people, the effect of all of which I see in the mirror everyday as I ponder – "*to shave or not to shave, that's the question.*" Here, in these few lines that follow, I attempt a word of thanks to all these individuals, who directly or indirectly have influenced me during the course of this research.

If I were to murmur cynically that "I owe nothing to no one, something to someone, everything to only one," - that would be Dr. Case. I lack words to shape my thanks for Dr. Case, who drew upon his infinite reserves of patience and wisdom, while tolerating my tantrums, to guide me not only within the bounds of academics, but, life in general during these two years. *As I pondered about all things in wonder, looking into the void yonder; I am only too glad that I had Dr. Case by my side to jack me up when I, oh!, committed the inadvertent blunder.*

I must thank Dr. Pratt for all his tolerance and patience with me during these past years, and I have enjoyed my professional association with him and I hope it will continue thus in the future. I also thank Dr. Kamath, who at all times, was ready to share his mind, thoughts, and his many books with me.

I am very thankful to Dr. Folks, but for embarrassing encounters with whom, I would never have dug so deep into the subject. And, so deep did I dig into the esoteric worlds of Galois Fields and Projective Geometry, the joy of learning which, have left my forehead saturated with teakwood varnish after all the many hours of banging it on the front desk. In fact, the ratio of teakwood varnish to flesh in my forehead is so high that it might as well be classified as furniture, so much so that I have to apply lemon oil polish on my forehead to maintain its luster, every

Sunday at 9ET/10PT. I am very pleased to say that knowing him for the short time that I did has been a privilege and a memorable experience, not to mention that it has taught me how to keep a placid expression, especially, when I am asked something about which I am totally clueless.

I would like to express my sincere gratitude to the Department of Industrial Engineering, but for whose kindness, my shoestring budget through all of graduate school would have been nonexistent. I must add that, knowing Dr. Koelling, both in his capacity as the Department Head and as a well-wisher has been a very memorable experience.

I shall cherish my experiences with Mrs. Patsy Coleman, who was always ready to share her motherly concern and advice with this home-sick, self-cooking, and malnourished kid, albeit sometimes with a little bit of asperity and that typical look that only women seem to be capable of. I thank you for everything, and especially, all the free Xerox copies that I have sneaked through.

I would like to thank and extend my appreciation to my friends, Say-Boon Goh and Gestur Petursson, who stood by me, both in times of sorrow and joy, spending endless hours listening to my incessant babble on Hadamard matrices, Orthogonal arrays and all my marital difficulties.

Last but not the least, and as has been in the traditions of formal protocol, I thank my parents, but for the joyous union of whom, this notorious, resource-consuming progeny of theirs would not have been conceived. Finally, I thank Eswar Sivaraman, but for the expert typing and cynical attitude of whom, this report could not have been created.

Eswar Sivaraman
Stillwater, Oklahoma.

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Chapter 1

SCREENING IN INDUSTRIAL EXPERIMENTATION

For "is" and "is-not" though with rule and line
And "up-and-down" by logic I define,
Of all that one should care to fathom, I
Was never deep in anything but - books and wine.

...Adapted from Omar Khayyam's "Rubaiyat"

1.0 INTRODUCTION

Designed experiments provide organized means for scientifically determining the relationships of inputs to outputs in a given process. A designed experiment involves purposeful changes to the inputs (factors) of a process in order to observe the corresponding changes in the outputs (responses) so as to understand and characterize how the inputs affect the response(s). The general situation considered is one in which there is a response (or, an output) variable which is thought to be dependent on controllable variables or inputs. This type of situation is of quite common occurrence, as may be seen from the following list of examples from widely different fields:

Table 1.1: Some Examples to Illustrate Relevance of Designed Experimentation

RESPONSE	INPUTS
1. Taste of Coffee	%Fat in milk, # of spoons of sugar, # of spoons of coffee powder used, etc.
2. Quality of an alloy - Hardness, Strength	Amounts of different metals, rate of cooling, temperature of alloying element addition, etc.
3. Torque applied to meet design specification for pre-load forces in high strength fasteners – from Bingham (1997).	• Turning surface (Nut/Head) • Lubricant (Dry/Lube) • Lube type (Aliphatic acid/Anti-Seize) • Sealant (Yes/No) • Sealant Viscosity (Thin/Thick) • Rotation (30/600 RPM) • Hardness of joint (Soft/Hard) • Nut plating (Cadmium/Dry film) • Finish (structure being fastened) - White Lacquer /Prime Aluminum • Washer (Bare/Cadmium plated) • Re-apply torque (Yes/No) • Bolt material (Titanium/CRES) • Nut geometry - Thin (hex)/Thick (12-point)

This list, which could be extended indefinitely, was intended to demonstrate the range of situations that have the same essential structure.

When an experiment involves several input variables, the effect of all such variables on a characteristic of interest, namely the response, may be investigated simultaneously by varying each factor (input variable), so that all or a suitable subset of all possible combinations of the input variables are considered for experimentation. An experiment in which this procedure is used is commonly known as a ***Factorial Experiment***. An experiment that involves all the possible treatment combinations is called a ***Full Factorial Experiment***.

It is frequently the situation, wherein several factors may be considered relevant, rather, significant in 'explaining' the response. It is common knowledge that brainstorming within a team in advance of any scientific study brings up a plethora of judiciously relevant factors (Table 1.1, example 3). In such situations, the scalpel of experience and maturity of judgement is used to further whittle down the list of factors to a pool of genuinely relevant factors that cannot be dismissed using rules of thumb or by word of experience alone. It is in such cases that designed experimentation bears tremendous relevance in providing an 'organized' method of study and analysis as opposed to 'hit-and-run-shop-floor-trial' methods which may seem attractive in prospect, but in reality, are of no use.

However, when the cost of experimentation is prohibitive, and economy of resources is preferred, an experimenter cannot afford[1] to examine all possible treatment combinations in detail, either at leisure or at pleasure. This idea is better brought home with an example. In the Torque/Pre-Load experiment (refer Table 1.1, example 3), where each 'relevant' factor is stated with 2 possible options, the total number of different experiments possible is 8192 (2^{13}). If it takes a conservative 1-minute to apply pre-load torque on one fastener joint, the total time taken for all experiments would be 5.69 days, working non-stop, the futility of which deserves no further elaboration. In such situations, experimenters take recourse in performing ***Screening Experiments***.

Screening experiments are intended for what their name implies, i.e., *screen* the set of 'known' factors to find out which factor(s) is truly significant in explaining the response and which others are just 'in-for the ride'. Thus, the purpose of the screening experiment is not so much to better the process as to determine which factors are essential for making improvements.

[1] 'Affording' may be interpreted in monetary terms, although, it is frequently the case, that time is the actual constraint.

Screening experiments allow the testing of many factors for their influence on the major process response(s); while the estimation of factor effects may often be imprecise, they allow the experimenter to identify the truly important factors from the many possible choices. A screening experiment usually helps conclude that only a small number of factors (usually 2-5) are truly significant in affecting the average response, the detailed analysis of which is carried out by following up with more refined experiments to completely model the effects of the 'selected' factors on the response(s). Also, the well designed screening experiment weeds out all factors affecting just the variance in the response, allowing the experimenter to set them at levels that will minimize the variation in subsequent experiments or process performance in general.

It is only appropriate that a formal definition be stated for a screening experiment:

> ***"A Screening experiment is an educated start towards understanding a process, supported with a lot of intuition and blind spots, a blend of facts noted and facts ignored, to expedite the identification of the 'vital few', i.e., the truly important factors affecting the desired response(s) from the many possible choices."***
>
> ....[Adapted from Crichton (1969)]

Peace (1993) summarizes the objectives of a screening experiment as being both short-term and long-term. The short-term objective not only is complementary to the latter, but also paves the way for its success. The long-term goal is to reduce process variability by optimizing the significant process variables. To achieve this, the short-term intent is to identify which factors should be optimized.

1.1 DESIGNED EXPERIMENTS - TERMINOLOGY AND SOME PRELIMINARIES

In order to avoid confusion and ambiguity, a list of terms relevant to designed experimentation, which will be used throughout this report, is presented below. It is hoped that the terseness of introduction and explanation would deem adequate. For further elaboration, readers are referred to Kempthorne (1952), Schmidt and Launsby (1994) or Anderson and McLean (1974), all of which are excellent references for the subject.

1.1.1 A **Factor** is a particular 'force' that is varied in the experiment at the will and under the control of the experimenter. A factor may sometimes be called an input or a controllable variable, but all references within this report will be restricted to 'factor(s)'. A factor may be *qualitative* or *quantitative*. A *quantitative factor* is one whose values can be measured on a numerical scale, e.g., amount of sugar, temperature, pressure, speed of rotation, etc. A *qualitative factor* is one whose values are not usually arranged in order of magnitude, e.g., Supplier A, B, C; the Good, Bad and Ugly coke varieties (for cast iron production); Yes/No, etc. The values of a qualitative factor cannot usually be measured on a numerical scale.

1.1.2 **Levels** are the various values at which a factor is tested, e.g., 3-coke varieties - G/B/U; temperature for degassing molten aluminum - 720/680/640 °C, etc.

1.1.3 A **Treatment Combination** is one of the possible combinations of levels of all factors under investigation.

1.1.4 An **Experimental Unit** is that entity on which a treatment combination is applied, e.g., a cup of coffee prepared with 5 spoons of sugar, 3 spoons of coffee powder, and 8% fat milk is an experimental unit.

1.1.5 A **Trial** or **Run** is the application of one treatment combination to one experimental unit. The terms, treatment combination, run and trial are however, used interchangeably in this report.

1.1.6 A **Plan** or **Design** is a *specified* set of treatment combinations.

1.1.7 A **Symmetrical Factorial Experiment** involves experimentation with factors each having the same number of levels. A factorial experiment in which at least one of the factors has its number of levels different from those of the other factors is called an **Asymmetrical** or a **Mixed Factorial** or a **Mixed Model Experiment**.

1.1.8 The **Effect** of a factor as has been alluded to earlier in the report, refers to the quantifiable change in the output response caused by a unit change in the quantity of the factor concerned, keeping all other factors and conditions constant. To better illustrate this, it is said that the effect of sugar is to 'sweeten' coffee. For example, if it can be better phrased that the 'effect' of a spoon of sugar is to increase the taste of coffee by 2 units, where the response (taste of coffee) is rated on a scale of 1-10, then the 'main-effect', of sugar is +2 units. The effects of factors are classified as main-effects or interactions, the explanation of which follow from the next definition.

1.1.9 The concept of **Degrees of Freedom** bears great relevance to the idea of a factor's effect as introduced above. A crude but intuitive explanation will be attempted here. For further details, readers are referred to Kempthorne (1952) or Anderson and McLean (1974).

A single factor at **N** levels can be tested once at all possible levels in **N** runs; let the output responses be termed Y_1, Y_2, ..., Y_N respectively. The input (only one) factor is termed X. Thus, with the N output values, a polynomial can be fitted to explain the data as shown below:

$$\mathbf{Y_1 = A_0 + A_1{*}X + A_2{*}X^2 + \ldots + A_{N-1}{*}X^{N-1}}$$

$$Y_2 = A_0 + A_1*X + A_2*X^2 + \ldots + A_{N-1}*X^{N-1}$$

.

.

.

$$Y_N = A_0 + A_1*X + A_2*X^2 + \ldots + A_{N-1}*X^{N-1}$$

It is easily noted that the order of the polynomial is N-1, which in statistical terms is referred to as N-1 degrees of freedom for the factor X. This means that, a factor at N levels has N-1 degrees of freedom, which in the vocabulary of designed experimentation means that a factor at N levels has N-1 effects. For e.g., a factor at 2 levels has 1 degree of freedom, i.e., one effect; likewise, a factor at 3 levels has 2 effects, and so on.

A factor at 2 levels (i.e., with one effect) is also referred to as having a 'linear' main effect, which is intuitively obvious if one stops and verifies that only a straight line can be plotted between 2 points (the responses for the 2 levels of the factor). Similarly, a factor at 3 levels is explained with a linear (e.g., A, B) and a quadratic (e.g., A^2, B^2) main effect. The term 'main-effect' refers to the effect attributable to the factor and that factor alone. When two or more factors interact in their effect on the response, an 'interaction' effect (e.g., AB, ABC, AB^2C, etc.) is defined.

An intuitive generalization of the concept of degrees of freedom leads to the useful idea, that for a full factorial experiment of 'n' factors at 2-levels each, all treatment combinations being experimented with in 2^n runs, the number of effects that can be estimated is 2^n-1. This is explained with an illustration as below:

Consider three 2-level factors, called A, B, and C.
Total # of treatment combinations = $2^n = 2^3 = 8$
Thus, total # of effects that can be estimated = $2^3 - 1 = 8 - 1 = 7$
The 7 effects that are estimated are:

- Linear 'main' effects : A, B, C
- Interaction effects : AB, AC, BC, ABC

1.1.10 Two factors (or input variables) are said to **interact** if one factor's effect on the response is dependent upon the level of the other. Examples drawn from the physical world include the case of alcohol and drugs, which when taken together compound in their disastrous effects than when taken alone. Consider the case of nitric acid and glycerin. Taken alone, they are just two chemical compounds, of interest only to the curious scientist. But when brought together, the term dynamite (Follet, 1982) needs no further elaboration. This is a classic example of an interaction effect. Similarly, the synergy exhibited by teams, pulling together is an example of an additive interaction effect.

The above definitions were intended to be a crude, but intuitive attempt at explaining some basic, but, nevertheless, very important concepts of designed experimentation. They are however, not intended, to replace a thorough and rigorous treatment, fuller understanding of which is highly pertinent before the study of designed experimentation can be attempted.

1.2 SOME ASPECTS OF FRACTIONAL REPLICATION

When a factorial experiment involves many factors, each of which is tested at several levels, economy of time and material may be attained by using only a fraction from all possible combinations of levels of the factors. Such a fraction may result in a loss of information on some interactions, but, if chosen properly, will allow the estimation of at least the main effects of all factors concerned. Following definitions stated earlier in Sections 1.0 and 1.1, it is obvious that such fractions constitute design plans for screening experiments. The technique for reducing the number of observations, by sacrificing information on selected interactions, is known as fractional replication.

Fractional replication is a natural outgrowth of the device of confounding, by which a complete replicate (full factorial design) is divided into several equally sized blocks. The interested reader is referred to Kempthorne (1952), Anderson and McLean (1974) or Fisher (1942) for an excellent treatment of the same. The higher the degree of fractionation, the greater is the number of interactions on which information is sacrificed. For the practical experimenter, who attempts screening trials, the pre-supposition is that interactions, if any, are negligible and are not considered relevant in the preliminary stages of scientific study, of which screening is such an important aspect. The general case of fractional replication deals with a $1/s^r$ replicate of the s^n experiment (full factorial plan for 'n' factors at s-levels each), in s^{n-r} runs where s is a prime or the power of a prime.

Screening experiments are undertaken using fractionated plans under the assumption that the interactions that have been confounded to create the plan are negligible. Often, industrial applications encounter usage of designs for screening trials under the assumption that two-way and higher order interactions (e.g., AB, ABC, AB^2C, etc.) are negligible. The primary objective of such screening experiments is to get an estimate of the main effects of all the factors being experimented with, so as to make judgmental inferences about which factors are truly significant.

1.3 FRACTIONAL FACTORIALS AND MAIN EFFECT PLANS FOR SCREENING

Experimental plans that allow the estimation of all main effects of a factorial experiment shall henceforth be referred to as main-effect plans. These plans may be particularly useful in preliminary studies on many factors when there is good reason to believe or assume that interactions among the factors are small, as is necessitated in screening experiments. When the cost of making an observation in a factorial experiment necessitates the use of fractional factorials, an important aspect of the design problem, when conducting screening trials, is to obtain reliable estimates of the important main effects with as few observations as possible.

It has been proved (Plackett and Burman, 1946) that experimental plans for which the maximum precision of estimation is attained are those which correspond to columns of an orthogonal matrix. Such plans that allow the estimation of all main-effects without correlation are termed orthogonal main effect plans. The extension of usage of orthogonal main effect plans for screening is straightforward and obvious.

The existing knowledge of orthogonal main effect plans is considerable but not exhaustive. The plans that are now available to the experimenter for screening designs are the standard Taguchi, Plackett-Burman type of designs (Schmidt and Launsby (1946); Plackett and Burman, 1946) relevant mainly for factors at two levels. For the general case of symmetrical factorial experiments, the construction of confounded plans using Galois field theory has been well elaborated in Kempthorne (1952) and Fisher (1942), extraordinary developments in which were presented by Addelman (1961, 1962a, 1962b). For all practical purposes, industrial experimenters rely on available catalogued designs (Connor and Zelen, 1959; Connor and Young, 1961; Lorenzen and Anderson, 1993; National Bureau of Standards - AMS #48, 1957) for symmetrical and asymmetrical factorials or consult software programs like SAS, RS/Discover for the construction of orthogonal or near orthogonal fractions to facilitate screening trails. A number of methods have been designed to generate needed orthogonal main-effect plans for the construction of symmetrical and asymmetrical fractions, a comprehensive review of which appear in Lorenzen and Anderson (1993), Cheng (1989), Raktoe *et al.* (1981), Dey (1985), and more recently, in Barton (1998), Bingham (1997), Meyer and Nachtsheim (1995), Wang and Wu (1991, 1992).

1.4 OVERVIEW OF RESEARCH

The major work on fractional factorial designs can be broadly classified into the following sub-topics: (Raktoe *et al.*, 1981; Dey, 1985)

(i) Study of orthogonal fractional factorial plans for symmetrical and asymmetrical factorials of resolution III, IV and V (Dey, 1985).

(ii) Study of optimality and construction of non-orthogonal fractions, with special emphasis on 2- and 3-level symmetrical factorials (Fedorov, 1972; Kiefer, (1959, 1974); Atkinson and Donev, 1992; Liao *et al.*, 1996; Meyer *et al.*, 1995).

(iii) Search models and search designs (Srivastava, 1975, 1976, 1977, 1980; Raktoe, 1981).

The objective of this research report is rather modest in comparison with some of the extraordinary developments in the sub-topics above, in the sense that it is relevant to only sub-topic (i) involving orthogonal asymmetrical factorial designs of resolution III.

1.5 RESEARCH OBJECTIVE

A new and unifying methodology for building mixed model orthogonal design plans incorporating any user-desired factor combination(s) in minimum number of runs will be developed. Design plans constructed thus are intended primarily for use in screening experiments. Relevant sub-objectives pertinent to this research effort are presented below.

1.5.1 Review existing methods for construction of Orthogonal Main Effect Plans (OMEPs). Specific tasks include:

- (i) Identify existing criteria for evaluation of design plans as may be relevant for screening experiments.
- (ii) Identify and track in chronological sequence, relevant developments in the construction of orthogonal design plans useful for designed experimentation.
- (iii) Present an index of existing OMEPs.

1.5.2 Develop a method for the construction of Orthogonal Linear Effect Plans[2] (OLEPs) applicable for any factor combination(s) in a minimum number of runs.
Specific tasks include:

- (i) Define vocabulary relevant to proposed *method of symmetric constructions* (MSC) and elaborate for ease of discussion and generality.

[2] Design plans that allow uncorrelated estimates of all linear effects, while higher order factor main effects are correlated with one another.

(ii) Present general propositions and suitable design templates relevant to the construction of OLEPs using MSC.

(iii) Present and detail a systematic walkthrough of MSC.

(iv) Catalogue and tabulate some useful design plans to highlight uses and advantages of MSC.

1.5.3 Present valid rules for replacing and/or collapsing of factor levels, to modify OLEPs to include higher order factors and also to allow for estimation of higher order factor main effects.

Specific tasks include:

(i) Define generic rules to modify lower level factors into higher level factors and vice-versa.

(ii) Describe techniques to adapt OLEPs constructed using the method of symmetric constructions to include any user-specified factor combination(s) and illustrate with suitable examples.

(iii) Present an elaborate index of modified OLEPs that may be used as screening designs involving at most 9 level factor(s) with guidelines for usage and further manipulations.

1.6 PLAN OF THIS REPORT

Chapter 2 is a review of all the existing techniques for the construction of orthogonal fractions as is relevant to this research effort. Chapter 3 includes brief discussions on number theory concepts and its relevance for designed experimentation. Chapter 4 is an exposition on *The Method of Symmetric Constructions*, its uses and extensions. Chapter 5 highlights applications and examples intended to augment the scope of Chapter 4. Chapter 6 concludes this report inasmuch as the scope of this undertaking is concerned, but includes recommendations for future research to extend the ideas introduced in Chapters 4 and 5.

Chapter 2

REVIEW OF THE LITERATURE

Ashes to Ashes, Dust to Dust,
If Death is all that is left for us,
I ask, then why all this fuss?

...Eswar

2.0 INTRODUCTION

It is expected that readers are aware of the basic principles underlying confounding and fractional replication, for much of what is discussed in the next section is based on these principles.

2.1 THE HISTORY OF ORTHOGONAL FRACTIONAL FACTORIALS

The literature concerning fractional replication is a direct extension of the work on confounding of factorial experiments. Confounded plans were originally suggested by Fisher (1926), practical implications of which were detailed by Yates (1933), including discussions on appropriate methods of analysis. Yates (1935) gave more illustrations of confounded plans and included discussions on the advantages of reducing block sizes using confounded plans. A very elegant treatment of the same concept appears in Kempthorne (1952), whose book is, thus far, the best and most concise treatment on the subject of designed experimentation.

Barnard (1936) made an enumeration of the confounded arrangements that are possible in a 2^n factorial experiment, wherein, he showed how the concept of generalized interaction may be used to construct fractionated plans. Yates (1933) expounded the importance of orthogonality in factorial experiments and included a detailed discussion of its practical implications. Since the number of treatments to be tried increases rapidly with the number of factors, the important concept of fractional replication, (i.e., trying only a subset of the treatments), whereby only one block of a confounded plan is considered, was proposed by Finney (1945). As a direct offshoot of this idea, Plackett and Burman (1946) introduced a class of plans, called multi-factorial plans, which accommodated a maximum number of factors and preserved only the main effects for *symmetrical factorial* experiments. This appears to be the earliest reference on the use of confounded plans for *industrial screening*. These multi-factorial plans actually constitute a class of orthogonal main effect plans (henceforth referred to as OMEPs) for symmetrical factorial experiments. They included a catalogue of OMEPs for symmetrical factorial experiments involving factors at two, three, five, or seven levels. These plans were based on Hadamard matrices, a comprehensive discussion of which is presented in Hall (1967), and Hedayat and Willis (1978). Paley (1933) first formulized the construction of Hadamard matrices, and derived several lemmas to allow for the existence of Hadamard matrices. Paley also derived the important result that the necessary condition for the existence of a Hadamard matrix of order N is that N be a multiple of four, N=1, 2 being trivial cases.

2.2 OMEPs DERIVABLE FROM HADAMARD MATRICES

The use of Hadamard matrices for the construction of OMEPs for asymmetrical factorial experiments has received considerable attention and has spawned several new approaches. Dey and Ramakrishna (1977) introduced a result for the construction of main effect plans for $4 \bullet 2^{2n-4}$ factor combinations in $2n$ runs, where n is a multiple of four such that a Hadamard matrix of order n exists. Chacko, Dey and Ramakrishna (1979) derived further extensions of the same idea to construct main effect plans for $4^3 \bullet 2^m$ experiments. They obtained a series of plans for $4^3 \bullet 2^{4n-10}$ experiments in $4n$ runs, where 'n' is a multiple of 4. Agrawal and Dey (1982) modified the plans for $4^3 \bullet 2^m$ in 4n runs to obtain OMEPs for $n \bullet 4^r \bullet 3^s \bullet 2^{3n-3(r+s)}$ experiments in $4n$ runs, where r and s are non-negative integers, $2 \leq r+s \leq 3$, $(r, s) \neq (0,0)$. Agrawal and Dey (1982) also derived another series of plans, using Hadamard matrices for $t \bullet 4 \bullet 2^{n-1}$ experiments in $2n$ runs (where n is a multiple of four). Also, Nigam and Gupta (1984), Cheng (1989) have derived several new classes of OMEPs for asymmetrical factorial experiments using Hadamard matrices, which in the interest of space limitations are not mentioned herein. The interested reader is referred to Raktoe *et al.* (1981), Dey (1985), and Raghavarao (1971) for a concise treatment of the subject of orthogonal fractional factorial designs.

2.3 OMEPs DERIVABLE THROUGH FINITE GEOMETRIES AND GROUP THEORY

The foundation of the general theory of confounded s^n factorial designs was developed by Bose and Kishen (1940), where s is a prime power (i.e., a prime number or a power of a prime number) through the use of Galois fields and related finite projective geometries. The interested reader is referred to Carmichael (1937), Stahl (1997) for an excellent introduction to the subject of group theory and allied concepts of abstract algebra. Bose (1947), in his epic paper, *"Mathematical Theory of the Symmetrical Factorial Design,"* definitively formalized the geometric foundations of symmetrical factorial designs employing the theory of finite projective geometry. Vajda (1967a, 1967b) has written two elegant monographs, giving a very comprehensive treatment of the mathematical foundations of experimental design.

Fisher (1942) developed a system of confounding for factors, each having two levels, whereby no main effects or two factor interactions were confounded with blocks using group theory. This system of confounding permits the estimation of all main effects when up to 2^n-1 factors, each at two levels are experimented with, in 2^n trials. Fisher (1945) further extended this concept to a generalized system of confounding to allow for the arrangement of $(s^n-1)/(s-1)$ factors, each at s levels, in s^n trials, where s is a prime power, without confounding any main effects. A large class of designs, popularized by Taguchi (see for e.g., Schmidt, 1994; Peace, 1993), which is in use for screening 2-level and 3-level factors or combinations thereof are based on either Fisher's principle of confounding or Hadamard matrices, which were discussed earlier. Kempthorne (1952) has made a great simplification of the underlying concepts for representing effects, interactions, confounding and the analysis of the general s^n factorial system. An index of useful plans that may be constructed using Fisher's principle of confounding is presented in Table 2.1.

Table 2.1: Index of Some OMEPs constructed using Fisher's Principle of Confounding

Number of Levels	Number of Factors	Number of Runs
2	3	4
2	7	8
2	15	16
2	31	32
2	63	64
3	4	9
3	13	27
3	40	81
4	5	16
4	21	64
5	6	25
7	8	49
8	9	64
9	10	81

Addelman and Kempthorne (1961) developed a method to augment a s^m $(m=(s^n-1)/(s-1))$ OMEP with s^n runs to generate a plan for a s^t OMEP in $2s^n$ runs, where $t = [2(s^n-1)/(s-1) - 1]$, in what may be viewed as a very ingenious extension of Fisher's theory of confounding. Some plans constructed through this procedure are an 18-run plan for a 3^7 experiment, a 54-run plan for 3^{25}, a 32-run plan for 4^9, and a 50-run plan for a 5^{11} experiment.

2.4 CONSTRUCTION OF SYMMETRICAL OMEPs BASED ON LATIN SQUARES

It is difficult to trace the origins of the use of Latin squares for constructing orthogonal plans. Yates (1933) details a few Latin square arrangements for field trials. Yates (1935) also gives some results on the efficiencies of complete randomization relative to randomized blocks (blocks confounded with some higher order interaction) and Latin Squares for field experiments.

A Latin square of side s (also termed LS of order s) is an arrangement of s symbols in s rows and s columns such that each symbol occurs in each row and each column only once. Two Latin squares of the same order are said to be orthogonal, if, when one is superimposed over the other, every ordered pair of symbols appears precisely once. A set of Latin squares is said to be a set of *mutually orthogonal Latin squares* (MOLS) if every pair of Latin squares in the set is orthogonal. It is known that the maximum number of MOLS for a Latin square of order s is $s-1$, when s is a prime power. Interested readers are referred to Raghavarao (1971). The concept of MOLS has been fully exploited in the construction of OMEPs for $s+1$ factors in s^2 runs, each factor occurring at s levels (s being a prime power). An index of OMEPs obtained through Latin squares is presented in Table 2.2.

Table 2.2: Index of Useful OMEPs Obtained through Latin Squares

Number of Levels	Number of Factors	Number of Runs
2	3	4
3	4	9
4	5	16
5	6	25
7	8	49
8	9	64
9	10	81

2.5 CONSTRUCTION OF OMEPs BASED ON ORTHOGONAL ARRAYS

Fractional factorial plans for symmetrical factorials are closely connected with *Orthogonal Arrays,* a modern convolution of the concept of *Hypercubes*. Rao (1946) introduced the concept of Hypercubes of strength d; since this concept is relevant for the construction of confounded plans, the following definition is presented. Let there be n factors, each of which may take on s values. Consider a subset of s^m factor combinations (out of a total of s^n possible combinations). This subset is called a hypercube of strength d and represented by (s^m, n, s, d) if all combinations of any d of the n factors occur an equal number of times ($=s^{m-d}$). The construction of hypercubes and relevant theories are based on concepts of projective geometry, the intricate details of which are not relevant to this discussion.

Rao (1946) showed that (i) a system of confounded plans that accommodated a maximum number of factors and preserved main effects and up to d-factor interactions could be constructed for the symmetrical factorial experiment if a hypercube of strength d existed, and (ii) hypercubes of strength two supplied confounded plans for some asymmetrical factorial experiments. Rao (1947) extended the definition of a hypercube of strength d to an orthogonal array of strength d. An orthogonal array of strength d consists of a subset of N treatment combinations from an s^n factorial experiment with the property that all s^d treatment combinations corresponding to any d factors chosen from n occur an equal number of times in the subset. It is useful to note that when N is of the form s^m, the orthogonal array is a hypercube of strength d. Rao (1947) noted that an orthogonal array of strength two could be used as an OMEP for a symmetrical factorial experiment. Rao (1947) utilized orthogonal arrays of strength d to construct (i) Multifactorial plans similar to those of the Plackett-Burman type, but leading to the estimation of main effects and up to d-factor interactions when higher order interactions are absent, (ii) block designs for symmetrical factorial experiments involving only a subset of the treatment combinations and preserving main effects and interactions up to a given order when higher order interactions are assumed to be absent, and (iii) a series of asymmetrical factorial plans derivable from arrays of strength two.

An index of useful plans that can be constructed by utilizing hypercubes of strength d is presented in Table 2.3.

Table 2.3: Confounded Plans for Symmetrical Factorial Experiments (from Rao, 1947)

Levels of a factor	Number of Runs	Strength, d	Maximum number of factors attainable
s (prime power)	s^m	2	$(s^n-1)/(s-1)$
2	2^m 2^4 2^5 2^6	3 4 4 4	2^{m-1} 5 6 8
3	3^3 3^4 3^4	3 3 4	4 10 5
4	4^3	3	6
5	5^3	3	6

Bose and Bush (1952) gave another series of orthogonal arrays of strength two (or equivalently, OMEPs). Suppose λ and s are both powers of the same prime, p. Then, it was shown by Bose and Bush (1952) that an orthogonal array of type ($N= \lambda s^2$, $n=\lambda s$, s, d=2) can be constructed. A plan constructed thus is a (27, 9, 3, 2), i.e., 3^9 OMEP in 27 runs. Bose (1947) formulated methods for attacking the problem of balancing and partial confounding for a class of symmetrical factorial experiments. By employing the theory of finite projective geometry, Bose also constructed confounded symmetrical plans which preserved all main effects and up to d-factor interactions when higher order interactions were absent.

Chakravarti (1956) considered the construction of an asymmetrical fraction by combining two or more corresponding symmetrical fractions, thus enabling the estimation of all main effects, and interactions among the factors. For instance, suppose an OMEP is desired for a $3^4 \bullet 2^7$ experiment. A plan may be derived by combining a 9-run OMEP for a 3^4 plan and a 8-run 2^7 OMEP to produce a 72-run plan for $3^4 \bullet 2^7$, which however, is far from being saturated or being economical.

2.6 OMEPs BASED ON CONDITION OF PROPORTIONAL FREQUENCIES

In a complete full factorial experiment, the levels of one factor appear equally often with each of the levels of any other factor, and, this condition is sufficient to provide uncorrelated estimates of the main effects. However, for OMEPs, the condition of equal frequencies, though sufficient, is not a necessary one. Plackett (1946) introduced the idea of proportional frequencies of levels and showed that the estimates of the main effects of a factorial experiment may be determined with maximum precision if the levels of any factor occur together in the plan with each of the levels of every other factor with *proportional frequencies*. Addelman (1961) consolidated the use of proportional frequencies in the construction of asymmetrical factorial plans. The idea of proportional frequencies may be formally stated as follows.

Let the two factors be A and B with r and s levels respectively. Suppose $n_{i\bullet}$ denotes the number of times the i^{th} level of A occurs in the plan, $n_{\bullet j}$, the number of times the j^{th} level of B occurs in the plan, and n_{ij}, the number of times the i^{th} level of A occurs with the j^{th} level of B in the plan, and n the number of runs in the plan. Then the proportional frequency condition may be stated as:

$$n_{ij} = n_{i\bullet}.n_{\bullet j}/n, \qquad i = 0, 1, 2, \ldots, r-1; \; j = 0, 1, 2, \ldots, s-1.$$

The proportional frequency condition helps primarily in obtaining OMEPs for an experiment with fewer numbers of levels from a plan with the next higher number of levels. Thus, from the OMEP for a 4^5 experiment in 16 runs, collapsing the levels of the 4-level factor to a 3-level factor will derive a 3^5 experiment in 16 runs by a many-to-one correspondence scheme, as shown below.

Levels of 4-level factor		Levels of 3-level factor
0		0
1		1
2	$\longrightarrow$	2
3		1

The designs constructed, thus, are also orthogonal, as they are based on the condition of proportional frequencies. Addelman (1962a, 1962b) first derived these results, and further applications for $2^n \bullet 3^m$ designs permitting estimation of two-factor interactions were illustrated by Margolin (1969). Addelman (1962a) also introduced the highly useful system of *replacement* and *collapsing* whereby a factor at $s=s_i^u$ levels may be collapsed into $(s-1)/(s_i-1)$ factors, each at s_i levels with s_I^u runs and vice-versa, an illustration of which is shown below:

Levels of 4-level factor		Levels of 2-level factor
0		000
1	$\xrightarrow{\text{Collapse}}$	011
2	$\xleftarrow[\text{Replace}]{}$	101
3		110

The above principle of collapsing (and equivalently replacement, when the opposite is done) has been used to obtain OMEPs for a $s_1^{t_1} \bullet s_2^{t_2} \bullet \ldots s_k^{t_k}$ factorial in s_1^n runs, where s_1 is a prime or a prime power, $s_1 > s_2 > s_3 \ldots > s_k$ and

$$\sum_{i=1}^{k} t_i < \left[\left(s_1^n - 1\right) / \left(s_1 - 1\right)\right]$$

This procedure is due to Addelman (1962a), wherein, both replacement and collapsing are used in such constructions. An example design built using this technique is a $2^3 \bullet 3^2 \bullet 4^2$ experiment in 16 runs, from a basic 4^5 experiment in 16 runs. The construction is detailed below:

- Collapse one 4-level factor into three 2-level factors using the correspondence:

0		000
1		011
2	$\longrightarrow$	101
3		110

- Collapse two 4-level factors into two 3-level factors using the correspondence:

$$\begin{array}{ccc} 0 & & 0 \\ 1 & \longrightarrow & 1 \\ 2 & & 2 \\ 3 & & 1 \end{array}$$

Note that this plan is based on proportional frequencies. Also, doubling the number of runs and the number of levels of one factor in an OMEP leads to plans of the type $t \bullet s^{m}$. These results were also derived by Addelman (1962a).

A trick to be used when mixed level designs are required and a few fractional interactions need to be estimated is to alter orthogonal main effect plans by combining the main effects having the product of the levels of the original main effects. For example, the AB interaction for two level factors A and B can be estimated by combining A and B into a four level factor, say C, to be used in an orthogonal main effect design. When data are collected, the 3 degrees of freedom for C must be broken down into the two main effect degrees of freedom (df) and the interaction df. Elaborate discussion of these ideas is presented in Lorenzen and Anderson (1993), a practical application of which is given by Bingham (1997).

2.7 OTHER USEFUL CONTRIBUTIONS

A very simple treatment of the subject of fractional factorials is presented by Box and Hunter (1961a, b), Youden (1961), and Fry (1961). The most important contribution by Box and Hunter (1961a, 1961b) was the idea of *Resolution* of a design plan. A fractional factorial design is said to be of Resolution R, if the smallest interaction in the identity group (same as the confounded block) is an R-factor interaction. As a consequence of this definition, in a Resolution R design, no p-factor interaction is aliased (i.e. confounded) with any other effect containing less than (R-p) factors. For instance, a Resolution III design is one in which no main effect is aliased with any other main effect, but main effects could be aliased with two-factor interactions. In this context, it is easy to note that OMEPs are designs of Resolution III, for they allow the estimation of main effects under the assumption that two-factor and higher order interactions are negligible. Also, these are designs that are very conducive for use in screening trials as part or preliminary experimentation.

Webb (1968) offered a generalization for the definition of Resolution. According to Webb, a fractional factorial is of Resolution (2R+1) if it permits the estimation of all effects up to R-factor interactions, when all effects involving (R+1) factors are assumed negligible. Further, a fractional factorial design is of Resolution 2R if it permits the estimation of all effects up to (R-1) factor interactions when all interactions involving (R+1) factors or more are assumed to be zero. Thus, a plan is of Resolution III, if it permits the estimation of all main effects under the assumption that all interactions are absent. Likewise, a Resolution IV design is one which permits the estimation of all main effects when all three-factor and higher order interactions are assumed negligible.

Lin (1986, 1987a, 1987b) introduced a novel procedure for the construction of mixed factorial experiments using the *Chinese Remainder Theorem*, which result in design plans similar to many of those discussed above.

Wang and Wu (1991) derived an approach for the construction of Orthogonal Arrays extending ideas proposed by Bose and Bush (1952), Addelman (1961a), Hadamard matrices and Kronecker sums, the elaboration of which is too detailed to be included here. Lorenzen and Anderson (1993) have catalogued an extensive listing of OMEPs built using the methods of Wang and Wu (1991), and all the techniques discussed above, for experiment combinations involving up to factors at six levels.

2.8 SUMMARY OF EXISTING OMEPs

An elaborate index of all available OMEPs for asymmetrical and symmetrical Factorials is presented in Lorenzen and Anderson (1993), and Dey (1985), with adequate instructions concerning the judicious use of the same for applications in screening experiments.

In Table 2.4, an index of all available orthogonal main effect plans for symmetrical factorial experiments requiring at most 81 runs is presented. All these plans are derivable from the general techniques described in this chapter. Plans for experiments with less number of factors than those given in Table 2.4 can be obtained by deleting an appropriate number of factors from a plan with more number of factors.

The task of presenting a catalogue of all possible orthogonal main effect plans for asymmetrical factorials, even with a fixed maximum number of runs, is enormous and would require too much of space. Instead, therefore, in Table 2.5, an index of hitherto known, basic asymmetrical orthogonal main effect plans requiring at most 50 runs is presented. Other plans may be derived by collapsing/replacing the factor(s) levels in these basic plans.

Table 2.4: Index of Orthogonal Main Effect Plans for Symmetrical Factorial Experiments

Experiment	# of Runs	Experiment	# of Runs	Experiment	# of Runs
2^3	4	2^{55}	56	4^6	25
2^7	8	2^{59}	60	4^9	32
2^{11}	12	2^{63}	64	4^{11}	50
2^{15}	16	2^{67}	68	5^6	25
2^{19}	20	2^{71}	72	5^8	49
2^{23}	24	2^{75}	76	5^{11}	50
2^{27}	28	2^{79}	80	6^8	49
2^{31}	32	3^4	9	7^8	49
2^{35}	36	3^7	16	7^{13}	81
2^{39}	40	3^{13}	27	8^9	64
2^{43}	44	3^{25}	54	9^{10}	81
2^{47}	48	3^{40}	81		
2^{51}	52	4^5	16		

Table 2.5: Index of Orthogonal Main Effect Plans for Asymmetrical Factorial Experiments

Experiment	# of Runs	Experiment	# of Runs	Experiment	# of Runs
$4 \bullet 2^4$	8	$8 \bullet 2^{24}$	32	$4^2 \bullet 2^{41}$	48
$3 \bullet 2^4$	12	$8 \bullet 4 \bullet 2^{21}$	32	$8 \bullet 2^{40}$	48
$6 \bullet 2^2$	12	$3^7 \bullet 2^{16}$	32	$6 \bullet 4 \bullet 2^{35}$	48
$4 \bullet 2^{12}$	16	$8 \bullet 4^2 \bullet 2^{18}$	32	$8 \bullet 6 \bullet 2^{31}$	48
$8 \bullet 2^8$	16	$8 \bullet 4^3 \bullet 2^{15}$	32	$4^8 \bullet 2^{23}$	48
$2 \bullet 3^7$	18	$8 \bullet 4^8$	32	$6 \bullet 4^4 \bullet 2^{26}$	48
$6 \bullet 3^6$	18	$3^{12} \bullet 2^{11}$	36	$4^{10} \bullet 2^{17}$	48
$6 \bullet 2^{14}$	24	$6 \bullet 3^{12} \bullet 2^2$	36	$6 \bullet 4^{11} \bullet 2^5$	48
$4 \bullet 3 \bullet 2^{13}$	24	$4 \bullet 3^{13}$	36	$4^{12} \bullet 3 \bullet 2^4$	48
$4 \bullet 2^{20}$	24	$4 \bullet 2^{36}$	40	$2 \bullet 5^{11}$	50
$6 \bullet 4 \bullet 2^{11}$	24	$5 \bullet 2^{28}$	40	$10 \bullet 5^{10}$	50
$4^3 \bullet 2^{22}$	32	$5 \bullet 4 \bullet 2^{25}$	40		

Chapter 3

THE BARE NECESSITIES

"One cannot escape the feeling that these mathematical formulae have an independent existence and an intelligence of their own, that they are wiser than we are, wiser even than their discoverers, and that we get more out of them than was originally put into them."

...Heinrich Hertz

3.0 INTRODUCTION

This chapter is intended as a primer to some of the elementary aspects of abstract algebra and its related applications in designed experimentation. The basic concepts of modular arithmetic are presented first followed by Fisher's theory of confounding to augment discussions initiated in chapter 2, such as is essential for understanding this report.

3.1 RUDIMENTS OF MODULAR ARITHMETIC

For any positive integer n, the two integers a and b are said to be *congruent modulo n*, and the notational representation is:

$$a \equiv b \ (mod\ n)$$

whenever n is a divisor of $(a\text{-}b)$. Thus $10 \equiv 4 \pmod 6$, $6 \equiv 0 \pmod 2$, $2 \equiv 14 \pmod 6$.

The operations of addition and multiplication can also be extended to *modular arithmetic* as illustrated below:

<u>ADDITION</u>	<u>MULTIPLICATION</u>
$10 + 16 \equiv 26 \equiv 2 \pmod 6$	$10 \bullet 16 \equiv 160 \equiv 4 \pmod 6$
$10 + 16 \equiv 4 + 4 \equiv 8 \equiv 2 \pmod 6$	$10 \bullet 16 \equiv 4 \bullet 4 \equiv 16 \equiv 4 \pmod 6$

Observe that when performing arithmetic modulo n, it suffices to consider the application of the arithmetic operations to the integer 0, 1, 2, ..., n-1 alone. The set of positive integers, when restricted to the set $\{0, 1, 2, \ldots, n\text{-}1\}$ is denoted by $\mathbf{Z}_n$. It is also referred to as the set of positive integers reduced modulo n.

Table 3.1 illustrates the addition and multiplication tables for $\mathbf{Z}_4$.

Table 3.1: Arithmetic modulo 4

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$\mathbf{Z_4,+}$

*	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

$\mathbf{Z_4,*}$

Interested readers are referred to Stahl (1997) for further details and applications of abstract algebra.

3.2 FISHER'S THEORY OF CONFOUNDING FOR SYMMETRICAL OMEPs

Fisher's theory of confounding, as has been alluded to in the previous chapter is widely used in the construction of OMEPs for s^m factor combinations in s^n runs, where $m=(s^n-1)/(s-1)$ and n is a positive integer. An elementary discussion extending the concepts of Section 3.1 is presented below and further details may be gleaned from Kempthorne (1952) or Fisher (1942).

This technique essentially augments a s^n factorial for n factors in s^n runs to include a total of m factors. Let the n factors of the s^n factorial be denoted by $X_1, X_2, \ldots, X_n$. The treatment combinations of the s^n factorial in factors $X_1, X_2, \ldots, X_n$ are first written down. The other $(m\text{-}n)$ factors' combinations are generated from these n columns by '*adding*' these columns in all possible ways over $\mathbf{Z}_s$, that is, by forming sums of the type $k_1X_1+k_2X_2+\ldots+k_nX_n$, where the k_i's are elements of $\mathbf{Z}_s$, and further, in each sum, the coefficient of the first factor is unity. This procedure will give the required plan, i.e., a total of m orthogonal columns in s^n runs, which can be used for screening m factors at s levels each.

This procedure for constructing OMEPs is illustrated by constructing a 9-run plan for a 3^4 (4 factors at 3 levels each) experiment by modifying a 3^2 full factorial. In this case, $s=3$, $n=2$, so that $m=(s^n-1)/(s-1) = 4$. The 9 treatment combinations of a 3^2 factorial in factors X_1, X_2 are first written down. The other two factors are given by X_1+X_2 and X_1+2X_2. Note that the coefficients 1 and 2 used in the *addition* of the factors X_1 and X_2 are the elements of $\mathbf{Z}_3$.

The treatment combinations for the 3^4 OMEP in 9 runs are shown in Table 3.2.

Table 3.2: A 3^4 OMEP in 9 Runs constructed using Fisher's Theory of Confounding

X_1	X_2	X_1+X_2	X_1+2X_2
0	0	0	0
0	1	1	2
0	2	2	1
1	0	1	1
1	1	2	0
1	2	0	2
2	0	2	2
2	1	0	1
2	2	1	0

Note that the addition of the elements of the columns conform to addition of integers in $\mathbf{Z}_3$ arithmetic, i.e., arithmetic modulo 3. Likewise, the construction of a 3^{13} OMEP in 27 runs (readers may also know this as the Taguchi L_{27}) is straightforward, the results of which are: X_1, X_2, X_1+X_2, X_1+2X_2, X_3, X_1+X_3, X_1+2X_3, X_2+X_3, X_2+2X_3, $X_1+X_2+X_3$, $X_1+X_2+2X_3$, $X_1+2X_2+X_3$, and $X_1+2X_2+2X_3$ respectively. Table 2.1 indexes some symmetrical OMEPs constructed using Fisher's theory of confounding.

3.3 ORTHOGONAL MATRICES: PRELIMINARIES AND RELEVANCE

An orthogonal design for an experiment can be defined as a way of collecting observations that will permit the experimenter to estimate and test for the various treatment effects and for interactions (if any) separately. The importance of orthogonality draws from the concept of multiple regression, wherein the estimate of β's (i.e., the factor main effects and interactions) for the model $\mathbf{Y}=\mathbf{X}\beta + \varepsilon$, are derived using the theory of least squares. The estimates of β are derived from $(\mathbf{X}^T\mathbf{X})^{-1}(\mathbf{X}^T\mathbf{Y})$ and the variance in the estimates for β is equal to $(\mathbf{X}^T\mathbf{X})^{-1}\sigma^2$, where σ^2 is the prediction variance[3], and it is desired that the estimates of the β's be uncorrelated with each other. This is equivalent to having the columns of independent variables in the $\mathbf{X}$ matrix uncorrelated with each other, so that $(\mathbf{X}^T\mathbf{X})$ is a symmetric matrix, which, equivalently is the definition of an orthogonal matrix.

The relevance of orthogonality as a property for designing an experiment arises largely from the efficiency of analysis and ease of interpretation of the individual estimates, and not so

[3] $\sigma^2 = SS_{Residuals}/(n-p) = \Sigma(y_i - y_{pred})^2/(n-p)$; n = #of observations, p = #of parameters being estimated.

much as being an exercise in mathematical manipulations. As an example, consider two independent variates, temperature and pressure, in a chemical study, each at two levels, arranged in a design plan as shown below:

Temperature (deg F)	Pressure (MPA)	Temp X Pressure
100	10	1000
100	15	1500
200	10	2000
200	15	3000

Desired model: $Y = b_{avg} + b_T \bullet T + b_P \bullet P + b_{TP} \bullet T \bullet P$

It is obvious that $(\mathbf{X}^T\mathbf{X})$ is not a symmetric matrix, though the design described above is a complete factorial, involving all treatment combinations. It is in situations like these, when a fixed variate is equally (or unequally) spaced in time or space, that the usual regression variates of the independent variables are replaced by orthogonal columns (also known as orthogonal polynomials). The orthogonal columns are so constructed that any column is independent over any other column, and effectively replaces a complex higher order polynomial regression equation to an additive linear model, expressed as functions of individual orthogonal linear forms. Each of the orthogonal linear forms, i.e., the columns in the **X** matrix represent the effects that they are used to estimate. Thus, the s-1 main effects that can be derived from a s level factor, may be represented by s-1 orthogonal linear columns in the **X** matrix. An example is presented below and readers are referred to Anderson and Bancroft (1952) for further details. Fisher and Yates (1957) present tables of orthogonal polynomials, i.e., linear forms for factors with upto 75 equally spaced levels.

A three level, equally spaced factor may be represented by its measurements, say $\mathbf{A} \equiv (0, 1, 2)$ and $\mathbf{A}^2 \equiv (0, 1, 4)$ or can be replaced by two orthogonal columns (-1, 0, 1) and (1, -2, 1) in the **X** matrix to represent the linear and quadratic effects of the factor. These orthogonal linear forms are so derived that the sum of individual values in the orthogonal columns is zero and the pair-wise product of any two columns sum to zero, which is essential to make the product $(\mathbf{X}^T\mathbf{X})$ symmetric.

Thus, the use of orthogonal polynomials to re-represent regression variates as an orthogonal matrix allows uncorrelated estimates of all effects as is possible from a regression model. This principle of orthogonality becomes more important for the design and evaluation of a screening experiment, largely because of the nature of decisions that are likely to be effected based on its results. Since the purpose of the screening experiment is to identify significant factors, based on estimates of their main effects, it is imperative that the screening experiment be orthogonal.

3.4 CONCLUDING REMARKS

The purpose of including such an elementary discussion was to introduce in advance the flavor of methods to come and establish a backbone for referencing and guiding the reader. The next chapter will introduce the vocabulary and elements necessary for the *Method of Symmetric Constructions* and references will be drawn from concepts presented in Sections 3.1-3.3. In brief, these are *the bare necessities*[4].

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[4] This metaphor was inspired by the song "The Bare Necessities of Life" from the movie - *The Jungle Book* (Walt Disney, 1967).

Chapter 4

THE METHOD OF SYMMETRIC CONSTRUCTIONS

"No more fiction for us: we calculate; but that we may calculate, we had to make fiction first."

...Nietzsche

"For, contrary to the unreasoned opinion of the ignorant, the choice of a system of enumeration is just a mere matter of convention."

...Blaise Pascal

4.0 INTRODUCTION

The *Method of Symmetric Constructions* has made possible the construction of Orthogonal Linear Effect Plans[5] (OLEPs) for $2^k \bullet s^p$ factor combinations in $s(1+k)$ runs, where s is any positive integer (≥ 2). This chapter introduces the basic components of this technique and details the notational and operational subtleties involved. Succeeding sections illustrate with examples relevant concepts and their interrelationship in the context of an OLEP.

4.1 THE VOCABULARY OF THE METHOD OF SYMMETRIC CONSTRUCTIONS

An OLEP consists of five basic elements[6]: build sets, constructs, addition sets, reflections and swaps. All the definitions described below pertain to the construction of a $2^k \bullet s^p$ design plan, involving 'k' 2-level factors and 'p' s-level factors in $s(1+k)$ runs.

Definition 1: A ***build set*** is a set of all ordered elements $\{a_i\}$, ($a_i = 0, 1, ..., s-1$) from $\mathbf{Z}_s$. Consider, for instance, an OLEP for a $2^3 \bullet 3^4$ factor combinations in $3(1+3)=12$ runs; here $s=3$. So, the *build set* in this context is (0, 1, 2) and is denoted by the symbol **B**.

(The reader is encouraged to verify for himself, that 12 runs are the theoretical minimum number of runs necessary for estimating the *main effects* alone for a $2^3 \bullet 3^4$ factor combination. Recall from Sections 1.1.8, 1.1.9 - how many total degrees of freedom, i.e., total number of effects are involved in a $2^3 \bullet 3^4$ combination; how many effects can be estimated in N runs?)

[5] Design plans that allow the complete estimation of all linear main effects while the higher order effects are slightly correlated. (refer Section 1.5.2)

[6] The author has created all these definitions. Any resemblance to concepts, dead or alive is purely incidental.

<u>Definition 2:</u> A ***construct*** is generated by adding one to all the elements of the build set, addition being performed in arithmetic modulo s. The *construct* will be denoted by the symbol **C**. Thus, for the example of a $2^3 \bullet 3^4$ OLEP cited in definition 1, the *build set* **B** was (0, 1, 2) and the *construct* is (1, 2, 0). The calculation is as follows: $0+1 \equiv 1$ *mod* 3; $1+1 \equiv 2$ *mod* 3; $2+1 \equiv 0$ *mod* 3;

<u>Definition 3:</u> An ***addition set*** is generated by adding two to all the elements of the construct. The *addition set* will be denoted by **A**. Thus, for the example cited in definition 1, the *construct* is (1, 2, 0) and the *addition set* is (0, 1, 2). The calculation is as follows: $1+2 \equiv 0$ *mod* 3; $2+2 = 4 \equiv 1$ *mod* 3; $0+2 \equiv 2$ *mod* 3;

<u>Definition 4:</u> A ***reflection*** is generated by taking a *mirror image* of the set in consideration (either a construct or an addition set or any set in general). Thus, the *reflection* of the addition set (0, 1, 2) is (2, 1, 0), which will be denoted by $\mathbf{R}^a$. The *reflection* of the construct (1, 2, 0) is (0, 2, 1) and will be denoted by $\mathbf{R}^c$.

<u>Definition 5:</u> The ***swap*** is generated by interchanging the $(s\text{-}1)^{th}$ and s^{th} elements (i.e., the last two elements), and then the i^{th} element with the $(s\text{-}1\text{-}i)^{th}$ element of the set in consideration (i = 1, 2, ... , $(s$-2)/2, if s is even, or i = 1, 2, ... , (s-3)/2 if s is odd). $\mathbf{S}^c$ will denote the swap operation performed on the *construct*, and $\mathbf{S}^a$, the swap operation performed on the addition set. This is illustrated below for both cases, i.e., when s is even and when s is odd. Note that the elements of the construct are numbered 1, 2, ... , s.

<u>Case 1:</u> s=5 (odd): The build set **B** is (0, 1, 2, 3, 4); The construct **C** is (1, 2, 3, 4, 0); The addition set is (3, 4, 0, 1, 2); $\mathbf{R}^c$ = (0, 4, 3, 2, 1); $\mathbf{R}^a$ = (2, 1, 0, 4, 3).

The swap performed on the construct and the addition sets are:

S. No	Construct, **C**	Swap, $\mathbf{S}^c$	Addition Set, **A**	Swap, $\mathbf{S}^a$
1	1	3	3	0
2	2	2	4	4
3	3	1	0	3
4	4	0	1	2
5	0	4	2	1

Case 2: s=6 (even): The build set is (0, 1, 2, 3, 4, 5); The construct is (1, 2, 3, 4, 5, 0); The addition set is (3, 4, 5, 0, 1, 2); $\mathbf{R}^c$=(0, 5, 4, 3, 2, 1); $\mathbf{R}^a$=(2, 1, 0, 5, 4, 3); The swap is performed thus:

S.No	Construct, **C**	Swap, $\mathbf{S}^c$	Addition Set, **A**	Swap, $\mathbf{S}^a$
1	1	4	2	5
2	2	3	1	0
3	3	2	0	1
4	4	1	5	2
5	5	0	4	3
6	0	5	3	4

An index of some useful constructs and swaps is presented in Table 4.1.

Table 4.1: Index of some useful Constructs and Swaps

s	Build Set, **B**	Construct, **C**	Swap of construct, $\mathbf{S}^c$	Addition Set, **A**	Swap of addition set, $\mathbf{S}^a$
3	(0, 1, 2)	(1, 2, 0)	(1, 0, 2)	(0, 1, 2)	(2, 1, 0)
4	(0, 1, 2, 3)	(1, 2, 3, 0)	(2, 1, 0, 3)	(3, 0, 1, 2)	(0, 3, 2, 1)
5	(0, 1, 2, 3, 4)	(1, 2, 3, 4, 0)	(3, 2, 1, 0, 4)	(3, 4, 0, 1, 2)	(0, 4, 3, 2, 1)
6	(0, 1, 2, 3, 4, 5)	(1, 2, 3, 4, 5, 0)	(4, 3, 2, 1, 0, 5)	(3, 4, 5, 0, 1, 2)	(0, 5, 4, 3, 2, 1)
7	(0, 1, 2, 3, 4, 5, 6)	(1, 2, 3, 4, 5, 6, 0)	(5, 4, 3, 2, 1, 0, 6)	(3, 4, 5, 6, 0, 1, 2)	(0, 6, 5, 4, 3, 2, 1)
8	(0, 1, 2, 3, 4, 5, 6, 7)	(1, 2, 3, 4, 5, 6, 7, 0)	(6, 5, 4, 3, 2, 1, 0, 7)	(3, 4, 5, 6, 7, 0, 1, 2)	(0, 7, 6, 5, 4, 3, 2, 1)

The *reflection* of the swapped set may be generated as a straightforward extension of definition 4, and the reflection thus generated will be denoted by $\mathbf{R}^{S_c}$ *or* $\mathbf{R}^{S_a}$ as appropriate.

4.2 USING THE VOCABULARY OF THE METHOD OF SYMMETRIC CONSRUCTIONS

To highlight the significance of the above definitions in the construction of OLEPs, the following propositions are stated without proof.

Proposition 1: It is possible to construct an orthogonal matrix $\mathbf{X}_{2s}$, in $2s$ runs as shown below:

$$\mathbf{X}_{2s} = \begin{bmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{B} & \mathbf{R}^c \end{bmatrix}$$

where, **B**, **C**, $\mathbf{R}^c$, are column vectors consisting of the elements of $\mathbf{Z}_s$, as per definitions 1, 2, 3, 4, and 5.

Example Illustration: Consider s=3. Here, $\mathbf{B}$ = (0, 1, 2); $\mathbf{C}$ = (1, 2, 0); $\mathbf{R}^c$ = (0, 2, 1). Then, using Proposition 1,

$$\mathbf{X}_{2s} = \mathbf{X}_4 = \begin{bmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{B} & \mathbf{R}^c \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \\ 2 & 0 \\ 0 & 0 \\ 1 & 2 \\ 2 & 1 \end{bmatrix}$$

The above matrix is orthogonal and the development of OLEPs from $\mathbf{X}$ matrices will be detailed in sections to follow.

Proposition 2: It is possible to construct an orthogonal matrix $\mathbf{X}_{4s}$, in $4s$ runs as shown below:

$$\mathbf{X}_{4s} = \begin{bmatrix} \mathbf{B} & \mathbf{C} & \mathbf{S}^c & \mathbf{A} \\ \mathbf{B} & \mathbf{R}^c & \mathbf{R}^s & \mathbf{C} \\ \mathbf{B} & \mathbf{C} & \mathbf{C} & \mathbf{R}^c \\ \mathbf{B} & \mathbf{R}^c & \mathbf{R}^c & \mathbf{R}^a \end{bmatrix}$$

Example Illustration: Here, $\mathbf{B}$ = (0, 1, 2); $\mathbf{C}$ = (1, 2, 0); $\mathbf{R}^c$ = (0, 2, 1); $\mathbf{A}$ = (0, 1, 2); $\mathbf{S}^c$ = (2, 1, 0). Then, using Proposition 2,

$$\mathbf{X}_{4s} = \mathbf{X}_{12} = \begin{bmatrix} \mathbf{B} & \mathbf{C} & \mathbf{S}^c & \mathbf{A} \\ \mathbf{B} & \mathbf{R}^c & \mathbf{R}^s & \mathbf{C} \\ \mathbf{B} & \mathbf{C} & \mathbf{C} & \mathbf{R}^c \\ \mathbf{B} & \mathbf{R}^c & \mathbf{R}^c & \mathbf{R}^a \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 2 & 0 & 2 & 2 \\ 0 & 0 & 2 & 1 \\ 1 & 2 & 0 & 2 \\ 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 2 & 2 \\ 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 0 \end{bmatrix}$$

The matrices thus constructed, using the above propositions and others that will be stated later, in $2^n s$ runs, will be denoted as '**X**' matrices of order $2^n s$. It is now appropriate to state a few additional definitions to supplement the ideas presented in propositions 1 and 2.

4.3 WHY IS IT CALLED THE METHOD OF SYMMETRIC CONSTRUCTIONS?

This section is intended to introduce in context, very intuitively, the reason for naming this technique *the method of symmetric constructions* and it is towards this end that the following definitions are presented.

Definition 6: There exists in **X** matrices of order $2^n s$ a *line of major symmetry* that divides the design plan into two symmetric halves. The line of major symmetry in an **X** matrix of order $2^n s$ will share $2^{n-1}s$ elements on either side of it. This is illustrated using the $\mathbf{X}_{12}$ matrix (i.e., $\mathbf{X}_{4s}$ matrix for s=3).

$$\mathbf{X}_{4s} = \mathbf{X}_{12} = \left[\begin{array}{cccc} \mathbf{B} & \mathbf{C} & \mathbf{S}^c & \mathbf{A} \\ \mathbf{B} & \mathbf{R}^c & \mathbf{R}^s & \mathbf{C} \\ \hline \mathbf{B} & \mathbf{C} & \mathbf{C} & \mathbf{R}^c \\ \mathbf{B} & \mathbf{R}^c & \mathbf{R}^c & \mathbf{R}^a \end{array}\right] = \left[\begin{array}{cccc} 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 2 & 0 & 2 & 2 \\ 0 & 0 & 2 & 1 \\ 1 & 2 & 0 & 2 \\ 2 & 1 & 1 & 0 \\ \hline 0 & 1 & 1 & 0 \\ 1 & 2 & 2 & 2 \\ 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 0 \end{array}\right]$$

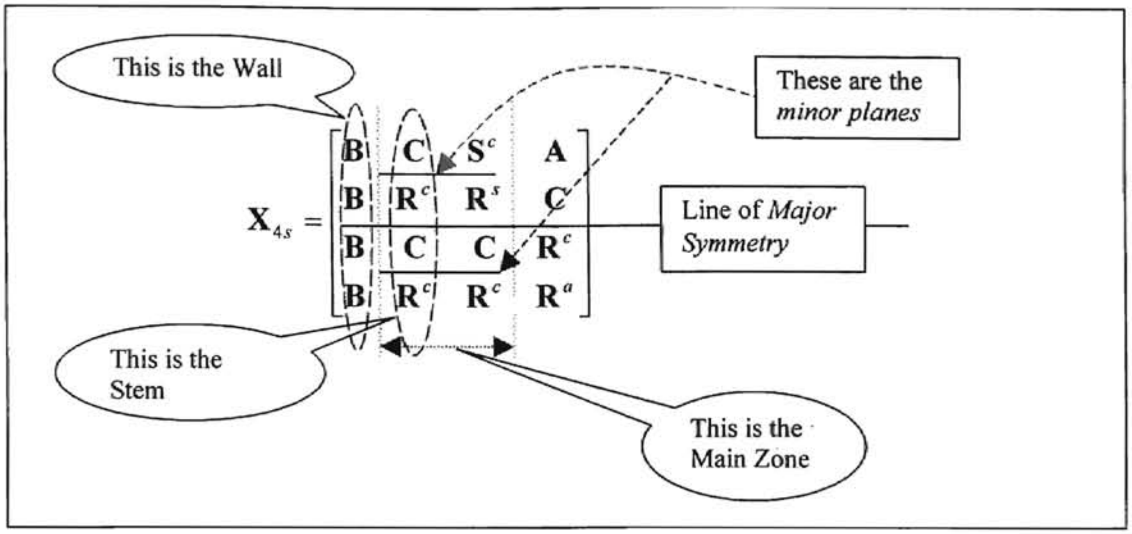
This is the Line of Major Symmetry

Definition 7: There exists in **X** matrices of order $2^n s$, ***minor planes*** about which the individual reflections are performed. Minor planes exist only when the order of the matrix $\geq 4s$. These minor planes segment the columns above and below the line of major symmetry and serve as pivotal points about which the reflections are performed.

Definition 8: There exists in all **X** matrices of order $2^n s$, a ***wall*** that constitutes the column of build sets, **B**, repeated 2^n times.

The ideas underlying definitions 6, 7, and 8 (and some more) are illustrated in Figure 4.1, wherein a $\mathbf{X}_{4s}$ has been dissected to show its component sections.

Figure 4.1: Visual Illustration to Support Definitions 6, 7, and 8



Some comments about Figure 4.1 that will be useful in subsequent discussions are:

(1) The swap performed in $\mathbf{X}_{4s}$ to generate the third column, is done keeping the lower half of the matrix same as the 2nd column, while the upper portion is replaced by swapping the construct and writing its reflection beneath it. This is termed a *manipulation*, the formal definition of which will be presented later.

(2) The swap and its reflection, (which constitutes a *swap-reflection* pair) that is observed in the 3rd column of $\mathbf{X}_{4s}$ is pivoted about the minor plane in the upper half of the matrix.

Figure 4.1 includes an additional term, the *main zone*, the definition and description of which follow:

<u>Definition 9:</u> Apart from the *Wall*, all the other columns in the **X** matrices are derived from individual **stems**, that form the basis for further development in their individual *zones*, the definition of which is presented next.

<u>Definition 10:</u> A **Zone** is defined in the matrix, within which all the columns are generated by manipulating the upper half of the corresponding stem and then progressively proceeding across the line of major symmetry. A **Main Zone** is defined wherein, the *stem* is the column vector $(\mathbf{C}\ \mathbf{R}^c\ \mathbf{C}\ \mathbf{R}^c \ldots \mathbf{C}\ \mathbf{R}^c)^T$. The reader is directed to note that the 2nd column, i.e., the column apart from the *Wall*, in both $\mathbf{X}_{2s}$ and $\mathbf{X}_{4s}$ match the above description.

Step-down Zones are generated by bringing two adjacent *minor planes* towards each other, stepping them down (up) by s units each and continuing till they coincide with each other or they merge with the *line of major symmetry*. Along with the *minor planes*, the *construct-reflection* (i.e., $\mathbf{C}$-$\mathbf{R}^{\mathbf{C}}$) pairs, pivoted about the corresponding minor planes are also stepped down, and this operation is performed symmetrically on either side of the *line of major symmetry,* making *minor planes* unique to each zone. The zones thus created are numbered 1, 2, 3, etc and the *main zone* is given a zone number of zero. However, before the discussion furthers into more details, it is necessary to introduce the idea of *manipulating* a *stem,* which is very useful for generating additional orthogonal columns.

Definition 11: The *stem* in a particular *zone* can be *manipulated* to generate other columns in the following ways:

(1) The lower half of the *stem* is kept constant and their corresponding swap-reflection pairs replace all the construct-reflection or addition-reflection pairs above the line of major symmetry. This will generate 2^{n-2-i} swap-reflection pairs, where i is the number of the *zone* to which the *stem* belongs and n is the exponent of 2 in the order of the $\mathbf{X}$ matrix (e.g., for the $\mathbf{X}_{4s}$, order = 2^2s, n=2). This is possible so long as the *minor planes* in the corresponding zones are distinct and different from the *line of major symmetry*. This step will be referred to as ***swapping*** and the column thus generated referred to as the ***swapped column***. This is the procedure for generating the 3rd column of $\mathbf{X}_{4s}$, and is described in Figure 4.2.

(2) Following *swapping*, if more than 1 swap-reflection pair is generated[7] above the *line of major symmetry*, ***switches*** can be performed as follows: Keep the first 2^{n-3-i} swap-reflection pairs above the *line of major symmetry* (Note: The order of the matrix is $2^n s$) in the *swapped column* constant. Permute the remaining 2^{n-3-i} swap-reflection pairs, as a block, with the 2^{n-2-i} construct-reflection (or addition-reflection pairs, as the case may be) pairs in the lower half of the column (i.e., below the *line of major symmetry*) to generate two additional columns. Such manipulations (swapping and

[7] This is possible when the *order* of the $\mathbf{X}$ matrix $\geq 8s$, propositions for which will be presented later in Section 4.4.

switches) in any *zone* will generate a maximum of 4 orthogonal columns in a *zone*, i.e., one stem, one swapped column and two switches.

Figure 4.2: Visual Illustration to support Definitions 9, 10 and 11

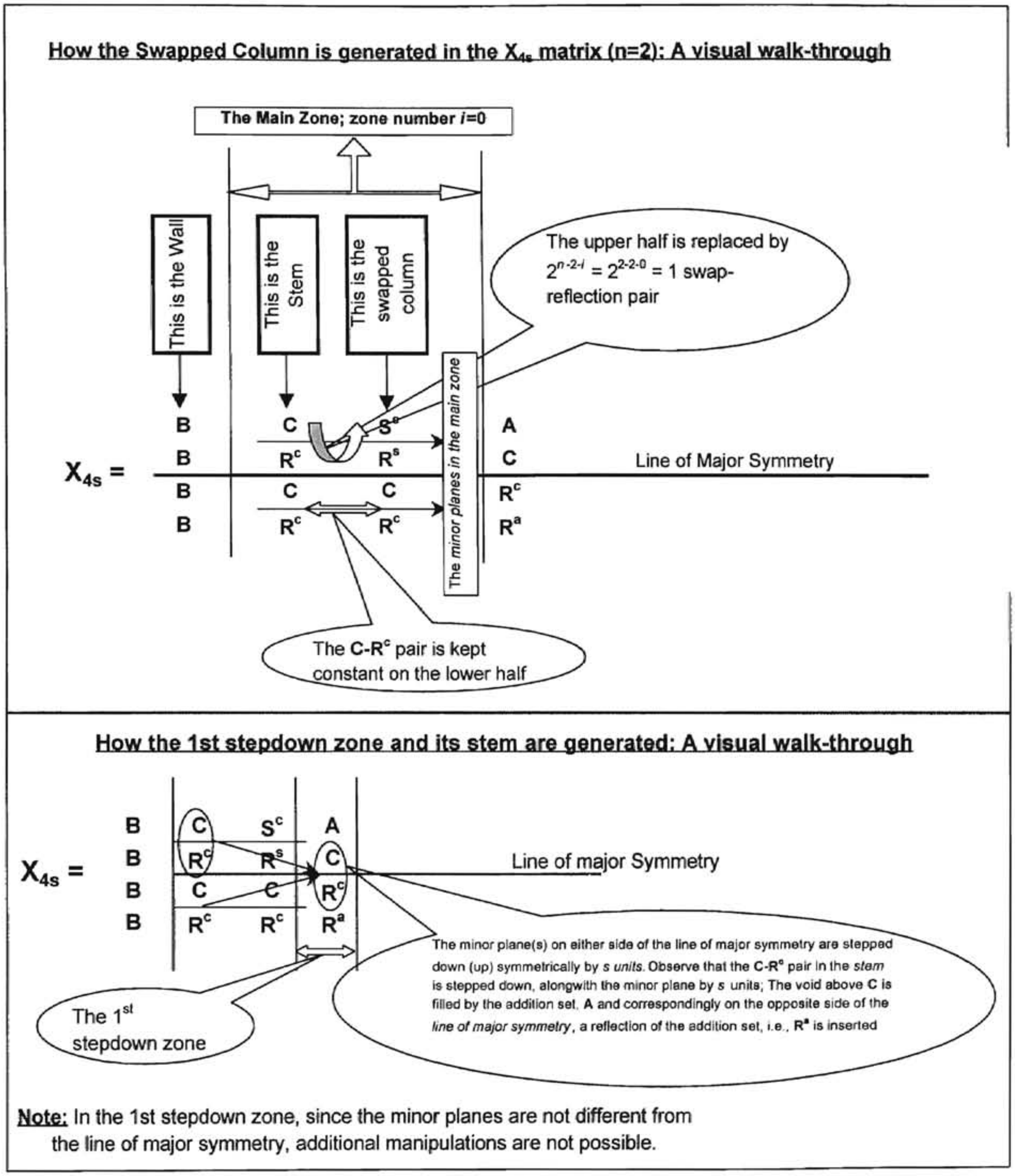


Figure 4.2 was intended to illustrate the basic notion of *stepping down, swapping* and their relevance in generating the corresponding **X** matrices.

The formulae presented below are relevant to determining the individual elements of an **X** matrix, relevant for the construction of an OLEP, and are suitable for algorithmic implementation.

For an X matrix of order $2^n s$:

The **Wall**, i.e., the column of build sets repeated 2^n times (from Definition 8) is first written down. Then the **Stem** for the *main zone*, i.e., the column of **C-R^C** pairs, is written down 2^{n-1} times, after which the following calculations may be performed.

- ***n*** = exponent of 2 appearing in order of the matrix. (for the $\mathbf{X}_{2s}$, $n = 1$; for the $\mathbf{X}_{4s}$, $n = 2$)
- ***i*** = zone identification # (for *main zone*, $i = 0$; $i = 1, 2, \ldots$ for step-down zones)
- $\boldsymbol{m_i}$ = Number of minor planes in zone i = 2^{n-1-i}
- $\boldsymbol{p_i}$ = Number of swap-reflection pairs in the swapped column for zone i = $\mathrm{int}(m_i/2)$
- If $\boldsymbol{p_i} = 0$, then a swapped column can't be generated, $\Rightarrow$ # of columns in zone i = $\mathbf{C_i} = 1$, i.e., the stem
- If $\boldsymbol{p_i} = 1$, then # of columns in zone i = $\mathbf{C_i} = 2$, i.e., the stem and the swapped column
- If $\boldsymbol{p_i}$ is greater than 1, then switches can be performed to manipulate the swapped column and generate two additional columns. $\Rightarrow \mathbf{C_i = 4}$

Figure 4.3 illustrates the use of these formulae in calculations relevant to the design of $\mathbf{X}_{2s}$ and $\mathbf{X}_{4s}$.

Figure 4.3: Design Calculations Relevant to $\mathbf{X}_{2s}$ and $\mathbf{X}_{4s}$

$$\mathbf{X}_{2s} = \left[\begin{array}{c|c} \mathbf{B} & \mathbf{C} \\ \hline \mathbf{B} & \mathbf{R}^c \end{array}\right]$$

n = 1;

m₀ = # of minor planes in the main zone = $2^{1-1-0} = 1$

$\mathbf{m_0} = 1 \Rightarrow$ minor planes coincide with the line of major symmetry.

$\Rightarrow$ can additional zones be created? - NO

$\mathbf{p_0} = \mathrm{int}(m_0/2) = 0 \Rightarrow$ number of columns in main zone = $\mathbf{C_0} = 1$

Total # of columns = $1 + C_0 = 1 + 1 = 2$

$$\mathbf{X}_{4s} = \left[\begin{array}{c|cc|c} \mathbf{B} & \mathbf{C} & \mathbf{S}^c & \mathbf{A} \\ \cline{2-3} \mathbf{B} & \mathbf{R}^c & \mathbf{R}^s & \mathbf{C} \\ \hline \mathbf{B} & \mathbf{C} & \mathbf{C} & \mathbf{R}^c \\ \cline{2-3} \mathbf{B} & \mathbf{R}^c & \mathbf{R}^c & \mathbf{R}^a \end{array}\right]$$

n = 2;

m_0 = # of minor planes in the main zone = $2^{2-1-0} = 2$

$p_0 = \mathrm{int}(2/2) = 0 \Rightarrow C_0 = 2$

m_1 = # of minor planes in the step-down zone 1 = $2^{2-1-1} = 1$

$p_1 = \mathrm{int}(1/2) = 0 \Rightarrow C_1 = 1$

Total # of columns = $1 + C_0 + C_1 = 1 + 2 + 1 = 4$

4.4 CONSTRUCTING HIGHER ORDER 'X' MATRICES

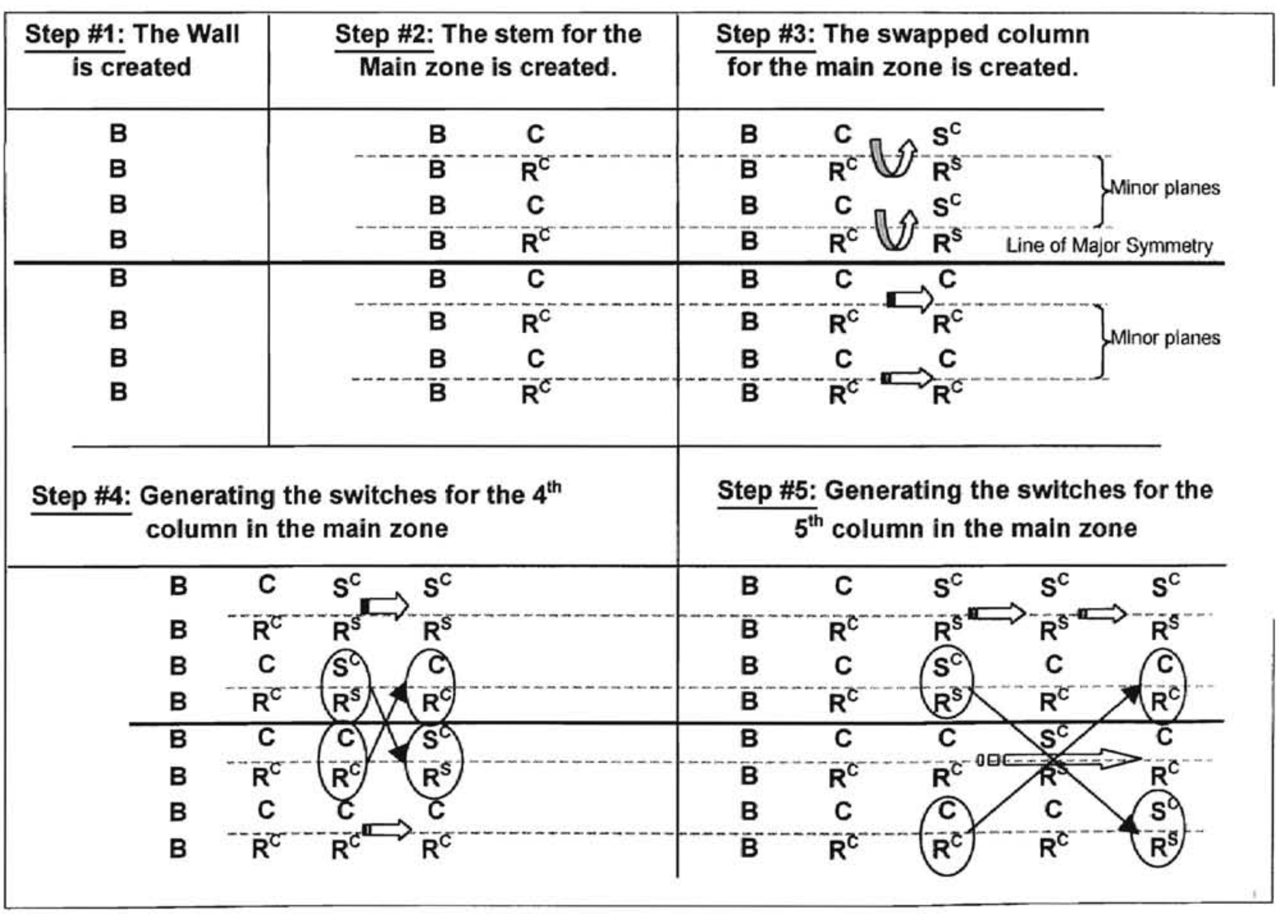
Proposition 3: It is possible to construct an orthogonal matrix $\mathbf{X}_{8s}$, in $8s$ runs. The following calculations are relevant to determine the operational subtleties involved.

- $n = 3$
- Create the *wall*, i.e., the column vector **[B B ... B]^T** ,the build sets repeated 2^n (= 8) times.
- Create the stem of the main zone, i.e., the column vector **[C R^c C R^c ... C R^c]^T**, the *construct-reflection pairs* repeated 4 ($=2^{n-1}$) times.
- m_0 = # of minor planes in the main zone = $2^{3-1-0} = 4$
- p_0 = # of swap-reflection pairs in the swapped column for main zone (i =0) = $2^{3-2-0} = 2$

  $\Rightarrow$ Can ***switches*** be performed in the *main zone*? - Yes. (since p_0>1)
- Total # of columns in main zone = 4 (=1 stem, 1 swapped column, 2 switches)

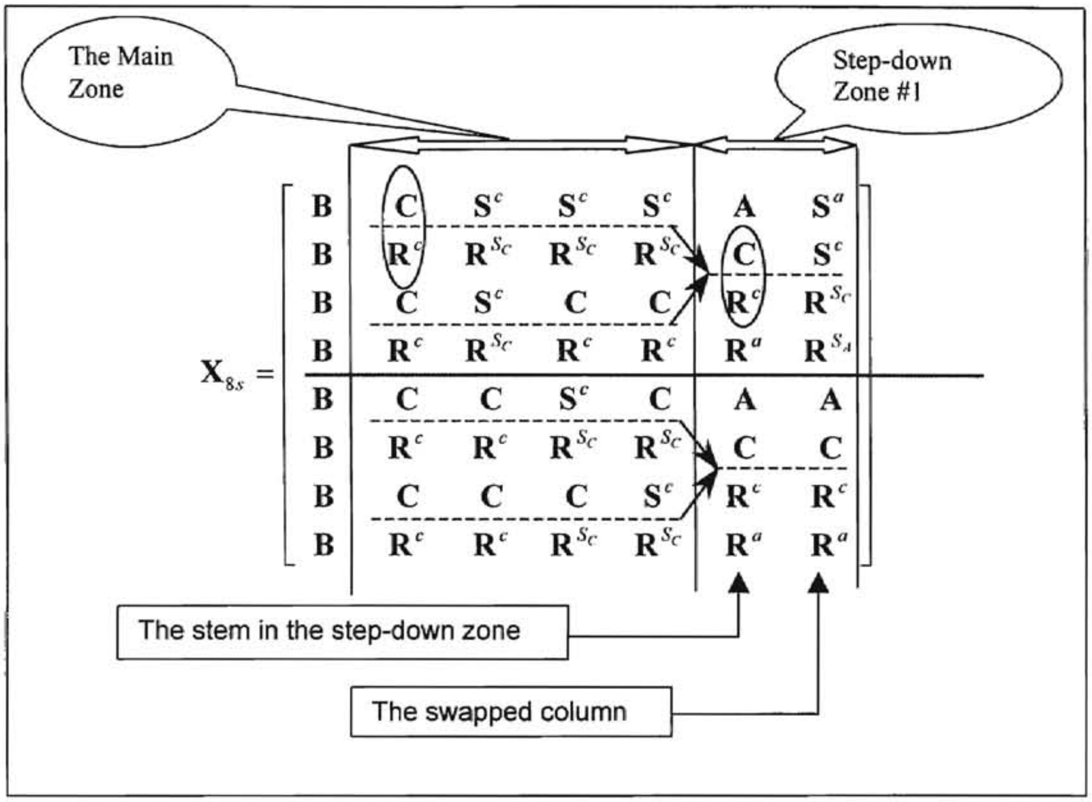
The steps involved in generating the columns for the wall and the main zone are detailed in Figure 4.4 below:

Figure 4.4: Steps in the Construction of the $\mathbf{X}_{8s}$ matrix - I



The 4 possible columns in the Main zone have been created. Since the 4 minor planes in the minor plane are distinct and different from the *line of major symmetry*, it is possible to create a step-down zone, and the swapped column for step-down zone #1, as is illustrated in Figure 4.5.

Figure 4.5: Steps in the construction of the X_{8s} matrix - II



Note: For the Step-down Zone #1, observe that the 2 adjacent $\mathbf{C}$-$\mathbf{R}^{C}$ pairs in the upper half of the main zone are brought together and merged as one and the voids above and below are filled with the addition set, **A** and its reflection $\mathbf{R}^{A}$.

The next logical step would be the construction of an $\mathbf{X}_{16s}$ matrix in 16*s* runs, the description of which is presented next.

Proposition 4: It is possible to construct an orthogonal matrix $\mathbf{X}_{16s}$ in 16*s* runs, using *the method of symmetric constructions* as was introduced in the previous sections. The matrix is presented in Figure 4.6 and necessary enumeration is included to bring out the specific details.

Figure 4.6: Design Template for the $\mathbf{X}_{16s}$ Matrix

$$\mathbf{X}_{16s} = \begin{array}{c} \begin{array}{cccc} \text{Wall} & \text{Main Zone} & \text{Step-down Zone \#1} & \text{Step-down Zone \#2} \end{array} \\ \left[\begin{array}{c|cccc|cccc|cccc}
\mathbf{B} & \mathbf{C} & \mathbf{S}^c & \mathbf{S}^c & \mathbf{S}^c & \mathbf{A} & \mathbf{S}^a & \mathbf{S}^a & \mathbf{S}^a & \mathbf{R}^a & \mathbf{R}^{S_A} & \mathbf{R}^{S_A} & \mathbf{R}^{S_A} \\
\mathbf{B} & \mathbf{R}^c & \mathbf{R}^{S_C} & \mathbf{R}^{S_C} & \mathbf{R}^{S_C} & \mathbf{C} & \mathbf{S}^c & \mathbf{S}^c & \mathbf{S}^c & \mathbf{A} & \mathbf{S}^a & \mathbf{S}^a & \mathbf{S}^a \\
\mathbf{B} & \mathbf{C} & \mathbf{S}^c & \mathbf{S}^c & \mathbf{S}^c & \mathbf{R}^c & \mathbf{R}^{S_C} & \mathbf{R}^{S_C} & \mathbf{R}^{S_C} & \mathbf{C} & \mathbf{S}^c & \mathbf{S}^c & \mathbf{S}^c \\
\mathbf{B} & \mathbf{R}^c & \mathbf{R}^{S_C} & \mathbf{R}^{S_C} & \mathbf{R}^{S_C} & \mathbf{R}^a & \mathbf{R}^{S_A} & \mathbf{R}^{S_A} & \mathbf{R}^{S_A} & \mathbf{R}^c & \mathbf{R}^{S_C} & \mathbf{R}^{S_C} & \mathbf{R}^{S_C} \\
\mathbf{B} & \mathbf{C} & \mathbf{S}^c & \mathbf{C} & \mathbf{C} & \mathbf{A} & \mathbf{S}^a & \mathbf{A} & \mathbf{A} & \mathbf{C} & \mathbf{S}^c & \mathbf{R}^a & \mathbf{C} \\
\mathbf{B} & \mathbf{R}^c & \mathbf{R}^{S_C} & \mathbf{R}^c & \mathbf{R}^c & \mathbf{C} & \mathbf{S}^c & \mathbf{C} & \mathbf{C} & \mathbf{R}^c & \mathbf{R}^{S_C} & \mathbf{A} & \mathbf{R}^c \\
\mathbf{B} & \mathbf{C} & \mathbf{S}^c & \mathbf{C} & \mathbf{C} & \mathbf{R}^c & \mathbf{R}^{S_C} & \mathbf{R}^c & \mathbf{R}^c & \mathbf{R}^a & \mathbf{R}^{S_A} & \mathbf{C} & \mathbf{R}^a \\
\mathbf{B} & \mathbf{R}^c & \mathbf{R}^{S_C} & \mathbf{R}^c & \mathbf{R}^c & \mathbf{R}^a & \mathbf{R}^{S_A} & \mathbf{R}^a & \mathbf{R}^a & \mathbf{A} & \mathbf{S}^a & \mathbf{R}^c & \mathbf{A} \\
\hline
\mathbf{B} & \mathbf{C} & \mathbf{C} & \mathbf{S}^c & \mathbf{C} & \mathbf{A} & \mathbf{A} & \mathbf{S}^a & \mathbf{A} & \mathbf{R}^a & \mathbf{R}^a & \mathbf{S}^c & \mathbf{R}^a \\
\mathbf{B} & \mathbf{R}^c & \mathbf{R}^c & \mathbf{R}^{S_C} & \mathbf{R}^c & \mathbf{C} & \mathbf{C} & \mathbf{S}^c & \mathbf{C} & \mathbf{A} & \mathbf{A} & \mathbf{R}^{S_C} & \mathbf{A} \\
\mathbf{B} & \mathbf{C} & \mathbf{C} & \mathbf{S}^c & \mathbf{C} & \mathbf{R}^c & \mathbf{R}^a & \mathbf{R}^{S_C} & \mathbf{R}^c & \mathbf{C} & \mathbf{C} & \mathbf{R}^{S_A} & \mathbf{C} \\
\mathbf{B} & \mathbf{R}^c & \mathbf{R}^c & \mathbf{R}^{S_C} & \mathbf{R}^c & \mathbf{R}^a & \mathbf{R}^c & \mathbf{R}^{S_A} & \mathbf{R}^a & \mathbf{R}^c & \mathbf{R}^c & \mathbf{S}^a & \mathbf{R}^c \\
\mathbf{B} & \mathbf{C} & \mathbf{C} & \mathbf{C} & \mathbf{S}^c & \mathbf{A} & \mathbf{A} & \mathbf{A} & \mathbf{S}^a & \mathbf{C} & \mathbf{C} & \mathbf{C} & \mathbf{S}^c \\
\mathbf{B} & \mathbf{R}^c & \mathbf{R}^c & \mathbf{R}^c & \mathbf{R}^{S_C} & \mathbf{C} & \mathbf{C} & \mathbf{C} & \mathbf{S}^c & \mathbf{R}^c & \mathbf{R}^c & \mathbf{R}^c & \mathbf{R}^{S_C} \\
\mathbf{B} & \mathbf{C} & \mathbf{C} & \mathbf{C} & \mathbf{S}^c & \mathbf{R}^c & \mathbf{R}^c & \mathbf{R}^c & \mathbf{R}^{S_C} & \mathbf{R}^a & \mathbf{R}^a & \mathbf{R}^a & \mathbf{R}^{S_A} \\
\mathbf{B} & \mathbf{R}^c & \mathbf{R}^c & \mathbf{R}^c & \mathbf{R}^{S_C} & \mathbf{R}^a & \mathbf{R}^a & \mathbf{R}^a & \mathbf{R}^{S_A} & \mathbf{A} & \mathbf{A} & \mathbf{A} & \mathbf{S}^a
\end{array}\right] \end{array}$$

This concludes the list of propositions that will be presented detailing the list of **X** matrices constructed using the *method of symmetric constructions*. The next section will discuss the construction of $2^k \bullet s^p$ OLEPs in $s(1+k)$ runs from the above defined **X** matrices.

4.5 CONSTRUCTION OF OLEPS USING THE X MATRICES

The construction of all relevant OLEPs discussed in this section is a direct extension of the **X** matrices defined and delivered in Sections 4.3 and 4.4 and they use the Fisher's theory of confounding (Section 3.2) as the underlying principle. The following propositions are now stated without proof.

Proposition 5: It is possible to construct a $2 \bullet s^2$ orthogonal matrix, ${}_s\mathbf{X}_{2s}$ in $2s$ runs as shown below:

$${}_s\mathbf{X}_{2s} = \begin{bmatrix} \mathbf{0} & \mathbf{B} & \mathbf{C} \\ \mathbf{1} & \mathbf{B} & \mathbf{R}^c \end{bmatrix}$$

where, **0** and **1** are column vectors of zeros and ones.

Proposition 6: It is possible to construct an $2^3 \bullet s^4$ orthogonal matrix, ${}_s\mathbf{X}_{4s}$ in $4s$ runs as shown below, where **0** and **1** are column vectors of zeros and ones.

$$
{}_s\mathbf{X}_{4s} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B} & \mathbf{C} & \mathbf{S}^c & \mathbf{A} \\ \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{B} & \mathbf{R}^s & \mathbf{R}^s & \mathbf{C} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{B} & \mathbf{C} & \mathbf{C} & \mathbf{R}^c \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{B} & \mathbf{R}^c & \mathbf{R}^c & \mathbf{R}^a \end{bmatrix}
$$

Both propositions essentially modify an **X** matrix into a $2^k \bullet s^p$ matrix through the following steps, stated generically for a **X** matrix of order $2^n s$.

- To the **X** matrix of order $2^n s$, append 'n' 2-level columns corresponding to the n factors in a 2^n factorial, i.e., write out a 2^n full factorial alongside the existing columns of the **X** matrix. This is illustrated for the $\mathbf{X}_{12}$ matrix (s=3). The $\mathbf{X}_{12}$ matrix is of order $2^2.3$, where n=2. Therefore, two additional 2-level columns will be appended to the existing $\mathbf{X}_{12}$ matrix as described in proposition 6.

$$
\mathbf{X}_{12} = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 2 & 0 & 2 & 2 \\ 0 & 1 & 0 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 & 0 & 2 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 2 & 2 & 2 \\ 1 & 0 & 2 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 2 \\ 1 & 1 & 1 & 2 & 2 & 1 \\ 1 & 1 & 2 & 1 & 1 & 0 \end{bmatrix}
$$

- Generate a total of 2^n-1 two level columns from the existing 'n' 2-level columns using Fisher's theory of confounding (Section 3.2). In the example of a $\mathbf{X}_{12}$ cited above, one additional 2-level column may be generated from the two existing columns using Fisher's theory of confounding. The complete matrix, i.e., the ${}_3\mathbf{X}_{12}$ matrix is shown in Figure 4.7.

Figure 4.7: Orthogonal Linear Effect Plan for a $2^3.3^4$ Factor Combination in 12 Runs

$$
{}_3\mathbf{X}_{12} = \begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 2 & 0 & 1 \\
0 & 0 & 0 & 2 & 0 & 2 & 2 \\
1 & 0 & 1 & 0 & 0 & 2 & 1 \\
1 & 0 & 1 & 1 & 2 & 0 & 2 \\
1 & 0 & 1 & 2 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 & 2 & 2 & 2 \\
1 & 1 & 0 & 2 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 & 0 & 2 \\
0 & 1 & 1 & 1 & 2 & 2 & 1 \\
0 & 1 & 1 & 2 & 1 & 1 & 0
\end{bmatrix}
$$

All the matrices constructed thus are orthogonal linear effect plans. They allow complete estimation of all linear main effects, while the higher order effects of the factors are correlated with each other. In the sections to come, methods will be described wherein, the problem of correlated higher order effects may be remedied. Next, additional propositions are presented, wherein the $\mathbf{X}_{8s}$ and the $\mathbf{X}_{16s}$ are modified into OLEPs based on the generic rules stated earlier.

<u>Proposition 7:</u> It is possible to construct a $2^7 \bullet s^7$ OLEP, ${}_s\mathbf{X}_{8s}$ in $8s$ runs as shown below, where, **0** and **1** are column vectors of zeros and ones.

$$
{}_S\mathbf{X}_{8s} = \begin{bmatrix}
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B} & \mathbf{C} & \mathbf{S}^c & \mathbf{S}^c & \mathbf{S}^c & \mathbf{A} & \mathbf{S}^a \\
\mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{B} & \mathbf{R}^c & \mathbf{R}^{S_c} & \mathbf{R}^{S_c} & \mathbf{R}^{S_c} & \mathbf{C} & \mathbf{S}^c \\
\mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{B} & \mathbf{C} & \mathbf{S}^c & \mathbf{C} & \mathbf{C} & \mathbf{R}^c & \mathbf{R}^{S_c} \\
\mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{B} & \mathbf{R}^c & \mathbf{R}^{S_c} & \mathbf{R}^c & \mathbf{R}^c & \mathbf{R}^a & \mathbf{R}^{S_A} \\
\mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{B} & \mathbf{C} & \mathbf{C} & \mathbf{S}^c & \mathbf{C} & \mathbf{A} & \mathbf{A} \\
\mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{B} & \mathbf{R}^c & \mathbf{R}^c & \mathbf{R}^{S_c} & \mathbf{R}^{S_c} & \mathbf{C} & \mathbf{C} \\
\mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{B} & \mathbf{C} & \mathbf{C} & \mathbf{C} & \mathbf{S}^c & \mathbf{R}^c & \mathbf{R}^c \\
\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{B} & \mathbf{R}^c & \mathbf{R}^c & \mathbf{R}^{S_c} & \mathbf{R}^{S_c} & \mathbf{R}^a & \mathbf{R}^a
\end{bmatrix}
$$

<u>Proposition 8:</u> It is possible to construct a $2^{15} \bullet s^{13}$ OLEP, ${}_s\mathbf{X}_{16s}$ in $16s$ runs as shown in Figure 4.8, where, **0** and **1** are column vectors of zeros and ones.

Figure 4.8: Design Template for the ${}_{s}X_{16s}$ Matrix in 16s Runs

$$
{}_{s}\mathbf{X}_{16s}=\begin{bmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\mathbf{B}&\mathbf{C}&\mathbf{S}^{c}&\mathbf{S}^{c}&\mathbf{S}^{c}&\mathbf{A}&\mathbf{S}^{a}&\mathbf{S}^{a}&\mathbf{S}^{a}&\mathbf{R}^{a}&\mathbf{R}^{S_A}&\mathbf{R}^{S_A}&\mathbf{R}^{S_A}\\
1&1&1&0&1&0&0&1&1&1&0&1&0&0&0&\mathbf{B}&\mathbf{R}^{c}&\mathbf{R}^{S_c}&\mathbf{R}^{S_c}&\mathbf{R}^{S_c}&\mathbf{C}&\mathbf{S}^{c}&\mathbf{S}^{c}&\mathbf{S}^{c}&\mathbf{A}&\mathbf{S}^{a}&\mathbf{S}^{a}&\mathbf{S}^{a}\\
1&1&0&1&0&1&0&1&1&0&1&0&1&0&0&\mathbf{B}&\mathbf{C}&\mathbf{S}^{c}&\mathbf{S}^{c}&\mathbf{S}^{c}&\mathbf{R}^{c}&\mathbf{R}^{S_c}&\mathbf{R}^{S_c}&\mathbf{R}^{S_c}&\mathbf{C}&\mathbf{S}^{c}&\mathbf{S}^{c}&\mathbf{S}^{c}\\
0&0&1&1&1&1&0&0&0&1&1&1&1&0&0&\mathbf{B}&\mathbf{R}^{c}&\mathbf{R}^{S_c}&\mathbf{R}^{S_c}&\mathbf{R}^{S_c}&\mathbf{R}^{a}&\mathbf{R}^{S_A}&\mathbf{R}^{S_A}&\mathbf{R}^{S_A}&\mathbf{R}^{c}&\mathbf{R}^{S_c}&\mathbf{R}^{S_c}&\mathbf{R}^{S_c}\\
1&0&1&1&0&0&1&1&0&1&1&0&0&1&0&\mathbf{B}&\mathbf{C}&\mathbf{S}^{c}&\mathbf{C}&\mathbf{C}&\mathbf{A}&\mathbf{S}^{a}&\mathbf{A}&\mathbf{A}&\mathbf{C}&\mathbf{S}^{c}&\mathbf{R}^{a}&\mathbf{C}\\
0&1&0&1&1&0&1&0&1&0&1&1&0&1&0&\mathbf{B}&\mathbf{R}^{c}&\mathbf{R}^{S_c}&\mathbf{R}^{c}&\mathbf{R}^{c}&\mathbf{C}&\mathbf{S}^{c}&\mathbf{C}&\mathbf{C}&\mathbf{R}^{c}&\mathbf{R}^{S_c}&\mathbf{A}&\mathbf{R}^{c}\\
0&1&1&0&0&1&1&0&1&1&0&0&1&1&0&\mathbf{B}&\mathbf{C}&\mathbf{S}^{c}&\mathbf{C}&\mathbf{C}&\mathbf{R}^{c}&\mathbf{R}^{S_c}&\mathbf{R}^{c}&\mathbf{R}^{c}&\mathbf{R}^{a}&\mathbf{R}^{S_A}&\mathbf{C}&\mathbf{R}^{a}\\
1&0&0&0&1&1&1&1&0&0&0&1&1&1&0&\mathbf{B}&\mathbf{R}^{c}&\mathbf{R}^{S_c}&\mathbf{R}^{c}&\mathbf{R}^{c}&\mathbf{R}^{a}&\mathbf{R}^{S_A}&\mathbf{R}^{a}&\mathbf{R}^{a}&\mathbf{A}&\mathbf{S}^{a}&\mathbf{R}^{c}&\mathbf{A}\\
1&1&1&1&1&1&1&0&0&0&0&0&0&0&1&\mathbf{B}&\mathbf{C}&\mathbf{C}&\mathbf{S}^{c}&\mathbf{C}&\mathbf{A}&\mathbf{A}&\mathbf{S}^{a}&\mathbf{A}&\mathbf{R}^{a}&\mathbf{R}^{a}&\mathbf{S}^{c}&\mathbf{R}^{a}\\
0&0&0&1&0&1&1&1&1&1&0&1&0&0&1&\mathbf{B}&\mathbf{R}^{c}&\mathbf{R}^{c}&\mathbf{R}^{S_c}&\mathbf{R}^{c}&\mathbf{C}&\mathbf{C}&\mathbf{S}^{c}&\mathbf{C}&\mathbf{A}&\mathbf{A}&\mathbf{R}^{S_c}&\mathbf{A}\\
0&0&1&0&1&0&1&1&1&0&1&0&1&0&1&\mathbf{B}&\mathbf{C}&\mathbf{C}&\mathbf{S}^{c}&\mathbf{C}&\mathbf{R}^{c}&\mathbf{R}^{a}&\mathbf{R}^{S_c}&\mathbf{R}^{c}&\mathbf{C}&\mathbf{C}&\mathbf{R}^{S_A}&\mathbf{C}\\
1&1&0&0&0&0&1&0&0&1&1&1&1&0&1&\mathbf{B}&\mathbf{R}^{c}&\mathbf{R}^{c}&\mathbf{R}^{S_c}&\mathbf{R}^{c}&\mathbf{R}^{a}&\mathbf{R}^{c}&\mathbf{R}^{S_A}&\mathbf{R}^{a}&\mathbf{R}^{c}&\mathbf{R}^{c}&\mathbf{S}^{a}&\mathbf{R}^{c}\\
0&1&0&0&1&1&0&1&0&1&1&0&0&1&1&\mathbf{B}&\mathbf{C}&\mathbf{C}&\mathbf{C}&\mathbf{S}^{c}&\mathbf{A}&\mathbf{A}&\mathbf{A}&\mathbf{S}^{a}&\mathbf{C}&\mathbf{C}&\mathbf{C}&\mathbf{S}^{c}\\
1&0&1&0&0&1&0&0&1&0&1&1&0&1&1&\mathbf{B}&\mathbf{R}^{c}&\mathbf{R}^{c}&\mathbf{R}^{c}&\mathbf{R}^{S_c}&\mathbf{C}&\mathbf{C}&\mathbf{C}&\mathbf{S}^{c}&\mathbf{R}^{c}&\mathbf{R}^{c}&\mathbf{R}^{c}&\mathbf{R}^{S_c}\\
1&0&0&1&1&0&0&0&1&1&0&0&1&1&1&\mathbf{B}&\mathbf{C}&\mathbf{C}&\mathbf{C}&\mathbf{S}^{c}&\mathbf{R}^{c}&\mathbf{R}^{c}&\mathbf{R}^{c}&\mathbf{R}^{S_c}&\mathbf{R}^{a}&\mathbf{R}^{a}&\mathbf{R}^{a}&\mathbf{R}^{S_A}\\
0&1&1&1&0&0&0&1&0&0&0&1&1&1&1&\mathbf{B}&\mathbf{R}^{c}&\mathbf{R}^{c}&\mathbf{R}^{c}&\mathbf{R}^{S_c}&\mathbf{R}^{a}&\mathbf{R}^{a}&\mathbf{R}^{a}&\mathbf{R}^{S_A}&\mathbf{A}&\mathbf{A}&\mathbf{A}&\mathbf{S}^{a}
\end{bmatrix}
$$

4.6 CONCLUDING REMARKS

The discussion on the *method of symmetric constructions* and the related construction of OLEPs is now complete. The next chapter will discuss techniques to modify the orthogonal linear effect plans so as to remedy the problem of correlated higher order effects and illustrate with examples, methods to incorporate any user specified factor combination(s).

Chapter 5

APPLICATIONS AND EXTENSIONS OF ORTHOGONAL LINEAR EFFECT PLANS

"Experiments are the only means of knowledge at everyone's disposal. The rest is poetry, and imagination."

...Max Planck

"But what has been said once, can always be repeated."

...Zeno

5.0 INTRODUCTION

This chapter is intended to extend the definition and scope of orthogonal linear effect plans (OLEPs) that were introduced in Chapter 4 and discuss applications, underlying assumptions, and advantages of *the method of symmetric constructions.*

5.1 MODIFYING ORTHOGONAL LINEAR EFFECT PLANS

The *method of symmetric constructions* was defined and described as a viable technique for the construction of OLEPs involving $2^k \bullet s^p$ factor combinations, whilst mentioning the drawback that the higher order main effects for the factors are correlated with each other. To better illustrate this, a ${}_3\mathbf{X}_{12}$ design, (i.e., a $2^3 \bullet 3^4$ OLEP in 12 runs) is dissected into its component factor main effect columns, using orthogonal linear transforms as is representative of a typical regression analysis in Figure 5.1. The results of the inter-intra column correlation coefficients of the matrix are also presented.

It is easily noted that the linear main effect(s) of the factors A - G are uncorrelated with each other while the quadratic effects, i.e., D^2, E^2, F^2, G^2 are correlated with each other and this is why they are called orthogonal linear effect plans. This research does not address de-merits of these correlated higher order effects since near-orthogonal arrays are indeed used widely in industry (Nguyen, 1996; Wang and Wu, 1991), primarily because of advantages of economic run size and usefulness of estimates derived therefrom in effecting decisions.

The orthogonal linear transforms that are used to represent the linear and quadratic coefficients for the 2-level and 3-level factors shown in Figure 5.1 and for other higher level factors are shown in Table 5.1. Fisher and Yates (1957) present complete tables of orthogonal transforms for up to 75-level equally spaced factors.

Figure 5.1: Inter-Intra Column Correlation Coefficient Calculations for the $_3X_{12}$ Matrix

$_3X_{12}$ =

A	B	C	D	E	F	G
0	0	0	0	1	1	0
0	0	0	1	2	0	1
0	0	0	2	0	2	2
0	1	1	0	0	2	1
0	1	1	1	2	0	2
0	1	1	2	1	1	0
1	0	1	0	1	1	0
1	0	1	1	2	2	2
1	0	1	2	0	0	1
1	1	0	0	0	0	2
1	1	0	1	2	2	1
1	1	0	2	1	1	0

=

A	B	C	D	D^2	E	E^2	F	F^2	G	G^2
-1	-1	-1	-1	1	0	-2	0	-2	-1	1
-1	-1	-1	0	-2	1	1	-1	1	0	-2
-1	-1	-1	1	1	-1	1	1	1	1	1
-1	1	1	-1	1	-1	1	1	1	0	-2
-1	1	1	0	-2	1	1	-1	1	1	1
-1	1	1	1	1	0	-2	0	-2	-1	1
1	-1	1	-1	1	0	-2	0	-2	-1	1
1	-1	1	0	-2	1	1	1	1	1	1
1	-1	1	1	1	-1	1	-1	1	0	-2
1	1	-1	-1	1	-1	1	-1	1	1	1
1	1	-1	0	-2	1	1	1	1	0	-2
1	1	-1	1	1	0	-2	0	-2	-1	1

Results of the inter-intra column correlation calculations for the $_3X_{12}$ matrix

	A	B	C	D	D^2	E	E^2	F	F^2	G	G^2
A	1										
B	0	1									
C	0	0	1								
D	0	0	0	1							
D^2	0	0	0	0	1						
E	0	0	0	0	-0.87	1					
E^2	0	0	0	0	-0.5	0	1				
F	0	0	0	0	0	0	0	1			
F^2	0	0	0	0	-0.5	0	1	0	1		
G	0	0	0	0	-0.43	0	0.87	0	0.87	1	
G^2	0	0	0	0	0.25	0	-0.5	0	-0.5	0	1

Table 5.1: Table of Orthogonal Transforms

		Ordered factor levels					
# of factor levels, s	Orthogonal Transform	0	1	2	3	4	
2	Linear	-1	1				
3	Linear	-1	0	1			
	Quadratic	1	-2	1			
4	Linear	-3	-1	1	3		
	Quadratic	1	-1	-1	1		
	Cubic	-1	3	-3	1		
5	Linear	-2	-1	0	1	2	
	Quadratic	2	-1	-2	-1	2	
	Cubic	-1	2	0	-2	1	
	Quartic	1	-4	6	-4	1	
6	Linear	-5	-3	-1	1	3	5
	Quadratic	5	-1	-4	-4	-1	5
	Cubic	-5	7	4	-4	-7	5
	Quartic	1	-3	2	2	-3	1
	Quintic	-1	5	-10	10	-5	1

Methods to modify and extend the OLEPs to include higher level factors will be discussed and in this regard, the following propositions are stated without proof.

Proposition 1: It is possible to replace a s-level factor and a 2-level factor with a 2s-level factor as shown below:

2-level	***s-level***		**2*s*-level**
0	0		0
0	1		1
0	2		2
.	.		.
.	.		.
0	*s*-2	Replace →	*s*-2
0	*s*-1	← Collapse	*s*-1
1	0		*s*
1	1		*s*+1
1	2		.
.	.		.
.	.		.
1	*s*-2		2*s*-2
1	*s*-1		2*s*-1

By an extension of the above proposition, it is also possible to replace a 2s level factor with a s-level and a 2-level factor. This is illustrated with the example of collapsing (and replacing) a 2-level and a 3-level factor into a 6-level factor in the following page.

2-level	*3-level*		*6-level*
0	0		0
0	1	Replace →	1
0	2		2
1	0	← Collapse	3
1	1		4
1	2		5

<u>Proposition 2:</u> It is possible to collapse a 2s-level factor into a k-level factor, where, $2s<k<s$ as illustrated below for the case of $s=3$.

2-level &	3-level		6-level		5-level		4-level				
0	0		0		0		0		0		0
0	1		1		1		0		1		1
0	2	***Replace*** →	2	***Collapse*** →	2	***Collapse*** →	1	or	2	or	1
1	0		3		2		2		2		2
1	1		4		3		2		3		3
1	2		5		4		3		3		3

The above propositions are based largely on Addelman's (1962a) principle of replacement and collapsing based on the concept of proportional frequencies.

The principle of replacement and collapsing of lower level factor into a higher order factor level can be used to modify OLEPs to include a mix of several factor levels and allow for flexibility in experimenting with a greater mix of factor levels. In addition, this principle of replacement and collapsing is necessary to modify the *swapped column*, the *switches* in the *main* or *step-down* zones of the OLEP into a higher order factor. This is because the swapped column and the switches are generated solely by a symmetric exchange of elements of the stem about the origin. Consequently, the quadratic effect (and other even higher order effects) for the factors represented by these columns will be completely correlated with the factor represented by the corresponding stem. For example, in Figure 5.1, observe that the quadratic effect of factor **F** is completely correlated with that of factor **E**, where factors **E** and **F** belong to the main zone of the $\mathbf{X}_{12}$ matrix. Since the *swapped column* and *switches* are possible only in **X** matrices of order≥4s, the following discussion details techniques to replace the switches with higher order factors.

Consider the example of a **X** matrix of order $2^n s$, $n \geq 2$. The *swapped column* and *switches,* are combined with the 2-level factors and are replaced by 2s-level factors to modify the OLEPs.

The 2-level columns chosen to combine with the s-level factor must be columns generated using Fisher's theory of confounding. This is illustrated in Figure 5.2 with the example of modifying the ${}_s\mathbf{X}_{4s}$ and ${}_s\mathbf{X}_{8s}$ matrices. The extension of this technique to the ${}_s\mathbf{X}_{16s}$ matrix is straightforward and obvious.

Figure 5.2: Techniques to Modify the ${}_s\mathbf{X}_{4s}$ and ${}_s\mathbf{X}_{8s}$ Matrices

Combine these two columns to generate a 2s-level factor

$$
{}_s\mathbf{X}_{4s} = \begin{bmatrix}
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B} & \mathbf{C} & \mathbf{S}^c & \mathbf{A} \\
\mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{B} & \mathbf{R}^s & \mathbf{R}^s & \mathbf{C} \\
\mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{B} & \mathbf{C} & \mathbf{C} & \mathbf{R}^c \\
\mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{B} & \mathbf{R}^c & \mathbf{R}^c & \mathbf{R}^a
\end{bmatrix}
$$

Combine these columns to generate four 2s-level factor columns

$$
{}_s\mathbf{X}_{8s} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{B} & \mathbf{C} & \mathbf{S}^c & \mathbf{S}^c & \mathbf{S}^c & \mathbf{A} & \mathbf{S}^a \\
1 & 1 & 1 & 0 & 1 & 0 & 0 & \mathbf{B} & \mathbf{R}^c & \mathbf{R}^{Sc} & \mathbf{R}^{Sc} & \mathbf{R}^{Sc} & \mathbf{C} & \mathbf{S}^c \\
1 & 1 & 0 & 1 & 0 & 1 & 0 & \mathbf{B} & \mathbf{C} & \mathbf{S}^c & \mathbf{C} & \mathbf{C} & \mathbf{R}^c & \mathbf{R}^{Sc} \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & \mathbf{B} & \mathbf{R}^c & \mathbf{R}^{Sc} & \mathbf{R}^c & \mathbf{R}^c & \mathbf{R}^a & \mathbf{R}^{SA} \\
1 & 0 & 1 & 1 & 0 & 0 & 1 & \mathbf{B} & \mathbf{C} & \mathbf{C} & \mathbf{S}^c & \mathbf{C} & \mathbf{A} & \mathbf{A} \\
0 & 1 & 0 & 1 & 1 & 0 & 1 & \mathbf{B} & \mathbf{R}^c & \mathbf{R}^c & \mathbf{R}^{Sc} & \mathbf{R}^{Sc} & \mathbf{C} & \mathbf{C} \\
0 & 1 & 1 & 0 & 0 & 1 & 1 & \mathbf{B} & \mathbf{C} & \mathbf{C} & \mathbf{C} & \mathbf{S}^c & \mathbf{R}^c & \mathbf{R}^c \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & \mathbf{B} & \mathbf{R}^c & \mathbf{R}^c & \mathbf{R}^{Sc} & \mathbf{R}^{Sc} & \mathbf{R}^a & \mathbf{R}^a
\end{bmatrix}
$$

This principle of replacement and collapsing is illustrated with an example. Figure 5.3 illustrates a $2^3 \bullet 3^4$, i.e., a ${}_3\mathbf{X}_{12}$ OLEP, (as was used in Figure 5.1) which has been modified into a $2^2 \bullet 3^3 \bullet 6^1$ OLEP.

Figure 5.3: Steps in Modifying a $2^3 3^4$ OLEP

Original $2^3 3^4$ OLEP in 12 runs

${}_3X_{12}$ =

A	B	C	D	E	F	G
0	0	0	1	2	0	1
1	1	0	0	0	0	2
0	1	1	1	2	0	2
1	0	1	2	0	0	1
0	0	0	0	1	1	0
1	1	0	2	1	1	0
0	1	1	2	1	1	0
1	0	1	0	1	1	0
0	0	0	2	0	2	2
1	1	0	1	2	2	1
0	1	1	0	0	2	1
1	0	1	1	2	2	2

Note: The columns of the original matrix have been re-arranged for sake of convenience

=

Modified $2^2 3^3 6$ OLEP in 12 runs

A	B	D	E	G	H
0	0	1	2	1	0
1	1	0	0	2	0
0	0	0	1	0	1
1	1	2	1	0	1
0	0	2	0	2	2
1	1	1	2	1	2
0	1	1	2	2	3
1	0	2	0	1	3
0	1	2	1	0	4
1	0	0	1	0	4
0	1	0	0	1	5
1	0	1	2	2	5

Note: Factor **H** generated by replacing factors **C** and **F**.

Substitution used:

		→	
0	0	→	0
0	1	→	1
0	2	→	2
1	0	→	3
1	1	→	4
1	2	→	5

This procedure of replacement and collapsing improves the orthogonality of the basic OLEP considerably and will be demonstrated with an example. However, the modified matrix remains an OLEP, since the even higher order effects of the modified columns remain correlated with each other, although to a much lesser degree. The improvement in the orthogonality of the basic design is demonstrated with a complete inter-intra column correlation analysis for the modified $2^2 \bullet 3^3 \bullet 6^1$ OLEP, and the results are presented in Figure 5.4. Also, observe that this matrix can (theoretically) be used to estimate 13 main effects (13=2*1+3*2+1*5) in 12 runs, making the design supersaturated.

Figure 5.4: Inter-Intra Column Correlation Coefficient Calculations for the Modified $_3X_{12}$ Matrix

$_3X_{12}$ =

A	B	D	E	G	H
0	0	1	2	1	0
1	1	0	0	2	0
0	0	0	1	0	1
1	1	2	1	0	1
0	0	2	0	2	2
1	1	1	2	1	2
0	1	1	2	2	3
1	0	2	0	1	3
0	1	2	1	0	4
1	0	0	1	0	4
0	1	0	0	1	5
1	0	1	2	2	5

=

A	B	D	D^2	E	E^2	G	G^2	H	H^2	H^3	H^4	H^5
-1	-1	0	-2	1	1	0	-2	-5	5	-5	1	-1
1	1	-1	1	-1	1	1	1	-5	5	-5	1	-1
-1	-1	-1	1	0	-2	-1	1	-3	-1	7	-3	5
1	1	1	1	0	-2	-1	1	-3	-1	7	-3	5
-1	-1	1	1	-1	1	1	1	-1	-4	4	2	-10
1	1	0	-2	1	1	0	-2	-1	-4	4	2	-10
-1	1	0	-2	1	1	1	1	1	-4	-4	2	10
1	-1	1	1	-1	1	0	-2	1	-4	-4	2	10
-1	1	1	1	0	-2	-1	1	3	-1	-7	-3	-5
1	-1	-1	1	0	-2	-1	1	3	-1	-7	-3	-5
-1	1	-1	1	-1	1	0	-2	5	5	5	1	1
1	-1	0	-2	1	1	1	1	5	5	5	1	1

Note: This is the modified $_3X_{12}$ matrix

Results of the inter-intra column correlation calculations for the modified $_3X_{12}$ matrix

	A	B	D	D^2	E	E^2	G	G^2	H	H^2	H^3	H^4	H^5
A	1												
B	0	1											
D	0	0	1										
D^2	0	0	0	1									
E	0	0	0	-0.87	1								
E^2	0	0	0	-0.5	0	1							
G	0	0	0	-0.43	0	0.866	1						
G^2	0	0	0	0.25	0	-0.5	0	1					
H	0	0	0	0	0	0	0	0	1				
H^2	0	0	-0.49	-0.09	0	0.189	0.164	-0.09	0	1			
H^3	0	0	0	0	0	0	0	0	0	0	1		
H^4	0	0	0.094	-0.49	0	0.982	0.85	-0.49	0	0	0	1	
H^5	0	0	0	0	0	0	0	0	0	0	0	0	1

Also, this modified $2^2\bullet3^3\bullet6^1$ OLEP may be further modified to incorporate a 12-level factor, or the 6-level factor can be collapsed to a five or four-level factor using proposition 2.

5.2 APPLICATIONS FOR OLEPs

The design plans that have been described in Chapters 4 and 5 are all intended for use in screening experiments, as is relevant for preliminary industrial experimentation. A very elaborate listing of OLEPs constructed using *the method of symmetric constructions* is included in Appendix A, which also includes guidelines for modifying and augmenting higher level factors with a basic $2^k\bullet s^p$ plan. However, it is only appropriate the assumptions underlying the use of these design plans be stated.

The most important assumption is that interactions, if any, among the factors are negligible, and only the main effects of the factors are to be estimated. The main effect estimates are then used to identify and select significant factors relevant for further study. Also, in Table 5.1, the orthogonal polynomials that have been presented are applicable only for equally spaced factors, or factor values which can be transformed into equally spaced variates (e.g., a logarithmic transformation). Orthogonal polynomials may however be derived for unequally spaced levels, although, the mathematical manipulations increase in complexity as the number of levels increase. Though the interpretation of a factor's higher order main effect(s) bears relevance only for quantitative factors, the design plans may be made to accommodate qualitative even level (e.g., 2-level, 4-level, etc.) factors. This is possible because the 2-level columns in the OLEPs are designed such that each level is tested at every other level for the other factors, and so an average estimate of its effect at each level will be complete and based on equal frequencies. Finally, the potential drawback of these OLEPs is that even higher order effects are correlated with each other. Depending on the experimenter's preference, this may or may not be considered relevant considered the widespread use of nearly orthogonal arrays for industrial experimentation.

The advantages, however, accrue from the fact that all the OLEPs incur the theoretically minimum number of runs possible and the possibilities for modifying and incorporating any user-specified factor combination using the propositions stated earlier are numerous, limited only by the ingenuity of the experimenter. Moreover, the method of symmetric constructions makes redundant earlier restrictions that factor levels need to be prime powers, as is made evident in the simplicity of construction of the $2^7\bullet6^7$ OLEP tabulated in Appendix A.

Apart from their intended use in screening experiments, it is envisaged that the OLEPs may be used for some additional uses as presented in the following page.

- Useful for Accelerated Reliability Testing and/or Performance Parameter (e.g., failure rate) estimation, wherein, additive linear models may be derived from logarithmic transformation of experimental data.
- These OLEPs may be useful as random balance designs, extending ideas proposed by Satterthwaite (1959), wherein the analysis is to declare the treatment combination with the highest (or desired) response as the winner (pick-the-winner) approach. Also, if the number of treatment combinations used in the experiment is more than the number of factors, a multiple regression analysis using only the linear main effects may be used assuming all the other effects and interaction as negligible.
- OLEPs would be ideal for screening in situations involving only qualitative factors, e.g., supplier-customer-machine evaluations, non-parametric modeling for marketing decisions involving qualitative customer-focused factors, coffee tasting, etc.

5.3 ANALYSIS TECHNIQUES FOR OLEPs

This section shall attempt to highlight possible techniques for analyzing the results from a screening experiment. Box and Meyer (1986) present a technique for analyzing unreplicated fractional factorial experiments, and Barton (1998) presents a novel approach for graphically summarizing the results from a fractional factorial experiment. These techniques are essentially intended for quantitative factors, although references are provided in Davies (1971), Anderson and McLean (1974), Hicks (1982), and Cox (1958) about analysis involving qualitative factors.

5.4 CONCLUDING REMARKS

This chapter presented techniques for modifying OLEPs and to incorporate any user-specified factor combination into a $2^k \bullet s^p$ OLEP, including generic rules for replacing and collapsing of factor levels to allow for generation of saturated mixed model design plans. Also, related issues regarding orthogonality of higher order effect estimates were discussed, and the advantages and assumptions underlying OLEPs were presented. This concludes the scope of this research undertaking and the next chapter summarizes the research effort complete with recommendations for future research.

Chapter 6

SUMMARY

"Whenever one lights upon more exact proofs, then we must be grateful to the discoverer; but, for the present, we must state what seems plausible."

...Aristotle

"What is beautiful, definite and the object of knowledge is by nature prior to the indefinite, the incomprehensible and the ugly."

...Nicomachus

6.0 SUMMARY

This thesis was concerned with orthogonal design plans for screening experiments, which would permit the estimation of all factor main effects, so as to allow for judgmental inference about (non) significance of a factor(s). To this end, a technique called the *method of symmetric constructions* for construction of design plans was developed which yielded:

(i) Uncorrelated estimates of all linear main effects

(ii) Slightly correlated estimates of higher even-order main effects.

The *method of symmetric constructions* has been elaborated in Chapter 4 and relevant terminology developed therein. Finally, the construction of $2^k \bullet s^p$ design plans in $s(1+k)$ runs has been detailed.

Techniques for modifying a $2^k \bullet s^p$ orthogonal linear effect plan to incorporate higher order factors have been discussed in Chapter 5, complete with suggestions for use and analysis of the design plans.

6.1 SCOPE FOR FUTURE RESEARCH

This section highlights possibilities for extending the scope of the ideas initiated in this report and to the subject of orthogonal screening in general. Ideas that merit further investigation include:

1. Establish valid rules for replacing and/or collapsing factor levels so as to maximize information content per observation.

2. Simulate and validate performance of OLEPs.

3. Undertake a comparative study to compare designs based on the principle of orthogonality and others based on criteria such as D-, E-, A-, V-, G-optimality, etc.

4. Investigate possibilities for modifying OLEPs to allow for estimation of main effects and some/all interactions.

5. Consolidate all available methods of construction for OMEPs including the *method of symmetric constructions*, as a computer program to generate design plans tailored to user's needs.

6. Attempt formal proofs for all the propositions stated in Chapters 4 and 5 so as to extend the *method of symmetric constructions* as a generic and robust technique, independent in its approach and method of solution.

7. Investigate the potential for using the concept of *random balance designs*, and establish precedent for use of OLEPs in 'pick-the-winner' solution methods.

An age or a culture is characterized less by the extent of its knowledge than by the nature of the questions it puts forward (Jacob, 1989). To this end, it is hoped that the *method of symmetric constructions*, while being just a man-made, conceptual shorthand to abstract and shape the author's ideas, will prove useful to interested researchers in the times to come. The merits, de-merits and applications of the technique are yet to be fully laundered, and inasmuch as the scope of this report is concerned, it deems necessary to conclude the report with the mention that - *so all things time will mend and so this report shall come to an end.*

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...Eswar

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Appendix A: Tables Of Useful Orthogonal Linear Effect Plans

Ah, my computations, people say,
Reduced the year to better reckoning? - Nay,
'Twas only striking from the calendar, what was
Unborn tomorrow and dead yesterday.

....Omar Khayyam

TABULATED ORTHOGONAL LINEAR EFFECT PLANS (OLEP) FOR 2-LEVEL AND 3-LEVEL FACTOR COMBINATIONS

	3X24				3X12		3X6		$2*3^2$		2^3*3^4			2^7*3^7
1	0	0	0	0	0	0	0	0	1	1	0	1	1	2
2	0	0	0	0	0	0	0	1	2	0	1	0	0	1
3	0	0	0	0	0	0	0	2	0	2	2	2	2	0
4	1	0	1	0	1	0	1	0	0	2	1	2	2	1
5	1	0	1	0	1	0	1	1	2	0	2	0	0	0
6	1	0	1	0	1	0	1	2	1	1	0	1	1	2
7	1	1	0	0	1	1	0	0	1	1	0	1	1	2
8	1	1	0	0	1	1	0	1	2	2	2	0	2	0
9	1	1	0	0	1	1	0	2	0	0	1	2	0	1
10	0	1	1	0	0	1	1	0	0	0	2	2	0	0
11	0	1	1	0	0	1	1	1	2	2	1	0	2	1
12	0	1	1	0	0	1	1	2	1	1	0	1	1	2
13	1	1	1	1	0	0	0	0	1	1	0	1	1	0
14	1	1	1	1	0	0	0	1	2	0	1	2	2	1
15	1	1	1	1	0	0	0	2	0	2	2	0	0	2
16	0	1	0	1	1	0	1	0	0	2	1	0	0	1
17	0	1	0	1	1	0	1	1	2	0	2	2	2	2
18	0	1	0	1	1	0	1	2	1	1	0	1	1	0
19	0	0	1	1	1	1	0	0	1	1	0	1	1	0
20	0	0	1	1	1	1	0	1	2	2	2	2	0	2
21	0	0	1	1	1	1	0	2	0	0	1	0	2	1
22	1	0	0	1	0	1	1	0	0	0	2	0	2	2
23	1	0	0	1	0	1	1	1	2	2	1	2	0	1
24	1	0	0	1	0	1	1	2	1	1	0	1	1	0

Associate with 6-level factor

Notes on usage and interpretation:

1. All the individual designs are enclosed within marked rectangular regions, which can be selected for use based on experimenter's needs.

2. All designs are named according to the following notation:
   sXk - These designs are of type $2^n.s^p$ requiring k trials
   - The **sXk** type designs involve 2-level and s-level factors

   For e.g.,
   3X12 is a design involving 2-level and 3-level factors requiring 12 runs.

3. The names for the individual designs appear near the top left corner of the rectangular region enclosing the design plan.
   Likewise, the designs that have been shown in the plan alongside are:
   3X24, 3X12, 3X6 the interpretation of which is straightforward.

4. The specifications for the corresponding OLEP is shown near the top right corner of the rectangular region enclosing the design plan.
   For e.g.,
   The specification for the OLEP named 3x12 is $2^3.3^4$, which is interpreted as a design for screening three 2-level and four 3-level factors

5. Association of factors: To collapse &/or replace

2	**2**	**4**
0	0	0
0	1	1
1	0	2
1	1	3

2	**3**	**6**	**5**	**4**
0	0	0	0	0
0	1	1	1	1
0	2	2 →	2 →	1
1	0	3	2	2
1	1	4	3	2
1	2	5	4	3

THE 3X48 DESIGN

TABULATED ORTHOGONAL LINEAR EFFECT PLAN FOR $2^{15}*3^{13}$ FACTOR COMBINATION IN 48 RUNS

	A	B	C	D	A+B	A+C	A+D	B+C	B+D	C+D	A+B+C	A+B+D	A+C+D	B+C+D	A+B+C+D	X1	X2	X3	X4	X5	X6	X7	X8	X9	X10	X11	X12	X13
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	2	2	2	1	1	1	1
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	2	0	0	0	1	1	1	1	0	2	2	2
3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0	2	2	2	2	0	0	0	2	0	0	0
4	0	0	0	1	0	0	1	0	1	1	0	1	1	1	1	0	0	2	2	2	1	1	1	1	2	0	0	0
5	0	0	0	1	0	0	1	0	1	1	0	1	1	1	1	1	2	0	0	0	2	0	0	0	0	2	2	2
6	0	0	0	1	0	0	1	0	1	1	0	1	1	1	1	2	1	1	1	1	0	2	2	2	1	1	1	1
7	0	0	1	0	0	1	0	1	0	1	1	0	1	1	1	0	1	1	1	1	0	2	2	2	1	1	1	1
8	0	0	1	0	0	1	0	1	0	1	1	0	1	1	1	1	2	0	0	0	2	0	0	0	2	0	0	0
9	0	0	1	0	0	1	0	1	0	1	1	0	1	1	1	2	0	2	2	2	1	1	1	1	0	2	2	2
10	0	0	1	1	0	1	1	1	1	0	1	1	0	0	0	0	0	2	2	2	2	0	0	0	0	2	2	2
11	0	0	1	1	0	1	1	1	1	0	1	1	0	0	0	1	2	0	0	0	1	1	1	1	2	0	0	0
12	0	0	1	1	0	1	1	1	1	0	1	1	0	0	0	2	1	1	1	1	0	2	2	2	1	1	1	1
13	0	1	0	0	1	0	0	1	1	0	1	1	0	1	1	0	1	1	1	1	0	2	0	0	1	1	1	1
14	0	1	0	0	1	0	0	1	1	0	1	1	0	1	1	1	2	0	2	2	1	1	1	1	2	0	2	2
15	0	1	0	0	1	0	0	1	1	0	1	1	0	1	1	2	0	2	0	0	2	0	2	2	0	2	0	0
16	0	1	0	1	1	0	1	1	0	1	1	0	1	0	0	0	0	2	0	0	1	1	1	1	0	2	0	0
17	0	1	0	1	1	0	1	1	0	1	1	0	1	0	0	1	2	0	2	2	2	0	2	2	2	0	2	2
18	0	1	0	1	1	0	1	1	0	1	1	0	1	0	0	2	1	1	1	1	0	2	0	0	1	1	1	1
19	0	1	1	0	1	1	0	0	1	1	0	1	1	0	0	0	1	1	1	1	0	2	0	0	1	1	1	1
20	0	1	1	0	1	1	0	0	1	1	0	1	1	0	0	1	2	0	2	2	2	0	2	2	0	2	0	0
21	0	1	1	0	1	1	0	0	1	1	0	1	1	0	0	2	0	2	0	0	1	1	1	1	2	0	2	2
22	0	1	1	1	1	1	1	0	0	0	0	0	0	1	1	0	0	2	0	0	2	0	2	2	2	0	2	2
23	0	1	1	1	1	1	1	0	0	0	0	0	0	1	1	1	2	0	2	2	1	1	1	1	0	2	0	0
24	0	1	1	1	1	1	1	0	0	0	0	0	0	1	1	2	1	1	1	1	0	2	0	0	1	1	1	1
25	1	0	0	0	1	1	1	0	0	0	1	1	1	0	1	0	1	1	1	1	0	0	2	0	1	1	1	1
26	1	0	0	0	1	1	1	0	0	0	1	1	1	0	1	1	2	2	0	2	1	1	1	1	0	0	2	0
27	1	0	0	0	1	1	1	0	0	0	1	1	1	0	1	2	0	0	2	0	2	2	0	2	2	2	0	2
28	1	0	0	1	1	1	0	0	1	1	1	0	0	1	0	0	0	0	2	0	1	1	1	1	2	2	0	2
29	1	0	0	1	1	1	0	0	1	1	1	0	0	1	0	1	2	2	0	2	2	2	0	2	0	0	2	0
30	1	0	0	1	1	1	0	0	1	1	1	0	0	1	0	2	1	1	1	1	0	0	2	0	1	1	1	1
31	1	0	1	0	1	0	1	1	0	1	0	1	0	1	0	0	1	1	1	1	0	0	2	0	1	1	1	1
32	1	0	1	0	1	0	1	1	0	1	0	1	0	1	0	1	2	2	0	2	2	2	0	2	2	2	0	2
33	1	0	1	0	1	0	1	1	0	1	0	1	0	1	0	2	0	0	2	0	1	1	1	1	0	0	2	0
34	1	0	1	1	1	0	0	1	1	0	0	0	1	0	1	0	0	0	2	0	2	2	0	2	0	0	2	0
35	1	0	1	1	1	0	0	1	1	0	0	0	1	0	1	1	2	2	0	2	1	1	1	1	2	2	0	2
36	1	0	1	1	1	0	0	1	1	0	0	0	1	0	1	2	1	1	1	1	0	0	2	0	1	1	1	1
37	1	1	0	0	0	1	1	1	1	0	0	0	1	1	0	0	1	1	1	1	0	0	0	2	1	1	1	1
38	1	1	0	0	0	1	1	1	1	0	0	0	1	1	0	1	2	2	2	0	1	1	1	1	2	2	2	0
39	1	1	0	0	0	1	1	1	1	0	0	0	1	1	0	2	0	0	0	2	2	2	2	0	0	0	0	2
40	1	1	0	1	0	1	0	1	0	1	0	1	0	0	1	0	0	0	0	2	1	1	1	1	0	0	0	2
41	1	1	0	1	0	1	0	1	0	1	0	1	0	0	1	1	2	2	2	0	2	2	2	0	2	2	2	0
42	1	1	0	1	0	1	0	1	0	1	0	1	0	0	1	2	1	1	1	1	0	0	0	2	1	1	1	1
43	1	1	1	0	0	0	1	0	1	1	1	0	0	0	1	0	1	1	1	1	0	0	0	2	1	1	1	1
44	1	1	1	0	0	0	1	0	1	1	1	0	0	0	1	1	2	2	2	0	2	2	2	0	0	0	0	2
45	1	1	1	0	0	0	1	0	1	1	1	0	0	0	1	2	0	0	0	2	1	1	1	1	2	2	2	0
46	1	1	1	1	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	2	2	2	2	0	2	2	2	0
47	1	1	1	1	0	0	0	0	0	0	1	1	1	1	0	1	2	2	2	0	1	1	1	1	0	0	0	2
48	1	1	1	1	0	0	0	0	0	0	1	1	1	1	0	2	1	1	1	1	0	0	0	2	1	1	1	1

TABULATED ORTHOGONAL LINEAR EFFECT PLANS FOR 2-LEVEL AND 4-LEVEL FACTOR COMBINATIONS

	4X32				4X16		4X8		$2*4^2$		2^3*4^4			2^7*4^7
1	0	0	0	0	0	0	0	0	1	2	3	2	2	0
2	0	0	0	0	0	0	0	1	2	1	0	1	1	3
3	0	0	0	0	0	0	0	2	3	0	1	0	0	2
4	0	0	0	0	0	0	0	3	0	3	2	3	3	1
5	1	0	1	0	1	0	1	0	0	3	1	3	3	2
6	1	0	1	0	1	0	1	1	3	0	2	0	0	1
7	1	0	1	0	1	0	1	2	2	1	3	1	1	0
8	1	0	1	0	1	0	1	3	1	2	0	2	2	3
9	1	1	0	0	1	1	0	0	1	1	0	2	1	3
10	1	1	0	0	1	1	0	1	2	2	3	1	2	0
11	1	1	0	0	1	1	0	2	3	3	2	0	3	1
12	1	1	0	0	1	1	0	3	0	0	1	3	0	2
13	0	1	1	0	0	1	1	0	0	0	2	3	0	1
14	0	1	1	0	0	1	1	1	3	3	1	0	3	2
15	0	1	1	0	0	1	1	2	2	2	0	1	2	3
16	0	1	1	0	0	1	1	3	1	1	3	2	1	0
17	1	1	1	1	0	0	0	0	1	2	3	1	1	3
18	1	1	1	1	0	0	0	1	2	1	0	2	2	0
19	1	1	1	1	0	0	0	2	3	0	1	3	3	1
20	1	1	1	1	0	0	0	3	0	3	2	0	0	2
21	0	1	0	1	1	0	1	0	0	3	1	0	0	1
22	0	1	0	1	1	0	1	1	3	0	2	3	3	2
23	0	1	0	1	1	0	1	2	2	1	3	2	2	3
24	0	1	0	1	1	0	1	3	1	2	0	1	1	0
25	0	0	1	1	1	1	0	0	1	1	0	1	2	0
26	0	0	1	1	1	1	0	1	2	2	3	2	1	3
27	0	0	1	1	1	1	0	2	3	3	2	3	0	2
28	0	0	1	1	1	1	0	3	0	0	1	0	3	1
29	1	0	0	1	0	1	1	0	0	0	2	0	3	2
30	1	0	0	1	0	1	1	1	3	3	1	3	0	1
31	1	0	0	1	0	1	1	2	2	2	0	2	1	0
32	1	0	0	1	0	1	1	3	1	1	3	1	2	3

Associate with 8-level factor

Association:

2	4	8
0	0	0
0	1	1
0	2	2
0	3	3
1	0	4
1	1	5
1	2	6
1	3	7

THE 4X64 DESIGN

TABULATED ORTHOGONAL LINEAR EFFECT PLAN FOR $2^{15} * 4^{13}$ FACTOR COMBINATION IN 64 RUNS

	A	B	C	D	A+B	A+C	A+D	B+C	B+D	C+D	A+B+C	A+B+D	A+C+D	B+C+D	A+B+C+D	X1	X2	X3	X4	X5	X6	X7	X8	X9	X10	X11	X12	X13
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	2	2	2	3	0	0	0	2	1	1	1
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	2	1	1	1	0	3	3	3	1	2	2	2
3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	3	0	0	0	1	2	2	2	0	3	3	3
4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3	0	3	3	3	2	1	1	1	3	0	0	0
5	0	0	0	1	0	0	1	0	1	1	0	1	1	1	1	0	0	3	3	3	1	2	2	2	3	0	0	0
6	0	0	0	1	0	0	1	0	1	1	0	1	1	1	1	1	3	0	0	0	2	1	1	1	0	3	3	3
7	0	0	0	1	0	0	1	0	1	1	0	1	1	1	1	2	2	1	1	1	3	0	0	0	1	2	2	2
8	0	0	0	1	0	0	1	0	1	1	0	1	1	1	1	3	1	2	2	2	0	3	3	3	2	1	1	1
9	0	0	1	0	0	1	0	1	0	1	1	0	1	1	1	0	1	2	2	2	0	3	3	3	1	2	2	2
10	0	0	1	0	0	1	0	1	0	1	1	0	1	1	1	1	2	1	1	1	3	0	0	0	2	1	1	1
11	0	0	1	0	0	1	0	1	0	1	1	0	1	1	1	2	3	0	0	0	2	1	1	1	3	0	0	0
12	0	0	1	0	0	1	0	1	0	1	1	0	1	1	1	3	0	3	3	3	1	2	2	2	0	3	3	3
13	0	0	1	1	0	1	1	1	1	0	1	1	0	0	0	0	0	3	3	3	2	1	1	1	0	3	3	3
14	0	0	1	1	0	1	1	1	1	0	1	1	0	0	0	1	3	0	0	0	1	2	2	2	3	0	0	0
15	0	0	1	1	0	1	1	1	1	0	1	1	0	0	0	2	2	1	1	1	0	3	3	3	2	1	1	1
16	0	0	1	1	0	1	1	1	1	0	1	1	0	0	0	3	1	2	2	2	3	0	0	0	1	2	2	2
17	0	1	0	0	1	0	0	1	1	0	1	1	0	1	1	0	1	2	1	1	3	0	3	3	1	2	1	1
18	0	1	0	0	1	0	0	1	1	0	1	1	0	1	1	1	2	1	2	2	0	3	0	0	2	1	2	2
19	0	1	0	0	1	0	0	1	1	0	1	1	0	1	1	2	3	0	3	3	1	2	1	1	3	0	3	3
20	0	1	0	0	1	0	0	1	1	0	1	1	0	1	1	3	0	3	0	0	2	1	2	2	0	3	0	0
21	0	1	0	1	1	0	1	1	0	1	1	0	1	0	0	0	0	3	0	0	1	2	1	1	0	3	0	0
22	0	1	0	1	1	0	1	1	0	1	1	0	1	0	0	1	3	0	3	3	2	1	2	2	3	0	3	3
23	0	1	0	1	1	0	1	1	0	1	1	0	1	0	0	2	2	1	2	2	3	0	3	3	2	1	2	2
24	0	1	0	1	1	0	1	1	0	1	1	0	1	0	0	3	1	2	1	1	0	3	0	0	1	2	1	1
25	0	1	1	0	1	1	0	0	1	1	0	1	1	0	0	0	1	2	1	1	0	3	0	0	2	1	2	2
26	0	1	1	0	1	1	0	0	1	1	0	1	1	0	0	1	2	1	2	2	3	0	3	3	1	2	1	1
27	0	1	1	0	1	1	0	0	1	1	0	1	1	0	0	2	3	0	3	3	2	1	2	2	0	3	0	0
28	0	1	1	0	1	1	0	0	1	1	0	1	1	0	0	3	0	3	0	0	1	2	1	1	3	0	3	3
29	0	1	1	1	1	1	1	0	0	0	0	0	0	1	1	0	0	3	0	0	2	1	2	2	3	0	3	3
30	0	1	1	1	1	1	1	0	0	0	0	0	0	1	1	1	3	0	3	3	1	2	1	1	0	3	0	0
31	0	1	1	1	1	1	1	0	0	0	0	0	0	1	1	2	2	1	2	2	0	3	0	0	1	2	1	1
32	0	1	1	1	1	1	1	0	0	0	0	0	0	1	1	3	1	2	1	1	3	0	3	3	2	1	2	2
33	1	0	0	0	1	1	1	0	0	0	1	1	1	0	1	0	1	1	2	1	3	3	0	3	2	2	1	2
34	1	0	0	0	1	1	1	0	0	0	1	1	1	0	1	1	2	2	1	2	0	0	3	0	1	1	2	1
35	1	0	0	0	1	1	1	0	0	0	1	1	1	0	1	2	3	3	0	3	1	1	2	1	0	0	3	0
36	1	0	0	0	1	1	1	0	0	0	1	1	1	0	1	3	0	0	3	0	2	2	1	2	3	3	0	3
37	1	0	0	1	1	1	0	0	1	1	1	0	0	1	0	0	0	0	3	0	1	1	2	1	3	3	0	3
38	1	0	0	1	1	1	0	0	1	1	1	0	0	1	0	1	3	3	0	3	2	2	1	2	0	0	3	0
39	1	0	0	1	1	1	0	0	1	1	1	0	0	1	0	2	2	2	1	2	3	3	0	3	1	1	2	1
40	1	0	0	1	1	1	0	0	1	1	1	0	0	1	0	3	1	1	2	1	0	0	3	0	2	2	1	2
41	1	0	1	0	1	0	1	1	0	1	0	1	0	1	0	0	1	1	2	1	0	0	3	0	1	1	2	1
42	1	0	1	0	1	0	1	1	0	1	0	1	0	1	0	1	2	2	1	2	3	3	0	3	2	2	1	2
43	1	0	1	0	1	0	1	1	0	1	0	1	0	1	0	2	3	3	0	3	2	2	1	2	3	3	0	3
44	1	0	1	0	1	0	1	1	0	1	0	1	0	1	0	3	0	0	3	0	1	1	2	1	0	0	3	0
45	1	0	1	1	1	0	0	1	1	0	0	0	1	0	1	0	0	0	3	0	2	2	1	2	0	0	3	0
46	1	0	1	1	1	0	0	1	1	0	0	0	1	0	1	1	3	3	0	3	1	1	2	1	3	3	0	3
47	1	0	1	1	1	0	0	1	1	0	0	0	1	0	1	2	2	2	1	2	0	0	3	0	2	2	1	2
48	1	0	1	1	1	0	0	1	1	0	0	0	1	0	1	3	1	1	2	1	3	3	0	3	1	1	2	1
49	1	1	0	0	0	1	1	1	1	0	0	0	1	1	0	0	1	1	1	2	3	3	3	0	1	1	1	2
50	1	1	0	0	0	1	1	1	1	0	0	0	1	1	0	1	2	2	2	1	0	0	0	3	2	2	2	1
51	1	1	0	0	0	1	1	1	1	0	0	0	1	1	0	2	3	3	3	0	1	1	1	2	3	3	3	0
52	1	1	0	0	0	1	1	1	1	0	0	0	1	1	0	3	0	0	0	3	2	2	2	1	0	0	0	3
53	1	1	0	1	0	1	0	1	0	1	0	1	0	0	1	0	0	0	0	3	1	1	1	2	0	0	0	3
54	1	1	0	1	0	1	0	1	0	1	0	1	0	0	1	1	3	3	3	0	2	2	2	1	3	3	3	0
55	1	1	0	1	0	1	0	1	0	1	0	1	0	0	1	2	2	2	2	1	3	3	3	0	2	2	2	1
56	1	1	0	1	0	1	0	1	0	1	0	1	0	0	1	3	1	1	1	2	0	0	0	3	1	1	1	2
57	1	1	1	0	0	0	1	0	1	1	1	0	0	0	1	0	1	1	1	2	0	0	0	3	2	2	2	1
58	1	1	1	0	0	0	1	0	1	1	1	0	0	0	1	1	2	2	2	1	3	3	3	0	1	1	1	2
59	1	1	1	0	0	0	1	0	1	1	1	0	0	0	1	2	3	3	3	0	2	2	2	1	0	0	0	3
60	1	1	1	0	0	0	1	0	1	1	1	0	0	0	1	3	0	0	0	3	1	1	1	2	3	3	3	0
61	1	1	1	1	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	3	2	2	2	1	3	3	3	0
62	1	1	1	1	0	0	0	0	0	0	1	1	1	1	0	1	3	3	3	0	1	1	1	2	0	0	0	3
63	1	1	1	1	0	0	0	0	0	0	1	1	1	1	0	2	2	2	2	1	0	0	0	3	1	1	1	2
64	1	1	1	1	0	0	0	0	0	0	1	1	1	1	0	3	1	1	1	2	3	3	3	0	2	2	2	1

TABULATED ORTHOGONAL LINEAR EFFECT PLANS FOR 2-LEVEL AND 5-LEVEL FACTOR COMBINATIONS

	5X40				5X20		5X10		$2{*}5^2$		$2^3{*}5^4$			$2^7{*}5^7$
1	0	0	0	0	0	0	0	0	1	3	3	3	3	1
2	0	0	0	0	0	0	0	1	2	2	4	2	2	0
3	0	0	0	0	0	0	0	2	3	1	0	1	1	4
4	0	0	0	0	0	0	0	3	4	0	1	0	0	3
5	0	0	0	0	0	0	0	4	0	4	2	4	4	2
6	1	1	0	0	1	0	1	0	0	4	1	4	4	3
7	1	1	0	0	1	0	1	1	4	0	2	0	0	2
8	1	1	0	0	1	0	1	2	3	1	3	1	1	1
9	1	1	0	0	1	0	1	3	2	2	4	2	2	0
10	1	1	0	0	1	0	1	4	1	3	0	3	3	4
11	1	0	1	0	1	1	0	0	1	1	0	3	1	4
12	1	0	1	0	1	1	0	1	2	2	4	2	2	0
13	1	0	1	0	1	1	0	2	3	3	3	1	3	1
14	1	0	1	0	1	1	0	3	4	4	2	0	4	2
15	1	0	1	0	1	1	0	4	0	0	1	4	0	3
16	0	1	1	0	0	1	1	0	0	0	2	4	0	2
17	0	1	1	0	0	1	1	1	4	4	1	0	4	3
18	0	1	1	0	0	1	1	2	3	3	0	1	3	4
19	0	1	1	0	0	1	1	3	2	2	4	2	2	0
20	0	1	1	0	0	1	1	4	1	1	3	3	1	1
21	1	1	1	1	0	0	0	0	1	3	3	1	1	3
22	1	1	1	1	0	0	0	1	2	2	4	2	2	4
23	1	1	1	1	0	0	0	2	3	1	0	3	3	0
24	1	1	1	1	0	0	0	3	4	0	1	4	4	1
25	1	1	1	1	0	0	0	4	0	4	2	0	0	2
26	0	0	1	1	1	0	1	0	0	4	1	0	0	1
27	0	0	1	1	1	0	1	1	4	0	2	4	4	2
28	0	0	1	1	1	0	1	2	3	1	3	3	3	3
29	0	0	1	1	1	0	1	3	2	2	4	2	2	4
30	0	0	1	1	1	0	1	4	1	3	0	1	1	0
31	0	1	0	1	1	1	0	0	1	1	0	1	3	0
32	0	1	0	1	1	1	0	1	2	2	4	2	2	4
33	0	1	0	1	1	1	0	2	3	3	3	3	1	3
34	0	1	0	1	1	1	0	3	4	4	2	4	0	2
35	0	1	0	1	1	1	0	4	0	0	1	0	4	1
36	1	0	0	1	0	1	1	0	0	0	2	0	4	2
37	1	0	0	1	0	1	1	1	4	4	1	4	0	1
38	1	0	0	1	0	1	1	2	3	3	0	3	1	0
39	1	0	0	1	0	1	1	3	2	2	4	2	2	4
40	1	0	0	1	0	1	1	4	1	1	3	1	3	3

Associate with 10-level factor

Association:

2	5	10
0	0	0
0	1	1
0	2	2
0	3	3
0	4	4
1	0	5
1	1	6
1	2	7
1	3	8
1	4	9

TABULATED ORTHOGONAL LINEAR EFFECT PLANS FOR 2-LEVEL AND 6-LEVEL FACTOR COMBINATIONS

	6X48				6X24		6X12		$2*6^2$		2^3*6^4			2^7*6^7
1	0	0	0	0	0	0	0	0	1	4	3	4	4	2
2	0	0	0	0	0	0	0	1	2	3	4	3	3	1
3	0	0	0	0	0	0	0	2	3	2	5	2	2	0
4	0	0	0	0	0	0	0	3	4	1	0	1	1	5
5	0	0	0	0	0	0	0	4	5	0	1	0	0	4
6	0	0	0	0	0	0	0	5	0	5	2	5	5	3
7	1	1	0	0	1	0	1	0	0	5	1	5	5	4
8	1	1	0	0	1	0	1	1	5	0	2	0	0	3
9	1	1	0	0	1	0	1	2	4	1	3	1	1	2
10	1	1	0	0	1	0	1	3	3	2	4	2	2	1
11	1	1	0	0	1	0	1	4	2	3	5	3	3	0
12	1	1	0	0	1	0	1	5	1	4	0	4	4	5
13	1	0	1	0	1	1	0	0	1	1	0	4	1	5
14	1	0	1	0	1	1	0	1	2	2	5	3	2	0
15	1	0	1	0	1	1	0	2	3	3	4	2	3	1
16	1	0	1	0	1	1	0	3	4	4	3	1	4	2
17	1	0	1	0	1	1	0	4	5	5	2	0	5	3
18	1	0	1	0	1	1	0	5	0	0	1	5	0	4
19	0	1	1	0	0	1	1	0	0	0	2	5	0	3
20	0	1	1	0	0	1	1	1	5	5	1	0	5	4
21	0	1	1	0	0	1	1	2	4	4	0	1	4	5
22	0	1	1	0	0	1	1	3	3	3	5	2	3	0
23	0	1	1	0	0	1	1	4	2	2	4	3	2	1
24	0	1	1	0	0	1	1	5	1	1	3	4	1	2
25	1	1	1	1	0	0	0	0	1	4	3	1	1	3
26	1	1	1	1	0	0	0	1	2	3	4	2	2	4
27	1	1	1	1	0	0	0	2	3	2	5	3	3	5
28	1	1	1	1	0	0	0	3	4	1	0	4	4	0
29	1	1	1	1	0	0	0	4	5	0	1	5	5	1
30	1	1	1	1	0	0	0	5	0	5	2	0	0	2
31	0	0	1	1	1	0	1	0	0	5	1	0	0	1
32	0	0	1	1	1	0	1	1	5	0	2	5	5	2
33	0	0	1	1	1	0	1	2	4	1	3	4	4	3
34	0	0	1	1	1	0	1	3	3	2	4	3	3	4
35	0	0	1	1	1	0	1	4	2	3	5	2	2	5
36	0	0	1	1	1	0	1	5	1	4	0	1	1	0
37	0	1	0	1	1	1	0	0	1	1	0	1	4	0
38	0	1	0	1	1	1	0	1	2	2	5	2	3	5
39	0	1	0	1	1	1	0	2	3	3	4	3	2	4
40	0	1	0	1	1	1	0	3	4	4	3	4	1	3
41	0	1	0	1	1	1	0	4	5	5	2	5	0	2
42	0	1	0	1	1	1	0	5	0	0	1	0	5	1
43	1	0	0	1	0	1	1	0	0	0	2	0	5	2
44	1	0	0	1	0	1	1	1	5	5	1	5	0	1
45	1	0	0	1	0	1	1	2	4	4	0	4	1	0
46	1	0	0	1	0	1	1	3	3	3	5	3	2	5
47	1	0	0	1	0	1	1	4	2	2	4	2	3	4
48	1	0	0	1	0	1	1	5	1	1	3	1	4	3

Associate with 12-level factor

Association:

2	6	12
0	0	0
0	1	1
0	2	2
0	3	3
0	4	4
0	5	5
1	0	6
1	1	7
1	2	8
1	3	9
1	4	10
1	5	11

TABULATED ORTHOGONAL LINEAR EFFECT PLANS FOR 2-LEVEL AND 7-LEVEL FACTOR COMBINATIONS

	7X56				7X28		7X14		$2*7^2$		2^3*7^4			2^7*7^7
1	0	0	0	0	0	0	0	0	1	5	3	5	5	3
2	0	0	0	0	0	0	0	1	2	4	4	4	4	2
3	0	0	0	0	0	0	0	2	3	3	5	3	3	1
4	0	0	0	0	0	0	0	3	4	2	6	2	2	0
5	0	0	0	0	0	0	0	4	5	1	0	1	1	6
6	0	0	0	0	0	0	0	5	6	0	1	0	0	5
7	0	0	0	0	0	0	0	6	0	6	2	6	6	4
8	1	1	0	0	1	0	1	0	0	6	1	6	6	5
9	1	1	0	0	1	0	1	1	6	0	2	0	0	4
10	1	1	0	0	1	0	1	2	5	1	3	1	1	3
11	1	1	0	0	1	0	1	3	4	2	4	2	2	2
12	1	1	0	0	1	0	1	4	3	3	5	3	3	1
13	1	1	0	0	1	0	1	5	2	4	6	4	4	0
14	1	1	0	0	1	0	1	6	1	5	0	5	5	6
15	1	0	1	0	1	1	0	0	1	1	0	5	1	6
16	1	0	1	0	1	1	0	1	2	2	6	4	2	0
17	1	0	1	0	1	1	0	2	3	3	5	3	3	1
18	1	0	1	0	1	1	0	3	4	4	4	2	4	2
19	1	0	1	0	1	1	0	4	5	5	3	1	5	3
20	1	0	1	0	1	1	0	5	6	6	2	0	6	4
21	1	0	1	0	1	1	0	6	0	0	1	6	0	5
22	0	1	1	0	0	1	1	0	0	0	2	6	0	4
23	0	1	1	0	0	1	1	1	6	6	1	0	6	5
24	0	1	1	0	0	1	1	2	5	5	0	1	5	6
25	0	1	1	0	0	1	1	3	4	4	6	2	4	0
26	0	1	1	0	0	1	1	4	3	3	5	3	3	1
27	0	1	1	0	0	1	1	5	2	2	4	4	2	2
28	0	1	1	0	0	1	1	6	1	1	3	5	1	3
29	1	1	1	1	0	0	0	0	1	5	3	1	1	3
30	1	1	1	1	0	0	0	1	2	4	4	2	2	4
31	1	1	1	1	0	0	0	2	3	3	5	3	3	5
32	1	1	1	1	0	0	0	3	4	2	6	4	4	6
33	1	1	1	1	0	0	0	4	5	1	0	5	5	0
34	1	1	1	1	0	0	0	5	6	0	1	6	6	1
35	1	1	1	1	0	0	0	6	0	6	2	0	0	2
36	0	0	1	1	1	0	1	0	0	6	1	0	0	1
37	0	0	1	1	1	0	1	1	6	0	2	6	6	2
38	0	0	1	1	1	0	1	2	5	1	3	5	5	3
39	0	0	1	1	1	0	1	3	4	2	4	4	4	4
40	0	0	1	1	1	0	1	4	3	3	5	3	3	5
41	0	0	1	1	1	0	1	5	2	4	6	2	2	6
42	0	0	1	1	1	0	1	6	1	5	0	1	1	0
43	0	1	0	1	1	1	0	0	1	1	0	1	5	0
44	0	1	0	1	1	1	0	1	2	2	6	2	4	6
45	0	1	0	1	1	1	0	2	3	3	5	3	3	5
46	0	1	0	1	1	1	0	3	4	4	4	4	2	4
47	0	1	0	1	1	1	0	4	5	5	3	5	1	3
48	0	1	0	1	1	1	0	5	6	6	2	6	0	2
49	0	1	0	1	1	1	0	6	0	0	1	0	6	1
50	1	0	0	1	0	1	1	0	0	0	2	0	6	2
51	1	0	0	1	0	1	1	1	6	6	1	6	0	1
52	1	0	0	1	0	1	1	2	5	5	0	5	1	0
53	1	0	0	1	0	1	1	3	4	4	6	4	2	6
54	1	0	0	1	0	1	1	4	3	3	5	3	3	5
55	1	0	0	1	0	1	1	5	2	2	4	2	4	4
56	1	0	0	1	0	1	1	6	1	1	3	1	5	3

Associate with 14-level factor

Association:

2	7	14
0	0	0
0	1	1
0	2	2
0	3	3
0	4	4
0	5	5
0	6	6
1	0	7
1	1	8
1	2	9
1	3	10
1	4	11
1	5	12
1	6	13

VITA

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