AN EXPLORATION OF COLLEGE STUDENTS' PROBLEM SOLVING BEHAVIORS WHILE VERIFYING TRIGONOMETRIC IDENTITIES: A MIXED METHODS CASE STUDY

By

BENJAMIN MARK WESCOATT

Bachelor of Science in Mathematics
Oklahoma State University
Stillwater, Oklahoma
2000

Master of Science in Mathematics
Oklahoma State University
Stillwater, Oklahoma
2007

Submitted to the Faculty of the
Graduate College of the
Oklahoma State University
in partial fulfillment of
the requirements for
the Degree of
DOCTOR OF PHILOSOPHY
July, 2013

AN EXPLORATION OF COLLEGE STUDENTS'

PROBLEM SOLVING BEHAVIORS WHILE

VERIFYING TRIGONOMETRIC IDENTITIES:

A MIXED METHODS CASE STUDY

Dissertation Approved:

Dr. Douglas B. Aichele

Dissertation Adviser

Dr. Christopher Francisco

Dr. James R. Choike

Dr. Melinda H. McCann

.

ACKNOWLEDGEMENTS

> Let no one deceive himself. If anyone among you thinks that he
> is wise in this age, let him become a fool that he may become wise.
> 1 Corinthians 3:18 ESV

There are many people at Oklahoma State who deserve my gratitude:

- To Dr. Doug Aichele, my advisor, for letting me be me and for leading me to the field of mathematics education.
- To Dr. Jim Choike, for being a tremendous mentor and a bang-up calculus 2 teacher.
- To Dr. Chris Francisco, for guiding me through the job hunt.
- To Dr. Mindy McCann, for opening up the world of statistics to me.
- To Dr. Alan Noell, for being my guardian angel and showing a genuine interest in all graduate students.
- To Shelly Downing and Angie McFarlin, for the support you've provided to me over the years.
- To Cynthia Francisco, for working with me early in my career.
- To Dr. Juliana Utley, for showing me how to do research.

I would also like to thank Valdosta State University for giving me my first faculty position, seeing something in me that I have a hard time seeing.

My sister, Robin Oblander, has helped me in various ways, too numerous to mention. Reflecting back, I realize how difficult it will be to begin paying her back.

Attending graduate school became much easier with the financial support of my parents, Mark and Dee Dee Wescoatt, but of course their most important support hasn't been financial. The thought of my imminent move away from Stillwater brings tears to my eyes and a sharp pain to my heart, knowing my parents' desires to have us closer to them. I love them very much.

Of course, my graduate school widow, Joanna, is the co-author of this all. Our three kids were born during our second stint in Stillwater. Graduate school is tough enough, and with three young children desiring and requiring your attention, it becomes that much tougher. Joanna managed to steer all of us through the tribulations. She is a very special woman. To my children, Sean, Gabriella, and little Alexandra (who is crying as I write this), you make my life so interesting, and I love you so much and am so proud of you!

During the last conversation I ever had with my younger sister, Katy, on a whim, I

mentioned I might go back to school to work on a graduate degree. At the time, I was an officer in the navy and had reached the point where I had to make a decision about my future career in the navy. There was a lot of turmoil in my life, and I really didn't know where I was headed.

My turmoil soon turned into a maelstrom, because six days after our conversation, she was dead, run over by a 16-year-old girl.

Over the past eight years of graduate school, I often thought on our conversation, trying in vain to reconstruct it. While many details eluded me, I never forgot my whimsical comment. It sustained me at times when I wondered if I was doing the right thing.

I offer this dissertation to the memory of my sister.

Katy Louise Wescoatt
17 November 1981 – 20 March 2004

Acknowledgements reflect the views of the author and are not endorsed by committee members or Oklahoma State University.

Name: BENJAMIN MARK WESCOATT

Date of Degree: JULY, 2013

Title of Study: AN EXPLORATION OF COLLEGE STUDENTS' PROBLEM SOLVING BEHAVIORS WHILE VERIFYING TRIGONOMETRIC IDENTITIES: A MIXED METHOD CASE STUDY

Major Field: Mathematics, with Specialization in Mathematics Education

Abstract: Topics in trigonometry have not been well-studied, especially with college-level students. Thus, despite providing a venue for important concepts such as notions of proof and algebraic skill, the process of verifying trigonometric identities, or VTI, has not been thoroughly explored. This study attempts to remedy this gap in the literature by exploring college students' concept images of VTI, providing a description of certain aspects of students' VTI concept images while focusing on students' problem solving and proof making behaviors during VTI.

Students viewed VTI in terms of proof construction and problem solving actions. Overall, students perceived the flow of VTI as a process of simplification involving simplifying acts. Viewing the flow in this way helped students monitor their actions; they tended to choose the complicated expression in order to simplify it or choose manipulative actions that would result in a perceived simpler expression. As for simplifying actions, students used a particular technique, dodging, to marginalize complicated function arguments, allowing them to choose appropriate actions once the cognitive complexity was removed.

As proof constructions, students generally believed VTI provided a verification of the truth or an explanation for the truth of the identity. The role VTI served depended on how students approached the identity prior to VTI, believing the identity to already be true or believing it could potentially be false. In turn, the knowledge students built as a result of the VTI process varied. Many students signaled the conclusion of VTI with a particular construction, the reflexive step. This construction acted as a sign, conveying information about the identity to the prover and readers of the VTI construction.

Students also indicated a preference for the visual nature of the VTI construction; they preferred formats that highlighted the manipulative steps of the construction. Overall, students believed a format that arranged the manipulated expressions in a vertical manner, forming columns, to clearly display the steps. When given the choice, students rated constructions of this format higher than constructions in which the steps were arranged in a horizontal format. Additionally, most students wrote their constructions in the columned format.

TABLE OF CONTENTS

Chapter Page

Chapter Page

LIST OF TABLES

LIST OF FIGURES

Figure Page

Figure Page

CHAPTER I

INTRODUCTION

Trigonometry

The roots of trigonometry as a mathematical subject lie in the practical. While trigonometric concepts originated in the study of celestial bodies, these ideas were later used for Earth-bound pursuits such as geography, navigation, and architecture. In the centuries that followed, mathematics became more systematized and symbolic, and the notion of function to describe trigonometry became more prevalent and important. The value of periodic functions, such as the trigonometric functions, in modeling phenomena also became apparent for the natural sciences (e.g., thermodynamics or electromagnetism). The work of Fourier further established the importance of trigonometry. Trigonometry was no longer just applications involving circles and triangles; it transformed into a discipline containing ideas fundamental for mathematics and science.

Conceptually, trigonometry offers a fertile ground for research, linking objects fundamental for geometry, e.g., the Pythagorean theorem, to objects fundamental for algebra, e.g., functions; in certain instances, students must coordinate ideas of triangles, ratios, and symbolic manipulation (Blackett & Tall, 1991). Yet, trigonometry has been under-studied. Some studies explored the medium through which students interacted with trigonometry while learning (e.g., Blackett & Tall, 1991; Choi-Koh, 2003). Other research described the understanding of trigonometric concepts and objects developed by students (e.g., Brown, 2005; Moore, 2009; Weber, 2005a).

Despite the potential for research, few studies have explored student conceptions and perceptions of trigonometric identities and verification of trigonometric identities. Delice (2002) investigated students' patterns of simplification of trigonometric expressions. However, the study did not explore why students chose certain identities to substitute or manipulations to perform; additionally, Delice assumed that students constructed no new knowledge during the simplification task, relying solely on their prior knowledge. Dugdale (1990) found that teaching treatments affected students' understandings of trigonometric identities; yet, these understandings were limited to how they related to graphs.

Having many important historical computational uses and contemporary uses in higher mathematics, trigonometric identities are equations involving trigonometric functions which are valid for all independent variable values in the common domain of the functions in the expressions. Even though computers and calculators have rendered some of the historical uses obsolete, verifying trigonometric identities (VTI) is important in contemporary curriculum (Common Core State Standards Initiative, 2010). While VTI is proof construction, VTI is also solving a problem since proof may be thought of as enhanced problem solving (Mamona-Downs & Downs, 2005). Thus, VTI involves engagement with problem solving, proof making, functions, and algebra, topics fundamental for mathematical learning (NCTM, 1989; NCTM, 2000). However, if VTI tasks do not facilitate sophisticated understandings, allowing students to *correctly* verify identities while maintaining incomplete or facile conceptual understandings, questions may arise about the value of VTI in the trigonometry curriculum. This study aims to provide insight on the role that VTI plays in the mathematical development of students.

Purpose Statement

To date, little research exists concerning student perceptions and behaviors involved in verifying trigonometric identities. The purpose of this study relates to describing aspects of a student's VTI concept image, focusing on students' problem solving and proof making behaviors during VTI. The research questions guiding the study are:

1. What are students' conceptions of VTI and to what extent do existing frameworks describe the conceptions?
2. What factors contribute to a student-perceived successful engagement in VTI?
    a. What does VTI accomplish from the student's perspective?
    b. What structural form does a VTI construction take?
    c. In what ways do the structure of the VTI construction and the VTI accomplishments interact?
3. In what ways do students' conceptions of VTI influence problem solving decisions?

The result of the study will be a description of how students verify identities and why students may verify the identities in the ways that they do.

Problem Context

While few studies have observed student understanding of trigonometric identities and verifying trigonometric identities, research has occurred on topics related to identities and verifying trigonometric identities. First, VTI is a task in problem solving. As described by Weber (2005b),

> a mathematical problem is a task in which it is not clear to the individual which mathematical actions should be applied, either because the situation does not immediately bring to mind the appropriate mathematical action(s) required to complete the task or because there are several plausible mathematical actions that the individual believes could be useful. (p. 352)

In VTI, the actions the students employ are the substitutions of identities; the identities may be previously known trigonometric identities or manipulated identities, i.e., identities formed through the algebraic manipulation of known identities. In most cases, demonstrating the equality of the two trigonometric expressions is not initially evident because it is not clear which identities to substitute. Hence, the verification task is a problem.

In order to verify the identities, students must decide which identities to substitute; this decision process is referred to as *strategic control* or *executive control* (Carlson & Bloom, 2005; Mamona-Downs & Downs, 2005). Mamona-Downs and Downs (2005) considered *cues* to be constructed knowledge configurations which connect knowledge to the problem solving environment. These cues act as mental triggers to access particular bits of knowledge. Hence, cues motivate decisions made through executive control. The term *resources* refers to "the conceptual understanding, knowledge, facts, and procedures used during problem solving" (Carlson & Bloom, 2005, p. 50). How resources serve as cues for problem solving decisions warrants further study (Mamona-Downs & Downs, 2005).

VTI is also a task in proof construction. That is, students show using a logical argument why one expression equals another expression; they verify that the proposed equality, or *theorem identity*, is true (CadwalladerOlsker, 2011). The theorems used by the students are the identities, which are statements of equality. After each identity substitution, the student must decide which theorem (identity) to use; usually, the choice is not apparent. Hence, the proof construction is also a task of enhanced problem solving (Weber, 2005b; Mamona-Downs & Downs, 2005).

Viewing VTI as a proof construction suggests exploration of student notions of trigonometric identities and equality. As noted by Weber (2001), studies have shown that inadequate understanding of the mathematical content of a theorem affects correct use of the theorem. In VTI, the theorems are the trigonometric identities, or the equalities involving trigonometric functions that are true for all appropriate independent variable values. Hence, student understanding of the theorems includes understanding of equivalence and the equal sign. Studies have found that students of all ages, even within the college ranks, may hold a weak notion of the equal sign, viewing it operationally rather than relationally (Kieran, 1981; Weinberg, 2010). Furthermore, weak notions of the equal sign and equivalence may affect performance in algebra (Knuth, Alibali, McNeil, Weinberg, & Stephens, 2005; Knuth, Stephens, McNeil, & Alibali, 2006). These weaknesses may in turn affect a student's ability to use the identities during identity verification.

Finally, Tall (2002) suggested that the process of proving a statement encapsulated the theorem as a concept. Once encapsulated, the mathematical object may be used to build new knowledge and prove more theorems. This encapsulation process might occur in VTI as students verify the equality statements, creating new identities. These new identities could then be used to verify further identities.

As the proof of trigonometric identities involves cycles of simplifying expressions and substituting in equivalent expressions while working with trigonometric functions, students' knowledge of algebraic manipulation and trigonometric functions is important. In general, students struggle with algebra (Tall & Thomas, 1991). They experience difficulties with notation (Booth, 1988; MacGregor & Stacey, 1997) and simplifying expressions (Booth, 1988; Tirosh, Even, & Robinson, 1998). They also have trouble viewing algebraic expressions as mathematical objects (Sfard & Linchevski, 1994). Moreover, students have difficulties with the concept of function (Breidenbach, Dubinsky, Hawk, & Nichols, 1992) and especially trigonometric functions (Blackett & Tall, 1991; Weber, 2005a). Additionally, tacit connections have been drawn between skills in algebraic manipulation and simplification of trigonometric expressions (Delice & Roper, 2006).

Definitions

In this section, some terms encountered in the study will be explained. However, not all terms will be defined; terms which develop from student understanding and data analysis will be described in the appropriate sections that follow.

Identity

The word *identity* means "the quality or condition of being the same in substance, composition, nature, properties, or in particular qualities under consideration." In the mathematical world, the sameness striven for is not of an outward appearance of mathematical symbols, but of the sameness in the underlying structure that the mathematical symbols describe. Hence, in mathematics, an *identity* is a tautologically true relation. For the purposes of this study, an identity is a mathematical sentence asserting the equivalence of two expressions. Thus, *theorem identity* refers to the given identity that students must verify as part of the VTI task.

Students usually first experience identities in an algebraic sense; an identity manifests as an equation containing variables which is always true for a suitably defined domain. For example, the equation $(x+1)^2 = x^2 + 2x + 1$ is true for any value of x since the right side of the equation is merely the left hand side after being expanded, or multiplied. Likewise, the equation

$$\frac{x+1}{x^2} = \frac{1}{x} + \frac{1}{x^2}$$

represents an identity as long as x is not zero, as neither side of the equation is defined at this value. Identities of these types are usually referred to as *algebraic identities* (Rickey & Cole, 1942).

A *trigonometric identity* (TI) is an identity involving trigonometric functions. During a standard trigonometry course, students encounter several basic TIs and basic skills with which the students are expected to become familiar. Grouping these so-called *fundamental identities and skills* in a hierarchy of a pedagogical nature results in the following TI types.

Notation Identity

A notation identity (NI) is a TI in which equality comes from agreed upon symbolic conventions. Students' initial exposure to the symbolic conventions occurs as part of the introduction of the trigonometric functions or ratios. Hence, students must learn and accept the new symbol usage. Examples of a NI are the de-emphasis of the formal function notation by dropping parentheses, e.g., $\sin x = \sin(x)$, and the exponential convention, e.g., $\sin^2 x = (\sin x)^2$.

Definition Identity

A definition identity (DI) is a TI in which equality is a consequence of the nature of the definitions of the trigonometric functions (e.g., Goldstein, 1994). The sine and cosine function are typically defined as the coordinates of a point on the unit circle. The remaining trigonometric functions may then be defined in terms of these initial two function, giving rise to the DIs. These are:

$$\tan x = \frac{\sin x}{\cos x},$$

$$\cot x = \frac{1}{\tan x},$$

$$\csc x = \frac{1}{\sin x},$$

and

$$\sec x = \frac{1}{\cos x}.$$

Manipulation Identity

A manipulation identity (MI) is a TI in which equality is a result of applications of arithmetic, field properties, algebra, etc. to an expression. For example,

$$\frac{\sin^2 x}{\cos x} + \cos x = \frac{\sin^2 x}{\cos x} + \cos x \cdot \frac{\cos x}{\cos x}$$

would be considered an MI. MIs are essential for the construction of new TIs as they result from the application of a property or an operation. In some sense, an MI could be thought of as an arithmetic identity (Kieran, 1981), or it could be conceived as an algebraic identity in which the objects of manipulation are trigonometric functions rather than variables.

Pythagorean Identity

The Pythagorean identity (PI) refers to the single identity which results from the combination of the Pythagorean Theorem and the sine and cosine functions:

$$\sin^2 x + \cos^2 x = 1.$$

Many texts list three identities when referring to PIs. The other two identities given relate the secant and tangent functions and the cosecant and cotangent functions. However, these two identities are more appropriately examples of another class of TIs, the *encapsulation identity*.

Encapsulation Identity

An encapsulation identity (EI) is a TI constructed from a combination of TI types in which the student accepts the equivalence of the two expressions without having to employ or re-employ VTI; the student does not need to be convinced. Either the student mentally "sees" the equality and does not need explicit convincing or the student does not need to be convinced again due to the equality being previously established.

Before providing examples of EIs, the term *encapsulate* will be explored. When a student encapsulates an idea, what is he doing? The prefix "en-" connotes *make* or *put in*. When paired with the word *capsule*, the word encapsulate is formed by appending the suffix "-ate," verbifying it. Thus, *encapsulate* literally means "to enclose or put in a capsule." To encapsulate also generally alludes to epitomizing or summarizing something. Now, *capsule*, meaning a little case or receptacle, derives from the Latin word *capsula*, a little box, case, chest, or repository. Capsula is the diminutive of *capsa*. Capsa may come from *capere*, which means "to take, grasp, lay hold, catch, undertake, be large enough for, or comprehend." Hence, capsule may be etymologically related to the words capable and captive. Thus, a student who encapsulates an idea grabs hold of the concept, touches it, examines it, manipulates it, and makes sense of it. He knows how it works. He can take it apart and put it back together again. He can then store it away and lay hold of it again when he needs it. Hence, a student who encapsulates an identity recognizes the equality of two expressions and is able to fluidly make use of this recognition.

One example of an EI is the following equation,

$$\tan^2 x = \frac{\sin^2 x}{\cos^2 x}.$$

This equation properly follows from a combination of notational, definition, and manipulation identities.

$$\tan^2 x = (\tan x)^2 \text{ (NI)}$$

$$= \left(\frac{\sin x}{\cos x}\right)^2 \text{ (DI)}$$

$$= \frac{(\sin x)^2}{(\cos x)^2} \text{ (MI)}$$

$$= \frac{\sin^2 x}{\cos^2 x} \text{ (NI).}$$

When a student accepts the validity of an identity such as

$$\tan^2 x = \frac{\sin^2 x}{\cos^2 x}$$

without needing to perform a verification of equality and is able to use the equality, he or she will be said to have *encapsulated* the identity.

In terms of the encapsulation notion, the DI

$$\tan x = \frac{\sin x}{\cos x}$$

is more accurately an *encapsulated* DI. In a right triangle with one of the non-right angles called x, the leg opposite x called b, the leg adjacent x called a, and the hypotenuse called h, the following definitions for the trigonometric ratios exist:

$$\sin x = \frac{b}{h},$$

$$\cos x = \frac{a}{h},$$

and

$$\tan x = \frac{b}{a}.$$

Noting that $b = h \sin x$ and $a = h \cos x$ results in

$$\tan x = \frac{h \sin x}{h \cos x}$$

$$= \frac{\sin x}{\cos x}.$$

The last two steps combined the definitions of sine and cosine with a manipulation identity, the "cancelling" of the *h* factor.

Other examples of EIs are the *encapsulated* Pythagorean identities (EPIs),

$$1 + \tan^2 x = \sec^2 x\,,$$

and

$$1 + \cot^2 x = \csc^2 x\,.$$

Similar to the PI, these identities develop from the right triangle and the Pythagorean theorem. However, other identities are needed to justify their equality. For example,

$$1 + \tan^2 x = 1 + \frac{\sin^2 x}{\cos^2 x}$$

$$= \frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$= \frac{1}{(\cos x)^2}$$

$$= \left(\frac{1}{\cos x}\right)^2$$

$$= (\sec x)^2$$

$$= \sec^2 x\,.$$

The other EPI may be verified in the following manner.

$$1 + \cot^2 x = 1 + (\cot x)^2$$

$$= 1 + \left(\frac{1}{\tan x}\right)^2$$

$$= 1 + \left(\frac{1}{\frac{\sin x}{\cos x}}\right)^2$$

$$= 1 + \left(\frac{\cos x}{\sin x}\right)^2$$

$$= \frac{\sin^2 x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x}$$

$$= \frac{\sin^2 x + \cos^2 x}{\sin^2 x}$$

$$= \frac{1}{\sin^2 x}$$

$$= \left(\frac{1}{\sin x}\right)^2$$

$$= (\csc x)^2$$

$$= \csc^2 x\,.$$

CHAPTER II

REVIEW OF LITERATURE

Overview

This study explores college students' perceptions of VTI. In order to provide a foundation for the explorations, this next section reviews literature pertinent to VTI and identities. However, before summarizing the literature, a brief history of trigonometry, highlighting some historical uses of trigonometric identities, will provide a context for the research and study. A review of some of the more recent curriculum changes that have affected the place of VTI in the curriculum will follow the historical summary. An exploration of pertinent literature follows the description of curricular changes. The chapter concludes with descriptions of theoretical frameworks through which to interpret the observations made in the study.

Historical Perspectives

Early History

As humans lifted their eyes upward, away from the Earth, curiosity drove their desire to measure the heavens. From these celestial origins sprung the seeds of the modern day study of trigonometry. The ancients envisioned a universe with the Earth as the center surrounded by a series of concentric spheres. In their observations of heavenly bodies and in their attempts to make sense of the heaven's structure, ancient Greek astronomers connected chord lengths with angular measure. Here was the establishment of a relationship between measurements of length and angle, foreshadowing the trigonometric functions to come millennia later.

To aid with astronomical calculations, the ancients constructed tables of chord values. Examples of tables of chord values were those constructed by Hipparchus and Ptolemy, who may have derived his from Hipparchus (Maor, 1998). While various geometric methods were used to find certain values, further techniques were necessary to generate accurate chord lengths for finer divisions of angular measure. For example, in his table, Ptolemy appears to have used formulas equivalent to the modern day addition and subtraction identities for sine and cosine,

$$\sin(x - y) = \sin x \cos x - \sin y \cos x\,,$$

and

$$\cos(x + y) = \cos x \cos y - \sin x \sin y\,.$$

Additionally, Ptolemy may have used a procedure equivalent to the sine half-angle formula,

$$\sin\frac{x}{2} = \sqrt{\frac{1-\cos x}{2}},$$

a formula known in the times of Archimedes (ibid).

Subsequent cultures, notably Indian and then Islamic, added to the study of trigonometry and constructed their own trigonometric tables. For example, the Indian astronomer Kamalākara knew identities equivalent to

$$\sin 3x = 3\sin x - 4\sin^3 x\,,$$

and

$$\cos 5x = 5\sin^4 x \cos x - 10\sin^2 x \cos^3 x + \cos^5 x\,,$$

using these to generate extremely accurate trigonometric tables (Van Brummelen, 2009). Thus, in the early days of trigonometry, trigonometric identities were important for the construction of trigonometric tables. Constructing new identities allowed for more values and greater accuracy within the tables.

Prior to the advent of logarithms, trigonometric identities were useful in a process called *prosthaphaeresis*, an amalgam of the Greek words *prosthesis* (*πρόσθεσις*), meaning addition, and *aphaeresis* (*ἀφαίρεσις*), meaning take away or subtraction. Appearing in western mathematics in the late sixteenth century, the algorithm allowed for products to be computed as sums and differences (ibid). Specifically, interested individuals such as astronomers or navigators could use the product-to-sum identities

$$\sin x \sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)],$$

and

$$\cos x \cos y = \frac{1}{2}[\cos(x-y) + \cos(x+y)],$$

transforming multiplication problems into computationally less strenuous addition or subtraction problems.

For example to compute 3.4311×21.104, first, one would shift the decimal place of each number to put it in the ranges of the trigonometric functions; that is,

$$3.4311 \times 21.104 = 1000 \times 0.34311 \times 0.21104.$$

A table of sine values gives

$$0.34311 = \sin(12°11'),$$

and

$$0.21104 = \sin(20°4').$$

Then, if someone utilized the sine product-to-sum identity, the cosines of the sum and difference of the angles would be needed and are (upon consulting a table of cosine values)

$$\cos(12°11' - 20°4') = \cos(20°4' - 12°11') = \cos(7°53') = .99055,$$

and

$$\cos(12°11' + 20°4') = \cos(32°15') = 0.84573.$$

Then, subtracting these numbers and dividing by two, per the identity, gives 0.7241. Thus,

$$3.4311 \times 21.104 = 1000 \times 0.7241 = 72.41.$$

The true product is 72.4099344, the error due to rounding in the table values. The invention of the logarithm brought upon the demise of prosthaphaeresis as a method due to greater precision and flexibility when using logarithms.

In addition to their usefulness in generating trigonometric tables and easing computations such as products, trigonometric identities motivated techniques to solve algebraic equations. This application owes its development to François Viète. (In fact, the use of algebraic methods in trigonometry is largely due to Viète.) Viète used his techniques to solve a polynomial of degree forty-five, impressing the French king Henry IV (Maor, 1998).

To illustrate the technique, one might consider the equation

$$x^3 - 3x - 1 = 0.$$

First, using a transformation via the substitution $x = 2y$, the equation becomes

$$4y^3 - 3y = \frac{1}{2}.$$

This equation resembles the trigonometric identity

$$4\cos^3 t - 3\cos t = \cos 3t\,,$$

where $y = \cos t$ and $1/2 = \cos 3t$. By solving for the function argument, $3t = 60° + 360° \cdot k$, where $k = 0,\ \pm 1,\ \pm 2, \ldots$. Thus, $t = 20° + 120° \cdot k$, meaning $x = 2\cos(20° + 120° \cdot k)$. Now, due to the periodicity of cosine, the cases for $k = 0$, $k = 1$, and $k = 2$ must be considered. Therefore, using a table of trigonometric values results in the solutions

$$\begin{aligned} x_1 &= 2\cos(20°) \\ &= 2(0.92969) \\ &= 1.87938, \\ x_2 &= 2\cos(140°) \\ &= 2(-0.76604) \\ &= -1.53208, \end{aligned}$$

and

$$x_3 = 2\cos(260°)$$
$$= 2(-0.17365)$$
$$= -0.3473.$$

Maor (1998) considered three developments in trigonometry to have “fundamentally changed the subject” (p. 198). First, Ptolemy’s table of chord values allowed for the practical use of trigonometry in applications. Second, the formulas of de Moivre and Euler transformed trigonometry from merely a tool for applications to an analytic science. Finally, the theorems of Fourier showed the power of trigonometry to mathematics and science. Trigonometric identities were fundamental for these movements. Ptolemy, and others, made extensive use of identities, in geometric form, to construct the trigonometric tables. Once algebra became more symbolic, the identities could be expressed in forms familiar to modern audiences, and trigonometry became more analytic. The de Moivre and Euler formulas, important for analysis, are essentially identities connecting complex numbers to trigonometric functions. With his solution to the heat equation and his belief that the techniques could extend to other functions, Joseph Fourier revolutionized mathematics and physics; the importance of trigonometric identities to Fourier analysis are the extensive use of identities for solving the integrals that are involved in the proofs and solutions.

1900-1959

While trigonometric identities ultimately found important usage in advanced mathematics, identities were still important in problem solving situations and so were part of the trigonometry curriculum. Articles appeared in mathematical journals offering pedagogical and curricular advice. Many of these articles lamented the difficulties that students encountered in trigonometry. Some of the scenarios may ring familiar to a contemporary audience. For example, in describing a curriculum for plane trigonometry, Granville (1910) discussed the need for symbol fluency, stating

You have probably met the boy who could prove

$$\sin X - \sin Y = 2\cos\tfrac{1}{2}(X+Y)\sin\tfrac{1}{2}(X-Y)$$

all right, but who was completely floored by

$$2\cos\tfrac{\alpha+\beta}{2}\sin\tfrac{\alpha-\beta}{2} = \sin\alpha - \sin\beta\,,$$

or who could not recognize as identical the expressions

$$2\sin^2\tfrac{x}{2} = 1-\cos x\,,\ \tfrac{1}{2}-\tfrac{1}{2}\cos A = \left(\sin\tfrac{1}{2}A\right)^2.$$ (p. 29)

Students struggled when using different letters for variables or switching the notation that was being used, perhaps signaling a reliance on procedural understanding and not conceptual understanding.

In another lamentation, Walker (1912) claimed, “And if a pupil’s algebra is so vague that the factors of $a^2 - b^2$ are ever in doubt, I hardly see why Trigonometry should be added to his repertoire of knowledge – or ignorance” (p. 375). Students’ struggles with algebra, and teachers’ lounge talk concerning these struggles, have existed for quite a while. These struggles with algebra were important in the realm of VTI as much of the manipulation occured via algebraic steps.

When outlining a course in numerical trigonometry, Mercer (1913) discussed what he perceived was a typical reaction to trigonometry. After the introduction of six mysterious ratios induced shock, and “before you had recovered from the shock, you plunged headlong into a set of exercises called identities, which to the mathematically inclined were amusing puzzles, but to the ordinary person were unfathomable mysteries calculated to choke him off at the very outset” (Mercer, 1913, p. 194). To some students, verifying an identity proved to be an enigma due to their not understanding what they were supposed to be doing or due to poor conceptions of functions or poor algebra skills.

The book *Plane Trigonometry* (Rosenbach, Whitman, & Moskovitz, 1937) provides an example of trigonometric identities in the curriculum. In the text, identities did not appear until chapter 5, titled *Fundamental Identities*. The text referred to these identities not as the fundamental identities but rather as the fundamental relations. To prove an identity, the book stated that there was no general method of procedure, but that "a thorough knowledge of the fundamental relations in their various forms is, however, essential" (Rosenbach, Whitman, & Moskovitz, 1937, p. 102). In other words, proving a trigonometric identity was a task in problem solving; moreover, to ensure success, conceptual knowledge should be well-connected (Lester, Jr., 1994). As general guidance, the book did suggest to transform the functions into sine or cosine and to transform the more involved side using known identities or both sides to a common form. However, the book provided no indication of what "involved" actually meant. Example identities to prove included the following:

$$\frac{\csc^2\theta - 1}{\sec^2\theta - 1} = \cot^4\theta\,,$$

$$(\tan\theta + \cot\theta)^2 = \sec^2\theta + \csc^2\theta\,,$$

and

$$\frac{1}{1-\cos\alpha} + \frac{1}{1+\cos\alpha} = 2(1+\cot^2\alpha).$$

In an effort to increase student success in mathematics, the National Council of Teachers of Mathematics published a pamphlet *How to Study Mathematics*. The pamphlet contained a fairly detailed section on how to prove identities, offering many suggestions. In one passage, Swain (1955) wrote:

> In proving an identity you wish to transform a given expression *without changing its value* until it is identical with a second given expression. This means you must (1) know what transformations can be made without changing the value and (2) watch the second expression to be sure any transformations you make are really making your first expression more like the second expression. (p. 29)

In essence, students needed to possess special knowledge for the problem and metacognitive awareness while solving the problem.

To emphasize this point, the pamphlet mentioned the need to understand which operations were permissible for transforming the expressions, listing substitution of basic identities and algebraic transformations. The need for algebraic skill received some emphasis: "If your algebra is rusty, get an algebra book and review the type of thing you need in trigonometry" (Swain, 1955, p. 29). Another suggestion the pamphlet gave was the need to watch the second expression, making the angles the same, the trigonometric functions the same, and the algebraic forms the same.

A final suggestion the pamphlet provided was to work on "seeing the connection between the trigonometric expressions and the corresponding algebraic ideas" (Swain, 1955, p. 31). To facilitate seeing the connection, students could consider replacing trigonometric functions by letters, performing the algebraic manipulations, and then inserting the trigonometric function back into the appropriate places. However, students were encouraged to move beyond explicitly substituting in letters: "Try, however, to develop the ability to think of the function as though it were a single letter and not depend on the actual substitution of a letter" (ibid). In other words, developing a rich understanding of the function as an object was an objective of VTI.

New Math

The decades that followed were a tumultuous time for mathematics and, specifically, trigonometry. Of this era, Maor (1998) stated,

> Two of the casualties of the New Math were geometry and trigonometry. A subject of crucial importance in science and engineering, trigonometry fell victim to the call for change. Formal definitions and legalistic verbosity – all in the name of mathematical rigor – replaced a real understanding of the subject. (p. xii)

Maor lamented the loss of the beautiful problems in trigonometry such as proving

$$\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\tan\beta\tan\gamma\ ,$$

with the condition that $\alpha + \beta + \gamma = 180°$ of course, a condition not included by Maor.

What occurred in this era to spawn Maor's lamentation? In 1959, the Commission on Mathematics of the College Entrance Examination Board (CEEB) issued a report calling for trigonometry to be folded into general mathematics courses with content spread over grades nine, eleven, and twelve (Moise, 1962). In essence, a separate trigonometry course should no longer exist; moreover, new topics would minimize the trigonometry content. The report also suggested a decreased emphasis on proving identities, shifting "analytic emphasis from identities to functional properties" (CEEB, 1959, p. 28). Additionally, the report called for using machines for calculations when possible rather than doing the calculations by hand. This use of technology would de-emphasize solving trigonometric equations by hand, work that relied on using identities.

Example texts from this "New Math" era are the series produced by the School Mathematics Study Group (SMSG). Chapter ten of the eleventh grade text *Intermediate Mathematics* was titled *Introduction to Trigonometry* and contained an introduction to identities: "Equations such as $\sin 2\alpha = 2\sin\alpha\cos\alpha$ are known as identities. They yield true statements no matter what angle or real number is substituted for α" (SMSG, 1961, p. 612). As for suggestions about how to prove the identities, the book offered the following suggestion: "There are no standardized methods for proving identities or solving equations. To prove an identity or to solve an equation often requires ingenuity and perseverance, and many methods must be devised to handle all the problems that arise" (SMSG, 1961, p. 613). Apparent from the quote was that the text intertwined proving trigonometric identities and solving trigonometric equations.

Examples from the exercises include the following proofs of identities:

$$\frac{2}{\csc^2 x} = 1 - \frac{1}{\sec 2x}\ ,$$

$$\frac{\cos\theta}{1+\sin\theta} = \frac{1-\sin\theta}{\cos\theta},$$

$$\frac{\cos^4\theta - \sin^4\theta}{1-\tan^4\theta} = \cos^4\theta,$$

and

$$\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin\alpha\cos\beta.$$

Of course, the last identity is a sum-product identity. In fact, the exercises contained all of the sum-product identities for the students to verify. This verification in the exercises was necessary as the text did not dedicate any more sections for identities beyond the basic identities.

SMSG intended the twelfth grade text *Elementary Functions* as a one semester course. The textbook treated the trigonometric functions in terms of waves and circular functions. In fact, the text refered to the functions as circular functions. Chapter 5, titled *Circular Functions* introduced the utility of the identity. From section nine, the text stated, "In the analysis of general periodic motions the product of two circular functions often appears, and the expression of a product as the sum or difference of two circular functions is quite useful" (SMSG, 1965, p. 272). Continuing this line of thought and introducing the definition of the identity, the text further explained,

> It is often useful to have some expression involving circular functions in more than one form. That is, we sometimes wish to replace one expression by another expression to which it is equal for all values of the variable for which both expressions are defined. A statement of this kind of relationship between two expressions is called an identity. (SMSG, 1965, p. 273)

The notion of identity appeared earlier in the chapter, in section six. This section introduced the addition formulas and developed the identities without calling them identities. In the exercises from this section, the students developed double angle identities, half angle identities, and tangent identities, all without calling them identities.

Exercises requested the students to derive the sum-product identities. Additional examples of “showing” something was an identity included

$$\frac{1+\cos\theta}{\sin\theta}+\frac{\sin\theta}{1+\cos\theta}=\frac{2}{\sin\theta},$$

and

$$1+\sin\alpha=\left(\sin\tfrac{1}{2}\alpha+\cos\tfrac{1}{2}\alpha\right)^2.$$

The text offered meager suggestions for accomplishing the proving task, merely suggesting trying to “transform one side into the other or both sides into identical expressions” (SMSG, 1965, p. 273).

The New Math curricula were not without contemporary critics. Seventy-five leading mathematicians signed their names to an open letter, published in *The American Mathematical Monthly*, expressing their concern about the reform efforts, including effects on trigonometry.

> Elementary algebra, plane and solid geometry, trigonometry, analytic geometry and the calculus are still fundamental, as they were fifty or a hundred years ago; future users of mathematics must learn all these subjects whether they are preparing to become mathematicians, physical scientists, social scientists or engineers, and all these subjects can offer cultural values to the general students. The traditional high school curriculum comprises all these subjects, except calculus, to some extent; to drop any one of them would be disastrous. (Ahlfors et al., 1962, p. 191)

In 1977, the Mathematical Association of America and the National Council of Teachers of Mathematics issued a statement, urging a return to a more traditional curriculum (Fey, 1978). This statement contributed to the demise of New Math as a movement. However, trigonometry would not remain immune to reform calls; in fact, the next challenge came from the National Council of Teachers of Mathematics.

The NCTM Standards and Recent Activity

The advent of the graphing calculator technology in the 1980s ushered in a contentious era concerning how the mathematics curriculum should utilize technology; in trigonometry, many of the historical uses of identities were rendered moot by the features offered by the new technology. In 1989, the National Council of Teachers of Mathematics (NCTM) published its *Curriculum and Evaluation Standards for School Mathematics* (the *Standards*). The *Standards* provided a framework for what learning should be occurring in students at the K-12 level. The effects of the standards would of course trickle up to the collegiate level as the students educated under the *Standards* graduated high school and started college.

The *Standards* contained fourteen curriculum standards for grades 9-12; trigonometry was standard nine. Within this standard, NCTM recommended scaling back trigonometric identity verification in the school-level curriculum, suggesting that only college-bound students should verify identities and that only the most basic identities should be verified; paper-and-pencil solutions of trigonometric equations, a topic that relied on trigonometric identities, was also suggested as a topic for de-emphasis (NCTM, 1989). VTI problems noted as being overly complex were artificially complicated identities "such as $\csc^6 x - \cot^6 x = 1 + 3\csc^2 x \cot^2 x$" (NCTM, p. 165).

These recommendations appeared to overlook the opportunities VTI provided for engaging with problem solving, proof making, functions, and algebra, ignoring the accompanying NCTM standards (NCTM, 1989; NCTM, 2000). The de-emphasizing of VTI ignored NCTM's own statement that VTI "improves [students'] understanding of trigonometric properties and provides a setting for deductive proof" (NCTM, 1989, p. 165). In a critique of the trigonometry standard and the standards overall, Wu (n.d.) wondered how average, non-college bound students were to understand identities such as $\sec^2 x = 1 + \tan^2 x$ if not given the opportunity to prove them.

The most recent movement in mathematics education has been the adoption of a common core curriculum by the separate state-level education departments across the United States. The study of functions is a domain of the Common Core State Standards; trigonometric functions and identities are included within this domain (Common Core State Standards Initiative, 2010). The Common Core State Standards suggests proving identities such as the Pythagorean identity and the addition and subtraction identities for sine, cosine, and tangent and using these identities to solve problems. The Common Core State Standards also include mathematical practice standards; these are areas of expertise that students should strive for in their mathematics. VTI could support areas such as forming viable arguments, making use of mathematical structure, and expressing regularity in reasoning. So, despite their practical origins for calculations, trigonometric identities and VTI are still relevant for today's technologically-enhanced student.

Related Issues and Studies

Equality and Equivalence

VTI depends on students substituting identities to transform mathematical expressions. Some of the identities may be identities previously known to the student. Other identities may be the result of the student performing an algebraic manipulation of a known identity and then using the resulting new identity. In either case, fundamental concepts which underpin VTI and identities are the notions of equivalence and the meaning of the equal sign. Kieran (1981) reviewed research into student uses of the equal sign and students' notions of equivalence and came to the conclusion that the notion that the equal sign means "do something" or merely separates the problem from the answer extends from elementary to college students. Students in elementary school viewed the equal sign as an operational symbol indicating that an action needed to be performed. Generally, they could not accept the lack of closure offered by an unevaluated expression. They felt the need to append an equal sign and try to perform an operation, usually incorrectly. In terms of equivalence, students had problems viewing both sides of an equation as representing the same number; they believed that the equal sign indicated replacing the expression with a unique result. For example, the expression $4 + 5 = 3 + 6$ could trouble the student as they would insist that the right hand side should be 9 rather than the unevaluated expression.

As students matured, they became more accepting of a relational meaning of the equal sign in addition to the operational meaning; they were more accepting of the equivalence of expressions indicated by the equal sign. However, the tendency to believe in operations on the left hand side and answer on the right hand side of the equation still persisted. While students' notions of the equal sign expanded, their misconceptions grew more sophisticated. An example of equal signs indicating answers was the string of equations,

$$1063 + 217 = 1280 - 425 = 1063. \text{ (Kieran, 1981, p. 320)}$$

While these types of errors occurred as students transitioned into pre-algebra and algebra courses, students did begin to expand their notions of equivalence and the equal sign in response to algebra topics.

Although college students generally held a relational view of the equal sign, they also developed a fairly sophisticated use of the equal sign as an operational symbol. In these instances, solving algebraic equations and calculus problems, the equal sign became a short-cut symbol rather than a true symbol of equivalence as the sign indicated a progression of steps. For example, Clement (as cited in Kieran, 1981, p. 324) noted the following work of a calculus student involved in finding a derivative:

$$\begin{aligned} f(x) &= \sqrt{x^2+1} \\ &= (x^2+1)^{1/2} \\ &= \tfrac{1}{2}(x^2+1)^{-1/2} D_x(x^2+1) \\ &= \tfrac{1}{2}(x^2+1)^{-1/2}(2x) \\ &= x(x^2+1)^{-1/2} \\ &= \frac{x}{\sqrt{x^2+1}}\ . \end{aligned}$$

Here, the equal sign links each step, indicating the flow of the problem rather than establishing any equivalence among the expressions.

Similar to Clement, Weinberg (2010) noted calculus students abusing the accepted meaning of the equal sign. Students generally accepted as correct run-on expressions such as "$f(x) = 2x^3 = f'(x) = 6x^2$" (Weinberg, 2010, p. 3). This more functional use of mathematical symbols, where in the previous example the equal sign indicated to the reader to move to the next step of the work, was also observed by Berger (2004) in a study of college students making sense of new mathematical symbols. She noticed that before fully comprehending the meaning of a new symbol, students tended to use the symbol as both an object with which to communicate mathematical ideas and an object on which to focus and organize mathematical ideas. The meaning of the symbol evolved from something having personal meaning for the student to something whose usage matched the usual usage of the larger mathematics culture. Through a semiotic analysis, Weinberg (2010) concluded that the meaning students ascribed to the equal sign focused more on it enabling them to solve the problem rather than it aligning with the accepted cultural meaning. However, use of personal meanings for symbols rather than cultural meanings at times lead to unproductive work.

These studies underscore the need for teachers to understand students' meanings of the equal symbol and with proper instruction, guide the students to culturally accepted and productive notions of the equal symbol and equality (Weinberg, 2010). Alibali, Knuth, Hattikudur, McNeil, and Stephens (2007) concluded that the earlier students acquired a relational understanding of the equal sign, the more successful they were at solving equivalent equation problems; a more sophisticated understanding equated to better performance. In a study of young students, Sherman and Bisanz (2009) showed that students exposed to equivalence in the context of non-symbolic problems were more successful with concepts and strategies in the context of symbolic equivalence problems. Sáenz-Ludlow and Walgamuth (1998) described a teaching treatment for third graders which transitioned students from a command or operational notion of the equal symbol to a relational symbol designating the equivalence of expressions.

Despite the importance of directly teaching for equality and equivalence, lessons in curriculum rarely focus on meaning, forcing students to develop their own meaning, usually an operational understanding, based upon their experience (McNeil & Alibali, 2005). Beyond their initial introductions in elementary school, little time is spent explicitly on the topics in later grades (Knuth, Alibali, McNeil, Weinberg, & Stephens, 2005). In a study involving elementary, seventh grade, and undergraduate students, McNeil and Alibali (2005) asked participants to define the equal sign and then rate given interpretations of the equal sign. McNeil and Alibali concluded that mathematical experience shaped meaning as elementary students interpreted the equal sign to indicate an answer, the seventh grade students utilized an operational meaning in the context of arithmetic and a relational meaning in an equivalence setting, and the college students interpreted the equal sign as relational. The study also demonstrated that the context of the equal sign may determine the meaning as evidenced by the variation in the seventh grade student responses.

Steinberg, Sleeman, and Ktorza (1991) studied eighth and ninth grade algebra students' knowledge of equivalence. They came to the conclusion that algebra students did not hold a good understanding of the concepts of equivalent equations. Students were asked to judge the equivalence of pairs of equations and provide a justification for their responses. The justifications fell into three broad categories. One strategy was based upon actually computing the solution for each equation in the pair and then comparing the answers or solving one equation and substituting it into the other equation. A second strategy depended on the students transforming the equations in the pair, but not solving them, attempting to make one look similar to the other. The third category contained incorrect solutions.

Of the first two strategies, students generally relied on one approach through all of their responses. Students who used the transformational approach had a significantly higher success rate when compared to students using the computational approach. However, while students acknowledged that transformations could be used to identify equivalence, almost a third of the students chose the computational approach, showing a reliance on procedural rather than conceptual approaches. Students who utilized the transformational approach more readily transferred their knowledge to unfamiliar situations such as equivalent equations involving numbers not appearing elsewhere in the equation. An example of this approach used on the survey was the pair of equations "$x + 2 = 5$ and $x + 2 - 99 = 5 - 99$" (Steinberg, Sleeman, & Ktorza, 1991, p. 113).

In a study concerning notions of equivalence for sixth, seventh, and eighth grade students, Knuth, Alibali, McNeil, Weinberg, and Stephens (2005) concluded that students' notions of equivalence affected their success in problem solving, the strategies they used in the problem solving process, and the justifications they provided for their solution process. As part of a prompt, students responded to the equation $3 + 4 = 7$, with an arrow pointing to the equal sign, answering the following questions:

a) The arrow above points to a symbol. What is the name of the symbol?
b) What does the symbol mean?
c) Can the symbol mean anything else? If yes, please explain. (Knuth et al., 2005, p. 70)

The student notions of the equal sign were coded as operational, relational, other, or no response. A large number of students, especially at the sixth grade level, held an operational understanding of the equal sign. At the eighth grade level, only 46% of students held the more sophisticated relational understanding.

Students also completed the equivalence statements $2 \times \_ + 15 = 31$ and $2 \times \_ + 15 - 9 = 31 - 9$ and explained their reasoning. The majority of the reasoning responses to the equivalence question were categorized as *recognize equivalence*, *solve and* *compare*, or *answer after the equal sign*. Students holding a relational understanding were more likely to view equivalence correctly and more likely to use the recognize equivalence strategy. Overall, the results showed students holding more sophisticated views as grade level increased; yet, a large portion of the students still held weaker notions of the equal symbol, affecting their ability to correctly interpret equivalence.

Notions of the equal sign and equivalence are very important to understanding algebra. As Kieran stated, “One of the requirements for generating and adequately interpreting structural representations such as equations is a conception of the symmetric and transitive character of equality – sometimes referred to as the ‘left-right equivalence’ of the equal sign” (as cited in Knuth et al., 2005, p. 69). The statement suggests that students need a relational understanding in order to make and understand algebraic transformations. The connection between equal sign understanding and skill in algebra was established by Knuth, Stephens, McNeil, and Alibali (2006). They showed that even when controlled for student ability as measured by standardized test scores, equal sign understanding was associated with performance on equation-solving algebra items; moreover, students holding a relational view were more likely to correctly solve the items.

In the study involving sixth through eighth grade students, participants responded to the same prompt regarding the equal sign as in Knuth et al. (2005). These responses were classified as relational, operational, other, or no response. Most students did not exhibit a relational understanding of the equal sign. Also, there was only a slight increase noted in students holding a relational understanding when comparing grade levels. Additionally, the participants were asked to state the value of *m* that would make the following number sentences true, $4m + 10 = 70$ and $3m + 7 = 25$. The correctness and the strategy were categorized. Strategies that were identified were *answer only*, *no response*, *guess and test*, *unwind*, *algebra*, and *other*. Eighth grade students with a relational view of the equal sign were more likely to use an algebraic strategy (*algebra*) while solving the problems. The findings of the Knuth et al. (2006) study are important in that they support the idea that a relational understanding of the equal sign is necessary to successfully operate on equations; equality and equivalence are very important for success in algebra.

Trigonometry and Algebra

As little research exists into student understanding of trigonometric concepts, not many studies exist related to verifying trigonometric identities. Dugdale (1990) explored high school students' reactions to an experimental teaching treatment of trigonometric identities. The treatment was based upon using computer activities to help students relate identities to their graphical representations. In her study, she compared 14 students who were presented trigonometric identities in a traditional spirit with a few additional graphing activities to a class of 16 students who were presented trigonometric identities using the experimental treatment.

The traditional treatment introduced trigonometric identity verification as exercises in symbolic manipulation. The focus was on procedures to use to demonstrate the equivalence of the expressions, and the exercises were routine repetitions highlighting the procedures. After a standard introduction to identities, students spent time graphing fundamental identities. In the experimental treatment, identities were introduced graphically. Students graphed their predictions for expressions on acetate slides covering the computer screens to verify the equivalence and then graphed the expression using the computer. Then, students were asked to justify algebraically, without instruction as to what "algebraically" meant, their predictions. The homework initially consisted of making graphical predictions for given functions. After a few days of graphing activities, class time was spent on formalizing algebraic manipulations to demonstrate equality. Comparing the two treatments, the experimental treatment spent less class time on the standard techniques for verifying trigonometric identities and more time using computer graphing. Also, whereas the traditional treatment began with a presentation of fundamental identities and useful procedures for verifying, the experimental treatment forced students to make their own decisions about what identities and procedures were useful to verify identities.

All students took the same posttest. Items on the posttest related to verifying identities algebraically and matching the graphs of identity expressions. Statistically, the students in the traditional and experimental treatments performed equally well on verifying trigonometric identities. Students in the experimental treatment performed significantly better than students in the traditional treatment on items requiring knowledge of identities and graphs.

Several interesting points emerged through observation of students in the traditional and experimental treatments. Initially, students in the experimental treatment showed more variety in their verification approaches and their work was not as clean when compared to the work of students in the standard treatment. Many students in the experimental treatment realized how to algebraically verify the identities by substituting reciprocal function definitions and then using algebraic manipulations. Some students used non-routine methods. One student related functions to the right triangle definition and would rewrite expressions in terms of side lengths; doing so allowed the student to connect her prior learning to the current topic. The student eventually convinced herself that this method was equivalent to the method other students were using and standardized her approach for verifying identities.

When students verify identities, they create new identities, perhaps encapsulating them, through substitution of known identities or through algebraic manipulations. Simplifying trigonometric expressions shares these aspects with the main difference being that students do not have a predefined target expression when simplifying as they do when verifying. Delice (2002) investigated students while they simplified trigonometric expressions. He found that students followed a certain pattern as they simplified the given expression. The first stage was the "recognition" stage. They began by reading the problem. Next, they recognized a certain form within the expression to be simplified. Finally, they recalled the particular identity, a known identity or one created through algebraic manipulation, that they could use. Once they recognized what to do, they transitioned to the “doing” stage. The students rewrote the expression by performing the proper substitution. Then, they cycled back through the recognize, recall, and rewrite phases until they believed they had fully simplified the given expression. However, students could have difficulty with simplifying the expression since one of the phases related to recognizing the equivalence of expressions; as demonstrated in other studies, students, even college students, struggled with ideas of equivalence.

Additionally, if students have poor algebra skills, they may have great difficulty. Sources of algebraic errors lie in several places. One source is a confusion of the function notation involved. As students engage new systems of notation, they tend to rely “on intuition and guessing, on analogies with other symbol systems they know, or on a false foundation created by misleading teaching materials … Their misinterpretations lead to difficulties in making sense of algebra” (MacGregor & Stacey, 1997, p. 15). So, as students learn the symbols for the trigonometric functions and the other notation involved, they may fall back on their knowledge of the algebraic notational system. Dangers exist with this interpretation as they may treat the symbol $\sin x$ as four distinct algebraic variables s, i, n, and x all multiplied together. Not understanding the symbolic notation of a system can be detrimental to solving problems involving objects represented by the system (Sajka, 2003).

Students’ troubles with the notation may also lie with their weak understanding of functions as objects. This failing in students is important for VTI; while verifying trigonometric identities, students, knowingly or unknowingly, treat the trigonometric functions as objects. Rather than evaluating the functions in a process-like fashion, students operate on the functions using algebraic manipulations and substitute equivalent expressions for the function. Trigonometric functions are examples of procepts (Weber, 2005a). A procept is an object, represented by a symbol, which has a dual nature as a process and a concept (Gray & Tall, 1994). For example, the sine function has the symbol “$\sin x$.” This symbol represents the actual object or concept of the sine of a number. The symbol also represents a process of constructing a ratio of numbers or locating a point on a unit circle based upon the definition of the sine function. Weber (2005a) suggested that students who viewed the sine function in the dual nature had a better understanding of trigonometric concepts.

In his study of students' object conceptions of trigonometric functions and instructional treatments, Weber (2005a) explored students' proceptual understanding of the sine function. Thirty-one students in a traditional college trigonometry course received typical instructor-led lectures. Forty students in the experimental trigonometry course received an experimental curriculum treatment during the first six weeks of the course, covering the same material from the same text as the traditional course. Weber developed the curriculum based upon a trajectory to encourage the development of a mathematical procept for the trigonometric functions. The experimental treatment covered five concepts and procedures: compute sine and cosine functions by way of the unit circle model; compute the tangent function using a graph; compute sine, cosine, and tangent functions through the use of right triangles; compute sine, cosine, and tangent functions using reference angles and the unit circle; and graph the three standard trigonometric functions. Each topic lasted two to three class periods. Students initially worked in groups to complete in-class activities; a classroom discussion followed. Standard homework problems were assigned from the textbook.

Using posttests and task-based interviews, the study seemed to indicate that the method of instruction may have an effect on students developing a proceptual view of trigonometric functions and a strong overall view of trigonometric functions. Students in the traditional course generally performed poorly on the posttest. In conjunction with the interviews, the results seemed to indicate that students in the traditional class did not hold a deep, rich understanding of trigonometric functions; the students did not view trigonometric functions as procepts. In contrast, students in the experimental course generally performed well on the posttest. The interviews seemed to indicate that these students viewed the trigonometric functions as procepts.

These results align with the results of previous studies. Students who tend to do poorly in mathematics tend to merely view things procedurally, while those who succeed are able to manipulate concepts or objects (Tall & Razali, 1993). Thus, successful trigonometry students tended to have a full proceptual view of trigonometric functions.

However, students generally have difficulty developing a full, rich conceptualization of functions and especially trigonometric functions. Sajka (2003) explored through procept theory how an average high school student understood the concept of function and to what extent the student held a fully developed view of function as a mathematical object. The student was interviewed while engaged in non-standard tasks involving functions. Difficulties in the task stemmed from misinterpretation of symbols in the functional notation and a limited proceptual view of function. Sajka concluded that the student's conception of function was not a fully developed mathematical object and was insufficient for solving the non-standard tasks. However, the function procept did contain elementary procepts and was in the process of developing into a complete procept.

In an investigation of high school students' understandings of the sine and cosine function, Brown (2005) found that students enrolled in an algebra II/trigonometry course held fragmented and incomplete views of the sine and cosine functions. The deficient understandings were due to lapses in prerequisite knowledge such as understanding of rotation angle and location on a unit circle, knowledge of coordinates and distance from the axes, and understanding the functions as both ratios and numbers. Due to these deficiencies, students had difficulties in connecting sine and cosine functions to definitions based upon unit circles or right triangles. Overall, many students did not attain an integrated understanding of the function, being unable to appreciate the connections existing among the various ways to define the functions. In other words, these students did not hold a full, rich conceptualization of the trigonometric functions.

Problems with algebraic understanding have been shown to affect ability to simplify trigonometric expressions and, in turn, VTI. In a study of Turkish and English trigonometry students, Delice and Roper (2006), found that Turkish students tended to score higher on problems that required them to simplify an algebraic expression and a form-equivalent trigonometric expression when compared to their English counterparts. However, both groups of students performed poorer when asked to simplify trigonometric expressions. This drop in performance could be due to the increase in the level of abstraction going from algebraic variables to trigonometric functions. They needed to negotiate new systems of notation as well as viewing the function as an object.

The Roles of Proof

When a textbook question prompts a student to "verify the given identity," what intended action is being requested of the student? Moreover, in what ways do students interpret this question? The word *verify* has its roots in the Latin word *vērus*, meaning *truth*; as a verb, *verify* derives from the Medieval Latin *vērificāre*, which means *make true*. Thus, from its Latin origins, verify implies an act of showing the truth of, leading to a modern definition of "to show to be true by demonstration or evidence; to confirm the truth or authenticity of; to substantiate" ("Verify," 2013). The important point of the etymological exercise is the revelation of the explicit relationship between *verify* and *truth*. When a question requests that students verify the identity, the question assumes that the identity is true by the nature of the command *verify*; students now must demonstrate the identity is in fact truly an identity. Implicit in the direction to verify is the impossibility that the identity is in fact *not an identity*. The purported equality is not a conditional equality; it is true for all input values within the functions' shared domains. This implied truth would be the meaning intended by the words; however, as words and phrases are interpretable by the observer, students may receive a differing message.

Now, what about those textbooks that phrase the prompt as "prove the given identity?" Does the act of proving also carry an implied quality of truth? Again exploring the etymology, *prove* has its roots in the Latin word *probus*. An etymological cousin of *prove* is the English word *probity*. In short, *prove* entails a sense of integrity, virtuousness, or goodness. There is a notion of correctness embedded in *prove*. In fact, a common definition of *prove* is "to establish as true; to make certain; to demonstrate the truth of by evidence or argument" ("Prove," 2013). One might imagine an everyday situation in which two individuals are arguing about a claim. Person A believes that Willie Mays is the greatest baseball player of all time. The skeptical person B, a New York Yankees fan and a Babe Ruth aficionado, retorts, "Prove it!" The skeptic wants the believer to back up the claim; while the skeptic may view the claim as false, uttering the word *prove* is a challenge to demonstrate the truth of the statement, not the falseness. Hence, *prove* and *true* are also directly related.

Thus, a textbook that prompts a student to either "prove" or "verify" an identity is requesting that the student show that the identity is in fact an identity. However, when a student actually verifies an identity, is the demonstration of truth all that is accomplished by the student? In other words, what role does proving a mathematical statement serve for the prover? De Villiers (1999) summarized the roles that proving a mathematical claim serves into six categories: verification, explanation, discovery, systematization, intellectual challenge, and communication.

For most practicing mathematicians, proof serves the role of verification (CadwalladerOlsker, 2011; Hersh, 1993). As limned by de Villiers (1999), proof (the result of proving) as verification describes the notion that a theorem is not accepted as a theorem until it has been proved to be right or correct; hence, the theorem is verified. Moreover, use of an unverified theorem is logically inconsistent. The verification process involves removal of doubt concerning the veracity of the theorem, and how this removal occurs is dependent on the mathematical culture the prover operates within (Harel & Sowder, 1998). For example, what a student may believe to be a convincing argument may not meet the standard of rigor for a teacher.

A common complaint about some proofs is that they may verify a claim, satisfying the burden of proof established by the culture, but the proof itself is not instructive about how or why the claim is true. As an example, consider an inductive proof to verify the sum of the first N terms of a geometric sequence. While the proof may satisfy the criteria established for proof by induction and thus “verify” the claim, a person may have questions lingering about the origin or derivation of the equation itself. A famous example of a non-explanatory proof, mentioned by de Villiers, is Appel and Haken’s “proof” of the four-color theorem. Halmos, as quoted by Hersh (1993), explained, “I do not find it easy to say what we learned from all that. We are still far from having a good proof of the Four-Color Theorem” (p. 393). That Appel and Haken’s proof provided no learning opportunities to understand why the theorem was true lessened the proof in Halmos’ and other mathematicians’ esteem. For them, good proofs should provide insight about the theorem (de Villiers, 1999). Thus, proof as explanation serves an important role. Proof as explanation relates more to the steps used in the verification and focuses on the question of exactly *why* the claim is correct. In comparison, *verification* focuses on the end result of the proof while *explanation* emphasizes the intermediate steps illuminating the mathematical inner-workings.

Proof as discovery, or creation (Knuth, 2002), describes how proof may “discover” or “create” mathematics previously unknown to the prover or the mathematical community through the use of deductive reasoning. A perhaps naïve example of proof as discovery is the following anecdote. While assessing some student work in beginning algebra, a teacher was attempting to explain (prove) through writing detailed steps and description the following algebraic identity for a student and expose some of the student’s algebraic misconceptions:

$$\frac{\frac{x}{y}-\frac{y}{x}}{\frac{1}{y}+\frac{1}{x}}=x-y.$$

Out of curiosity, the teacher played with the operations and quickly arrived at another algebraic identity:

$$\frac{\frac{x}{y}-\frac{y}{x}}{\frac{1}{y}-\frac{1}{x}}=x+y.$$

This realization was quickly followed by replacing x and y with $\sin x$ and $\cos x$ after noting the shared algebraic structure of the fractions and the trigonometric functions to "discover" from the teachers perspective two new trigonometric identities involving the six basic trigonometric functions combined in a fairly simplistic way:

$$\frac{\tan x - \cot x}{\sec x + \csc x} = \sin x - \cos x \ ,$$

$$\frac{\tan x - \cot x}{\sec x - \csc x} = \sin x + \cos x \ .$$

This discovery was satisfying for the teacher, having previously "discovered" a similar identity through exploration:

$$\frac{\sec x + \csc x}{\tan x + \cot x} = \sin x + \cos x \ .$$

The teacher's next step was to equate the last two identities and cross multiply to create a new identity:

$$\sec^2 x - \csc^2 x = \tan^2 x - \cot^2 x \ .$$

This led to two more identities that closely resembled the original identities, but with squares "inserted":

$$\frac{\sec^2 x - \csc^2 x}{\tan^2 x - \cot^2 x} = \sin^2 x + \cos^2 x \ ,$$

$$\frac{\tan^2 x - \cot^2 x}{\sec^2 x - \csc^2 x} = \sin^2 x + \cos^2 x \, .$$

These led to the final observation:

$$\left(\frac{\sec^2 x - \csc^2 x}{\tan^2 x - \cot^2 x}\right)^n = \left(\frac{\tan^2 x - \cot^2 x}{\sec^2 x - \csc^2 x}\right)^n = 1.$$

At this point, the teacher was amused to derive such complicated disguises for 1. However, the main thrust of the above process was that while the teacher was originally "proving" something for a student, this "proof" helped the teacher create some identities unknown to him.

De Villiers (1999) described the systematization function of proof as collecting the various definitions, postulates, and axioms into a deductive system. Through proof, the list of necessary statements in a system is whittled down to arrive at a core that fully explains the ideas of the system. Proof can expose circular arguments and unknown assumptions. Additionally, proof can provide a global view of the hidden structure that gives rise to the properties of the system.

Due to its nature as a human endeavor, proof may be thought of as a communication of mathematical knowledge to others. As Harel and Sowder (1998) described, proof involves the processes of ascertaining and persuading. Ascertaining concerns the removal of the doubts the prover has about the truth of a mathematical statement while persuading involves convincing of others that the mathematical statement is true. To persuade an audience, the proof constructed by the prover will serve this persuading function, communicating the mathematical ideas to others reading the proof. Moreover, the proof disseminates the mathematical knowledge contained in the proof to the mathematical community so that others might build upon this newfound knowledge.

Finally, a proof can be an intellectual challenge. In another anecdote, an undergraduate student struggled to prove that the minimum number of moves required in the Tower of Hanoi challenge was $2^n - 1$, where n was the number of disks to move. Over the years, this failure nagged at this student. Finally, 13 years later, the student, now in graduate school, accepted the challenge again and ventured again to prove the equation. Through a more mature exploration of the problem, the student discovered how the equation worked and was then able to build an induction argument. The elation experience was similar to that felt upon solving a Rubik's cube or noting the "trick" of a New York Times crossword puzzle. In the end, The Tower of Hanoi proof satisfied an intellectual curiosity.

The de Villiers categories of roles of proof are not necessarily mutually exclusive (de Villiers, 1999) or exhaustive (CadwalladerOlsker, 2011). However, they do provide a general framework for analyzing conceptions of proof. Knuth (2002) explored in-service secondary teachers' conceptions of proof by analyzing them through a framework similar to de Villiers', omitting the intellectual challenge aspect. As motivation, he believed that an informed conception of proof "must include a consideration of proof in each of these roles" (Knuth, 2002, p. 381). All of the teachers in the study indicated the belief that a proof verifies a claim, three-fourths of the teachers believed proof communicated ideas, and half displayed an understanding that mathematics may be created and/or systematized through proof. However, no teachers explicitly stated that a proof could offer insight as to why a mathematical statement is true. These findings led Knuth to describe the teachers' conceptions as diverse, yet lacking in the view of proof as allowing for understanding, a role that should naturally occur in a classroom setting (CadwalladerOlsker, 2011). In a sense, the learning opportunities that proof could provide may be absent for the students.

Proof and Problem Solving

As previously stated, VTI is proof construction embedded within a problem solving task. In the context of VTI, the theorems to be proved are the given trigonometric identities; proving the identity involves the construction of one equivalent expression after another until a string of equalities links the initial expression to the target expression. Moreover, an identity is not an identity "until it has been verified to be such by the construction of a proof" (CadwalladerOlsker, 2011, p. 39). However, some students resist the necessity to prove the identity, believing the identity is already true since the problem is asking them to verify the truth. This resistance may be due to students' conception of proof, or proof scheme.

A student's proof scheme "consists of what constitutes ascertaining and persuading for [the student]" (Harel & Sowder, 1998, p. 244). Harel and Sowder (1998) classified three main proof schemes, external conviction, empirical, and analytic; external conviction schemes relate to students not relying on intrinsic properties of the problem, empirical schemes depend on the student using facts or sensory information, and analytic schemes utilize logical deduction. Many of these proof schemes may be applicable to students' conceptions of VTI. The external conviction proof scheme has three subcategories. In the ritual scheme, the visual appearance of the argument provides evidence for truth. Proof is due to a source external to the problem in the authoritarian scheme; for example, students may depend on the book or the teacher to give them procedures to memorize. Students using a symbolic scheme will detach symbols from their contextual meaning and then use some sort of symbolic reasoning to prove the proposition. In the inductive scheme, a subcategory of the empirical scheme, students test the theorem for special cases and from these several cases make a broad generalization and declare the theorem to be true.

Stylianou, Chae, and Blanton (2006) investigated the relationship between a student's proof scheme and his problem solving patterns. They noted that students within each of the three broad proof schemes followed similar problem solving patterns. This in turn impacted the students' abilities to successfully complete the proof. For example, students with an external conviction scheme, while in an exploration phase of problem solving, would generally ritually manipulate symbols. Students with analytic schemes generally spent more time in the analysis phase of problem solving. They would set goals, use symbols for definitions, explore the definition, and link new insights back to the original problem.

While proof scheme refers to a student's beliefs concerning proving a statement, how the student actually goes about constructing a proof is important to consider. A factor to consider in proof and problem solving is the knowledge a student brings to the situation (Weber, 2001). As previously defined, a resource refers to all the conceptual knowledge and skills used in solving a problem (Carlson & Bloom, 2005). A concept similar to resources is *schema*. In the realm of problem solving, the term schema has been used to mean "a cluster of knowledge that contains information about core concepts, the relations between these concepts and knowledge about how and when to use these concepts" (Chinnappan, 1998). In the setting of trigonometric and geometric problems, Chinnappan (1998) found that high-achieving students accessed more sophisticated and varied schema with greater frequency. The kernel of knowledge about how and when to use concepts, or in other words, which facts or theorems to apply at a given point in the problem, is referred to as strategic knowledge (Weber, 2001). Strategic knowledge guides the actions in problem solving. In some sense, strategic knowledge is related to the notion of cues, the knowledge configurations connecting knowledge to the problem solving environment; cues act as triggers to the relevant pieces of knowledge (Mamona-Downs & Downs, 2005). They serve a similar purpose, guiding the application of knowledge in a particular problem solving setting. Without sufficient strategic knowledge, students have difficulty in constructing proofs, even though the students demonstrate they hold the relevant knowledge (Weber, 2001).

Due to the nature of the objects involved and the goals, how VTI proofs are constructed by students may become somewhat procedural for students. The danger is that while proofs are generally considered opportunities to construct knowledge, how the student constructs the proof influences the knowledge the student can develop (Weber, 2005b). VTI proofs could be considered proceptual proofs in that symbols representing objects are manipulated in order to prove the identity statements (Tall, 2002). If VTI is procedural in nature for the student, absent of little conceptual thought, students may not develop deep, connected understanding of the objects involved.

Weber (2005b) outlined types of proof construction and the learning opportunities the successful proof provided. In a procedural proof, students use an external source rather than intrinsic properties of the objects to infer a procedure. An example would be a student looking for a proof of a similar concept to modify and emulate. While the student may gain practice in standard proof techniques, emulating procedures allows for students to avoid deductive reasoning and understanding underlying concepts. A second type of proof production is a syntactic proof. Students using this method apply established theorems and rules of logic to deduce the statement to be proved. This method provides students opportunities to better understand the theorems and how to apply the theorems. However, students do not necessarily need to use intuition, maintaining a facile understanding of the problem. Using a semantic proof method, students will use intuition to represent concepts embedded within the problem and guide them toward a proof construction. In doing so, students develop an intuitive understanding of the validity of the theorem and develop representations for mathematical concepts.

Word Clouds: A Research Tool

A new method of data analysis, deriving from Web 2.0 internet sites such as blogs and social media sites, is the *tag cloud*, also sometimes referred to as a *content cloud* or a *word cloud*. In their original setting, tag clouds were "visual presentations of a set of words, typically a set of 'tags' selected by some rationale, in which attributes of the text such as size, weight, or color are used to represent features of the associated terms" (Rivadeneira, Gruen, Muller & Millen, 2007, p. 995). Tags are human-generated keywords used to categorize information found on websites; they provide a summary of the content for users of the site, serving as a sort of table of contents. As an example, blogs use tag clouds to summarize the content of posts, with the words in the cloud being keywords generated by the blogger. Tags for popular or frequently visited content on a website may appear with a bolder color or a larger font size in the tag cloud. Additionally, within the tag cloud, the individual tags can be hyperlinked to the resources and content on areas of the site labeled with the particular tag (Rivadeneira et al., 2007; Bateman, Gutwin, & Nacenta, 2008).

The image and video hosting site Flickr.com implemented tag clouds early on. Users could tag an image with a common keyword. Then, this image could be accessed along with other images tagged with the same keyword. An example of a cloud (Figure 1) is the all-time most popular tags, accessed from www. flickr.com/photos/tags/. Site users can click on a word in the cloud; the keyword is hyperlinked to images that have been tagged with that keyword.

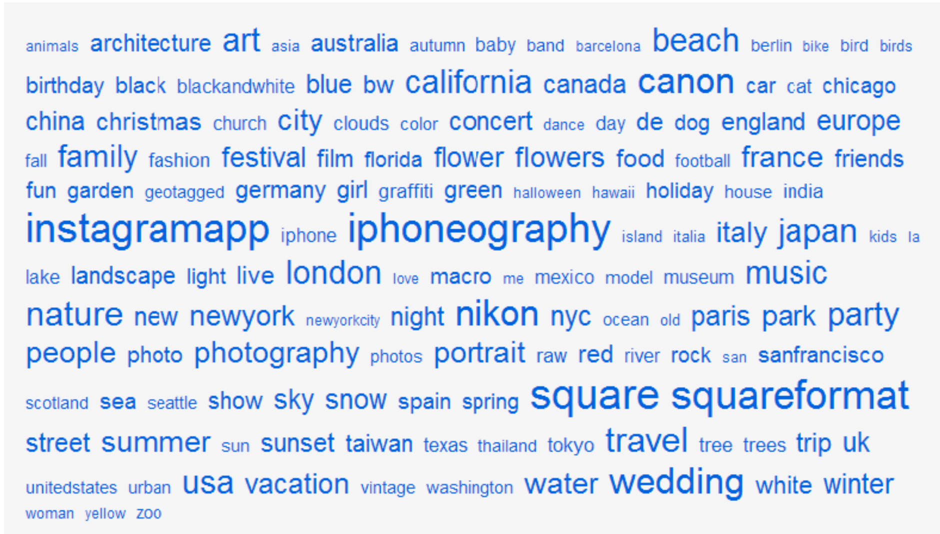


Figure 1. Tag cloud of all-time most popular tags on Flickr.com (screen capture accessed from www.flickr.com/photos/tags/ on 3Apr2013).

Another website that implemented tag clouds is the mega online retailer Amazon.com. Customers of the site could tag an item that they were considering purchasing or had purchased with a keyword. For example, if someone purchased the book *Pride and Prejudice and Zombies*, a zombie-Jane Austen mash-up by Seth Grahme-Smith, the tags “zombie” and “parody” might be applied. Then, the customer would be able to access other items tagged as “zombie” or “parody” by other customers. Likewise, a customer clicking on the tag “zombie” could then be directed to a listing now containing *Pride and Prejudice and Zombies*. The tag clouds used by Amazon.com were multidimensional in that the most frequently used keywords appeared in larger font size while the more recently used keywords appeared in bolder or darker font.

Rivadeneira et al. (2007) proposed a scheme for user tasks that tag clouds can support. Users can engage in *searching* for a specific concept on the site. Using the tag cloud allows for navigation to the desired content. A specific concept does not need to be known a priori; thus, tag clouds allow users to engage in *browsing* for general content. While *gisting*, or *impression forming*, users are able to form a general impression of the content of the site, observing prevalent

Figure 2. Tag cloud of most popular tags on Amazon.com (screen capture accessed from www.amazon.com/gp/tagging/cloud on 3Apr2013).

concepts while at the same time noting underlying themes. Finally, tag clouds can provide users the ability for *matching*, or *recognizing*, sets of data. In other words, the tag clouds may provide identifying features unique to the content of the site; users may be able to narrow down what the underlying concept of the data possibly is through exploration of the associated tag cloud.

While tag clouds originated with web-based applications, the categories of Rivadeneira et al. have broad appeal as the tasks associated with the categories are not bound necessarily to websites. The underlying content that generates the tag cloud may be any data set. Because of this quality, tag clouds are useful as a research tool. For example, researchers may generate clouds to *search* for the presence of a specific idea in the data. On the other hand, if limited knowledge exists of a phenomenon and the research is of the exploratory type, then researchers may *browse* the clouds and observe emerging themes. Clouds can also be used to form a quick impression or get the *gist* of the data. Finally, clouds may be used to distinguish and *recognize* the underlying phenomenon being observed.

For example, in comparing the clouds from Amazon.com (Figure 2) and Amazon.co.uk (Figure 3), similarities and differences may be observed. The tag *1080p* seems to indicate the customers in North America and the United Kingdom share a common interest in high-definition HDTVs. Similarly, overlaps appear to exist in genres such as “historical fiction” and “science

Figure 3. Tag cloud of most popular tags on Amazon.co.uk (screen capture accessed from www.amazon.co.uk/gp/tagging/cloud on 3Apr2013).

fiction." On the other hand, the tag "christianity" is much more prominent for North American customers in comparison to United Kingdom customers; additionally, while "christian fiction" has some prominence in North American, it is wholly absent in the United Kingdom cloud. Thus, researchers may be able to form initial impressions or generate confirmatory evidence for a study, such as a comparison of the religiosity of two cultures.

While tag clouds are formed from any type of data, this study uses the phrase *word cloud* to specifically refer to clouds formed from textual data sets. Thus, a word cloud is a visualization generated from a text, with the words used in the text creating a picture. The more frequently a word is used, the larger the word appears in the picture relative in size to the least frequently used word in the cloud. Thus, anything that can be analyzed through a content analysis can be visualized with a word cloud (Cidell, 2010). Used as a research technique, word clouds can quickly show what the phenomenon being studied is about, leading to researchers forming generally impressions (Gottron, 2009). Additionally, word clouds can quickly reveal differences among ideas in selections of written or spoken texts through a visual inspection and comparison of the pictures, illustrating any emerging themes (McNaught & Lam, 2010; Williams, Parkes & Davies, 2013).

Despite the inherent potential, word clouds have been used sparingly in research thus far. Cidell (2010) proposed word clouds as a method for exploratory analysis in geographic information systems. The method consisted of generating word clouds using public meeting transcripts and green building articles. For each data source, the word clouds were then visually mapped to represent the geographic location from which the transcripts or articles originated. Then, the clouds were analyzed for within and across themes; a location was explored for what matters most to that region based upon prominence of words in the cloud and then certain words were explored across regions to compare the prominence. Results from the meeting transcripts were triangulated with comments made at the meetings; in this way, the word clouds served a confirmatory role, supporting the content analysis of the meeting transcripts. Cidell concluded that using the word clouds illuminated the differences in regional attitudes and commented that the word clouds suggested many avenues for future research into the issues being researched. Overall, Cidell maintained that the method of word clouds "offers the potential of combining content analysis, visualisation and qualitative GIS" (p. 522).

In an action research study, Baralt, Pennestri, and Selvandin (2011) explored utilizing word clouds in teaching foreign languages. Through a series of stages, the researchers generated word clouds from students' revisions of compositions. After each draft, the instructor presented students with a word cloud formed from combined compositions of the class. A discussion occurred among the students and the instructor using the prominent words from the cloud, discussing issues such as a need to use more vocabulary words. Following this cycle, the researchers concluded that analyzing the word clouds led to a more varied vocabulary, a broader use of verb tenses, and greater grammatical accuracy. Moreover, students held a positive perception of the use of word clouds, and the instructor believed the word clouds placed the emphasis of the class discussions on the students, creating a student-centered environment.

Williams, Parkes, and Davis (2013) used word clouds to gain an initial overview of aspects from an induction program in management education. Students responded to prompts exploring their views about what they enjoyed the least and most and about what they found useful in the induction program. The raw data were used to generate an initial word cloud.

Finding the resultant cloud uninformative, the survey responses were classified as negative and positive. Then word clouds were created from these two categories. Creating these thematic clouds allowed the data to be presented in a meaningful way, within their original context. Finally, common phrases were categorized into general themes, and word clouds were generated from these themes. The researchers concluded that the word clouds were powerful tools for preliminary research, allowing the data to be quickly analyzed.

Finally, McNaught and Lam (2010) discussed two studies in which word clouds served different roles. The first study explored human factors affecting the comments of participants in focus-group meetings. Before an in-depth analysis of the transcripts of the meetings, the transcripts (with minimal corrections) were used to create clouds. Cursory observations of the clouds led the researchers to note important differences among the meetings, providing a preliminary understanding of what occurred in the meetings and directing attention to issues that needed follow-up studies. In the second study, the researchers used comments from five participant blogs about the usage of eBooks to form five separate word clouds. Before inputting the text, researchers made minor modifications to the text to maintain a sense of context; this step was taken as the themes within the blogs were important to the researchers. For example, spaces after the word "not" were removed to retain a negative connotation of an idea. The word clouds were then compared in order to confirm previous analysis of students' perceptions of eBooks.

In mathematics education research, utilization of word clouds as a complementary tool has been virtually nonexistent. While not strictly research, Nickell (2012) described displaying word clouds in her ninth-grade geometry class as a means of introducing mathematical vocabulary. Prior to the lesson, she would display a word cloud of vocabulary words, and students would then discuss familiar and unfamiliar terms. In addition to introducing students to the vocabulary, the word clouds served as overviews of the upcoming lessons. Nickell believed students began to rely on the word clouds as learning aides, boosting their confidence. At the end of the semester, students and Nickell generated their own word clouds for topics covered throughout the semester with the direction that the larger the word appears in the cloud, the more frequent and more important it was for the class.

Word clouds do have their inherent limitations. Bateman et al. (2009) found that larger font size or larger font weight for words exerted the strongest influence on individuals exploring word clouds. Thus, due to how clouds are generally constructed, larger or bolder words may distract the observer from underlying themes that are not mentioned as frequently, thus biasing the analysis. Additionally, Cidell (2010) pointed out that the sizes of words are relative to the frequencies of the other words in the text; the implication of this is that when comparing word clouds, similarly sized words do not imply that the word was mentioned the same number of times in the different passages. To circumvent this limitation and allow for better cloud comparison, several researcher have suggested techniques such as parallel tag clouds (Collins, Viegas, &Wattenberg, 2009), seam carving (Wu, Provan, Wei, Liu, & Ma, 2011), and word storms (Castella & Sutton, 2013).

Also related to the usage of frequencies to build the cloud image, McNaught and Lam (2010) suggested that word clouds should only be used to analyze the actual spoken (transcribed) or written word of participants. Using word clouds for field notes or researcher summaries would be less powerful as it would reveal information about the researcher and not necessarily the participants. Another drawback highlighted by McNaught and Lam was that word clouds remove the words out of their contexts. Thus, words being prominently displayed in a picture implies nothing about the importance of the word to the phenomena being investigated. Instead, prominence of a word merely suggests in-depth textual analysis of how the word was used in conjunction with ideas, serving as an initial thematic framework.

Therefore, due to the limitations of word clouds, McNaught and Lam (ibid) suggested that the role word clouds serve in research should be limited to a complimentary research tool. Specifically, they believed that word clouds are best used as:

- A tool for preliminary analysis, quickly highlighting main differences and possible points of interest, thus providing a direction for detailed analyses in following stages; and
- A validation tool to further confirm findings and interpretations of findings. The word clouds thus provide an additional support for other analytic tools. (p. 631)

Theoretical Framework

Concept Image

The purpose of this study related to describing aspects of a student's VTI concept image. The VTI concept image includes any actions, resources, or other concepts students associate with VTI. It encompasses imagery invoked as the student thinks about VTI or engages in VTI. Any formal or personal definitions students hold concerning VTI are a component of the concept image. In short, a student's VTI concept image is "the total cognitive structure" the student associates with VTI (Tall & Vinner, 1981).

Problem Solving

In this study, VTI was viewed as an enhanced problem in proof construction. The Multidimensional Problem-Solving (MPS) Framework (Carlson & Bloom, 2005) framed students' VTI behavior. The MPS Framework characterizes problem solving in terms of four behavioral phases: orientation, planning, executing, and checking. The framework also describes four attributes of problem solving (resources, heuristics, affect, and monitoring) and explains their roles during the phases. In their study of expert mathematical problem solvers from which the framework emerged, Carlson and Bloom (2005) noted some areas for further study; one such area was the role played by well-connected conceptual knowledge. Carlson and Bloom found that the richness of the conceptual knowledge affects a problem solver's ability to access useful concepts and procedures. This issue involving executive control appears to be related to strategic knowledge; even if a problem solver knows all of the necessary concepts, theorems, and procedures, the attempt at problem solving may still end in failure (Weber, 2001).

Roles of Proof

As VTI is viewed as proof construction, the VTI process serves a role. These roles were classified based upon the so-called de Villiers Roles of Proof framework. Through this framework, VTI could serve one of six roles: verification, explanation, discovery, systematization, intellectual challenge, and communication. While the main point of the study was not to classify students' perceptions of VTI using the framework, students' VTI concept images were described using the language of the categories. Additionally, students were not pigeonholed into categories. That is, a student who discussed VTI as serving a verifying role could also believe VTI served an explanatory role.

A Learning Framework

Also, in this study, identities were understood to be the theorems in the VTI proof constructions and that proving these theorems could lead to encapsulated identities from which further knowledge could be constructed, implying VTI provided learning opportunities for students (Weber, 2005b). The learning principles within the *How Students Learn* framework framed student learning and understanding (Donovan & Bransford, 2005). The first learning principle states that people learn through the engagement of their prior understanding and preconceptions. An example of this principle is the reliance of students engaged in new mathematical notation systems on prior notation systems with which they are familiar (MacGregor & Stacey, 1997). Thus, if students engaged in VTI were accessing faulty conceptions in order to understand VTI, the knowledge they build would in turn be faulty.

The second learning principle posits that understanding occurs when factual knowledge is acquired within conceptual frameworks. In other words, facts are connected in a way to facilitate easy retrieval for application. This principle implies that algebraic skill, trigonometric function knowledge, and other concepts related to VTI become connected into a schema while students engage in reflection while solving VTI problems. When this knowledge schema is required to solve a problem, a cue embedded with the schema will trigger the use of the schema in the problem setting (Mamona-Downs & Downs, 2005). That students learn within conceptual frameworks would become quite significant if students possessed a deficient framework.

The third learning principle claims that students use metacognition in order to self-monitor their learning. Through the verification of identities, students construct new knowledge through the encapsulation of identities. As they verify identities, students may engage in metacognition as they consider what the trigonometric expression presently looks like and what they want the expression to look like. In a sense, students would be assessing if they were where they needed to be in order to solve the problem. They then use cues to trigger their next VTI action (Mamona-Downs & Downs, 2005). Students not using any sort of metacognitive strategies may rely on shallow understanding and procedures, plodding through the problem mindlessly and thwarting any attempt to develop understanding.

.

CHAPTER III

METHODOLOGY

This study used mixed methods research. Mixed methods research is appropriate for research in which "the limitations of one method can be affected by the strength of the other method, and the combination of quantitative and qualitative data provide a more complete understanding of the research problem than either approach by itself" (Creswell & Plano Clark, 2011, p. 8). In other words, mixed methods research allowed for more than one method to be used in the study to enhance understanding of the phenomenon being researched (Creswell & Plano Clark, 2011).

Mixed Methods Definition

Creswell and Plano Clark (2011) described mixed methods as research in which the researcher

> collects and analyzes persuasively and rigorously both qualitative and quantitative data (based on research questions); mixes (or integrates or links) the two forms of data concurrently by combining them (or merging them), sequentially by having one build on the other, or embedding one within the other; gives priority to one or to both forms of data (in terms of what the research emphasizes); uses these procedures in a single study or in multiple phases of a program of study; frames these procedures within philosophical worldviews and theoretical lenses; and combines the procedures into specific research

designs that direct the plan for conducting the study. (p. 5)

The following sections explain how the design of the study shared some of these characteristics.

Study Design

The research design of the study was a variant of the mixed methods embedded design. A mixed methods case study allows for the collection of both quantitative and qualitative data within the framework of investigating a single phenomenon. "The embedded design is a mixed methods approach where the researcher combines the collection and analysis of both quantitative and qualitative data within a traditional quantitative research design or qualitative research design" (Creswell & Plano Clark, 2011, p. 90). For this study, the traditional research design was a case study.

A case study is "a detailed, intensive study of a particular contextual, and bounded, phenomena that is undertaken in real life situations" (Luck, Jackson, & Usher, 2006, p. 104). A case study is most useful when the study questions are of the "how" and "why" nature and the focus of the study is on a unit of analysis (Yin, 2003). The current study investigated aspects of students' problem solving behaviors during VTI, focusing on students' problem solving and proof making behaviors during VTI in order to explore students' VTI concept images. The unit of analysis for the study was a classroom of students engaged in the phenomenon of the verification of a trigonometric identity. Case studies are also concerned with a phenomenon which is bounded in some sense by space and time. The class unit concerning verification of an identity had a definite beginning and ending; hence, it was bounded temporally. Students engaged in VTI were also bounded spatially; the verification occurred within the student and was transmitted to a piece of paper.

A case study was an appropriate strategy to utilize for this study. As stated by Yin (2003), "A case study is an empirical inquiry that investigates a contemporary phenomenon within its real-life context, especially when the boundaries between phenomenon and context are not clearly evident" (p. 13). For the situation of VTI, isolating the verification from the student would be impossible. The conceptual understandings that motivated the problem solving actions could not be ignored. Also, while students were engaged in VTI, they were constructing new knowledge and were not static in their understandings (Weber, 2005b). Thus, VTI would be difficult to remove from its context as a student endeavor. Furthermore, Yin (2003) stated that

> The case study inquiry copes with the technically distinctive situation in which there will be many more variables of interest than data points, and as one result relies on multiple sources of evidence, with data needing to converge in a triangulating fashion, and as another result benefits from the prior development of theoretical propositions to guide data collection and analysis. (pp. 13-14)

Many conceptual understandings may influence a student engaged in VTI. Collecting data from various sources and focusing on components that relate to VTI allowed for a triangulation of the data analysis results without having to worry whether or not all of the conceptual understandings had been accounted for. In other words, the point was not to state and test all of the variables that influenced a students' VTI concept image. Instead, the result of the study was descriptive in nature.

Another benefit of using a case study approach was the ability to integrate both quantitative and qualitative methods. Luck, Jackson, and Usher (2006) suggested that a case study facilitates the use of mixed methods due to the inherent need to completely analyze a phenomenon. Using both quantitative and qualitative methods in the case study provided a deeper, richer description; the two data types complemented each other, the one type strengthening where the other type had weaknesses. Additionally, utilizing solely quantitative or qualitative methods did not completely address the research questions; rather, the mixed methods case study allowed for a practical approach to studying the phenomenon (Creswell & Plano Clark, 2011). As the study explored student behaviors, perceptions, and understandings related to VTI, the research questions contained elements best addressed by both quantitative and qualitative methods.

Within the case study, qualitative methods were prioritized. Merriam (2002) described qualitative research as an attempt to "understand and make sense of phenomena from the participant's perspective" (p. 6). Furthermore, qualitative researchers try to "understand the meaning people have constructed about their world and their experiences" (Merriam, 2002, p. 4). The emphasis of this study was to provide a description of students' VTI concept images. Qualitative methods were appropriate as the research questions focused on why students performed certain actions and how they understood the concepts involved in VTI. To do so, students' thoughts during VTI about the objects in VTI needed to become apparent. In other words, this study attempted to understand VTI from the point of view of the student.

Study Description

The data for the study was collected in four stages. Data were collected during the first week of class, throughout the class unit on trigonometric identities culminating in a semester exam that contained VTI items, with thorough participant interviews taking place after this semester exam, and from VTI items on the final exam. Additionally, contextual data about the classroom setting and class sessions were collected from the instructor. The different data types collected for the case study allowed for a rich description of the phenomenon being observed. Each data source added a layer to this description. Where one source allowed for a description of a certain feature, another type gave a varied but complimentary report of the feature. The following section will provide an overview of the data collection and analysis processes.

Setting

This case study involved students enrolled in a college trigonometry class at a large Midwestern university. The researcher was the instructor of the section. The instructor met with the students two times per week, on Tuesday and Thursday, for 75-minute class sessions. The classes were conducted in a somewhat traditional fashion, with class sessions generally consisting of instructor-led lecturing using material from the textbook *Analytic Trigonometry with Applications* (Barnet, Ziegler, Byleen, & Sobecki, 2008). The unit on trigonometric identities, taken from chapter 4 of the aforementioned textbook, comprised 6 class periods. While the class was lecture-driven, during the third and fourth class periods of the VTI unit, students worked in pairs verifying identities on a worksheet as the instructor roamed the room, answering student questions and posing questions to the students. As the students verified the identities, they were asked to converse with one another, explaining each step being taken and providing the motivation for implementing that particular step.

Student performance relating to the final grade in the class was assessed using weekly quizzes (a mixture of in-class and take-home), three semester exams, and one final exam. Even though homework was not a component of the final grade per se, daily homework was assigned, collected, assessed, and returned the next class period to monitor student performance and provide a venue for feedback to the students. However, class participation was measured by attendance and completion of the homework assignments; furthermore, a class participation score was available to students as bonus points for the final class grade. Therefore, attempting and completing homework assignments was induced through the offering of bonus points.

Data Collection Instruments

Biographical information. As a component of the course, students provided basic biographical information using the Math 1613: Biographical Information sheet (*Appendix B*). The biographical information included age, year of study, degree program, reason for pursuing the degree, previous high school mathematics courses taken, and previous college mathematics courses taken. The students provided the information to facilitate the instructor in better acquainting himself with his students. All students returned the completed sheet by the third class session. This information was maintained by the instructor and not returned to the students.

Algebra knowledge exam. As part of initial preparation for the course, all students in the section took an Algebra Knowledge Exam, or AKE, (*Appendix C*). Students were provided 25 minutes at the end of the fourth class period and were then asked to complete the remaining questions as a homework assignment. They were instructed to work the problems without any

Table 1

Class Calendar of Topics Covered and Assessments Collected

	Tuesday	Thursday
Week 7	Sections 4.1 & 4.2 Supplement: Algebraic Simplification	Section 4.2 Supplement: Verifying Identities
Week 8	Sections 4.2 & 4.3 Supplement: Assessing Verification	Section 4.3 Quiz 7
Week 9	Section 4.4 In-class Quiz: Verifying Identities	Section 4.4 Homework Quiz 8 Writing Prompt: Verifying Identities
Week 10	Semester Exam 2	
Week 17	Final Exam	

other resources other than their own logic and ingenuity; in other words, they were asked to refrain from talking to friends, asking tutors, or referring to textbooks and online resources. The purpose of the AKE within the course structure was to help students identify their readiness for taking a trigonometry course and their weaknesses that needed to remediation. Moreover, students had performed poorly on a similar diagnostic quiz given the second class period; thus, the AKE would provide further areas of remediation for the students. After scanning them to form an electronic file for further analysis and scoring the exams, the AKEs were returned to the students.

The AKE consisted of problems that assessed basic algebraic skills pertinent to success in a trigonometry course. Examples of the skills were arithmetic with fractions, factoring trinomial expressions, multiplying expressions, and simplifying expressions with exponents. The AKE also contained items that attempted to assess students' understanding of a function with a focus on a function as an object. Examples of these items were knowledge of the inverse function and factoring expressions involving functions.

Worksheet prompts. Following the timeline detailed in Table 1, students completed supplemental worksheet activities (*Appendices D-J*) designed to enhance student understanding of trigonometric identities and VTI. All students were required to complete the worksheets in order to receive credit for attendance and participation for that class period. The worksheets were either in-class activities or assigned as additional homework. The worksheets were composed of prompts that explored students' notions of equivalence, identities, VTI, and problem solving cues. The intent of the prompts in the context of the VTI unit was to induce reflection by the students on these topics. The completed worksheets of the participants were collected and scanned as electronic files, prior to assessment, for future analysis.

Homework. During the class unit on trigonometric identities, students worked problems from the textbook that related to verifying trigonometric identities and using identities to solve problems. Prior to grading, the homework assignments from each participant were scanned to form electronic files for future analysis. The graded assignments were then returned to students in order to provide critical feedback.

Semester exam. Students in the course took a semester exam during the tenth week of the semester. The exam contained items where students were requested to verify trigonometric identities (*Appendix K*) and use trigonometric identities to solve problems. The problems were representative of the concepts that the unit on trigonometric identities covered. Prior to grading, the participant responses to exam items that related to verifying or using trigonometric identities were scanned to form electronic files for future analysis.

Formative interviews. During the unit on trigonometric identities, students visited with the instructor during office hours. Some of these visits were drop-in visits during office hours, while others were arranged via email requests at the student's initiative. A few were email conversations or brief meetings after the conclusion of the class period. Meetings with participants that involved trigonometric identities or VTI notions were logged on a template (*Appendix M*) by the instructor using a naturalistic or ethnographic method. That is, the instructor attempted to reconstruct and write down portions of the conversation that emphasized student notions of trigonometric identities and VTI after the participant left. The focus in logging was to capture the important themes of the conversation with a verbatim transcription when appropriate and possible. These interviews were referred to as formative since they occurred while the student was still attempting to understand concepts; the intent of logging these interactions was to record student learning and understanding in a more naturalistic setting.

Teacher journal. Throughout the class unit on trigonometric identities, the instructor maintained a teaching journal using a template (*Appendix N*). The journal recorded aspects of the class during the unit. Through recall immediately after the conclusion of the class period, the instructor provided an overview of the topics covered on each day of class and the manner in which the content was presented. Additionally, student questions and responses were detailed. Furthermore, the instructor reflected on his teaching and the student reactions and gave his impressions of student learning and understanding.

Summative interviews. After the second semester exam was graded and returned to students, the researcher interviewed some of the participants. The interviews were referred to as summative interviews as they occurred after the conclusion of the unit and the exam over the topics. During these task-based interviews, student problem solving behaviors while engaged in VTI were observed. Also, students' notions related to trigonometric identities and VTI were explored. Participants verified trigonometric identities and were encouraged to think aloud as they solved the problem. A point of emphasis was for the participants to describe why they were performing a particular VTI action. Each interview lasted approximately one and a half hours. These interviews were audio-recorded, and the researcher created a transcription from the recording. Additionally, the participants generated written artifacts; they were encouraged to write down all their steps and not erase work they believed to be incorrect.

The analysis of the AKE and the VTI items and the informal analysis of the formative interviews and teacher journal influenced the development of the interview protocol (*Appendix O*). The task-based questions (*Appendix Q*) for the interview protocol were developed using results from previous investigations into VTI and were slightly modified to reflect results of the previously mentioned analyses of participant work.

Final exam. Students in the course took a comprehensive final exam during the sixteenth academic week of school. The exam contained questions related to topics from the entire course; specifically, the exam contained one VTI item (*Appendix L*), requiring students to verify a trigonometric identity. Prior to grading the final exam, the participant responses to the VTI item were scanned to create electronic files. Collecting data from the final exam helped to contextualize student learning in the course.

Participants

Study participants. In the second week of the class, during the beginning of the fourth class period, a member of the mathematics department faculty visited the class and invited the students to take part in the study. While the faculty member read the recruitment script (*Appendix T*) to the students and subsequently collected the signed or unsigned consent forms (*Appendix S*), the researcher waited in the hallway outside the classroom. This recruitment procedure was used to minimize the appearance of coercion. Of the 44 students enrolled in the section, 43 were present during the recruitment, and 33 of the 43 students agreed to participate in the study. (The student who was absent dropped the course the following day.)

After receiving the consent forms from the faculty member, the researcher created a master file for those students who signed the consent forms. The names of the students were entered in a random fashion. That is, the names were not listed in the master file in any recognizable way. Then, each student was assigned the code 12Sp#, where “12” stood for the calendar year 2012, “Sp” represented the spring semester, and # reflected where the student’s name appeared in the random list. For example, the code 12Sp1 correlated with the first student on the list, 12Sp32 correlated with the thirty-second student on the list, and so forth. In total, the codes ranged from 12Sp1 to 12Sp33. By the end of the semester, 32 of the 33 participants were still active in the class, having taken all of the exams. However, none of the participants dropped from the course; student 12Sp18 failed to participate on the first semester exam and the final exam.

As self-reported on the *Biographical Information* sheet, the median age of these participants at the beginning of the semester was 19. According to class standing based upon credit hours, 19 of the participants were freshmen, 10 were sophomores, 3 were juniors, and 1 was a senior. As for intended degree programs, 12 participants were in a pre-medical or pre-veterinary program, 6 participants were in an engineering program, 5 participants were in a biological sciences program, 1 participant was in an economics program, 1 participant was in a teacher education program, and the remaining 8 participants were either in non-technical programs or undecided. All 33 participants had at least the equivalent of Algebra 2 while in high school, while 30 participants took a geometry course. Additionally, 21 participants reported taking a trigonometry/pre-calculus course. Furthermore, 4 participants took a calculus course while in high school. Finally, all 33 participants took college algebra either through concurrent enrollment in high school or while a college student.

Summative interview participants. After the semester exam containing VTI items was graded and returned to students, participants were elicited to undergo more extensive interviews with the researcher. The elicitation occurred via an email message (*Appendix U*) to all participants; the email was addressed to the researcher with the participants being blind cc'd. In this way, participants' identities were shielded from each other. Of the 33 participants in the study, 10 assented to further their participation in the study through the interviews. However, 1 of the respondents seldom attended class and infrequently turned in assignments; thus, further participation of this student was declined by the researcher. One other student failed to arrange an interview time with the researcher. Therefore, 8 students participated in the interviews according to the timeline in Table 2. What follows is a short characterization of each interview participant based upon self-reported biographical information; information about their

Table 2

Calendar of Participant Summative Interviews

	Monday	Tuesday	Wednesday	Thursday	Friday
Week 14			Katie	Alan Amber	Charles Helen Maria
Week 15	Bella				
Week 16			Cooper		

performance on assessment items may be found in *Appendix A*. Overall, interview participants were considered to be above average students.

Alan. Alan was a 20 year old freshman. He was an engineering technology major. In high school, he took algebra I, algebra II, geometry, and pre-calculus. In college, he took college algebra.

Amber. Amber was a 19 year old sophomore. She was an animal science major. In high school, she took algebra II, geometry, trigonometry, pre-calculus, calculus, and college algebra. In college, she took college algebra and an accounting course.

Bella. Bella was an 18 year old sophomore. She was a pre-vet animal science major. In high school, she took algebra II, geometry, trigonometry, and pre-calculus. In college, she took college algebra.

Charles. Charles was a 22 year old senior. He was a pre-med major. In high school, he took algebra I, algebra II, and geometry. In college, he took college algebra.

Cooper. Cooper was an 18 year old freshman. He was an engineering major. In high school, he took algebra I, algebra II, and geometry. In college, he took college algebra and chemistry I.

Helen. Helen was an 18 year old sophomore. She was an elementary education major. In high school, she took algebra II, geometry, trigonometry, pre-calculus, and college algebra. In college, she took college algebra.

Katie. Katie was an 18 year old freshman. She was a pre-med animal science major with the intention to attend dental school. In high school, she took algebra I, algebra II, geometry, and pre-calculus. In college, she took college algebra (after being enrolled in calculus for a week).

Maria. Maria was an 18 year old junior. She was a zoology major. In high school, she took algebra I, algebra II, geometry, and pre-calculus. In college, she took college algebra and a statistics course.

In an attempt to attenuate instructor selection bias, the researcher anticipated participant selection would be guided by the data collected from assessment instruments, with attention being given to exceptional cases. If possible, participants who had demonstrated large increases in their conceptual understanding as measured by the AKE and VTI assessments would be selected; if this criterion could not be met, participants would have been selected to represent the spectrum of VTI success. One noted limitation of the study was that the self-selection of participants might be skewed toward a particular type of student; students who volunteer might be more apt to enjoy mathematics and view themselves as being proficient at mathematics. However, participants matching the desired criteria did not assent to being interviewed. Therefore, interview participants ended up being selected based upon convenience; that is, except discriminating against the participant who seldom participated in class, those who volunteered were interviewed.

Data Analysis

Data from the various collection instruments were analyzed in a variety of ways. Rather than describing the general methods of analysis based upon data type, i.e., quantitative or qualitative, this subsection will describe the methods of analysis used with each of the data collection instruments.

AKE. Each item on a student's AKE was assessed for correctness and was given a score of 1, 0.5, or 0. A score of 1 was for a correct response. A score of 0.5 was for an incorrect response that contained a minor error not related to the algebraic content being assessed in the item; the minor error rendered what would have been a correct answer incorrect. For example, a student had the following as her answer to item 3a.:

$$\frac{1-(z-1)}{(z-1)(z+1)}\,.$$

The directions indicated that the final expression should be written in lowest terms. As the student did not completely simplify the expression by performing the addition in the numerator, yet the main thrust of the problem was to assess the ability to add fractions, the response was assessed a score of 0.5.

A score of 0 was assessed to an incorrect response not assessed a score of 0.5. As an example, in answering item 2c., which requested students to factor the expression $1 - y^2$, a student answered:

$$-(y^2 - 1).$$

While nothing was mathematically incorrect with the statement, as the typical expected response from a prompt to factor is:

$$(1 - y)(1 + y),$$

the response was assessed a score of 0.

After assessing each student's AKE, student performance on each individual AKE item was recorded in an electronic spreadsheet and summed to arrive at a student's AKE score. Additionally, students self-assessed their performance on the AKE and self-assessed their mathematical abilities on a linear scale from 1 to 5. Their ratings were also placed into the electronic spreadsheet. Overall mean scores for the AKE, each AKE item, and the self-assessments were calculated.

Quiz and exam items. For the semester and final exam VTI items and the VTI quiz problems, VTI item assessment scores for each participant were recorded in an electronic spreadsheet. Overall means scores were calculated.

Supplement: Verifying identities. Students returned this completed supplemental homework activity on the second day of the VTI unit. Students responded in writing to the following prompt located at the beginning of the activity: "In your own words, what does it mean to ***verify*** that A = B?" At the end of the assignment, students responded to a similar prompt: "When someone asks you to verify a trigonometric identity, what does this mean to you?" The student responses on the homework supplement were scanned to electronic files and were subsequently transcribed. These transcriptions were analyzed to determine whether or not the student responses indicated that students believed verifying was a form of proof. To answer this question, student responses were coded using the de Villiers Roles of Proof categories (de Villiers, 1999), as similarly utilized by Knuth (2002); if a student's response mentioned an aspect of proof, the student would be taken to be speaking of VTI in terms of a proof. Of the 27 participants who returned the homework assignment, 25 responded to the first prompt and 25 responded to the second prompt; students did not necessarily respond to both prompts. While the possibility existed for a response to be coded in multiple categories, as the purpose of the analysis was to ascertain whether or not students viewed VTI as proof, responses were generally coded for the main thrust of the response. Moreover, responses tended to be short enough to warrant inclusion in solely one category.

Supplement: Assessing verification. Participants returned the completed supplemental homework activity on the third day of the VTI unit. The purpose of the activity was to deepen students' understanding of what constituted verification of identities. In the instructions of the activity, students were reminded that VTI was an argument to convince the prover and other readers of the verification of the veracity of the theorem identity. The students were also reminded that outside readers should be able to follow the logic within the argument.

Students were then directed to read through five verifications of the following identity,

$$\frac{\tan y + 1}{\sec y} = \sin y + \cos y.$$

Each VTI construction was different in the layout of the steps and varied in the explicit manipulation steps shown. Moreover, each construction was correct in the sense that no algebraic or trigonometric errors were made in the manipulations.

After reading the verification, students assessed the quality of the logic and presentation found within the verification on a scale of 0 to 5. Upon assessing the VTI construction, students then justified their assessment, providing a written reason for their assigned score. By analyzing the assessment scores and the score justifications, insight could be provided into what students believe to be convincing VTI constructions. Taken as a whole, the analysis would show more of the picture of students' VTI conceptions. The analysis used combined quantitative and qualitative techniques, with results of one type of analysis informing the direction the next analysis took; the analysis of the data from this instrument represented a mixed methods approach.

Of the 33 participants, 25 returned the supplemental homework activity. Conceivably, students could assign a score from the interval 0 to 5; thus, the scores were treated as interval data. The means of the assessment score were computed for each VTI construction. However, while the data were intended to be on an interval, only one student assigned a non-integer score, assessing one of the verification as a 4.5. Students appeared to treat their assessments as ordinal data rather than interval data, where their assessment scores were assumed to be related to choices, for example, "Perfect," "Excellent," "Good," "Marginal," "Bad," and "Unacceptable." In a preliminary data analysis of the assessment scores, frequencies were tallied of the assessment scores for each VTI construction. To be conservative, the score of 4.5 was treated as a score of 4.

The explanations of the scores were also visually explored using word clouds. Each student explanation of their assessment was transcribed into a master spreadsheet matrix so that explanations from each student formed the rows and responses about a particular VTI construction formed the columns. These transcriptions included not only words, but mathematical symbols written in the student explanations. For each VTI construction, all of the responses were copied and pasted into a word processing document. Symbols and trigonometric function names were then manually scrubbed from this document. The rationale for removing function names was to focus the analysis of the explanations on the descriptive words used in conjunction with the assessment scores. In general, function names were used in describing particular manipulations or steps and were not used to describe the rationale behind an assessment score. Furthermore, misspellings of words were corrected to ensure all occurrences of a word were accurately counted.

For each VTI construction, the edited words were copied and pasted to the website Wordle.net in order to create a word cloud for the VTI construction. Using the options Wordle provided, all words were changed to lower-case to ensure all instances of a word were counted together. Additionally, the maximum number of words to consider in the cloud was set to 100, thus focusing attention on the more frequently used words. Finally, the option to order words alphabetically was selected so that when clouds were visually compared, differences relative to the location in the cloud would be more visually striking.

Wordle.net allowed two permanent output options for the generated word clouds. The clouds could either be directly printed, or they could be saved to the website's cloud gallery. No option existed to directly save the clouds from the site to a specified file format. As the clouds in the gallery would become public, this option was rejected. Thus, the clouds were printed; however, using Adobe Acrobat, the clouds could be printed directly as portable document format (PDF) files. Therefore, each generated cloud was printed to an electronic PDF file. These files were then not only saved for future analyses but also printed for an immediate preliminary visual inspection.

A visual comparison among the clouds was made to quickly observe differences in the word clouds. The differences were noted, prominence of words based upon relative size and a shifting of the focal points on the cloud images qualitatively that made the clouds appear different. These observations were used to create a rough initial thematic framework for an in-depth qualitative exploration within each VTI construction across the student responses to observe the connections among the prominent words and develop an overall description of the ways students explained their assessments. Overall, the means and histograms helped establish a difference in the manner students assessed the constructions. The word clouds visually confirmed these differences based upon how students explained their assessments and then provided the initial framework through which to interpret the explanation of each construction assessment.

To further establish the observations concerning the means, frequencies, and word clouds, statistical tests were implemented to test the differences between pairings of VTI construction assessments. As the concern of the analysis was to establish if students viewed the verifications differently and not to determine the size of the discrepancy, if it existed, the data were treated as being non-parametric. In this way, concerns over small sample size and normality assumptions could be obviated. However, independence of assessment scores could not be assumed; in fact, in their explanations of scores, some students indicated assigning similar scores across items due to the constructions appearing similar. That is, some student assessments displayed a dependence on one another. Thus, the non-parametric equivalent of a paired t-test, the Wilcoxon signed rank test, was used.

In coordination with the word clouds, the results of the statistical tests indicated the direction of the in-depth textual analysis of the explanations students provided for their assessments of each VTI construction. The explanations of the assessment scores were analyzed in pairings suggested by the word clouds and the quantitative analyses. Each pairing of explanations from an individual participant was read. While the explanations were explored through the initial framework suggested by the word clouds, the thrust of the analysis was toward explaining why students assessed the constructions differently. Therefore, as commonalities across descriptions were noted, they were jotted down. Then, by cycling through the explanations, themes evolved, and explanations were categorized into the themes. More details of the analysis process will be provided along with the results in the next chapter.

Supplement: Algebraic simplification. In preparation for the first class period in the VTI unit, students completed the homework supplement titled "Algebraic Simplification." The intent of the activity was to provide students a venue to practice their algebraic manipulations in advance of using manipulation skills while engaged in VTI. The very beginning of the activity contained the prompt, "In your own words, what does it mean to simplify?" The participants' responses to this question were analyzed in order to determine a consensus on the trigonometry classroom community's definition of what "simplify" means, since this concept is socio-cultural in nature.

The written participant responses to the prompt were transcribed to a spreadsheet in order to form a master file linking participant to comment. Minimal grammatical and spelling corrections occurred during the transcription process. The text from the spreadsheet was then copied and pasted into a word processing document to form a text document. This text in turn was copied and pasted to Wordle.net, and a word cloud was generated. The same parameters used in prior clouds were implemented to form the cloud. The same general procedure used to analyze the *Assessing Verification* supplement was then utilized; analysis of the word clouds informed the textual analysis of responses. Themes emerged from the textual analysis and were identified.

Writing prompt: Verifying identities. During the first five minutes of the sixth class in the VTI unit, students responded to a writing prompt. They were requested to complete the following phrase to form a simile, "To me, verifying a trigonometric identity is like." Then, following another prompt, students explained the intended meaning of their simile using several clarifying sentences. The directions for the prompt explained the meaning of a simile and provided a basic example. The intent of the writing prompt was to allow insight into students' conceptions of VTI.

Each student offered unique similes and explanations that contained powerful notions; however, the point of the analysis was to describe patterns across the responses. While the analysis could take several directions, as a simile is constructed using the comparative phrase *is like*, analysis of the similes focused on the *essence* of VTI for each student, or what VTI *is* for each student. In coding the responses, the simile was assumed to be a vehicle for VTI, eliciting the meaning VTI held for the student. Thus, in thinking about the similes, students activated their conceptions of VTI. Accordingly, the explanations of the similes were then interpreted as references to the students' conceptions of VTI, providing insight of the essence of VTI. These interpretations occurred within the simile's domain, with the particular imagery of the simile being used to color the possible interpretations of the meanings.

Thirty-two of the thirty-three participants were present during class and completed the prompt. One student provided two distinct similes; hence, thirty-three distinct responses were analyzed. The responses were analyzed using an open coding procedure. Each response was read, focusing on the essence of VTI provided by the simile explanation, and as commonalities emerged across the responses, notes were jotted down. Another reading of the responses occurred to begin separating the responses into categories of themes based upon the noted commonalities. After cycling through the responses several times to ensure that the responses in each theme shared a common feature, descriptions of the categories were written using the commonalities and the categories were named based upon the shared features. Then, the participant responses were removed from their categories, and the themes were set aside. After allowing several weeks to elapse, the process of coding the responses began anew, following the same coding procedure as before. After this second attempt at coding, the resulting new themes were retained.

Summative interviews. The summative interviews of the eight participants were analyzed using an open-coding format. At each stage of the analysis, which started during the actual interviews, preconceived notions held by the researcher were identified in an attempt to control any researcher bias. During the interviews, as participants responded to the interview prompts or follow-up questions, notes relating to interesting or common ideas held by the participants were jotted down. After all of the interviews were conducted, each interview was transcribed, with every verbal utterance included. Again, during the transcription process, notes were jotted down and were used to clarify or add to the previous ideas that had been noted. In this way, the themes of each interview evolved and established a basic thematic framework of students' concept images of VTI.

The thematic framework was further molded and strengthened in a cyclical fashion. After transcription, each interview was individually read through the lens of the evolving framework, with the themes undergoing revisions based upon any new ideas or re-interpreted meanings of participants' responses. Then each interview underwent an in-depth analysis, with passages of the interview being classified according to the thematic framework. However, the framework was continually modified when induced by the participants' responses to ensure that the themes captured the essence of the participants' responses. Thus, the thematic framework developed internally for each participant and also across the participants.

CHAPTER IV

FINDINGS AND INTERPRETATIONS

This study explored students' concept images of verifying trigonometric identities. The phenomena were explored using a case study methodology, employing various data collection instruments. The collected data were analyzed using quantitative, qualitative, and when appropriate, a cyclical mixing of techniques. In this chapter, the results of the analyses will be discussed and interpreted. The exploration of data and the subsequent interpretations were shaped by the general research questions. These questions were:

1. What are students' conceptions of VTI and to what extent do existing frameworks describe the conceptions?
2. What factors contribute to a student-perceived successful engagement in VTI?
    a. What does VTI accomplish from the student's perspective?
    b. What structural form does a VTI construction take?
    c. In what ways do the structure of the VTI construction and the VTI accomplishments interact?
3. In what ways do students' conceptions of VTI influence problem solving decisions?

The chapter begins by considering the nature of VTI before exploring the mechanics of VTI and the resulting accomplishments of VTI. Included in these explorations is a discussion of guiding strategies used by students in constructing VTI proofs and the visual appearance of VTI.

The Dual Nature of VTI

During the first five minutes of the sixth class in the VTI unit, students completed the activity *Writing Prompt: Verifying Identities*. They were requested to complete the following phrase to form a simile, "To me, verifying a trigonometric identity is like." Then, following another prompt, students explained the intended meaning of the simile using several clarifying sentences. Through open coding, focusing on the essence of VTI, six themes evolved describing the participants' VTI similes: *Proving*, *Manipulating*, *Surmounting*, *Adhering*, *Meandering*, and *Affecting*. What follows is a description of each theme with examples of students' similes.

Similes involving *Proving* (10 responses) generally suggested that VTI was a process that served one of the roles of proof. Two subthemes existed under this category related to roles of proof, *Verification* (3 responses) and *Explanation* (7 responses). Students that described VTI in an explanatory fashion used similes that illuminated or explained the why of an event within the simile. One student described VTI as a court case, explaining, "In court you must give reasoning behind why you believe something to be true. You gather up many facts to prove one thing is another, or true… In the end, you should end up with the right verdict!" For another student, the Proving simile manifested as figuring out a magic trick. "When someone performs a magic trick it's obvious what they did you're just not sure how. When they tell you the secret both sides of the trick make sense because you know how they're doing it and why they end with certain results." With the Proving similes, some level of acceptance existed concerning a phenomenon, such as the innocence of the client for the defense attorney or the acknowledged duping due to the magical prestige. Then, VTI was compared to a process providing an insight as to the existence of the phenomenon. Thus, the Proving simile concerned an aspect of revelation or explanation; a mountain of evidence showed why the accused was innocent or the mechanics of the legerdemain of the illusion are explained. These similes got at the root or heart of the matter and answered the question "Why?". In other words, VTI explained why an identity was an identity.

For students viewing VTI as verification, the responses described a process revealing the truth of a situation. One student commented, “To me, verifying a trigonometric identity is like proving a superhero is good. Just like how superheroes are thought to be law breakers, Identities may sometimes look to be faulse [sic]. However, most always the super heroes are good and always Identities are true.” Another student described VTI as a puzzle, explaining, “You know all the pieces are there and that the two equal each other but you have to prove it by working it or by putting the puzzel [sic] together to prove all the pieces do make a puzzel [sic]! Proof!” The verification similes indicated that VTI was a process that revealed a known truth; everyone knows superheroes are good and everyone knows pieces compose a puzzle. Yet, VTI made these truths known and certain. Related to this idea was the notion of constancy in the face of uncertainty; appearances were deceiving. Despite the appearances of villainy, the superhero was always the good guy; even though the pieces were a jumbled mess, they did in fact compose a picture. At its heart, a constant quality existed, whether it was *goodness* or *picture*.

Manipulating (6 responses) encompassed those similes that suggested achieving a goal through a manipulative process; this process could be a physical or a mental achievement. To achieve this end result, the manipulation of parts or pieces found within the simile domain occurred. The goals suggested by the similes fell into three categories, with two participant responses in each category. One goal involved attempting to make something similar in appearance. As one student put it, “To me, verifying a trigonometric identity is like trying to solve a Rubik’s cube. In order to get each side to be the same, they must be changed and adjusted.” For another student, VTI was like “Trying to rearrange all of the ‘pieces’ on one side so they look exactly like the other side.” That is, VTI was like a puzzle. Both of these students envisioned VTI as moving parts of a whole around to gain a visual equality.

Another goal involved a metaphorical unlocking. As one student stated, VTI was like solving a puzzle. “You just need to find the right piece and twist it just the right way, and then everything falls into place. Once you've got the right piece in the right spot, the rest is easy.”

This explanation evoked an imagery of tumblers within a lock falling into place to make straight the way for opening the lock. Another student was more literal, describing VTI as "trying to pick a lock. You have to read in very closely and use high understanding and logic to decode and unlock it, and I am not very good at it." Both of these similes emphasized moving parts around to achieve a goal. In some sense, this goal involved managing a barrier. But through the similes, maneuvering around the barrier was not accomplished in a willy-nilly manner; strategy or cognition was utilized. Just the right puzzle piece was located and properly placed in order to facilitate the solving of the puzzle. Likewise, thought must be taken in opening the lock. Trying random combinations or jiggling the pick was unlikely to open the lock; the correct combination of moves was necessary.

Finally, a third goal mentioned in manipulating was a process of an object moving from a complex to a simple form while emphasizing the unchanging essence of the underlying material composing the object. For example, a student stated that VTI was like "trying to translate a different language and finding that it means the same thing in English." Another student believed that VTI was similar to a giant tree. "You can cut it down and make a toothpick. Most trigonometric function [sic] seem to show a very simple version is equal to a more complex problem. They are both made up of the same parts and values." Both examples involved the changing of an object, from a tree to a bunch of toothpicks and from a word in one language to a word in another language. In each simile, the object initially existed in a more complicated form; a tree and a foreign word were in some senses more complex than a toothpick and the word translated into English. Moreover, the appearances of the beginning objects changed. However, there was an implied underlying essence that did not change. The tree and the toothpicks were composed of the same material. The two words that looked dissimilar described the same notion.

Similes coded as *Surmounting* (5 responses) involved a situation in which a barrier needed to be overcome before the goal was reached. For example, one student stated, "To me, verifying a trigonometric identity is like football. I never understand which plays they're using but it always ends up with a goal (answer). In the midst of the problem, I get mixed up and turned around but eventually find a way out." In another response, a student described VTI as "finding your way out of the woods. There are several different ways to get to where you want to go, just like there are several ways to verify an identity. But as long as you stay focused and pay attention, you'll eventually find your way home, or you will eventually get a correct verification." The barrier existed because of a state of "confusion" brought on by cognitive difficulty or by the possibility of multiple choices. However, a single goal existed in the similes. In the suggestion that VTI was like a tricky maze in that "Sometimes you get to the end on the first try, and sometimes you have to start over and take a different route," different routes were possible, all of them dead-ends save one. The maze-solver needed to weed out the possibilities until finding the one leading to the end.

Some similes explained VTI as *Adhering* to rules (3 responses). As an example, a student wrote, "To me, verifying a trigonometric identity is like stadium jumping a horse. In stadium jumping, you jump a set of jumps in a certian [sic] order as fast as you can." For another student, VTI was like playing chess: "There are moves you have to make to win in chess like trapping your opponents [sic] king so that if it moves, it dies and you win. The same thing kind of applies to verifying idendities [sic]. You have to make certain moves to get the sides to where they cannot be simplified any more (a move). If they equal each other, then you have verified the identity or won the game." If the rules of the game were followed, then the desired goal was achieved; likewise, if the rules the student imposed on VTI were satisfied, the identity would be verified. In the first simile, the steps, substituting identities, occur in a defined order. An element of speed also exists in this simile; for that student, VTI should be executed in a rapid, flawless manner, suggesting satisfaction was gained through speedy implementation. In the second simile, existing combinations of moves were needed to achieve the goal. For the student, these combinations equated to the simplification of the sides in VTI; a visual equality in VTI equated to winning the chess game.

Meandering similes (4 responses) implied a lack of cognitive effort in reaching an end. One student described VTI as eating from a jar of red M&M's, stating, "You always know what you're gonna get." For another student, VTI was like "going to Walmart and trying to pick a toothbrush! With these, you have lots of trial and error and guessing. You don't always know what 'brand' to use and sometimes you have to just take a chance and hope that it was the best decision." A third student explained VTI like life in that "It usually all works out in the end." In each simile, some conclusion was reached despite a lack of any real input. These similes suggested little to no cognitive effort from the students. VTI might involve a high level of guessing or perhaps VTI, with a predetermined outcome, presented no cognitive challenge.

Several responses related to feelings or emotions VTI induced in students; hence, these similes were coded as *Affecting* (5 responses). One student wrote, "To me, verifying a trigonometric identity is like working with annoying people. Its [sic] a pain in the ass." Another student likened VTI to "shooting a paper ball into space. No matter how hard I try I can never figure it out." These similes tended to force the focus away from the essence of VTI and directed attention to the frustrations students experienced with VTI. The sources of the frustration tended to stem from an inability to understand how to verify an identity or to view VTI as a worthwhile activity. As one student explicated, for him VTI was like "orally consuming nothing but air when you're starving. Explaining something that is a known fact seems pointless."

Analyzing the commonalities in the themes, Manipulating, Surmounting, and Adhering similes explained VTI in terms of the mechanics of how verification was accomplished. These similes treated VTI as a process; VTI was a problem to be solved. Moreover, the Affecting simile addressed an attribute of problem solving that students needed to manage in order to be successful. On the other hand, the Proving similes described VTI in terms of its nature as proof construction. Thus, students could perceive VTI as a problem solving process, or, they could viewed it in terms of proof construction.

Trying to categorize a student as one who believed that VTI was solely a problem to solve or solely a proof construction would not be possible based upon this prompt; the activity suggested writing one simile. Given the opportunity to construct more similes, students very likely would have produced similes that varied in the facets of VTI described. The important point was that students evidenced beliefs in VTI as problem solving and as proof construction. Therefore, problem solving and proof frameworks could be legitimately used to analyze student attempts at solving VTI problems.

Cutting across the themes, many similes also contained a notion of equivalence in them. That this notion was present was of no surprise when considering the objects involved in the VTI process, namely identities. These similes could directly equate objects in a process, such as the student who related trees and toothpicks. Also, for another student, the goodness of a superhero remained a constant throughout his or her story. The similes could also imply that the VTI process resulted in a known or expected outcome; thus, in a sense, outcomes were equating expectations. Examples of this equivalence were those students who built court cases to show innocence. They began with the assumption of innocence; the case proved this assumption true. The student who orally consumed air believed in the futility of VTI due to already knowing the outcome. While identities are the theorems of VTI and thus are important in their own right, this current study did not delve further into students' concept images of trigonometric identities. Thus, this issue should be studied further in the future.

VTI as Simplification

Student Conceptions of Simplification

When comparing two mathematical expression, identifying which expression is "simpler" is not straightforward and a somewhat subjective task. For example, in comparing the two equivalent expressions $\sin 2x$ and $2 \sin x \cos x$, one student may state that $\sin 2x$ is simpler as it has a visually smaller physical size. On the other hand, a student uncomfortable with function arguments and double-angled trigonometric functions may be troubled by $\sin 2x$ and thus

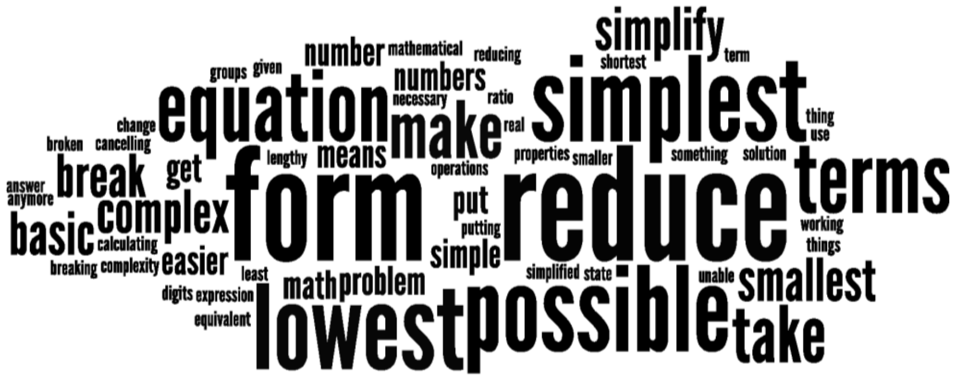


Figure 4. Initial word cloud of students' explanations of the meaning of "simplify."

consider $2 \sin x \cos x$ to be the simpler expression. While this issue of which expression students found to be simpler was not explored in this study, the general notion of students' conceptions of simplifying was.

In preparation for the first class period in the VTI unit, students completed the homework supplement titled *Algebraic Simplification*. The intent of the activity was to provide students a venue to practice their algebraic manipulations in advance of using manipulation skills while engaged in VTI. The very beginning of the activity contained the prompt, "In your own words, what does it mean to simplify?" The participant responses to this question were analyzed in order to determine a consensus on the trigonometry classroom community's definition of what "simplify" meant, since this concept was socio-cultural in nature.

Of the 33 participants in the study, 24 participants completed the prompt. The written participant responses to the prompt were transcribed to a spreadsheet in order to form a master file linking participant to comment. Minimal grammatical and spelling corrections occurred during the transcription process. Because the text was going to be used in an analysis that depended upon frequency, having a word spelled correctly was desired in order to ensure that the word was counted the proper number of times. The text from the spreadsheet was then copied and pasted into a word processing document to form a text document. This text in turn was

Figure 5. Final word cloud of students' explanations of the meaning of "simplify."

copied and pasted to Wordle.net, and a word cloud (Figure 4) was generated. The same parameters used in prior clouds were implemented to form the cloud.

After a brief scanning of the cloud, looking for prominence of words and similar ideas, a new text document was created from the original text document with the following changes. First, some students used the word "simplify" as a signal phrase in their responses, e.g., "To simplify, …" Thus, the word had little relevance for the explanations of what it meant. Next, as "break" had some prominence, the similar phrases "breaking" and "be broken" were altered to "break" in order to accurately capture the same idea. For the same reason, the word "reducing" was edited to "reduce." Finally, while enacting these changes, the observation was made that Wordle.net filtered the word "down" out of the cloud, treating it as a common *stop word*. However, for the purposes of their descriptions, "down" was used as a descriptive adverb in a significant way. Thus, a tilde was used to form the phrases "break~down" and "reduce~down," forcing Wordle.net to use the phrases in the cloud formation.

A cursory analysis of the word cloud (Figure 5) revealed several potential themes. First, the participants spoke of simplify in comparative states of existences, using words such as "lowest," "simplest," and "smallest." Size played a role in this existence, by virtue of the root word "small," "short," and "length." Actions such as "break down," "reduce," and "reduce down" were apparent in the cloud, suggesting a process. Along with these words, other prominent notions, such as "basic," "possible," and "complex" needed to be explored to identify the context in which they were used and to determine their relationships to the prominent words "form," "equation," and "terms." Thus, this question suggested a line of inquiry for an in-depth textual analysis.

While initial and informal textual analyses of the participant responses actually occurred during the transcription and cloud formation processes, a more in-depth analysis occurred using the final word cloud as a guide. One result of this analysis related to usage of the word "equation." From the point of view of a teacher, the usage of the word "equation," resulting in its prominence in the cloud, bordered on being a malaprop; in every instance, the word "expression" would have been the appropriate word as it would convey the intended meaning in the response. Regardless of why students tended to misuse the word, no further analysis was attempted.

Having the notion of "to bring down or diminish to a smaller number, amount, quantity, extent, etc., or to a single thing; to bring down to a simpler form," ("Reduce", 2013), the word "reduce" is very prominent in the cloud. Thus, an analysis of the participant responses was undertaken to explore how students were using this idea as it related to simplifying. Students described "to simplify" as "to reduce" in eight distinct comments out of 24 total responses. Of these eight, one student directly equated simplifying with reducing, while another student explained simplifying as reducing the complexity. The other six comments were all related by the idea that simplifying was the act of reducing to some base state of existence with a notion of finality associated to it. For example, students stated that to simplify meant:

> To take an equation and reduce its math properties until you cannot reduce it anymore. (12Sp11)
>
> Reduce problem to the lowest, simplest form possible. (12Sp10)

In these examples, simplifying entailed changing the state to something that could no longer be changed; the primitive existence of the expression was reached.

The concept of reducing related to another prominent cloud feature, that of "break down." Although to break down may elicit several interpretations, students used it in the sense of decomposing something to simpler components. A directional quality of the act of simplifying was emphasized through pairing the word "break" with "down"; in the same manner of reducing, breaking down lead to a low-level, base state of existence for the expression. Once this simplest or basic form was reached, nothing further could be done. For example, students described what simplifying meant in the following way:

> Breaking an equation down to the most basic form possible. (12Sp12)
>
> To break down to its most basic state. (12Sp19)
>
> To put in the simplest terms possible; an expression is unable to be broken down further once simplified. (12Sp16)

Thus, for these students, simplifying was akin to the atomism of Leucippus and Democritus; a point would be reached where the expression could not be further broken down and simplified.

While not specifically always using the phrases "reduce" or "break down" to describe the act, several comments described reaching a most primitive state, making the "form" of the expression more basic, simpler, or smaller. Examples of these comments were:

> To get to the most basic form. (12Sp13)
>
> Change it to the simplest form. (12Sp15)
>
> Simplify: to take an equation or number to its smallest or shortest form. (12Sp21)

In total, eight comments referred to the form of the expression dictating when the expression was simplified. Moreover, when writing of a "simplest" form, some students might have been thinking of the complexity in terms of the physical size of the expression, as suggested by the student who wrote:

> To take a lengthy or complex equation and use mathematical operations to put the equation in its least complex form. (12Sp24)

In this comment, the student linked the length of the expression with a progression of the expression to its simplest form.

Students also referred to simplifying as bringing the expression to its lowest or simplest "terms" (5 comments). Some students explained simplifying in the following way:

Reducing to the lowest terms necessary. (12Sp22)

Putting things in lowest terms. (12Sp33)

Colloquially, terms generally mean the components of something, so "lowest terms" might be taken to imply reaching a state that can be broken down no further. Also, some students might have believed simplest and lowest to be synonymous, as suggested by the following comment:

To break something down into its lowest/simplest terms. (12Sp18)

Furthermore, bringing an expression to lowest terms, to make it simpler, was connected to the physical size of the expression. As a student explained:

To make more simple by cancelling terms out. (12Sp28)

Cancelling terms out reduced the matter composing the expression, thus making the expression smaller in size.

The size of the expression was an underlying theme across many of the responses. In many instances, the size described the final state of the expression. As one student wrote:

Make smaller. (12Sp25)

However, as previously discussed, the notion of "making small" is embedded in the phrases "break down" and "reduce," as each phrase described simplifying as an action of taking the expression to a more basic state. The size of the expression may not necessarily refer to the physical size, but to a cognitive load. The more complex an expression is, the larger that expression is. Thus, simplifying shrinks the cognitive load of the expression for that student. While the cognitive size may be related to factors based upon the actual components used and not depend on the size, e.g., a student may perceive $\sin x$ as being cognitively smaller than $\sec x$ due to familiarity with the sine function, students definitely linked complexity of an expression to the physical size.

To summarize, participants within the classroom community viewed the act of simplifying as a process of taking an expression to its most basic state in order to reduce the perceived size (physical or cognitive) of the expression. As some students wrote, simplifying meant:

> To reduce the #'s that you are working down to the lowest possible digits to make the math easier. (12Sp4)
>
> To take a complex thing and make it easier. (12Sp8)

This result was not groundbreaking or unexpected and aligned with a broader accepted meaning of simplifying. However, since being mathematically simplified holds a subjective meaning, establishing what students meant when using the word was important for the study since simplifying acts played an important role in VTI.

The reason exploring student perceptions of simplifying was important emerged in some of the decisions students made in VTI and may be related to VTI concept images. As an example, in the *Assessing Verification* supplemental activity, one student (12Sp28) assessed VTI constructions #2 and #4 to be a 5 and a 4, respectively, giving the following explanations of the scores:

> VTI #2: Perfect. ***Did the complicated side first*** and is neat.
>
> VTI #4: Perfect ***except didn't do the complicated side***.

The bolded passages indicated that the student held an explicit conception regarding VTI; the starting expression should be the one that was deemed the more complicated by the student. Another student (12Sp14) assessed the same constructions as a 0 and a 5, respectively, explaining the scores in the following manner:

> VTI #2: Very complicated with many unnecessary fractions made. ***Should've started with the other side because there was more to work with***.

VTI #4: Easy to read and ***using the right side made the reasoning easier to understand and see the steps***.

This student held the opposite view regarding the complexity of the expressions as compared to the other student. Furthermore, this student explained the desire to start with the complicated side; doing so allowed for a clearer vision of the VTI construction itself. The sheer volume of fractions in the construction of VTI #2 caused the construction to be complicated.

What specifically caused the complication for this student was unclear. The student may have experienced confusion due to the fractions. Perhaps the complexity was due to the presence of several fractions creating a larger expression than the beginning expression. Or, perhaps it was a combination of these reasons. Regardless, the student's comment implied that in VTI, the complicated expression, the one with more to work with, should be chosen in order to simplify it, not cause it to become more complicated. In creating a more complicated expression, the reasoning within the VTI steps became difficult to see and understand.

Thus, in these students' explanations of the comments, an underlying theme emerged concerning VTI and simplifying. Simplifying helped to reduce the components of the steps, allowing for easy understanding of the VTI process. Additionally, simplifying was an expectation for VTI; the overall action of VTI was one of compression. This dual purpose of simplifying became apparent in the participant interviews.

The Drive to Simplify

The concept of simplifying was a major theme, cutting across participants and unifying other thematic elements. In this upcoming subsection, the different ways that simplifying manifested in interactions with participants will be described. Before proceeding to the manifestations, the origins of simplifying will be discussed. Working with expressions in smaller forms was a natural tendency for students such as Cooper. Cooper believed this tendency had extra-mathematical origins.

INTERVIEWER: Where do you think that comes from, this desire for making it simple?

COOPER: Uh, from being a kid I think, you know. From sitting there and looking at it and going, why can't life just be mud puddles and, you know, different things like that. Uh, whenever you have something smaller to deal with, it, it just makes it, makes you think, you're less stressed about it. … Which one's easier sounding, building some little tiny little, you know, shed, or something, or building a huge gigantic house? … It's easier to think of, try to make it smaller situations. I know it's kind of a huge exaggerated of, ah, example, but it's, it's still kind of the same concept. Smaller is better because, uh, it's less for you to think about it, honestly. … It's not as complex looking, honestly. It's just what it brings down to. It's not complex very well. I mean, if you simplify it, it's not too complex.

Thus, the desire for a simple life created a drive to simplify problems.

COOPER: Whenever you're looking at some huge word problems, and you, it just looks very complex. But when you break it down, you realize, okay, well these numbers, there's, you know, there's only a few things in here that are actually relevant to the situation. Yeah, you, you, you try to shrink it down less, the less, if you can cut out like, you know, if you can focus on these smaller parts and make it kind of a simpler form, a simpler layout, it's a lot more easier to look at. … It's just the basic, you know, wants to make everything simpler and easier. What's the easiest way, you know, work smart, not hard type situation? … I think it's something that's just, uh, bred into people. I think they're born with it honestly. I mean, simpler is easier. It's just how I think it's always kind of an instinct.

For Cooper, ways that he simplified problems, cutting out what did not matter and focusing on smaller parts was an innate quality. Thus, in a sense, students had a "drive to simplify."

The Flow of VTI

The flow of a VTI problem was predicated on simplification. Students tended to describe VTI as a condensing process. In addition to the natural desire for a simpler state, this perception of VTI flow received support from the students' experiences in mathematics. Bella expressed this sentiment in describing her preferred method for verifying identities.

> BELLA: I do it like, you start with the big equation, you get it smaller and smaller and smaller. … That's just math in general.

Bella believed that math problems tended to flow from big to small. Helen described some experiences from her algebra courses.

> HELEN: That's what we're taught to do is something times something, you know, a small number is outside of a big parentheses number, you automatically distribute it. So that seems like it breaks it down and makes it simpler.

Through certain exercises designed to reinforce mathematical properties, such as practicing the distributive property, Helen formed a belief about mathematics and what it was supposed to be. This belief had a profound effect on her. Upon finishing VTI6, she emoted.

> INTERVIEWER: Do you get satisfaction from, uh, once you've actually finished it?
>
> HELEN: Uh, I love coming to, like a con-, like, that's why I love algebra. Because it's like huge equations and like you finally come to one and, like, that's so cool.

Her satisfaction derived from shrinking a large situation down; she enjoyed simplifying.

Simplification had a similar affective influence on Charles's conception of VTI. He explained that he believed identities were useful for simplifying a problem, making it easier to use and understand.

> CHARLES: Um, when I look at, like a, like a big nasty looking type of problem, uh, I try and like simplify it, I guess. And it helps me to know what to do. Like, I don't know. It just, when I look at something, it, I freak out. And if it's, gets simpler, then I freak out less.

Out of necessity, Charles was obligated to simplify expression. Not doing so would leave him in a state of confusion concerning the problem. This drive to simplify colored his beliefs of

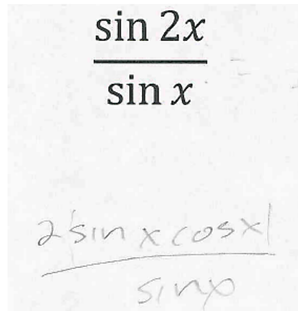


Figure 6. Charles's "answer" to VTI5, finding an equivalent expression.

mathematics. In solving VTI5 (Figure 6), Charles struggled with basic algebra, causing hesitation with his steps. The question prompted him to find expressions equivalent to the given expression

$$\frac{\sin 2x}{\sin x}.$$

Through much discussion, he finally arrived at an expression. However, upon arriving at his "answer," Charles felt much discomfort.

INTERVIEWER: Okay, so have you answered the question?

CHARLES: No. I don't think so.

INTERVIEWER: You don't think so? So why don't you think you've answered the question?

CHARLES: Cuz I would say, cuz it looks, uh, I mean, this looks simpler than this.

Since Charles viewed the beginning expression as visually smaller than his "answer," Charles felt his answer was not simpler and thus could not be a legitimate answer. Charles confirmed this belief.

INTERVIEWER: Do you think then that for your answer, it needs to be simpler than what you started with?

CHARLES: Generally, yeah usually you do, yeah. … Like, uh, if this is true, then I would say, you know, it has to be like one, like this equals tangent or something, you know what I mean.

Therefore, for Charles to believe he had answered this question correctly, he believed the equivalent expression should have been smaller, like a single tangent function. A belief such as this could handcuff Charles when attempting to verify identities. For one, he would need to begin with the larger expression in order to end up at the smaller one. Moreover, during the course of verifying identities, he might shy away from using identities that enlarged the expression, even temporarily, due to the complex state being introduced; doing so would run counter to his drive to simplify.

Students tended to view VTI in terms of simplification, describing VTI as a condensing process. When asked to explain what it meant to her to have verified an identity, Helen described VTI using the language of simplification.

> HELEN: Well, you're breaking, you're either breaking it down or condensing it to something smaller, um, to make sure that one equals the other.

Thus for Helen, in order to show the equality, she needed to break down the expression to a more basic state, a smaller expression. Condensing the expression down was perceived to be an easier, more natural movement. While discussing his solution to VTI6, Alan used the word "unravel"; he explained how "unravel" described what VTI was for him.

> ALAN: You're just trying to, uh, strip apart, piece by piece, to get to where they're equal, where they're the same, so. I guess you're just trying to, the way I like to think of it is just you're trying to whittle it down, to. That's why I start with the most complicated side to begin with. … So you're kind of taking off the layers I guess you could say, unraveling them, to eventually get to its, to the more s-, to the simpler of the two sides.

Thus, for Alan, VTI involved removing components of the expression, reducing its size to what he considered the simpler form.

This view of VTI as a condensing action had an impact on how VTI proceeded. As Alan mentioned, he was compelled to choose what he deemed the more complicated expression in order to begin his manipulations. He then simplified, or unraveled, the expression down to its basic form. Thus, students felt beginning with the more complicated expression was easier.

> ALAN: I like to go with the most complicated side, I th-, it's easier to condense things down than to try to build them up.
>
> AMBER: Normally it's easier to go to, from complicated to simple than simple to complicated.

Why students believed it easier to condense rather than build up may have related to the drive to simplify; building up ran counterintuitive to the natural tendency to shrink. By building an expression up, complexity was reintroduced into the expression. The complexity may have involved size, but it also included multiplicity in paths.

> ALAN: It's easier to see where you need to go when there's a single end result than trying to build it up to a big complex mess. … Yeah, there's lots of ways to expand that out, but there's few ways to compress that down to get to that.

Alan connected simplicity and complexity with the number of manipulations associated with the expression. Beginning at the simpler state, too many choices were available to manipulate the expression. Compressing had a more limiting nature to it.

While Alan appeared to mean that a large expression offered fewer possible paths, perhaps what he meant was that larger expressions offered fewer plausible solution paths to the smaller form. That is, while one could conceivably perform an infinite number of different manipulations on an expression, only a limited number of those would achieve the desired compressing action. However, which manipulation to use was not always apparent, and the number of plausible steps to take was more difficult to whittle down. Thus, with fewer plausible choices, the decisions were easier to make when going from big to small; the cognitive loads for the students were reduced. Therefore, when students spoke of picking the more complicated side because there was "more to work with," they meant that having more to work with provided them with more information, thus, limiting the scope of manipulations. Having more to work with allowed them to recognize and then connect the expression in a more unique way to existing, known identities.

Helen and Cooper seemed to echo this sentiment. In describing her decision to choose the expression on the left side of the equation in VTI2 to begin manipulating, Helen mentioned the issue of choosing the large expression.

> HELEN: Usually if they're bigger, that means that they're broken down into simpler steps. So, simple things like one minus cosine squared theta condenses into something smaller. … If I had started with the right side, you'd have to unfold, and figuring out a whole lot more formulas that wouldn't come to mind right off the bat for me.

Thus, in the larger expression, Helen could see the pieces that would match with a known identity, resulting in a smaller expression. Starting with the smaller expression and unfolding, or expanding to, the larger expression necessitated Helen to consider many possible identities containing that smaller expression.

While solving VTI4, Cooper began his solution by picking the starting expression to begin manipulating.

> COOPER: I'm going to probably go with the left side because it does seem a lot more to work with. … Whenever you have a hu-, a bigger situation, you can, a lot more that you can change, and a lot more to manipulate, really. It's just whenever you have a bigger situation, there's a lot more to man-, manipulate than there is with a, a smaller amount of stuff to change.

Thus, Cooper believed that choosing the larger expression allowed for more choice when choosing identities to implement or manipulations to enact.

> COOPER: The smaller n-, the smaller side is, it's specific, it's a more specific situation. Whereas the left side, when it's a lot more broader. … If you have more identities and stuff that you can use, such as the left side has more ways. Like for the one, you could change that into, uh, sine squared x plus cosine squared if you wanted to.

For Cooper, bigger expressions represented a more general situation compared to the smaller expression representing a specific situation; for Cooper, VTI solutions followed a flow from general to specific, which was the natural or simpler flow for the problem.

> COOPER: If you have a more complex problem that you can work with and just break down, it does make it easier than to start out with something that's already broken down and it's already to that point. And, it's, it's a, it's a little more easier to do that that way instead of trying to take this thing that's broken down and expand it, and expand it some more to reach the other side.

For Cooper, he believed beginning with a large expression and breaking it down into a smaller expression was easier than starting with the more specific instance and working backward, building it back up. In other words, VTI problems flowed from complex to simple, with the VTI actions serving the role of simplification. Moreover, an expression that existed in its base state was very difficult to manipulate further; this related to students' perceptions of simplification taking an expression to its lowest possible form, one that could be broken down no further.

Picking a Side

Going with the flow. Already displayed in the students' comments was how the notion of simplification controlled the picking of the side. For example, Alan stated that he picked the complicated side because he would then whittle it down, or unravel it, to the simplified state. Students also believed that starting with a more complex expression would provide more material with which to work and match identities. Additionally, the plausible choices for manipulation or substitution actions would be limited. Thus, due to their view of what VTI was and their recognition that simplifying an expression was easier, students picked the more complicated or larger expression. Hence, the starting expression was chosen for its potential to manipulate; students perceived the expression existed in a complex state and could be manipulated down to a lower state.

Identity recognition. Some students chose the more complicated looking side because they believed it gave them more opportunities to match it to known identities. While solving VTI2, Charles picked the left side to begin manipulating.

CHARLES: I always look at the, the one that looks like bigger, more, like, one that has more stuff on it.

INTERVIEWER: And why is that?

CHARLES: I don't know. Just to, if I f-, I want to see if I can recognize anything. And like, you know, the one minus cosine squared theta, I've recognized that, so I'd look that up in the book.

In choosing the bigger expression, Charles believed he gave himself a greater chance to recognize an identity that he could use. Once he thought he noticed an identity, he could use it to simplify the expression down to the target.

A more complicated side based upon the size of the expression was not always identified by the student. For example, students had a difficult time deciding which side was more complicated in VTI9 as each side involved the same two functions. The difference between the sides was the operation involved; on the left side, the two functions were multiplied, while on the right side, the two functions were subtracted. Alan discussed why he chose the expression on the right side.

ALAN: Now this one, since the two sides are roughly similar, um, like one side doesn't stick out as being simpler as, than the other one. But, usually in these circumstances, I'll go with whichever one has the minus or the, the subtraction or the addition sign.

INTERVIEWER: Why is that?

ALAN: Um. Because there are, um, more identities, I guess, that you can work with that involve, kind-, the ones that stick out to me are the Pythagorean identities that you could use. Uhn, and all of those in some way involve an addition or subtraction sign.

Thus, Alan began by considering which side was the complicated side. However, even though that venture failed, he still relied on a criteria base on multiplicity, or size; he chose the expression he considered would potentially have more identities associated with it. Thus, in a sense, he was still choosing the bigger expression.

Flow Control

Beyond choosing the beginning expression, students used the flow of VTI to monitor their progress and help them overcome barriers. Cooper used the concept of VTI as simplifying to control other actions during VTI. While solving VTI4, Cooper commented about an action he made.

> COOPER: I'm going to take notice that cosine squared, there is a, there's a identity for one minus cosine squared. And that, that's to shrink it down and try to change the problem up a little bit and bring it down to a simpler form.

While he noticed an identity he could use, his ultimate justification for using that identity was the shrinking action that using the identity would impart. This controlling aspect of simplification provided Cooper satisfaction with his answer when his actions led to simplified forms. While solving VTI5, Cooper explained his rationale for his actions.

> COOPER: Since this is basically like saying two times sine x times cosine x, I'm going to go ahead and cancel off the sine x's. … You kind of shrank down and simplified the problem a little bit.

Being in a simplified form also provides finality to his answer. Cooper was probed about whether multiple solutions to VTI5 existed.

> INTERVIEWER: Since you're not seeing another identity for sine of two x, you're not thinking of anything else that it could equal?
>
> COOPER: Yeah, I'm just kind of leaving it as is. And it's in a pretty simplified form.

While his reliance on identities restricted his actions, a simplified form indicated to Cooper when an answer was reached. Even though Cooper was empowered through this simplified result, the drive to simplify problems restricted the possibilities for him, only allowing for solutions that narrowed the expression down to a smaller form.

Maria also used the flow of the VTI problem to monitor her progress. While solving VTI6, she encountered a decision among three equal expressions for $\cos 2x$.

> INTERVIEWER: How do you decide which path to take?
>
> MARIA: I just try to find the one that looks like it could, um, help me simplify it, I guess. I mean, just the one that has the terms closest to the ones I already have, um, to try to condense it a little bit.

Maria had originally substituted in $\cos^2 y - \sin^2 y$. She further explained her reservations about what she had written down.

> MARIA: Once I wrote it down, I guess in some ways it just looks a lot more complicated than it did in my head.

Thus, upon reflection, Maria believed the expression was not becoming simpler, creating doubt in her mind about her progress in the problem. During VTI7, Maria again encountered the same issue.

> MARIA: Now I'm wondering if doing this is the right idea. … Just because there's a lot of tangent going on and I don't know what to do with it.
>
> INTERVIEWER: So, is it that you just don't see it simplifying down, in a sense?
>
> MARIA: Right.

As Maria manipulated the expression (Figure 7), she expected the sheer volume of the tangent function to be lessened.

> MARIA: Originally I wanted all of the tangent combined into one. … Into just like one term. And that wasn't going to happen. … I mean at first, I was honestly just trying to get rid of the tangents. … Cuz they tend to be a bit more complicated then sines and cosines.

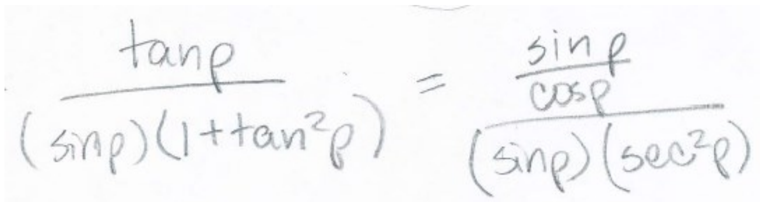


Figure 7. Maria and too many tangent functions.

Not only were the number of tangent functions not reducing, but tangent represented a function that Maria found more complicating in comparison to the sine and cosine functions. Thus, she wanted to remove it from the expression in order to simplify the expression. Interestingly, even after initially searching the common identity sheet, Maria did not immediately use the Pythagorean identity to remove $1 + \tan^2 \rho$ from the expression, reducing the number of tangent functions. Only after substituting in $\sec^2 \rho$ did she realize it was a plausible action due to the reciprocal nature of the secant and cosine functions. Perhaps the visual clutter of too many tangent functions and the cognitive clutter of the uncomfortable tangent function obscured her vision of the VTI path.

In discussing her preferred strategies in VTI, Amber commented on her tendencies to keep the expressions in simple forms.

> AMBER: The more simple I can keep it, the better. … Because, yeah, keep it simple, stupid. Um, um, because of the fact that once it gets huge, there's more stuff to keep track of.

Amber acknowledged the cognitive difficulty of having to manage a complex situation. She further explained how she hopped around in her work, operating on small pieces, in an attempt to maintain her focus and not be overwhelmed by the enormity of the expression. The ability to simplify was used as a cue for when to hop.

> AMBER: I mean, that's just kind of the way it works, if I can simplify stuff, I'll simplify it. And then I'll keep simplifying it until I can't do anything more with it. And then go

> back to, okay, this goes here in the problem and then go back and start, is there anything else that can be simplified?

While she was speaking about solving mathematics problems in general, Amber explained how this process played out in VTI.

> AMBER: Um, I've also noticed that if I'm, feel like I can't move on one side, I'll start working on the other side, even if they're eventually I see something later and, oh, I can actually do this, and start back on the other side. So, I mean, if I can't move on one side, I'll go to the other. And then if I can't move on that side, then I'll go back to the other side and s-, sometimes see something else I could do.

Overall, to explain the flow of VTI for Amber, she picked a starting expression by identifying the most complicated expression as she believed it easier to simplify down than start with the simpler expression. She then used simplifying actions until she became stuck or the expression was in simplest terms. At that point, she moved to the expression on the other side of the equation and repeated the process of simplification. She cycled through the process until she believed she had verified the identity. Through this process, the notion of simplifying played a controlling role.

The Technique of Dodging

The word "dodge" may typically evoke a notion of evasion through deceit. However, "dodge" does hold slightly different connotations. As a verb, dodge can mean to minimize or reduce the intensity of a presentation ("Dodge, v.", 1976); as a noun, dodge can suggest a technique to effect an end with increased effectiveness ("Dodge, n.", 1976). Through these meanings then, the term *dodging* will be taken to mean "an action during VTI that a student took to marginalize a feature of the problem in order to enhance their chances of a successful VTI conclusion." Dodging was a technique used by students to simplify VTI problems.

Default to *x*. Students dodged using two specific techniques. The first technique was *Default to x*. Defaulting to x occurred when the arguments of the trigonometric functions forming the expression were anything other than the variable x. In these instances, when a student defaulted to *x*, he or she, upon seeing the function's argument, either mentally saw an *x* in the argument's place, physically wrote an *x* while writing the VTI solution, or verbally spoke *x* when discussing the solution. The default to *x* did not need to be a purposeful effort by the student as demonstrated by Helen.

> HELEN: And one over sine squared *x* is cosecant squared theta. Er, I mixed *x* and theta.

An abstraction to defaulting to *x* was when students were able to view a complicated function argument as a single entity without specifically using an *x*; in other words, students viewed the argument as a single object.

When pressed for a variable, students generally favored *x*. However, if a student viewed the function argument as a single object, the student was said to default to *x*; the variable *x* did not need to be explicit. In fact, many of the students referred to mentally viewing the argument as an object or replacing it with an *x*. For example, Maria confessed to this while solving VTI4.

> INTERVIEWER: How do you see that in your head? Do you see a two alpha minus one?
>
> MARIA: It's *x*.
>
> INTERVIEWER: Do you, so in your head, do you, do you see an *x*?
>
> MARIA: So, everything in the parentheses is just an *x*.

Pressed further, Maria admitted to defaulting to *x*, on a previous problem, VTI2. In that problem, the function argument was the symbol θ; while thinking *x*, she actually wrote θ when writing her solution.

***The preferred variable,* x.** During the interviews, most of the participants demonstrated an inordinate preference for the letter x. After accidentally writing an *x* instead of a *y* for the function argument, Katie explained her slipup.

> KATIE: I always have *x* in my head, so I'm bad about writing *x*. … I would rather just write *x* just cuz it's natural for me to write *x*. … I'll just write *x* without, like even thinking.

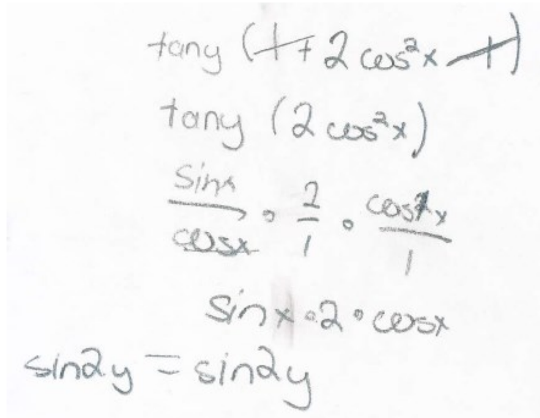


Figure 8. Bella defaulting to *x* on VTI6.

This phenomenon was not unique to Katie. Several of the other participants also described a reliance on *x* as the variable of choice. For Bella, *x* creeped into work unconsciously on VTI6 (Figure 8).

> BELLA: I'm always used to using *x*'s when I get in a hurry or when I'm really thinking about something.

Thus, when she was distracted by other aspects of a problem, her mind drifted to her default variable. When presented with a choice, the variable *x* was the first in line for Maria.

> MARIA: *x* is just the first thing that pops into my head whenever there's any variable to look for.

Finally, Amber had a violent and humorous interjection to the use of the Greek letter rho as the variable in the function argument in a problem.

> AMBER: Dangit! Why can't you use *x*?

Thus, given their druthers, students would have every variable be an *x*.

***Origin of preference for* x.** Students holding *x* as the variable of choice had their preferences rooted in their prior education; in a sense, *x* was the universal variable in mathematics. For Cooper, the comfort with the variable *x* derived from extensive experience and its usage in the introduction of the variable concept.

> INTERVIEWER: Okay. Why *x*? Why not *y* or *w* or *p*?

COOPER: We've always used x.

INTERVIEWER: So we, is that algebra class?

COOPER: It's fourth, fifth grade, you know. Everything's just kind of always been in terms of x and. You, you, that's just usually what you always find. You know, the original first problems are like, you know, x plus three equals two.

Thus, x was the universal variable due to its curricular primacy; it was the first variable learned. Using the variable x seemed to take Cooper back to those childhood days when life was much easier and filled with mud puddles to stomp in instead of complicated equations to solve.

This domination by x continued to higher levels of schooling. Bella described the usage of x as a continuing process.

BELLA: It's just what has always been in an equation before. Like, throughout my high school career, it's always been x something equals this, this, and this.

Moreover, Amber implied that her high school teacher played a role in her preference for x.

AMBER: My math teacher that I actually had for eighth, ninth, and tenth grade, eh, he told us like if we saw a variable in the book that wasn't x, just, you could change to x if you wanted to. … So, I mean, that's where my kind of love for x I guess you could say would, yeah, kind of stems from. … So, that's kind of where I'm like automatically kind of geared towards oh, let's just use x.

Thus, due to early educational experiences, students developed a strong preference for variable x. Using the variable x allowed students to return to a comfortable setting, using a symbol very familiar to them, simplifying the situation for them.

***Rationale for defaulting to* x.** While somewhat natural, students believed defaulting to x was a mathematically legitimate technique for solving the problems. This legitimacy was due to the marginalized nature of not only the choice of letter for the variable, but the function argument itself in the VTI process. While defaulting to x, students removed the emphasis from the symbol being used, instead viewing the argument as just some object. Alan explained this view of the argument when solving VTI4. In this verification, the function argument was $2\alpha - 1$. Referring to that argument, Alan observed the following.

ALAN: You could think of that as just one thing.

Maria expressed a similar sentiment while working the same problem.

MARIA: What is in the parentheses is just one term.

Amber went further with her marginalization of the argument.

AMBER: You aren't going to do anything with it. It's just kind of there for the problem.

With these comments, students indicated that the argument did not matter in that particular situation; Amber acknowledged that the argument was there for purposes of being mathematically correct. Bella expanded the object notion of the argument and addressed the arbitrary choice of the letter used for the variable.

BELLA: You don't really have to know what it is to understand how the problem works. … The whole sine squared *x* or whatever just represents one number. It doesn't matter what, whether it's *x* or theta or gamma or sigma or anything like that. It's still sine.

The choice of the letter or symbol had no effect on the function itself. Additionally, Bella indicated viewing not just the argument as an object, but the function itself as an object. The emphasis in VTI was on the function, not the variable. Amber expounded why the argument could be marginalized, especially in VTI.

AMBER: You're not actually verifying that tangent *y* plus cotangent *y* over cosecant *y* times secant *y*, um, is equal to one because of *y*. You're not thinking it's because of that variable. You're thinking of, in terms of the trig functions. … As long as every variable in that expression or equation is the same, then it sh-, it's going to be true if you verify it.

Thus, the veracity of the identity had no dependence on the choice of letter in the argument or the argument itself. What mattered were the functions involved in the identity. In a sense, by defaulting to *x*, students were acknowledging the nature of an identity, that it was true for *all* inputs from the domain; identities did not have a conditional nature.

***Benefits in defaulting to* x.** Students indicated real benefits in defaulting to *x*. While reviewing a student's attempt at verifying an identity in VTI10, Cooper defaulted to *x*. The identity being verified involved the variable λ as the function argument. This choice of letter became a confounding issue for Cooper.

> COOPER: I'm just going to use *x* cuz that's, that works. It's going crazy. … They make things a little more, just, more of a, well, mouthful I guess you could say. So trying to remember it all in terms of *x* kind of makes it simpler.

Thus, for Cooper, the variable *x* was simpler than the variable λ. Cooper explained why he found the choice of variables complicated and why he used *x*.

> COOPER: It's simpler is because the, the, the amount of time it takes to say *x*, you know. It rolls off the tongue, *x*. And then you got lambda, which is kind of a time consuming. … And you know, it's very quick, not really any thought to get across to it. And sometimes, I mean, it might throw it off, somebody, if they're sitting there trying to remember the name for this Greek letter. And they forget that it's just a placeholder, … as long as they're not a constant, they have an actual, you know, an, like an actual amount that they're, in, in there for. Then, they're, they're, you can change them.

The simplicity in *x* came from knowing what it was; it did not confound the issue by requiring one to remember the name of the letter. Thus, defaulting to *x* removed a superficial but, from the student's point of view, real barrier, not knowing the name of the symbol. Using somewhat complicated symbols posed real cognitive barriers for students. Alan and Katie also indicated similarly superficial reasons for using *x*; Alan commented that writing *x* simplified writing the symbol of the argument, while Katie mentioned that *x* was easier to say.

Beyond simplifying the surface features of the variable, students believed defaulting to *x* simplified the actual VTI problem. Maria and Bella both addressed this aspect. For example, when asked if she physically wrote a complicated argument, such as that found in VTI4, as an *x*, Bella explained the situations that would cause her to do so.

> BELLA: I do occasionally. But sometimes, if it's just like something like this simple of an identity, I don't. But if it gets really complicated, then I would.

Thus, defaulting the argument to *x* simplified a complicated identity, allowing her to more easily solve the problem.

In Cooper's response, he mentioned unnecessary cognitive barriers that students may encounter with an unfamiliar symbol, commenting that these students may forget that the symbol being used had no bearing on the verification of the identity. This issue was brought up by Katie as well in her explanation of VTI7. Not initially recognizing the interchangeability of the symbols created difficulties for Katie as she solved the problem.

> KATIE: I know these identities, I know them as *x*. … That's just how it's in my head, is *x*. So then when I see this, I kind of have to reali-, realize that that rho is the same thing as an *x*. … I should probably know that rho is equal to *x*, er, that rho and *x* are interchangeable.

Katie had the basic identities memorized with *x* as the default argument. In some instances, she needed to consciously change a function argument to an *x* in order to match the expression to the corresponding one she had memorized. Katie also used this technique to reveal to herself that she could use certain identities. For example, while solving VTI4 (Figure 9), Katie manipulated a Pythagorean identity off to the side of her main body of work.

> KATIE: Um, well, I was just trying to, like, as a side note, I guess I would just say one minus cosine squared *x* equals si-, sine, equals sine squared *x*. Ehm, so that just shows me, like, ah, *x* is just my general, eh, variable I use, I guess, just showing me that one minus cosine squared *x* equals sine squared *x*. But then when I used it in an equation, then I would use, I would use *x* as what was actually written.

Figure 9. Katie's scratch work on VTI4.

Cooper experienced a barrier similar to Katie's. Defaulting to *x* was a necessary explicit step in order to recognize identities and solve the problem. While solving VTI6, Cooper defaulted to *x* throughout the problem.

> COOPER: Whenever I don't see *x*, I kind of ignore the identities for a moment until I look at it and go, oh, it's the same thing. It's the same thing as saying cosine two *x* or cosine two *y*. You know, it's the same idea.

For Cooper, the notion of *x* as the universal variable controlled his actions. Without consciously reminding himself of the equivalence of the symbols, actions became invisible. However, after acknowledging the interchangeability, Cooper fluidly moved between different symbols as the argument. In a similar way to how he worked VTI6, Cooper ended up defaulting to *x* throughout VTI7; however, he initially had a mixture of variables. He described manipulating the right side of the equation.

> COOPER: Um, so that's cosine over sine. And then the, the tangent, I'm going to do the same thing with that, go ahead and turn that to sine over cosine, *x*. Er, yeah, I need to go ahead and change that one over sine *p* to *x* so I can keep them all the same, the variables looking.

Thus, Cooper wrote most of the variables as *x*, realized one of them was written as a *p*, and so went back, erased the *p*, and replaced it with an *x* (Figure 10). He did this act without much comment; he viewed the symbols as being interchangeable.

At the conclusion of the verification, having manipulated the expression on the right side to a $\cos x$, Cooper made one final substitution.

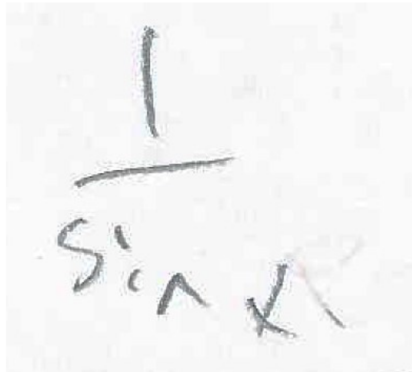


Figure 10. Cooper erasing *p* and changing it to *x*.

COOPER: And that just leaves me with cosine *x*. So, cosine *x* is what I get. And then, you know, I, I see that I can, you know, *p*, just switch out the *p*. And then that does equal cosine *p*. [Garble] just different variables.

Thus, without much fanfare, Cooper seamlessly switched the *x* back to *p* (Figure 11), highlighting the marginalized nature of the argument in this problem. Of interest was that Cooper did feel compelled to exactly visually match the end expression on the right side to the target expression on the left side; this action probably stemmed from Cooper's belief about how VTI was accomplished.

Figure 11. Cooper switching back to variable *p*.

While defaulting to *x* was necessary for some students in order to make connections to identities, students also defaulted to *x* in order to note similarities that aided their problem solving efforts. For example, VTI4 was the same identity as VTI2 with a change in the function argument; VTI2 used a θ while VTI4 used $2\alpha - 1$ for the argument. After defaulting to *x*, most students commented on the similarities of the problems. For example, Alan commented on the initial doubts he had while solving VTI4.

ALAN: Well, just from looking at it off the bat, it's pretty much the same as the one just before. The only difference is that there are multiple things that cosine squared is in the parentheses. Um, so pretty much just the first thing I think of when I see this is would I be able to do the same things if it was just one? Um, and as far as I know, um, it's pretty much, you can do it the same way as before.

Treating the complicated function argument as a single object allowed Alan to match the identity to one he previously verified. From that point, he followed the same verification steps used on the previous problem. Several other students experience this same phenomenon while solving VTI4.

BELLA: I realized that there was the exact same thing over there. And there wasn't like a theta or an *x* or anything. Like this, this little number is theta. … It just clicked in my head that there wasn't like a theta right there or anything like that. It was, this whole thing equals theta.

MARIA: Well, it's the same relationship as in the previous problem. … Uh, because, what is in the parentheses is just one term.

Defaulting to *x* allowed students to match the identity to previously verified identities; thus, students were able to simplify the problem for themselves.

This simplification process went beyond merely matching expressions to established identities. Katie indicated that the more complex argument initially was a barrier.

KATIE: I picture this as just one thing, as this could be in parentheses, just like that is. … It was the same things as the problem before, just instead of theta, it was two alpha minus one. … I didn't recognize that at first. I mean, just cuz there's more, more going on.

Defaulting to *x* removed visual clutter within the expression. The clutter of the argument played a role in how Amber proceed in VTI4. She began by manipulating the expression on the right side of the equality.

AMBER: I was looking at the left hand side and while it's basically one minus cosine squared alpha, essen-, if you take out the two alpha minus one, I mean, it's basically alpha, if you, or x, if you let x equal to x, two alpha minus one, which is a s-, the Pythagorean identity is just my, in my mind I guess it looks more complicated than that. Like at first glance, it looks more complicated.

Before defaulting to x, the sheer size of the argument caused Amber to shy away from the left side as she initially did not recognize the Pythagorean identity, an identity with which Amber indicated she was intimately familiar and preferred. Not recognizing the identity induced Amber to begin manipulating the right expression with a reciprocal identity, another identity familiar to Amber. Therefore, Amber and many of the other students relied on defaulting to x in order to reveal hidden structures, allowing them to note similarities to other verification problems and known identities. Writing or thinking of the expressions with variable x brought the problem back to a comfortable situation for them, utilizing a comfortable variable, one in use since the students were introduced to the concept of variable in middle school.

An extreme version of defaulting to x occurred as Charles attempted to verify VTI4 and encountered difficulties. His behavior was probably influenced by a supplemental homework activity. On the supplement *Algebraic Simplification*, in order to assist students with their algebraic manipulations on trigonometric functions and help them view functions as objects, students completed some exercises in which they replaced functions with letters and then performed the manipulation. For example, to reduce the expression,

$$\frac{\sin^2 x}{\sin x},$$

the suggestion was made to replace $\sin x$ with the letter a and then simplify it in the following way,

$$\frac{a^2}{a} = a.$$

With this activity in his background, upon reaching a barrier in VTI4 concerning the functions involved, Charles commented on his desire to bring terms down to simpler forms.

> CHARLES: I like thinking of it as like you know, you put instead of like, you make it more algebra, so you put like one minus *x* squared, try and think of it as an algebra equation instead of like an identity type of equation.
>
> INTERVIEWER: So, you can visualize and see it as one minus *x* squared?
>
> CHARLES: Usually when I see something like this, I try and stay away from what I don't really know, which is, you know, the identities, and go to like *x*, or, substitute, like, a letter in there.

Thus, when Charles encountered something he believed he did not understand, such as trigonometric functions and identities, he desired to be in a more comfortable, simpler environment. He defaulted the functions to *x* in an attempt to remove himself to an algebraic environment rather than the perceived more complex trigonometric environment.

Omit *x*. The second manner in which student dodged the function argument was by *Omitting x*. When students omitted *x*, they mentally suppressed the function argument, did not write the function's argument while writing the VTI solution, or did not verbalize the function's argument when explaining the solution. For example, while solving VTI4 and explaining his actions, Alan verbally omitted *x*.

> ALAN: It's pretty much just one over, you take the one minus cosine squared, you change that to sine squared.

Rather than vocalizing the argument, $2\alpha - 1$, Alan just spoke the function names.

In a similar way to defaulting to *x*, omitting *x* was a simplifying action. While solving VTI2 and explaining his thought process, Cooper omitted *x* (Figure 12).

> COOPER: I like to just ignore that one unless there's multiple variables in the situation because it's, it's something else that you're trying to keep in your mind, but it really

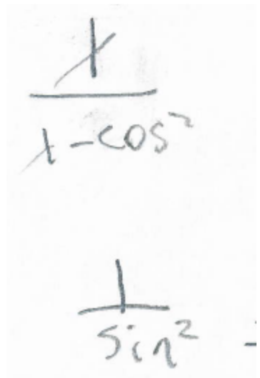


Figure 12. Cooper omitting *x* on VTI2.

> doesn't play a whole lot of importance unless there's multiple variables in it. And as long as you put it in your final answer, it does, it, it makes, I mean, it's still right. It's just that I like, the less in my mind is the less to think about. … It makes the problem simpler sounding.

Omitting the argument simplified the problem for Cooper. In his opinion, as long as the argument did not affect which identities to use, he could omit the argument without repercussion to the actual solution as the argument did not affect the identities to use for this particular VTI problem. At face value, omitting *x* appeared to be a conceptual problem regarding students' conceptions of functions. However, Cooper fully understood that the argument was an important aspect of the function.

> COOPER: It's not holding up too much importance for me at the, at that point in time. It's, it's more of a side note that's not really too important. It's, it's, it's still important, don't get me wrong. It's just, whenever it, it's less to think about, so it makes it kind of a less, more complex problem.

While Amber did not exhibit the phenomenon of omitting *x* during her interview, upon prompting, she confessed to mentally omitting *x*.

> AMBER: You get so caught up in the sines and cosine that sometimes the *x* just kind of disappears in your head. Because you're thinking, uh, you're, you're, cuz you're, all the

identities, like the *x* is, is basically saying there's a variable there. And it, I mean, and you can't have a trig function without a variable. … I've caught myself thinking like, tangent squared plus cotangent squared is equal to this. Or sine, or sine squared plus cosine squared is equal to one. I'm not thinking of cosine square *x*.

Thus, even though students may not have spoken the argument or written the argument, they did recognize that the argument was important to the notion of function. However, for purposes of VTI, students believed the argument could be marginalized by omitting it.

Omitting *x* did not occur merely to simplify the speaking process or because it could be marginalized for the VTI problem. The technique also aided the problem solving process. While solving VTI4, Cooper omitted *x* in a unique way. While speaking "one minus cosine squared," Cooper drew a box around the same expression on the paper, effectively cutting out the function argument $2\alpha - 1$ (Figure 13). Cooper explained why he did this.

COOPER: It gives it a focal point. … So, I mean, I, I, I'm kind of a visual person when it comes to math. I like to make sure I have like, when I put a box around it, I immediately look at that and go, okay, this is what I'm trying to remember to work with.

Thus, Cooper was able to focus on what he believed to be the important part of the expression, the function. Doing so served a practical purpose. Initially, the argument confused Cooper.

COOPER: I was like angle sum identity or something like that. Wherever, you know, you got sine of something, sine of parentheses *x* plus *y*, or something like that. It kind of made me think about that. It did, cuz there was, it did kind of throw me off for a second. But, I realized that both sides had the, had the same thing and that it really wasn't any much of a effect of the actual, original, what you're trying to verify.

Thus, marginalizing the argument through its omission clarified the situation for him, simplifying it to the point where he recognized he could use a Pythagorean identity.

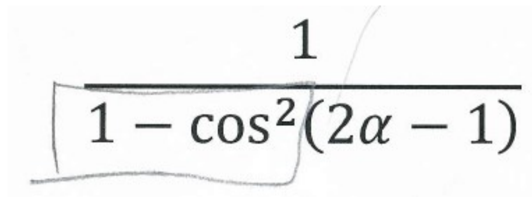


Figure 13. Cooper omitting *x* by boxing the function name.

While he was marginally successful in his efforts, Charles needed to omit *x* in order to reduce the confusion of the argument in VTI4.

> CHARLES: Um, I think that it, that I would have to distribute first. You know, like, but I don't know how. Like, I wouldn't know what to do for this. Like, I see, like the one over one minus cosine, like we did. And then the cosecant.
>
> INTERVIEWER: So, are you saying it looks similar to the one we just did?
>
> CHARLES: Yeah.

Charles's natural tendency was to treat the function name symbol, *cos*, as an object to distribute through the parentheses of the argument. However, by omitting *x*, he noted that the problem resembled the form of problem VTI2. Doing so created cognitive dissonance for Charles, causing him to hesitate before committing a grievous mathematical error. Although he needed a bit of coaching to fully understand that the complex argument in VTI4 was like the simpler argument in VTI2, thus leading him to verify the identity in VTI4 based upon his verification in VTI2, initially omitting *x* planted the seeds of doubt concerning his potential "malstep."

Students marginalized the argument as it remained an abstraction in the VTI problem. For solving VTI problems, only the functions mattered. Thus, in a sense, students focused their attention on the function, viewing the functions as objects to be operated on. Furthermore, the veracity of an identity depended upon what functions were involved and not the arguments; after all, identities were valid for all input values. Hence, beyond domain, identities had no dependence on the argument values. Cooper expanded this idea.

COOPER: You can always just replace it with *x* if you want to. … It's just not to really focus on that. Cuz the main focus of the problem is what's next to it, like trying to get that idea. … It's just kind of a placeholder. You can replace it with *x*, *y*, *z*, *q*, *p*, if you want. It's just a placeholder. … Unless it's actually a constant. And then, then they have meaning.

Therefore, dodging was an important technique for some students, serving to simplify their situation in order to solve the problem.

The two dodging techniques led to the same end, marginalizing the function argument for the student. At times, students mixed both techniques. While solving VTI4 (Figure 14), Helen encountered the same issue that Cooper did, initially believing the form of the argument would necessitate using a difference identity. This initial belief caused a lot of hesitation while solving the problem.

INTERVIEWER: So what was, what was bothering you about this problem?

HELEN: Because I was thinking that I was going to have to do one of those huge ones of. I'm thinking of a different chapter that we just finished doing. That I didn't like. That was like alpha, like cosine and then alpha minus beta. … I hadn't seen that as, okay, as in, cuz I was working the left side, and I saw one minus cosine squared equals, in my mind, sine squared. But then when I realized I'm making it a lot more complicated than it is.

INTERVIEWER: So I noticed when you're saying it, it's, it's written one minus cosine squared of two alpha minus one. But I noted you're saying one minus cosine squared equals sine squared, and you're not really saying the two alpha minus one.

HELEN: Um, yeah because, I'm not mentioning that because I didn't really do anything with it. … To me, like, whenever, yeah, it's true. Whenever there's like, uh, parentheses with kind of, things are inserted that just look a little weird, I try, which is probably not

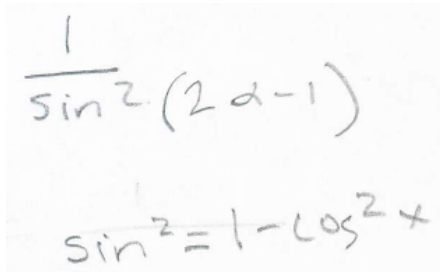


Figure 14. Helen dodging in scratch work on VTI4.

> good, but to ignore it just because it can get me a lot more mixed up than I need to be. Um, so since I realized that nothing has anything to do with it. Because, like here, I didn't even incorporate it in the equation. … I just kind of plugged it in at the last minute, which I guess could be bad because I could forget about it. But it didn't have any meaning as to I didn't have to foil it out or anything.

By dodging the function argument, Helen was able to overcome the barrier presented by the complicated looking argument. By ignoring the argument, she could focus on the aspect of the expression that did matter for VTI. Thus, for Helen, the argument was marginalized to a high degree; it was a superficial feature of the problem, one that had no bearing on the problem as evidenced by her recognizing she could reinsert the proper argument at the end of her solution with no apparent ill effects toward her solution.

Therefore, dodging was a valuable simplifying technique employed by students during VTI. Through dodging, students marginalized an aspect of the expression that presented a complexity due to the physical size of the argument or due to uncomfortable variable choice. Dodging the argument allowed students to focus on the features of the expression that did matter for VTI, namely the functions. The students were then able to recognize proper identities to use or actions to take. In using this technique, students acknowledge the special property that identities had; identities were true for all input values, having no dependency on the choice of input. Therefore, in general, students did not need to worry about the argument, and it could be pushed aside.

Comfort Functions

The physical size of an expression did not always signal students to simplify the expression down to a smaller form. Some students considered certain trigonometric functions to be simpler than other functions. The functions typically named by students as easier functions were sine and cosine, although tangent was sometimes included in this listing; thus, sine, cosine, and tangent formed a family of "comfort" functions. That is, students felt comfortable working with those functions. Cotangent, secant, and cosecant were usually thought to be more complicated, forming a family of uncomfortable functions. Katie explained why she converted tangent using a quotient identity, involving sine and cosine, rather than a reciprocal identity, involving cotangent.

> KATIE: I'm more used to using sine and cosine, and it's more common to me, and so I think it's the first thing that comes to my head, and it's usually easier to use sine and cosine in a problem rather than one over cotangent.

Cooper related his first reaction to looking at an identity.

> INTERVIEWER: When you read it, what did you think about it?
>
> COOPER: Confusing. … Whenever you're throwing, whenever there's a lot of cosecants … I usually kind of, kind of, you know, make sure I [garble], take a step back and look at the bigger problem, whenever I see a cosecant or something like that. Cuz, there's a little more to them that's not in the sine. … You try to bring them back into the sines and cosines terms.

Both Katie and Cooper either avoided the uncomfortable functions or used identities to remove the uncomfortable functions.

A potential origin for comfort functions was alluded to by Katie.

KATIE: It's just like, that's one of the first things that, like sine and cosine. I can a-, I always know what those things can translate to or whatever.

She was used to working with the functions; therefore, she felt she understood them better and understood the identities associated with them better. Sine and cosine are usually the first trigonometric functions students encounter in a high school curriculum. In the college trigonometry class that formed the current study, the comfort functions were the initial trigonometric functions the student encountered. Thus, students attached special significance to the comfort functions due to their primacy in the curriculum and the volume of problems students worked involving the sine and cosine functions

Charles's comfort family was small, consisting of sine and cosine. He explained this family when addressing why he converted tangent using a quotient identity.

CHARLES: I usually have an easier time with sine and cosine. … They're easier to work with than like tangent.

INTERVIEWER: Why do you think they're easier to work with?

CHARLES: Um, cuz they usually work well together. Like I've noticed, uh, that sine and cosine generally work out a lot easier than like a tangent type thing. … Cuz I don't know much about tangent compared to sine and cosine problems.

Charles believed he understood his comfort functions better than the other trigonometric functions. This belief induced him to convert expressions via identities to expressions containing sine and cosine functions.

At times, based upon the form of the expression students were manipulating or the form of the target expression, students would strategically use identities to convert an expression to one composed of sine and cosine functions. While solving VTI7 (Figure 15), Bella's first step was to use reciprocal and quotient identities to convert the expression. In this instance, her motivation for doing so was predicated on the form of the target.

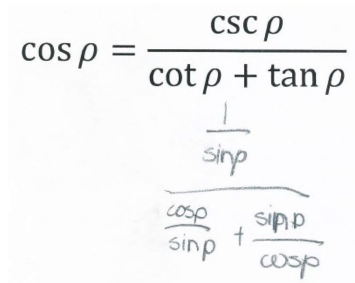


Figure 15. Bella manipulating the expression in VTI7 to sine and cosine functions.

BELLA: Like, this has a cosine in it. And there's not a single cosine in here. Then that's kind of a signal that you need to change some stuff around. But if this had like a tangent over here, or a cotangent or a cosecant, then I might not be so willing to change stuff around.

INTERVIEWER: Even if there were something over there on the left hand side, would there be situations where you would just say, ah, forget it, and change everything to sines and cosines?

BELLA: Mm-hm. Cuz if, I think it's a lot, you can do a lot easier cancellations, stuff like that, if you switch it over to cosines and sines. And if you had to, you could just switch it right back.

Even though she did not utilize the notion of comfort functions to motivate her choice, Bella acknowledged that she found operating with sine and cosine functions to be an easier task. More importantly, she indicated the fluidity existing among the functions. As a first step, she could manipulate the expression to be composed of sines and cosine, then she could perform her operations, and finally, she could use identities to introduce the appropriate functions in order to match the target expression.

Students recognized that many of the identities introduced in the curriculum involved sine and cosine. Alan explained some of the general strategies he employed in VTI.

ALAN: There are just some, I guess, some basic steps like the whole sine and cosine that's, those are the two big ones. Like most of the identities are broken up into sine and cosine, or you can break them up. … And it's I find to be the simplest way.

Maria and Charles expressed a similar reason for turning to the sine and cosine functions.

MARIA: There's a lot more ways to relate sine and cosine, usually, than there are to relate sine to tangent.

CHARLES: I think it's easier to find stuff on sine and cosine.

Thus, in addition to having been introduced to the comfort functions first, in addition to having worked many problems involving the comfort functions and therefore feeling a better understanding of them, students recognized that they knew more identities involving the comfort functions. Thus, through experience in solving VTI problems, they found working in terms of sine and cosine to be easier.

In defaulting to their comfort functions, student believed they achieved the desired simplified form. At times, returning an expression to comfort functions physically enlarged an expression. By using identities to change expressions into the comfort function family, students believed the problems became easier to understand and solve. While solving VTI2, Cooper commented on using a reciprocal identity.

COOPER: It helps to kind of put everything on the same table, same, same situation. Like, uh, cuz, I mean, since this is a cosine squared over here, um, I really don't want to deal with a cosecant something that can be changed to a simpler form. And, well I, it's not really a simpler form, it's kind of more of a conf-, like a di-, a bigger form of it. But it helps you, it makes it easier to see the same, the similarities between the problems. … Whenever I see that one, I like to make them a, kind of, this is one of the few times I like to make something bigger. It just kind of makes it a little more easier to get to that answer. Cuz, it, it, it does kind of make it a little, like, bigger, but it doesn't make it more confusing. It makes it easier to notice what it is.

Thus, while he connected the notion of simpler with the physical size, Cooper also explained the need for a bigger form to help solve the problem. Cooper temporarily allowed for enlargement of

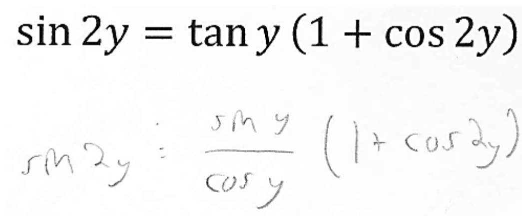


Figure 16. Alan's first step on VTI6.

the expression as he then believed this helped with solving the problem. This enlargement allowed him to understand the problem and note similarities within the expressions and match the functions. While changing the functions to the same family enlarged the perceived physical size of the expression, doing so appeared to decrease the cognitive size for Cooper, removing the number of trigonometric functions within the expressions.

> COOPER: Cuz you're looking at it, and whenever you look at the problem, you see on the left side, you have a one on top of something. Well, I mean, it makes it easier if you can look on the right side and go, well I know there's, you can make a one on that side above it. … It helps building up a similarity whenever you can have a basic similar-, similarity like that. Then you can kind of flow through it a little bit better.

Thus, reducing the number of functions allowed for Cooper to more easily match up his working expression with the target expression.

While solving VTI6 (Figure 16), Alan's first step was to transform the tangent function using a quotient identity. Alan explained his rationale for making this substitution.

> ALAN: Since there's sine and cosine and a tangent, um, I usually just like to get rid of the tangent and get it all as sines and cosines. … It'd be easier just to have, it'd make it more, a little less variation in what's in there and kind of, that might help make it a little bit more uniform in trying to find a way to, yeah, kind of simplify it a little bit more, and just get sines and cosines.

By converting to sines and cosines, Alan reduced the size of the expression on the right in the equation. He believed having fewer functions would be easier for him in solving the problem.

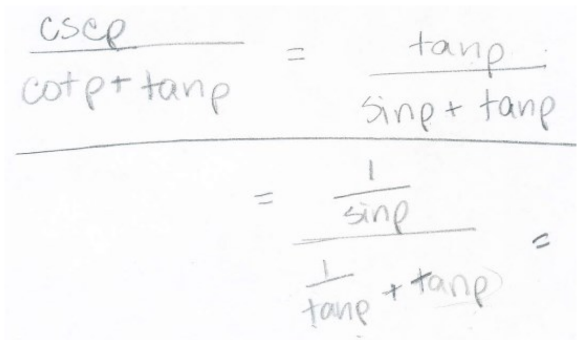


Figure 17. Maria's first step on VTI7, converting to simpler functions.

Similar to Cooper, Alan was able to reduce the cognitive load.

While students used the physical size of an expression to indicate that it entailed "more to work with" and was thus more complicated, the complication was also extended to the types of functions within the expression. Because of this, students also used comfort functions to pick their starting expression. Maria expressed this notion when explaining why she chose the expression on the right side in VTI7 (Figure 17).

> MARIA: In my brain, it's always easier to deal with sines, cosines, and tangents than it is to deal with the, like, the cotangent and cosecant, or secant. … I've always seen those as like the more basic ones, I guess. … So I always try to get everything there and then start trying to solve it. … Try to get them to sines, cosines, and tangents.

Maria did not choose the right side because it contained the "simpler" functions, namely, sine, cosine, and tangent. She chose the right side because it did not contain the simpler functions. It contained what she considered to be more complex functions. Then, as a first step, after a false start due to incorrect algebra, she used identities to convert the expression from one containing complex functions to one containing simple functions. Thus, the notion of simplifying extended to function types and directed the manner in which Maria picked her starting side.

Simplification played many important roles in VTI for students. As a process, some students viewed the flow of VTI problems in terms of simplifying an expression. This view guided actions students took, such as choosing a starting expression, and acted as a monitor for whether or not a particular action would be beneficial in solving the problem. Generally, students preferred actions that condensed the expression down to the target expression. Simplification also described specific problem solving actions that students took. Dodging, which simplified problems by marginalizing parts of the expression, allowed for students to overcome cognitive barriers and realize the appropriate actions to take. By marginalizing the function arguments, students decreased the physical or cognitive size of the expressions they manipulated. With the removal of the visual or cognitive clutter, they could see which identities to use or match the current problem to known solutions. Finally, reducing expressions to comfortable functions from the students' perspectives simplified the expressions in a sense. In rewriting expressions in terms of comfort functions, students believed they could better recognize which steps to take. Moving toward the comfort functions decreased the cognitive size of the expression, simplifying it, and aided students in construction the solution path.

VTI as Proof

As suggested by participant similes, one view students held of VTI was that VTI was a process of proof construction. By analyzing responses to supplemental homework prompts, students' views regarding the role VTI served in its capacity as a proof emerged. Additionally, through exploration of responses in the summative interviews, the manner in which successful VTI completion became more apparent. The following section describes how VTI served as a proof for the participants.

The Meaning of "Verify"

Exploration of students' notions of the concept of "verifying" occurred directly by asking students to consider the meaning of the word. During the second class period in the VTI unit, the instructor led a class-discussion on the meaning of the word "verify," posing the question to the class. Amber suggested that "verify" meant to "demonstrate." The instructor supported the response and added that when verifying, the students needed to show why the equation was true using manipulations, mentioning that showing their reasoning was important.

Prior to this discussion on the second day, students had returned the completed *Supplement: Verifying Identities* homework activity at the beginning of class. Students responded in writing to the following prompt located at the beginning of the activity: "In your own words, what does it mean to ***verify*** that A = B?" At the end of the assignment, students responded to a similar prompt: "When someone asks you to verify a trigonometric identity, what does this mean to you?" In the subsequent analysis, participant responses were classified according to the de Villiers Roles of Proof framework (1999). That is, a proof may verify, explain, discover, systematize, provide intellectual challenge, or communicate.

To begin, a couple of points need to be clarified. The word *show* appeared in several comments. This phenomenon was quite natural as in VTI, students were eventually writing something down, showing their work. Furthermore, although VTI could occur mentally, the mental activity would still show, or demonstrate, something to the prover. Conceivably, these statements indicating *showing* could be thought to indicate a form of communication and thus be coded as *proof as communication*. However, unless an external individual was explicitly mentioned in the responses, none was assumed for the purpose of coding; the prover could be referring to showing himself or herself. Coding for communication in this manner stays true to the description of communication as the dissemination of knowledge to others or the removal of doubt in others (persuading) (Harel & Sowder, 1998).

Another point pertains to students' abuse of mathematical terminology. Students frequently used the words *identity* and *equation* when referencing one of the equivalent expressions. For example, one student wrote, "To verify an identity, I am establishing that the different identities mean the same," while another student commented, "To prove that the equation is really equal to one another." In both instances, use of the word *expression* instead of *identities* or *equation* would make more sense. Assuming students meant *expression* is not a large leap and is well-founded based upon anecdotal experiences interacting with students.

With the first prompt, students were provided a general context to consider what verifying meant. Nine of the responses to the first prompt described verifying as a process of *Explanation*. To be considered Proof as Explanation, the response needed to indicate that the verification provided some measure of insight into the reason for the equality of the expressions. Common words to indicate this belief were *show how*, *show why*, and *explain*. For example, one student explained, "To verify that A=B you must show how A=B," while another student stated that verifying is to "show in steps why one equals the other." Other students responded that verifying involved *showing the steps* that demonstrate the equality, using phrases such as, "To show the steps that make A=B," or "To explain and show the steps of how I concluded that A=B." In these instances merely mentioning the word *show* was not sufficient for inclusion in the *Explanation* category; the responses needed to explicitly mention that the steps show the quality of sameness for the expressions, providing insight for the equality. While *show* carries a sense of "exhibition" or "demonstration" about it, unless the response indicated what was being demonstrated or shown, such as a *why* or a *how*, no assumption was made.

Responses coded as proof as *Verification* (seven responses) implied that verifying identities involved the *verification* of the identity. In other words, verifying an identity means that the identity is not a false identity. Statements indicating a verification process tended to focus on the truth of the proposed identity. Some of the participants explicitly allowed for the falsity of the identity. One student wrote, verifying is "to see if the value of A equals the value of B." With the word *if*, the student implied that the identity may not be true. For some students, verification was an examination process. Examples of this focus are "To test it's rightness," and "Check and make sure whats [sic] stated is true." Other statements focused on the essence of the equality; as one student explained, "To verify an equation, I am proving that the proposed equation is true." As a final example, a student responded that verifying was "to show that

A=B." This statement was coded literally; what was being exhibited was that the two quantities were equal, and so the statement was coded as a verification.

While the remaining nine responses could not be coded using the Roles of Proof framework, two of the responses indicated that verifying meant proving. Therefore, 18 of the 25 responses clearly indicated students equated verifying with proving. In analyzing the remaining seven responses, five of them were coded as *Not Coded* for not properly addressing the prompt. As the prompt requested the students' views on what *verify* meant, the responses should have contained an element of an activity or process since *verify* is a transitive verb. However, four responses focused on the equality aspect of the statement A=B. For example, one student wrote, "Quantity A is equal to quantity B." Perhaps these students misread the prompt in the sense of "What does it mean that A=B?" In other words, these responses were restating the statement to be proved; thus, no attempt was made to code the responses beyond this aspect. The fifth response not coded was nonsensical.

Even though the final two uncoded responses not properly fitting into the Roles of Proof framework may not have addressed a proof function, they did share a common theme and were of interest.

> Make the two #'s on each side of the equation true and equal (12Sp4)
>
> Simplify both sides until the value A is the same as the value B. (12Sp23)

While each contained an element of sameness of the expressions and coding the responses as indicating a proof role was tempting, the focus of the responses was not placed on determining the sameness or showing why the expressions were the same. Instead, each of these responses indicated how the student would accomplish showing that the two expressions were the same; the emphasis was placed on the actions. Thus, the responses did not focus on what verification accomplished but on how verification was achieved. The second response was particularly interesting as a specific action, *simplify*, was named.

The second prompt on the worksheet provided students a particular context from which to respond, that of trigonometric identities. While responses generally described VTI in terms of *verification* or *Explanation*, a subtle subtheme evolved across these categories, indicating actions to be taken in VTI. The criteria used to code the responses to the second prompt followed from that which developed through coding the responses to the first prompt with only two responses deemed not coded. One of these responses was an affective response to VTI ("A lot of homework problems to solve" (12Sp29)), while the other response focused solely on the mechanics of VTI and the usage of identities ("That I could use an identity that looks different to verify another" (12Sp22)). Interestingly, the responses for these participants to the first prompt were also not coded using the Roles of Proof framework.

Twelve responses indicated that VTI provided *Verification*. For example, one student replied that VTI meant "To check if they are equivalent to each other" (12Sp19). Thus, verifying involved making sure that the expressions described the same object. In a more descriptive statement, a student wrote, "To work out both sides of the equation until they are alike or no other steps can be taken to either prove a statement true or not" (12Sp16). In responses to the second prompt, students began indicating how the verification would be accomplished in more detail. For example, beyond just saying "using steps," this student spoke of operating in some fashion on the expressions.

Eleven responses evidenced the belief in VTI as *Explanation*. As one student explained, "It means that you must prove the trig identity by showing the reasons why one thing equals another" (12Sp14). A similar response was, "To show step by step how one trig identity is equal to another and to explain each step" (12Sp33). Again, some of the responses also showed how students believed they would accomplish the verification, for example, responding "It means to reduce the trig functions to lowest possible forms and figure out why they are identical" (12Sp10). For this student, VTI would be accomplished through the simplification of the expressions.

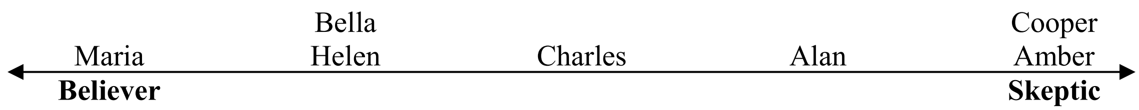


Figure 18. Spectrum of students' beliefs regarding the theorem identity.

The Veracity of the Theorem Identity

As one of the roles that a proof could play was the verification of the theorem, the proof of a theorem could serve to transform an individual's view of the theorem from a state of skepticism to a state of belief. Another role served by proof was that of explanation. In this instance, the proof clarified why a theorem could possibly be true. A proof could play both of these roles at the same time; an individual may not be convinced about the veracity of a claim until constructing the proof, and during the course of proving the claim, observed why the claim has to be true.

To explore the role VTI played for students in transforming their beliefs, during the summative interviews, participants were probed concerning their views regarding the theorem identity. Of interest was the students' approach to the theorem identity during VTI, that is, how students would treat the purported identity. Students could believe the identity to already be true; if this belief existed for the student, then VTI would necessarily serve a different role for the student than if the student was skeptical about the veracity of the theorem identity. If the student believed the theorem identity had the potential for being false, VTI could play a role in removing the skepticism the student had. Thus, the students' attitudes toward the theorem identity could limit the effect VTI had on the identity for the students.

Upon analysis of several points in each summative interview, the participants' beliefs appeared to span a spectrum of possibilities (Figure 18). These beliefs ranged from a firm belief in the veracity of the theorem identity prior to the VTI process on one end, the *Believers*, to doubt in the truth of the identity until VTI showed otherwise on the other end, the *Skeptics*. To best understand how the initial view of the theorem identity and VTI related, a discussion for each participant will occur. While students responded to direct questions regarding their views, these responses were triangulated with their responses to other questions in the interviews and to their responses in supplemental homework activities in order to define their beliefs.

Maria. Based upon all of her responses Maria seemed to be a staunch believer in the theorem identity. Prior to her engaging in VTI6, she believed the theorem identity to already exist as a legitimate identity. When asked to explain her view of the purported theorem identity prior to verifying it, she described a relationship that she saw.

> MARIA: Okay, so starting from the other side, I guess you could say that, uhm, you know, tangent of *y* and one plus cosine of two, er, two *y*, uh, they're kind of components of sine of two *y*.

Thus, Maria perceived the identity in terms of the relationship being described; she viewed the relational nature of the equal sign. After verifying the identity, Maria again described her view of the identity.

> INTERVIEWER: When you see a problem like that, are you ever skeptical about whether that equation is actually true or not. Or do you, like, oh, they're asking me to verify, so they've got to be true. And, um, so you see it as being a true equation before you actually show it?
>
> MARIA: Uh, if it's asking me to verify it, then I do see it as being true before I start. … So that usually gets me into the mentality of there is a way to do this, I just have to find it.

Maria viewed VTI as a process of finding the correct path. Already believing the theorem identity to be true, effectively holding the identity as empowered with equality prior to VTI, bolstered Maria's efforts in finding the equality path; she knew the path existed, so through persistence, she could find the connection between the expressions in the identity.

Her belief regarding the theorem identity influenced what impact a valid VTI construction could have. While reviewing the construction in VTI10, Maria commented on the series of intermediate equalities formed by appending the unchanged target expression to the manipulated expression via an equal sign on each step.

> INTERVIEWER: Do you think they really mean at this second step, do you think they're saying that these two quantities are equal? These two expressions are equal? Then on the third line, that these two expressions are equal?
>
> MARIA: Um. Well yeah. I mean, because, they're not ever, I guess, changing, uh, the actual value of what's written there. Cuz they're all, they're all equal to each other. Just a different way of writing it.
>
> INTERVIEWER: Okay. So, so these are all true equations, then. They're all equal to each other?
>
> MARIA: Um. Yeah. I would say so.
>
> INTERVIEWER: Why are they equal? Why can we say that they're all equal to each other?
>
> MARIA: Because we're using identities. … You know, we know, well we know that, you know, what we're given in identities is true. And as long as we stick to what we know is true, then, you know, applying it here would make this true.

Maria believed the purported identity to be true; it stated a relation between equivalent expressions. Therefore, applying legitimate algebraic manipulations or using valid, known identities would not affect the equality of the different expression being written. Since she started in a state of equality, she remained in a state of equality.

The source of her conviction regarding the theorem identities never became apparent through her responses in the interview. Although she believed that the point of VTI was to show why the expressions were equal, doing so was not as one student described VTI in simile, "orally consuming nothing but air when you're starving. Explaining something that is a known fact seems pointless." Maria perceived a value in VTI. Responding to the prompt on the *In-class*

Quiz: Verifying Identities supplement to explain the purpose of VTI in a trigonometry course, she made this perception clear.

> To learn our way around them so we can understand how they fit and work together to later apply them to realistic situations. (12Sp17)

Thus, Maria felt that VTI strengthened her understanding of identities. While VTI seemingly did not affect her view of the theorem identity, it did teach her more about the system of trigonometry. She also spoke to this belief when listing the perceived benefits of VTI during the interview.

> MARIA: Uh, just to help you see relationships between things, you know, that you know are related. But if you know, like I mean, not necessarily just in math. … Uh, but even, you know, in life, you know. When you know that two things are related, if, you know, learn that relationship really well, you can see how things change when you change one variable, or when you, um, just make one slight tweak to something, you know, it can change everything.

For Maria, VTI provided a conviction beyond trigonometry and mathematics.

Helen. Students tending toward the believer end of the spectrum perceived the theorem identity to be true before the application of VTI. These students consistently and clearly indicated a positive conviction regarding the veracity of the theorem identity. This belief existed for the students unless they were confronted with an enormity of proof to convince them otherwise. Thus, VTI would not serve a verifying role for them personally. In these instances, VTI usually served another role, such as explaining why the theorem identity had to be true through outlining the manipulation steps. Helen explained this notion.

> HELEN: Verify means that it’s already true, but you need to show how it’s true.

Thus, Helen perceived the command “verify” embedded in VTI to hold its literal meaning. Similar to Maria, this conviction helped Helen as she solved VTI problems. Originally, Helen considered herself a skeptic.

HELEN: I used to think that verify meant, like, prove that it's right, as in it could also be wrong. … But for the longest time, I thought that I would get to a crossroads and be like, oh, I can't go any further. So, especially at the beginning of my homework, I would like stop in the middle of it, um, and just, oh, I can't go any farther, so it must not be right.

As long as the possibility existed that the theorem identity could be false, Helen could not overcome certain barriers. However, her evolved view of what "verify" meant provided persistence for her during VTI constructions.

INTERVIEWER: Okay. So you're already taking it then as this is a true equality.

HELEN: Yes, so it has to equal it somehow.

INTERVIEWER: There's not a possibility it could be false. That there's not a possibility they're trying to trick me.

HELEN: Uh, yeah. And so, yeah, and so, therefore when I get stuck, I know that it's not the problem's fault. That means I need to reroute and try it a different way. … But I thought that proving something means that it could also be wrong. … If it was worded as in it could be wrong, I would give up a lot sooner. Because I'd be like, oh it's not working out, so it's just wrong, and so it's not an identity or whatever.

Helen's belief in the veracity of the theorem identity allowed her to focus on showing the steps of the VTI construction without having any lingering doubts. She knew that as long as she kept attempting the construction, eventually she would find the path.

Although Helen was a believer, she did not appear as fanatical of a believer as Maria seemed. That is, while Maria used the assumed equality of the theorem identity to establish the equality of expressions flowing from it, Helen hesitated to do so. In reviewing the hypothetical construction in VTI10, Helen appeared perturbed by the equality formed at each step by rewriting the unchanged expression next to the manipulated expression. While the format bothered her, she mentioned the issue of the legitimacy of the equality.

HELEN: Technically, I don't even think that that is, well right in a sense of writing like this is equal to this because right now you haven't proven that. I mean that's what the whole point of the equation is, is to prove that. So you're saying that these are equal, so wouldn't you have to prove that that equals that? … You have to show the previous steps because in the end, you see how that's true, but you sh-, show how this equals this first before this equals this.

Thus, despite believing the theorem identity was true, Helen still believed that the identity needed to be verified before using the equality to form implications. Then, once the identity was established, she could draw conclusions about the truth of these "intermediate" equalities formed. In a sense, Helen refrained from using a theorem until she had verified it for herself; she needed to see why it was true before accepting it into her system of identity theorems.

Bella. In discussing her solution to VTI 2, Bella expressed her belief in the theorem identity.

BELLA: Um, I'm not, I'm hardly ever skeptical that it's equal. I think it's more of the, I need to show that that, this, the one over one cosine squared theta equals cosecant squared theta.

INTERVIEWER: Okay. So, you see that you got to show that. But starting off, are you saying this is equal?

BELLA: Yeah.

Because Bella viewed the identity as already true, VTI for her did not involve a process of removing her doubt. She had none. Instead, VTI construction was a process of showing why the equality had to be an identity; it served an explanatory role.

Bella formed her conviction regarding the theorem identity due to her successful experiences in VTI. Prior to solving VTI7, Bella reiterated that she viewed the purported theorem identity as true. She explained how she came to her conviction regarding the theorem identity.

BELLA: I can see how one thing would lead to another, which would lead to another, which would lead to another. Which would end up solving the identity. … So I never really questioned whether or not it was actually equal. I just kind of assumed unless I proved otherwise.

INTERVIEWER: Did you ever get to a point in working problems where it didn't work out, or did it usually kind of work out for you?

BELLA: It usually worked out for me.

Thus, due to her past success in verifying identities, Bella did not usually question whether the theorem identity was true or not. Instead, the focus of VTI was on showing the manipulation steps. While Bella mentioned the possibility of proving the theorem identity otherwise, this sentiment differed from that held by the skeptics. Based upon their responses, the skeptics, when not seeing the correct path, would be skeptical or assume that the identity could be wrong. Bella would need a convincing argument to dissuade her from her belief about the identity.

Although she appeared fairly entrenched in her belief, she did not adhere to the dogma that Maria did, assuming the veracity of the theorem prior to VTI and then using this consequence to infer other result. Bella presented herself as a believer in the vein of Helen. VTI8 presented a hypothetical student construction in which the student "crossed the equal sign," multiplying both sides of the equation by the same quantity. Bella took issue with this approach based upon the grounds of VTI serving the role of verification. In her view, the theorem identity could not logically be treated as a true equality until it was established as an equality. Bella explained this concept.

BELLA: Like I said, you don't know if they're equal or not. You're supposed to show that they're equal. I mean, if they're equal, that's fine. If the [garble] says these are equal, then you can do that. … But you don't know that they're equal yet. … So, I was always taught not to do that.

INTERVIEWER: So, is your problem then that you were taught not to do that? … Or, is your problem in that, ooh, we haven't established they're equal yet, so we can't do that.

BELLA: Yeah, a little bit of both. More so the, the, you're, they're not equal yet, so you can't do that.

From this exchange, Bella appeared to be a skeptic, not believing that the identity was true. However, Bella did not state she personally believed the identity had a possibility to be wrong. While she believed in the identity, she needed to formally establish it as an identity before using the properties of the identity in a problem. She made this point clearer later in the interview while discussing VTI10.

BELLA: But I haven't shown that in my own head that they're equal. I'm trusting that they're equal.

She could believe in the equality prior to showing the equality. However, she did not feel empowered to use the consequences of the theorem identity until she verified it personally.

Bella further clarified this concept when she differentiated between the notions of "assert" and "assume" in the realm of VTI. Bella was prompted to speculate about the beliefs of a hypothetical student concerning intermediate equalities formed by writing the unchanged target expression for each step.

INTERVIEWER: Are they asserting that this equality is true when they write this line down?

BELLA: I think they're assuming that it's equal. But they can't say for sure because they haven't shown it's equal. But this is equal to this, so, if they're assuming that this is equal to this, then, then this would be assumed equal to this.

Thus, if a student assumed the equality of the identity and used it, then any equality that resulted would exist in a tenuous, unproven state. "Truth" claims stemming from the theorem identity prior to the theorem's verification were not valid.

INTERVIEWER: What's the difference between asserting that it's true or assume that it's true.

BELLA: Like this whole entire problem is asserting that it's true. This, this bottom right here. Cuz, so tangent this equals that. Or, this equals that. That's asserting. … But this is, until you get to this point, you're just assuming.

Therefore, prior to reaching the final construction step of VTI, the theorem identity remained in a tenuous, unproven state. However, once the VTI construction was completed successfully, the VTI construction acted as an assertion for the veracity of the theorem identity and empowered it to be used.

INTERVIEWER: This is speaking to when you do the problems as well, when you get to this bottom line where you write tangent squared lambda minus sine squared lambda equals tangent squared lambda minus sine squared lambda, does then that kind of retroactively say these are assertions? Or is it still assumptions? These, these previous steps.

BELLA: They're assertions, but only once you've proven that they are.

Once VTI was successful, the full implications of the theorem identity could be realized. Any intermediate equalities formed in the process were necessarily true assertions.

INTERVIEWER: So it's only until you've shown it, then you can say yes, then all of these things are true in fact.

BELLA: Mm-hm. Yep.

Therefore, VTI served a powerful role in a perceptual shift for Bella. While she assumed theorem identities were in fact true due to her successes in verifying them, claiming she was hardly skeptical that they could be false, if she were presented with evidence to not believe in a particular theorem identity, she would concede the falsehood. A successful VTI construction removed the seeds of doubt.

Perhaps a more apt description of Bella was that she was not staunch in her beliefs. Every theorem identity was not all-powerful until it was verified true. While she professed to not be skeptical, Helen appeared to view the theorem identity in a similar manner. In fact, when describing what VTI accomplished, Helen used language suggestive of VTI as establishing an equality.

> INTERVIEWER: To you, what does it mean that you've verified a trig identity?
>
> HELEN: Verified that they're equivalent to each other. … You're either breaking it down or condensing it to something smaller, um, to make sure that one equals the other.

For Helen, VTI provided some assurance about the equality; it "made sure" that the identity was true. Moreover, immediately after describing herself as a believer, Helen used the word "assume" in relation to the veracity of the identity.

> HELEN: Whenever I see the problem like to start with, like, I automatically assume it's true and I know that that seems like common sense, but, that was hard for me to realize.

For Bella and Helen, they approached a VTI problem with an assumption about the theorem, that it was true. VTI would then transform this assumption to an assertion. Once they could assert that the theorem was true, they were unencumbered in using it to create new mathematics, such as the formation of other equalities.

Exploring the meanings of the words "assume" and "assert" clarified the views of Helen and Bella. The word "assume" generally implies accepting or taking on without any other stipulations. Thus, when Bella and Helen assumed the theorem identity to be true, they were accepting its truth without proof. However, the acceptance had no real strength to it. Only until the identity was verified could they make use of it because the verification gave the identity teeth. The identity could now assert itself and be of use, with the word "assert" bearing the sense of forcefulness in it.

Charles. On the American science fiction television series *The X-Files*, the main protagonist, FBI agent Fox Mulder, had a poster hanging in his cubicle. Below a picture of a purported flying saucer were the words “I Want to Believe.” This phrase described Charles’s relationship with the theorem identity. While Bella’s success with her VTI constructions bolstered her conviction in the theorem identity, Charles’s inability to successfully construct verifications due to his failings with basic algebra inhibited and discouraged his beliefs.

While constructing a solution to VTI2, Charles rewrote the unchanged target expression alongside the manipulated expressions, placing an equal sign between the expressions.

> INTERVIEWER: Are you saying that this left hand side is equal to the, what’s on the right hand side? Is that what you’re saying by writing, with the equal sign there?
>
> CHARLES: Kind of. Yeah. Yeah, I, I mean, I think, like throughout the whole problem, I mean, I guess I think about it like before I even work out that that is an identity. So I’m trying to, um, figure out why. So it’s equal the whole time. And, like I think, that’s my reasoning for it.

Thus, Charles approached the VTI problem with the belief that the identity was true. The VTI construction showed the reasoning, explaining why the expressions were equal. He furthered this sentiment when he explained what verifying an identity meant to him.

> CHARLES: Um, that I got it right. That’s it’s done, like that that can be, you could substitute, you know, like the beginning of the equation in for the other side that was different, and it’d still be the same.

Therefore, Charles appeared to behave in the way that Helen and Bella did; once the identity was verified, he could use it to create new implications through its power of equality.

Furthermore, Charles needed to feel satisfied that the identity was actually verified before using it.

> CHARLES: If I didn’t get it right, then I wouldn’t risk putting it in something else. Cuz I’d probably get it wrong. So, I don’t know.

Charles was not empowered to use an unverified identity, feeling that doing so would lead to an incorrect result. He had no confidence in the identity prior to a successful verification. Thus, although Charles made an earlier claim that he believed in the veracity of the theorem identity prior to VTI, he actually did not always have this belief.

INTERVIEWER: Okay. So you come to it, and they say verify this equality is true. And you're coming to it assuming, well, this is true already.

CHARLES: Yeah. … And if, I mean, if I can't figure out why, then, then I assume it's wrong, or that I'm wrong, so.

Charles appeared to be a believer. However, upon being pressed, Charles revealed he was not quite at the end of the spectrum with the other believers. Because of his mathematical deficiencies, Charles never developed a confidence in his ability to verify identities. Thus, even though he claimed to believe the theorem identity was true, he allowed that he could prove the identity to be incorrect. As a result, his belief did not provide the comfort in problem solving the way that Helen's or Maria's beliefs did.

While solving both VTI6 and VTI7, Charles struggled with some of the algebraic manipulations. However, he eventually reached what he considered to be a successful result. For example, he described his reactions upon finishing VTI6.

INTERVIEWER: You got what you wanted on your last step. … To you, does that say anything about your preceding steps that you took? That you felt a little bit of hesitation about. Is this the right thing to do?

CHARLES: Um, I wouldn't, I mean, I see it, you know, I got, er, I got to the right answer, but I don't know if what I think is actually right.   .. So, it doesn't really make me feel better about what I've done.

INTERVIEWER: Okay. And do you feel this identity is verified then?

CHARLES: Yes, for me, yeah.

Charles described a similar reaction upon completing VTI7. Even though he believed he had verified the identity, the doubts he had about his algebraic work could not be assuaged by the supposed empowerment of the theorem identity.

Although Charles claimed to believe in the truth of the identity prior to VTI, he also mentioned the belief that the identity could be wrong. Either he was wrong, or the identity was wrong. As his algebra skills were poor, Charles must have remained in a tenuous state regarding the theorem identity. VTI could not remove any doubts he held as he had no faith in his ability to construct a legitimate argument. Moreover, even though he might reach a "successful" conclusion for the construction, he received no consolation regarding questionable algebraic manipulations, casting doubt on the whole process. Thus, his self-doubts severely limited the empowering of the identity, inhibiting him from using it to develop mathematical knowledge. Because he believed that the identity could be wrong and because he believed his verification could be wrong due to algebraic malsteps, he could not in good conscience use a "verified" identity in another problem for fear of creating an incorrect implication.

Alan. From his responses and the way he approached the VTI problems, Alan appeared to be a skeptic of the theorem identity, allowing for the possibilities that it could be true or false.

> INTERVIEWER: To you, what does it mean that you have verified an identity?
>
> ALAN: Um, to me, um, it's pretty much just to prove that what you, what had, you had in the beginning is actually true, like that is, those two sides are equal to each other. … So, that's pretty much what it means to verify, just uh, prove that what you had at the beginning was actually true.

Through the use of the word "actually," Alan seemed to indicate that he held doubts about the identity. Although, perhaps he viewed the identity as Bella did, with an assumption about the equality and thus VTI transformed the assumption into an assertion.

> INTERVIEWER: Okay. Um, what do you believe you have accomplished after verifying an identity?
>
> ALAN: Um, I felt, uh, well, besides that I have verified an identity, um, I guess to show that kind of either prove or disprove what is stated at the beginning. Um, I don't know,

that's kind of the way that I think about it, is trying to show whether or not that, whatever was said at the beginning was true or not.

However, upon further probing, Alan clarified that for him verification either proved or disproved the theorem identity. Thus, Alan approached the theorem identity with a bit of skepticism.

Alan's skepticism of an identity probably derived from whether or not he could immediately see a plausible path for verifying it. For example, relative to other identities in the interview, VTI2 could be considered easy in that verification could occur in two manipulation steps using very basic identities.

ALAN: Actually with this one, I was able to determine it was true from the very beginning. Um, just because what I know of, I was able to pretty much do that all in my head before, um, actually writing it out.

INTERVIEWER: Okay, so, you saw it, and immediately you were like, oh, of course, this is going to be one over sine squared, that's cosec-, so of course it's true.

ALAN: Yeah.

INTERVIEWER: Is that true for all identities that you see, when it says verify that, and they give you the equation?

ALAN: Oh, no. Um, this is, um, relatively simple I would guess. Um, I know that's pretty up to interpretation. … But, to me, this is a pretty easy one to do. Now, if it's a more complex one, or one that doesn't actually have really known identities just immediately that shine out to me, um, I would actually, most of the time I would actually have to go through it and write out the steps. But this particular one, I was able to see right away.

Thus, for identities perceived to have more complex verifications, Alan could not summarily proclaim it as true. Alan explained this thought after verifying VTI6.

INTERVIEWER: Okay, so there's, there was a little bit of skepticism in you?

ALAN: Yeah. Um, if it's not just completely obvious to me that it is true, I am, there is some, a little bit of skepticism.

However, even for the simpler ones, Alan described a mental verification process; he believed the identity true after noting plausible steps, showing the path that displayed the verification steps. Thus, when asked about his view of an identity, separating his view about its veracity was difficulty due to the possibility that his responses may have been colored by a mental verification performed on "easier" identities. Alan's discussion at the beginning of his attempt at VTI9 highlighted this possibility. He stated that he was skeptical about whether or not the theorem identity was true and explained his reasoning.

ALAN: Um, just because I'm trying to think of ways that I could actually, could go about doing it, but there's really, most of the ways I think of don't really seem to work out too well.

Therefore, Alan related skepticism about the identity to the ability to see the path to take. However, the fact that he allowed for the identity to possibly be false marked him as a skeptic.

Although he was a skeptic, Alan differed from Charles in that he was quite successful in verifying identities. Thus, he would not remain in a state of limbo regarding the status of the theorem identity. Because of arriving at a definite conclusion regarding the identity, he could gain conviction about the identity and use its implications. For example, Alan tended to write the unchanged target expression, relating it to the manipulated expression via an equal sign. In doing this step, he created statements of equality.

INTERVIEWER: At the beginning, you said you had some, you, you were holding some skepticism back about whether it's truthful. If you have that skepticism, do you have skepticism about all these other equalities th- you, that you have written.

ALAN: Well, yeah, um, until I, until I actually get what I, like what, get it to be equal, there's going to be a little bit of skepticism.

As a skeptic, Alan had no chance to begin with an assumed equality and build new intermediate equalities from it. Bella described the intermediate equalities as being assumed until transforming to assertions upon successful VTI. However, since Alan started with skepticism, the intermediate equalities he wrote carried the skepticism with them.

> ALAN: Each step I go, the, my skepticism about it being correct gets whittled down each time, until I actually show that they are equal. And then that kind of shows to me that all the steps before would be equal as well.

Once he successfully completed VTI, the skepticism or doubt about the equalities was removed. Therefore, VTI not only convinced him about the theorem identity, it showed him the truth about additional identities being formed through the VTI process.

Alan needed to be thoroughly convinced by his verifications before being empowered to use the identity to build new mathematical knowledge. One example of this phenomenon was not believing that a "successful" verification of an identity sanctioned questionable algebra. For example, while constructing the verification for VTI9, Alan doubted some of his algebraic manipulations of fractions, wondering aloud about possibly breaking up a fraction to achieve a desired result.

> INTERVIEWER: So suppose you encounter that, you know, like, ooh, can I do this? Well, if you take one path, yeah, well like maybe I can do this, and it gets you to where you're going, do you ever say, well that must be the right th-, that algebra must be correct if it's getting me to where I want to go?
>
> ALAN: Uh, no. If, something like this, um, I'm not exactly sure, I'm not just going to play with the algebra.
>
> INTERVIEWER: So, you're still ha-, uh, you're still a little bit reserved about, ooh, maybe this is still wrong?
>
> ALAN: Yeah, um. Because it might actually end up that this identity isn't true.

For Alan, because he did not begin the problem with a firm conviction about the veracity of the theorem identity, achieving a desired result did not remove the doubt he had about his algebraic skill. Thus, he could not use a successful result of the verification to sanction his "messing with the rules of algebra."

Cooper. While believers approached a theorem identity with the belief that it was correct before a successful application of VTI, some skeptics approach theorem identities in an inverse manner. For example, Cooper brought a healthy amount of skepticism to a VTI problem.

> COOPER: I'm not going, well, obviously, you know, this obviously equals this. … This is wrong until proven right, honestly, is kind of how I look at it. … I don't, like you know, sit there, and like, and try to keep, prove it wrong, you know. But there's, but I, I, I, I try to prove it right.

Thus, Cooper placed the theorem identity in a state of complete doubt; he then needed to build an argument in order to move the identity to a state of acceptance. Cooper continued to describe this argumentative process.

> COOPER: I'm not necessarily putting full, full heartedly believing into it. Nah, I'm, I'm sitting back going okay, well, I'm going to have to prove this to myself. Like an argument when you're arguing with someone, you're, they're trying to prove a point to you. You're trying to prove a point. So, you know, you're trying to prove something, you're going to prove this is a, and then when you do hit that end point, you go, okay, well, okay, it does.

Thus, Cooper believed a successful VTI outcome established the veracity of the theorem identity.

The removal of Cooper's skepticism extended to the intermediate equalities that some students formed in their VTI constructions. Cooper commented on the formation of intermediate equalities in the hypothetical student construction in VTI10.

INTERVIEWER: Do you think they're actually saying that on each step, cuz they have this equal sign that separates them. Do you think they're saying that's equal to that. That's equal to that. That's equal to that. At that point?

COOPER: Well, you, you might want to s-. Hmm. That's, I don't know what you'd, [garble], you might have to use maybe a different thing. But right now, I guess you could be saying, maybe like, put an equal question mark on it, if you would. I mean maybe that would be a better situation to put it. Cuz they, cuz you're not sure if they do equal each other right now.

Cooper noted a logical flaw in the equality structure formed in the student's construction, proposing the use of a different symbol to suggest that at that point in the written argument, the implication was not that the expressions were equal. Thus, the equal sign appeared to play a dynamic role. When initially written in the construction, it signaled another purported identity. For Cooper, the equal sign usage would be the same usage as in the theorem identity; it represented a proposed relational idea that needed to be proved. He amplified this notion when supposing the thought process of the hypothetical student.

COOPER: They're claiming that that's kind of what, what they're going off of, is that it's supposed to be equal. But, however, if they come to the end, and they go, okay, well that's not equal, so then, maybe go back and cross them out. Or just, you know, say that this does not equal, that these are not, you know, right.

INTERVIEWER: So if they come to the end and get something, does that say some-, er, if they come to the end and get something that's equal, like they do in this case.

COOPER: Then, yeah, these do, I'd say these do equal each other, each step of the way.

Therefore, a successful VTI construction empowered not only the theorem identity, but the intermediate equalities formed in the construction.

Amber. Amber was a fairly strong skeptic of the theorem identity.

INTERVIEWER: Do you consider it true?

AMBER: No. Because you're telling me to verify it. So there's a possibility that it's not.

INTERVIEWER: Okay. So there's, so you're saying there is some sort of hesitation or skepticism?

AMBER: Right, I mean, there's a lot of skepticism in the beginning.

Amber hinted at the source of her skepticism. In a discussion of what she believed inhibited successful verification, Amber mentioned that simple algebraic mistakes would throw her off. In turn, these errors impacted her perception of the theorem identity.

AMBER: If you tell me it's an identity, and it's not, and I cannot, and I proved it, and I cannot, or I haven't proved it and I cannot make any further moves besides going back and checking, oh, did I make a mistake? Then I'm wondering what's wrong with the problem.

Thus, for Amber, past experiences during VTI construction cast doubts in her mind about the identity. These failures caused her to bring a skeptical view to the construction process. Like Charles, Amber questioned the problem if she could not verify the identity and thus could not derive perseverance like Helen or Maria could.

Whether Amber was skeptical about the theorem identity may have depended upon whether she could envision a solution path on first glance. This idea emerged as she described her work in VTI2.

AMBER: If I can't those, if I don't see those steps in my head, or if they don't look the same in the beginning, … I'm going to do everything that I can to basically prove that it might be right, but if it doesn't, then it's not, r-, an identity.

Thus, if she did not immediately see the verification path, Amber doubted the veracity of the identity and felt that verification would show her whether or not the theorem identity was correct.

Due to her approach to the theorem identity, Amber limited the possible growth in mathematical knowledge that she could experience using the theorem identity. In a discussion of using identities she had previously "verified," Amber explained her tendency to hesitate using the new identity.

AMBER: I make stupid, little simple stupid mistakes. And if I were, that would just compound into more points if I made a mistake on the previous one. And then, and, and, and then I used it. And then it was wrong beforehand and I didn't know that.

Amber was aware of her propensity for making mistakes, that if uncaught, would nullify an identity. Because Amber was a skeptic, believing the theorem identity could in fact be incorrect, using an identity that she had "verified" could be tantamount to using a false theorem to prove a claim. She hesitated using a result that appeared to be correct as she was not fully convinced it was correct. Therefore, for students such as Amber, verifying the identity did not necessarily encapsulate the identity as a theorem to be used to construct new mathematical knowledge.

At the beginning of the interview, Amber hinted at her lack of empowering identities to their fullest by use of VTI. The first interview prompt, VTI1, was not a typical VTI problem, lacking a command to "verify." Instead, the prompt asked, "Do you consider the following equation to be an identity?" The equation that was displayed was

$$\tan^2 x = \frac{\sin^2 x}{\cos^2 x} .$$

Amber initially wanted to believe it to be an identity due to recognizing the definition identity for tangent and the fact every function was squared. However, she hesitated in fully accepting it as an identity because of the square; she remained unsure of the effect of the square on the definition identity although she stated "her gut" told her yes.

INTERVIEWER: How could you convince yourself? How could you convince yourself that you gut's right? That it is an identity?

AMBER: Seeing it, basically, stated as, "This is an identity."

Thus, for Amber, her first reaction was to consider an equation to be an identity only if an authority proclaimed it to be one. She did not want to rely on herself to determine it. She did not think to use VTI in order to remove the doubts she had. Only after being prompted did she engage in VTI.

Katie. Regarding her belief in the theorem identity, Katie's convictions could not be ascertained from her responses to interview questions and activity prompts; some of her responses appeared to have been tainted by the interviewer questions, and the responses could not be clarified by other instances from the interview. However, her responses indicated a general approach to VTI and its impact on her perceptions. She commented on the intermediate equality statements formed in the hypothetical construction in VTI3.

INTERVIEWER: How do you know each one of those is equal to cosecant squared theta?

KATIE: I guess you don't until you get to the final step. But, if this, like, if this is just another way of stating the left hand side, and this equals it, then that also has to equal it. And this is also just another way of stating the left hand side, er, cosecant squared theta. So, therefore it'd have to equal cosecant squared theta all the way through.

INTERVIEWER: For you, is that then a conclusion you can make at, once you've gotten your last line, then are you saying then, really you can go back and fill everything else in and say well then, this one would have to equal cosecant squared theta, and the other expression would also have to equal cosecant squared theta.

KATIE: Yes.

INTERVIEWER: So that's something you can say after you've done the verification?

KATIE: Yeah.

Katie conceded that she could not claim the intermediate equalities as being true when they were initially written within the construction. Once the verification was successfully completed, by virtue of the equalities used, the intermediate equalities would have to be true. However, Katie believed reaching a successful verification was necessary to make truth claims about the intermediate equalities.

KATIE: Cuz, I mean, there's many times that I could mess up in my work so therefore it wouldn't, you know, I could make a mistake, and then it wouldn't be equaling it. So I like to make sure I'm doing the whole thing correctly before I, I guess say, yeah, each step equals cosecant squared theta.

Katie feared that without having a positive result for the verification, she may have inadvertently written down an intermediate equality that was not true. However, once she did construct the verification correctly, she would accept the intermediate equalities as truth statements.

The Reflexive Step

Based upon their VTI constructions, students believed they reached a successful for various reasons. Some students in the class concluded their VTI constructions with an obviously true equality. For example, on a homework assignment from the textbook, a student (12Sp31) provided the following verification (Figure 19).

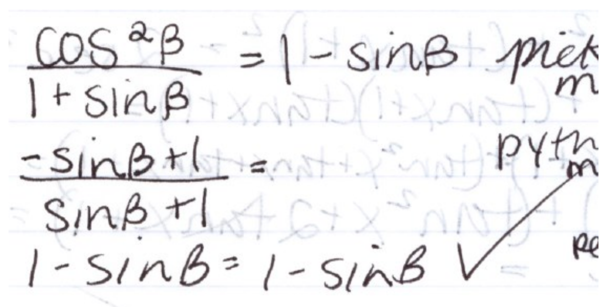


Figure 19. Student VTI construction displaying the reflexive step.

This step, a visual tautology, was defined as the *reflexive step*. This section will explain what the reflexive step is, provide motivations for its use, and explore how students used it in VTI constructions and the role it played in the proof.

The word "reflexive" has a similar root as the word "reflect," which derives from the classical Latin *reflectere*, which in turn means "to bend back, to turn round, to retrace one's steps, turn back" ("Reflect," 2013). In language, reflexive verbs are words in which the actor of the verb acts on itself; thus, the subject and object of the statement are the same. For example, the

Spanish command *siéntate* literally means “You seat yourself” when translated to English. In mathematics, a reflexive relation is a relation that is true when an element from a set is related to itself. For example, the relation “ > ” (greater than) is not a reflexive relation on the real numbers since a number is not greater than itself. However, the relation “ ≥ “ (greater than or equal to) is a reflexive relation on the real numbers since although a number is not greater than itself, it is equal to itself. Thus, for all real numbers a, $a \geq a$. So, when students wrote a step such as

$$1 - \sin\beta = 1 - \sin\beta$$

for a step in their VTI construction, they were writing a reflexive equality since it is patently true. In fact, as an identity, not only is it true for all input values, in other words, based upon the function nature, but it is visually true. The two sides visually are identical and thus the same. Therefore, in some senses, the reflexive step represents an augmented identity, one true on multiple dimensions.

Of the eight students participating in the summative interviews, 6 of them displayed the reflexive step at some point in their work during the interviews. Alan, Amber, and Bella used the reflexive step for each of the 5 VTI constructions (VTI2, VTI4, VTI6, VTI7, and VTI9). Cooper displayed the reflexive step on every VTI construction he attempted; due to time constraints, he did not attempt VTI9. Charles used the reflexive step for 3 constructions, not using it on VTI4 due to difficulties with the problem and not attempting VTI9 due to time constraints. Finally, Katie utilized the reflexive step on VTI2 and VTI4. Helen and Maria did not use a reflexive step in their constructions; they tended to present their steps in a horizontal layout, making a reflexive step prohibitive. The constructions of the 6 students who did use a reflexive step tended to have a vertical visual flow.

When taking into account the twenty-six VTI constructions from homework problems, quizzes, and exams, Amber clearly favored use of the reflexive step, using it on twenty-four constructions for a 92.3% completion rate (e.g., Figure 20). Alan and Bella utilized the reflexive

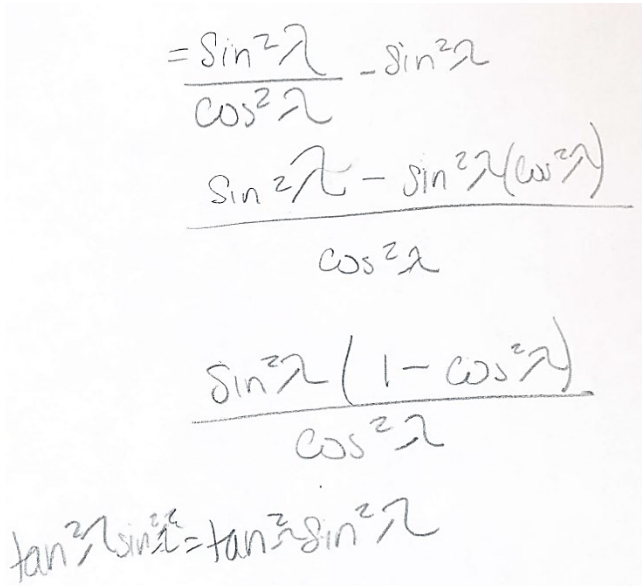


Figure 20. Amber's reflexive step construction for VTI9.

step as a construction technique at the next highest rates; for Alan, he had a 50% rate (thirteen out of twenty-six) while Bella had a 42.3% completion rate (eleven out of twenty-six). Perhaps students used the reflexive step at a higher rate in the interviews since they were encouraged to work the problems as they would on exams. With the presence of their instructor as the researcher, students may have felt compelled to provide the teacher with what they perceived to be what the teacher wanted to see.

Students used the reflexive step in their VTI constructions to signal that the VTI construction was completed and that the identity had been verified. When Amber verified identities, she allowed herself to manipulate both sides of the equation. She explained this process when discussing her solution to VTI2.

> AMBER: I started on the left hand side with this one. If I got down to the left hand side where I couldn't work with it anymore, but it still did not look like the right hand side, I'd start working on the right hand side while leaving the left hand side where I left it at. … In my mind, it's verified once they look, I mean, once they are the exact same on either side.

Once Amber noticed that she had expressions that looked the same on either side of the equation, she wrote the reflexive step, indicating to her that the identity had been verified.

Alan spoke extensively on the reflexive step. In responding to why he wrote

$$\csc^2 \theta = \csc^2 \theta$$

as the final step of his construction on VTI2, he suggested several themes for why students utilize the reflexive step.

> ALAN: To me that's just verifying that I was able to get the correct answer and to show whoever is looking at this that if you eventually get to the steps, you will get what equal the cosecant squared equals cosecant squared. It's pretty much to me just to show that I'm done and to just, kind of help clarify that I was, I went through all the steps and got to an end point where ones, the left side was equal to the right side, so just to kind of put any doubt out of peoples' mind that I was able to.

Thus, the reflexive step served as a sign of completion that the construction was completed, and the identity had been verified. Moreover, the reflexive step was a public sign; it was to indicate to someone else that the verification had been successful. From Alan's perspective, the reflexive step served to remove the doubt from peoples' minds about the veracity of the VTI construction.

Bella also mentioned the role that the reflexive step played for readers of the construction.

> BELLA: If I just left it with that, it wouldn't make sense to people who are, like, reading my work. … Like, you really didn't finish it. … It's just like leaving *x* equals five plus four. … It's just like summing it all up, bringing it all together.

For Bella, writing the reflexive step was the natural conclusion to a VTI construction; it indicated the final answer. In her view, readers of the construction expected to read a reflexive step at the end. The step acted like a last paragraph of a paper, summarizing what the reader had just read. Not writing the reflexive step would leave the reader in suspense, waiting for the expected

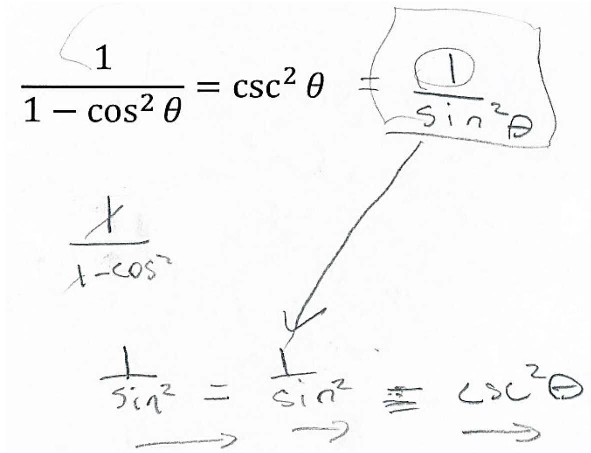


Figure 21. Cooper building the reflexive step on VTI2.

simplified conclusion. By comparing the reflexive step's omission to having an answer that was not added, Bella treated the reflexive step as the conclusion to a simplification problem, bolstering the view of a VTI construction as a simplifying process.

Alan also believed that the reflexive step added a sense of clarity. He probably meant this in a public sense; the reflexive step clarified for the reader that the construction was successful. However, several students mentioned that the reflexive step offered clarity in a private way. Cooper expressed this sentiment. In his work, he circled expressions and drew arrows to physically form the reflexive step (e.g., Figure 21).

> COOPER: I place it down there just to make, make sure that everything kind of, kind of equaled out. I like to, you know, be able to follow through my work plan and kind of see where I'm going with it. … I always like to double check myself and make sure that that last check on it to make sure it's the same idea and same, it does equal each other.

For Cooper, building the reflexive step helped guide his work and provided a check for himself that he had completed the verification. Thus, reaching the reflexive step caused him to reflect on the status of the problem and whether he was progressing in a successful manner. Reaching the reflexive step indicated to Cooper that he had reached his desired destination.

> COOPER: I like to make sure to place it all in a type of a flow so I can kind of see where it's going and making sure it's following the … the, where, where it's supposed to, the end line.

Thus, the reflexive step had a visual quality to it, showing where the problem was headed.

Placing the two expressions next to each other, in close visual proximity, was important to some students. When asked about not being allowed to write the reflexive step, Alan responded that he would be a bit uneasy.

> ALAN: I could probably do it, but I just like doing it the way that I've always done it. … I would have to look back to make sure, to just, uh, clarify where I'm trying to go, but it just helps me to do it this way.

Thus, similar to Cooper, Alan believed that the reflexive step clarified his progress in the solution. Furthermore, he preferred not having to look back to the original problem to perform this visual check; instead, he desired writing the same expression on either side of the equal sign.

Cooper depended on this visual proximity in the reflexive step to provide assurance about the status of his VTI construction.

> COOPER: It's to put, give yourself a back hold, like a foothold, like to make sure that you're not going to fall off the bandwagon. And you know, mess up. … It helps to have them right by each other to make sure they are the same. And it doesn't really matter how small it is. It's still a nice, you know, mind holder, and to make-sure-you-got-the-problem-right habit. Right by each other.

Thus, the reflexive step played a decisive role in Cooper's verification. Seeing the same exact expressions next to each other left no doubt in his mind. As he put it colloquially, the reflexive step gave him a "foothold," or a firm foundation, concerning the identity. He could rely on reaching the reflexive step to show that the identity was verified.

Inherent in the idea of using the reflexive step as a check of the progress of the VTI construction was the notion of "looking back" to the original problem, the identity to be verified.

Students monitored their progress in the answer by looking at their current manipulated expression and then looking at their target expression. As the meaning of the word "reflect" suggests, writing the reflexive step encapsulated this idea of "looking back." Bella had mentioned the idea of the reflexive step summarizing the VTI construction. Cooper mentioned this idea as well.

> COOPER: I mean, you can look at the problem, and it's just right there. And you can see it and just, you don't have to look anywhere else. You don't have to go back up here to one side, part of the paper and look down somewhere else. That's, it makes it simpler.

The punch line of the construction was contained in the reflexive step. Writing the step tied the problem together. While she did not use the reflexive step in her work, Maria speculated about the utility of doing so after being shown VTI3.

> MARIA: Well, I think it's just, like, their result on the left and the, you know, you take it back to the original problem to show that, uh, they really are equal to each other.

Thus, Maria believed that the reflexive step provided a conclusion to the construction, indicating to a reader that the original problem had been successfully solved.

In VTI8, students read a VTI construction in which someone else "verified" the identity in VTI7 by multiplying both sides of the equation with the denominator of the expression to the right of the equal sign. In terms of VTI construction, students are generally taught not to do this "crossing of the equal sign."

> BELLA: I was always taught to don't go across the equal sign cuz you don't know they're equal yet. … You're supposed to show they're equal. … they're not equal yet, so you can't do that.

Thus, until an identity was verified as an identity, Bella believed she could not treat it as an equality. However, some students actually used this strategy to verify identities in their homework. One of the students (12Sp7) used the strategy of "crossing the equal sign" in several of his constructions on the early homework assignments. For example, as the initial

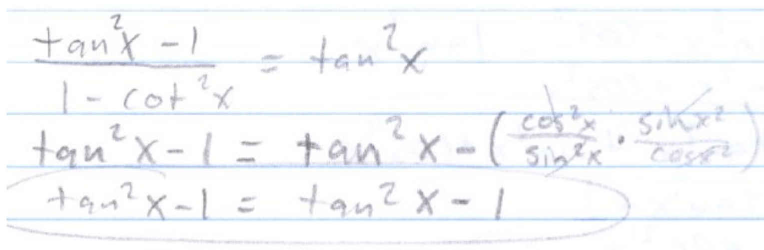


Figure 22. 12Sp7 crossing the equal sign, reaching a reflexive step.

step in one problem (Figure 22), he began by multiplying both sides of the equation by the denominator of the left expression. After some manipulation on the right side, he wrote and then circled the reflexive step, indicating he had successfully completed the verification.

Amber also had a tendency to cross the equal sign in some of her early constructions in homework assignments (e.g., Figure 23). As one example, Amber began by subtracting the right expression from both sides, redefining the right expression to 0. After several manipulation steps, her construction ended successfully when she reached this new target expression, and she wrote the reflexive step, $0 = 0$, a patently true equality.

For some students, performing this manipulation of crossing the equal sign was undesirable since it affected the reflexive step. In the construction in VTI8, the reflexive step

$$\frac{1}{\sin\rho} = \frac{1}{\sin\rho}$$

was constructed rather than the anticipated

$$\cos\rho = \cos\rho\ .$$

Alan commented on the ramifications of this.

> ALAN: One of the things, um, that I've learned and kind of picked up is to not go over the equal sign. … That kind of mixes it up, and kind of, I guess I could say muddies up where you're actually trying to get. Because instead of having a clear cosine rho that you're trying to get to, you're now messing it up.

However, reaching a different reflexive step than anticipated only provided a minor nuisance; as a reader of the construction, Alan was merely confused.

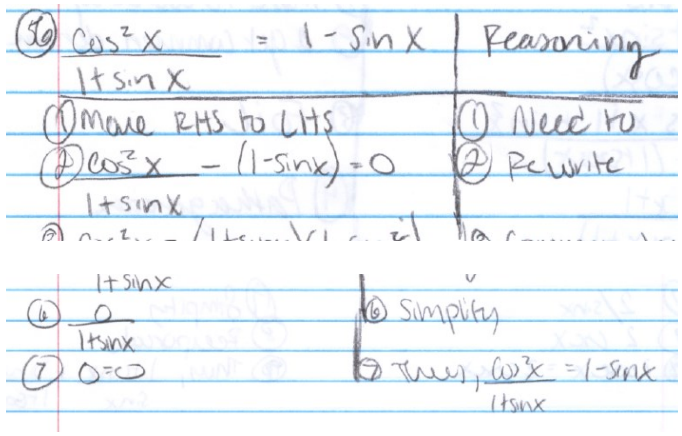


Figure 23. Amber crossing the equal sign, reaching the reflexive step.

ALAN: It does show that you could get the two sides equal to each other. … So yeah, I'm convinced that they verified it. But, in a very, in a not clear way.

Thus, reaching any reflexive step was enough for Alan to show that the identity was verified.

Not all students were as forgiving as Alan was. Cooper believed that the given identity had not been verified.

COOPER: They might have verified some other things. … Cuz when you look at the bottom, you see one over sine *p* equals one over sine *p*. Well, that's good and all, but, there's no one over sine *p* up here.

Reaching a reflexive step verified something for Cooper, but since the original expression was not present in the reflexive step, since the expected construction was not built, the identity remained unverified. When Cooper constructed his reflexive step, he pulled the target expression downward, via his circles and arrows. In trying to do this step while checking the work in VTI8, Cooper encountered dissonance as the one over sine *p* did not match up with any of the expressions in the theorem identity. Katie had a similar reaction to VTI8.

KATIE: They haven't showed that cosine equals this. They've showed that this times this equals cosecant.

Just like Cooper, Katie believed an identity had been verified, but not the original one. The identity she believed was verified was the equation formed after intermingling the two sides of the equation,

$$\cos\rho \cdot (\cot\rho + \tan\rho) = \csc\rho\ .$$

She explained her reasoning.

> KATIE: When I look at a problem after I've verified it, I would just look at the top and the bottom and see if they're equal to each other. Cuz that's what I feel like getting to verify is the point of. But then when I look at this, cosine rho is not equal to one over sine rho.

Katie checked her VTI constructions by connecting her expressions at the end back to the original problem. For this particular construction, Katie noticed that what was contained in the reflexive step did not visually match or could not match via a simple identity to the original problem. Thus, in reading the reflexive step in the construction in VTI8, she looked back until she found an expression that matched, via a simple reciprocal identity, to what was contained in the reflexive step. Therefore, the reflexive step served as a signal, indicating to Katie what identity had been verified.

While students found the reflexive step useful, students could overly rely on it to indicate a successful completion of the VTI construction. As Alan indicated in his response to the VTI8 prompt, noting that a reflexive step was reached indicated that the construction was successful. However, in the actual construction, beyond the questionable intermingling of sides, the manipulations used were mathematically sound. In response, to homework and exam problems, students constructed the reflexive step to conclude their constructions even with blatant incorrect manipulations present in their intermediate steps. One example was already displayed (Figure 19), the student (12Sp31) placing a check mark next to the

$$1 - \sin\beta = 1 - \sin\beta$$

even though the prior step contains a simplification error and the reflexive step could not mathematically follow. For that student, with the use of the check mark, reaching the reflexive

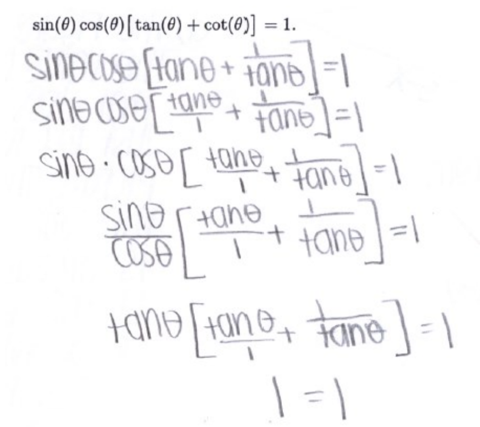


Figure 24. Reflexive step of 12Sp6, with incorrect steps, on the final exam.

step seemed to indicate a success. Another student (12Sp6) made several error while attempt to verify the identity on the final exam (Figure 24). However, the reflexive step was reached, probably forced, to indicate that the construction was concluded.

During his interview, Charles struggled with several VTI constructions. In particular on VTI6, he hesitated on several steps, questioning whether he was performing legitimate manipulations (Figure 26). After much prompting, he finished his construction.

INTERVIEWER: Do you feel this identity is verified then?

CHARLES: Yes, for me, yeah. … But I don't know if they're actually, if it actually, like, like this part, especially.

The part Charles referred to was whether

$$\sin y\,(2\cos y)$$

was equal to

$$2\sin y\cos y\ .$$

However, he still maintained that the identity was verified.

INTERVIEWER: So, could you just briefly explain why you believe it's verified?

CHARLES: Cuz I set both sides equal.

INTERVIEWER: But, you still have some hesitation about whether your work is correct?

CHARLES: About, yeah, like, the work in between.

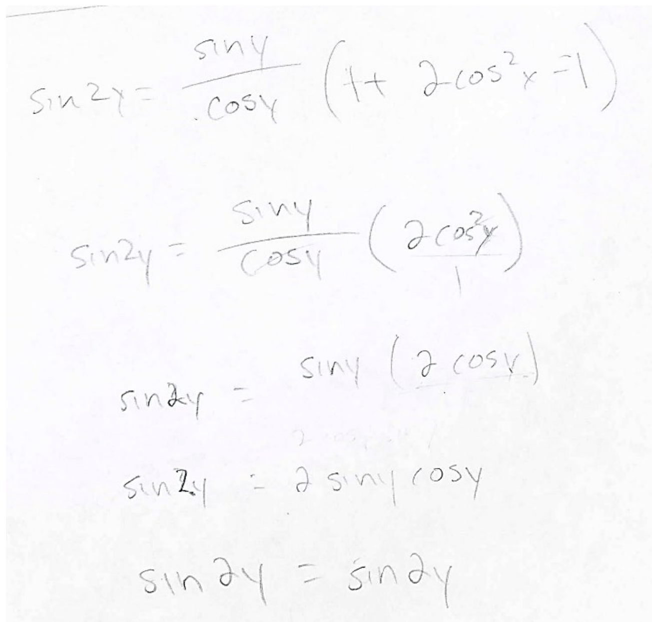


Figure 25. Charles's construction on VTI6.

Thus, reaching the reflexive step provided Charles with the conviction that he verified the identity. This conviction did not depend on the manipulations he performed. He was unsure about whether or not the steps were legitimate, even upon reaching a desired outcome, the reflexive step. Therefore, for Charles, the intermediate manipulation steps and the reflexive step did not always depend on each other. Moreover, a successful or completed VTI construction may have depended solely on the reflexive step.

The reflexive step and proof schemes. Students' usage of the reflexive step connects to some larger ideas in mathematics. One concept is the manner in which people go about constructing a proof. Students' use of the reflexive step, especially when connected to incorrect manipulations, had a somewhat ritualistic quality to it. Harel and Sowder (1998) described a ritualistic proof scheme as a scheme relying on the ritual and form of the argument in order to remove the doubt about the veracity of the claim being argued. Thus, how a proof looks on the surface or that fact that certain steps were implemented, not the content of the steps, has an impact on whether a claim is true or not.

A proof scheme is everything the proof constructor does to remove the doubts the constructor and a reader of the proof have regarding a claim being made. For example, Alan mentioned he wrote the reflexive step to remove doubts of the readers. Thus, he believed a reader of his construction, upon seeing the reflexive step, would believe that his construction was correct and the theorem identity was verified.

Charles believed he verified the identity because he completed the ritual of writing down manipulation steps, culminating in writing the reflexive step. He mentioned having hesitation about his manipulation steps. Thus, his conviction regarding the theorem identity could not depend upon the correctness of the steps. However, he still believed the identity to be verified since he reached the reflexive step. Other students, as evidenced by their homework solution, ended their constructions with the reflexive step despite the impossibility of reaching it due to errors in their steps. However, they went through the ritual of writing steps, culminating in the reflexive step.

The reflexive step as a sign. Students used the reflexive step to signal a successful completion of the identity verification. According to the basic semiotics of Charles Sanders Peirce, a sign is composed of three parts. The *signifier* is the embodiment of the sign, the *object* is what is begin signified, and the *interpretant* is the meaning or translation the observer attaches to the sign regarding the object (Atkin, 2013). For example, suppose a man notices smoke pouring from the chimney of his neighbor's house. He thinks, "My neighbor lit a fire." Here, the smoke signifies the presence of a fire, the object. The man interprets this sign to mean his neighbor started a fire in the fireplace. While the man had that particular interpretation, other individuals may come to different conclusions, such as, "My neighbor's house is burning down!" Also, other signs, such as the smell of smoke, may have led a person to the same conclusion, "My neighbor lit a fire."

Of importance is that the sign is determined by the object. This determination is based upon certain aspects of the object. For a sign to successfully signify an object, the sign must adhere to certain restrictions placed upon it by features of the object. Referring to the example of the smoke as a sign, fire generates smoke; thus, smoke as a sign of fire naturally adheres to features of the object. The object, fire, imposes characteristics on the sign, the smoke, in order for the sign to be successful at signaling the presence of smoke. This connection must be represented by the sign (ibid).

Furthermore, the interpretant depends on the sign in that the interpretant is determined by the sign. The sign points the observer to certain aspects of the object. As stated by Atkin (2013), "The sign determines an interpretant by using certain features of the way the sign signifies its object to generate and shape our understanding" (1.3 The Interpretant). In the example of the smoke and fire, the smoke forces the man to recall that fire generates smoke, reminding him of the connection between the sign and the object.

For students engaged in VTI construction, the reflexive step appeared to behave as a sign, in the sense of Peirce, for students. For example, in the VTI construction, the object being signified would be the theorem identity, and the written reflexive step would be the signifier, or sign. Now, to be a successful sign, the signifier should be shaped by the object. As the object is a statement of equality, a natural sign could be another statement of equality. Additionally, the interpretant should depend on the sign. The desired meaning to be communicated concerning the theorem identity would be that the identity is true. Thus, what better way to impart this meaning than by a statement of equality that is absolutely true? Without a doubt, an object is equal to itself. In writing the reflexive step, the students seemed to say, "See, you know this reflexive equality has to be true. So then, you know my verification of the theorem identity is finished and correct."

Reaching an equality that the student held absolutely no doubt about meant that the student or a reader should have no doubts remaining about the theorem identity. For example, on the final exam, a student (12Sp11) constructed a verification culminating in a reflexive step (Figure 26). During the VTI unit, students were encouraged to explicitly explain their steps in

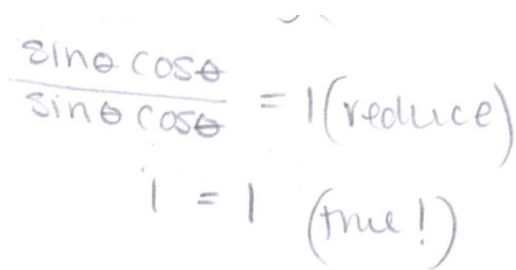


Figure 26. VTI construction on final exam for 12Sp11.

order for them to begin thinking metacognitively about their VTI constructions. Hence, the student wrote “reduce” next to the penultimate step, indicating that the expression on the left was going to be reduced down to “1.” By writing an emphatic “true!” next to the reflexive step, the student appeared to indicate that since the reflexive step was obviously true, the construction was successfully completed.

Another example illustrating this idea also occurred on the final exam. Another student (12Sp32) provided the following verification (Figure 27).

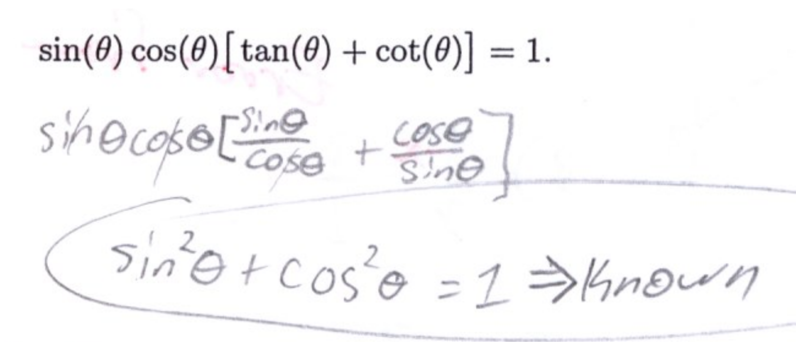


Figure 27. VTI construction on final exam for 12Sp32.

The student began by manipulating the expression on the left, using quotient identities to convert the tangent and cotangent functions. He indicated cancellation of the denominators by drawing lines through the associated functions. This led him to writing

$$\sin^2\theta + \cos^2\theta = 1 ,$$

the Pythagorean identity. While the student did not use a reflexive step, he ended his construction by writing the word “known” next to his final equation. In other words, seeing that final

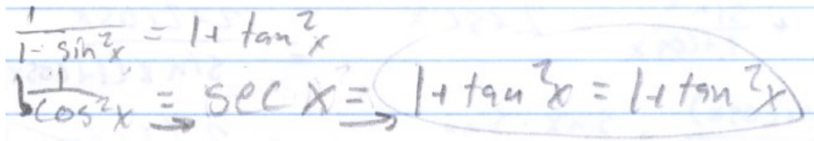


Figure 28. 12Sp7 constructing the reflexive step.

equation, and reaching what the student considered an obvious truth, the Pythagorean identity, the construction was completed successfully.

In this previous example, what the "1" proceed from was unclear. Perhaps the "1" was a result of the Pythagorean identity, and then the student mentally connected this "1" with the "1" in the original problem. Another possibility was that the left side expression was written, the student wrote the "1" from the original equation, pulling it down, and then determined that the resulting equation was true, prompting the writing of "known," and signaling the successful completion of the verification. However, when students wrote the reflexive step, the evidence suggested that students were making a mental comparison between the manipulated expression and the target expression, and when they noted a match, they would then physically write it down. While verifying an identity on one of the homework assignments, a student (12Sp7) manipulated the left side of the equation, displaying the steps in a horizontal layout (Figure 28). This layout created a string of equal expressions, culminating in the same expression as that found on the right side of the equation,

$$1 + \tan^2 x \ .$$

However, the construction did not stop once the target expression was reached. The student rewrote the expression again, creating a redundant step in the string. Seemingly, upon reaching the target expression, the student looked back to the original problem, noted the intended target expression, and then ritualistically wrote this expression from the original equation next to the same expression that had been achieved through manipulation. Thus, the student created a reflexive step; he circled it, indicating that the construction was successfully completed.

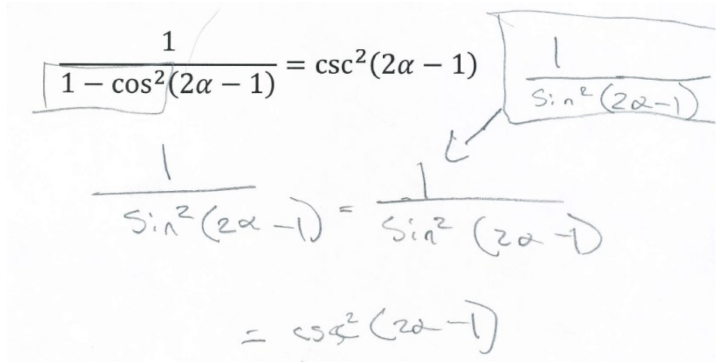


Figure 29. Cooper dragging the expression to form the reflexive step on VTI4.

At times in his construction, Cooper was quite explicit with this comparison, circling the target and drawing an arrow down form the reflexive step, almost as if he were dragging the expression downward to form the construction. On VTI4, Cooper wrote what he described as a "side note" next to the target expression; he applied a reciprocal identity to create a modified target. Then, in his actual construction, he began manipulating the left expression. After noting that he reached his modified target, Cooper "dragged" the target down to form the reflexive step (Figure 29). In doing so, Cooper placed the manipulated expression and the target expression in close proximity, allowing him to check his construction. Reaching the reflexive step signaled to him that his construction was correct.

During a VTI construction on a homework assignment, Amber employed a strategy she previously indicated using in VTI; she manipulated the expressions on both sides of the theorem identity until she achieved the same expression from both manipulations (Figure 30). The resulting VTI construction was similar to Cooper's effort on VTI4. Beginning with the left expression, she manipulated it, using two successive identities, to $\sec^2\theta$. Then, she used a Pythagorean identity to convert the right-side expression to $\sec^2\theta$. Her final step of the construction was to write the reflexive step,

$$\sec^2\theta = \sec^2\theta \ ,$$

formed by connecting the two manipulated expression. Doing so led her to conclude that the original identity had been verified, as indicated by her note. Thus, a sign, the reflexive step, that

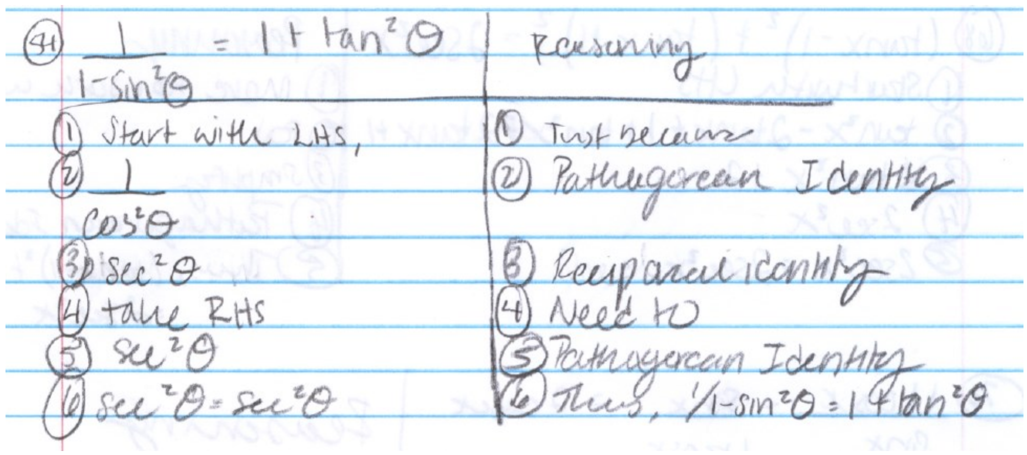


Figure 30. Amber creating the reflexive step by matching expressions.

reflected the quality of the object to which the sign pointed, the theorem identity, seemed to be translated as signaling the veracity of the theorem identity due to the sign being a blatantly true equality.

The Building of Knowledge

As stated by CadwalladerOlsker (2011), the role of proof as verification empowered the theorem, i.e., “a theorem is not a theorem until it has been verified to be such” (p. 39). Thus, even though a person may have held a strong conviction about the truth of a claim, the claim could not legitimately be implemented to build new mathematical knowledge until it had been verified. For example, claims made under unproven mathematical hypotheses, such as the Riemann hypothesis, will remain in a tenuous state, conditionally true, until the particular hypothesis is verified, at which point the claims will be legitimized.

This notion of empowering the theorem played out in various ways among the believers and skeptics. A student’s belief or skepticism in the theorem identity depended on his or her perception of VTI problems. Put differently, skeptics believed that in VTI problems, the theorem identity could be a bum identity; the problem could be attempting to trick the student in asking them to verify a falsehood. The problem might as well have asked the student to verify that $1 + 1 = 3$. On the other hand, believers accepted the premise of the problem. However, although they accepted the premise, believers such as Helen and Bella still kept the theorem identity in a tenuous state for the sake of the problem. The result of this was that although a student held a positive conviction about the theorem identity, the student would not feel comfortable using the full power of the identity until it had been verified. Thus, the student maintained a measure of mathematical integrity. The idea of integrity in verification extended to the skeptic as well. Alan mentioned hesitating using a successful VTI outcome to justify questionable algebraic steps within the verification; in a sense, he believed the identity had not yet been empowered as questions still lingered about the legitimacy of the verification.

Mathematical integrity could also have a limiting power. Tall (2002) suggested that the process of proving a statement encapsulated the theorem as a concept. Once encapsulated, the mathematical object could then be used to build new mathematical knowledge and prove more theorems. However, students such as Charles mentioned hesitating using identities that they had proved for fear that they had made a mistake in the verification and ended up proving an identity that was not actually an identity. Thus, Charles's initial skepticism regarding the theorem identity and his awareness in his propensity for mistakes prohibited VTI from encapsulating the theorem identity as a new theorem to be used to build new mathematical knowledge.

For some of the students, successful verification implied truths about intermediate equalities developed during the construction process. As an example, Amber mentioned the power that a successful VTI had while discussing her solution to VTI2.

> AMBER: If you verify it, then you can match up, if you have left the left side true, if the left side is the true left side and the right side's the true right side, you can take any step between and match up. Or you could take, like, the, if like, okay, say if I started working down the left hand side and then I stopped and started working down the right hand side, if I hadn't ever crossed the equal sign, then, and once I've verified that, then I could take step three from the left hand side and match it up from step two from the right hand side. And they'll still be the same.

Once the theorem identity was verified, Amber was able to make a claim that all of the intermediate equalities that could be formed through linking expressions to the left of the equal sign with expressions to the right of the equal sign were true. Before she verified the theorem identity as true, Amber could not consider this possibility due to her skepticism in the theorem identity. As the identity could be false, linking the expressions to form intermediate equalities could end in mathematical disaster. However, success in VTI empowered her to construct additional truth claims.

Visual Preferences in VTI

Comparing VTI Constructions

In preparation for the third class period in the unit on VTI, students completed a supplemental homework activity, outside of class, titled *Assessing Verification*. In the activity, students were directed to read through five verifications of the following identity

$$\frac{\tan y + 1}{\sec y} = \sin y + \cos y.$$

Each VTI construction was different in the layout of the steps and varied in the explicit manipulations shown. Moreover, each construction was correct in the sense that no algebraic or trigonometric errors were made in the manipulations.

After reading the verification, students assessed the quality of the logic and presentation found within the verification on a scale of 0 to 5. Upon assessing the VTI construction, students then justified their assessment, providing a written reason for their assigned score. Of the 33 participants, 25 returned the supplemental homework activity. Conceivably, students could assign a score from the interval 0 to 5; thus, the scores were treated as interval data. The means of the assessment score were computed for each VTI construction (Table 3). A superficial analysis of the means seemed to indicate differences among how students perceived the VTI constructions, with construction #4 most preferred and construction #3 least preferred. VTI constructions #1 and #5 appeared to be assessed similarly; construction #2 fell somewhere

Table 3

Means of Participants' Assessment Scores of VTI Constructions

Construction	VTI #1	VTI #2	VTI #3	VTI #4	VTI #5
Mean	3.62	4.16	2.32	4.56	3.80

between construction #4 and constructions #1 and #5.

However, while the data were intended to be on an interval, only one student assigned a non-integer score, assessing one of the verifications as a 4.5. Students appeared to treat their assessments as ordinal data rather than interval data, where their assessment scores were assumed to be related to choices, for example, "Perfect," "Excellent," "Good," "Marginal," "Bad," and "Unacceptable." In a preliminary data analysis of the assessment scores, frequencies were tallied of the assessment scores for each VTI construction (Table 4). To be conservative, the score of 4.5 was treated as a score of 4. The frequency counts seemed to indicate that the same differences highlighted by the observation of the means existed in how students perceived the VTI constructions. However, the scores for VTI constructions #2 and #4 appeared similar. Thus, the frequency counts suggested three groupings of assessment scores based upon similarities: VTI #3 alone, VTI #1 and #5, and VTI #2 and #4.

To confirm the results of the analysis of the mean scores and the frequencies, a preliminary visual comparison of the word clouds associated with the explanations of VTI construction assessment scores was conducted in order to detect differences among the

Table 4

Frequencies of Assessed Scores Per Construction Item

	VTI #1	VTI #2	VTI #3	VTI #4	VTI #5
Score of 0	1	1	3	0	0
Score of 1	2	1	3	1	1
Score of 2	1	1	10	0	2
Score of 3	7	3	3	1	6
Score of 4	5	3	4	5	8
Score of 5	9	16	2	18	8

constructions. This exploration was also undertaken to inform further explorations into student preferences of VTI constructions via in-depth analyses of the explanations of the assessment scores. The rationale for the word cloud exploration was the belief that participants' explanations of scores would differ if the constructions were assessed differently. However, the possibility existed that explanations differed even if the constructions were rated the same; this possibility existed due to the differences in the constructions. The following comparisons were made: construction #3 and #4, constructions #1 and #4; constructions #4 and #5; and constructions #3 and #5. The visual comparisons of the pairings of clouds will be featured in more detail after discussions of statistical considerations.

While the statistical analyses suggested which clouds to pair, some comparisons were also based upon intrinsic distinctions or commonalities between constructions; glancing at the constructions would suggest certain constructions should be assessed similarly. VTI constructions #1 and #4 contained two main differences. While both of them were arranged in a vertical fashion, forming columns with the expressions, construction #1 repeated the unchanged expression on the left side of the equation to form an equation at each step; construction #4 omitted the unchanged expression until the final line. Additionally, construction #1 contained four manipulative steps, and construction #4 contained five manipulative steps; construction #4 explicitly showed the step

$$\sin y \cdot \frac{1}{\cos y} = \frac{\sin y}{\cos y}$$

before progressing to $\tan y$ while construction #1 merely showed

$$\sin y \cdot \frac{1}{\cos y} = \tan y\,.$$

Other than this one step and the inclusion of the unchanged expression, the steps of the construction were identical. However, participants appeared to assess the constructions differently.

The difference between construction items #4 and #5 was the manner in which the steps were arranged. The same exact algebraic manipulations and trigonometric substitutions were used on each step; however, the steps of construction #4 were arranged in vertical columns while the steps of construction #5 were arranged in horizontal rows. Construction #4 also ended in a reflexive step:

$$\frac{\tan y + 1}{\sec y} = \frac{\tan y + 1}{\sec y}\ .$$

Despite containing the same steps, students assessed construction #4 to be of a higher quality than construction #5.

As the mean assessment score for construction #4 was the highest and the mean score for construction #3 was the lowest, these word clouds were compared to help determine what students preferred while reading VTI constructions. The constructions themselves held not much in common. Construction #3 was arranged horizontally while construction #4 was vertically. Construction #3 began with manipulation of the left expression while construction #4 manipulated the expression on the right first.

Statistical tests were implemented to further establish differences noted in the comparison of the word clouds. The tests were based on determining whether or not students viewed the constructions differently by considering the scores that students assessed each construction. As the concern of the analysis was to establish if students viewed the verifications differently and not to determine the size of the discrepancy, if it existed, the assessment score data were treated as being non-parametric. In this way, concerns over small sample size and normality assumptions could be obviated. However, independence of assessment scores could not be assumed; in fact, in their explanations of scores, some students indicated assigning similar scores across items due to the constructions appearing similar. Thus, some student assessments displayed a dependence on one another. Hence, the non-parametric equivalent of a paired *t*-test, a two-tailed Wilcoxon signed rank test, was used.

Table 5

Results of Wilcoxon Signed Rank Tests for All Pairings of Assessment Scores

Pair Tested	W	Unadjusted *p*-value
#1 v #2	55.5	$p > 0.2$
#1 v #3	28	$0.001 < p < 0.005$
#1 v #4	16	$0.005 < p < 0.01$
#1 v #5	67	$p > 0.2$
#2 v #3	20	$p < 0.001$
#2 v #4	41.5	$p > 0.2$
#2 v #5	75	$p > 0.2$
#3 v #4	0	$p < 0.001$
#3 v #5	0	$p < 0.001$
#4 v #5	5	$0.001 < p < 0.005$

If students assessed each construction similarly, then the difference in pairings of the scores for each student should obviously be zero. The Wilcoxon signed rank test analyzed whether or not the median change in assessment scores from one construction to the next across all students was significantly different from zero. Ten separate tests were conducted based upon pairing the assessments scores in all unique combinations. The results of the analyses are found in Table 5.

To control the familywise error rate (FWER), a Bonferroni correction was used to adjust the significance level for determining the rejection of the null hypothesis for each test. The FWER is the probability of making at least one Type I error when considering all ten of the Wilcoxon signed rank tests together, not just individually. The Bonferroni correction is a conservative procedure that ensures that the FWER $\leq \alpha$. For purposes of the procedure, the number of members in the family was taken to be $m = 10$. Thus, null hypotheses from individual tests were rejected when $p_i \leq \alpha/10$, where p_i represented the *p*-value of the *i*th pairing tested.

Overall, students considered VTI construction #3 to be the poorest construction as the scores for every pairing involving construction #3 were significantly different at the Bonferroni adjusted significance level of 0.005. What students deemed to be the best construction could not be clearly determined as intertwining of results occurred. For example, VTI constructions #2 and #5 were not significantly different, and VTI constructions #2 and #4 were not significantly different; however, VTI constructions #4 and #5 were significantly different at the Bonferroni adjusted significance level of 0.005. Although the results of the statistical tests could not lead to broader conclusions, analyses of the word clouds associated with the explanations of the assessments revealed differences in student perceptions regarding the constructions. Coupling these results with the results of the Wilcoxon signed rank tests for the individual pairings suggested that students viewed some constructions more favorably than others. As suggested by the initial analyses of word clouds, the statistical analyses, and similarities or differences in actual construction format, the following pairings of constructions were engaged in deeper explorations: #3 versus #4; #1 versus #4; #4 versus #5; and #3 versus #5. A discussion of the results of the individual tests and the subsequent exploration of the explanations of assessment scores follows.

Comparing VTI constructions #3 and #4. When considering how students assessed constructions #3 and #4, the median change in assessment scores was significantly different from zero at the Bonferroni adjusted significance level of 0.005. Thus, students were assessing the constructions differently. A quick visual analysis of the word clouds associated with the explanations revealed how students were assessing the constructions differently.

Construction #3. Exploring the cloud for construction #3 (Figure 31) revealed the notions of "steps," "confusing," "follow," and "understand" to have prominence. Some minor themes of "reasoning," and "hard" were also present. With these initial themes as a framework, each assessment explanation for the participants' scoring of construction #3 was examined. The overriding sentiment in the explanations was that construction #3 was difficult to follow and understand, causing confusion and leading to not being able to follow the reasoning of the overriding sentiment in the explanations was that construction #3 was difficult to follow and understand, causing confusion and leading to not being able to follow the reasoning of the construction. In the 25 explanations, 10 references having a negative connotation were made

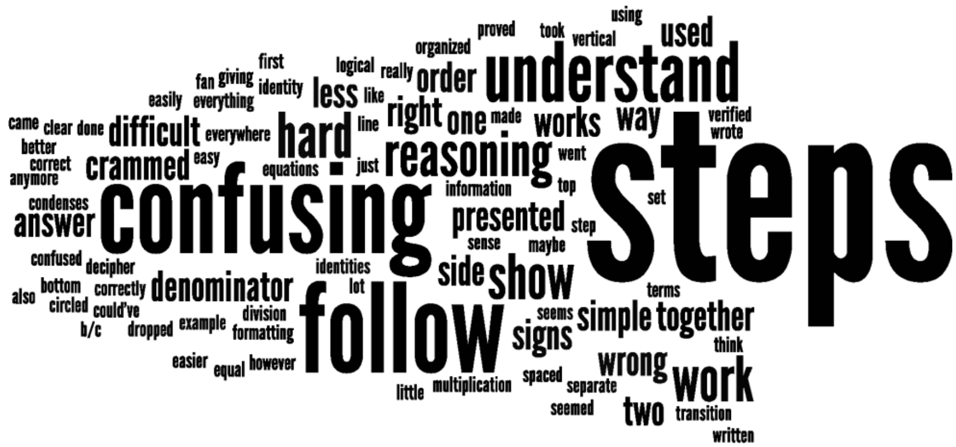


Figure 31. Word cloud of explanations of scores for VTI construction #3.

concerning the format or organization of the construction. Some students believed that a horizontal layout made following the reasoning of the steps difficult. For example, one student made the following comment:

> Steps presented from side to side are less easy to understand than steps presented from top to bottom. (12Sp2)

Another student made a similar statement:

> All the work was crammed together so it was difficult to follow the line of reasoning. (Alan)

Having the work in a horizontal layout appeared to impede the students ability to discern the steps being taken.

Another 6 references were made about the steps of the construction being confusing. As one student explained:

> The reasoning is hard to understand. (12Sp20)

While commenting on confusing steps, a student offered a possible source of the confusion:

> Steps were confusing to follow. The work is right but everything is crammed together. (12Sp18)

Thus, the confusion students felt while reading construction #3 may have stemmed from the

Figure 32. Word cloud of explanations of scores for VTI construction #4.

format being used. Interestingly, four students claimed that the verification was incorrect. A student commented:

> Dropped +1 in transition to terms of $\sin(x) + \cos(x)$. Also had wrong multiplication in denominator. Giving wrong answer. (12Sp7)

Therefore, the format appeared to impact how students assessed construction #3. In some instances, students claimed steps were missing or incorrect; perhaps the horizontal format obscured students' ability to perceived and follow the steps, resulting in a downgrading of the assessment score.

Construction #4. Comparing the cloud of #4 to the cloud of #3, an immediate difference was apparent. The concept of "easy" was very prominent in the cloud of #4 (Figure 32) while it was almost absent in the cloud of #3. In a quick search of the explanations for construction #3, the words "easy," "easily," and "easier" each appeared once. However, in each instant, the words were used in a negative or comparative sense.

To identify what student found easy in construction #4, an in-depth textual analysis occurred. The analysis was informed based upon the other prominent concepts appearing in the cloud, "reasoning," "follow," and "understand." The text file for the explanations of the assessments of construction #4 was modified in order to completely capture notions. Thus,

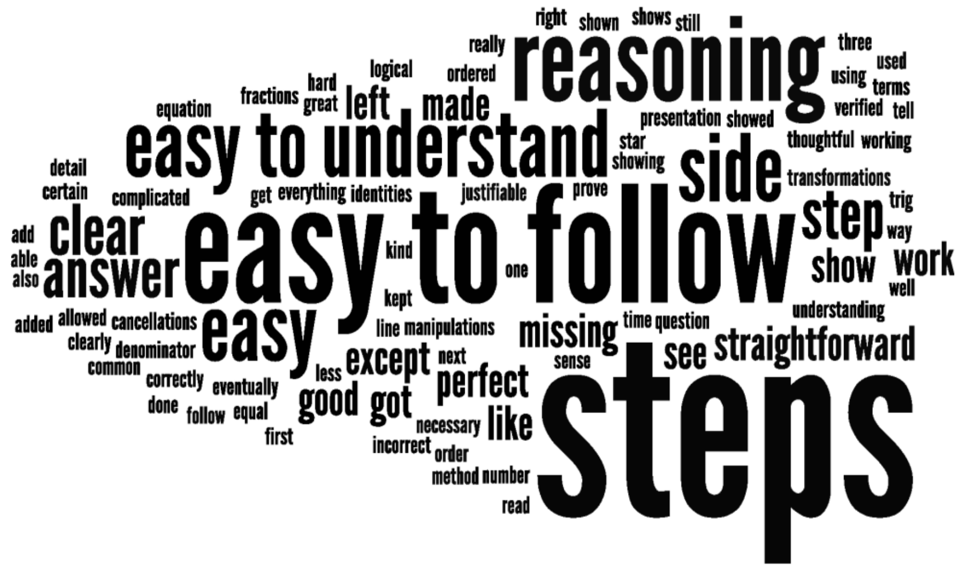


Figure 33. Altered word cloud of explanations of scores for VTI construction #4.

instances of the word "easier" were edited to "easy" as long as doing so did not change the implied meaning in the explanation. From the analysis of the text, students related the notion of "easy" to the notions of "understand" and "follow." Thus, the text document was further modified to form the phrases "easy~to~follow" and "easy~to~understand." Through the insertion of the tilde, Wordle.net would accept these phrases as a word for cloud formation purposes. The new text document was fed to Wordle.net in order to generate a new cloud (Figure 33).

Thus, in comparison to the difficulties in following the steps in construction #3, student believed construction #4 presented the VTI proof in a form that was easier to follow and understand. While the textual analysis did not always clearly reveal why students found it easier to follow, based upon the complaints some students made about the horizontal appearance of construction #3 and some comments made about construction #4, a hypothesis was made that students found the vertical layout of steps in construction #4 easier to follow. Some students explicitly stated or implied that the format allowed them to easily follow the steps.

Easy to understand and also straight forward presentation. (12Sp3)

Easy to follow. Ordered work method. Great manipulations. (12Sp12)

Figure 34. Word cloud of explanations of scores for VTI construction #1.

In contrast with #3, students generally did not mention confusing or missing steps while explaining their assessments of construction #4.

> The work was clear and no steps were left out so it was easy [to see] the line of reasoning. (Alan)

In fact, as Alan complained that the work appeared crammed in construction #3, the conclusion that format affect the score seemed plausible.

Comparing VTI constructions #1 and #4. When considering how students assessed constructions #1 and #4, the median change in assessment scores was significantly different from zero at the Bonferroni adjusted significance level of 0.01. While the significance could be considered marginal, due to the conservative nature of the Bonferroni correction, the significance could in fact be stronger. For purposes of analysis, the result of the inference test was treated as being significant. Thus, students appeared to be assessing these constructions differently. A quick analysis of the word clouds associated with VTI constructions #1 and #4 revealed some qualitative differences in how students explained their assessments.

Exploring the cloud of construction #1 (Figure 34) and comparing it to the cloud of construction #4, the notion of “easy” appeared diminished in prominence. To better understand why students assessed the constructions differently, two different analyses of the explanations of score assessments were implemented. In the first analysis, each student explanation of his or her score assessments for VTI constructions #1 and #4 was classified as containing a positive quality, which would elevate the perception of the construction, or a negative element, which would diminish the assessment of the construction. For example, the following comments were classified as being positive:

> VTI #1: Its [sic] clear what they did and the identities used prove the identity. (12Sp4)
>
> VTI #1: Very clear and easy to follow. Great example. (12Sp29)
>
> VTI #4: Can tell how they got the answer. (12Sp13)
>
> VTI #4: It shows very easy detail on how to eventually equal the left side. (12Sp30)

The vast majority of explanations classified as positive were scored a 5 by the students; however, two positive comments corresponded to scores of 4. No positive comments were associated with a score of 3 or below. Explanations categorized as containing a negative element corresponded to assessments scores ranging from 0 to 4. Examples of comments classified as negative are:

> VTI #1: I don't think it's clear in showing why [writes 2nd step to 3rd step]. It seems like it skipped a step. (12Sp15)
>
> VTI #1: The original equation was not verified. (12Sp20)
>
> VTI #4: Kind of hard time understanding certain steps. (12Sp29)

While most explanations could be classified as containing solely positive or negative elements, four explanations possessed both positive and negative qualities. For example, one student wrote:

> VTI #1: It's straightforward + simple, but the last step takes a little extra thinking to understand. (12Sp17)

Another student commented:

> VTI #4: Perfect except didn't do the complicated side. (12Sp28)

Unsurprisingly, none of these mixed comments received a score of 5.

Table 6

Contingency Table of Negative and Positive Comments for VTI Constructions #1 and #4

	VTI #1	VTI #4
# of Negative comments	15	6
# of Positive comments	12	21

After classifying each comment, frequencies were calculated. Students generally explained their scores of the assessments of construction #4 in more favorable terms; 4 comments were negative, 2 comments were mixed, and 19 comments were fully positive. On the other hand, for the explanations of the scoring of construction #1, 13 comments were negative, 2 comments were mixed, and 10 comments were positive. If each instance of a positive or negative comment is counted, that is, if the comments that were considered mixed are place in both the positive and negative category, a contingency table may be generated (Table 6).

Since the first analysis only indicated a difference in the explanations based upon positive or deprecating terminology used, further analysis of the assessment explanations was conducted with the intent to identify thematic differences beyond positive or negative in the statements. As the purpose of the analysis was related to why students viewed the constructions differently, only explanations corresponding to the situation in which a student assessed VTI constructions #1 and #4 with different scores were considered for this analysis. Thus, 16 pairs of comments were included in the analysis; the other 9 pairs of comments corresponded to an equal assessment score for the constructions. Of these 16 pairs, 14 involved construction #4 being assessed with a higher score. Special attention was paid to specific mention of the discrepancy in the number of steps used. Moreover, any similarities in the explanations were noted and set aside, with the focus on elements resulting in a diminishing of the score.

Broad themes of discrepancies between the pairings of explanations evolved from the analysis. Two students explained that *format* affected their assessment. As one student wrote:

VTI #1: Started with a good sign, but putting it equal to the opposite side on every line made it confusing.

VTI #4: Easy to read and using the right side made the reasoning easier to understand and see the steps. (12Sp14)

Here, the student described confusion while reading the construction arising due to the format used to display the steps. This confusion contrasted with the ease of reading the steps in construction #4. Another student commented:

VTI #1: Easy to follow and done correctly. Don't have to keep writing the left side of the equation. Should have done the more complicated side.

VTI #4: Perfect except didn't do the complicated side. (12Sp28)

In both explanations, the sense that the steps are good was present as well as a complaint about the expression chosen to begin manipulating. However, the explanation to construction #1 included an additional critique of the inclusion of the unchanged expression on each line.

During the summative interviews, some students actually commented on writing the unchanged expression on each line. During earlier constructions, such as VTI2 and VTI4, Katie and Bella wrote the unchanged target expression, yet on subsequent constructions failed to do so. Katie explained her rationale for not rewriting the target during her construction for VTI7.

KATIE: I guess that gives it more, there's just more writing in there, and it makes it more jumbled and more confusing. So I think kind of the less, the most simple I can make it.

For Katie, not writing the target during this construction offered a visual simplicity. Rewriting the target would have presented a jumbled appearance, causing confusion.

Katie also referenced the confusion caused by rewriting the target while reviewing the hypothetical student construction in VTI10.

KATIE: It does, I mean, it does make it a little confusing cuz like at, I mean at first, you don't know what side you're looking at. So you're going constantly back and forth, I guess. But then, it just, I just think it makes it more clear when you have the steps you're working and not showing each time it equals the other side.

Thus, Katie believed that omitting the target expression would have offered clarity in the visual presentation, allowing her to more easily follow the steps in the verification. Other students also expressed confusion by the construction in VTI10.

> MARIA: It got me confused for a second. … I wasn't really sure what I was looking at until I realized that it was just repeating the same thing.

While Maria was able to quickly overcome her confusion due to the rewriting of the unchanged target expression, Helen needed to take more extreme measures in order to understand the construction in VTI10.

> HELEN: That was confusing to me.
>
> INTERVIEWER: Why was it confusing?
>
> HELEN: Because I work in the aspect of left to right, top to bottom. And so I kept wanting to go from this equals this, and like working like that. But they're working like this. … And so I'd rather just put this once at the end. .. Because, then I mean, I had to block it out because it was too distracting.
>
> INTERVIEWER: I see, yeah, I saw your hand over the top of that.

In order to perceive the identity as being verified by the construction, Helen had to visual remove what she considered a distraction by covering the column of target expressions with her hand. With the distraction visible, she remained confused about the validity of the verification.

Returning to the supplemental activity constructions VTI#1 and VTI#4, as the prior pairing of comments from 12SP28 highlighted, some students indicated displeasure for the expression chosen to manipulate. While this comment existed in both of the explanations, three pairings included a critique of the beginning expression in the explanation for construction #1 but not in the explanation for construction #4. An example of this phenomenon was the following pairing:

VTI #1: This is true but if you had broke $\frac{\tan y+1}{\sec y}$ up, there would be an answer with no fractions.

VTI #4: Steps were all good, kept in terms of answer. (12Sp7)

Thus, this student suggested manipulating the other expression with the expressed intent to avoid having to manipulate too many fractions, presumably avoiding the addition of fractions that occurred by using the other expression. This difficulty with the addition of fractions manifested in another pairing of explanations.

VTI #1: Should have used other side, it was more simple. When they made a common denominator, I was a bit confused.

VTI #4: Perfect steps, easy to follow. (12Sp21)

The final pairing that critiqued the choice of beginning expressions did not express an issue with the addition of fractions; more alarming, the student believed a step was incorrect.

VTI #1: Should have started with left side. Also, $\frac{1}{\sec y} \neq \cos$.

VTI #4: Easy to follow. Ordered work method. Great manipulations. (12Sp12)

Interestingly, these students did not express a concern with VTI construction #4. Instead, they highly praised the construction, assessing it with a 5 in each instance. Thus, students found fault with the steps of construction #1 but not construction #4 despite the constructions sharing the same steps. Perhaps the visual clutter of rewriting the unchanged target expression in construction #1 obscured the actual steps from the view of the students, causing the students to view the steps differently.

Another reason given for differing assessments was because of the steps used in the two constructions. In the two instances in which students assessed construction #1 higher, the explanation indicated that the steps were the culprit.

VTI#1: It's clear what they did and the identities used prove the identity.

VTI#4: Used a few less steps to show & prove it. (12Sp4)

In fact, VTI#4 included one additional step in comparison to VTI#1. The other student expressed confusion with the steps used in VTI#4.

VTI#1: Very clear and easy to follow. Great example.

VTI#4: Kind of had hard time understanding certain steps. (12Sp29)

While this student expressed confusion with construction #4, the confusion with the construction was generally directed in the opposite direction. For example, for one student, the constructions appeared to be night and day.

VTI#1: $\cos y = \frac{1}{\sec y}$ But confused here, don't know what's going on!

VTI#4: Easy to understand! Good steps and reasoning! (12Sp31)

For the first construction, the student assigned it a score of 1, while the student scored construction #4 as a 5. The students did not express what about the two constructions was so drastically different to warrant such a difference in the scores. As a point of comparison, the same student assessed construction #3 with a score, a 2, explaining:

VTI#3: Confusing! The steps seemed out of order maybe! (12Sp31)

Thus, the student found the construction to be equally confusing, but assessed it with a higher score compared to construction #1.

One possibility that could also help explain the difference in scores was that students may have learned from the constructions as they read through them. Therefore, for a student like 12Sp31, the first construction was confusing, resulting in a low score, but after reading several other constructions, the student had a better grasp on the concepts in the construction. Another possibility could be an inherent fluctuation in assessing due to the subjective nature of assessment. Students were not using a fixed rubric set by an external authority and instead were relying on their own internal rubric based on what they preferred in a VTI construction or based on their own mathematical understanding. Hence, the above student (12Sp31) believed

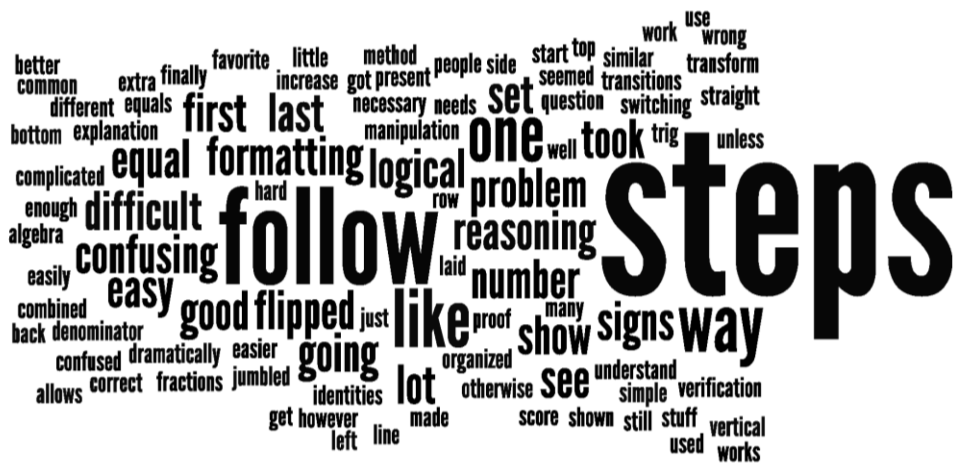


Figure 35. Word cloud of explanations of scores for VTI construction #5.

construction #1 “felt” like a 1 and believed construction #4 “felt” like a 2, while also believing them to both be confusing.

Comparing VTI constructions #4 and #5. When considering how students assessed constructions #4 and #5, the median change in assessment scores was significantly different from zero at the Bonferroni adjusted significance level of 0.005. Thus, students assessed the constructions differently. A quick analysis of the word clouds associated with VTI constructions #4 and #5 revealed some qualitative differences in how students explained their assessments. As occurred in the cloud for construction #1, the notion of “easy” in the cloud for #5 (Figure 35) appeared diminished in comparison to the cloud for #4. Additionally, “understand” became insignificant in relation to the other ideas in the cloud. Also, “formatting” gained prominence, being absent in the cloud of construction #4.

To better understand why students scored VTI construction items #4 and #5 differently, an in-depth analysis of the explanations was conducted. The explanations of each construction item were compared for each participant to explore the differences between the comments. As the purpose of the analysis was related to why students viewed the constructions differently, only explanations corresponding to the situation in which students assessed VTI constructions #4 and #5 with different scores were considered for this analysis. Consequently, 14 pairs of explanations were compared; of these 14 pairs, 13 of them involved VTI construction #4 being assessed with a higher score. During the analysis, each comment was compared against its paired comment. Common elements were noted and set aside; any differences between the comments were highlighted. The focus was on positive or negative comments appearing in one explanation and not appearing in the other. Additionally, attention was paid to the negation of an explanation for one assessment score appearing as the explanation for the other assessment score.

Two themes describing the differences evolved from this comparison; one student's response contained elements of both themes as contributing to the lower assessment. The first theme, f*ormat*, related to the layout of the steps. Of the 14 explanations analyzed, ten of them involved the format theme. Explanations containing the format theme generally indicated a diminishing of the assessment score due to the horizontal arrangement of the steps in construction #5. This arrangement led to a disorganized appearance in the students' esteems. For example, one student's pairing of comments was the following:

VTI #4: Easy to understand and also straight forward presentation.

VTI #5: Better to present steps top to bottom. (12Sp4)

Here, the student indicated a preference for a vertical presentation of the steps; when the steps were arranged in a column, the student was better able to understand. Another student believed the vertical presentation help in identifying what steps were being used:

VTI #4: Clear + straightforward. All necessary steps shown.

VTI #5: Steps are good, but it's easier to see the transitions in number 4. (12Sp17)

For this student, while the steps were present in both constructions, the layout in construction #5 obscured the steps. Even when a student could follow the steps, the arrangement of the steps could affect the score:

VTI #4: Easy to read and using the right side made the reasoning easier to understand and see the steps.

VTI #5: Took very few steps and reasoning seemed easy to understand; however set up was confusing with the many equal signs on one row. (12Sp14)

Another student explicitly stated the effect of the format on the assessment:

VTI #4: The reasoning is clear and thoughtful. It is easy to follow.

VTI #5: While not as difficult to follow as #3, the formatting is difficult to follow. Different formatting would increase the score dramatically. (12Sp21)

Thus, the manner in which the steps in the VTI construction affected how students critically viewed the construction.

While not necessarily apparent, the format of the construction may have influenced the second theme related to the differences in how students view the VTI constructions. Namely, five students mentioned issues with the *steps* of the construction. Comments coded as *steps* explained that steps were missing or possibly confusing. One student wrote:

VTI #4: Easy to follow.

VTI #5: Not enough proof or explanation of steps. (12Sp24).

Another student stated:

VTI #4: Used a few less steps to show & prove it.

VTI #5: They didn't use a lot of steps to show their reasoning. (12Sp4)

While these students assessed construction #5 lower due to a perceived lack of steps in comparison, the two constructions actually contained the exact same steps. Although the students did not explain the scores further, a possible explanation for the perceived discrepancy with the steps resided in the comments of another student:

VTI #4: The work was clear and no steps were left out so it was easy the line of reasoning [sic].

VTI #5: The work was not layed [sic] out in a way that made it easy to follow and they left out steps.

Thus, in addition to disliking the format of the steps, the student claimed the steps were missing in VTI construction #5 while at the same time claiming that no steps were left out of VTI construction #4. Perhaps, the format obscured the actual steps being used, causing the student to believe fewer or different steps were being used.

Comparing VTI constructions #3 and #5. The final in-depth comparison made between constructions occurred between constructions #3 and #5. Both of these constructions had a horizontal format, with the steps forming a linear equality string. In previous analyses, students appeared to dislike this horizontal format, with the format mentioned by 10 distinct students (with 9 students criticizing the layout in both #3 and #5) in a deprecatory way in the explanations of the assessment scores. However, the median change in assessment scores was significantly different from zero at the Bonferroni adjusted significance level of 0.001. Thus, students actually assessed the two constructions differently.

A further textual analysis was conducted to determine what about the constructions suggested the different assessments despite the constructions having the same horizontal presentation, a format disliked by the students. A procedure similar to that used in the comparison of constructions #4 and #5 was utilized; the explanation pairs corresponding to scoring differences were analyzed. Thus, 18 total pairs of explanations were explored. In each instance, construction #3 was assessed with the lower score.

Overall, why students would assess the constructions differently was not apparent based on explanation pairs. The notion of "steps" appeared to affect the scores. However, construction #5 was rated lower in comparison to construction #4, the top-rated construction. Thus, students necessarily needed to lace their explanations of #5 with downgrading remarks. Thus, separating the comments for constructions #3 and #5 was somewhat difficult due to them containing similar complaints.

As a result, the analysis was further limited in scope to the pairs in which construction #5 was assessed a score of 5. Doing so reduced the number of pairs analyzed to 6 and would in a sense treat construction #5 as a constant; students would not make deprecatory comments about construction #5 for those assessed a 5. In all but one instance, the gap between the scores was 3 or more. For the exception, the student (12Sp1) assessed construction #3 as a 4 and construction #5 as a 5.

VTI#3: It made sense, but the way it was set up was confusing.

VTI#5: I didn't like the way it was set up, but all the necessary steps were shown.

Thus, while the student disliked the format of the construction in each instance, she mentioned that she perceived all of the necessary steps in VTI#5. She did not claim this about construction #3, but also mentioned she found that construction confusing, a concept she did not assign to construction #5.

In four of the pairs, the notion of "steps" was mentioned in a negative way by the student to explain the lower score for construction #3. Two of the students believed construction #3 contained an incorrect step. One of these students (12Sp7, in a comment previously mentioned in the discussion on construction #3) believed that incorrect multiplication occurred in the denominator. Another student (12Sp10) also mentioned a problem with the denominators. The student assigned scores of 0 and 5 to the constructions and commented on the use of denominators.

VTI#3: I don't understand where the denominator went where I circled it. I don't think this one works.

VTI#5: This is my favorite way. Not a lot of algebra, just some common denominator stuff. Good!

The student did not comprehend that a reciprocal identity was used to remove the denominator, explicitly circling the troubling steps as seen in Figure 36. Perhaps the use of the division symbol ÷ created confusion in the student's mind. In two other explanations, students made negative remarks regarding use of the symbol.

The final pair, with a score of 0 for construction #3 versus the 5 assessed to construction #5, expressed a general sense of confusion regarding construction #3.

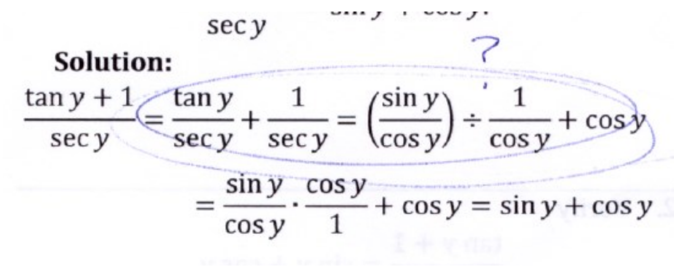


Figure 36. 12Sp10 circling the "misunderstood" step in VTI construction #3.

> VTI#3: Seems way too confusing to work out and be simple.
>
> VTI#5: It was simple like one before. (12Sp30)

In the explanation of the one before, construction #4, the same student stated:

> VTI#4: It shows very easy detail on how to eventually equal the left side. (12Sp30)

Thus, the student believed that construction #5 was clear in its presentation of the steps. In contrast, the student felt that construction #3 was just not simple enough. The same student, in scoring construction #2 to be a 2, explained:

> VTI#2: Dividing all the sins and cos make it difficult to make simple than it could or should be.

Perhaps students felt that construction #3 was just too complicated-looking. In comparison, even though construction #5 had the horizontal layout, students may not have believed it to look too complicated. This perception may have stemmed from use of the division symbol as stated by a few students. However, why students viewed the constructions differently was not completely clear.

The Visual Format of VTI

During the summative interviews, students displayed a preference in the visual format of the construction. While Helen and Maria used a horizontal layout, forming a row, or a chain of equalities, the other six participants, Alan, Amber, Bella, Charles, Cooper, and Katie, used a predominantly vertical arrangement in their manipulation steps, arranging the steps to form a column and allowing for the possible creation of intermediate equalities by rewriting the unchanged target expression. In fact, during the assessing activity, Bella rated all of the constructions in a vertical format as a 5 and the constructions in a horizontal format as a 3. She explained the grouping of assessment scores similarly. For example:

> VTI#4: Logical + easy to follow.
>
> VTI#5: Logical, but not as organized.

Thus, Bella viewed the verifications in a vertical format as being easier to follow and more organized.

While discussing the student construction in VTI11, some of the students explicitly stated their preference for the column layout. For example, as Alan considerer the proposed VTI solution, he exposed his VTI conceptions.

> ALAN: Well, uh, it looks like, like going from there to there, they did take out some steps, but, it is possible what they did. Pretty much what I'm noticing, it's just the layout. Um, it's very, it's not very clear to follow.

For Alan, the successful VTI should have included a demonstration of correct steps presented in a clear format. Clarity was important and necessary in demonstrating, in a public fashion, how the manipulation steps progressed from on to the other.

> ALAN: My whole thing is if you're trying to prove that, then the clearer you can make it, um, the more obvious that what you're doing, um, that the steps actually equal each other, that the easier it would be for yourself and also for whoever is looking at it.

Not showing all of the steps along the path to equality obscured the path for Alan, making equality harder to realize. Alan indicated that he believed his preferred format, the columns, provided a clear presentation in order to comprehend the steps.

> ALAN: I think it helps, the, the way that I do it, um, I think it helps. It's clear, and, for me it is, it's clear. It's step by step. Each step shows. And it helps you to get the idea that all those steps are actually equal to that.

Moreover, Alan believed that his format also highlighted the new knowledge being created, the equality of the steps.

Throughout his constructions in the interview, Alan usually rewrote the unchanged target expression. By writing the expressions, the target and the manipulated, in two columns joined by the equal sign, he emphasized the equality.

> ALAN: It helps you to get the idea that all those steps are actually equal to that. It's just a different form. … All those little steps are actually equal to each other. … Who's to say there's not going to be a point where you're going to work a problem where it's basically like that but you start out with something like this instead of that.

In VTI, Alan verified the theorem identity; from that point, he was empowered to use that identity to solve problems. However, he also recognized that the intermediate equalities also represented identities that could be used to solve problems. For Alan, VTI not only empowered the theorem identity but the intermediate equalities as well. His visual format made this concept apparent.

While Alan preferred the vertical layout of steps, he indicated that the preference should remain a private matter. He commented on the lack of equal signs in the student's construction in VTI11.

> ALAN: If you're able to follow it, I mean. It's all pretty much the whole layout is up to whoever is doing it to be able to follow their own work. And if they're able to follow it going by that then I don't think there should be anything against being able to do it that way.

Alan explained that the layout used was a result of how the student was able to understand his or her path to equality. For Alan, as long as students clearly showed the proper steps indicating the path, then the VTI would be successful.

When the presentation format does not match student preferences, students may become confused about the construction. Commenting on the hypothetical construction in VTI3, Maria exposed her preference for the horizontal format in VTI.

MARIA: I had to think about how this equals that because it wasn't, because of the way it was shown, I guess.

The presentation of the solution did not match her expectations or preferences, causing a hesitation in her acceptance of the solution. Maria preferred a linear format for her VTI solutions. The rationale that the visual format stemmed from how the student best understood the VTI process was echoed by Maria.

MARIA: It kind of follows my train of thought that way. It's easier to keep it together. … I think it's easier to see the relationships that way, if you can look at them next to each other. Um, as opposed to, you know, looking at it like this, you have to think about it again. Oh well, you know, they're looking at the denominator in each one. Uh, there, I guess I think it's just easier to see, cuz it just goes straight across.

INTERVIEWER: What relationships are you referring to?

MARIA: The Pythagorean identity in the denominator of the fraction.

Her preference related to the ability to observe the identities being used. The columns format obscured these actions as it did not follow the natural flow; she preferred work that flowed left to right, much as reading a sentence.

MARIA: I think just because I'm a very visual person, and I need to see it. I mean, like over here, like, I need to see the relationships as I'm going along. … I could see them right next to each other, you know. But if you're going like this, you kind of have to go back up to the beginning.

To feel comfortable, Maria desired a visual proximity with the expressions. The linear format provided this proximity. In the columns format, the proximity of expressions next to each other, side by side, was lost. With the expressions side by side, she could easily note the differences by making a visual comparison. Also, statements of equality, such as trigonometric identities on an identity sheet, are typically placed in a linear format. Perhaps, in order to view expressions as equal, Maria needed to see them placed in their usual visual format.

However, Maria was not completely beholden to the format. If she could follow the steps and see the manipulations being made, she believed the identity to be verified. For example, Maria rated the column construction in VTI construction #4 as a 5, higher than the 4 she assessed VTI construction #5, a linear format. She explained her assessments on the activity.

> VTI#4: Clear + straightforward. All necessary steps shown.
>
> VTI#5: Steps are good, but it's easier to see the transitions in number 4.

Thus, being able to see the steps took precedence over the format.

Helen also preferred the horizontal format in her own constructions. During her interview, she became confused while viewing constructions written in a vertical fashion. She revealed her preference while reviewing the hypothetical student construction in VTI10, commenting on the rewriting of the target expression.

> HELEN: But to me, that was confusing to me. … Because I work in the aspect of left to right, top to bottom. And so I kept wanting to go from this equals this, and like working like that.

Upon approaching the problem, Helen wanted to read the construction as she normally worked. When the construction was not presented in her preferred format, she experienced confusion.

Although she preferred a horizontal presentation of steps, Helen could accept constructions presented in a vertical format in the same way Maria could. In fact, Helen assessed all of the constructions from the *Assessing Verification* activity a 5, except for construction #3, due to correct steps being used in those constructions. For example, she explained why she assessed VTI construction #4 as a 5.

> VTI#4: Because in steps 2-5, the trig identities allowed transformations + cancellations in order to get $\frac{\tan y+1}{\sec y}=\frac{\tan y+1}{\sec y}$.

Since she believed the steps were correct and led to the desired reflexive step, she accepted the verification as being good. On the other hand, she assessed VTI#3 as a 2 because she believed

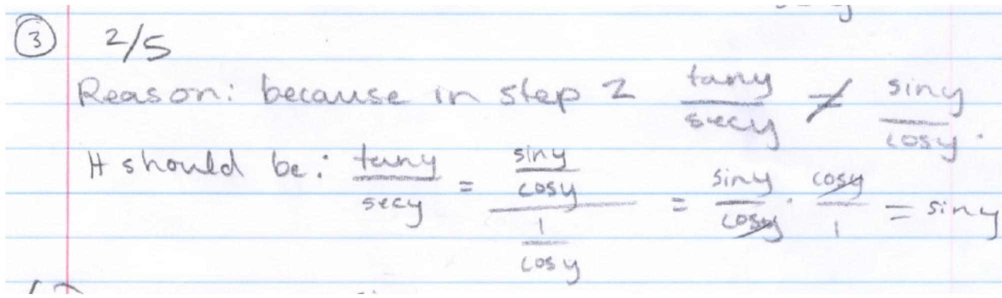


Figure 37. Helen's explanation of her score for VTI construction #3.

the verification contained an incorrect step. Her reasoning appears in Figure 37. Helen appeared to visually compare the forms of the different expressions in order to determine the transitions. She looked at the same location in each expression to determine whether or not the manipulation steps were correct and failed to look beyond to notice what manipulations actually occurred. The explanations for her other assessments, as already evidenced, all mentioned the correctness of the steps being used. Perhaps the layout caused Helen to fail to fill the void in the missing step by mentally checking for the missing step. Apparently, after believing in the error in VTI construction #3, Helen checked some steps in VTI construction #4 and convinced herself that these steps were correct, as seen in Figure 38.

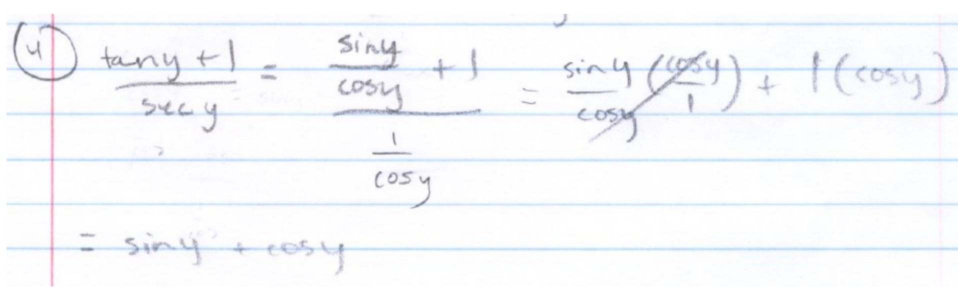


Figure 38. Helen checking VTI construction #4.

Thus, being able to clearly view the steps in the construction was important to the students. Students believed that certain layouts facilitated seeing the steps more easily. Those in a vertical, column-like format were deemed easier to follow, facilitating the viewing of the manipulation steps. Alcock and Inglis (2010) stated that the visual format of expressions should be taken into account in the presentation of proofs. They claimed that equations in a horizontal layout could be visually confusing; in those situations, a vertical layout would be more appropriate. This claim, while not substantiated by Alcock and Inglis, seemed to be validated by the students who assessed the VTI constructions. Additionally, Alcock and Inglis stated that layout of the proof may also affect the logic of the argument. A vertical layout created visual space between the beginning and end, separating what was being proved from the conclusion and logically disconnecting the parts of the argument. Again, while the claim was only hypothesized by Alcock and Inglis, the claim may have received some validation by students such as Maria, who claimed to prefer the horizontal format as it highlighted the transitions and strengthened the relational aspect, emphasizing the equality of all of the expressions.

Furthermore, not all columned constructions were perceived equally. As already noted VTI construction #4 was assessed higher than VTI construction #1. Two main differences existed between the actual constructions. VTI construction #4 contained one additional step. However, this fact was not indicated by many students. Another difference was that VTI construction #1 included the unchanged target expression. As students felt VTI construction #4 offered a clearer presentation of the steps, perhaps the unchanged target expression created visual clutter. At one point in the summative interview, Helen had to cover the column of unchanged expressions in order to block out the distractions and perceive the correctness of the construction. Removing this visual clutter may be similar to students who used the dodging technique of omitting x, a process students indicated made the problem easier. Research into visual perceptions indicates that visual clutter leads to an increase in errors of judgment (e.g., Baldassi, Megna, & Burr, 2006). Perhaps results of this research could explain how students perceived steps to be missing or incorrect in the VTI constructions that might be considered by the students to be visually cluttered.

Ideal VTI: A Score of 5

In order to establish the qualities that the students desired in a good VTI construction, the explanations for the constructions assessed a score of 5 were explored; as the initial step of the

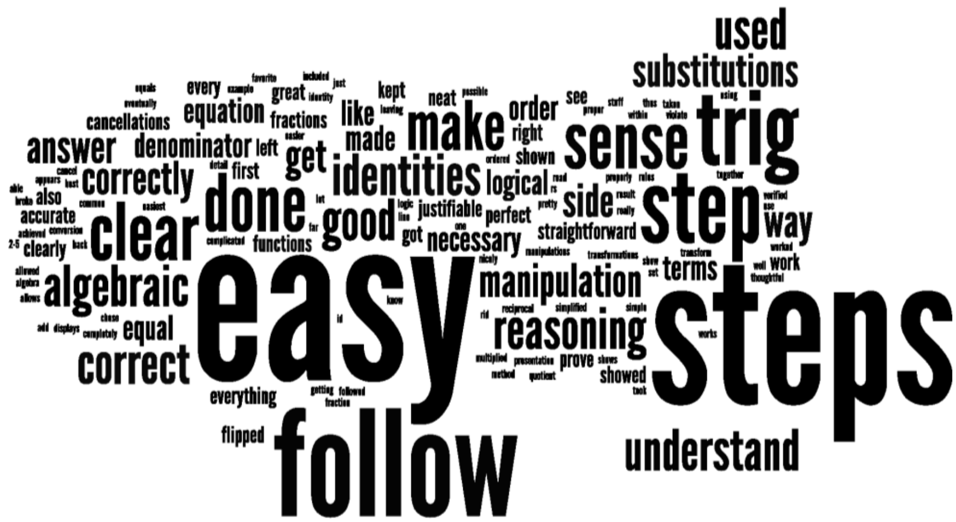


Figure 39. Initial word cloud of student explanations of assessment score of 5.

analysis, word clouds of these explanations were generated. First, a master document was created by copying and pasting all explanations associated with assessment scores of 5 from the assessment matrix to a blank word document. Minor editorial changes were made to the text. Spelling and grammatical mistakes were corrected in order to accurately count words; also, names of trigonometric functions and parenthetical researcher comments were scrubbed. Additionally, phrases were altered in order to best reflect the content of the student comments and align them with more prominent words. Thus, the phrase “wasn’t hard” was replaced with the word “easy.” Furthermore, phrases suggesting an assessment score was the same as a previous assessment score due to the belief that the VTI constructions were similar or the same were cut and replaced with the explanation of the previous assessment. For example, one student wrote, “Unless I am wrong, this is like question number 1 & 4.” This phrase was removed and the phrase “Steps were all good, kept in terms of answer” was used in its stead, being the explanation of the previous VTI construction score. These changes were made to ensure that the full meanings of the student explanations were captured in the word cloud. The edited text was then copied and pasted at Wordle.net and the initial word cloud was generated. The same cloud parameters used with previous word clouds were implemented with this cloud (Figure 39).

Table 7

Altered Words in Word Cloud of Assessment Score of 5

Altered word:	Altered to:
Accurate, Correctly, Proper, Properly	Correct
Easier, Easiest	Easy
Great, Perfect	Good
Followed	Follow
Clearly	Clear

After a cursory examination of the cloud, several words were noted to feature prominently in the cloud. Furthermore, synonyms and variants of the prominent words were also observed. As a result of these observations, a new text document of the explanations for a score of 5 was created as an alteration of the previous document to combine similar themes and ideas. The changes of words that were made are listed in Table 7. In addition to the grouping of similar words, if a word was used in a compound formation to describe two different ideas, the word was duplicated; this editorial decision retained the original meaning of the explanation while ensuring the word was counted properly when forming the cloud. This decision affected two phrases, changing "Good steps and reasoning" to "Good steps and good reasoning" and changing "Reasoning is easy to follow and understand" to "Reasoning is easy to follow and easy to understand." Finally, in the original cloud formation, by replacing phrases signaling one construction was similar to another construction with the other constructions assessment explanation, some duplication of ideas and words occurred. This overlap was corrected by removing the phrase "done correct" from three explanations in which this idea was captured by the added phrase. The edited text was again copied and pasted to Wordle.net, and the new word cloud (Figure 40) was generated using the same cloud parameters as before.

After another cursory examination of the cloud, a final text document was created by further altering the student explanations. In addition to the prior edits, commonly occurring phrases were formed by placing a tilde (~) between the words of the phrases; using the tilde was a

Figure 40. Altered word cloud of student explanations of assessment score of 5.

way to force Wordle.net to "see" the phrase as a single word when generating the cloud. By first perusing the cloud and then the accompanying text document, commonly repeated phrases were noted. Thus, "easy" was connected with the words "follow," "understand," and "read" to form the Wordle phrases "easy~to~follow," "easy~to~understand," and "easy~to~read." In a similar fashion, "good" and "steps" were conjoined to form the phrase "good~steps." Two slight alterations to student comments were made to generate this phrase. The first change was superficial in nature, changing "steps were all good" to "good steps." The second change was more substantial, but it still retained the gist of the explanation; the word "manipulations" was altered to "steps" as the context of the comment indicated the two were synonymous for this student.

After the editorial changes, the new text document was fed to Wordle.net and a new cloud was created that contained the conjoined words. The same parameters as used in previous word clouds were initially used. However, after noticing that the phrases that were formed from the prior editorial decisions were difficult to see, the "Horizontal" option for text direction was switched to "Any Which Way" in order to increase the prominence of the phrases. The resulting cloud (Figure 41) represented a description of assessment scores for ideal VTI constructions.

As VTI construction #4 was assessed with the highest mean score, the word clouds of #4

Figure 41. Final word cloud of student explanations of assessment score of 5.

and Score of 5 were compared. The text document of participants' explanations of their assessments for VTI construction #4 was prepared in a similar way that the text document for Score of 5 in order to generate a word cloud under the same parameters; the same word alterations were effected, and the same word combinations were conjoined to form phrases within the clouds.

The intent in forming the Score of 5 cloud was to determine what students perceived to be the ideal VTI construction from the standpoint of a reader of the construction. Through comparison of this cloud to the cloud of #4, and through reflection on participants' responses in the interviews, a consensus formed about what a VTI construction should be. From the students' points of view, a VTI construction consisted of manipulation steps that were presented clearly so that they were easy to follow and understand. The VTI construction displayed the path to follow, demonstrating how one expression equaled the other expression. If the steps were not clearly presented so that they were perceived to be possibly missing, or if steps were in fact missing, the construction was not held in high esteem by the students. As a practical image, in the form of a simile, a VTI construction was a path through a forest. A good path would be clearly marked, with the steps clearly visible and the tree limbs cleared off the path so that a person could be sure of which steps to take. Conversely, a poor path would have tree branches blocking the path and would be poorly marked so that a person would be unsure about which steps to take.

When considering the student construction on VTI11, the five participants who answered the prompt all mentioned that not enough steps were demonstrated. For Amber, VTI meant showing the correct algebraic manipulations and identity substitutions and was very much a private endeavor; she did not prepare her solutions for an external reader. When reviewing the VTI work of another student, she commented on the lack of showing steps.

> INTERVIEWER: So is that too much of a leap for your tastes?
>
> AMBER: Yes, a very big leap for my tastes. Because of the fact if you go, okay, I see what they did. Cuz it'd be sine over cosine and cosine over sine. Yeah, uh, that is a big leap for my tastes because of the fact that I would write the intermed-, the reciprocal identity before going to the fraction, er, before simplifying the fraction.

However, not showing the steps did not preclude the identity from being verified in her mind.

> AMBER: As long as they're using the identities correctly and not, um, using an identity that isn't, doesn't work in the situation, I mean they're not using the identities incorrectly. And if all the algebra is correct, then yes, I would say it's proven.

Thus, Amber focused on the correctness of the steps rather than a logical and clear presentation of the steps. If she believed all of the steps were mathematically correct, she believed the identity to be verified.

> AMBER: I know somebody might actually, like I mean like you or, might actually look at it, or whatever, I never think of like, oh, can he actually follow this. Because like once I'm in the mode of solving a pro-, like a problem, eh, I'm just going. … Once I'm satisfied that I didn't make any mistakes and which normally I don't go back and double check which could be a problem, um, but um, but once I'm satisfied, like I didn't make any mistakes or um, that I'm basically, I'm basically good with that problem. I mean once I say yes, then just move on.

In her own verifications, once Amber believed her manipulations to be correct and she reached the desired conclusion, which for her was the reflexive step, Amber believed she had verified the identity.

While reviewing VTI11, Cooper reacted in a manner similar to Amber, mentioning the omission of steps at the beginning of the verification. He was prompted to provide a score assessment of the construction.

> INTERVIEWER: If you had to assess this on a scale of zero to ten, what score would you give it?
>
> COOPER: Mm. I don't know, how many steps did they skip between a and b? There's a lot of steps. Cuz you're, I mean, that's kind of like, you're not really verifying it if you skipped a whole lot of steps. For all we know, they could have just said, okay, well, I know that I can change this, and eventually this one. But that one, I don't know.
>
> INTERVIEWER: Okay. So, is that then a big thing for verifying, is showing those, [garble], those steps?
>
> COOPER: Showing the work to get to your steps. Well, I mean, that's the whole point if you're verifying. You got to show what you're doing to change it in each step. … It's kind of an important step. I mean there's maybe some small minor ones. But as far as this one goes, that's just too major of a jump. … I wouldn't like give them zero points, cuz, I mean, they still kind of proved it. But at the same time, I'm not going to give them, you know, like eight or nine or full credit cuz it's kind of a, it's still not enough to show that, they what they did.
>
> INTERVIEWER: So, what number do you give it?
>
> COOPER: Maybe, I don't know, maybe a five. … They did kind of, they did half the work I guess you could say. Cuz, I mean, yeah, that top half. That's a huge chunk that they did miss.

Thus, Cooper's focus was not necessarily on the format for purposes of assessing the construction. Instead, he was concerned about the steps. If the construction provided some

Figure 42. Word cloud of student assessments of VTI construction #4 (Score of 5 version).

correct steps, then the identity would still be verified; skipping steps did not preclude the identity being verified. However, from the point of view that the construction should convince a reader that the identity was verified, the construction should provide all of the details demonstrating the equality.

A quick comparison of the clouds for a score of 5 and for VTI construction #4, altered along the parameters for the score of 5 cloud (Figure 42), revealed similarities in the prominent words and phrases. Students indicated the belief that the construction in #4 was easy to follow and easy to understand, the steps were good, and the reasoning was apparent. These themes, while not necessarily of the same prominence, still figured significantly in the score of 5 cloud. Thus, VTI construction #4 appeared to represent an ideal VTI construction for many of the students.

However, the final VTI construction to be discussed, VTI construction #2, also represented a possible ideal construction. Based upon the distribution of the frequency counts of the assessment scores, VTI constructions #2 and #4 looked similar. In fact, the median change in assessment scores was not significantly different from zero. Therefore, a word cloud for the explanations of the assessments scores for VTI construction #2 was generated in accordance with the procedure used to generate the altered cloud for #4.

Figure 43. Word cloud of student assessments of VTI construction #2 (Score of 5 version).

Comparing the cloud for VTI construction #2 (Figure 43) to the score of 5 cloud and the VTI construction #4 cloud, the notion of “easy to follow” appeared diminished in importance. Additionally, the notions of “understand” and “reasoning” are wholly absent from the cloud, while the concept of “easy” as a standalone theme almost disappeared. Thus, a possible conclusion was that students actually viewed VTI construction #4 more favorably in that it offered a clearer presentation to follow and understand the steps. Thus, while this issue would need to be explored further, of the presented choices, VTI construction #4 seemed to represent the ideal VTI construction as perceived by the students.

CHAPTER V

CONCLUSION

This study explored students' concept images of verifying trigonometric identities. In this chapter, the interpretations of the findings will be summarized. The summary will provide a general description of aspects of the VTI concept image through a discussion of plausible answers to the research questions that guided the study. These questions were:

1. What are students' conceptions of VTI and to what extent do existing frameworks describe the conceptions?
2. What factors contribute to a student-perceived successful engagement in VTI?
    a. What does VTI accomplish from the student's perspective?
    b. What structural form does a VTI construction take?
    c. In what ways do the structure of the VTI construction and the VTI accomplishments interact?
3. In what ways do students' conceptions of VTI influence problem solving decisions?

The chapter will conclude with suggestions for future directions and areas needing clarification and further research.

VTI Concept Image

Question 1: Conceptions of VTI

Students appeared to view VTI in terms of problem solving and proof. When thinking of

VTI as a process of proof construction, students commented on the roles that VTI served, according to the de Villiers Roles of Proof framework. Some students stated that the purpose of verifying trigonometric identities was to check whether or not the identity was in fact an identity. In other words, VTI served a verification role. Other students indicated that verifying an identity was about showing why the theorem identity had to be an identity. For these students, VTI was serving more of an explanatory role. Students did not strictly adhere to one specific role. Many students spoke of VTI in terms of both verification and explanation. Thus, defining a student in terms of his or her beliefs about a single role for VTI was difficult if not impossible in this current study. In viewing VTI as a problem to solve, students discussed strategies they used to construct the solution to the VTI problem. Overall, students viewed VTI as a problem in simplification. This notion of simplification extended to describe the VTI process itself and also the problem solving actions implemented.

Question 2a: VTI Accomplishments

From the students' perspectives, a successful VTI solution accomplished many purposes. Based upon a student's view of the theorem identity during the process of VTI, students were classified based upon their beliefs in the veracity of the identity, prior to beginning the process of VTI, as being skeptics or believers. These initial views implied what VTI could accomplish for students. For believers such as Helen and Bella, believing the identity to be true prior to VTI, the process of VTI empowered the identity, elevating the identity to theorem status and opening the gates for it to be legitimately used to solve other problems. For skeptics, VTI removed any doubt they may have had concerning the theorem identity. VTI also created additional, intermediate equalities that some students viewed as being true upon successful VTI completion.

Question 2b: The Form of VTI

Students preferred a visual format for VTI constructions that reduced visual clutter and clearly displayed manipulation steps. The favored format consisted of displaying the manipulated steps in a column. Some students wrote the target expression next to the manipulated expression, joining them at each line with an equal sign. Many students concluded the verification using a reflexive step. All of these visual constructions served a purpose for the student.

Question 2c: The Implications of the Structure

The format of the construction interacted with VTI and had implications for the students. For students who rewrote the unchanged target expression, creating the intermediate expressions seemed to create and emphasize additional equalities that would be implied true once VTI was successfully completed. The reflexive step appeared to behave as a sign, signaling when VTI was completed and indicating that the theorem identity was either explained true or verified true, depending on the role VTI was serving for the student.

Question 3: Problem Solving Decisions

Specific strategies used by students were predicated on simplification. For example, in order to choose the expression to begin manipulating, student chose the expression they perceived that they could more easily condense down or simplify to a more basic form. This tendency was due to the perception that simplifying was an easier endeavor than “building up” an expression. Students believed the simplification path was in a sense set as it offered a limited number of plausible manipulations of the expression. In contrast, many plausible manipulations existed to build the expression up. Therefore, choosing what was dubbed the “more complicated side” to begin manipulating simplified the students’ decisions due to limiting the number of choices.

Through dodging, students used techniques to simplify the arguments of the functions being manipulated. In defaulting to x or omitting x, students marginalized perceived complicated function arguments or arguments involving unfamiliar symbols, removing visual or cognitive clutter. The marginalization allowed the students to focus on what was important for showing the veracity of the identities, the actual functions. Students realized that they could marginalize the argument for purposes of solving the problem.

The flow of the VTI problem, from complicated to simple, guided students as they faced choices. Some students used identities to convert expression to more familiar, comfortable functions. In doing this action, students were again reducing clutter, this time of a cognitive nature. Other students expected the expression to simplify in form. Thus, when they chose identities to simplify the expression and the expression did not simplify in the expected manner, the students backed off the problem and reconsidered their decisions.

Implications and Future Directions

The notions of skeptics and believers in the theorem identity depended on how students viewed the identity in the realm of verifying it. That is, students may have actually believed the identity to already be established as an identity, but because of their conceptions of what "verify" meant, they approached the identity as not true. Thus, VTI would show that the theorem identity was in fact true. The difference is subtle; thus, the issue not apparent was if students actually were or were not convicted about the theorem identity prior to VTI. That is, did students state that the identity was unproven out of a sense of mathematical integrity or professionalism yet in their minds viewed it as true since it was stated as an equality? This issue was difficult to separate in the interview and was not adequately approached. Thus, for future research, better methods need to be used to explore this issue and gain a better understanding of it.

Furthermore, Tall (2002) suggested that proof could encapsulate theorem for students into usable objects. Some students appeared to balk at using identities that they had established, needing an additional authority to empower the identity. Thus, further exploration of students' notions of identities should occur. This exploration can take place using the current data.

Students appeared to view VTI in terms of simplification, describing the flow of the problem as a simplifying act. This desire to make the problem smaller served as a guide for several students. However, it may have also impeded a student's problem solving attempts. As students move on to higher mathematics, such as calculus, many times solving a problem involves representing expressions in visually larger forms. An example would be using partial fraction decomposition in order to integrate a function. Closer to trigonometry would be using trigonometric identities to "enlarge" an expression, visually or cognitively, in order to represent it as something a student knows how to integrate. The issue needing exploration is how the desire to have simplified expressions possibly conflicts with the need to represent an expression in a visually larger form in order to solve problems.

Related to the issue of simplification is the question of what forms students actually consider simpler. From a pedagogical standpoint, knowing that students trend toward desiring a simpler form has implications in how to facilitate the mathematical growth in students. According to Cooper, this desire is inherent. This drive to simplify could be harnessed to expand students' views of what simple entails. If students always view visually smaller expressions as simpler, students could be guided to place the notion of simpler with the context of the current problem. Thus, students could pick representations that best fit the situation and make solving the problem easier. Moreover, methods such as partial fraction decomposition could become vibrant, gaining a rationale, rather than appearing as just another method to learn. Mathematics could become more connected for students rather than appearing as a collection of discrete things to memorize.

Further work needs to continue in how students' conceptions of VTI related to the strategies they used in constructing the proof. At times, students' constructions appeared to have a ritualistic nature, such as using the reflexive step to signal a successful completion. However, students did not always appear thoroughly convinced that the identity was verified despite claiming to be finished.

Related to the reflexive step was students' idiosyncratic use of the equal sign throughout the VTI construction. While literature typically labels students' conceptions of the equal sign as relational or operational, students appeared to be using the equal sign to signal concepts not of a relational or operational nature. Further analysis of the data could reveal other ways to classify usage or perhaps to refine the definitions for relational and operational.

One of the shortcomings of the interview portion of the study was that the majority of the students could be considered above-average students. Because of the self-selection by the "good" students to take part in the summative interviews, conclusions could not be made about whether or not VTI conceptions could be connected to results of assessments such as VTI item scores and course grades. However, the explorations in this study do provide basic frameworks from which to conduct smaller studies. For example, a study would consist of identifying students' beliefs concerning the theorem identity and then determining if student performance could be tied to these beliefs.

While students favored the column layout for VTI constructions, the actual source of this preference was unclear. It could be an artifact from the textbook or a format dictated by a high school teacher. While the source of the preference could be explored, what would be of more importance would be to investigate why students believed the format was clearer. Was it due to a familiarity with the format, or was it because they can actually see the expressions better due to a reduction in clutter. Students reported steps being incorrect or missing in the VTI constructions that had horizontal layouts. If this was true, then format could be a learning barrier that should be negotiated by the students with the teacher. From a pedagogical standpoint, if students miss information due to clutter, then instructors should consider how to best present information visually to remove unnecessary distractions. This issue is not limited to VTI but to any mathematics presented in the classroom on chalkboards, Smart Boards, overhead projectors, or texts.

REFERENCES

Ahlfors, L. V., Bacon, H. M., Bell, C., Bellman, R. E., Bers, L., Birkhoff, G., … Wittenberg, A. (1962). On the mathematics curriculum of the high school. *American Mathematical Monthly, 69*(3) 189-193.

Alcock, L. & Inglis, M. (2010). Visual considerations in the presentation of mathematical proofs. *Seminar.net – International Journal of Media, Technology and Lifelong Learning, 6*(1), 43-59.

Alibali, M. W., Knuth, E. J., Hattikudur, S., McNeil, N. M., & Stephens, A. C. (2007). A longitudinal examination of middle school students' understanding of the equal sign and equivalent equations. *Mathematical Thinking and Learning, 9*(3), 221-247.

Atkin, A. (2013). Peirce's theory of signs. In E. N. Alta (Ed.), *The Stanford Encyclopedia of Philosophy, Summer 2013 Edition*. Forthcoming URL http://plato.stanford.edu/archives/sum2013/entries/peirce-semiotics/

Baldassi, S., Megna, N., & Burr, D. C. (2006). Visual clutter causes high-magnitude errors. *PLoS Biol, 4*(3), e56. doi:10.1371/journal.pbio.0040056

Baralt, M., Pennestri, S., & Selvandin, M. (2011). Action research using wordles to teach foreign language writing. *About Language Learning & Technology, 15*(2), 12-22.

Barnett, R. A., Ziegler, M. R., & Byleen, K. E. (2006). *Analytic trigonometry with applications* (9th ed.). Hoboken, NJ: John Wiley & Sons, Inc.

Barnett, R. A., Ziegler, M. R., Byleen, K. E., & Sobecki, D. (2008). *Analytic trigonometry with applications* (10th ed.). Hoboken, NJ: John Wiley & Sons, Inc.

Bateman, S., Gutwin, C., & Nacenta, M. (2008, June). Seeing things in the clouds: the effect of visual features on tag cloud selections. In *Proceedings of the nineteenth ACM conference on Hypertext and hypermedia* (pp. 193-202).

Berger, M. (2004). The functional use of a mathematical sign. *Educational Studies in Mathematics, 55*(1/3), 81-102.

Blackett, N., & Tall, D. O. (1991). Gender and the versatile learning of trigonometry using computer software. In F. Furinghetti (Ed.), *Proceedings of the 15th conference of the International Group for the Psychology of Mathematics Education* (Vol.1, pp. 144–151). Assisi, Italy: PME.

Booth, L. (1988). Children's difficulties in beginning algebra. In A. F. Coxford (Ed.), *The Ideas of Algebra, K-12 (1988 Yearbook)* (pp. 30-32), Reston, VA: NCTM.

Breidenbach, D., Dubinsky, E., Hawks, J., & Nichols, D. (1992). Development of the process conception of function. *Educational Studies in Mathematics, 23*(3), 247–285.

Brown, S. A. (2005). The trigonometric connection: Students' understanding of sine and cosine. (Unpublished doctoral dissertation). Illinois State University, USA.

CadwalladerOlsker, T. (2011). What do we mean by mathematical proof? *Journal of Humanistic Mathematics, 1*(1), 33-60.

Carlson, M. P., & Bloom, I. (2005). The cyclic nature of problem solving: An emergent multidimensional problem-solving framework. *Educational Studies in Mathematics, 58*(1), 45-75.

Castella, Q., & Sutton, C. (2013). Word storms: Multiples of word clouds for visual comparison of documents. *arXiv preprint arXiv:1301.0503v1*.

Chinnappan, M. (1998). Schemas and mental models in geometry problem solving. *Educational Studies in Mathematics, 36*(3), 201-217.

Choi-Koh, S. S. (2003). Effect of a graphing calculator on a 10th-grade student's study of trigonometry. *The Journal of Educational Research,* 96(6), 359-369.

Cidell, J. (2010). Content clouds as exploratory qualitative data analysis. *Area*,*42*(4), 514-523.

Collins, C., Viegas, F. B., & Wattenberg, M. (2009). Parallel tag clouds to explore and analyze faceted text corpora. In *Visual Analytics Science and Technology, 2009. VAST 2009. IEEE Symposium on Visual Analytics Science and Technology* (pp. 91-98).

Common Core State Standards Initiative. (2010). *Common Core State Standards for mathematics*. Retrieved from http://www.corestandards.org/assets/CCSSI_ELA%20Standards.pdf

College Entrance Examination Board. (1959). *Program for college preparatory mathematics: Report*. New York, NY: CEEB.

Creswell, J. W., & Plano Clark, V. L. (2011). *Designing and conducting mixed methods Research* (2nd ed.). Thousand Oaks, CA: SAGE Publications.

Delice, A. (2002). A model of students' simplification of trigonometric expressions. In S. Goodchild (Ed.), *Proceedings of the British Society for Research into Learning Mathematics, 22*(1&2), 19-24.

Delice, A., & Roper, T. (2006). Implications of a comparative study for mathematics education in the English education system. *Teaching Mathematics and Its Applications: an International Journal of the IMA,* 25(2), 64-72.

De Villiers, M. (1999). *Rethinking proof: With the Geometer's Sketchpad.* Emeryville, CA: Key Curriculum Press.

Dodge. (1976). In *Webster's third new international dictionary of the English language unabridged.* Springfield, MA: G. & C. Merriam Company.

Donovan, M. S., & Bransford, J. D. (Eds.). (2005). *How students learn: Mathematics in the classroom.* Washington, D.C.: The National Academies Press.

Dugdale, S. (1990). Beyond the evident content goals part III. An undercurrent-enhanced approach to trigonometric identities. *The Journal of Mathematical Behavior, 9*(3), 233-287.

Fey, J. T. (1978). Change in mathematics education since the late 1950's – ideas and realization, U.S.A. *Educational Studies in Mathematics, 9*(3), 339-353.

Goldstein, L. J. (1994). *Precalculus and its applications*. Dubuque, IA: Wm. C. Brown Publishers.

Gottron, T. (2009). Document word clouds: Visualising web documents as tag clouds to aid users in relevance decisions. In *Research and Advanced Technology for Digital Libraries* (pp. 94-105).

Granville, W. A. (1910). On the teaching of the elements of plane trigonometry. *The American Mathematical Monthly, 17*(2), 25-31.

Gray, E. M., and Tall, D. O. (1994). Duality, ambiguity, and flexibility: A proceptural view of simple arithmetic. *Journal for Research in Mathematics Education, 25*(2), 115-141.

Harel, G. & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. In E. Dubinsky, A. Schoenfeld and J. Kaput (Eds.), *Research in Collegiate Mathematics Education, III* (pp. 234-283). Providence, RI: American Mathematical Society.

Hersh, R. (1993). Proving is convincing and explaining. *Educational Studies in Mathematics,24*(4), 389-399.

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics, 12*(3), 317-326.

Knuth, E. J. (2002). Secondary school mathematics teachers' conceptions of proof. *Journal of Research in Mathematics Education, 5*(1), 61-88.

Knuth, E. J., Alibali, M. W., McNeil, N. M., Weinberg, A., & Stephens, A. C. (2005). Middle school students' understanding of core algebraic concepts: Equivalence & variable. *Zentralblatt für Didaktik der Mathematik, 37*(1), 68-76.

Knuth, E. J., Stephens, A. C., McNeil, N. M., & Alibali, M. W. (2006). Does understanding the equal sign matter? Evidence from solving equations. *Journal for Research in Mathematics Education, 37*(4), 297-312.

Lester, Jr., F. K. (1994). Musings about mathematical problem-solving research: 1970 – 1994. *Journal for Research in Mathematics Education, 25*(6), 660-675.

MacGregor, M., & Stacey, K. (1997). Students' understanding of algebraic notation: 11-15. *Educational Studies in Mathematics, 33*(1), 1-19.

Mamona-Downs, J. & Mamona, M. (2005). The identity of problem solving. *The Journal of Mathematical Behavior, 24*(3-4), 385-401.

Maor, E. (1998). *Trigonometric delights.* Princeton, NJ: Princeton University Press.

McNaught, C., & Lam, P. (2010). Using Wordle as a supplementary research tool. *The Qualitative Report, 15*(3), 630-643.

McNeil, N. M., & Alibali, M. W. (2005). Knowledge change as a function of mathematics experience: All context are not created equal. *Journal of Cognition and Development, 6*(2), 285-306.

Mercer, J. W. (1913). The teaching of numerical trigonometry. *The Mathematical Gazette, 7*(108), 193-197.

Merriam, S. B. (2002). Introduction to qualitative research. In S. Merriam (Ed.), *Qualitative research in practice: Examples for discussion and analysis* (pp. 3-17). San Francisco, CA: Jossey-Bass.

Merriam, S. B. (2002). Assessing and evaluation qualitative research. In S. Merriam (Ed.), *Qualitative research in practice: Examples for discussion and analysis* (pp. 18-39). San Francisco, CA: Jossey-Bass.

Moise, E. (1962). The new mathematics programs. *The School Review, 70*(1), 82-101.

Moore, K. C. (2010). The role of quantitative and covariational reasoning in developing precalculus students' images of angle measure and central concepts of trigonometry. In *Proceedings of the 13th Conference on Research in Undergraduate Mathematics Education*. Raleigh, NC.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for* *school mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

Nickell, J. (2012). Word clouds in math classrooms. *Mathematics Teaching in the Middle School*, *17*(9), 564-566.

Prove, v. Def. I.1. (2013). In *OED Online*. Oxford University Press. Retrieved from http://www.oed.com/view/Entry/153398isAdvanced=false&result=2&rskey=kRV6iK

Reduce, v. Def. I.2a. (2013). In *OED Online*. Oxford University Press. Retrieved from http://www.oed.com/view/Entry/160503?rskey=mY1LVm&result=2&isAdvanced=false #eid

Reflect, v. (2013). In *OED Online*. Oxford University Press. Retrieved from http://www.oed.com/view/Entry/160912#eid26120048

Rickey, F. A. & Cole, J. P. (1942). *Plane and spherical trigonometry*. New York, NY: The Dryden Press.

Rivadeneira, A. W., Gruen, D. M., Muller, M. J., & Millen, D. R. (2007, April). Getting our head in the clouds: toward evaluation studies of tagclouds. In *Proceedings of the SIGCHI conference on Human factors in computing systems* (pp. 995-998).

Rosenbach, J. B., Whitman, E. A., & Moskovitz, D. (1937). *Plane trigonometry: With tables*. Boston, MA: Ginn and Company.

Sáenz-Ludlow, A., & Walgamuth, C. (1998). Third graders' interpretations of equality and the equal symbol. *Educational Studies in Mathematics, 35*(2), 153-187.

Sajka, M. (2003). A secondary school student's understanding of the concept of function: A case study. *Educational Studies in Mathematics, 53*(3), 229-254.

School Mathematics Study Group. (1961). *Intermediate mathematics: Student's text, part II*. New Haven, CT: Yale University Press.

School Mathematics Study Group. (1965). *Elementary functions: Student's text*. Stanford, CA:

SMSG.

Sfard, A., & Linchevski, L. (1994). The gains and the pitfalls of reification: The case of algebra. *Educational Studies in Mathematics, 26*(2/3)*,* 191-228.

Sherman, J., & Bisanz, J. (2009). Equivalence in symbolic and nonsymbolic contexts: Benefits of solving problems with manipulatives. *Journal of Educational Psychology, 101*(1), 88-100.

Steinberg, R. M., Sleeman, D. H., & Ktorza, D. (1991). Algebra students' knowledge of equivalence of equations. *Journal for Research in Mathematics Education, 22*(2), 122-121.

Strauss, A. & Corbin, J. (1998). *Basics of qualitative research. Techniques and procedures for developing grounded theory*. Thousand Oaks, CA: Sage Publications.

Stylianou, D., Chae, N., & Blanton, M. (2006). Students' proof schemes: A closer look at what characterizes students' proof conceptions. In S. Alatorre, J. Cortina, M. Sáiz, & A. Méndez, (Eds.), *Proceedings of the 28th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 54-60). Mérida, México: Universidad Pedagógica Nacional.

Swain, H. (1955). *How to study mathematics: A handbook for high school students*. Washington, D.C.: National Council of Teachers of Mathematics.

Tall, D. (2002). Differing modes of proof and belief in mathematics. In F.-L. Lin (Ed.), *Proceedings of the International Conference on Mathematics,* Taipei, Taiwan, National Taiwan Normal University, pp. 91-107.

Tall, D., & Razali, M. R. (1993). Diagnosing students' behavior in learning mathematics. *International Journal of Mathematical Education, 24*(2), 209-222.

Tall, D., & Thomas, M. (1991). Encouraging versatile thinking in algebra using the computer. *Educational Studies in Mathematics, 22*(2), 125-147.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational studies in mathematics*, *12*(2), 151-169.

Tirosh, D., Even, R., & Robinson, N. (1998). Simplifying algebraic expressions: Teacher awareness and teaching approaches. *Educational Studies in Mathematics, 35*(1), 51-64.

Van Brummelen, G. (2009). *The mathematics of the heavens and the earth: The early history of trigonometry.* Princeton, NJ: Princeton University Press.

Walker, W. J. (1912). On the teaching of elementary mathematics in secondary schools. *The Mathematical Gazette, 6*(100), 371-375.

Warren, C. A. B., & Karner, T. X. (2010). *Discovering qualitative methods: Field research, interviews, and analysis*. New York, NY: Oxford University Press.

Weber, K. (2001). Student difficulty in constructing proofs: The need for strategic knowledge. *Educational Studies in Mathematics, 48*(1), 101-119.

Weber, K. (2005a). Students' understanding of trigonometric functions. *Mathematics Education Research Journal, 17*(3), 91-112.

Weber, K. (2005b). Problem-solving, proving, and learning: The relationship between problem-solving processes and learning opportunities in the activity of proof construction. *The Journal of Mathematical Behavior, 24*(3-4), 351-360.

Weinberg, A. (2010). Undergraduate students' interpretations of the equals sign. In *Proceedings of the 13th Conference on Research in Undergraduate Mathematics Education*. Raleigh, NC.

Williams, W., Parkes, E. L., & Davies, P. (2013). Wordle: A method for analysing MBA student induction experience. *The International Journal of Management Education, 11*, 44-53.

Wu, H. (n.d.). The mathematics education reform: What is it and why should you care? Retrieved from http://www.math.berkeley.edu/~wu/reform3.pdf

Wu, Y., Provan, T., Wei, F., Liu, S., & Ma, K. L. (2011). Semantic-preserving word clouds by seam carving. *Computer Graphics Forum, 30*(3), 741-750.

Verify, Def. 2. (2013). In *OED Online.* Oxford University Press. Retrived from http://www.oed.com/view/Entry/222511?redirectedFrom=verify

Yin, R. K. (2003). *Case study research: Design and methods*. Thousand Oaks, CA: Sage Publications.

APPENDICES

Appendix A

Summative Interview Participants Profile

	AKE Score (%) (Quartile)	AKE Self-Rating	Mathematical Ability Self-Rating	Exam 2 VTI Score (%)	Quiz 7 Score (%)	Quiz 8 Score (%)
Alan	98 (4th)	4	4.5	97.26	90	100
Amber	80 (4th)	4	5	86.3	90	95
Bella	84 (4th)	4	4	98.6	100	100
Charles	16 (1st)	1	2	56.16	20	70
Cooper	56 (3rd)	2.6	3	93.15	90	95
Helen	74 (4th)	2	4	95.9	90	100
Katie	54 (3rd)	2.5	3	94.5	90	75
Maria	62 (3rd)	3	4	98.63	80	30
Class Mean	---------	2.65	3.45	85.97	72.6	78.9

Appendix B

Math 1613: Biographical Information

1. Name:

2. Age:

3. Please circle your year of study:

   Freshman **Sophomore**
   Junior **Senior**
   Other

4. Please list your academic major.

5. Why did you choose this major?

6. Please list all mathematics and mathematics-based courses taken in high school.

7. Please list all mathematics and mathematics-based courses taken or currently enrolled in at the college level.

8. If you were a mathematical object or symbol, what would you be and why?

Appendix C

Name:______________________________

Algebra Knowledge Exam

Please answer each question, showing all your work and reasoning in the provided space. Do not use any resources except for your mind. Try your best!

1. Perform the following operations and write your answer in lowest terms.
   a. $\frac{1}{2} + \frac{2}{3} =$
   b. $\frac{1}{x} + \frac{2}{x} =$
   c. $\frac{1}{x} - \frac{2}{y} =$
   d. $\frac{2}{3} \cdot \frac{1}{3} =$
   e. $\frac{x}{y} \cdot \frac{z}{x^2} =$
   f. $\frac{1}{x} \div \frac{1}{x+1} =$
   g. $\sqrt{2} \cdot (-\sqrt{2}) =$
   h. $x^{\frac{1}{2}} \cdot x^{\frac{2}{3}} =$
   i. $\sqrt{4+4} =$
   j. $\left(y^{\frac{2}{3}}\right)^{\frac{1}{4}} =$

2. Factor the following expressions.
   a. $x^2 - 5x - 6$
   b. $a^2 + 3a + 2$
   c. $1 - y^2$

3. Perform the following operations and write the answer in lowest terms.

a. $\frac{1}{1-z^2} - \frac{1}{1+z} =$

b. $\frac{b+3}{b^2+3b+2} \cdot \frac{b+2}{b} =$

c. $(x+y)^2 =$

4. Find the values of z that make the following equations true.

a. $(4z-3)+(1-2z)=0$

b. $z^2+3z+3=1$

5.

a. Simplify the following expression to lowest terms.

$$\frac{c^2-4}{c^2+4c+4}$$

b. If

$$g(c) = \frac{c^2-4}{c^2+4c+4},$$

find $g(3)$.

6. Simplify the following expression and write the answer in lowest terms.

$$\frac{x^{-2}y^3(x+y)^2}{(x+y)^3y^{-2}x^{-1}}$$

7.

a. Simplify the following expression to lowest terms:

$$\frac{\frac{1}{x}+y}{\frac{1}{x}-y}$$

b. If

$$f(x)=\frac{\frac{1}{x}+y}{\frac{1}{x}-y},$$

find $f(2)$.

8. Given that

$$(x^2-5)^2-2(x^2-5)+3$$
$$=(f(x))^2-2(f(x))+3,$$

find an equation for $f(x)$ in terms of x.

9. Verify that

$$(x+1)^3=(x+1)^2(x-1)+2(x+1)^2$$

for all x.

Self-assessment

1. **On the scale below, place an X on the location that best represents how you believe you performed on the Algebra Knowledge Exam.**

Outstanding Good Average Poor Very Poor

--------5------------------------4------------------------3------------------------2------------------------1--------

Please explain your choice:

2. **On the scale below, place an X on the location that best represents your mathematical ability.**

Outstanding Good Average Poor Very Poor

--------5------------------------4------------------------3------------------------2------------------------1--------

Please explain your choice:

Instructor Summary and Recommendations

Appendix D

Supplement: Algebraic Simplification

Name:

Objective: In this activity, you will practice simplifying algebraic and trigonometric expressions.

❖ Simplify

In many instances, mathematical terms derive their meanings from everyday words. In your own words, what does it mean to ***simplify***?

❖ Simplifying: Algebraic Expressions

What ***simplify*** means in mathematics is somewhat relative, at times relying on an authority figure, such as a teacher, to provide the definition. Perhaps we can take ***simplify*** to mean ***make smaller*** or ***reduce the clutter***.

Simplifying an algebraic expression is related to actions such as addition, multiplication, factoring, and cancelling factors. As an example,

$$\frac{x^2+x}{x} = \frac{x(x+1)}{x} = \frac{x}{x} \cdot \frac{x+1}{1} = x+1.$$

So, by simplifying,

$$\frac{x^2+x}{x} = x+1.$$

❖ Practice

Simplify the following algebraic expressions by adding, multiplying, factor, cancelling etc.

1. $\frac{x^2-y^2}{x-y} =$

2. $\frac{x}{y} \cdot \frac{y^2+y}{x} =$

3. $\frac{x}{y} + \frac{y}{x} =$

4. $z + \frac{1}{x} =$

5. $x \cdot \frac{y}{x^2+x} =$

6. $\frac{x^2+3x+2}{x+1} =$

7. $\frac{\frac{x}{y}+z}{\frac{x}{y}-z} =$

❖ Simplifying: Trigonometric – as – Algebraic Expressions

Simplifying a trigonometric expression is important in working with trigonometric functions. In order to help us reduce the clutter, we can concentrate on the form of the trigonometric expression. To help, we can be very explicit by actually rewriting the trigonometric expression as an algebraic expression with a matching form.

For example, the expression $\sin^2 x$ has the form "an object squared." So instead of $\sin^2 x$, we can write a^2.

❖ Practice

Simplify the following trigonometric expressions by:

- first, rewriting it as a matching algebraic expression,
- then simplifying the algebraic expression,
- and finally converting back into the appropriate trigonometric form.

For example, to simplify $\sin^2 x/\sin x$, we would have

$$\frac{\sin^2 x}{\sin x} \quad \to \quad \frac{a^2}{a} = a \quad \to \quad \sin x,$$

so,

$$\frac{\sin^2 x}{\sin x} = \sin x\,.$$

1. $\frac{\sin^2 x - \cos^2 x}{\sin x - \cos x}$

2. $\tan x + \frac{1}{\sec x}$

3. $\frac{\frac{\sin x}{\cos x} + \cot x}{\frac{\sin x}{\cos x} - \cot x}$

❖ Simplifying: Trigonometric Expressions

Recall that trigonometric functions are objects. Explicitly rewriting the trigonometric expression as an algebraic expression emphasizes this; you think of $\sin x$ as some object a or $\cos x$ as some object b.

But if you remember that trigonometric functions are objects, things that can be added, multiplied, and raised to exponents for example, you don't need to explicitly rewrite the trigonometric expression as an algebraic expression.

For example, thinking of $\sin x$ as an object,

$$\frac{\sin^2 x + \sin x}{\sin x} = \frac{\sin x(\sin x + 1)}{\sin x} = \frac{\sin x}{\sin x} \cdot \frac{\sin x + 1}{1},$$

which equals $\sin x + 1$.

❖ Practice

Simplify the following trigonometric expressions without first rewriting as an algebraic expression.

1. $\csc x + \csc x =$

2. $\frac{1}{\cot x} + \cot x =$

3. $\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} =$

Appendix E

Supplement: Verifying Identities

Name:

Objective: In this activity, you will learn about verifying trigonometric identities.

❖ Verify

In your own words, what does it mean to ***verify*** that $A = B$?

❖ Algebraic Manipulations and Trigonometric Substitutions

State whether the following transformations involve an algebraic manipulation or a trigonometric substitution.

1. $\sec x \sin x = \dfrac{1}{\cos x} \cdot \sin x$

2. $\dfrac{1}{\cos x} \cdot \sin x = \dfrac{\sin x}{\cos x}$

3. $\dfrac{\sin x}{\cos x} = \tan x$

❖ Showing Equality

Consider the following equalities.

1. Why does
$$\csc^2 x + \sec^2 x = \frac{1}{\sin^2 x} + \frac{1}{\cos^2 x} \ ?$$

2. Why does
$$\frac{1}{\sin^2 x} + \frac{1}{\cos^2 x} = \frac{\cos^2 x}{\sin^2 x \cos^2 x} + \frac{\sin^2 x}{\sin^2 x \cos^2 x} \ ?$$

3. Why does
$$\frac{\cos^2 x}{\sin^2 x \cos^2 x} + \frac{\sin^2 x}{\sin^2 x \cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\sin^2 x \cos^2 x} \ ?$$

4. Why does
$$\frac{\cos^2 x + \sin^2 x}{\sin^2 x \cos^2 x} = \frac{1}{\sin^2 x \cos^2 x} \ ?$$

5. Why does
$$\frac{1}{\sin^2 x \cos^2 x} = \frac{1}{\sin^2 x} \cdot \frac{1}{\cos^2 x} \text{ ?}$$

6. Why does
$$\frac{1}{\sin^2 x} \cdot \frac{1}{\cos^2 x} = \csc^2 x \cdot \sec^2 x \text{ ?}$$

7. What can you say about $\csc^2 x + \sec^2 x$ and $\csc^2 x \cdot \sec^2 x$? Explain.

❖ Verifying Trigonometric Identities

When someone asks you to verify a trigonometric identity, what does this mean to you?

Appendix F

Supplement: Assessing Verification

Name:

Objective: In this activity, you will assess examples of verifying trigonometric identities, gaining a deeper understanding of what constitutes verification.

❖ Logical Arguments

When you verify a trigonometric identity on an exam or homework problem, you are writing down an argument to convince yourself and others of the equality. A reader of the verification should be able to follow and understand the logic of the argument.

Read through the following verifications of the same identity. Judge the quality of the logic and the quality of the presentation. State your overall score, on a scale of 0 to 5, where 0 is the lowest score, and 5 is the highest score; also, provide a justification for your score.

❖ Practice

1. Verify

$$\frac{\tan y + 1}{\sec y} = \sin y + \cos y.$$

Solution:

$$\frac{\tan y + 1}{\sec y} = \sin y + \cos y$$

$$\frac{\tan y + 1}{\sec y} = \sin y + \frac{1}{\sec y}$$

$$\frac{\tan y + 1}{\sec y} = \frac{\sin y \cdot \sec y + 1}{\sec y}$$

$$\frac{\tan y + 1}{\sec y} = \frac{\sin y \cdot \frac{1}{\cos y} + 1}{\sec y}$$

$$\frac{\tan y + 1}{\sec y} = \frac{\tan y + 1}{\sec y}$$

Score: /5

Reason:

2. Verify

$$\frac{\tan y + 1}{\sec y} = \sin y + \cos y.$$

Solution:

$$\frac{\tan y + 1}{\sec y} = \frac{\frac{\sin y}{\cos y} + 1}{\frac{1}{\cos y}}$$

$$= \frac{\frac{\sin y}{\cos y} + \frac{\cos y}{\cos y}}{\frac{1}{\cos y}}$$

$$= \frac{\frac{\sin y + \cos y}{\cos y}}{\frac{1}{\cos y}}$$

$$= \frac{\frac{\sin y + \cos y}{\cos y} \cdot \frac{\cos y}{1}}{\frac{1}{\cos y} \cdot \frac{\cos y}{1}}$$

$$= \frac{\frac{\sin y + \cos y}{1}}{1}$$

$$= \sin y + \cos y$$

Score: /5

Reason:

3. Verify

$$\frac{\tan y + 1}{\sec y} = \sin y + \cos y.$$

Solution:

$$\frac{\tan y + 1}{\sec y} = \frac{\tan y}{\sec y} + \frac{1}{\sec y} = \left(\frac{\sin y}{\cos y}\right) \div \frac{1}{\cos y} + \cos y$$

$$= \frac{\sin y}{\cos y} \cdot \frac{\cos y}{1} + \cos y = \sin y + \cos y$$

Score: /5

Reason:

4. Verify

$$\frac{\tan y + 1}{\sec y} = \sin y + \cos y.$$

Solution:

$$\frac{\tan y + 1}{\sec y} = \sin y + \cos y$$

$$= \sin y + \frac{1}{\sec y}$$

$$= \frac{\sin y \cdot \sec y + 1}{\sec y}$$

$$= \frac{\sin y \cdot \frac{1}{\cos y} + 1}{\sec y}$$

$$= \frac{\frac{\sin y}{\cos y} + 1}{\sec y}$$

$$\frac{\tan y + 1}{\sec y} = \frac{\tan y + 1}{\sec y}$$

Score: /5

Reason:

5. Verify

$$\frac{\tan y + 1}{\sec y} = \sin y + \cos y.$$

Solution:

$$\sin y + \cos y = \sin y + \frac{1}{\sec y} = \frac{\sin y \cdot \sec y + 1}{\sec y}$$

$$= \frac{\sin y \cdot \frac{1}{\cos y} + 1}{\sec y} = \frac{\frac{\sin y}{\cos y} + 1}{\sec y}$$

$$= \frac{\tan y + 1}{\sec y}$$

Score: /5

Reason:

Appendix G

Math 1613.008 Name:_______________________________
Spring 2012
In-class Quiz 7

Make sure to clearly show your work and reasoning. You will receive a 0 on a problem if you show no work or reasoning.

1. (4 pts) Is $\cos(x) = 1 - \sin(x)$ an identity? Please explain your response.

2. (6 pts) Verify the following trigonometric identity:

$$\csc^2 \theta + \sec^2 \theta = \csc^2 \theta \cdot \sec^2 \theta$$

   a. State the expression you will start with and provide your reason for choosing this expression.

   b. Using algebraic manipulations and trigonometric identities, transform your beginning expression to the other given expression. State your reasoning for each step you take.

Work	**Reasoning**

Appendix H

In-Class Quiz: Verifying Identities Name:

This is an attendance and participation quiz. Completion of the quiz constitutes attendance and participation for the day.

1. Verify the following identity. Please provide a justification for each step indicating why you chose that particular step.

$$\tan x + \cot x = \sec x \csc x$$

2. In your opinion, what is the purpose of verifying trigonometric identities in a trigonometry course?

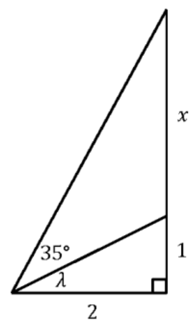
Appendix I

Math 1613.008
Spring 2012
Homework Quiz 8

Name:__________________________________

This homework quiz is due at the beginning of class on Thursday, March 8. You should provide complete solutions to each problem in the spaces provided. You may seek help from anyone or anything, but only ***your*** ideas and understandings should be written on this paper.

1. (4 pts) In this question, you will determine the length x in the figure below by following the steps.



   a. Use the figure to find the exact value of $\tan(\lambda)$.
   b. Use the addition formula for tangent to find the exact value of $\tan(\lambda + 35°)$.
   c. Use the figure to express $\tan(\lambda + 35°)$ in terms of x.
   d. Equate your equations for $\tan(\lambda + 35°)$ and solve the resulting equation for x. Answer to two decimal places

Don't forget about page 2

2. (5 pt) Verify that the proposed identity is true for all appropriate x and y.

$$\tan(x-y)=\frac{\cot y-\cot x}{\cot x\cot y+1}$$

3. (1 pt) Over the past several weeks, you have verified numerous identities during class, in homework assignments, and on quizzes. ***In your opinion, what has been the point of you verifying these identities?*** (Please provide an honest, thought-out response, not just a short, flippant comment. Thanks.)

Appendix J

Writing Prompt: Verifying Identities Name:

This is an attendance and participation prompt. Completion of the prompt constitutes attendance and participation for the day.

A ***simile*** is an expression in the English language in which one thing is said to be like another thing. The following simile was said by the main character in the movie *Forrest Gump*:

Life was like a box of chocolates.

The character then provided an explanation of his simile:

You never know what you're gonna get.

Please complete the following phrase to form a simile:

To me, verifying a trigonometric identity is like

____

____.

Using several sentences, please clearly explain your simile.

Appendix K

Exam 2 VTI Item

6. (10 pts) Verify that the following equation is an identity.

$$\frac{\tan y + \cot y}{\csc y \cdot \sec y} = 1$$

Appendix L

Final Exam VTI Item

Verify the identity

$$\sin(\theta)\cos(\theta)\,[\,\tan(\theta) + \cot(\theta)\,] = 1\,.$$

Appendix M

Student/Teacher Interaction Log

Date:	**Topics discussed:**
Student(s):	**Interaction type:**
Describe interaction:	
Areas noted for student understanding improvement:	**Areas noted for discussion facilitation improvement:**
Reflection:	

Appendix N

Teacher Journal Entry

<table>
<tr><td>Date:</td><td>Topics discussed in class:</td></tr>
<tr><td colspan="2">Describe general classroom action:</td></tr>
<tr><td colspan="2">Describe general classroom discourse:</td></tr>
<tr><td>Areas noted for student understanding improvement:</td><td>Areas noted for discussion facilitation improvement:</td></tr>
<tr><td colspan="2">Reflection:</td></tr>
</table>

Appendix O

Interview Protocol

[Welcome the participant to the interview.]

Introductory and Establishing Questions

Thank you for assisting me with my project. I really appreciate your willingness.

1. How has your semester been going?
2. Do you have any big plans for the summer?

Preliminaries

Before we begin, I want to go over a few things.

1. First, let me know at any time, and I can stop the interview.
2. Also, I will be taking some notes during the interview. I would be more than happy to show you what I wrote down after the interview is over.
3. Last, in my transcription of our interview, I will use a pseudonym for you known only to me.
4. If you don't have any questions, I'm going to begin recording.

[Begin recording.]

Preliminary Questions

Now, the main goal is to talk about verifying identities, but before doing that, I'd like to ask you some general questions.

1. How would you characterize yourself as a math student?
2. Tell me a little bit about trigonometric identities. What are they? How have you used them?

Questions Regarding Verifying Trigonometric Identities

Now, let's look at verifying some trigonometric identities. I'm interested in the way you go about solving these problems. I'm going to give you questions on sheets of paper. I ask that you carefully read each question aloud. If you need any clarification, please ask. If you feel you need additional information, let me know. I also have a sheet with some common identities if you need them. Please solve the problems as if you were solving them on an exam and write down any necessary work on the sheet of paper. Additionally, as you solve the problem and write your solution, please tell me what you are thinking. Don't just tell me what you are doing; tell me why you are doing it.

[Provide each prompt; what is listed is the particular problem.]

1. (VTI1) Do you consider the following equation to be an identity?

$$\tan^2 x = \frac{\sin^2 x}{\cos^2 x}$$

2. (VTI2) Verify that:

$$\frac{1}{1 - \cos^2 \theta} = \csc^2 \theta$$

3. I was wondering if you could help me out. I've noticed that some students have verified identities similar to this [provide VTI3]. Does their work seem convincing to you? Why do you think they wrote $\csc^2 \theta = \csc^2 \theta$ at the last step?

4. (VTI4) Verify that:

$$\frac{1}{1 - \cos^2(2\alpha - 1)} = \csc^2(2\alpha - 1)$$

5. To you, what does it mean that you have verified a trigonometric identity? What do you believe you have accomplished after verifying an identity?

6. (VTI5) What does the following expression equal? Does it equal anything else?

$$\frac{\sin 2x}{\sin x}$$

7. (VTI6) Verify that:

$$\sin 2y = \tan y\,(1 + \cos 2y)$$

8. (VTI7) Verify that:

$$\cos \rho = \frac{\csc \rho}{\cot \rho + \tan \rho}$$

   a. Before you solve the problem, what does that equation mean to you?

   b. Now that you have solved the problem, what does that equation mean to you?

9. I need some more help. Let me show you some work from another student. [Provide VTI8]. Does this work seem convincing to you? Why?

10. (VTI9) Verify that:

$$\tan^2 \lambda \cdot \sin^2 \lambda = \tan^2 \lambda - \sin^2 \lambda$$

11. I've noticed some things while looking at work of other students, and I was hoping you could help me out. [Provide VTI10]. Is the work convincing to you? Why do you think this student kept writing down the expression on the right side of the equal sign?

12. While verifying identities, what has been the biggest factor in preventing your success? What about for your friends or others in the class?

13. What are your preferred strategies that you use to verify an identity? In general terms, please describe how you go about verifying an identity.

14. Let me show you a way that a student verified the identity on the exam. [Provide VTI11]. What is your opinion of the work?

15. I've noticed on some student work that people are not writing down the variable with the function. For example, instead of writing sin x, they just write sin. Why do you think this is the case?

[I anticipate asking probing questions similar to the following during questions 2, 4, 7, 8, and 10.]

a. Why did you start where you did?
b. What led you to take that step?
c. Why did you choose that particular action?
d. What within the problem guided you?
e. How would you describe the general flow of the problem?

Conclusion

Well, that's about it. Thank you so much for your time. I appreciate it..

1. After having verified a bunch of identities in class and just now, what do you believe are the benefits to you of doing the verifications?
2. Before I let you go, is there anything you would like to add?

[Turn off recording.]

Again, I appreciate you stopping by and giving up your time to help me. I was hoping it would be okay with you if once I did a little bit of the analysis of our interview, if I could get your opinion about my impressions?

Appendix P

Common Trigonometric Identities

Pythagorean Identities	$\sin^2 x + \cos^2 x = 1$ $\cot^2 x + 1 = \csc^2 x$ $\tan^2 x + 1 = \sec^2 x$
Double-angle Identities	$\sin 2x = 2 \sin x \cos x$ $\cos 2x = \cos^2 x - \sin^2 x$ $= 1 - 2\sin^2 x$ $= 2\cos^2 x - 1$
Angle sum Identities	$\sin(x + y) = \sin x \cos y + \sin y \cos x$ $\cos(x + y) = \cos x \cos y - \sin x \sin y$ $\tan(x + y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$
Angle difference Identities	$\sin(x - y) = \sin x \cos y - \sin y \cos x$ $\cos(x - y) = \cos x \cos y + \sin x \sin y$ $\tan(x - y) = \dfrac{\tan x - \tan y}{1 + \tan x \tan y}$

Appendix Q

Summative Interview Prompts

VTI1

Do you consider the following equation to be an identity?

$$\tan^2 x = \frac{\sin^2 x}{\cos^2 x}$$

VTI2

Verify that:

$$\frac{1}{1-\cos^2\theta}=\csc^2\theta$$

VTI3

Verify that:

$$\frac{1}{1 - \cos^2\theta} = \csc^2\theta$$

Student work:

$$\frac{1}{1-\cos^2\theta} =$$

$$\frac{1}{\sin^2\theta} =$$

$$\csc^2\theta = \csc^2\theta$$

So, $\frac{1}{1-\cos^2\theta} = \csc^2\theta$

VTI4

Verify that:

$$\frac{1}{1-\cos^2(2\alpha-1)} = \csc^2(2\alpha-1)$$

VTI5

What does the following expression equal?

$$\frac{\sin 2x}{\sin x}$$

VTI6

Verify that:

$$\sin 2y = \tan y\,(1 + \cos 2y)$$

VTI7

Verify that:

$$\cos \rho = \frac{\csc \rho}{\cot \rho + \tan \rho}$$

VTI8

Verify that:

$$\cos\rho = \frac{\csc\rho}{\cot\rho + \tan\rho}$$

Student work:

$$\cancel{\cos\rho = \frac{\csc\rho}{\cot\rho+\tan\rho}}$$

$$\cos\rho \cdot (\cot\rho + \tan\rho) = \csc\rho$$

$$\cos\rho \cdot \left(\frac{\cos\rho}{\sin\rho} + \frac{\sin\rho}{\cos\rho}\right) = \frac{1}{\sin\rho}$$

$$\cos\rho \cdot \left(\frac{\cos^2\rho}{\sin\rho\cdot\cos\rho} + \frac{\sin^2\rho}{\sin\rho\cdot\cos\rho}\right) = \frac{1}{\sin\rho}$$

$$\cos\rho \cdot \left(\frac{\cos^2\rho+\sin^2\rho}{\sin\rho\cdot\cos\rho}\right) = \frac{1}{\sin\rho}$$

$$\cancel{\cos\rho} \cdot \left(\frac{1}{\sin\rho\cdot\cancel{\cos\rho}}\right) = \frac{1}{\sin\rho}$$

$$\frac{1}{\sin\rho} = \frac{1}{\sin\rho}$$

VTI9

Verify that:

$$\tan^2 \lambda \cdot \sin^2 \lambda = \tan^2 \lambda - \sin^2 \lambda$$

VTI10

Verify that:

$$\tan^2 \lambda \cdot \sin^2 \lambda = \tan^2 \lambda - \sin^2 \lambda$$

Student work:

$$\tan^2 \lambda \cdot \sin^2 \lambda = \tan^2 \lambda - \sin^2 \lambda$$

$$\tan^2 \lambda \cdot (1 - \cos^2 \lambda) = \tan^2 \lambda - \sin^2 \lambda$$

$$\tan^2 \lambda - \tan^2 \lambda \cdot \cos^2 \lambda = \tan^2 \lambda - \sin^2 \lambda$$

$$\tan^2 \lambda - \frac{\sin^2 \lambda}{\cos^2 \lambda} \cdot \cos^2 \lambda = \tan^2 \lambda - \sin^2 \lambda$$

$$\tan^2 \lambda - \sin^2 \lambda = \tan^2 \lambda - \sin^2 \lambda$$

$$\text{So, } \tan^2 \lambda \cdot \sin^2 \lambda = \tan^2 \lambda - \sin^2 \lambda$$

Verify that:

$$\frac{\tan y + \cot y}{\csc y \cdot \sec y} = 1$$

Student work:

a. $\frac{\tan y+\cot y}{\csc y\cdot\sec y}$

b. $\dfrac{\frac{\sin^2 y}{\sin y\cos y}+\frac{\cos^2 y}{\sin y\cos y}}{\frac{1}{\sin y}\cdot\frac{1}{\cos y}} \rightarrow \dfrac{\frac{\sin^2 y+\cos^2 y}{\sin y\cos y}}{\frac{1}{\sin y\cos y}}$

$$\frac{\frac{1}{\sin y\cos y}}{\frac{1}{\sin y\cos y}} \rightarrow \frac{1}{\sin y\cos y}\cdot\left(\frac{\sin y\cos y}{1}\right)$$

$$= 1$$

c. $\frac{\tan y+\cot y}{\csc y\cdot\sec y} = 1$

Appendix R

Oklahoma State University Institutional Review Board

Date: Tuesday, January 17, 2012

IRB Application No AS11136

Proposal Title: An Exploration of College Students' Problem Solving Behaviors While Verifying Trigonometric Identities: A Mixed Methods Case Study

Reviewed and Processed as: Expedited

Status Recommended by Reviewer(s): Approved **Protocol Expires: 1/16/2013**

Principal Investigator(s):

Ben Westcoatt
432 Math Science
Stillwater, OK 74078

Doug Aichele
401 MS
Stillwater, OK 74078

The IRB application referenced above has been approved. It is the judgment of the reviewers that the rights and welfare of individuals who may be asked to participate in this study will be respected, and that the research will be conducted in a manner consistent with the IRB requirements as outlined in section 45 CFR 46.

☒ The final versions of any printed recruitment, consent and assent documents bearing the IRB approval stamp are attached to this letter. These are the versions that must be used during the study.

The reviewer(s) had these comments:
Thank you for a well-prepared application!

As Principal Investigator, it is your responsibility to do the following:

1. Conduct this study exactly as it has been approved. Any modifications to the research protocol must be submitted with the appropriate signatures for IRB approval.
2. Submit a request for continuation if the study extends beyond the approval period of one calendar year. This continuation must receive IRB review and approval before the research can continue.
3. Report any adverse events to the IRB Chair promptly. Adverse events are those which are unanticipated and impact the subjects during the course of this research; and
4. Notify the IRB office in writing when your research project is complete.

Please note that approved protocols are subject to monitoring by the IRB and that the IRB office has the authority to inspect research records associated with this protocol at any time. If you have questions about the IRB procedures or need any assistance from the Board, please contact Beth McTernan in 219 Cordell North(phone: 405-744-5700, beth.mcternan@okstate.edu).

Sincerely,

Shelia M. Kennison

Shelia Kennison, Chair
Institutional Review Board

Oklahoma State University Institutional Review Board

Date:	Monday, March 19, 2012	Protocol Expires:	1/16/2013
IRB Application No:	AS11136		
Proposal Title:	An Exploration of College Students' Problem Solving Behaviors While Verifying Trigonometric Identities: A Mixed Methods Case Study		
Reviewed and Processed as:	Expedited **Modification**		

Status Recommended by Reviewer(s) **Approved**

Principal Investigator(s):

Ben Westcoatt
432 Math Science
Stillwater, OK 74078

Doug Aichele
401 MS
Stillwater, OK 74078

The requested modification to this IRB protocol has been approved. Please note that the original expiration date of the protocol has not changed. The IRB office MUST be notified in writing when a project is complete. All approved projects are subject to monitoring by the IRB.

☒ The final versions of any printed recruitment, consent and assent documents bearing the IRB approval stamp are attached to this letter. These are the versions that must be used during the study.

The reviewer(s) had these comments:

The modification request to change the interview protocol and interview prompts is approved.

Signature :

Shelia Kennison, Chair, Institutional Review Board

Monday, March 19, 2012
Date

Appendix S

INFORMED CONSENT FORM

Project Title:
An Exploration of College Students' Problem Solving Behaviors While Verifying Trigonometric Identities: A Mixed Methods Case Study

Investigator:
Benjamin Wescoatt, PhD student, Mathematics Department, Oklahoma State University

Purpose:
The purpose of this study relates to describing how a student verifies trigonometric identities, focusing on aspects of a student's problem solving behavior during verification. Specifically, this study will emphasize describing the cues for the decisions students make while verifying, describing the resources students utilize while verifying, and describing how verification looks. Connecting these aspects, the study will also describe the extent that the aspects affect students' success in verifying trigonometric identities.

Procedures:
As part of the trigonometry course, students will submit a Biographical Information sheet, take an Algebra Knowledge Exam, complete homework assignments and supplementary activity worksheets, take semester and final exams, and interact with the instructor during class time and during office hours. Participants in the study will allow the researcher to use their responses in the coursework related to verifying trigonometric identities to answer the study's questions. Participant course grades on assignments are not factors in the study.

Following completion of the second semester exam, the researcher will elicit via email 6 volunteers to participate in a one-on-one interview with the researcher. During the interview, the participant will verify trigonometric identities and answer questions related to his or her understanding of trigonometric identities and verifying trigonometric identities. The participants will write problem solution attempts on provided paper. The interview will be audiotaped and will last about 1 to 1.5 hours and will be conducted at a time convenient to both researcher and participant; the location of the interview will be a room within the MSCS building.

Risks of Participation:
There are no known risks associated with this project which are greater than those ordinarily encountered in daily life.

Benefits:
Through reflection on knowledge of mathematical tasks and objects, students may increase their mathematical understanding. Knowledge of student understanding and behavior will benefit educators as educators make decisions concerning pedagogy and curriculum development.

Confidentiality:
Signed consent forms will be collected and stored separately from any research data. Each completed instrument will be identified with a code; names will be removed from student work. An electronic document connecting student name to code will be created and maintained in a password-protected folder on a password-protected computer. This file will be erased upon completion of data collection and analysis. After selection of interview participants, the researcher will provide a pseudonym for the interview participants. Only the researcher will have knowledge of which participant the pseudonym belongs to. An electronic document connecting student name to pseudonym will be maintained in a password-protected folder on a password-protected computer. This file will be erased upon completion of data collection and analysis. Only the researcher will have access to the

document.

The researcher will scan and make electronic files of collected coursework. These electronic files will be stored in password-protected folders on a password-protected computer.

The participant interviews will be recorded on a digital voice recorder. These digital recordings will be uploaded into a password-protected folder on a password-protected computer. The original recording on the device will be erased after successful uploading. Electronic transcripts of the interviews will be maintained in a password-protected folder on a password-protected computer. Written problem solutions generated during the interview will have any identifying features removed after the interview is completed. Hard copies of all data will be maintained in a secure location. All electronic data will be backed up in a password-protected file on a password-protected computer. All hard copies of data will be destroyed by August 31, 2013, and all electronic data will be destroyed by August 31, 2017. Access to student data will be restricted to the researcher and his advisor.

The records of this study will be kept private. Any written results will discuss group findings and will not include information that will identify you. Research records will be stored securely and only researchers and individuals responsible for research oversight will have access to the records. It is possible that the consent process and data collection will be observed by research oversight staff responsible for safeguarding the rights and wellbeing of people who participate in research.

Compensation:
None.

Contacts:
Benjamin Wescoatt, Oklahoma State University, 432 MSCS, 405-744-2307, wescoatt@math.okstate.edu

Douglas B. Aichele, Oklahoma State University, 410 MSCS, 405-744-5688, aichele@math.okstate.edu

If you have questions about your rights as a research volunteer, you may contact Dr. Shelia Kennison, IRB Chair, 219 Cordell North, Stillwater, OK 74078, 405-744-3377 or irb@okstate.edu.

Participant Rights:
Participation in the current research study is entirely voluntary. You are free to decline to participate and may stop or withdraw from the study at any time. You may consent to allow your coursework to be used and refuse to be interviewed; giving consent now does not obligate you to be interviewed later. There is no penalty for refusing or withdrawing your participation.

Signatures:
I have read and fully understand the consent form. I sign it freely and voluntarily. A copy of this form has been given to me.

_______________________ ______________
Signature of Participant Date

I certify that I have personally explained this document before requesting that the participant sign it.

_______________________ ______________
Signature of Researcher Date

Appendix T

Researcher's Recruitment Script

Ben Wescoatt, your instructor, is conducting a study to collect information related to verifying trigonometric identities, a topic you will start covering about 7 weeks from now. He is interested in your understandings of trigonometric identities, how you verify trigonometric identities, and what motivates the decisions you make while verifying trigonometric identities. He believes that the knowledge you provide will be beneficial to other students and instructors as we best try to help students understand mathematics.

As part of the course, you will work homework problems, work supplemental worksheets, take exams, and interact with your instructor in and out of the classroom. Ben is asking your permission to use your coursework to help answer the study's questions. Note that Ben is not requesting that you do any extra work or assignments. He would merely like to use the work that you will already be doing as a natural part of the course.

Additionally, after the second semester exam, he would like to conduct one-on-one interviews with six individuals. During your interview, lasting from 1 to 1.5 hours, he will ask you to verify some trigonometric identities. He will also ask questions related to your understanding of trigonometric identities and verifying trigonometric identities. After the second exam, Ben will contact you if he wants to interview you. While he hopes you accept, you may refuse his request. In other words, even though you let him use your coursework for his study, you always have the option to decline his request for an interview.

To be clear, Ben is not requesting to use your course grades in the study. Also, participation or non-participation in the study will not affect your course grade. If you feel that your participation or non-participation is affecting your grade, please contact me.

Your confidentiality is important to Ben. Any electronic files or hard copies of data pertinent to the study will have your name removed and replaced with a code. Only he will have access to the code key. After all of the data has been collected and analyzed, he will destroy the code key. For the write-up of the study, he will use pseudonyms for all participants. Only he will know who the pseudonym represents.

[At this time, pass out the Informed Consent Form.] If you wish to participate in this study, please follow along. [Describe the Informed Consent Form.] If you choose to allow your instructor to use your work in his study and possibly interview you, please sign and date the Informed Consent Form. You are not obligated to take part in the study, and even after giving consent, you may withdraw from the study or refuse to be interviewed without any fear of being penalized.

Appendix U

Interview Recruitment Email

Subject: Trigonometric Identities Study: Interview

Dear _____________,

Thank you for participating in my study on verifying trigonometric identities by letting me use your coursework. I am writing to you to ask for your help again.

I am very interested in your understandings involving trigonometric identities. To learn more, I would like to interview you as you verify trigonometric identities, recording the audio of the session with a digital voice recorder. The interview will last around 1 to 1.5 hours and will take place on campus.

Your participation in the interview will be confidential. I will not tell others about your participation or non-participation. Additionally, I will not use your real name in the transcription of the interview.

Although I hope you agree to be interviewed, please do not feel pressured into saying yes.

If you are willing to let me interview you, just respond to this email, and I will begin to make arrangements on a time and place that best matches your schedule. If you decline to be interviewed, please let me know as well. I know that you are busy, so I value the time you can provide to me.

I hope to hear from you soon.

--
Sincerely,
Ben Wescoatt

VITA

Benjamin Mark Wescoatt

Candidate for the Degree of

Doctor of Philosophy

Thesis: AN EXPLORATION OF COLLEGE STUDENTS' PROBLEM SOLVING BEHAVIORS WHILE VERIFYING TRIGONOMETRIC IDENTITIES: A MIXED METHOD CASE STUDY

Major Field: Mathematics, with Specialization in Mathematics Education

Biographical:

Personal Data:
Born 20 October 1977, Chico, California.

Education:
Completed the requirements for the Doctor of Philosophy in Mathematics, with Specialization in Mathematics Education at Oklahoma State University, Stillwater, Oklahoma in July, 2013.

Completed the requirements for the Master of Science in Applied Mathematics at Oklahoma State University, Stillwater, Oklahoma in May,.2007

Completed the requirements for the Bachelor of Science in Mathematics at Oklahoma State University, Stillwater, Oklahoma in December, 2000.

Experience:
Graduate Teaching Assistant, Department of Mathematics, Oklahoma State University, Stillwater, Oklahoma, August 2005 to July 2013.
Division Director, Officer Mathematics and Physics, Navy Nuclear Power School, Goose Creek, South Carolina, March 2004 to March 2005.
Instructor, Officer Mathematics and Physics, Navy Nuclear Power School, Goose Creek, South Carolina, July 2002 to March 2005.
Instructor, Enlisted Mathematics, Navy Nuclear Power School, Goose Creek, South Carolina, July 2001 to July 2002.
Commissioned Officer, United States Navy, December 18, 2000 to June 18, 2007.

Professional Memberships:
Mathematical Association of America
Research Council on Mathematics Learning