A STUDY OF REFLECTING SEQUENCES

FOR LABELED CHAINS

By

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CHAPTER I

PRELIMINARIES

Graph traversal is fundamental for many graph algorithms. Universal traversal sequences provide a traversal strategy for graphs in limited space complexity. The study of universal traversal sequences was introduced by Cook [Ale78]. For a family of edge-labeled graphs, universal traversal sequences traverse all vertices starting at any vertex of each graph in the family. The edge-labeled graphs are regular undirected graphs, all edges incident with each vertex have unique labels. Lower and upper bounds on lengths of universal traversal sequences for regular undirected graphs have been established [Tom90] [BT95] [DF96a]. Good length bounds translate into good time bounds for traversing graphs.

Reflecting sequences were introduced in [Tom90] for proving length lower bounds for universal traversal sequences. Reflecting sequences provide end-to-end traversal in labeled chains. For a labeled chain. like edge-labeled graphs, two edges incident with every interior vertex have unique labels in $\{0,1\}$. Each complete end-to-end traversal is considered a reflection.

The bridge between reflecting sequences and universal traversal sequences is circumnavigation sequences, which were introduced in [BRT89]. Circumnavigation sequences provide cyclic traversal in labeled cycles. The labeled cycles are regular undirected graphs with regularity of 2. A circumnavigation starting at a vertex v is a traversal that returns to v moving in the same direction in which it last exits v.

Reflecting sequences, circumnavigation sequences, and universal traversal sequences are related:

1. A tradeoff exists between the circumnavigation frequency/cycle order and regularity/graph order, and
2. One circumnavigation traversal in a $2n$-vertex labeled cycle behaves like a two reflecting traversals in a n-vertex labeled chain.

The length lower bounds computation for universal traversal sequences involves two reductions:

1. Reducing length lower bounds for universal traversal sequences to that for circumnavigation sequences - motivated by the possible tradeoff between the regularity/circumnavigation frequency and the order of the graph.
2. Reducing length lower bounds for circumnavigation sequences to that for reflecting sequences - motivated by the possible tradeoff between circumnavigation frequency/reflecting frequency and the order of the graph.

The universal traversal sequences for undirected graphs and their variants [Tom90] are introduced below.

1.1 Universal Traversal Sequences for Undirected Graphs

For positive integers d and n such that $d < n$, let $\mathcal{G}(d,n)$ be the set of all connected, d-regular, n-vertex, edge-labeled, undirected graphs $G = (V, E)$. For every edge $\{u, v\} \in E$, there are two labels $l_{u,v}$ and $l_{v,u}$ on both endvertices. For every vertex $u \in V$,

$\{l_{u,v} | \{u,v\} \in E\} = \{0, 1, \cdots, d-1\}$. For each edge-labeled graph $G \in \mathcal{G}(d,n)$, a sequence $U \in \{0, 1, \cdots, d-1\}^*$ traverses a unique sequence of vertices from every staring vertex v_0 in V. If all vertices of G are visited at least once by U, U is said to traverse G starting at v_0. A sequence U is called a universal traversal sequence (UTS) for $\mathcal{G}(d,n)$ if U traverses each $G \in \mathcal{G}(d,n)$ starting at any vertex in G. We let $U(d,n)$ denote the shortest length of UTSs for non-empty $\mathcal{G}(d,n)$, and define $U(d,n) = U(d,n+1)$ in case $\mathcal{G}(d,n)$ is empty. Note that $\mathcal{G}(d,n)$ is not empty if and only if dn is even [BRT89]. Figure 1 shows an example of an edge-labeled graph and two example traversal sequences starting at vertex 0.

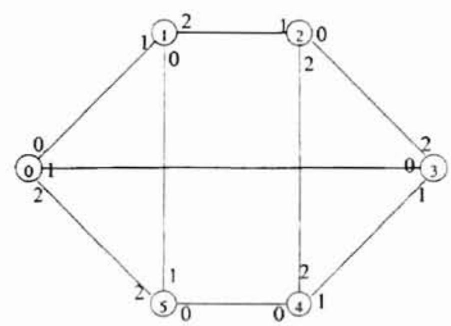

Figure 1. A 6-vertex 3-regular edge-labeled graph with two traversal sequences (02010 and 01212220).

1.2 Circumnavigation Sequences for Labeled Cycles

For edge-labeled cycles $C \in \mathcal{G}(2,n)$, a sequence $U \in \{0,1\}^*$ is said to circumnavigate C t times starting at v_0 if there are at least t times at which the traversal returns to v_0 moving in the same direction in which it last exits v_0. More precisely, U circumnavigates C t times if and only if there exist $0 \leq i_1 < i_2 \leq i_3 < i_4 \leq \cdots \leq i_{2t-1} < i_{2t} \leq |U|$ such that

1. $v_0 = v_{i_1} = v_{i_2} = \cdots = v_{i_{2t}}$,

2. $v_l \neq v_0$ for all i_{2j-1} and $1 \leq j \leq t$, and

3. $v_{i_{2j-1}+1} \neq v_{i_{2j}-1}$, for all $1 \leq j \leq t$.

A sequence U is a t-circumnavigation sequence for $\mathcal{G}(2, n)$ if U circumnavigates each $C \in \mathcal{G}(2, n)$ t times starting at any vertex in C. Let $C(t, n)$ denote the shortest length of t-circumnavigation sequences for $\mathcal{G}(2, n)$. Figure 2 shows an example of an edge-labeled cycle and an example 2 times circumnavigation sequence starting at vertex 0.

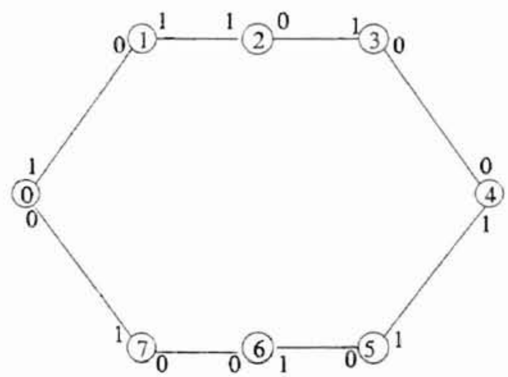

Figure 2. An 8-vertex edge-labeled cycle with a traversal sequence (1100100110010011010110).

1.3 Reflecting Sequence for Labeled Chains

A labeled chain of length n is a graph G with vertex set $V(G) = \{0, 1, \cdots, n\}$ and edge set $E(G) = \{\{i, i+1\} | 0 \leq i \leq n-1\}$. An edge-labeling is defined as follows: Every edge $\{i, i+1\} \in E(G)$ has two labels, $l_{i,i+1}$ and $l_{i+1,i}$ such that

1. $l_{0,1} = l_{n,n-1} = \{0, 1\}$.

2. $l_{i,i-1}$ and $l_{i,i+1}$ from a partition of $\{0,1\}$ for all $1 \leq i \leq n-1$

Let $\mathcal{L}(n)$ be the set of all labeled chains of length n. A labeled chain $G \in \mathcal{L}(n)$ can be identified with the sequence $\alpha = \alpha_1\alpha_2 \cdots \alpha_{n-1} \in \{0, 1\}^{n-1}$ where $l_{i,i+1} = \{\alpha_i\}$

for $1 \leq i \leq n-1$. The sequence α_i is called the label of G. A sequence $U \in \{0,1\}^*$ determines a unique traversal sequence of vertices $(v_0, v_1, \cdots, v_k)$ with starting vertex $v_0 = 0$. For $t \geq 0$, a sequence $U \in \{0,1\}^*$ is said to reflect t times in $G \in \mathcal{L}(n)$ if the endvertices n and 0 are visited alternately by U at least t times. More precisely, there exist $0 < j_1 < j_2 < \cdots < j_t \leq |U|$ such that $v_{j_{2k-1}} = n$ for all $1 \leq k \leq \lceil t/2 \rceil$ and $v_{j_{2k}} = 0$ for all $1 \leq k \leq \lfloor t/2 \rfloor$. A sequence U is a t-reflecting sequence for $\mathcal{L}(n)$ if U reflects t times on each $G \in \mathcal{L}(n)$. We let $R(t,n)$ denote the shortest length of t-reflecting sequences for $\mathcal{L}(n)$. Figure 3 shows an example of an edge-labeled chain and an example 2 times reflecting sequence starting at vertex 0.

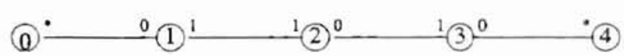

Figure 3. An edge-labeled 100 chain (end labels are denoted by *) with a traversal sequence (111100111110).

The current best known lower bounds on $U(d,n)$ [DF96a] is :

$$U(d,n) = \begin{cases} \Omega(n^{log_7 19}) & \text{if } d = 2 \\ \Omega(d^{2-log_7 19} n^{1+log_7 19}) & \text{if } 3 \leq d \leq \frac{n}{17}+1 \end{cases}$$

Since the study of reflecting sequences for labeled chains of various lengths serves as an important vehicle for length lower bounds for universal traversal sequences, the thesis work focused on the combinatorial nature of reflecting sequences.

CHAPTER II

LENGTH LOWER BOUNDS FOR REFLECTING SEQUENCES

The length lower bounds for UTSs, circumnavigation sequences, and reflecting sequences are closely related [Tom90]. Their relationships are explained below, describing how a good lower bound on $R(t,n)$ translates to a good lower bound on $U(d,n)$.

2.1 Relating U, C, and R

The first theorem below shows the relationship of lower bound between lengths of UTSs and circumnavigation sequences. Notice the tradeoff between the number of circumnavigations and the order of the cycle.

Theorem 1 [BRT89]: Let $d \geq 3$ be an integer and n be a multiple of $8(d-1)$, then

$$U(d,n) \geq \frac{d}{2} C\left(\frac{(d-2)n}{4} + 2, \frac{n}{8(d-1)}\right).$$

The next theorem relates circumnavigation sequences for labeled cycles to reflecting sequences for labeled chains. The underlying idea is to project a cycle onto a chain. Suppose that a labeled chain of length n has a label $\alpha = \alpha_1\alpha_2\cdots\alpha_n$. For $\alpha \in \{0,1\}^*$, define $\overline{\alpha}$ to be the string that results from reversing α and then complementing its bits. For example, if $\alpha = 01000$, then $\overline{\alpha} = 11101$. Construct a $2n$-cycle whose clockwise label from its start vertex is $0\alpha0\overline{\alpha}$. The correspondence between the cirnumnavigation in the labeled cycle and the reflection in the labeled chain is illustrated in the example in Figure 4.

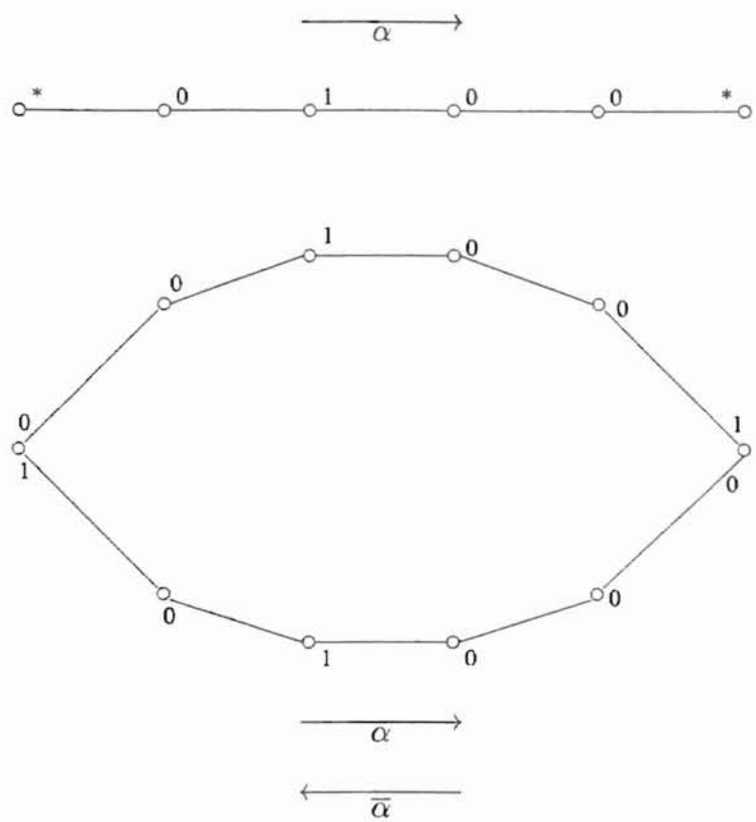

Figure 4. An example of correspondence between the labeled chain and labeled cycle

Theorem 2 [BRT89]: For any positive integers t and n,

$$C(t, 2n) \geq R(2t, n).$$

By combining Theorems 1 and 2, we relate the length lower bounds of universal traversal sequences and reflecting sequences. Again, notice the tradeoff between the number of reflections and the length of the chain.

Theorem 3 [BRT89]: Let $d \geq 3$ be an integer and n be a multiple of $16(d-1)$, then

$$U(d, n) \geq \frac{d}{2} R\left(\frac{(d-2)n}{2} + 4, \frac{n}{16(d-1)}\right).$$

2.2 Recurrence for $R(t, n)$

The following recurrence illustrates a tradeoff between the length of labeled chains and the frequency of reflections. Hence, it suffices to study lengths lower bounds for reflecting sequences on short labeled chains.

Theorem 4 [BRT89]: For all positive integers t, m, and n,

$$R(t, mn) \geq R(R(t, m), n).$$

The next theorem shows that we only need to find constant c, r, and t to obtain the length lower bounds of reflecting sequences. The large $log_c r$ translates into a good lower bound.

Theorem 5 [DF96a]: Suppose that there exists a positive integer c and positive reals $r \geq 2$ and k such that for every positive integer t, $R(t, c) \geq rt - k$. Then for all positive integers t and n that is an integral power of c,

$$R(t, n) \geq (t - k)n^{log_c r}.$$

By Theorems 3, 4, and 5, a lower bound on $U(d, n)$ can be obtained.

Theorem 6 [DF96a]: Suppose that there exists a positive integer $c \geq 2$ and positive reals $r \geq 2$ and k such that for every positive integer t, $R(t, c) \geq rt - k$, then

$$U(d, n) = \begin{cases} \Omega(n^{log_c r}) & \text{if } d = 2 \\ \Omega(d^{2-log_c r} n^{1+log_c r}) & \text{if } 3 \leq d \leq \frac{n}{17} + 1. \end{cases}$$

CHAPTER III

ANALYTICAL APPROACHES

In order to obtain the length lower bounds of reflecting sequences via the recurrence in Theorem 4, the basic lower bounds of short chains are needed. Tompa [Tom90] showed that $R(t,3) \geq 4t$ and $R(t,4) \geq 6t$ using a marking scheme on a hypothetical reflecting sequence. In Tompa's marking scheme, a mark is placed on a pair $\alpha\beta$ (where $\alpha, \beta \in \{0,1\}$) beginning at an even index when the pair causes the last exit from vertex 1 in a complete forward (vertex 0 to vertex n) traversal. For the chain length 3, since a pair has at most one mark, the lower bound is obtained easily. For the chain length 4, a pair may have at most two marks. Since $|\mathcal{L}(4)| = 2^3$, the lower bound is obtained by considering all cases how marks are placed on pairs. The marking scheme yields a lower bound of $R(t,n) \geq tn^{log_4 6}$.

In another marking scheme [BT95], two kinds of marks, called an opening mark and a closing mark, are used. The Buss-Tompa marking scheme was used to obtain length lower bounds of reflecting sequences for labeled chains of length 5. An opening mark is similar to a simple mark in Tompa's marking scheme. In addition, a closing mark on a pair is to delimit a complete left-to-right traversal. If an opening mark or a closing mark is not placed on a pair, an opening debt or a closing debt, respectively, is placed on the pair. This marking scheme improved the lower bound to $R(t,n) \geq tn^{log_5 10}$.

3.1 Tompa's Marking Scheme

Consider a traversal sequence $S \in \{0,1\}^*$ in a labeled chain in $\mathcal{L}(n)$ with label $\alpha = \alpha_1\alpha_2\cdots\alpha_{n-1}$, where $\alpha_i \in \{0,1\}$ for $i = 1, 2, \cdots, n-1$. Within every complete forward traversal from vertex 0 to vertex n induced by S, the last exit from vertex 1 enroute to vertex n corresponds to a pair-substring $\alpha_1\alpha_2$ beginning at an even index on S. We place a mark M_α on the pair $\alpha_1\alpha_2$. An example is depicted in Figure 5.

M_{001}	M_{101}		M_{111}		
M_{000}	M_{100}		M_{110}	M_{011}	
00	10	00	11	01	10

Figure 5. An example of marks on a traversal sequence for labeled chains in $\mathcal{L}(4)$.

The following two theorems show the optimal bounds on the length of reflecting sequences for $\mathcal{L}(3)$ and $\mathcal{L}(4)$, using Tompa's marking scheme.

Lemma 7 [Tom90]: $R(2t-1,3) \geq 8t$, for all positive integers t.

Proof: Let U be a $(2t-1)$-reflecting sequence for $\mathcal{L}(3)$. For each labeled chain with label α in $\mathcal{L}(3)$, U induces at least t complete forward traversals from vertex 0 to vertex 3 in the labeled chain. We mark each of the first t complete forward traversals according to Tompa's marking scheme. Since $|\mathcal{L}(3)| = 2^2 = 4$, a total of $4t$ marks are placed on U (at even indices). Also, none of these marks share the same even index. Thus, $|U| \geq 8t$. ∎

Theorem 8 [Tom90]: $R(t,3) \geq 4t$, for all positive integers t.

Proof: $R(2t,3) \geq R(2t-1,3) + 3 \geq 8t + 3$. Thus, $R(t,3) \geq 4t$. ∎

The bound in Lemma 7 and hence the one in Theorem 8 is tight. The string 1101(10001

$101)^t$ is a $2t$-reflecting sequence for $\mathcal{L}(3)$.

Lemma 9 [Tom90]: $R(2t-1,4) \geq 12t$, for all positive integers t.

Proof: Let U be a $(2t-1)$-reflecting sequence for $\mathcal{L}(4)$. For each labeled chain with label α in $\mathcal{L}(4)$, U induces at least t complete forward traversals from vertex 0 to vertex 4 in the labeled chain. We mark each of the first t complete forward traversals according to Tompa's marking scheme. First, we consider the four labeled chains: 000, 001, 100, and 101, each of which yields marks that begin substrings matching a regular pattern, as shown in Table 1.

Table 1: Substrings of U beginning at marks

label of chain	substring of U must match
000	00(10)*0
001	00(00)*1
100	10(10)*0
101	10(00)*1

A substring of 00 or 10 beginning at an even index on U is called a pair. For these four chains, there are $4t$ marks on the pairs 00 and 10. Despite the fact that two of these marks may share an even index, there $4t$ marks account for at least $3t$ distinct occurrences of the pairs 00 and 10. We notice the following constraints on marked pairs:

1. Any pair with two marks must be followed immediately by another occurrence of either 00 or 10 (see Table 1).
2. Two pairs consecutive in U, one marked "000" and one marked "101", must be followed immediately by a third occurrence of either 00 or 10 (see Table 1).
3. Two pairs with the same mark must be separated by a distance of at least 8, to account for two reflections.

To show the marks-to-pairs density of at most $\frac{4}{3}$, we argue that at least 3 pairs can be charged to every 4 distinct marks from the labeled chains with labels 000, 001, 100, and 101 as follows.

1. For the case of two consecutive pairs in U, each with two marks: By constraint 1, an immediate third occurrence is either 00 or 10 pair. By constraint 3, this third pair can not be marked. Hence these three pairs can be charged to these four marks.
2. For the case of a pair with two marks followed immediately by a pair with a single mark 000: By constraint 2, an immediate third occurrence is either 00 or 10 pair.
   a. If the third pair has a mark, that must be 001. By constraint 3, these three pairs can be charged to these four marks.
   b. If the third pair has no mark, there must be a pair P with a single mark 001 in U. These two consecutive pairs and P can be charge to four marks. Notice that, if P immediately follows a pair with two marks, P is not charged to these preceding two marks since P has been charged to the other marks already.
3. For the case of a pair with two marks followed immediately by a pair with a single marks 001: If the 001 mark has already been used elsewhere (case b), then these two pairs are charged to these two marks. Otherwise, there must be a pair P with a single mark 000, and these three pairs can be charged to these four marks (P cannot immediately follow a pair with two marks, which has been handled in case 2).
4. For the case of a pair with two marks followed immediately by a pair with either single mark 100 or 101: This is handled as in case 2 and 3.

5. The remaining cases are that a pair with two marks followed by an unmarked pair, or a pair with a single mark. In these cases, each pair can be charged to one mark.

Similarly, the other four marks 010, 011, 110, and 111 account for at least 3t distinct occurrences of the pairs 01 and 11. Since these two sets of pairs ({00, 10} and {01, 11}) are disjoint, we have $|U| \geq 12t$. ∎

Theorem 10 [Tom90]: $R(t, 4) \geq 6t$, for all positive integers t.

Proof:

Case 1: $t = 2k - 1$ for some integer k. Then

$$R(t, 4) = R(2k - 1, 4) \geq 12k = 6t + 6.$$

Case 2: $t = 2k$ for some integer k. Then

$$R(t, 4) = R(2k, 4) \geq R(2k - 1, 4) \geq 12k = 6t.$$

∎

3.2 The Buss-Tompa Marking Scheme

Tompa's marking scheme is improved in [BT95]. We consider a motivating example on $(2t - 1)$-reflecting sequences U for $\mathcal{L}(5)$. For example, to traverse the chain 0000, the sequence U must contain a substring matching 00(01+10)*00 starting at an even index. The substring of length 2 beginning at an even index in U is called a pair. The first 00 pair is said to start the traversal of 0000 and the second 00 pair is said to finish the traversal of 0000. Since there are at least t forward traversals of 0000, U must contain at least $2t$ occurrences of the pair 00. Similarly, there must be at least $2t$ occurrences of 01, 10, and 11. These pairs, which correspond to starting and finishing forward traversals of chain 0000, 0101, 1010, and 1111, are called base pairs. A pair of U is called nonbase if it is not a base pair. This argument shows that $R(t,5) \geq 8t$. Since $log_5 8 < log_4 6$, there is no improvement over the result when using Tompa's making scheme. A new idea in the Buss-Tompa marking scheme is their introduction of opening mark, closing mark, opening debt, and closing debt. In the following argument, eight chains of the forms $\alpha\beta\alpha\beta$ and $\alpha\beta\overline{\alpha}\overline{\beta}$, where $\alpha, \beta \in \{0,1\}$, are considered.

If an $\alpha\beta$ pair finishes a traversal of either $\alpha\beta\alpha\beta$ or $\overline{\alpha}\overline{\beta}\alpha\beta$, the $\alpha\beta$ pair has a closing mark; otherwise, the $\alpha\beta$ pair has a closing debt. If an $\alpha\beta$ pair starts a traversal of $\alpha\beta\alpha\beta$ or it is the last $\alpha\beta$ pair during a traversal of $\alpha\beta\overline{\alpha}\overline{\beta}$, the $\alpha\beta$ pair has an opening mark; otherwise, the $\alpha\beta$ pair has an opening debt.

Table 2 shows the substring requirements for a reflecting sequence for $\mathcal{L}(5)$ corresponding to all labeled chains. In addition to base pairs, we plan to use nonbase pairs of a reflecting sequence to obtain a better length lower bound. First we derive a relationship between the number of nonbase pairs and of debts in a reflecting sequence, then a lower bound on the number of debts.

Table 2: Substrings of U traversing chains from left to right

chain label	substring of U must match
0000	00(01+10)*00
0011	00(00+10)*11
0101	01(00+11)*01
0110	01(01+11)*10
1001	10(00+10)*01
1010	10(00+11)*10
1100	11(01+11)*00
1111	11(01+10)*11
$\alpha\beta\alpha\beta$	$\alpha\beta(\alpha\overline{\beta}+\overline{\alpha}\beta)^*\alpha\beta$
$\alpha\beta\overline{\alpha}\overline{\beta}$	$\alpha\beta(\alpha\beta+\overline{\alpha}\beta)^*\overline{\alpha}\overline{\beta}$
$\overline{\alpha}\overline{\beta}\alpha\beta$	$\overline{\alpha}\overline{\beta}(\alpha\overline{\beta}+\overline{\alpha}\overline{\beta})^*\alpha\beta$

Theorem 11 [**BT95**]: Let U be a $(2t-1)$-reflecting sequence for $\mathcal{L}(5)$, where $t \geq 1$. The number of nonbase pairs in U is exactly half the number of debts in U.

Proof: For $\alpha, \beta \in \{0, 1\}$, U embeds substrings of the forms (starting at even indices): $\alpha\beta(\alpha\overline{\beta}+\overline{\alpha}\beta)^*\alpha\beta$, $\alpha\beta(\alpha\beta+\overline{\alpha}\beta)^*\overline{\alpha}\overline{\beta}$, $\overline{\alpha}\overline{\beta}(\alpha\overline{\beta}+\overline{\alpha}\overline{\beta})^*\alpha\beta$, corresponding to forward traversals of $\alpha\beta\alpha\beta$, $\alpha\beta\overline{\alpha}\overline{\beta}$, $\overline{\alpha}\overline{\beta}\alpha\alpha\beta$, respectively (see Table 2). From Table 2, we can see that no $\alpha\beta$ pair can both start a traversal of $\alpha\beta\alpha\beta$ and be the last $\alpha\beta$ pair during a traversal of $\alpha\beta\overline{\alpha}\overline{\beta}$ (at most one opening mark on a pair), and no $\alpha\beta$ pair can finish a traversal of both $\alpha\beta\alpha\beta$ and $\overline{\alpha}\overline{\beta}\alpha\beta$ (at most one closing mark on a pair). So, the number of marks plus the number of debts on any $\alpha\beta$ pair is two. Since U induces at least t forward traversals of each chain, there are at least $4t$ marks on $\alpha\beta$: $2t$ for traversals of $\alpha\beta\alpha\beta$, t for traversals of $\alpha\beta\overline{\alpha}\overline{\beta}$, and t for traversals of $\overline{\alpha}\overline{\beta}\alpha\beta$.

Suppose that U has n nonbase $\alpha\beta$ pairs and a total of d debts on $\alpha\beta$ pairs. By noting that there are (first) $4t$ marks on $\alpha\beta$ pairs, the total of marks and debts on $\alpha\beta$ pairs is $4t+d$, and by noting that there are $2t$ base $\alpha\beta$ pairs, the total is $2(2t+n)$—the number of marks and debts on an $\alpha\beta$ pair (base or nonbase) is 2. These give us that $4t+d = 2(2t+n)$, hence $n = \frac{d}{2}$. ∎

The next two lemmas describe the marking and debting on consecutive $\alpha\beta$ pairs. The proofs follow immediately from Table 2.

Lemma 12 [**BT95**]: Let $\alpha, \beta \in \{0, 1\}$. If an $\alpha\beta$ pair has a closing mark, then the next $\alpha\beta$ or $\overline{\alpha}\overline{\beta}$ pair to its left must have an opening mark. If an $\alpha\beta$ pair has an opening mark, then the next $\alpha\beta$ or $\overline{\alpha}\overline{\beta}$ pair to its right must have a closing mark.

Lemma 13 [**BT95**]: Let $\alpha, \beta \in \{0, 1\}$. If an $\alpha\beta$ pair has a closing debt and the next $\alpha\beta$ or $\overline{\alpha}\overline{\beta}$ pair to its left exists, then it must have an opening debt. If an $\alpha\beta$ pair has an opening debt and the next $\alpha\beta$ or $\overline{\alpha}\overline{\beta}$ pair to its right exists, then it must have a closing debt.

In the marking scheme in [Tom90], we consider only the t forward traversals of each chain induced by a $(2t-1)$-reflecting sequence. To improve the length lower bounds of reflecting sequences, backward (right-to-left) traversals are also considered. We introduce the notion of interval that captures the constraints between two forward traversals. For $\alpha, \beta \in \{0,1\}$, an $\alpha\beta\alpha\beta$ interval is a substring of U that begins with the $\alpha\beta$ pair that finishes a traversal of $\alpha\beta\alpha\beta$, and ends with the $\alpha\beta$ that starts the next traversal of $\alpha\beta\alpha\beta$.

The following lemma shows a lower bound on the number of debts in an $\alpha\beta\alpha\beta$ interval; and by Theorem 11, a lower bound of the number of nonbase pairs is obtained.

Lemma 14 [BT95]: For $\alpha, \beta \in \{0,1\}$, each $\alpha\beta\alpha\beta$ interval contains at least two debts on $\alpha\beta$ and/or $\overline{\alpha}\overline{\beta}$ pairs.

Proof: Each $\alpha\beta\alpha\beta$ interval must contain a substring of the regular pattern $\overline{\beta}\overline{\alpha}(\beta\overline{\alpha} + \beta\overline{\alpha})^*\overline{\beta}\overline{\alpha}$ beginning at an odd index to account for a backward traversal of $\alpha\beta\alpha\beta$. It is rewritten as $x\overline{\beta}(\overline{\alpha}\beta)^*\overline{\alpha}\overline{\beta}((\alpha\overline{\beta})^*\alpha\beta(\overline{\alpha}\beta)^*\overline{\alpha}\overline{\beta})^*(\alpha\overline{\beta})^*\overline{\alpha}y$ where $x, y \in \{0,1\}$, beginning at an even index. Consider the leftmost $\overline{\alpha}\overline{\beta}$ pair, not including the pair matching $x\overline{\beta}$.

L1. If it has a closing debt, then by Lemma 13, the next $\alpha\beta$ or $\overline{\alpha}\overline{\beta}$ pair to its left must have an opening debt. Note that an $\alpha\beta\alpha\beta$ interval ends with an $\alpha\beta$ pair, so there are at least two debts within the interval.

L2. If it has a closing mark, it must finish a traversal of $\overline{\alpha}\overline{\beta}\overline{\alpha}\overline{\beta}$. Otherwise, it would finish a traversal of $\alpha\beta\overline{\alpha}\overline{\beta}$, and this is not possible since the regular languages denoted by $x\overline{\beta}(\overline{\alpha}\overline{\beta})^*\overline{\alpha}\overline{\beta}$ and by $\alpha\beta(\alpha\beta + \overline{\alpha}\beta)^*\overline{\alpha}\overline{\beta}$ are disjoint.

Next, consider the rightmost $\overline{\alpha}\overline{\beta}$ pair, not including the pair matching $\overline{\alpha}y$.

R1. If it has an opening debt, then by Lemma 13, the next $\alpha\beta$ or $\overline{\alpha}\overline{\beta}$ pair to its right must have a closing mark. Note that an $\alpha\beta\alpha\beta$ interval ends with an $\alpha\beta$ pair, so there are at least two debts within the interval.

R2. If it has an opening mark, it must start a traversal of $\overline{\alpha}\overline{\beta}\overline{\alpha}\overline{\beta}$. Otherwise, it would be the last $\overline{\alpha}\overline{\beta}$ pair in a traversal of $\overline{\alpha}\overline{\beta}\alpha\beta$, and this is not possible since the regular languages denoted by $\overline{\alpha}\overline{\beta}(\alpha\overline{\beta})^*\overline{\alpha}y$ and by $\overline{\alpha}\overline{\beta}(\overline{\alpha}\overline{\beta})^*\alpha\beta$ are disjoint.

Suppose that the $\alpha\beta\alpha\beta$ interval contains fewer than two debts on $\alpha\beta$ and/or $\overline{\alpha}\overline{\beta}$ pairs. From cases L2 and R2, there must be an $\overline{\alpha}\overline{\beta}\overline{\alpha}\overline{\beta}$ interval within the $\alpha\beta\alpha\beta$ interval. By an analogous argument, there must be an $\alpha\beta\alpha\beta$ interval within the $\overline{\alpha}\overline{\beta}\overline{\alpha}\overline{\beta}$ interval. Since no $\alpha\beta\alpha\beta$ interval contains another $\alpha\beta\alpha\beta$ interval, a contradiction arrives. Therefore, the $\alpha\beta\alpha\beta$ interval contains at least two debts on $\alpha\beta$ and/or $\overline{\alpha}\overline{\beta}$ pairs. ∎

Now we obtain the length lower bound of reflecting sequences for $\mathcal{L}(5)$ using Theorem 11 and Lemma 14.

Theorem 15 [BT95]: $R(t,5) \geq 10t$ for all positive integers t.

Proof: Consider a $(2t-1)$-reflecting sequence U for $\mathcal{L}(5)$. The sequence U contains $k-1$ 0000 intervals and $k-1$ 0101 intervals (all base pair can be seen from these intervals). By Theorem 14, U has at least $2k-2$ debts on 00 and/or 11 pairs, and at least $2k-2$ debts on 01 and/or 10 pairs. The total of $4k-4$ debts account for $2k-2$ nonbase pairs by Theorem 11. Since there are $8k$ base pairs (k forward traversals of each chain), there are at least $10k-2$ pairs. Thus, $R(2k-1,5) \geq 2(10k-2) = 20k-4$. To show that $R(t,5) \geq 10t$, we consider two cases:

Case 1: $t = 2k - 1$ for some integer k. Then

$$R(t, 5) = R(2k - 1, 5) \geq 20k - 4 = 10t + 6.$$

Case 2: $t = 2k$ for some integer k. Then

$$R(t, 5) \geq R(t - 1, 5) + 5 = R(2k - 1, 5) + 5 \geq 20k + 1 = 10t + 1.$$

∎

The next theorem shows that the Buss-Tompa marking scheme improved the length lower bound of reflecting sequences from $tn^{log_4 6}$ to $tn^{log_5 10}$.

Theorem 16 [BT95]: $R(t, n) \geq tn^{log_5 10}$.

Proof: Immediately by combining Theorem 5 with Theorem 15. ∎

The best upper bound of length reflecting sequences for $\mathcal{L}(5)$ is given in next theorem.

Theorem 17 [BT95]: $R(t, 5) \leq 12t + O(1)$.

Proof: The string $(000010110111001001011110)^{t+1}$ is a $2t$-reflecting sequence for $\mathcal{L}(5)$. ∎

The bounds in Theorems 8 and 10 are tight because, in each case, a matching upper bound is demonstrated by exhibiting a t-reflecting sequence of repeating form—uv^t for some $u, v \in \{0, 1\}^*$. However, the lower and upper bounds in Theorem 15 do not match. The upper bound of Theorem 17 cannot be improved with the type of simply repeating sequence.

Theorem 18 [BT95]: Let $c < \frac{4}{3}$. If $|P|$ is even and $P^{\lfloor ct \rfloor}$ is a $(2t - 1)$-reflecting sequence for $\mathcal{L}(5)$ for all positive integers t, then $|P| \geq 24$.

Proof: Suppose that $P^{\lfloor ct \rfloor}$ is $(2t-1)$-reflecting sequence for $\mathcal{L}(5)$. There are t forward traversals of each of the chains 0000, 0011, and 1100, so $4t$ marks are on 00 pairs altogether. Therefore, there exists a copy of P (including a cyclic shifting of P) has at least $\lceil 4t/\lfloor ct \rfloor \rceil \geq \lceil 4/c \rceil \geq 4$ marks on 00. First notice that P cannot have only one 00 pair by Theorem 11. We show that P cannot have exactly two 00 pairs. Suppose the contrary that P has only two 00 pairs with four marks. Then P must be a rotation of (01+10+11)*11(01)*00(01+10)*00(10)*11 (see Table 2). In this case, P^* has no occurrence of 00(01+10)*00 beginning at an odd index (no backward traversal of 1111). Because of this contradiction, P must contain at least three 00 pairs.

Similarly, P must contain at least three occurrences of the pairs 01, 10, and 11. Therefore, P contains at least 12 pairs. ∎

CHAPTER IV

COMPUTATIONAL APPROACHES

In order to apply Theorems 5 and 6, we need to determine c, r, and k such that $R(t, c) \geq rt - k$, that is, the length of a t-reflecting sequence for $\mathcal{L}(c)$ is at least $rt - k$. The next theorem shows that it suffices to consider only the case when the number of reflections is odd.

Theorem 19 [DF96a]: For all positive integers t, c and positive reals r, k, if $R(2t - 1, c) \geq 2rt - k$ for every positive integer t, then $R(t', c) \geq rt' - k$ for every positive integer t'.

To show that $R(2t-1, c) \geq 2rt-k$ in Theorem 19, we modify a technique in [FDO94], which assigns a minimum of $2^{c-1}t$ "marks" to various positions in a hypothetical $(2t-1)$-reflecting sequence S (there are 2^{c-1} different labeled chains and t marks for each labeled chain of length c).

4.1 Marks

There are 2^{c-1} different labeled chains of length c. Let $C_i \in \mathcal{L}(c)$ have label $i \in \{0^{c-1}, 0^{c-2}1, 0^{c-3}1^2, \cdots, 1^{c-1}\}$. As C_i is traversed by a hypothetical $(2t-1)$-reflecting sequence S, put a mark M_i on the bits in S that correspond to the last exit from vertex 1 in a complete forward traversal. For example, if $c = 4$, there are eight chains 000, 001, 010, 011, 100, 101, 110, and 111. Traversing C_{000} with $S = 0110010001111001$

visits vertices 0, 1, 0, 1, 2, 3, 2, 3, 4, 3, 2, 1, 0, 1, 2, 3, 2; hence, we place the mark M_{000} corresponding to the chain with label 000 at the 4th and 14th bits in S.

Two different kinds of "mark" are defined below, depending on the completion of the forward traversal:

1. Closed Marks: If S traverses chain C_i from vertex 1 to vertex n without returning to vertex 1, closed marks of M_i are put on that bit of S that correspond to the last exit from vertex 1 in a complete forward traversal.
2. Open Marks: If a bit of S that correspond to the last exit from vertex 1 and S doesn't make a complete forward traversal of chain C_i, an open mark is put on that bit. Even if an open mark M_i is not closed on S, M_i is a potential closed mark because the mark might be closed on SS' where $S' \in \{0,1\}^*$

Two open marks on S are suffix-inconsistent if there does not exist any sequence $S' \in \{0,1\}^*$ such that both open marks become closed on SS'. Figure 6 shows an example of a traversal sequence for labeled chains for $\mathcal{L}(5)$ with marks M_i, where $i \in \{0000, 0001, \cdots, 1111\}$. Marks M_{0000}, M_{0010}, M_{0011}, M_{1000}, and M_{1011} are closed and the other marks are open marks.

			M_{1111}
M_{0011}	M_{1011}		M_{1110}
M_{0010}	M_{1010}		M_{1101}
M_{0000}	M_{1000}		M_{1100}
00	10	00	11

Figure 6. A traversal sequence for labeled chains for $\mathcal{L}(5)$ with marks.

For a $(2t-1)$-reflecting sequence S for $\mathcal{L}(c)$, S makes at least t forward traversals for all labeled chain in $\mathcal{L}(c)$. There are at least $2^{c-1}t$ marks on S. To show that $R(2t-1, c) \geq 2rt - k$, it suffices to show that the "marks-to-bits" density for all but a short suffix of S is at most $\frac{2^{c-1}t}{2rt} = \frac{2^{c-1}}{2r}$.

Two properties of marks are immediate:

1. All marks on S are put at even bit-position.
2. The two same marks are put at least $2c$ bits apart from.

Let $\rho_1(S')$ denote for the marks-to-bits density of a subsequence S' of S. From property 2, we obtain that $\rho_1(S') \le \frac{2^{c-1}}{2c}$ for every subsequence S' of S. This implies that if $S = S_1 S_2$ such that $\rho_1(S_1) \le \frac{2^{c-1}}{2r}$ and $|S_2| \le r'$ for some positive real r and integer r', then the number of marks on S_1 is at least $2^{c-1}t - |S_2|\rho_1(S_2) \ge 2^{c-1}t - r'\frac{2^{c-1}}{2c}$. Thus,

$$|S| \ge |S_1| \ge (2^{c-1}t - r'\frac{2^{c-1}}{2c})/\rho_1(S_1) \ge 2^{c-1}(t - \frac{r'}{2c})/\frac{2^{c-1}}{2r} = 2rt - \frac{rr'}{c}.$$

Here, the supposition in Theorems 5 and 6 will be satisfied and we can see that the length of the suffix S_2 is bounded by some r', which is not dependent on t.

By property 1, S can be regarded as a sequence of "pairs" of bits following the first single bit. The focus is on "marks-to-pairs" density instead of on marks-to-bits, and our task is to prove that marks-to-pairs density of every $(2t-1)$-reflecting sequence S for $\mathcal{L}(c)$ is at most $\frac{2^{c-1}}{r}$ for all but a short suffix of S.

A tree T, called quadtree, can be constructed to divide marked S into segments, each of which, except for the last segment, has a marks-to-pairs density at most $\frac{2^{c-1}}{r}$. For the marked $(2t-1)$-reflecting sequence S for $\mathcal{L}(c)$, we use T to segment S as follows:

1. Discard the first bit of S.
2. Starting at the root of T, go down the branches of T, whose labels correspond to S, until a leaf of T is reached (at depth d).

3. Once reached to a leaf, there exists a prefix of S of length at most $2d$ that has a mark-to-pairs density at most $\frac{2^{c-1}}{r}$. Discard the prefix from S and repeat at step 2 using the remainder sequence.

At step 2, when the remainder of S runs out before reaching a leaf, that means, we have found the final segment of S. If such a tree exists, the supposition in Theorems 5 and 6 is satisfied.

4.2 Building the Quadtree

For a given chain length c and an upper bound ρ_2 on marks-to-pairs density, the quadtree is built in a depth-first fashion. First, generate 00-branch and find an upper bound of the marks-to-pairs density for 00. If that density is greater than ρ_2, extend the tree by appending 00 at that vertex. Now we consider the sequence 00 00, and calculate an upper bound on the marks-to-pairs density on every prefix of 00 00. If no prefix has lower marks-to-pairs density than ρ_2, extend the tree and consider 00 00 00 as well; otherwise, 00 00-branch becomes a leaf and examine the 00 01-branch next. The quadtree is completed when the enumeration process terminates. The method of building the quadtree is summarized by the following algorithm [DF96a].

ALGORITHM Build_Quadtree(c, p)

1. Initialize S to the pair 00;

2. (* Only examine 00-branch by symmetry *)

while $S \neq 01$ do

2.1. Try to find a prefix of S whose marks-to-pairs density is at most ρ_2;

2.2. If no such prefix exist, then

2.2.1. Extend the quadtree by appending the pair 00 to S;

else

2.2.2. (* The sequence S represents a leaf *)

Replace S with the pair-sequence no longer than S and following S in the lexicographic ordering of pair-sequence;

END Build_Quadtree

Figure 7 illustrates the Quadtree built by given $c = 4$ and $\rho_2 = \frac{3}{2}$.

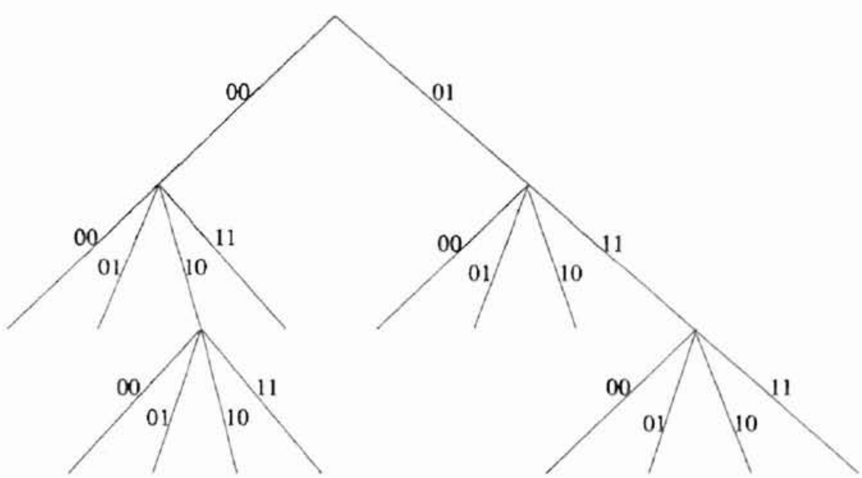

Figure 7. A quadtree for $\mathcal{L}(4)$ by given a marks-to-pairs density of $\frac{3}{2}$. The other two subtrees at the root with 10 and 11 are symmetric to 01-subtree and 00-subtree, respectively.

4.3 Inconsistent and Inconsistency Graphs

In Step 2.1 in the Build_Quadtree algorithm, we calculate the marks-to-pairs density of each prefix. We obtain the maximum number of marks for each chain in $\mathcal{L}(c)$ by assuming the forward traversal of the chain with the traversal sequence S. Yet, among several open marks, they may not co-exist. We construct an inconsistency graph to solve the maximum number of open marks which can co-exist. Let $G_c(S)$ denote an inconsistency graph for the traversal sequence S, whose vertex set represents all possible open marks on S and edge set represents all pairwise inconsistencies between open marks on S. Thus, an upper bound on the marks-to-pairs density of S is computed by the number of closed marks on S and the independence number of $G_c(S)$.

We define the inconsistency graph G_c as follows. The vertex set $\{(C_\alpha, u) | \alpha \in \{0^{c-1}, 0^{c-2}1, \cdots, 1^{c-1}\}$ and $u \in \{2, 3, \cdots, c-1\}$ is odd$\}$, in which a vertex (C_α, u) represents all possible open marks on S. The edge set E of G_c represents pairwise inconsistencies between all possible pairs of $((C_\alpha, u), (C_\beta, v)) \in V(G_c) \times V(G_c)$ from one trivial and two non-trivial sources.

One trivial sources are $((C_\alpha, u), (C_\alpha, v)) \in E(G_c)$ for $(C_\alpha, u), (C_\alpha, v) \in V(G_c)$ with $u \neq v$. The other two sources are suffix-inconsistency and prefix-inconsistency defined below.

For $\alpha, \beta \in \{0^{c-1}, 0^{c-2}1, \cdots, 1^{c-1}\}, u, v, u', v' \in \{0, 1, \cdots, c\}$, and a traversal sequence $S \in \{0, 1\}^*$. Let $((C_\alpha, u), (C_\beta, v)) \xrightarrow{S} ((C_\alpha, u'), (C_\beta, v'))$ denote that S induces two traversals: in C_α from u to u' and in C_β from v to v'. In particular, $((C_\alpha, u), (C_\beta, v)) \xrightarrow{S;(1,c)}$ means interior traversal, that is, they don't enter and leave vertices 1 and c.

$(C_\alpha, u), (C_\beta, v) \in \mathcal{L}(c) \times \{2, 3, \cdots, c-1\}, (C_\alpha, u)$ and (C_β, v) are suffix consistent if there exists a traversal sequence $S \in \{0, 1\}^*$ such that $((C_\alpha, u), (C_\beta, v)) \xrightarrow{S;(1,c)} ((C_\alpha, c), (C_\beta, v'))$ for some $v' \in \{2, 3, \cdots, c\}$ or $((C_\alpha, u), (C_\beta, v)) \xrightarrow{S;(1,c)} ((C_\alpha, u'), (C_\beta, c))$ for some $u' \in \{2, 3, \cdots, c\}$.

$(C_\alpha, u), (C_\beta, v) \in \mathcal{L}(c) \times \{2, 3, \cdots, c-1\}, (C_\alpha, u)$ and (C_β, v) are prefix consistent if there exists a traversal sequence $S \in \{0,1\}^*$ such that $((C_\alpha, 1), (C_\beta, 1)) \xrightarrow{S} ((C_\alpha, u), (C_\beta, v))$

The pair (C_α, u) and (C_β, v) are suffix inconsistent (prefix inconsistent) if they are not suffix consistent (prefix consistent, respectively).

4.4 The Inconsistency Graph G_9 for $\mathcal{L}(9)$

In order to obtain a lower bound on $R(t, 9)$, we need to understand the structure of the (suffix-)inconsistency graph, denoted by G_9, and solve the associated Maximum Independent Set Problem.

The vertex set of G_9, $V(G_9)$, is $\{(C_i, u) | i \in \{0, 1, \cdots, 2^8 - 1\}$ and $u \in \{3, 5, 7\}$. The edge set $E(G_9)$ can be determined using one of the following methods:

1. A characterization theorem for suffix-inconsistency in [DF96a]:

   Let $\delta(u, v) = (c - u) - (v - 1)$, the differences of distances of u and v from vertices c and 1 in C_α and C_β, respectively. Let $\alpha(i, j)$ denote $\alpha_i \alpha_{i+1} \cdots \alpha_j$ if $i \leq j$, and the empty sequence otherwise, and let $\overline{\alpha(i,j)}$ denote the reversal of the component-wise complement of $\alpha(i, j)$. We define $NR_{c,\alpha,\beta}$ as follows:

$$NR_{c,\alpha,\beta}(i) = \begin{cases} \beta(2, c-1-i) = \overline{\alpha(2, c-1-i)} & \text{if } i \in \{0, 1\}, \\ (\beta(2, c-1-i) = \overline{\alpha(2, c-1-i)}) \vee & \\ \exists j_i \in \{2, 3, \cdots, c-1-i\} (\beta_{j_i} = \alpha_{2+c-1-i-j_i} \wedge NR_{c,\alpha,\beta}(i-2)) & \text{if } i \geq 2. \end{cases}$$

Theorem 20 [DF96a]: Let $(C_\alpha, u), (C_\beta, v) \in \mathcal{L}(c) \times \{2, 3, \cdots, c-1\}$, we have:

i. If $\delta(u, v) < 0$, then (C_α, u) and (C_β, v) are suffix-consistent.

ii. If $\delta(u,v) \geq 0$, then (C_α, u) and (C_β, v) are suffix-inconsistent if and only if $NR_{c,\alpha,\beta}(\delta(u,v))$.

An implementation of the theorem is given in Appendix B.1.

2. An equivalent condition for suffix-inconsistency for $\mathcal{L}(c)$ is that for all traversal sequence $S \in \{0,1\}^*$ such that $((C_\alpha, u), (C_\beta, v)) \xrightarrow{S} ((C_\alpha, c), (C_\beta, v'))$ for some $v' \in \{0, 1, \cdots, c\}$ $(((C_\alpha, u), (C_\beta, v)) \xrightarrow{S} ((C_\alpha, u'), (C_\beta, c))$ for some $u' \in \{0, 1, \cdots, c\}$) if the traversal in C_α from u to c (in C_β from v to c, respectively) does not transit through vertex 1 and c, then, the traversal in C_β from v to v' (in C_α from u to u', respectively) must visit vertex 1.

An implementation of this detection is given in Appendix B.2

When viewing G_9 as three layer of vertices:

layer-u vertices of the form (C_i, u), where $i \in \{0, 1, \cdots, 2^8 - 1\}$ and $u \in \{3, 5, 7\}$, we obtain the adjacency structure in $E(G_9)$ as follows.

1. Partitioning layer-3 into eight 32-order clusters, $\{V3_i \mid i = I, II, \cdots, VIII\}$

$$
\begin{aligned}
V3_I &= \{C_i | 0 \leq i \leq 15, 128 \leq i \leq 143\} \times \{3\} \\
V3_{II} &= \{C_i | 16 \leq i \leq 31, 144 \leq i \leq 159\} \times \{3\} \\
V3_{III} &= \{C_i | 32 \leq i \leq 47, 160 \leq i \leq 175\} \times \{3\} \\
V3_{IV} &= \{C_i | 48 \leq i \leq 63, 176 \leq i \leq 191\} \times \{3\} \\
V3_V &= \{C_i | 64 \leq i \leq 79, 192 \leq i \leq 207\} \times \{3\} \\
V3_{VI} &= \{C_i | 80 \leq i \leq 95, 208 \leq i \leq 223\} \times \{3\} \\
V3_{VII} &= \{C_i | 96 \leq i \leq 111, 224 \leq i \leq 239\} \times \{3\} \\
V3_{VIII} &= \{C_i | 112 \leq i \leq 127, 240 \leq i \leq 255\} \times \{3\}
\end{aligned}
$$

Within $\{V3_i \mid i = I, II, \cdots, VIII\}$, there are four complete $K_{32,32}$- bipartitions:

$(V3_I, V3_V)$, $(V3_{II}, V3_{VI})$, $(V3_{III}, V3_{VII})$, $(V3_{IV}, V3_{VIII})$

which provide a four-combination of 32-order cluster choices for a maximum independent set in G_9.

2. Partitioning each 32-order clusters into form 8-order clusters (a total of 32 8-order clusters), $\{V3_i \mid i = 1, 2, \cdots, 32\}$:

$$
\begin{aligned}
V3_1 &= \{C_i|0 \leq i \leq 3, 128 \leq i \leq 131\} \times \{3\} \\
V3_2 &= \{C_i|4 \leq i \leq 7, 132 \leq i \leq 135\} \times \{3\} \\
V3_3 &= \{C_i|8 \leq i \leq 11, 136 \leq i \leq 139\} \times \{3\} \\
V3_4 &= \{C_i|12 \leq i \leq 15, 140 \leq i \leq 143\} \times \{3\} \\
V3_5 &= \{C_i|16 \leq i \leq 19, 144 \leq i \leq 147\} \times \{3\} \\
V3_6 &= \{C_i|20 \leq i \leq 23, 148 \leq i \leq 151\} \times \{3\} \\
V3_7 &= \{C_i|24 \leq i \leq 27, 152 \leq i \leq 155\} \times \{3\} \\
V3_8 &= \{C_i|28 \leq i \leq 31, 156 \leq i \leq 159\} \times \{3\} \\
V3_9 &= \{C_i|32 \leq i \leq 35, 160 \leq i \leq 163\} \times \{3\} \\
V3_{10} &= \{C_i|36 \leq i \leq 39, 164 \leq i \leq 167\} \times \{3\} \\
V3_{11} &= \{C_i|40 \leq i \leq 43, 168 \leq i \leq 171\} \times \{3\} \\
V3_{12} &= \{C_i|44 \leq i \leq 47, 172 \leq i \leq 175\} \times \{3\} \\
V3_{13} &= \{C_i|64 \leq i \leq 67, 192 \leq i \leq 195\} \times \{3\} \\
V3_{14} &= \{C_i|68 \leq i \leq 71, 196 \leq i \leq 199\} \times \{3\} \\
V3_{15} &= \{C_i|72 \leq i \leq 75, 200 \leq i \leq 203\} \times \{3\} \\
V3_{16} &= \{C_i|76 \leq i \leq 79, 204 \leq i \leq 207\} \times \{3\} \\
V3_{17} &= \{C_i|112 \leq i \leq 115, 240 \leq i \leq 243\} \times \{3\} \\
V3_{18} &= \{C_i|116 \leq i \leq 119, 244 \leq i \leq 247\} \times \{3\} \\
V3_{19} &= \{C_i|120 \leq i \leq 123, 248 \leq i \leq 251\} \times \{3\} \\
V3_{20} &= \{C_i|124 \leq i \leq 127, 252 \leq i \leq 255\} \times \{3\} \\
V3_{21} &= \{C_i|48 \leq i \leq 51, 176 \leq i \leq 179\} \times \{3\} \\
V3_{22} &= \{C_i|52 \leq i \leq 55, 180 \leq i \leq 183\} \times \{3\} \\
V3_{23} &= \{C_i|56 \leq i \leq 59, 184 \leq i \leq 187\} \times \{3\} \\
V3_{24} &= \{C_i|60 \leq i \leq 63, 188 \leq i \leq 191\} \times \{3\} \\
V3_{25} &= \{C_i|80 \leq i \leq 83, 208 \leq i \leq 211\} \times \{3\} \\
V3_{26} &= \{C_i|84 \leq i \leq 87, 212 \leq i \leq 215\} \times \{3\} \\
V3_{27} &= \{C_i|88 \leq i \leq 91, 216 \leq i \leq 219\} \times \{3\} \\
V3_{28} &= \{C_i|92 \leq i \leq 95, 220 \leq i \leq 223\} \times \{3\} \\
V3_{29} &= \{C_i|96 \leq i \leq 99, 224 \leq i \leq 227\} \times \{3\} \\
V3_{30} &= \{C_i|100 \leq i \leq 103, 228 \leq i \leq 231\} \times \{3\} \\
V3_{31} &= \{C_i|104 \leq i \leq 107, 232 \leq i \leq 235\} \times \{3\} \\
V3_{32} &= \{C_i|108 \leq i \leq 111, 236 \leq i \leq 239\} \times \{3\}
\end{aligned}
$$

Within $\{V3_i \mid i = 1, 2, \cdots, 32\}$, there are 16 complete $K_{8,8}$- bipartitions:

$(V3_1, V3_{20})$, $(V3_2, V3_{24})$, $(V3_3, V3_{28})$, $(V3_4, V3_8)$, $(V3_5, V3_{32})$, $(V3_6, V3_{12})$, $(V3_7, V3_{16})$, $(V3_9, V3_{18})$, $(V3_{10}, V3_{22})$, $(V3_{11}, V3_{26})$, $(V3_{13}, V3_{19})$, $(V3_{14}, V3_{23})$, $(V3_{15}, V3_{27})$, $(V3_{17}, V3_{29})$, $(V3_{21}, V3_{30})$, $(V3_{25}, V3_{31})$

which provide a refined combination of 8-order cluster choices inherited from 1. for a maximum independent set in G_9.

3. Within $\{V3_i \mid i = 1, 2, \cdots, 32\}$, there are 64 complete $K_{2,2}$- bipartitions (for simplicity, we abbreviate $(C_i, 3)$ as 3_i):

$$\begin{array}{lll}
(\{3_0,3_{128}\},\{3_{127},3_{255}\}), & (\{3_1,3_{129}\},\{3_{63},3_{191}\}), & (\{3_2,3_{130}\},\{3_{95},3_{223}\}) \\
(\{3_3,3_{131}\},\{3_{31},3_{159}\}), & (\{3_4,3_{132}\},\{3_{111},3_{239}\}), & (\{3_5,3_{133}\},\{3_{47},3_{175}\}) \\
(\{3_6,3_{134}\},\{3_{79},3_{207}\}), & (\{3_7,3_{135}\},\{3_{15},3_{143}\}), & (\{3_{16},3_{144}\},\{3_{123},3_{251}\}) \\
(\{3_{17},3_{145}\},\{3_{59},3_{187}\}), & (\{3_{18},3_{146}\},\{3_{91},3_{219}\}), & (\{3_{19},3_{147}\},\{3_{27},3_{155}\}) \\
(\{3_{20},3_{148}\},\{3_{107},3_{235}\}), & (\{3_{21},3_{149}\},\{3_{43},3_{171}\}), & (\{3_{22},3_{150}\},\{3_{75},3_{203}\}) \\
(\{3_{23},3_{151}\},\{3_{11},3_{139}\}), & (\{3_{32},3_{160}\},\{3_{125},3_{253}\}), & (\{3_{33},3_{161}\},\{3_{61},3_{189}\}) \\
(\{3_{34},3_{162}\},\{3_{93},3_{221}\}), & (\{3_{35},3_{163}\},\{3_{29},3_{157}\}) & (\{3_{36},3_{164}\},\{3_{109},3_{237}\}) \\
(\{3_{37},3_{165}\},\{3_{45},3_{173}\}), & (\{3_{38},3_{166}\},\{3_{77},3_{205}\}), & (\{3_{39},3_{167}\},\{3_{13},3_{141}\}) \\
(\{3_{48},3_{176}\},\{3_{121},3_{249}\}), & (\{3_{49},3_{177}\},\{3_{57},3_{185}\}), & (\{3_{50},3_{178}\},\{3_{89},3_{217}\}) \\
(\{3_{51},3_{179}\},\{3_{25},3_{153}\}), & (\{3_{52},3_{180}\},\{3_{105},3_{233}\}), & (\{3_{53},3_{181}\},\{3_{41},3_{169}\}) \\
(\{3_{54},3_{182}\},\{3_{73},3_{201}\}), & (\{3_{55},3_{183}\},\{3_{9},3_{137}\}), & (\{3_{64},3_{192}\},\{3_{126},3_{254}\}) \\
(\{3_{65},3_{193}\},\{3_{62},3_{190}\}), & (\{3_{66},3_{194}\},\{3_{94},3_{222}\}), & (\{3_{67},3_{195}\},\{3_{30},3_{158}\}) \\
(\{3_{68},3_{196}\},\{3_{110},3_{238}\}), & (\{3_{69},3_{197}\},\{3_{46},3_{174}\}), & (\{3_{70},3_{198}\},\{3_{78},3_{206}\}) \\
(\{3_{71},3_{199}\},\{3_{14},3_{142}\}), & (\{3_{80},3_{208}\},\{3_{122},3_{250}\}), & (\{3_{81},3_{209}\},\{3_{58},3_{186}\}) \\
(\{3_{82},3_{210}\},\{3_{90},3_{218}\}), & (\{3_{83},3_{211}\},\{3_{26},3_{154}\}), & (\{3_{84},3_{212}\},\{3_{106},3_{234}\}) \\
(\{3_{85},3_{213}\},\{3_{42},3_{170}\}), & (\{3_{86},3_{214}\},\{3_{74},3_{202}\}), & (\{3_{87},3_{215}\},\{3_{10},3_{138}\}) \\
(\{3_{96},3_{224}\},\{3_{124},3_{252}\}), & (\{3_{97},3_{225}\},\{3_{60},3_{188}\}), & (\{3_{98},3_{226}\},\{3_{92},3_{220}\}) \\
(\{3_{99},3_{227}\},\{3_{28},3_{156}\}), & (\{3_{100},3_{228}\},\{3_{108},3_{236}\}), & (\{3_{101},3_{229}\},\{3_{44},3_{172}\}) \\
(\{3_{102},3_{230}\},\{3_{76},3_{204}\}), & (\{3_{103},3_{231}\},\{3_{12},3_{140}\}), & (\{3_{112},3_{240}\},\{3_{120},3_{248}\}) \\
(\{3_{113},3_{241}\},\{3_{56},3_{184}\}), & (\{3_{114},3_{242}\},\{3_{88},3_{216}\}), & (\{3_{115},3_{243}\},\{3_{24},3_{152}\}) \\
(\{3_{116},3_{244}\},\{3_{104},3_{232}\}), & (\{3_{117},3_{245}\},\{3_{40},3_{168}\}), & (\{3_{118},3_{246}\},\{3_{72},3_{200}\}) \\
(\{3_{119},3_{247}\},\{3_{8},3_{136}\}) & &
\end{array}$$

which provide the final choices for layer-3 vertices in a maximum independent set in G_9.

4. Partitioning layer-5 into 32 8-order clusters, $\{V5_i \mid i = 1, 2, \cdots, 32\}$:

$$\begin{array}{l}
V5_1 = \{C_i | 0 \le i \le 3, 128 \le i \le 131\} \times \{5\} \\
V5_2 = \{C_i | 4 \le i \le 7, 132 \le i \le 135\} \times \{5\} \\
V5_3 = \{C_i | 8 \le i \le 11, 136 \le i \le 139\} \times \{5\} \\
V5_4 = \{C_i | 12 \le i \le 15, 140 \le i \le 143\} \times \{5\} \\
V5_5 = \{C_i | 16 \le i \le 19, 144 \le i \le 147\} \times \{5\} \\
V5_6 = \{C_i | 20 \le i \le 23, 148 \le i \le 151\} \times \{5\}
\end{array}$$

$$
\begin{aligned}
V5_7 &= \{C_i|24 \le i \le 27, 152 \le i \le 155\} \times \{5\} \\
V5_8 &= \{C_i|28 \le i \le 31, 156 \le i \le 159\} \times \{5\} \\
V5_9 &= \{C_i|32 \le i \le 35, 160 \le i \le 163\} \times \{5\} \\
V5_{10} &= \{C_i|36 \le i \le 39, 164 \le i \le 167\} \times \{5\} \\
V5_{11} &= \{C_i|40 \le i \le 43, 168 \le i \le 171\} \times \{5\} \\
V5_{12} &= \{C_i|44 \le i \le 47, 172 \le i \le 175\} \times \{5\} \\
V5_{13} &= \{C_i|64 \le i \le 67, 192 \le i \le 195\} \times \{5\} \\
V5_{14} &= \{C_i|68 \le i \le 71, 196 \le i \le 199\} \times \{5\} \\
V5_{15} &= \{C_i|72 \le i \le 75, 200 \le i \le 203\} \times \{5\} \\
V5_{16} &= \{C_i|76 \le i \le 79, 204 \le i \le 207\} \times \{5\} \\
V5_{17} &= \{C_i|112 \le i \le 115, 240 \le i \le 243\} \times \{5\} \\
V5_{18} &= \{C_i|116 \le i \le 119, 244 \le i \le 247\} \times \{5\} \\
V5_{19} &= \{C_i|120 \le i \le 123, 248 \le i \le 251\} \times \{5\} \\
V5_{20} &= \{C_i|124 \le i \le 127, 252 \le i \le 255\} \times \{5\} \\
V5_{21} &= \{C_i|48 \le i \le 51, 176 \le i \le 179\} \times \{5\} \\
V5_{22} &= \{C_i|52 \le i \le 55, 180 \le i \le 183\} \times \{5\} \\
V5_{23} &= \{C_i|56 \le i \le 59, 184 \le i \le 187\} \times \{5\} \\
V5_{24} &= \{C_i|60 \le i \le 63, 188 \le i \le 191\} \times \{5\} \\
V5_{25} &= \{C_i|80 \le i \le 83, 208 \le i \le 211\} \times \{5\} \\
V5_{26} &= \{C_i|84 \le i \le 87, 212 \le i \le 215\} \times \{5\} \\
V5_{27} &= \{C_i|88 \le i \le 91, 216 \le i \le 219\} \times \{5\} \\
V5_{28} &= \{C_i|92 \le i \le 95, 220 \le i \le 223\} \times \{5\} \\
V5_{29} &= \{C_i|96 \le i \le 99, 224 \le i \le 227\} \times \{5\} \\
V5_{30} &= \{C_i|100 \le i \le 103, 228 \le i \le 231\} \times \{5\} \\
V5_{31} &= \{C_i|104 \le i \le 107, 232 \le i \le 235\} \times \{5\} \\
V5_{32} &= \{C_i|108 \le i \le 111, 236 \le i \le 239\} \times \{5\}
\end{aligned}
$$

Between the groups of $\{V3_i \mid i = 1, 2, \cdots, 32\}$ and $\{V5_i \mid i = 1, 2, \cdots, 32\}$, there are 32 complete $K_{8,8}$-bipartitions:

$(V3_1, V5_{20})$, $(V3_2, V5_{24})$, $(V3_3, V5_{28})$, $(V3_4, V5_8)$ $(V3_5, V5_{32})$, $(V3_6, V5_{12})$,
$(V3_7, V5_{16})$, $(V3_8, V5_4)$, $(V3_9, V5_{18})$, $(V3_{10}, V5_{14})$, $(V3_{11}, V3_{22})$, $(V3_{12}, V5_6)$
$(V3_{13}, V3_{19})$, $(V3_{14}, V5_{23})$, $(V3_{15}, V5_{27})$, $(V3_{16}, V5_7)$, $(V3_{17}, V5_{28})$, $(V3_{18}, V5_9)$
$(V3_{19}, V5_{13})$, $(V3_{20}, V5_1)$, $(V3_{21}, V5_{29})$, $(V3_{22}, V5_{10})$, $(V3_{23}, V5_{14})$, $(V3_{24}, V5_2)$
$(V3_{25}, V5_{31})$, $(V3_{26}, V5)$, $(V3_{27}, V5_{15})$, $(V3_{28}, V5_3)$, $(V3_{29}, V5_{21})$, $(V3_{30}, V5)$
$(V3_{31}, V5_{25})$, $(V3_{32}, V5_5)$

5. Within $\{V5_i \mid i = 1, 2, \cdots, 32\}$, there are 64 complete $K_{2,2}$-bipartitions (for simplicity, we abbreviate $(C_i, 5)$ as 5_i):

$(\{5_0, 5_{128}\}, \{5_{127}, 5_{255}\})$, $(\{5_1, 5_{129}\}, \{5_{63}, 5_{191}\})$, $(\{5_2, 5_{130}\}, \{5_{95}, 5_{223}\})$
$(\{5_3, 5_{131}\}, \{5_{31}, 5_{159}\})$, $(\{5_4, 5_{132}\}, \{5_{111}, 5_{239}\})$, $(\{5_5, 5_{133}\}, \{5_{47}, 5_{175}\})$
$(\{5_6, 5_{134}\}, \{5_{79}, 5_{207}\})$, $(\{5_7, 5_{135}\}, \{5_{15}, 5_{143}\})$, $(\{5_{16}, 5_{144}\}, \{5_{123}, 5_{251}\})$
$(\{5_{17}, 5_{145}\}, \{5_{59}, 5_{187}\})$, $(\{5_{18}, 5_{146}\}, \{5_{91}, 5_{219}\})$, $(\{5_{19}, 5_{147}\}, \{5_{27}, 5_{155}\})$
$(\{5_{20}, 5_{148}\}, \{5_{107}, 5_{235}\})$, $(\{5_{21}, 5_{149}\}, \{5_{43}, 5_{171}\})$, $(\{5_{22}, 5_{150}\}, \{5_{75}, 5_{203}\})$
$(\{5_{23}, 5_{151}\}, \{5_{11}, 5_{139}\})$, $(\{5_{32}, 5_{160}\}, \{5_{125}, 5_{253}\})$, $(\{5_{33}, 5_{161}\}, \{5_{61}, 5_{189}\})$
$(\{5_{34}, 5_{162}\}, \{5_{93}, 5_{221}\})$, $(\{5_{35}, 5_{163}\}, \{5_{29}, 5_{157}\})$ $(\{5_{36}, 5_{164}\}, \{5_{109}, 5_{237}\})$
$(\{5_{37}, 5_{165}\}, \{5_{45}, 5_{173}\})$, $(\{5_{38}, 5_{166}\}, \{5_{77}, 5_{205}\})$, $(\{5_{39}, 5_{167}\}, \{5_{13}, 5_{141}\})$
$(\{5_{48}, 5_{176}\}, \{5_{121}, 5_{249}\})$, $(\{5_{49}, 5_{177}\}, \{5_{57}, 5_{185}\})$, $(\{5_{50}, 5_{178}\}, \{5_{89}, 5_{217}\})$
$(\{5_{51}, 5_{179}\}, \{5_{25}, 5_{153}\})$, $(\{5_{52}, 5_{180}\}, \{5_{105}, 5_{233}\})$, $(\{5_{53}, 5_{181}\}, \{5_{41}, 5_{169}\})$
$(\{5_{54}, 5_{182}\}, \{5_{73}, 5_{201}\})$, $(\{5_{55}, 5_{183}\}, \{5_9, 5_{137}\})$, $(\{5_{64}, 5_{192}\}, \{5_{126}, 5_{254}\})$
$(\{5_{65}, 5_{193}\}, \{5_{62}, 5_{190}\})$, $(\{5_{66}, 5_{194}\}, \{5_{94}, 5_{222}\})$, $(\{5_{67}, 5_{195}\}, \{5_{30}, 5_{158}\})$
$(\{5_{68}, 5_{196}\}, \{5_{110}, 5_{238}\})$, $(\{5_{69}, 5_{197}\}, \{5_{46}, 5_{174}\})$, $(\{5_{70}, 5_{198}\}, \{5_{78}, 5_{206}\})$
$(\{5_{71}, 5_{199}\}, \{5_{14}, 5_{142}\})$, $(\{5_{80}, 5_{208}\}, \{5_{122}, 5_{250}\})$, $(\{5_{81}, 5_{209}\}, \{5_{58}, 5_{186}\})$
$(\{5_{82}, 5_{210}\}, \{5_{90}, 5_{218}\})$, $(\{5_{83}, 5_{211}\}, \{5_{26}, 5_{154}\})$, $(\{5_{84}, 5_{212}\}, \{5_{106}, 5_{234}\})$
$(\{5_{85}, 5_{213}\}, \{5_{42}, 5_{170}\})$, $(\{5_{86}, 5_{214}\}, \{5_{74}, 5_{202}\})$, $(\{5_{87}, 5_{215}\}, \{5_{10}, 5_{138}\})$
$(\{5_{96}, 5_{224}\}, \{5_{124}, 5_{252}\})$, $(\{5_{97}, 5_{225}\}, \{5_{60}, 5_{188}\})$, $(\{5_{98}, 5_{226}\}, \{5_{92}, 5_{220}\})$
$(\{5_{99}, 5_{227}\}, \{5_{28}, 5_{156}\})$, $(\{5_{100}, 5_{228}\}, \{5_{108}, 5_{236}\})$, $(\{5_{101}, 5_{229}\}, \{5_{44}, 5_{172}\})$
$(\{5_{102}, 5_{230}\}, \{5_{76}, 5_{204}\})$, $(\{5_{103}, 5_{231}\}, \{5_{12}, 5_{140}\})$, $(\{5_{112}, 5_{240}\}, \{5_{120}, 5_{248}\})$
$(\{5_{113}, 5_{241}\}, \{5_{56}, 5_{184}\})$, $(\{5_{114}, 5_{242}\}, \{5_{88}, 5_{216}\})$, $(\{5_{115}, 5_{243}\}, \{5_{24}, 5_{152}\})$
$(\{5_{116}, 5_{244}\}, \{5_{104}, 5_{232}\})$, $(\{5_{117}, 5_{245}\}, \{5_{40}, 5_{168}\})$, $(\{5_{118}, 5_{246}\}, \{5_{72}, 5_{200}\})$
$(\{5_{119}, 5_{247}\}, \{5_8, 5_{136}\})$

6. Between the groups of $\{V3_i \mid i = 1, 2, \cdots, 32\}$ and $\{V5_i \mid i = 1, 2, \cdots, 32\}$, there are complete 128 $K_{2,2}$-bipartitions:

$(\{3_0, 3_{128}\}, \{5_{127}, 5_{255}\})$, $(\{5_0, 5_{128}\}, \{3_{127}, 3_{255}\})$, $(\{3_1, 3_{129}\}, \{5_{63}, 5_{191}\})$
$(\{5_1, 5_{129}\}, \{3_{63}, 3_{191}\})$, $(\{3_2, 3_{130}\}, \{5_{95}, 5_{223}\})$, $(\{5_2, 5_{130}\}, \{3_{95}, 3_{223}\})$
$(\{3_3, 3_{131}\}, \{5_{31}, 5_{159}\})$, $(\{5_3, 5_{131}\}, \{3_{31}, 3_{159}\})$, $(\{3_4, 3_{132}\}, \{5_{111}, 5_{239}\})$
$(\{5_4, 5_{132}\}, \{3_{111}, 3_{239}\})$, $(\{3_5, 3_{133}\}, \{5_{47}, 5_{175}\})$, $(\{5_5, 5_{133}\}, \{3_{47}, 3_{175}\})$
$(\{3_6, 3_{134}\}, \{5_{79}, 5_{207}\})$, $(\{5_6, 5_{134}\}, \{3_{79}, 3_{207}\})$, $(\{3_7, 3_{135}\}, \{5_{15}, 5_{143}\})$
$(\{5_7, 5_{135}\}, \{3_{15}, 3_{143}\})$, $(\{3_{16}, 3_{144}\}, \{5_{123}, 5_{251}\})$, $(\{5_{16}, 5_{144}\}, \{3_{123}, 3_{251}\})$
$(\{3_{17}, 3_{145}\}, \{5_{59}, 5_{187}\})$, $(\{5_{17}, 5_{145}\}, \{3_{59}, 3_{187}\})$, $(\{3_{18}, 3_{146}\}, \{5_{91}, 5_{219}\})$
$(\{5_{18}, 5_{146}\}, \{3_{91}, 3_{219}\})$, $(\{3_{19}, 3_{147}\}, \{5_{27}, 5_{155}\})$, $(\{5_{19}, 5_{147}\}, \{3_{27}, 3_{155}\})$
$(\{3_{20}, 3_{148}\}, \{5_{107}, 5_{235}\})$, $(\{5_{20}, 5_{148}\}, \{3_{107}, 3_{235}\})$, $(\{3_{21}, 3_{149}\}, \{5_{43}, 5_{171}\})$
$(\{5_{21}, 5_{149}\}, \{3_{43}, 3_{171}\})$, $(\{3_{22}, 3_{150}\}, \{5_{75}, 5_{203}\})$, $(\{5_{22}, 5_{150}\}, \{3_{75}, 3_{203}\})$
$(\{3_{23}, 3_{151}\}, \{5_{11}, 5_{139}\})$, $(\{5_{23}, 5_{151}\}, \{3_{11}, 3_{139}\})$, $(\{3_{32}, 3_{160}\}, \{5_{125}, 5_{253}\})$
$(\{5_{32}, 5_{160}\}, \{3_{125}, 3_{253}\})$, $(\{3_{33}, 3_{161}\}, \{5_{61}, 5_{189}\})$, $(\{5_{33}, 5_{161}\}, \{3_{61}, 3_{189}\})$
$(\{3_{34}, 3_{162}\}, \{5_{93}, 5_{221}\})$, $(\{5_{34}, 5_{162}\}, \{3_{93}, 3_{221}\})$, $(\{3_{35}, 3_{163}\}, \{5_{29}, 5_{157}\})$
$(\{5_{35}, 5_{163}\}, \{3_{29}, 3_{157}\})$, $(\{3_{36}, 3_{164}\}, \{5_{109}, 5_{237}\})$, $(\{5_{36}, 5_{164}\}, \{3_{109}, 3_{237}\})$
$(\{3_{37}, 3_{165}\}, \{5_{45}, 5_{173}\})$, $(\{5_{37}, 5_{165}\}, \{3_{45}, 3_{173}\})$, $(\{3_{38}, 3_{166}\}, \{5_{77}, 5_{205}\})$
$(\{5_{38}, 5_{166}\}, \{3_{77}, 3_{205}\})$, $(\{3_{39}, 3_{167}\}, \{5_{13}, 5_{141}\})$, $(\{5_{39}, 5_{167}\}, \{3_{13}, 3_{141}\})$

$(\{3_{48}, 3_{176}\}, \{5_{121}, 5_{249}\})$, $(\{5_{48}, 5_{176}\}, \{3_{121}, 3_{249}\})$, $(\{3_{49}, 3_{177}\}, \{5_{57}, 5_{185}\})$
$(\{5_{49}, 5_{177}\}, \{3_{57}, 3_{185}\})$, $(\{3_{50}, 3_{178}\}, \{5_{89}, 5_{217}\})$, $(\{5_{50}, 5_{178}\}, \{3_{89}, 3_{217}\})$
$(\{3_{51}, 3_{179}\}, \{5_{25}, 5_{153}\})$, $(\{5_{51}, 5_{179}\}, \{3_{25}, 3_{153}\})$, $(\{3_{52}, 3_{180}\}, \{5_{105}, 5_{233}\})$
$(\{5_{52}, 5_{180}\}, \{3_{105}, 3_{233}\})$, $(\{3_{53}, 3_{181}\}, \{5_{41}, 5_{169}\})$, $(\{5_{53}, 5_{181}\}, \{3_{41}, 3_{169}\})$
$(\{3_{54}, 3_{182}\}, \{5_{73}, 5_{201}\})$, $(\{5_{54}, 5_{182}\}, \{3_{73}, 3_{201}\})$, $(\{3_{55}, 3_{183}\}, \{5_{9}, 5_{137}\})$
$(\{5_{55}, 5_{183}\}, \{3_{9}, 3_{137}\})$, $(\{3_{64}, 3_{192}\}, \{5_{126}, 5_{254}\})$, $(\{5_{64}, 5_{192}\}, \{3_{126}, 3_{254}\})$
$(\{3_{65}, 3_{193}\}, \{5_{62}, 5_{190}\})$, $(\{5_{65}, 5_{193}\}, \{3_{62}, 3_{190}\})$, $(\{3_{66}, 3_{194}\}, \{5_{94}, 5_{222}\})$
$(\{5_{66}, 5_{194}\}, \{3_{94}, 3_{222}\})$, $(\{3_{67}, 3_{195}\}, \{5_{30}, 5_{158}\})$, $(\{5_{67}, 5_{195}\}, \{3_{30}, 3_{158}\})$
$(\{3_{68}, 3_{196}\}, \{5_{110}, 5_{238}\})$, $(\{5_{68}, 5_{196}\}, \{3_{110}, 3_{238}\})$, $(\{3_{69}, 3_{197}\}, \{5_{46}, 5_{174}\})$
$(\{5_{69}, 5_{197}\}, \{3_{46}, 3_{174}\})$, $(\{3_{70}, 3_{198}\}, \{5_{78}, 5_{206}\})$, $(\{5_{70}, 5_{198}\}, \{3_{78}, 3_{206}\})$
$(\{3_{71}, 3_{199}\}, \{5_{14}, 5_{142}\})$, $(\{5_{71}, 5_{199}\}, \{3_{14}, 3_{142}\})$, $(\{3_{80}, 3_{208}\}, \{5_{122}, 5_{250}\})$
$(\{5_{80}, 5_{208}\}, \{3_{122}, 3_{250}\})$, $(\{3_{81}, 3_{209}\}, \{5_{58}, 5_{186}\})$, $(\{5_{81}, 5_{209}\}, \{3_{58}, 3_{186}\})$
$(\{3_{82}, 3_{210}\}, \{5_{90}, 5_{218}\})$, $(\{5_{82}, 5_{210}\}, \{3_{90}, 3_{218}\})$, $(\{3_{83}, 3_{211}\}, \{5_{26}, 5_{154}\})$
$(\{5_{83}, 5_{211}\}, \{3_{26}, 3_{154}\})$, $(\{3_{84}, 3_{212}\}, \{5_{106}, 5_{234}\})$, $(\{5_{84}, 5_{212}\}, \{3_{106}, 3_{234}\})$
$(\{3_{85}, 3_{213}\}, \{5_{42}, 5_{170}\})$, $(\{5_{85}, 5_{213}\}, \{3_{42}, 3_{170}\})$, $(\{3_{86}, 3_{214}\}, \{5_{74}, 5_{202}\})$
$(\{5_{86}, 5_{214}\}, \{3_{74}, 3_{202}\})$, $(\{3_{87}, 3_{215}\}, \{5_{10}, 5_{138}\})$, $(\{5_{87}, 5_{215}\}, \{3_{10}, 3_{138}\})$
$(\{3_{96}, 3_{224}\}, \{5_{124}, 5_{252}\})$, $(\{5_{96}, 5_{224}\}, \{3_{124}, 3_{252}\})$, $(\{3_{97}, 3_{225}\}, \{5_{60}, 5_{188}\})$
$(\{5_{97}, 5_{225}\}, \{3_{60}, 3_{188}\})$, $(\{3_{98}, 3_{226}\}, \{5_{92}, 5_{220}\})$, $(\{5_{98}, 5_{226}\}, \{3_{92}, 3_{220}\})$
$(\{3_{99}, 3_{227}\}, \{5_{28}, 5_{156}\})$, $(\{5_{99}, 5_{227}\}, \{3_{28}, 3_{156}\})$, $(\{3_{100}, 3_{228}\}, \{5_{108}, 5_{236}\})$
$(\{5_{100}, 5_{228}\}, \{3_{108}, 3_{236}\})$, $(\{3_{101}, 3_{229}\}, \{5_{44}, 5_{172}\})$, $(\{5_{101}, 5_{229}\}, \{3_{44}, 3_{172}\})$
$(\{3_{102}, 3_{230}\}, \{5_{76}, 5_{204}\})$, $(\{5_{102}, 5_{230}\}, \{3_{76}, 3_{204}\})$, $(\{3_{103}, 3_{231}\}, \{5_{12}, 5_{140}\})$
$(\{5_{103}, 5_{231}\}, \{3_{12}, 3_{140}\})$, $(\{3_{112}, 3_{240}\}, \{5_{120}, 5_{248}\})$, $(\{5_{112}, 5_{240}\}, \{3_{120}, 3_{248}\})$
$(\{3_{113}, 3_{241}\}, \{5_{56}, 5_{184}\})$, $(\{5_{113}, 5_{241}\}, \{3_{56}, 3_{184}\})$, $(\{3_{114}, 3_{242}\}, \{5_{88}, 5_{216}\})$
$(\{5_{114}, 5_{242}\}, \{3_{88}, 3_{216}\})$, $(\{3_{115}, 3_{243}\}, \{5_{24}, 5_{152}\})$, $(\{5_{115}, 5_{243}\}, \{3_{24}, 3_{152}\})$
$(\{3_{116}, 3_{244}\}, \{5_{104}, 5_{232}\})$, $(\{5_{116}, 5_{244}\}, \{3_{104}, 3_{232}\})$, $(\{3_{117}, 3_{245}\}, \{5_{40}, 5_{168}\})$
$(\{5_{117}, 5_{245}\}, \{3_{40}, 3_{168}\})$, $(\{3_{118}, 3_{246}\}, \{5_{72}, 5_{200}\})$, $(\{5_{118}, 5_{246}\}, \{3_{72}, 3_{200}\})$
$(\{3_{119}, 3_{247}\}, \{5_{8}, 5_{136}\})$, $(\{5_{119}, 5_{247}\}, \{3_{8}, 3_{136}\})$

7. Between the groups of $\{V3_i \mid i = 1, 2, \cdots, 32\}$ and $\{(C_i, 7) | i = 1, 2, \cdots, 255\}$ (for simplicity, we abbreviate $(C_i, 7)$ as 7_i), there are complete 128 $K_{2,2}$-bipartitions:

$(\{3_{0}, 3_{128}\}, \{7_{127}, 7_{255}\})$, $(\{7_{0}, 7_{128}\}, \{3_{127}, 3_{255}\})$, $(\{3_{1}, 3_{129}\}, \{7_{63}, 7_{191}\})$
$(\{7_{1}, 7_{129}\}, \{3_{63}, 3_{191}\})$, $(\{3_{2}, 3_{130}\}, \{7_{95}, 7_{223}\})$, $(\{7_{2}, 7_{130}\}, \{3_{95}, 3_{223}\})$
$(\{3_{3}, 3_{131}\}, \{7_{31}, 7_{159}\})$, $(\{7_{3}, 7_{131}\}, \{3_{31}, 3_{159}\})$, $(\{3_{4}, 3_{132}\}, \{7_{111}, 7_{239}\})$
$(\{7_{4}, 7_{132}\}, \{3_{111}, 3_{239}\})$, $(\{3_{5}, 3_{133}\}, \{7_{47}, 7_{175}\})$, $(\{7_{5}, 7_{133}\}, \{3_{47}, 3_{175}\})$
$(\{3_{6}, 3_{134}\}, \{7_{79}, 7_{207}\})$, $(\{7_{6}, 7_{134}\}, \{3_{79}, 3_{207}\})$, $(\{3_{7}, 3_{135}\}, \{7_{15}, 7_{143}\})$
$(\{7_{7}, 7_{135}\}, \{3_{15}, 3_{143}\})$, $(\{3_{16}, 3_{144}\}, \{7_{123}, 7_{251}\})$, $(\{7_{16}, 7_{144}\}, \{3_{123}, 3_{251}\})$
$(\{3_{17}, 3_{145}\}, \{7_{59}, 7_{187}\})$, $(\{7_{17}, 7_{145}\}, \{3_{59}, 3_{187}\})$, $(\{3_{18}, 3_{146}\}, \{7_{91}, 7_{219}\})$
$(\{7_{18}, 7_{146}\}, \{3_{91}, 3_{219}\})$, $(\{3_{19}, 3_{147}\}, \{7_{27}, 7_{155}\})$, $(\{7_{19}, 7_{147}\}, \{3_{27}, 3_{155}\})$
$(\{3_{20}, 3_{148}\}, \{7_{107}, 7_{235}\})$, $(\{7_{20}, 7_{148}\}, \{3_{107}, 3_{235}\})$, $(\{3_{21}, 3_{149}\}, \{7_{43}, 7_{171}\})$
$(\{7_{21}, 7_{149}\}, \{3_{43}, 3_{171}\})$, $(\{3_{22}, 3_{150}\}, \{7_{75}, 7_{203}\})$, $(\{7_{22}, 7_{150}\}, \{3_{75}, 3_{203}\})$

$$
\begin{array}{lll}
(\{3_{23},3_{151}\},\{7_{11},7_{139}\}), & (\{7_{23},7_{151}\},\{3_{11},3_{139}\}), & (\{3_{32},3_{160}\},\{7_{125},7_{253}\}) \\
(\{7_{32},7_{160}\},\{3_{125},3_{253}\}), & (\{3_{33},3_{161}\},\{7_{61},7_{189}\}), & (\{7_{33},7_{161}\},\{3_{61},3_{189}\}) \\
(\{3_{34},3_{162}\},\{7_{93},7_{221}\}), & (\{7_{34},7_{162}\},\{3_{93},3_{221}\}), & (\{3_{35},3_{163}\},\{7_{29},7_{157}\}) \\
(\{7_{35},7_{163}\},\{3_{29},3_{157}\}), & (\{3_{36},3_{164}\},\{7_{109},7_{237}\}), & (\{7_{36},7_{164}\},\{3_{109},3_{237}\}) \\
(\{3_{37},3_{165}\},\{7_{45},7_{173}\}), & (\{7_{37},7_{165}\},\{3_{45},3_{173}\}), & (\{3_{38},3_{166}\},\{7_{77},7_{205}\}) \\
(\{7_{38},7_{166}\},\{3_{77},3_{205}\}), & (\{3_{39},3_{167}\},\{7_{13},7_{141}\}), & (\{7_{39},7_{167}\},\{3_{13},3_{141}\}) \\
(\{3_{48},3_{176}\},\{7_{121},7_{249}\}), & (\{7_{48},7_{176}\},\{3_{121},3_{249}\}), & (\{3_{49},3_{177}\},\{7_{57},7_{185}\}) \\
(\{7_{49},7_{177}\},\{3_{57},3_{185}\}), & (\{3_{50},3_{178}\},\{7_{89},7_{217}\}), & (\{7_{50},7_{178}\},\{3_{89},3_{217}\}) \\
(\{3_{51},3_{179}\},\{7_{25},7_{153}\}), & (\{7_{51},7_{179}\},\{3_{25},3_{153}\}), & (\{3_{52},3_{180}\},\{7_{105},7_{233}\}) \\
(\{7_{52},7_{180}\},\{3_{105},3_{233}\}), & (\{3_{53},3_{181}\},\{7_{41},7_{169}\}), & (\{7_{53},7_{181}\},\{3_{41},3_{169}\}) \\
(\{3_{54},3_{182}\},\{7_{73},7_{201}\}), & (\{7_{54},7_{182}\},\{3_{73},3_{201}\}), & (\{3_{55},3_{183}\},\{7_{9},7_{137}\}) \\
(\{7_{55},7_{183}\},\{3_{9},3_{137}\}), & (\{3_{64},3_{192}\},\{7_{126},7_{254}\}), & (\{7_{64},7_{192}\},\{3_{126},3_{254}\}) \\
(\{3_{65},3_{193}\},\{7_{62},7_{190}\}), & (\{7_{65},7_{193}\},\{3_{62},3_{190}\}), & (\{3_{66},3_{194}\},\{7_{94},7_{222}\}) \\
(\{7_{66},7_{194}\},\{3_{94},3_{222}\}), & (\{3_{67},3_{195}\},\{7_{30},7_{158}\}), & (\{7_{67},7_{195}\},\{3_{30},3_{158}\}) \\
(\{3_{68},3_{196}\},\{7_{110},7_{238}\}), & (\{7_{68},7_{196}\},\{3_{110},3_{238}\}), & (\{3_{69},3_{197}\},\{7_{46},7_{174}\}) \\
(\{7_{69},7_{197}\},\{3_{46},3_{174}\}), & (\{3_{70},3_{198}\},\{7_{78},7_{206}\}), & (\{7_{70},7_{198}\},\{3_{78},3_{206}\}) \\
(\{3_{71},3_{199}\},\{7_{14},7_{142}\}), & (\{7_{71},7_{199}\},\{3_{14},3_{142}\}), & (\{3_{80},3_{208}\},\{7_{122},7_{250}\}) \\
(\{7_{80},7_{208}\},\{3_{122},3_{250}\}), & (\{3_{81},3_{209}\},\{7_{58},7_{186}\}), & (\{7_{81},7_{209}\},\{3_{58},3_{186}\}) \\
(\{3_{82},3_{210}\},\{7_{90},7_{218}\}), & (\{7_{82},7_{210}\},\{3_{90},3_{218}\}), & (\{3_{83},3_{211}\},\{7_{26},7_{154}\}) \\
(\{7_{83},7_{211}\},\{3_{26},3_{154}\}), & (\{3_{84},3_{212}\},\{7_{106},7_{234}\}), & (\{7_{84},7_{212}\},\{3_{106},3_{234}\}) \\
(\{3_{85},3_{213}\},\{7_{42},7_{170}\}), & (\{7_{85},7_{213}\},\{3_{42},3_{170}\}), & (\{3_{86},3_{214}\},\{7_{74},7_{202}\}) \\
(\{7_{86},7_{214}\},\{3_{74},3_{202}\}), & (\{3_{87},3_{215}\},\{7_{10},7_{138}\}), & (\{7_{87},7_{215}\},\{3_{10},3_{138}\}) \\
(\{3_{96},3_{224}\},\{7_{124},7_{252}\}), & (\{7_{96},7_{224}\},\{3_{124},3_{252}\}), & (\{3_{97},3_{225}\},\{7_{60},7_{188}\}) \\
(\{7_{97},7_{225}\},\{3_{60},3_{188}\}), & (\{3_{98},3_{226}\},\{7_{92},7_{220}\}), & (\{7_{98},7_{226}\},\{3_{92},3_{220}\}) \\
(\{3_{99},3_{227}\},\{7_{28},7_{156}\}), & (\{7_{99},7_{227}\},\{3_{28},3_{156}\}), & (\{3_{100},3_{228}\},\{7_{108},7_{236}\}) \\
(\{7_{100},7_{228}\},\{3_{108},3_{236}\}), & (\{3_{101},3_{229}\},\{7_{44},7_{172}\}), & (\{7_{101},7_{229}\},\{3_{44},3_{172}\}) \\
(\{3_{102},3_{230}\},\{7_{76},7_{204}\}), & (\{7_{102},7_{230}\},\{3_{76},3_{204}\}), & (\{3_{103},3_{231}\},\{7_{12},7_{140}\}) \\
(\{7_{103},7_{231}\},\{3_{12},3_{140}\}), & (\{3_{112},3_{240}\},\{7_{120},7_{248}\}), & (\{7_{112},7_{240}\},\{3_{120},3_{248}\}) \\
(\{3_{113},3_{241}\},\{7_{56},7_{184}\}), & (\{7_{113},7_{241}\},\{3_{56},3_{184}\}), & (\{3_{114},3_{242}\},\{7_{88},7_{216}\}) \\
(\{7_{114},7_{242}\},\{3_{88},3_{216}\}), & (\{3_{115},3_{243}\},\{7_{24},7_{152}\}), & (\{7_{115},7_{243}\},\{3_{24},3_{152}\}) \\
(\{3_{116},3_{244}\},\{7_{104},7_{232}\}), & (\{7_{116},7_{244}\},\{3_{104},3_{232}\}), & (\{3_{117},3_{245}\},\{7_{40},7_{168}\}) \\
(\{7_{117},7_{245}\},\{3_{40},3_{168}\}), & (\{3_{118},3_{246}\},\{7_{72},7_{200}\}), & (\{7_{118},7_{246}\},\{3_{72},3_{200}\}) \\
(\{3_{119},3_{247}\},\{7_{8},7_{136}\}), & (\{7_{119},7_{247}\},\{3_{8},3_{136}\}) &
\end{array}
$$

4.5 Solving the Embedded Maximum Independent Set Problem in G_9

For computing the independence number of an induced subgraph G of G_9, we consider "ladder subgraphs" L of G_9 of the form:

$$V(L) = \{u_i \mid i \leq i \leq l\} \cup \{v_i \mid i \leq i \leq l\} \text{ and}$$

$$\begin{aligned} E(L) = \; & \{\{u_i, u_{i+1}\} \mid 1 \leq i \leq l-1\} \cup \{\{v_i, v_{i+1}\} \mid 1 \leq i \leq l-1\} \cup \\ & \{\{u_i, v_{i+1}\} \mid 1 \leq i \leq l-1\} \cup \{\{v_i, u_{i+1}\} \mid 1 \leq i \leq l-1\}. \end{aligned}$$

The independence number of G is given by $\max\{|V(G) \cap \{u_{i_1}, v_{i_1}, u_{i_2}, v_{i_2}, u_{i_3}, v_{i_3}, \cdots\}| \mid$ $i_1, i_2, i_3, \cdots \in \{1, 2, \cdots, l\}$ and $i_j + 1 < i_{j+1}$ for $j = 1, 2, \cdots\}$.

CHAPTER V

CONCLUSIONS

In this thesis, we study two approaches that prove the length lower bounds of reflecting sequences for labeled chains. Improving the length lower bounds of reflecting sequence results in improving those of universal traversal sequences, which provide a traversal strategy for graphs in limited space complexity.

For short labeled chains, the lower bounds of reflecting sequences are obtained by analytical approaches. Tompa's marking scheme is used for the labeled chains of lengths 3 and 4, and the Buss-Tompa marking scheme is used for that of length 5. In Tompa's marking scheme, only forward traversals on the chains are considered, and marks are placed on a hypothetical reflecting sequence. Tompa's marking scheme yield a lower bound of $R(t,n) \geq tn^{log_4 6}$. In the Buss-Tompa marking scheme, backward traversals are also considered in addition to forward traversals, and the new notions of debts and interval are introduced. The Buss-Tompa marking scheme improved the length lower bound of t-reflecting sequences from $tn^{log_4 6}$ to $tn^{log_5 10}$.

A computational approach is used to improve the length lower bounds of universal traversal sequences further. This approach applies Tompa's marking scheme. Two kinds of marks, open marks and closed marks are introduced. We consider inconsistencies between open marks since some of them may not co-exist on any reflecting sequence. The inconsistency graph that contains information of such inconsistencies between open marks is used for constructing a quadtree. The computational approach show a length lower bound of $R(t,n) \geq tn^{log_7 19}$ using labeled chains of length 7.

We obtain the suffix inconsistency graph of labeled chains of length 9. The structure of the graph is complicated because there are $2^{9-1} = 256$ chains with different labels. We need to understand the structure of the graph well in order to get an efficient construction of the quadtree.

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APPENDIX A: GLOSSARY

$\alpha\beta\alpha\beta$ Interval A substring of a reflecting sequence that begins with the $\alpha\beta$ pair that finishes a traversal of $\alpha\beta\alpha\beta$, and ends with $\alpha\beta$ pair that starts the next traversal of $\alpha\beta\alpha\beta$, where $a\,\beta \in \{0,1\}$.

Base Pairs The pairs which correspond to starting and finishing left-to-right traversal of the chains with labels $\alpha\beta\alpha\beta$ for $\alpha, \beta \in \{0,1\}$.

Circumnavigation Sequence A traversal command for labeled cycles.

Closed Marks If S traverses chain C_i from vertex 1 to vertex n without returning to vertex 1, closed marks of M_i are put on that bit of S that induces to exit from vertex 1.

Closing Marks If an $\alpha\beta$ pair finish a traversal of either $\alpha\beta\alpha\beta$ or $\alpha\beta\overline{\alpha}\overline{\beta}$, the $\alpha\beta$ pair has a closing mark.

Closing Debts If an $\alpha\beta$ pair doesn't have a closing mark, the $\alpha\beta$ pair has a closing debt.

Complete Bipartite Graph $K_{m,n}$ The graph with m left vertices, n right vertices, and an edge joining every left vertex to every right vertex.

Edge Labeling Place a unique label on each endpoint of an edge.

Finishing Traversal The pair matching the second $\alpha\beta$ in $\alpha\beta(\alpha\overline{\beta} + \overline{\alpha}\beta)^*\alpha\beta$ is said to finish a (left-to-right) traversal of $\alpha\beta\alpha\beta$.

Labeled Graph A graph in which each endpoint of an edge is labeled by edge labeling.

Labeled Chain A chain in which each endpoint of an edge incident with every interior vertex have a unique label.

Labeled Cycle Labeled graphs with regularity of 2.

Lower Bound The Ω notation gives a lower bound for a function to within a constant factor. We denote $f(n) = \Omega(g(n))$ if there are positive constants n_0 and c such that $n > n_0$ and the value of $f(n)$ always lies on or above $cg(n)$.

Marks-to-Bits Density The number of marks per bit.

Marks-to-Pairs Density The number of marks per pair.

Marks-to-pairs density = 2∗(marks-to-bits density).

Nonbase Pairs The pairs which are not base pairs.

Open Marks If a bit of S induces to exit vertex 1 and S doesn't make traversal reach neither vertex 1 nor vertex n on chain C_i, an open mark is put on that bit.

Opening Marks If an $\alpha\beta$ pair starts a traversal of $\alpha\beta\alpha\beta$ or it is the last $\alpha\beta$ pair during a traversal of $\alpha\beta\overline{\alpha}\overline{\beta}$, the $\alpha\beta$ pair has an opening mark.

Opening Debts If an $\alpha\beta$ pair doesn't have an opening mark, the $\alpha\beta$ pair has an opening debt.

Recurrence In order to accomplish a task, use itself with some part of the task.

Reflecting Sequence An end-to-end traversal command on labeled chains.

Reflection A complete endvertex-to-endvertex traversal on a labeled chain.

Regular Graph A graph in which every vertex has the same degree of regularity.

Regularity The number of edges connected to a vertex in the regular graph.

Starting Traversal The pair matching the first $\alpha\beta$ in $\alpha\beta(\alpha\overline{\beta} + \overline{\alpha}\beta)^*\alpha\beta$ is said to start a (left-to-right) traversal of $\alpha\beta\alpha\beta$.

Traversal Sequence A traversal command for labeled graphs.

Undirected Graph A graph whose edges are unordered pairs of vertices.

Universal Traversal Sequence A particular traversal sequence that makes traversal visit all vertices at least once staring at any vertex for each graph in the set of d-regular, n-vertex, edge-labeled, undirected graphs.

Upper Bound The O notation gives a upper bound for a function to within a constant factor. We denote $f(n) = O(g(n))$ if there are positive constants n_0 and c such that $n > n_0$ and the value of $f(n)$ always lies on or below $cg(n)$.

APPENDIX B: PROGRAM LISTING

B.1 Implementation of Finding Inconsistencies (Method I)

```
/*
 find inconsistencies of open marks by using the theorem in [DF96a]
*/
#include <stdio.h>

#define CHAIN_LEN 9
#define MAX 256
#define CONSIS 0
#define INCONSIS 1


void init_table(int t[][MAX])
{
 int i, j;

 for(i=0;i<MAX;i++)
   for(j=0;j<MAX;j++)
     t[i][j] = 0;
}


void int_to_bin(int n, int a[])
{
 int i = CHAIN_LEN-1, j;

 do{
   a[i] = n % 2;
   n = n/2;
   i--;
 }while(n != 0);

 for(j=i;j>=1;j--)
   a[j] = 0;
}


int NR(int alpha[], int beta[], int delta)
```

```
{
  int i, j, end1, end2;
  int flag = 1, flag2 = 0;
  int s, t;

  end1 = CHAIN_LEN-1-delta;
  j = end1;

  if(delta<=1){
    for(i=2; i<= end1; i++, j--)
      if(beta[i] != !alpha[j])
       return CONSIS;
    return INCONSIS;
   }

  else{
    for(i=2; i<= end1; i++, j--)
      if(beta[i] != !alpha[j])
        flag = 0;

    if(flag)
      return INCONSIS;
    else{
      end2 = end1+2;
      for(t=2;t<=end1;t++)
        if(beta[t] == alpha[end2-t])
          flag2 = 1;

      if(flag2 && NR(alpha, beta, delta-2))
        return INCONSIS;
      else
        return CONSIS;
    }
  }
}


void make_table(int t[][MAX], int delta)
{
 int i, j;
 int alpha[CHAIN_LEN], beta[CHAIN_LEN];

 for(i=0;i<MAX;i++){
   int_to_bin(i, alpha);
```

```
    for(j=0;j<MAX;j++){
      int_to_bin(j, beta);

      t[i][j] = NR(alpha, beta, delta);
    }
  }
 }


void print_table(int table[][256], int numchains)
{
  int cont, i, j;

  for(i=0;i<numchains;i++){
    cont = 0;
    printf("\n%d\n",i);
    for(j=0;j<numchains;j++){
      if(table[i][j]){
        if(cont)
          printf(",%d",j);
        else
          printf(" - %d",j);
        cont = 1;
      }
      else{
        if(cont)
          printf("\n");
        cont = 0;
      }
    }
  }
}


main()
{
 int table33[MAX][MAX], table35[MAX][MAX], table37[MAX][MAX];
 int i, j, s;

 init_table(table33);
 make_table(table33,4);
 print_table(table33, MAX);

 init_table(table35);
```

```
 make_table(table35,2);
 print_table(table35, MAX);

 init_table(table37);
 make_table(table37,0);
 print_table(table37, MAX);
}
```

B.2 Implementation of Finding Inconsistencies (Method II)

```
/*
 find inconsistencies of open marks by traversing each chain
*/
#include <iostream.h>

#define MAXCHAINLEN 256
#define chainlen 9

typedef struct{
  char size;
  short int chainnum;
  char labels[MAXCHAINLEN];
  char currentstate;
} DFA;

void setdfa(int size, int dfanum, int initstate, DFA *dfap)
{
  int i;

  dfap->size = size;
  dfap->chainnum = dfanum;
  dfap->currentstate = initstate;

  for(i=size-1; i>0; i--){
    dfap->labels[i] = dfanum % 2;
    dfanum = dfanum / 2;
  }
}

void setstate(int initstate, DFA *dfap)
{
  dfap->currentstate = initstate;
}


int state(DFA *dfap)
{
  return dfap->currentstate;
}
```

```
int dfasize(DFA *dfap)
{
  return dfap->size;
}


void movebit(int bit, DFA *dfap)
{
  int oldcurrent = dfap->currentstate;

  if(dfap->currentstate == dfap->size)
    (dfap->currentstate)--;
  else if(dfap->currentstate == 0)
    (dfap->currentstate)++;
  else if(dfap->labels[dfap->currentstate] == bit)
    (dfap->currentstate)++;
  else
    (dfap->currentstate)--;
}

int consistentopenpair(DFA *dfap1, DFA *dfap2,
                int visited[MAXCHAINLEN+1][MAXCHAINLEN+1])
{
  int state1, state2, i, consistent, n = dfasize(dfap1);

  state1 = state(dfap1);
  state2 = state(dfap2);

  if(!visited[state1][state2]){
    visited[state1][state2] = 1;
    if(state1 == 1 || state2 == 1)
      return 0;

    if(state1 == n || state2 == n)
      return 1;
    consistent = 0;

    for(i=0; i<2; i++){
      movebit(i, dfap1);
      movebit(i, dfap2);
      consistent |= consistentopenpair(dfap1, dfap2, visited);
      setstate(state1, dfap1);
      setstate(state2, dfap2);
    }
```

```
    return consistent;
  }
  else
    return 0;
}


void print_table(int table[][256], int numchains)
{
  int cont, i, j;

  for(i=0;i<numchains;i++){
    cont = 0;
    cout <<"\n"<<i<<"\n";;
    for(j=0;j<numchains;j++){
      if(table[i][j]){
        if(cont)
          cout <<','<<j;
        else
          cout <<" - "<<j;
        cont = 1;
      }
      else{
        if(cont)
          cout <<"\n";
        cont = 0;
      }
    }
  }
}

void make_table(int table[][256], int numchains, int v1, int v2)
{
  DFA dfa1, dfa2;
  int dfanum1, dfanum2, i, j;
  int consistent;
  int visited[MAXCHAINLEN+1][MAXCHAINLEN+1];

  for(dfanum1=0; dfanum1<numchains; dfanum1++)
    for(dfanum2=0; dfanum2<numchains; dfanum2++){

      for(i=0; i<=chainlen; i++)
        for(j=0; j<=chainlen; j++)
```

```
          visited[i][j] = 0;

      setdfa(chainlen, dfanum1, v1, &dfa1);
      setdfa(chainlen, dfanum2, v2, &dfa2);

      consistent = consistentopenpair(&dfa1, &dfa2, visited);

      table[dfanum1][dfanum2] = (!consistent);
  }
}

main()
{
  DFA dfa1, dfa2;
  int dfanum1, dfanum2, state1, state2, consistent;
  int i, numchains;

  int table33[MAXCHAINLEN][MAXCHAINLEN];
  int table35[MAXCHAINLEN][MAXCHAINLEN];
  int table37[MAXCHAINLEN][MAXCHAINLEN];
  int table55[MAXCHAINLEN][MAXCHAINLEN];

  cout << "Computing Tables for chainlength = 9\n";

  for(i=chainlen, numchains=1; i>1; i--)
    numchains *= 2;

  cout << "Computing (3,3) state pair table...";
  make_table(table33, numchains, 3, 3);
  print_table(table33, numchains);


  cout << "Computing(3,5) state pair table...";
  make_table(table35, numchains, 3, 5);
  print_table(table35, numchains);

  cout << "Computing(3,7) state pair table...";
  make_table(table37, numchains, 3, 7);
  print_table(table37, numchains);

  cout << "Computing(5,5) state pair table...";
  make_table(table55, numchains, 5, 5);
  print_table(table55, numchains);
}
```

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