CASE STUDIES OF INSTRUCTIONAL PRACTICES IN PROOF-BASED MATHEMATICS LECTURES

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CASE STUDIES OF INSTRUCTIONAL PRACTICES IN

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may your curiosity and love of learning never fail.

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Abstract: This multi-case study investigates the teaching practices of four instructors who were teaching undergraduate level proof-based mathematics courses using lecture methods. Both interview data with the instructors and several video observations of each classroom were gathered throughout the course of a semester. The data analysis techniques were primarily qualitative, but included some quantitative methods such as frequency counts and percentages to give an overall picture of the instructors' teaching. Analysis occurred in several phases, and used multiple units of analysis including the proof presentations, examples used in proof presentations, the class period, and the individual instructor questions.

The first and second research questions addressed the pedagogical moves that the instructor makes during proof presentations, and the instructors' allocation of class time. The instructors spent between 35% and 70% of their class time presenting proofs. The proof presentation techniques that were identified in the interviews were *outline, examples, logical structure,* and *context.* At least one of these strategies was observed in 94% of the proof presentations in the video data. Three of the four instructors expected active engagement in 95% of their proof presentations, while the fourth expected active engagement in 50% of his proof presentations. The proportion of class time spent on interactive lecture ranged from 26% to 62%.

The third and fourth research questions addressed the uses, types, and timing of examples during the instructors' proof presentations. Examples used during the observed proof presentations were used to create a framework that describes the uses of examples, types of examples, and chronological placement of examples in their proof presentations.

The fifth and sixth research questions addressed the questions posed by the instructors. The question rates ranged from 0.69 to 1.81 questions per minute, and that the percentage of higher order questions ranged from 30.1% to 54.2%. The percentage of questions to which students responded ranged from 35% to 52%. When restricting to only questions which were answered by students, if was found that in all four cases the percentage of answered questions that were higher-order matched the percentage of asked questions that were higher order. Thus, students were answering a variety of types of questions.

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CHAPTER I

INTRODUCTION

Instructional practices at the undergraduate level have been largely unexamined despite repeated calls for such studies (Harel & Fuller, 2009; Harel & Sowder, 2007; Speer, Smith, & Horvath, 2010). Many of the studies that focus on teaching practice have occurred in lower division courses like calculus, where students are expected to be able to do computations and applications (Bressoud, 2011; Epstein, 2007; Speer & Wagner, 2009; Speer, 2008; Wagner, Speer & Rossa, 2007). In advanced mathematics courses, the content shifts to formal mathematics, and undergraduates are expected to be able to comprehend and write mathematical proofs. This transition is notoriously difficult for undergraduates (Grassl & Mingus, 2007; Larsen, 2009; Larsen & Zandieh, 2008; Selden & Selden, 2003; Tall 1997; Tall, 2008), but the ability to construct original proofs and the ability to read and validate proofs of others are essential skills for competency in modern mathematics.

The way that formal mathematics is presented in the university classroom could have a profound impact on how undergraduate and graduate math majors learn to read and write mathematical proofs, and therefore impact their success or failure in their training as mathematicians. Because the content and expectation of the students is quite different, the teaching of mathematics at this level may need to be examined separately.

Although there has been an increase in efforts to reform undergraduate instruction, lecture is still widely used (Armbruster, 2000; Bressoud, 2011; McKeachie & Svinicki 2006). Studies that examine teaching often focus on where the instructor's presentation method lies on the continuum of lecture to reform (McClain & Cobb, 2001; Sawada, Piburn, Judson, Turley, Falconer, Binford & Bloom, 2002; Steussey, 2006), but this emphasis on the presentation style tends to gloss over subtle features of teaching practice, namely, differences that may occur within a lecture format. A masterful lecturer may use examples, give summaries, check for student understanding, or make connections between different topics (McKeachie & Svinicki, 2006). Lecture can also be interactive, incorporating lots of questions that guide students through the material (Bagnato, 1973). In short, there may be significant variation among lecturers.

In advanced mathematics courses that are taught using lecture methods, attending to the presentations of the proofs in class is an important element of students' understanding of mathematical proof (Weber, 2004). Previous studies of teaching practice at this level have focused on proof presentations (Fukawa-Connelly 2012a; Fukawa-Connelly 2012b; Weber, 2004), while other studies have employed interview methods to gain instructors' perspectives on their in-class proof presentations (Alcock, 2010; Weber, 2011). This study will contribute to the current body of literature on teaching advanced mathematics by investigating the allocation of class time, the example usage, and the questions posed by instructors in multiple case studies of instructors' lectures. A foundation of knowledge about the range of current teaching practices in traditionally taught proof-based mathematics courses is necessary to interpret the existing research on teaching and learning proof (Fukawa-Connelly, 2012a), to inform curriculum development and future research (Speer, et. al, 2010), and to ultimately improve student learning.

1.1 Statement of Purpose and Research Questions

The purpose of this multi-case study is to examine the teaching practices of four mathematics instructors who were teaching different proof-based mathematics courses using lecture methods. In particular this study will investigate pedagogical tools that they use when presenting proofs in class, how they allocate time within lectures, how they use examples in conjunction with their proof presentations, and how they use questions in their lectures. The research questions are:

1. What pedagogical moves do instructors plan to use to help students understand their proof presentations, and how often do they use these moves?
2. How do instructors allocate their class time in traditionally taught proof-based undergraduate courses?
3. What types of examples do instructors use in presentations of theorems and proofs in an upper-division proof-based mathematics course? When do these examples occur chronologically in relation to the presentation of theorems or proofs?
4. What are the instructors' pedagogical purposes for the different types of examples when presenting the statement of a theorem or a proof?
5. How often do instructors who are teaching advanced mathematics using lecture methods interact with their students by asking questions?
6. What types of questions are asked by instructors who are teaching advanced mathematics using lecture methods, and what types of responses are expected of students?

1.2 Research Design Overview

Under the oversight of the University's Institutional Review Board, I recruited faculty members in the mathematics department who were teaching proof-based mathematics courses during a particular school year to participate in my study. Four instructors agreed to participate, all of whom were tenured faculty members with many years of teaching experience. The four participants were all well respected by their colleagues, and three of the four have won teaching awards. The study consists of four interrelated case studies (Stake, 2005) to document different aspects of the participants' teaching. Pseudonyms were given to all participants based on the courses that they were teaching.

Dr. A taught Introduction to Modern Algebra, a junior-level course which, according to the course catalog, covers introduction to set theory and logic, elementary properties of rings, integral domains, fields, and groups. Dr. C taught the senior level Introduction to Analysis (Advanced Calculus) course, covering properties of the real numbers, sequences and series, limits, continuity, differentiation and integration. Dr. G taught the senior level course in Geometry, which is an axiomatic development of Euclidean and non-Euclidean geometries. Dr. N taught the senior level Number Theory course which covers divisibility of integers, congruences, quadratic residues, distribution of primes, continued fractions, and the theory of ideals. The Introduction to Modern Algebra course is a required prerequisite for Number Theory, and recommended for Geometry and Advanced Calculus.

Both interview and observation data were collected from the four participants. An interview was conducted at the beginning of the semester. In this 1-hour interview, the participants were asked to describe what they do to help the students understand the proofs that they present in class. Video observation data were collected from each participant's classrooms six or seven times throughout the semester. Observations were purposefully chosen to be spread throughout the semester, to be convenient for the instructor, and to avoid exam days. Since there was no noticeable variation in teaching style across different instruction days for each participant, the data can be considered *saturated* (Glaser & Strauss, 1967) in regard to the teaching methods used, therefore providing a fair snapshot of each instructors' teaching.

In every observation across all four cases, the instructors were standing at the board while the students are sitting in rows in desks. There were instances of back-and-forth discussion between the instructor and students, but there was no evidence of any of the instructors initiating student-to-student interactions in class. All of the participants self-reported that they used lecture methods, although Dr. C and Dr. G preferred to call their method “modified lecture” because of their attempts to engage their students.

The data were analyzed using several different methods to address the first research question. The interview data were analyzed using the *constant comparative method*, which is a qualitative analysis technique that is associated with *grounded theory* (Glaser & Strauss, 1967). The interviews gave the contextual and perceptual information needed to describe the setting in each of the four cases. From the interview data, four *proof presentation strategies* were identified, and levels of *expected engagement* were constructed. Each proof presentation was coded according to the proof presentation strategies that were used and the levels of expected engagement. This analysis showed that the instructors were using examples to varying degrees in their proof presentations, and that they were engaging the students in their proof presentations. These findings led to the development of research questions 3-6.

The second research question was addressed by the construction of timelines for each observation that showed when the instructor was presenting definitions, statements of theorems, proofs, examples, or homework problems. The timelines also indicated when the instructor was speaking and when the student was speaking. This analysis of the observation data gives an overall picture of how the instructors utilize their time in class.

The third and fourth research questions address how instructors use examples in conjunction with their proof presentations. To investigate this matter, examples were identified in the proof presentations and the constant comparative method was employed to develop categories of examples with similar properties. The different types of examples that occurred during the participants’ proof presentations were then synthesized with existing example types in the literature to establish the pedagogical intentions of each example type. For categories that did not match with types of examples found in the literature, the pedagogical intentions were formed using inferences from the interview and observation data. The examples were then organized into a coherent descriptive framework that is grounded in observation and interview data. This framework illustrates when and how the different types of examples are used in the proof presentations.

Upon the completion of the analysis of the interview data, the construction of the timelines, and the analysis of the instructors' example usage, another one-hour interview was conducted as a member check. This occurred approximately a year after the completion of the observations. In the follow-up interview the participants discussed the results of data analysis. The participants were presented with the categories that appeared in their interviews and the timelines that described the way that they utilized their time in class. They were presented with my descriptions of their classroom examples and excerpts of the transcripts from their teaching and were asked to comment on several instances when they used examples in proofs. They were asked to comment on the framework for example usage in proof presentations and the hypothesized intentions for the different types of examples that they used. They were also asked to describe the ways in which they interacted with their students, although a full analysis of their questions had not yet been completed.

The fifth and sixth research questions address the frequency and types of instructor questions. An analysis of the instructor questions was done using modifications of existing frameworks for question analysis. Each question was assigned to a level of Anderson's Revised Bloom's Taxonomy (Anderson & Krathwohl, 2001; Tallman & Carlson, 2012) to determine the *cognitive engagement* required to answer the question. The questions were also coded according to the *expected response type*, which was a category adapted from two sources: the Teaching Dimensions Observation Protocol (CCHER, 2009) and Mehan's (1979) types of questions. Questions that did not require a verbal response were coded Rhetorical or Comprehension, and if the student was supposed to give a response, the response type was coded as Choice, Product, Process, or Meta-process. Percentages of the different question types were calculated, and percentages of the types of questions that were answered by the students were also computed.

The results will be presented using the interviews as a backdrop for interpreting the results of the analysis of the instructors' use of examples in proof presentations and their use of questions to interact with their students during their lectures.

1.3 Researcher Perspectives and Assumptions

Since the methods of data collection and analysis in this study are qualitative, the assumptions and theoretical orientation of the researcher must be made explicit from the outset of the study. My experiences as a student in undergraduate and graduate level mathematics have guided my inquiry into the teaching of advanced mathematics. All of the advanced mathematics courses that I have taken have been taught using lecture methods, and I have been successful in all of my coursework. My primary assumption in regard to this study is that teaching mathematics using lecture methods needs to be examined more closely. I believe that in the effort to reform mathematics teaching, researchers and reformers alike have criticized traditional instruction without carefully examining the variation that occurs within lecture methods. An in-depth understanding of what instructors do in the classroom when teaching advanced mathematics can lay the groundwork for the interpretation of results about students' proof-writing abilities, and can inform curriculum development and professional development programs.

In this study, I will examine the ways in which the instructors use examples and how they ask questions. Investigating with examples has been helpful to me in my own coursework, and although I have not taught a proof-based course, I do use examples in my own teaching. Also, as a student I frequently asked questions and participated in class discussions, and in my teaching I ask questions and interact with my students.

Although I do believe that using examples and interacting with my instructors benefitted me as a student, I also believe that their use in the classroom and implications for student learning are highly nuanced. Because this study did not collect data from the students, I cannot make any claims about how the instructors' use of examples or questions affected student learning. This study merely catalogues the use of examples in proof presentations and the frequency and types of questions used thus giving a foundational understanding of the variation that can take place among different lecturers in proof-based mathematics courses.

1.4 Rationale and Significance

This study addresses a gap in the literature by exploring current teaching practices of mathematicians when presenting proof in undergraduate mathematics lectures, namely, the pedagogical tools that they use in proof presentations, how they utilize their class time, ways in which they use examples in their proof presentations, and a multi-dimensional analysis of the instructors' questions. The participants of this study are four instructors teaching different upper-division proof courses at the undergraduate level. All of these instructors taught using some variation of lecture, meaning that the instructor was primarily standing at the board presenting the material while the students were taking notes. This knowledge of teacher practices in traditionally taught advanced mathematics courses lays a foundation of understanding about teaching practices upon which future studies can build.

1.5 Summary of Findings

Four proof presentation strategies were identified from the participants' interviews: outline, example, logical structure, and context. The presentation strategies that occurred in each proof presentation were recorded, and it was found that only four of the 64 proofs did not use any of the identified strategies. Also, examples were used by the instructors to varying degrees. One instructor did not use examples in his proof presentation, while the other three used examples in approximately half of their presentations.

The interview data were also used to identify levels of expected engagement. It was found that three of the four participants expected students to actively contribute to 95% of their proof presentations. The fourth instructor expected active contribution for 50% of his proof presentations. Thus, the instructors were engaging their students in the majority of their proof presentations.

The allocation of class time was analyzed, and it was found that the instructors spent between 35% and 70% of their class time presenting proofs. Across all four cases, the largest proportions of class time were spent on proofs and examples. The amount of time spent on straight lecture vs. interactive lecture was also analyzed. It was found that the instructors spent between 26% and 62% of their class time on interactive lecture.

The examples used during proof presentation were categorized, and the kinds of examples were synthesized with example types that occurred in the literature. In the analysis, a new kind of example surfaced. *Metaphorical examples* are used to compare an unfamiliar mathematical structure to a different (more familiar) mathematical structure via metaphor. The example types and the pedagogical purposes of the examples were organized into a descriptive framework of the instructors' example usage in their proof presentations. This framework describes when and how the examples were used to motivate and support the presentation of the theorem/proof pair.

An analysis of the instructor questions showed that the instructors frequently engaged their students by asking questions. The rate of instructor questions ranged from 0.69 to 1.81 questions per minute. A more thorough analysis of the question types revealed that between 30% and 54% of the questions asked were higher-order. An analysis of the questions that were actually answered by the students revealed that the students were answering a variety of questions, including higher-order questions.

CHAPTER II

REVIEW OF THE LITERATURE

The purpose of this multi-case study is to examine the teaching practices of four mathematics instructors who were teaching different proof-based mathematics courses using lecture methods. In particular this study will investigate pedagogical tools that they use when presenting proofs in class, how they allocate time within lectures, how they use examples in conjunction with their proof presentations, and how they use questions in their lectures. To carry out this study, it was necessary to conduct a critical review of the current literature pertaining to this topic. This review took place throughout each phase of the study, and was refined throughout the entire process.

To conduct this literature search, multiple sources were used such as books, journal articles, dissertations, and conference proceedings. Sources were located initially by using ERIC and ProQuest, and additional sources were found by referencing the bibliographies of papers that I had previously located and by word-of-mouth referrals from colleagues in the field with whom I have discussed my work. Four main bodies of literature were considered pertinent to this study: the teaching and learning of mathematical proof, teaching practices of university mathematics instructors, the use of examples in teaching mathematics, and the use of questions in teaching in general and mathematics in particular.

This chapter will begin by providing a brief history of Research in Undergraduate Mathematics Education. Because this study is an investigation of teaching practices in advanced mathematics lectures, the study must be framed by results in two foundational research areas: research on the teaching and learning of mathematical proof, and research on teaching practices at the collegiate level. Then, studies that also investigate teaching practices in proof-based courses will be reviewed. Finally, since this study concentrates on teaching practices as they relate to example usage and instructor questions, the literature pertaining to those topics will be reviewed.

Throughout the review, the literature is synthesized to point out gaps, and each section concludes with a summary that focuses on the research implications for this study. The final section of the literature review is a summary that will illustrate how the findings in the literature bear on this study and contribute to the conceptual framework within which this study is situated.

2.1 A Brief History of Research in Undergraduate Mathematics Education

Mathematics is one of the oldest and most respected disciplines. As long as mankind has been "doing" mathematics, mathematics education has also been present. Learning mathematics has long been considered part of becoming an educated member of society. Above the entrance to Plato's academy was written "None but geometers enter here." It is well known that Euclid's Elements was one of the first widespread mathematics textbooks even before the printing press. As civilizations become more industrialized and more people were able to receive an education, mathematics has remained a pillar of the education system.

Although mathematics education has a long tradition, mathematics education research is a relatively young field (Selden & Selden, 1993). In the United States, The National Council of Teachers of Mathematics was formed in 1920 to support and equip mathematics teachers. Russia's launching of Sputnik in the late 1950's caused the U.S. government to increase funding for math and science education research. The majority of the research was on teaching K-12 mathematics, though there were some projects directed at the teaching of college calculus, such as the famous Harvard Calculus Consortium. Since the 1980's, the field of research in undergraduate mathematics education has blossomed. Some of the first studies were of the form of "teaching experiments" or were extensions of theories that existed in K-12 mathematics education. Schoenfeld's well-attended talk at the AMS/MAA meeting in 1990 sparked the loose organization of mathematicians who were interested in education research (Selden & Selden, 1993). The Research in Undergraduate Mathematics Education (RUME) special interest group of the Mathematical Association of America was founded in 2000.

Within RUME, there are several different research areas. Since Calculus and transition to proof are both areas of difficulty for undergraduates, there are large groups working in both of those areas. There are also groups working on teaching differential equations, number theory, and preparing graduate student teaching assistants, among other topics.

Mathematics education research is essentially social research, but like mathematics, there are two main purposes: Pure and Applied. Pure mathematics education research seeks to understand the nature of mathematical thinking, teaching, and learning. It is often exploratory, and combines methods from education, psychology, sociology, and cognitive science. Applied mathematics education research takes results from pure research and uses those understandings to improve mathematics instruction (Schoenfeld, 2000).

Another difference between research in mathematics and research in mathematics education is summed up in Henry Pollak's statement, "there are no proofs in mathematics education" (Schoenfeld, 2011, p. 47). Because mathematics education is essentially social research, it cannot have the certainty that proof affords. However, there are alternative methods of gaining confidence in the results of education research. Studies in education can have *descriptive* power, *explanatory* power or *predictive* power. It is also important to take into account the *scope* of educational research studies when interpreting results (Schoenfeld, 2011).

2.1.1 Implications for this Study. This study is investigating teacher practices, which can best be classified as pure research because the goal is to merely describe and catalog the phenomenon of teaching. Since RUME is a much younger field than K-12 mathematics education research, the foundation of research-based knowledge of teacher practices had not been as thoroughly constructed (Speer, et. al, 2010). My research will address this gap in the literature by examining four interrelated case studies of teacher practice in proof-based mathematics at the university level by investigating the pedagogical moves that instructors use during their in-class proof presentations.

One aspect of my research is the development of a framework describing when and how examples are used by instructors to motivate and support their proof presentations. This framework has descriptive and explanatory power, as it is grounded in the interview and observation data as well as the literature.

This study will also examine the ways in which instructors use questions in their advanced mathematics lectures. The questions asked by the instructor will be analyzed using existing taxonomies that have proven to be useful in K-12 mathematics education. In particular, I have extended Anderson's Revised Bloom's Taxonomy (Anderson & Krathwohl, 2001) to encapsulate different types of questions that occur in proof-based mathematics courses.

2.2 Research on the Teaching and Learning of Proof

Early research addressing teaching and learning mathematical proof focused on investigating student understanding. Students encounter variations of proof at different times throughout their schooling. As early as elementary school, students may be expected to form case arguments (Maher & Martino, 1996), and to reason about integers and prime numbers (Zazkis & Liljedahl, 2004). It is therefore important for pre-service elementary school teachers to have a basic understanding of proof that is consistent with the mathematical community. However, in a large scale study of pre-service elementary teachers, Martin & Harel (1989) found that more than half of their subjects accepted arguments as valid proofs which would generally not be accepted by mathematicians.

Variations of proof also appear at the secondary level, particularly in Geometry. Bell, (1976) asked 32 fifteen year old students to construct two proofs. He found that over half of the students failed to construct a proof, and those who encountered some success seemed to prefer checking cases over proving generalities. There were many studies in the UK that assessed the effectiveness of Britain's National Curriculum by investigating secondary students' understanding of proof (Coe & Ruthven, 1994; Healey & Hoyles, 2000). These studies found that students seemed to prefer example-based arguments, although they recognized that their teachers preferred the rigor of more general arguments (Healey & Hoyles, 2000). It may be that secondary students are not taught to value proof, because even their teachers demonstrate an inadequate understanding of what constitutes proof and do not view proof as a tool for learning mathematics (Knuth, 2002).

Rigorous proof is generally an important part of the undergraduate curriculum for mathematics majors. This section will address the major difficulties that undergraduate students have with proof as found in the literature, and then will review some of the different teaching strategies and innovative curricula that address teaching mathematical proof.

2.2.1 Student (Mis)Understanding of Mathematical Proof. My research focuses on proof as it appears in upper-division undergraduate mathematics courses, which are populated by mostly mathematics and secondary mathematics education majors. Current research studies have documented several difficulties that these students have with mathematical proof. Although the students' effort is always a factor (Wu, 1999), research has found four major reasons that students struggle with reading and writing proofs (Weber, 2001). We'll review some of the major results.

2.2.1.a. Mathematical language barrier. For students to begin a proof, they must first understand the statement that they are trying to prove. This means that they must read the statement and make sense of the symbols and language. Selden and Selden (1995) found that undergraduates in a bridge to proof course were unable to "unpack" logical statements using quantifiers. Watkins (1979) suggests that special instruction may be necessary to teach students "Mathematical English."

This communication barrier is a major hindrance to proof writing. If students fail to interpret the mathematical language correctly, they will be unable to understand the statement, state the definitions, or begin to structure a proof (Moore, 1994). Proofs in an undergraduate level bridge to proof course were analyzed, and it was found that, on average, over 70% of a proof at that level can be constructed by using the assumptions and associated definitions (Savic, 2011). This suggests that students who are unable to state and use definitions, or who are unable to unpack the assumptions in the statement that they are trying to prove will have little success in proof writing.

2.2.1.b. Insufficient understanding of mathematics content. An important distinction between the formal mathematical definition and the images evoked in the students' mind was made by Tall and Vinner (1981). The *concept image* is the total cognitive structure in a student's mind that has been built up over the years, including all mental pictures and associated properties. It may be incomplete and may contain conflicting ideas. The *concept definition* is the formal mathematical definition that has been accepted by the mathematical community.

A student may struggle to write a proof if they do not have a flexible understanding of the mathematical concepts with which they are working. In other words, their concept image differs from the accepted concept definition. The literature contains many examples of students struggling with the content in advanced mathematics courses. In abstract algebra, students struggle with the concepts of normality, cosets, and quotient groups (Asiala, Dubinsky, Mathews, & Oktac, 1997). Nardi (2000) provides evidence that first year undergraduates at Oxford had difficulties with the foundational concepts of Group Theory. Moore (1994) found that students in a bridge to proof course had "little intuitive understanding of the concepts" (p. 251). In their work on student understanding of definitions, Edwards & Ward (2004) found that many students do not use definitions the way mathematicians do, even when the student can correctly state and explain the definitions, and even in the apparent absence of any other course of action.

2.2.1.c. Students' beliefs about mathematics and proof. Students' belief systems concerning mathematics may also have an impact on their success (Moore, 1994; Schoenfeld, 1985; Schoenfeld, 1989; Solomon, 2006). Solomon (2006) summarizes her findings in this way: "[Undergraduates'] beliefs about the nature of mathematics as a matter of certainty, rule-following, isolation, abstraction and lack of creativity differ little from those identified by researchers into school mathematics. Again in correspondence with school research, their beliefs about learning mathematics emphasize speed and fixed ability" (p. 389). Students often do not view themselves as a mathematical authority, but are "consumers of others' mathematics" (Schoenfeld, 1988). This view of mathematics leaves students feeling powerless, bound by the experts' list of rules.

Students' beliefs may also affect their ability to solve problems and write proofs. What constitutes evidence in the students' eyes may not be considered a proof to a mathematician; and a rigorous proof may not be convincing to the student (Harel & Sowder, 1998; Mills, 2010). Inexperienced students often attack problems with a *naive empiricism*; even if they are able to make deductive arguments, they don't think to use that method (Schoenfeld, 1985). Students also tend to focus on checking the logic line-by-line rather than looking at the overarching argument in a proof (Selden & Selden, 2003; Mills, 2010).

2.2.1.d. Lack of strategic knowledge. Thus far, I have given several reasons that students may struggle with proof. They may have difficulty with the language and notation, the concepts themselves, or maybe they are still learning what their instructors mean by 'proof.' But what if a student does understand the concepts, and does understand what a proof should look like, but is still unable to construct a proof? Weber (2001) identifies another reason for student failure: a lack of *strategic knowledge.* The undergraduates in his study displayed the factual knowledge of the abstract algebra material, but when they were presented with statements to prove, they failed.

They were not sure what theorems were useful or when it was appropriate to use definitions or manipulate symbols. In contrast, doctoral students were able to quickly choose the strategy that would lead to a proof.

In another study Weber (2006) outlined a strategy for proving theorems involving group isomorphisms. A computer programmed with the strategy was able to complete proofs of most of the statements, and students who were taught the strategy were able to prove significantly more theorems. So, it seems that when teaching students to be successful in writing original proofs, we must consider not only content and logic, but also strategies for proving in that particular area of mathematics.

2.2.2 Reform Efforts for Teaching Proof. The previous section presented several reasons why students struggle with mathematical proof. There have also been efforts on the part of instructors and researchers to address student learning of proof. This section will outline some of the most important interventions.

The Moore Method has been a famous example of reform since R. L. Moore himself taught at the University of Texas from the 1920's to the 1970's. He gave the students a list of definitions, axioms, and theorems, and had the students prove the theorems on their own and present them in class. Students were not allowed to consult other texts or other mathematicians. Current modified Moore Method courses have been toned down, though the emphasis is still on the students producing the mathematics (Krantz, 1999).

An inquiry based abstract algebra curriculum was developed by Leron and Dubinski (1995). They used ISETL computer programming to allow the students to explore structures in abstract algebra, and included group discussions. Their teaching style used activities, class discussion, and lots of examples. Larsen (2009) created a curriculum for teaching group theory that is based on *guided reinvention* (Freudenthal, 1991) of the concepts. The instructor serves as a guide to help the students attend to the properties of the algebraic structures that are mathematically important.

Some small-scale interventions have also been reported, which may be able to be adapted to classroom settings. Weber (2006) taught a group of students a strategy for proving statements about group isomorphisms. The students, in an interview setting, were able to prove significantly more theorems after the intervention. Selden & Selden (2003) interviewed students, asking them to read student-produced proofs and check them for correctness. At first, the students' validations were no better than chance, but throughout the interview process, their ability to verify proofs increased significantly. This gives evidence that the skill of reading and verifying mathematical proofs can be taught.

2.2.3 Implications of Research on Teaching and Learning Proof. Research has shown that students struggle with proof for various reasons, whether it is difficulty with the notation, content, beliefs about proof that are inconsistent with the mathematical community, or a lack of strategic knowledge (Weber, 2001). Although there have been efforts to reform instruction at the advanced mathematics level, traditional lecture is still the primary method of delivery. Thus, a research-based understanding of classroom teaching practices in traditionally taught proof courses is critical for interpreting these results about students' difficulties with proof (Fukawa-Connelly, 2012a).

2.3 Research on Teaching Practice of University Teachers

Lecture has long been the tradition in university teaching. Even today, it is the dominant style used in undergraduate Calculus (Bressoud, 2011), and this trend likely continues up through advanced mathematics courses as well. In a large-scale study designed to investigate the effectiveness of Inquiry Based Learning (IBL) mathematics courses at the University of Colorado Boulder, it was found that in non-IBL courses, the students spent 87% of their class time listening to the instructor talk (Laursen, Hassi, Kogan, Hunter, & Weston, 2011). Another large scale study in Geoscience education found that across all academic levels (graduate, honors, major, and non-major), instructional practices were teacher centered (Markley, Miller, Kneeshaw, & Herbert, 2009).

Although lecture is still widely used (Armbruster, 2000), few studies investigate in detail the teaching practices of instructors using primarily lecture methods. As Krantz (1999) points out, a masterful lecturer may include many different pedagogical moves to connect to his or her audience. Instructors can use examples, give summaries, check for student understanding, or make connections between different topics (McKeachie & Svinicki, 2006). Lecture can also be interactive, incorporating lots of questions that guide students through the material (Bagnato, 1973). In advanced mathematics some instructors may rely on direct instruction but may still ask a lot of questions and encourage the students to ask questions and supply examples (Moore, 1994; Fukawa-Connelly, 2012a).

The traditional lecture style of teaching mathematical proof should be investigated more closely in order to catalog the strategies that professors are using. Speer, Smith, & Horvath (2010) claim that although the “effects of instructional activities have been examined… the actions of the teachers using those activities have not” (p. 101). They conducted an extensive literature search for published articles addressing collegiate teaching practice in mathematics, only to find five articles that fit their criteria. All of the five articles were case studies with one faculty member as a participant, and all used observation and interview data to analyze the instructor’s teaching practices. Only one of the articles mentioned was an analysis of a proof-based course (Weber, 2004).

2.3.1 Implications of Research on Collegiate Teaching Practices. Although studies investigating teaching practice have proved to be foundational in K-12 mathematics education, there are not many studies that exist at the level of advanced undergraduate mathematics (Speer, et.al., 2010). Those that do exist are often single case studies, and many of these studies focus on non-traditional teaching methods. Large-scale studies on teaching practices have found that lecture is the dominant style of instruction, and that in traditional courses students spend over 80% of their time listening to the instructor speak (Armbruster, 2000; Laursen, et.al., 2011). This study adds to the existing literature by providing a larger study that incorporates four case studies of mathematics instructors teaching proof-based courses with traditional lecture methods in four different mathematics content areas.

Speer et al. (2010) presented a framework for analyzing teaching practice. The seven dimensions of their framework are: (a) allocating time within lectures, (b) selecting and sequencing content within lessons, (c) motivating specific content, (d) asking questions, using wait time, and reacting to student responses, (e) representing mathematical concepts and relationships, (f) evaluating completed teaching and preparing for the next lesson, and (g) designing assessments and evaluating student work. This study will address the allocation of class time, the use of examples to motivate specific content and represent mathematical concepts and relationships, and the types of instructor questions in proof-based mathematics courses.

2.4 Proof Presentations in Advanced Undergraduate Mathematics Classes

In a traditionally taught undergraduate course, students generally are expected to learn the material by attending lectures, reading the textbook, and doing the homework. If writing proofs is an expectation, then these three avenues for student learning are the ways in which the socio-mathematical norm of proof writing is established. Harel and Sowder (2007) state that the goal of instruction in these courses is to help students develop an understanding of proof consistent with the mathematical community. This is accomplished in part by the instructor modeling the mathematical behavior of proof writing in class (Fukawa-Connelly, 2010).

Recent studies have found that instructors in advanced mathematics classes spend roughly half of their class time presenting proofs in class, on average (Weber, 2004; Mills, 2011). Therefore, the presentation of proof in class is a vital part of developing students' understanding of proof in a traditional classroom.

Despite its importance in developing young mathematicians, few studies have investigated instructors' in-class proof presentations. Mejia-Ramos & Inglis (2009) performed a literature search in the top seven journals that have a history of publishing research in undergraduate mathematics education. They found 102 research papers addressing writing, reading, and understanding of proof by undergraduates, but none of the tasks in these papers were focused on proof presentation, either by instructors or by students. There are a few studies that are focused on investigating the proof presentations of instructors in lectures (Fukawa-Connelly, 2012a; Fukawa-Connelly, 2010; Weber, 2004; Mills, 2011; Mills, 2012), and a study focused on students' proof presentations in an inquiry-based abstract algebra class (Fukawa-Connelly, 2012b).

This section contains literature that lies at the intersection of research on teaching practice and research on proof. Since presenting proof has been referred to as modeling the mathematical behavior of proof writing (Fukawa-Connelly, 2010), this section will first outline the strategies that mathematicians and successful graduate students employ when reading and writing proofs. Next, there will be a summary of studies that investigate instructors' pedagogical views in regard to proof presentations. Lastly, there will be a presentation of studies examining the teaching practices of instructors when presenting proofs.

2.4.1 Proof Writing Strategies of Experts. I have argued that a major goal of proof presentation is for the instructor to model the mathematical behavior of mathematicians when reading and writing proofs. Therefore, studies describing experts' or successful graduate students' practices when constructing and reading proofs can inform our understanding of the behavior that the instructors are attempting to model.

It is important to note that proofs are valued by mathematicians for multiple reasons, not just to determine the truth or falsity of mathematical statements (Rav, 1999). A recent study showed that when mathematicians read published mathematical proofs in their research area, they are often not checking for the correctness of the proof. Rather, they tend to focus on the overarching ideas and methods in the proof, and often understand different steps in the proof by applying the ideas to specific examples (Mejia-Ramos & Weber, in press). They also check to see if the proof has a legitimate structure before proceeding with line-by-line validation (Weber, 2008). The differences between how experts and novices read mathematical proofs was investigated using eye-movement data (Alcock & Inglis, 2012). They found that novices tend to focus on external features of proofs (i.e. embedded equations), while mathematicians read by jumping back and forth between different lines looking for the overall structure.

When writing proofs, mathematicians and advanced doctoral students tend to use empirical evidence to convince themselves of the truth or falsity of claims (Inglis, Mejia-Ramos & Simpson, 2007). They also may use examples to investigate definitions or structure a proof (Alcock, 2010; Weber, 2011).

Thus, if instructors are modeling their own behavior when presenting proofs in class, they are likely trying to teach students to value both overarching ideas as well as line-by-line verification, and they are likely to use examples in various ways.

2.4.2 Instructors' Pedagogical Views of Proof Presentations. There have been several studies that investigate mathematics faculty members' pedagogy in regard to proof presentation (Weber, 2011; Alcock, 2010; Yopp, 2011; Hemmi, 2010). These studies have used interviews of faculty members to investigate why and how they teach proof, and have described how these mathematicians talk about their intentions and pedagogical perspectives.

Two of the studies investigated the reasons that the participants presented proofs to their students. These studies found that math instructors present proofs to develop their students' ability to write proofs on their own (Yopp, 2011) or when it illustrates a new proving technique (Weber, 2011). Other reasons for presenting proofs included cultural reasons, to expose students to proof, or to illustrate the truth of a theorem.

Alcock (2010) identified four modes of thinking that the faculty members aim to teach their students by presenting proofs. *Instantiation* is used to "understand a mathematical statement by thinking about its referent objects," (p. 69) and includes thinking about examples or images. *Structural thinking* is a way of reducing abstraction and making use of the logical structures to drive the construction of the proof. *Creative thinking* includes experimenting with examples with the hope to generalize or attempting to construct a counterexample. The goal of *critical thinking* is to check for the correctness of each line of a proof.

When considering a written proof that will be presented to students, mathematicians believe that adding introductory and concluding sentences to clarify the strategy of the proof improves the quality of the proof. They also believe that avoiding unnecessary equations and calculations and formatting the necessary equations so that they are on their own line improves the proof (Lai, Weber, & Mejia-Ramos, 2012).

Although these studies showed that the instructors take their students' level of experience into account (Lai, et. al, 2013), they seemed to lack an arsenal of strategies for helping students understand their proof presentations (Weber, 2011; Alcock, 2010; Harel & Sowder, 2009).

2.4.3 Studies of Teaching Practice in Proof Presentations. Weber's (2004) case study of an instructor's teaching practice in introductory analysis lectures was highlighted by Speer, et al, (2010) as a model study of teaching practice. This instructor was able to clearly articulate his goals and beliefs about teaching analysis and student learning from his years of experience, and he chose a teaching style that was consistent with his beliefs. He made an effort to reveal the reasoning behind the proof construction so that students could learn to construct original proofs themselves. Weber (2004) identified three distinct proof presentation styles that were used by the instructor: *logico-structural*, *procedural*, and *semantic*. Although the instructor was unaware of research about teaching mathematical proof, the strategies that he used appeared to be designed to teach some of the cognitive skills discussed in the literature.

Another study examines the pedagogical choices of an instructor, Dr. Tripp, in an abstract algebra course (Fukawa-Connelly, 2010; Fukawa-Connelly, 2012a). Several *pedagogical content* *tools* (Rasmussen & Marrongelle, 2006) were identified, in particular the instructor 'modeled mathematical behaviors' such as proof writing, definition exploration, and example generation. Although Dr. Tripp self-identified as a traditional instructor, she used a significant amount of dialogue with her students when presenting proof (Fukawa-Connelly, 2012a). This *proof presentation with dialogue* pattern began with an instructor question that was directed at the entire class, which solicited a student response. Then, Dr. Tripp would either ask another question or comment on the response. She was observed to use a *funneling* pattern (Wood, 1994) which is a questioning sequence that begins with a higher-order question but reduces the cognitive load with each successive question. The outcome of this is that the questions that students ultimately answered were merely factual. These questioning sequences, however, were used by Dr. Tripp to model the mathematical thinking that is required to write the proof. This study gives an existence proof that university mathematics professors do not always use a "pure telling" method of proof presentation.

2.4.4 Implications of Research on Proof Presentations. Recent studies have focused on their pedagogical views of proof presentation (Alcock, 2010; Harel & Sowder, 2009; Hemmi, 2010; Lai, Weber, & Mejia-Ramos, 2012; Weber, 2011; Yopp, 2011). Though these studies describe how faculty members claim to help students comprehend their proof presentations, they do not describe what is actually happening in the classroom. Hemmi (2010) acknowledges that "the relationship between talk and reality is complex. The mathematicians' talk about proof and the teaching and learning of proof is considered as shedding light on the practice, but not as an objective description of it" (p. 272). This study will both allow the instructors to shed light on their practice via interviews, as well as provide a description of their practices in the classroom.

Investigations of instructional practice in traditionally taught proof-based mathematics courses have been single case studies (Fukawa-Connelly, 2010; Fukawa-Connelly, 2012a; Weber, 2004). This study contributes by providing four interrelated case studies of mathematics instructors who are teaching in different content areas.

2.5 Example Usage in Mathematical Proof

It has been shown in the literature that mathematicians use examples in their own work in various ways (Alcock, 2010; Inglis, et al, 2007; Weber, 2011). Since the presentation of proof in class is an opportunity for the instructor to model the mathematical behavior of proof writing, the instructor is likely to use examples. This section outlines the literature on how examples are used by experts, how they have been used by students and teachers in the classroom, and then presents the rationale for the creation of my framework for example usage in proof presentations.

2.5.1 Defining Example. The term "example" has had varied meanings in the literature. Watson & Mason (2005) say that an "example" is "anything from which the learner may generalize" (p. 3). Their use of example is learner-dependent, allowing the learner to construct examples that may not be mathematically accurate. Others have taken the learner out of the picture and refer only to a mathematical requirement in their definition of example. Zazkis & Leikin (2008) use example to mean an "instance, illustration, case, or element of a mathematical idea, object, process, or class." Alcock & Weber (2010) use "example" in a much more restricted way, to mean "a mathematical object satisfying the definition of some concept."

In this study, I will consider a mathematical object an example if it has two properties: it must be specific and concrete as opposed to general and abstract. Specificity is a mathematical requirement; the object must represent a particular element of a larger class. Concreteness implies that students at this level must be able to either compute with or investigate properties of the mathematical object. Thus, concreteness is concerned with the accessibility of the mathematical object to the learner. Therefore, I will use the following definition: *An example is a specific, concrete representative of a class of mathematical objects, where the class is defined by a set of criteria.*

It is possible that a mathematical object can be mathematically specific but not concrete for a particular group of students. In an introductory analysis class, a function from the power set of the natural numbers to the real numbers can be defined by $f(A) = 0.a_1 a_2 a_3 \cdots a_n \cdots$ where $a_n = 0$ if $n \notin A$ and $a_n = 1$ if $n \in A$. This is a specific function, but most students at that level cannot investigate its properties because it is still too abstract to be accessible to them. However, if the instructor were to choose a specific, subset A , such as $A = \{1,2,3\}$ and show how $f(A) = 0.11100000 \cdots$, then the object is much more likely to be concrete to the students and would therefore be classified as an example.

It is also possible that a mathematical object could not be mathematically specific enough to be classified as an example. In a presentation of a proof about the surjectivity of a composition of functions, an instructor in this study drew and labeled three "blobs" on the board to represent the three sets, arrows to represent the maps, and dots to represent elements of the set. Throughout the proof, he referred to the diagram. The diagram served as an alternative representation of the general proof and could not be said to specify a member of a class of functions with a given property. Therefore, it did not fit my definition of example and was said to be a *generic diagram*. This paper will investigate instructors' uses of examples, and therefore the generic diagrams that appeared in the data will not be included in the results.

Mathematical objects have multiple representations such as numerical, algebraic, or pictorial. In the same manner, examples may be represented in multiple ways. In this study, examples will not be classified by their representation, but rather according to their type and pedagogical use.

2.5.2 Uses of Examples by Experts. Non-deductive reasoning often plays a critical role in experts' mathematical argumentation, however experts are generally very clear about whether or not an argument is a deductive proof (Inglis, Mejia-Ramos, and Simpson, 2007). Experts may use examples to varying degrees, and his or her research area or personal preference may influence the degree to which examples are used (Alcock & Inglis, 2008).

A mathematician interviewed in Alcock's (2010) study stated that when he is presented with a new definition, he immediately begins to generate examples and non-examples to help him understand the definition. In another interview study, a mathematician described how he would choose a particular example that he would work side-by-side with the general proof (Weber, 2011). This strategy is similar to Rowland's (2002) *generic example*, which is a particular example whose steps mirror the general proof.

Experts' uses for examples when testing and proving conjectures were described in a framework based on the responses of mathematicians in an online survey (Lockwood, Ellis, Dogan, Williams, & Knuth, 2012). This framework encompassed several types of examples, uses of examples, and example-related strategies. Since the framework is aimed to describe their example-related activity when determining the truth or falsity of a conjecture, some of the categories do not apply directly to proof presentations of known theorems in the classroom. The uses of examples that may apply to in-class proof presentations are: make sense of the situation, proof insight, generalize, and understand the statement of the claim.

2.5.3 Uses of Examples in the Classroom. Harel and Sowder (1998) point out the dominance of *empirical* and *inductive* proof schemes among students, such as proof by example. Thus, examples can serve as a powerful tool for convincing students of mathematical arguments, but Harel and Sowder (1998) caution that "the empirical scheme should for mathematics students at some stage fill only confirming and conjecturing roles" (p. 277). Thus, one goal as students mature in their mathematical learning is to help them move towards the analytical proof schemes.

Student-generated examples have been the topic of many research studies in undergraduate mathematics education. Dahlberg and Hausman's (1997) found that students who spontaneously generated examples when given the new definition developed a more sophisticated concept image and were more able to complete proving tasks associated with the definition. Watson and Shipman (2008) had two groups of junior high students using learner generated examples to explore a new mathematical idea. They found improvements across all levels of mathematical abilities, even among the low-achieving group of students. Harel (2001) observed number theory students explore with examples to notice patterns which they formed into conjectures. Some actually extracted a process from the pattern that would generalize into a deductive proof.

Mason and Watson (2001) encourage a particular type of example generation in the classroom. They define *boundary examples* to be examples that "distinguish between having or not having a specified property." While boundary examples are often described by Mason and Watson as examples that help students to solidify the boundary around a particular mathematical definition, they also toy with the idea of boundary examples of a theorem or technique. They conjecture that complete understanding of a theorem or technique is directly related to one's ability to construct boundary examples.

Iannone, Inglis, Mejia-Ramos, Simpson & Weber (2011) called into question whether example generation would improve a student's ability to prove conjectures associated with a new definition. Using the same definition and proving tasks as Dahlberg and Hausman (1997) and a much larger sample size, they gave half of the students example generation tasks and the other half a sampling of worked examples to read. They found no significant difference in the two groups' ability to prove or disprove the related conjectures. This suggests that using example generation tasks may be more nuanced than previously thought, but it still supports the importance of examples for students who are learning a new concept.

Studies of pre-service teachers show that they are largely unconscious of their example choices (Zodik & Zaslavsky, 2008) and need specific guidance about the role and nature of mathematical examples and need to be made aware of common pitfalls in example selection (Rowland, 2008). When choosing examples, pre-service elementary teachers showed a strong preference for examples that they believed would make sense to their students, often without considering the mathematical correctness of the example (Zazkis & Leikin, 2008).

Pictures and diagrams are often an integral part of exemplification (Michner, 1978). Stylianou (2002) observed mathematicians solving problems, and found that they tended to construct their visual representations in steps, using what she called “structured visual qualitative exploration.” When teaching undergraduate analysis, a professor was observed presenting a diagram alongside a proof which he filled in step-by-step, modeling the behavior mathematicians (Weber, 2004).

Example spaces (Goldberg & Mason, 2008) are the set of examples that can be brought to mind for a particular mathematical object, together with associations and construction methods. Fukawa-Connelly, Newton, & Shrey (2011) explored the example space of a mathematical group in an abstract algebra class by identifying the mathematical purpose that each example served. They identified four purposes for examples: exemplifying a definition, creating/refining a definition, exploring a conjecture, or illustrating a proof.

The literature has shown that accessible examples can serve to introduce mathematical concepts, while diagrams also prove to be important organizational tools. Thus, the use of examples in proof presentations may be a natural way to connect the abstraction of mathematical concepts to students’ natural tendency to lean on examples.

2.5.4 Purpose of the Framework for Example Usage in Proof Presentations. The literature contains an array of examples and ways that they could be used at different levels of instruction; however, there are no studies that focus explicitly on describing the ways that instructors use examples when presenting proofs in class. Section 4.3.1 of this paper will organize the examples used in proof presentations into a coherent descriptive framework that is grounded in observation and interview data, and integrated with results from the literature.

The construction of this framework can be compared and contrasted to a social model, which is a consistent, dynamic system that represents the relationships between certain components and the behavioral outcomes that they produce for a particular social phenomenon (Schoenfeld, 2011). Social models are used to describe, explain, and ultimately predict behavior of individuals or groups of people.

The framework that I have developed is an organizational structure that illustrates the relationship between different types of examples and the presentation of theorems and proofs in mathematics lectures. My framework serves to describe the uses of examples that emerged from my observation data, and provides some insight into the pedagogical intentions of the instructors via interviews and synthesis with the literature. The framework is not intended to predict the behavior of instructors, but rather to describe and explain the roles that examples play in instructors' proof presentations.

2.5.5 Implications of Research on Example Usage. Although it has been argued that students' natural tendency to produce empirical arguments should be discouraged (Harel & Sowder, 1998), research about how mathematicians reason with examples suggests that examples can be useful and appropriate for exploring and proving conjectures (Inglis, Mejia-Ramos, & Simpson, 2007), and may even lead to the production of a deductive proof (Harel, 2001).

The literature has shown that examples have been used in various ways in the classroom by both teachers and students, and example generation tasks have been shown to be an effective tool for helping prospective teachers to think about their example usage in class (Rowland, 2008; Zodik & Zaslavsky, 2008). Similar tasks may also be useful for helping mathematics professors think about how they could use examples in proof presentations.

This research seeks to provide a framework for organizing the uses and types of examples that are used by instructors when presenting mathematical proofs in class. Although a framework for experts' example usage when testing and proving conjectures has been constructed (Lockwood, et al, 2012), their uses of examples when presenting proofs to students has not been investigated. When presenting proofs, instructors may model the ways that they use examples in their own understanding of the mathematics content.

2.6 Teacher-Student Interaction in the Classroom

Since mathematics learning can be viewed as an inherently social process, the ways in which instructors interact with their students has a great impact on student learning (Nickerson & Bowers, 2008). Research has shown that many of the questions posed by teachers in mathematics classrooms tend to require no more than factual responses (Sahin & Kulm, 2008; Fukawa-Connelly, 2012a). This study will examine the questioning used by four different instructors who are teaching proof-based advanced mathematics courses at the undergraduate level.

This section will describe the literature on interaction patterns in the classroom and taxonomies that are used for analyzing instructor questions. Then, I will present the rationale for the multi-dimensional approach that I have used to analyze the instructors' questions.

2.6.1 Interaction Patterns in Mathematics Classrooms. In grade-school traditional mathematics classes, a typical questioning pattern involves the teacher *initiating* with a question, the student *responding,* and the teacher *evaluating* the response and proceeding to the next question (Mehan, 1979). Similar results were found in adult education as well (Medina, 2001). These IRE questioning sequences are designed to evaluate whether or not the student knows the answer and often do not require the student to be involved in any mathematical thinking to participate. Elementary students have been seen to participate in class discussions by merely following the teacher's linguistic and contextual cues (Voigt, 1989). Sahin & Kulm (2008) found that teachers asked more factual questions than higher-order, and that guiding questions were rarely used. Although there are some who claim that asking higher-order questions can be linked to student achievement (Gall, 1970), others are hesitant to make such claims (Winne, 1979).

In Weber's (2004) case study of an introductory analysis course, "most of the lectures consisted of Dr. T writing definitions, examples, proofs, and occasionally diagrams on the blackboard and the students studiously copying Dr. T's writing into their notebooks. Students asked questions only infrequently and rarely participated in class discussions" (Weber, 2004 p.118). Thus, Dr. T rarely interacted with his students using questioning and discussion.

A contrasting case study showed that an abstract algebra instructor frequently used questions to devolve responsibility to students when presenting proofs (Fukawa-Connelly, 2012a). Dr. Tripp used rhetorical questions to model the mathematical thinking involved in creating the proof's structure, and she also asked a large number of questions that solicited student feedback. She often used a funneling pattern in her questioning (Wood, 1994). She would begin by asking a higher-level question but then she asked several successive questions in a row until the final question required merely a factual response or re-statement of something she previously said. Though she did devolve some responsibility for proof writing to students, the majority of students' answers stated the next part of the proof or the next algebraic step.

In summary, the research shows that questioning patterns in traditionally taught mathematics classes may reduce the cognitive demand on students and often merely require factual responses.

2.6.2 Taxonomies for Analyzing Questions. There are many different ways to classify questions. Some classification schemes classify questions by the expected products, such as whether the questions require the student to make a choice, give factual information, give reasons for their thinking, or justify their thinking (Wood, 1999; Mehan, 1979). Mehan's four types of questions are:

Choice- those that dictate that a student choose an answer from a short list of options (yes/no questions are classified as *choice*).

Product- those that require the student to provide factual responses

Process- those that call for students interpretations

Meta-Process- those that ask students to reflect upon their thinking and justify their answers

This type of classification is useful for questions that are intended to elicit responses. Although this is a reasonable way to catalog question types, the expected response type does not necessarily capture the cognitive processes required to answer the question. For example, a highly conceptual question could be presented in a multiple-choice format.

There may be questions that instructors pose that do not necessarily elicit responses. Rhetorical questions have been used to model the mathematical thinking involved in structuring a proof (Fukawa-Connelly, 2012a). Instructors may also ask comprehension questions, which are general questions that check for student understanding (CCHER, 2009). Comprehension questions, such as "Do you understand?" or "Does that make sense?" can often be answered using non-verbal cues such as facial expressions or a head nod.

The categories of cognitive processes in Anderson's Revised Bloom's Taxonomy are: remember, understand, apply, analyze, evaluate, and create (Anderson & Krathwohl, 2001). One problem with analyzing questions in this way is that the cognitive process needed by the student cannot be directly observed, and so the researcher must make inferences (Gall, 1970). Nonetheless, Bloom's Taxonomy has been used heavily in K-12 mathematics education research with similar results: that instruction and assessment commonly emphasize remembering and reciting facts and that *higher-order* (apply understanding and above) questions are less frequent (Anderson & Krathwohl, 2001).

Bloom's Taxonomy is designed to analyze questions in all subject areas and is not specific to mathematics. Because of this, when Tallman & Carlson (2012) used Anderson's

Revised Bloom's Taxonomy to analyze Calculus 1 examinations, they made a slight adaptation to the "remember" category. They argued that students in Calculus 1 could apply a procedure without demonstrating understanding of the mathematics that they were using. Therefore, they parsed the "remember" category into "remember" and "recall and apply a procedure." They found that 85% of the items coded on 150 Calculus 1 final exams could be solved by retrieving rote knowledge from memory or recalling and applying a procedure. Thus, their findings were consistent with studies examining questioning and assessment at the K-12 level.

Bloom's taxonomy has been used to investigate the types of questions that appear on computational mathematics examinations at the undergraduate level. This study differs in two distinct ways. First of all, the nature of the mathematics in these case studies is abstract as opposed to computational and may therefore influence the types of questions asked by the instructors. Secondly, this study investigates questions that were verbally posed by instructors in class, not written questions that have been constructed specifically for use on examinations.

2.6.3 Rationale for a Multi-Dimensional Approach. Gall (1970) points out that there are several types of questions that do not fit well into the existing taxonomies. I have already noted that the response type of a question may not give a complete representation of the cognitive engagement required to answer the question. Thus, this study will consider the expected response types and cognitive engagement as separate dimensions, which will allow for a more in-depth classification of the question types.

Previous studies of teaching practice in proof-based mathematics courses have not focused explicitly on instructor questions, although research in K-12 mathematics teaching suggests that the types of questions asked by instructors when teaching proof is at a higher level than when instructors are teaching computational mathematics (Thompson, et. al., 1994). Weber (2004) found that the analysis instructor in his case study did not frequently interact with his students. Fukawa-Connelly (2012a) describes an abstract algebra instructor's questions as "high level" and others as "factual," noting that students primarily answered only factual questions.

This study contributes by providing more specific and detailed descriptions of the questions that were asked in lectures.

Past research has noted that in traditionally taught courses instructors tend to use a funneling pattern of questioning, where the instructor would begin with a higher order question and ask successively lower order questions (Wood, 1994; Fukawa-Connelly, 2012a). Thus, I recorded for each instructor question whether or not the question was linked to another question and whether or not the question was actually answered by a student. This will allow for a separate analysis of the questions that were actually answered by students.

2.6.4 Implications of Research on Interactions. In traditionally taught courses, it has been shown that the IRE interaction pattern is prevalent, and that questions tend to be lower-order. The questions that appear to be higher-order are often part of a funneling sequence of questioning, so that the students end up answering a lower-order question in the end. This study will investigate the type of questions that are asked by instructors in four proof-based mathematics courses that are taught in a traditional style. Using a two-dimensional taxonomy for assessing the level of questioning, along with a record of linked questions and student responses, this study will give a more in-depth look at instructor questions in advanced mathematics courses.

2.7 Summary and Conceptual Framework

The review and critique of the literature has led to the development of a conceptual framework, which serves as a map of the territory being investigated (Miles & Huberman, 1994). "Development of a conceptual framework posits relationships and perspectives vis-à-vis the literature reviewed, thereby providing the conceptual link between the research problem, the literature, and the methodology selected for your research" (Bloomberg & Volpe, 2012, p. 86). This section will summarize the literature review, applying it directly to the research problem and methods used to investigate the research questions.

Speer et al (2010)'s model for assessing teacher practice includes: (a) allocating time within lectures, (b) selecting and sequencing content within lessons, (c) motivating specific content, (d) asking questions, using wait time, and reacting to student responses, (e) representing mathematical concepts and relationships, (f) evaluating completed teaching and preparing for the next lesson, and (g) designing assessments and evaluating student work. Research in Undergraduate Mathematics Education is a much younger field than research in K-12 mathematics education (Selden & Selden, 2003), and thus a foundational research-based understanding of teacher practice at the undergraduate level has not yet been as firmly established (Speer, et al, 2010). This study will address this gap in the literature by investigating several different aspects of teaching practices in proof-based mathematics lectures: the allocation of class time among different types of content, motivating and representing mathematical concepts and relationships through examples, and questions posed by the instructors.

The transition to proof-based courses at the undergraduate level is notoriously difficult for students. The literature has identified several contributing factors to students' difficulties including difficulties with the mathematical language and notation (Selden & Selden, 1995; Thurston, 1994), insufficient understanding of the definitions and concepts involved (Edwards & Ward, 2004; Moore, 1994), beliefs about mathematics and proof that are inconsistent with the mathematical community (Solomon, 2006), or a lack of strategic knowledge about the domain's proof techniques (Weber, 2001). Although the participants in this study were unfamiliar with the research on students' difficulties with proofs, many of the pedagogical strategies that they employed in their proof presentations were addressing these issues.

Although there have been some reform efforts in proof-based mathematics courses (Leron & Dubinsky, 1995; Larsen, 2009), traditional lecture is still widely used in undergraduate education (Armbruster, 2000; Bressoud, 2011). Despite this, researchers have not identified the varying pedagogical practices that may occur among instructors who identify as traditional. Thus, a research-based understanding of teacher practice in traditionally taught proof based courses is necessary to "support and explain results of studies of students' proof-writing abilities" (Fukawa-Connelly, 2012a). The studies that do describe aspects of teacher practice in proof-based traditional mathematics courses are single case studies (Fukawa-Connelly, 2010; Fukawa-Connelly, 2012a; Weber, 2004). This study adds to the existing literature by providing a multi-case study, and by combining observation data with interviews of the instructors.

Since Mejia-Ramos and Inglis (2009) identified proof presentations as an area needing further research, there have been many studies focusing on instructors' pedagogical perspectives of their proof presentations (Alcock, 2010; Weber, 2011; Yopp, 2011; Hemmi, 2010). There have also been case studies describing different aspects of in-class proof presentations (Weber, 2004; Fukawa-Connelly, 2010; Fukawa-Connelly, 2012a). The rationale for the investigation of proof presentations is that the instructors' proof presentations are the primary way that the instructor models the mathematical behavior of proof writing to his students. This study will provide further justification for research on proof presentations by documenting the large proportion of class time that instructors spend on proof presentations.

Instructors in mathematics classes have been shown to use examples in various ways. When proving, students tend to prefer empirical arguments (Harel & Sowder, 1998). This tendency to reason with examples is not inherently problematic, in fact, expert mathematicians have been shown to use examples when formulating and testing conjectures and when constructing and verifying proofs (Inglis, et al, 2007; Lockwood, et al, 2012; Weber, 2011). Also, instructors claim that they use examples as a pedagogical tool in their proof presentations (Alcock, 2009; Weber, 2011). This study will document the when and how the four participants used examples in their proof presentations, providing a coherent framework that is grounded in observation and interview data and informed by the literature.

Questions posed by instructors at the K-12 level are understood to be primarily factual (Anderson & Krathwohl, 2001; Sahin & Kulm, 2008), with the exception of exchanges in which mathematical proofs are offered (Thompson, et al, 1994). In undergraduate level proof-based mathematics courses, some instructors do not interact with their students using questions (Weber, 2004), while others ask a variety of questions, including some higher-order questions (Fukawa-Connelly, 2012a). Higher order questions were often followed by a sequence of narrowing questions so that the students eventually responded to a factual question (Fuakwa-Connelly, 2012a). This study will give an analysis of the types of questions asked by instructors in proof-based mathematics classes as well as an analysis of the questions that were actually answered by the students. The level of the questions will be determined by a two-dimensional framework that is based on Anderson's Revised Bloom's Taxonomy (Anderson & Krathwohl, 2001) and Mehan's (1979) four types of questions.

CHAPTER III

METHODOLOGY

The purpose of this multi-case study is to examine the teaching practices of four mathematics instructors who were teaching different proof-based mathematics courses using lecture methods. In particular this study will investigate pedagogical tools that they use when presenting proofs in class, how they allocate time within lectures, how they use examples in conjunction with their proof presentations, and how they use questions in their lectures. The initial research questions were questions 1 and 2. The analysis of those questions led to the identification of two pedagogical moves, examples and questioning, which are the focus of research questions 3-6.

1. What pedagogical moves do instructors plan to use to help students understand their proof presentations, and how often do they use these moves?
2. How do instructors allocate their class time in traditionally taught proof-based undergraduate courses?
3. What types of examples do instructors use in presentations of theorems and proofs in an upper-division proof-based mathematics course, and when do these examples occur chronologically in relation to the presentation of theorems or proofs?

4. What are the instructors' pedagogical uses for the different types of examples when presenting the statement of a theorem or a proof?
5. How often do instructors who are teaching advanced mathematics using lecture methods interact with their students by asking questions?
6. What types of questions are asked by instructors who are teaching advanced mathematics using lecture methods, and what types of responses are expected of students?

This chapter describes the study's research methodology and has sections that discuss the following areas: (a) rationale for research approach, (b) description of the research sample, (c) summary of information needed, (d) overview of research design, (e) methods of data collection, (f) analysis and synthesis of the data, (g) ethical considerations, (h) limitations of the study.

3.1 Theoretical Perspective and Rationale

The epistemology underlying this study is constructionism, which is often referred to as radical constructivism (Von Glasersfeld, 1995). Crotty (1998) states that constructionism is, "the view that all knowledge, and therefore all meaningful reality as such, is contingent upon human practices, being constructed in and out of interaction between human beings and their world, and developed and transmitted within an essentially social context" (p. 43). Therefore, the researcher constructs his or her understanding of the data by engaging in interaction with the data and participants throughout the analysis process. The epistemology of constructionism adopts the point of view that it is impossible to describe social phenomena purely objectively, because any such description is subject to the researcher's interpretations. Thus, to maintain trustworthiness, researchers using this epistemology must be upfront about their perspectives, and must clearly articulate all of the steps that were used in data collection and analysis.

The constructionist epistemology provided a lens through which to analyze my data. In particular, it allowed me to proceed with the analysis in an iterative manner. Von Glasersfeld states that, “Empirical facts, from the constructivist perspective, are constructs based on regularities in a subject’s experience. They are viable if they maintain their usefulness and serve their purposes in the pursuit of goals” (Von Glasersfeld, 1995, p. 128). Thus, throughout the multiple phases of data analysis, I was constructing my results from the regularities that were occurring in the data. This process allowed me to identify areas of focus as I pursued my research questions.

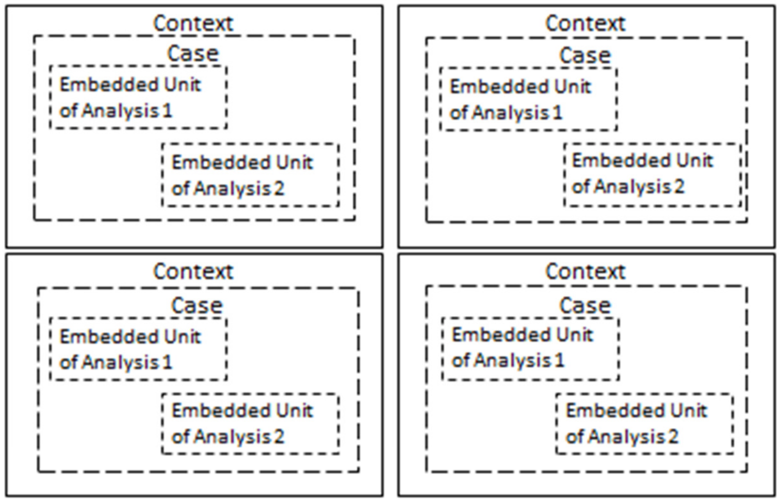
3.2 Research Design

This study is a descriptive multi-case study, as it seeks to describe the phenomenon of teaching proof-based mathematics courses in the context of real classrooms. Case study research is defined as “the in-depth study of one or more instances of a phenomenon in its real life context that reflects the perspective of the participants involved in the phenomenon” (Gall, Gall, & Borg, 2007, p. 447). According to Yin (2003), “case study research is the method of choice when the phenomenon under study is not readily distinguishable from its context” (p. 4). Since the phenomenon of teaching proof cannot be divorced from other social aspects of the classroom the context is necessary for interpreting the results of this study.

Case study research is commonly used to investigate teacher practice. Several recent studies that investigate teaching mathematics at the advanced level have been case studies (Weber, 2004; Fukawa-Connelly, 2012a; Fukawa-Connelly, 2012b), and multi-case studies at the K-12 level have also been conducted to investigate teacher practice (Zodik & Zaslavsky, 2008).

Yin (2009) lists two major components of case study designs. Case studies can be single-case studies or multi-case studies. They can also have one unit of analysis or several embedded units of analysis. This study is what Yin (2009) calls a Multi-Case Embedded Design. A diagram of this type of design is shown in Figure 1.

Figure 1: *Multi-Case Embedded Design* (Yin, 2009; p. 46)



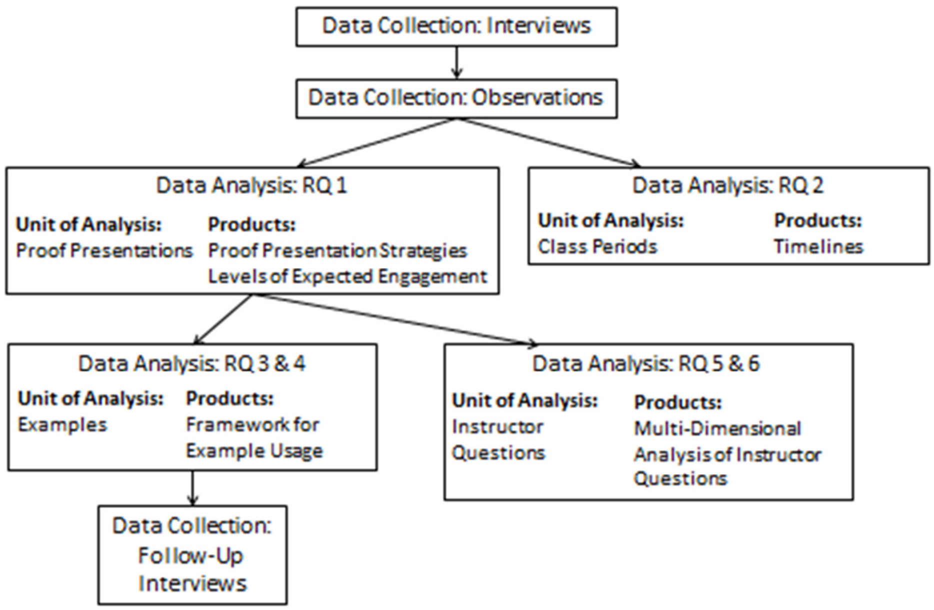
This study is comprised of four interrelated case studies of mathematics instructors who are teaching upper-division proof-based mathematics courses. The embedded units of analysis are the proof presentations, the allocation of class time, the examples used in proof presentations, and the questions asked by the instructors. The context for each case is extremely important for interpretation of the results, but there is logic for drawing cross-case conclusions. In the past, some have compared multiple cases to multiple responses in a survey, that is, cross-case comparison follows “sampling” logic. This is a mistaken analogy because the selection of the cases is usually not done using statistical sampling methods. Rather, the logic for cross-case comparisons can be compared to the logic of a scientist who does multiple experiments. The scientist may conduct an experiment and draw a conclusion. Then, the scientist tries to replicate that experiment to provide robustness to his conclusions. Multi-case studies follow a similar logic. Cases can be selected to predict similar results, or to predict contrasting results for anticipatable reasons (Yin, 2009).

Qualitative research is used to explore and describe social phenomena, and is often used to investigate the complexities of the sociocultural world (Bloomberg & Volpe, 2012). Teacher practice in the classroom is a complex social phenomenon which cannot be fully described by quantitative methods, thus, my data collection methods and theoretical perspective were essentially qualitative.

My study design, however, is an emergent mixed methods design. The quantitative aspect of the study occurred after the study was underway, thus, the design is called “emergent” (Creswell & Plano-Clark, 2011). The quantitative parts of the study were conducted by quantizing the qualitative data and computing descriptive statistics to provide a more in-depth picture of the instructional practices. Because the qualitative and quantitative methods were mixed at the data analysis phase, and no quantitative data were collected, the study is considered *embedded* in qualitative methodology, and can be described as “QUAL-quan” (Creswell & Plano-Clark, 2011).

I began my study by investigating the first two qualitative research questions, and the conclusions that were drawn from those research questions led to other, more quantitative research questions. The first research question addressed the pedagogical moves that the instructors make when they are presenting proofs in class. The conclusions showed that the instructors used examples in their proof presentations to varying degrees, and that they interacted with their students frequently by asking questions during their proof presentations. This analysis led to the identification of two areas of instructional practice that I would further explore: use of examples and instructor questions. In particular, the fifth and sixth research questions which deal with the frequency and types of questions that are asked by the instructor required a significant amount of quantitative analysis, such as frequency counts and percentages.

Figure 2: *Research Design Diagram*



The research design diagram in Figure 2 outlines the flow of the data collection and analysis. The interview and observation data were collected. The interview and observation data were used to address the first research question, and the observation data were used to address the second research question. The conclusions drawn from the analysis of proof presentations led to the development of the 3rd-6th research questions, addressing the examples used in proof presentations and the instructor questions. Upon the completion of the data analysis of examples used in proof presentations, follow-up interviews were conducted.

3.3 Information Needed to Conduct the Study

This multi-case study investigates the teaching practices of four university teachers who are teaching proof-based upper-division mathematics courses. In particular, this study will focus on the tools that they identify for proof presentations, the ways that they allocate their class time, the different examples that are used in proof presentations, and the types of questions that they ask their students. Contextual information describes the culture and environment of the setting, and is critical for case study research. This information was found by using the university's course catalog, records of student enrollment, and interview data from the participants. Since the data contain interviews, perceptual information about the participants' experiences in teaching proof-based courses shapes the interpretation of the data. The contextual and perceptual information is presented in Section 3.4: Research Participants and Settings.

Table 1: Information Needed for the Study

	What the Researcher Requires	Methods
Contextual	Organizational background, course descriptions, enrollment information.	Course catalog, university records, interviews
Perceptual	Participants' descriptions of their experiences teaching proof courses.	Interviews
Research Question 1	Participants' views on the pedagogical tools that will help students to understand their proof presentations.	Interviews
Research Question 2	Analysis of the time allotted for various course components including presenting definitions, theorems, proofs and examples.	Observations
Research Question 3	Examples used in proof presentations and the context in which the examples were used.	Observations
Research Question 4	Participants' reasons for using different types of examples.	Interviews and inferences drawn from observations
Research Question 5	Counts of the number of questions asked by instructors, total class time, and student responses.	Observations
Research Question 6	Analysis of instructor questions using the Cognitive Process dimension and Expected Response Type dimension	Observations and inferences drawn from observations

Six research questions were formulated, and the information needed to answer these questions is outlined in the table. In addition, an ongoing review of the literature was conducted to provide the theoretical grounding for the study.

3.4 Summary of Data Collection and Analysis Procedures

The following summarizes each step that was used to carry out this research. An in-depth description of each stage of the research will follow.

1. Preceding the collection of data, a literature review was conducted to frame the study with existing results and to justify the purpose for this research. The literature review was ongoing throughout the entire research process, examining the contributions of others in the areas of mathematical proof, teacher practice at the university level, example usage in mathematics classrooms, and instructor questions.

2. I acquired approval from the Institutional Review Board to proceed with the research. The IRB application required me to be explicit about all data collection and analysis methods, and to ensure that this study would meet the standards for the protection of human subjects including informed consent documents and confidentiality.

3. Research participants were identified by looking at the departmental teaching assignments. All instructors who were teaching proof-based undergraduate level mathematics courses were contacted in person and asked to participate in the study. Three participants volunteered to participate in the study.

4. Upon recruitment, a one-hour interview was scheduled with each participant. At the time of the interview, directions for data analysis had not yet been determined, so the interview questions were intentionally broad. The participants were asked to describe how they present a

proof in class, and what they do to help the students understand a presented proof. To help them focus, they were asked to describe a particular proof that they might present in their content area.

5. The interview data were analyzed using the constant comparative method to determine the different pedagogical moves that the instructors would make during proof presentations to help their students understand their presentation. The two overarching themes that emerged were *expected engagement* and *proof presentation strategies.* The four proof-presentation strategies that were identified from the interview data were *outline, examples, logical structure,* and *historical context.*

6. Throughout the course of the semester, approximately every two weeks, video-taped observations of each classroom were conducted. The camera was focused on the instructor and the chalkboard. Dates for the observations were discussed in advance with the instructor to avoid collecting data on exam days. Six or seven observations of each faculty member were made throughout the semester.

7. Initial analysis of the video data included making detailed logs of each observation that described the activities of the instructor, identifying and transcribing all instances of proof presentations, and looking for instances when the instructors were using the different pedagogical tools that they mentioned in their interviews. This initial analysis showed that the instructors, on average, were using half of their class time on proof presentations. This finding further justifies the importance of investigating proof presentations in advanced mathematics courses.

8. Although the data were initially collected as a pilot study, the data were so rich that my advisor and I decided that the data were sufficient for my dissertation study. Thus, a proposal was defended and accepted by her committee.

9. Another participant was recruited, and an initial interview was conducted with this participant. In this case, the instructor was teaching a course with a distance-learning student, and

so all of the lectures were video-taped by the campus IT staff. In this case, a sampling of the video-taped lectures was used. This sample was comparable to the sampling from the other instructors. Proof presentations in this data were transcribed.

10. The observation data were analyzed to determine the incidents in which examples were used in conjunction with proof presentations. Examples were categorized using the constant comparative method, and then categories were compared to the types of examples that were identified in the literature. The types of examples that matched with examples found in the literature were named in a method consistent with the literature, and categories that were not found in the literature were identified as well.

11. A framework for example usage was constructed by determining the timing of when the examples were presented in conjunction with the theorem/proof pair and the pedagogical intentions of the instructor.

12. Timelines of each observation were constructed to record the content to which class time was allotted, and the source of the verbalization in the classroom. The categories for content were Definition, Theorem, Example, Proof, and Homework Problem. The categories for verbalization were Students Speaking and Instructor Speaking. These timelines can give a visual representation for how class time was spent.

13. The four participants were contacted again to participate in follow-up interviews to serve as a member-check. The participants were shown their timelines, the framework for example usage, and transcripts of their observations. They were asked specific questions about particular uses of examples, and asked to comment on their timelines and the framework. They were also asked to comment on the ways in which they questioned their students, although a detailed analysis of their questions had not yet been completed.

14. The observation data were viewed again, and all of the questions asked by the instructor were transcribed into an Excel spreadsheet. Multiple dimensions were recorded for each question, including the time in which the question was asked, whether or not the question was linked to another question, whether or not there was a student response, the cognitive engagement required to answer the question, and the type of response expected of the student. The cognitive engagement dimension was an adaptation of Anderson's Revised Bloom's Taxonomy (Anderson & Krathwohl, 2001), and the expected response type was a modification of Mehan's (1979) four types of questions.

3.5 Definitions

3.5.1 Proof-Based Mathematics Course: A *proof-based mathematics course* is a course in which students are expected to comprehend written proofs and create original proofs for themselves.

3.5.2 Traditional Lecture: In this study, I will use the phrase *traditional lecture methods* to refer to teaching methods in which the instructor is the primary mathematical authority. In traditional classrooms the instructor is typically standing at the board presenting material while the students are sitting in rows in their desks.

3.5.3 Proof: "The process an individual employs to remove or create doubts about the truth of an observation" (Harel & Sowder, 1998). For this study a mathematical presentation was coded as a proof if one of these three indicators was present: the instructor himself said or wrote the word "proof" in reference to the presentation, the instructor provided partial or complete justification for a claim, or the instructor worked a homework problem that required justification.

3.5.4 Proof Presentation: A proof presentation contains both the statement of the claim that is to be proved, the proof itself, and any comments or examples that precede or follow the theorem/proof pair that are related to the proof. The beginning and ending of the presentation were determined by natural breaks in the dialogue.

3.5.5 Example: A specific, concrete representative of a class of mathematical objects, where the class is defined by a set of criteria.

3.5.6 Generic Diagram: A picture that is used to guide and structure a proof that lacks the specificity to be considered an example.

3.5.7 Question: Any utterance that had the grammatical form of a question as well as an utterance that were intended to elicit a student response were considered questions.

3.6 Research Participants and Settings

A purposeful sampling was used to select the participants in this study. In particular, criterion sampling (Bloomberg & Volpe, 2012) was used to select faculty members who were teaching proof-based upper-division mathematics courses at a large comprehensive research university. Four faculty members agreed to be interviewed and video-taped as they taught periodically throughout the semester. All four professors are tenured, have many years of teaching experience, and are well respected by their colleagues. Three of the four have won awards for their teaching. The four courses observed were Introduction to Modern Algebra, Number Theory, Geometry, and Introduction to Modern Analysis (Advanced Calculus). To protect the anonymity of the participants, masculine pronouns will be used for all of the participants throughout this paper regardless of their gender.

The observation data revealed that in all four classrooms the instructors were standing at the board talking while the students were sitting in rows in their desks taking notes. Although there were varying degrees of interactions between the teacher and students, instances of student-to-student interactions were not observed. When asked what instructional method that they used in class, Dr. A and Dr. N said that they used lecture, while Dr. C and Dr. G were both more comfortable referring to their style as “modified lecture,” because they attempt to involve students. Several of the instructors said that they did not use inquiry methods, which they described as “leaving them [the students] with open-ended problems.”

The interview data were used to present a summary of each participant’s views about teaching proof, which will provide a setting for the four case studies. In the interview, the participants were asked their purpose for presenting proof in class and what they do when they present proof in class to help the students understand. These questions were rather broad, and did not specifically focus on examples or questioning methods. The interview data were coded in an iterative process using the constant comparative method (Glaser & Strauss, 1967). Interview questions can be found in Appendix A, and the codes from the analysis are in Appendix B.

3.6.1 Dr. A’s Algebra Class. Introduction to Modern Algebra is a course that serves two purposes at this university. It is both an introduction to proof, and covers either group theory or ring theory, depending on the choice of textbook. The university course catalog says that this course covers an introduction to set theory and logic, elementary properties of rings, integral domains, fields, and groups. The class consisted of 24 students, and because students who are minoring in mathematics often take this course there is a more diverse range of majors. There were six math education majors, eight math majors, six engineering majors, two computer science majors, one geography major, and one chemistry major. There was one sophomore, and the rest were approximately half juniors and half seniors.

During the first week of classes, I sat down with Dr. A for our initial interview. The codes that he mentioned most frequently were: student understanding, level of audience, drawing pictures, and students should think about it at home. He seemed to be very focused on student understanding of the concepts. He said, “I try very hard to say to the students, 'What does the statement mean?' 'What do we have to prove?' Ok, 'What do we have to do to get this thing?'” He said that he likes to “not do stuff about proving things, I like to prove something and then talk about why I'm doing it in the middle of the proof. Of, um… I like to weave in, um, fundamentals, foundational things, into the proof." He expressed a desire to get into the algebra content and show the methods of proof within the content rather than teaching methods of proof first.

When presenting the material in class, he mentioned several times that he doesn't like to write out all of the details, because he believes that the students should get ideas from the lecture, and then go home and work out the details on their own. He also believes that there is value in showing students the thinking behind constructing a proof, and says that when presentations are "too slick" that they are often too fast for students to fully comprehend. He said that he enjoys the give and take with students, and that he likes to "just talk math and figure out what's true and what's not true and what we can understand, as opposed to the formal teaching and all."

3.6.2 Dr. C's Advanced Calculus Class. The course catalog says that the introduction to modern analysis course covers properties of the real numbers, sequences and series, limits, continuity, differentiation and integration. There were 9 students in the class: five were math education majors, two were math majors, and two were engineering majors. Eight were seniors, and one was a junior.

Dr. C's interview took place after the semester was already over, because the observation data collected from Dr. C was archival. Dr. C had taught a section with a distance learning student, and so all of the class periods were video-taped and already on file. The interview codes with the highest frequency were: examples, time constraints, asking for student input, proof is a means of verification, draw pictures, importance of applications, and proof is a means for communication.

He talked of being aware of where the students are by asking a lot of questions and getting the students to talk. He says that he likes to "have some sense that people are processing on what I am saying and with me before I move on." He likes to include applications of results so that the students see the need for the mathematics and how it is situated among other mathematical areas.

When presenting a proof, Dr. C talked a lot about the need to prepare students minds for formal proof. "I try to prepare their minds for proof, and usually that happens by example and computation… Proof is the means by which we verify interesting facts that we observe. Sometimes facts are observed by calculation, and by patterns in lots of examples, and the proof verifies the facts." So, this way of thinking about proof verifying the facts that we can discover by doing examples may influence his style of proof presentation.

He also talked about showing the thinking behind the proof construction, and the value of a non-linear presentation. Dr. C likes to involve the students in proof presentations. He says, "I ask them to help me. I think I always ask them to help me, and I think that I have, well, I don't know. Hopefully I always ask them to help me, it depends on how much time we have. If I'm running short, I might not." So, he does value student interaction, although he realizes that the time constraints of the classroom may not always allow for it.

3.6.3 Dr. G's Geometry Class. The university catalog describes this course an axiomatic development of Euclidian and non-Euclidian geometries. The class enrollment consisted of 9 students: four math education majors, four math majors, and one engineering major. Eight of the students were seniors, and one was a junior.

The interview took place during the first week of classes. The codes that Dr. G mentioned most frequently were: level of audience, write out details, ask questions to students, wait for responses, interact with students, historical significance, skip details of difficult proofs, and comment about proof structure.

Dr. G emphasized that he wants the students to be interacting with him throughout his lecture. He said "I insist throughout, throughout my lecture that the students respond to me… I just won't, I won't let them sit there. I just won't let them do it. So eventually, I just insist that I get some kind of response. You can say 'I don't know' if you absolutely have to, but you have to respond somehow." He wants to have feedback from his students, possibly so that he can adjust the presentation of the material to the level of his audience.

He also talked a lot about the historical significance of some of the famous proofs in geometry. He said that he would sometimes present a more difficult famous proof that he doesn't expect the students to be able to reproduce. "Sometimes there are proofs that are just too far, the class is not ready, they're more advanced and so you might say something about the theorem, but either skip the details, or maybe skip the proof all together, but basically if it's something that I think the class has a chance of following then I'll do it. I think you should take every opportunity to do real mathematics in math classes."

He also said that when he presents a proof that is within the students' ability, he makes a lot of comments about the structure of the proof and often asks students to tell him the next step as he walks through the proof. He also said that he would write out all of the details of the proof on the board.

3.6.4 Dr. N's Number Theory Class. The university course catalog says that this course covers divisibility of integers, congruences, quadratic residues, distribution of primes, continued fractions, and the theory of ideals. There were 14 total students in the class: seven were math education majors, six were math majors, and there was one engineering major. Thirteen of the students were seniors, and there was one junior.

The interview also took place during the first week of the semester. The codes that had the highest frequency in Dr. N's interview were: level of audience, problem solving, ask for student input, ask questions, dealing with student responses, group projects, logic, and application.

Dr. N clearly wanted to make sure that his teaching style addressed the needs of the students. He talked about how the last few times he has taught Number Theory he has skipped the difficult proof of quadratic reciprocity and instead talked about how the principles of quadratic reciprocity can be applied to credit card transactions. He said, "I've tried to make it more applied, just to have a bigger audience, so, I'll put in to, in undergraduate number theory, I'll put in a unit on cryptography, and other applications of number theory." He also advocated doing group projects in Number Theory. These projects consist of giving the student groups a pattern to explore, with the hopes that they will discover a pattern and be able to come up with a theorem. He said that they often find all kinds of different patterns that he didn't expect.

Another emphasis that Dr. N made was that all of mathematics is just problem solving. "I don't think of them as proofs, I think of them as solutions to problems. And, I think that that's what math is all about is solving problems, and, uh, a carefully reasoned, step by step solution is a proof. And, so when I'm presenting proofs, I'm presenting proofs to harder problems, so if I never go there, the students are never going to learn how to solve harder problems. And, uh, anyways, I view the focus of math is solving problems." He referenced the work of Polya (1945), and talked about how he likes to incorporate Polya's methods of asking questions to lead a student through the problem-solving process. Dr. N expressed that he couldn't always teach in the style of Polya because of class size and time constraints, but that he believes it is what should be done.

He said that he likes to spend time in a class discussion brainstorming how to attack a proof. He starts by presenting the theorem, "I try to read the fact that we're trying to uncover, or the theorem, and I say, 'What's important here? What do you think of when you read this?' And, I ask for ideas, what relevant theorems might be true and things like that… And then we discuss whether or not they really are relevant or not, you know. And, uh, so, um, if I have time, I like to engage in that sort of thing. You know, that's what you do in real life when you're trying to solve a problem, is you try to think of things which are relevant to what you're doing, and you try to find things and piece them together into the proof of what you're trying to do." He also mentioned that he is not worried about making mistakes in class, because that can show the students what solving problems in real life looks like.

Dr. N lamented that the students aren't taught basic logic anymore. He talked about how he has to incorporate teaching basic logic to the students as he is presenting proofs in class. Another presentation tool that he mentioned was pattern generalization. He said that he likes to do some numerical examples before presenting a theorem, in hopes that the students will be able to guess the statement of the theorem.

3.7 Data Collection Methods

Participants were solicited from among the instructors who were teaching proof-based mathematics courses at the undergraduate level at a Midwestern research institution during the 2010-2011 school year. Four experienced, tenured faculty members agreed to participate in the study. The courses that they taught were Abstract Algebra, Number Theory, Geometry, and Advanced Calculus (Introduction to Modern Analysis). The instructors and their courses were described in detail in the previous sections.

Multiple types of data were collected to achieve triangulation (Creswell & Plano-Clark, 2007; Bloomberg & Volpe, 2012). The data sources are an initial interview with each participant, six to seven video-taped observations of each participant's teaching, and a follow-up interview upon completion of the data analysis.

In the initial interview, the participants were asked to describe what they do to help the students understand a proof that they present in class. The interview method was used because it allows the participants to voice their own perspective of teaching advanced mathematics, and their comments as well as the existing literature are used to frame the data analysis. Because the interview was designed to frame the analysis of the observations, the initial interview questions were intentionally broad, asking the instructors to describe their experiences teaching proof-based courses, their philosophical stance on why we should ask students to do proofs, and why we should present proofs in class. Then they were asked how they decide whether or not to present a proof in their lectures, and what strategies they use when presenting a proof to help their students understand. Finally, they were asked how they assess student understanding of the proofs that they present in class. The initial interview questions can be found in Appendix A.

Table 2: Observation Data Collection

	Algebra	Adv Calc	Geometry	Num Thry
Instructor	Dr. A	Dr. C	Dr. G	Dr. N
# Students	24	9	9	14
# Instruction Days	44	29	44	44
# Observations	7	7	6	6
Total time observed (min)	323	536	250	264

Dates for video observations were purposefully chosen to be approximately every two weeks of the semester, to be convenient for the instructor, and to avoid exam days. The number of instruction days and observations are recorded in Table 2. Three of the courses met three days a week for 50 minutes, while the fourth met for 75 minutes per class period. All four courses had 2-3 mid-term exams and a comprehensive final exam, as well as periodic homework assignments.

There was no noticeable variation in teaching style across different instruction days for an individual participant, and so the data can be considered *saturated* in regard to the teaching methods used (Glaser & Strauss, 1967), therefore providing a fair snapshot of each instructors' teaching.

Some of the video was collected by the university's IT department, but the majority of the video was collected by either myself or a hired technician. Technical difficulties in the data collection include one instance when the video camera battery died during an observation of Dr. G's class, and another instance when part of the video file was corrupted in an observation of Dr. N's class.

Follow-up interviews were conducted in the spring semester of 2012. These interviews served as a member check, so the interview procedure varied slightly for each participant. The overall structure of the follow-up interview was to first allow the participant to comment about their thoughts on example usage and student interaction, then to share with the participant the analysis of their observation data, and then ask for any additional comments or explanations as needed. These were one-hour interviews that took place upon completion of data analysis of the observations, about a year after the completion of observation data collection. In the follow-up interview, the participants were presented with my descriptions of their classroom examples, excerpts of the transcripts from their teaching, and the timelines that showed how they partitioned their class time and the amount of time that the instructor and students were speaking. The participants were asked to comment on several instances when they used examples in proofs. They were also asked to comment on the framework for example usage in proof presentations and the hypothesized intentions for the different types of examples that they used. Although a detailed analysis of instructor questions had not yet been completed, they were also asked about their interactions with their students. The follow-up interview questions can be found in Appendix C.

3.8 Data Analysis and Synthesis

Qualitative data can be analyzed in several ways. The researcher can begin with a hypothesis and the data can be quantized and analyzed using quantitative methods, or the researcher can use the data to develop theory by taking notes and making memos, without explicitly coding the data. The *constant comparative method* combines an explicit coding procedure with the style of theory development (Glaser & Strauss, 1967). The researcher begins by coding each incident in his data into as many categories as possible as the categories emerge. This could be done through memo writing or by chunking the data onto cards. Then he compares the incidents that appear in each category so that the properties of the category begin to emerge. The language used may come directly from the data, or may be a description of the category that the researcher creates. Over time, the researcher compares each incident to the description of the categories that have emerged. This will force the researcher to choose the most important characteristics of an incident, so that the same situation is not used over and over again in different categories.

Data analysis occurred in several phases. The interview coding employed the constant comparative method (Glaser & Strauss, 1967) to identify themes in the data. For this study, the interviews were chunked into segments, and segments pertaining to the pedagogical tools that the instructors were using in proof presentations were extracted. These chunks were placed on index cards and sorted into piles. Two main themes emerged: comments that were focused on specific proof presentation strategies, and comments that referred to different ways that the instructors would interact with their students. Within those two themes, quotes pertaining to particular strategies were grouped into piles, and quotes pertaining to particular ways of interacting were grouped into piles. By comparing quote to quote, the characteristics of each category became clearer. Then the quotes were compared with the characteristics of each category as they developed.

The most frequent proof presentation strategies that were mentioned by the participants were chosen to use for the analysis of the observations. The interaction categories pertained more to the expectation of the instructor than the actual actions of the students, and thus the name for this theme developed into *expected engagement.* The different categories fell easily into a hierarchy, from least expected engagement to greatest expected engagement, which is presented in Table 4.

The frequencies of the codes as well as excerpts from the interviews were used to describe the contextual and perceptual information provided in Section 3.4. The interview data were also used to identify *proof presentation tools* and levels of *expected engagement* that were mentioned by the participants. The observation data were viewed, activities were categorized, and all instances of proof presentations were transcribed. Each proof presentation was coded for the *proof presentation tools* that were used, and the level of *expected engagement.*

The next phase was a more in-depth investigation of one particular proof presentation tool: the use of examples. The examples that were used in the proof presentations were analyzed using codes developed from the literature and new codes were developed and codes were modified throughout the analysis process. The examples were organized based on when they occurred within the presentation of the theorem/proof pair into a descriptive framework that describes the ways in which examples are used in proof presentations.

The final phase of analysis of the observation data was a further exploration into the ways that the instructors engaged their students by asking questions. Each instructor question was transcribed and analyzed across several dimensions: the cognitive engagement, expected response type, student response, and whether or not the question was linked to another question. A separate analysis was conducted, restricting to only questions that were answered by the students. This was done to investigate the cognitive level of the questions that were actually answered by the students.

3.8.1 Interview. The interviews were semi-structured, addressing what the instructors do when they present proofs in class, why they make those choices, and what they do to help students understand their presentation of proofs in class. The interview questions can be found in Appendix A. The interview data analysis served two purposes: to provide the settings for the case studies by describing the participants' teaching philosophies, and to identify themes with which to analyze the observation data.

The interview data were analyzed to give an overall impression of each participant's views on teaching proof. The interview data were transcribed and broken into chunks which were sorted into groups to identify themes, using the constant comparative method (Glaser & Strauss, 1967). The themes that occurred in this analysis are given in Appendix B. The frequency with which these themes appeared in the interview data gives an impression of what each participant values when teaching proof-based courses. Excerpts from the interview data were also used to describe the settings for the case studies, as presented in Section 3.5.

Table 3: Coding Scheme for Proof Presentation Strategies

	Description	Interview Segments
Outline	Discussed the ideas of the proof before writing out the proof	"Probably I'll start with kind of an outline. And then after I sort of, hopefully, get the plan down, then I'll say, 'Ok, now we're gonna write down the details.'"
Examples	Uses examples to motivate the theorem or proof, or to support students' understanding of the proof	I have to play with that statement in my brain to make sense of it. Well, how do I make sense of it? I start looking at examples. So, that's why I do that a lot.
Logical Structure	Discuss the logical structure of the statement or the proof, specific references to logic during a proof presentation	"And, so then I make the statement on the board, and I point out if it's an if-then statement, or if it's a, both, if-and-only-if statement, and I talk about that, what we have to prove then."
Context	Instructor points out the historical significance of the proof, or places the proof in context of the mathematical content area.	I don't know, I think everybody who is going to teach geometry in high school should *see* this. This is something that they should be aware of.

Next, the types of *expected engagement* and *proof presentation strategies* that appeared in the analysis of the interview data were used for the analysis of the observations. Table 3 summarizes the four proof presentation strategies that were mentioned the most frequently by the participants, along with excerpts from the interview data that give an impression of the types of comments made by the participants.

Table 4: Coding Scheme for Expected Engagement

	Description	Interview Segments
1	Professor does not request active contribution during the proof presentation, and does not appear to engage students.	(none)
2	Professor does not request active contribution during the proof presentation, but based on monitoring, non-verbal communication with students, or feedback from closed-ended questions.	"I really try to make eye-contact and try to, you know, wake them up and make sure that they are following me." "Well, first of all, I just try and watch 'em."
3	Professor expects students to contribute some factual information during the proof presentation.	"I'll say something like, well, 'this triangle is congruent to that one, why? Tell me why this triangle is congruent to that one. What's the reason that this one is congruent to that?"
4	Professor expects students to contribute some factual information and some key ideas of the proof during proof presentation	"I may discuss why I chose the next step, or give them the opportunity to suggest the next step"
5	Professor expects students to generate the majority of the ideas for the proof during the proof presentation.	"I try to read the fact that we're trying to uncover, or the theorem, and I say, 'What's important here? What do you think of when you read this?' And, I ask for ideas, what relevant theorems might be true and things like that." "I will ask them to help me set up a strategy for what the proof is going to be"

Table 4 summarizes the five levels of expected engagement sorted from least interaction to most interaction with excerpts from the interview data that are representative of the codes. The categories created from the interview analysis were used to analyze each proof in the observation data, which will be described in the following section.

3.8.2 Initial analysis of observation data. The analysis of the observation data also took place in multiple phases. Detailed logs of each observation were created, which recorded the time, activities, examples used, and descriptions of what was happening in each chunk of class time. Each definition and theorem were included in full, and descriptions of the examples and proof presentations were also included, as well as some of the interactions that occurred between the instructor and students. A sample log from each instructor can be found in Appendix D. Instances of proof presentation were identified using the following indicators:

- Instructor wrote a formal proof on the board, and called it a proof himself.
- Instructor provided partial or complete justification for an argument.
- Instructor worked problems involving a proof from the students' homework, or on the same level.

Because the homework problems required proof or some kind of formal justification, I also counted homework problems that were worked or discussed in class as proof presentations. Then transcriptions were then made of each instance of proof presentation, beginning with the introduction to the statement of the claim and ending after the proof at a natural break in the dialogue. Next, counts of the frequency of proof presentation and amount of time of proof presentation were computed to give a broad picture of how frequently proofs are presented in class.

Once the transcriptions were complete, I read through each observation again, taking note of the amount of time spent on each proof presentation. The times were analyzed to determine the amount of class time spent on proof presentations, the number of proof presentations per 50 minute class period, and the average amount of time spent on each proof presentation.

The next phase of analysis used each proof presentation as the unit of analysis. Four *proof presentation strategies* were identified from comments made by the professors in the interview data. The four strategies are *outline, examples, logical structure,* and *context.* Descriptions of the four categories are listed in Table 3. For each proof presentation, I noted which of the four proof presentation strategies occurred. Some of the presentations contained none of these identified strategies and some contained multiple strategies, in fact, there were proof presentations in my data that contained all four.

Next, each proof presentation was classified as one of the five levels of *expected engagement.* The categories for expected engagement were constructed from the interview data and are presented in Table 4.

3.8.3 Identifying and coding examples in proof presentations. Three of the participants frequently used examples in their proof presentations. This section will discuss my definition of "example," methods for identifying instances of examples in the data, the creation of categories of examples, and the creation of my framework for example usage.

3.8.3 a. Identifying Examples in the Data. The first step in analyzing the usage of examples was to go through all of the observation transcripts and pick out instances that satisfied my definition of example. Often, I needed to go back to the actual video footage to see what was taking place in the proof.

Figure 3: *Example usage in an observation of Dr. C's lecture.*

Pf #	Theorem	Example	Type?
C.9.1	Proves that $P(A) \cup P(B) \subseteq P(A \cup B)$	No examples used	
C.9.2	Absolute value proof. $\lVert x\rvert - \lvert y\rVert \leq \lvert x-y\rvert$	No examples used	
C.9.3	Proves the trichotomy law for the field of rational functions with a particular ordering.	To explain a partial ordering. Ordering on subsets of *N*, by *A*<*B* if *A* is contained in *B*, and {1,2}, {3,4} are not comparable. Example to illustrate the fact that R has no zero divisors, prompted by student question. "You know that (x-2)(x-3)=0 means x=2 or x=3"	Start-up example Instantiation of sub-claim
C.9.4	Prove that a subset of a finite set is finite.	No examples used	

I also looked at the detailed log of what was happening in the classroom to determine if there were any examples near the transcribed proofs that may have been associated with the proof in some meaningful way. I took notes on the ways that examples were used in and around each proof by making a table for each observation. The tables had three columns: the statement of the theorem to be proved, a description of the example(s) used, and notes about the type of example. A sample table is presented in Figure 3. Throughout the coding process and as the categories developed, I would refer to the tables for an overall picture of the examples that were used.

3.8.3 b. Creation of Coding Categories. First, I identified all of the presentations of theorems or proofs that have included examples. I then viewed these portions of the video data multiple times, read through the transcripts, and noted the characteristics of each example, including the timing of the examples in relation to the presentation of the theorem/proof, the use of the example, and the interaction between the instructor and students (e.g. whether the example spontaneously generated in response to a student misconception). Constant comparison between the incidents of examples and the properties of each category as they developed helped to solidify the characteristics of the different types of examples.

Some of the types of examples conformed to types of examples mentioned in the literature. In these cases, I used terminology consistent with the literature. The categories that were linked to example types in the literature were: start-up examples (Michner, 1979), pattern generalization examples (Harel, 2001; Rowland & Bills, 1999), boundary examples (Mason & Watson, 2001), instantiation of a claim or definition (Alcock, 2010), and generic examples (Rowland, 2002).

Some minor changes were made to the categories that were linked to the literature. The name *pattern generalization example* was changed because the purpose is slightly different from

Harel's (2001) two types of pattern generalization, in which the students are attempting to use examples to construct a proof of a given claim. In the context of this study, the examples are used to lead up to the statement of the claim, which is similar to activities given to students in studies on conjecturing (Morselli, 2006). Thus, the name was changed to *pattern exploration* to reflect the exploratory nature of these examples that lead up to the statement of the claim. It has been shown that students do not have a broad range of example generation strategies (Iannone, et al, 2011), and that students who choose examples methodologically are better able to communicate their conjectures and even construct some proofs (Morselli, 2006). When an instructor uses pattern exploration examples, he is modeling a mathematically mature way to select examples that could lead to the formulation of theorems.

Some of the types of examples did not match any of the types of examples found in the literature, so new categories were created in these cases. In particular, there were some examples that were instantiating, but were not instantiating the claim or a particular definition. In my data, I found examples that were instantiating sub-claims, notation, or just general mathematics concepts. The different types of instantiation were used for different purposes. Instantiation of a sub-claim is used to support students' understanding of a proof, while instantiations of definitions, notation, or concepts were used to support students' understanding of the underlying mathematical concepts.

Another change was the addition of a code for *metaphorical examples*. One problematic example occurred when Dr. A was trying to discourage students from counting the identity element in S_4 multiple times. The students were counting four "one-cycles." Dr. A then commented that there were just multiple ways to write the same element, and he compared it to the fractions ½ and $^{30}/_{60}$, stating that these are not two different fractions. At first, this example was coded as instantiation, but it didn't seem to fit into that category because the example was comparing the similarities between two different and distinct mathematical structures:

permutations on a finite set and equivalence classes of fractions. Thus, a *metaphorical example* is an example that is used to compare one mathematical structure to a different (more familiar) mathematical structure. The name was chosen because the example is serving as a *linking metaphor,* linking arithmetic to other branches of mathematics (Lakoff & Nunez, 2000).

Once the categories were formed, I began the process of delimiting the categories and assigning pedagogical intentions to the example types. Glaser and Strauss (1967) say that "some comparisons at this point can be based on the literature of other professional areas (p. 110)." I was able to link some of the categories to types of examples found in the literature on mathematics teaching and learning. This allowed me to form hypotheses about the pedagogical intention of the instructor in the context of the lecture. For categories that did not fit with example types that appeared in the literature, I also formed pedagogical intentions based on the properties of each category and the interview data.

The timing of the example in relation to the presentation of the theorem and proof was an important factor in determining the pedagogical purpose of the example. Examples will be said to *motivate* a particular mathematical concept if they occur before the concept is presented, and *support* the concept if the example is given during or after the concept is presented.

The resulting categories with their pedagogical intentions are:

Warm-Up Examples occur before the statement of the claim and serve to prepare the students minds for the claim.

Pattern Exploration Examples also occur before the statement of the claim and help the students to generalize the statement of the claim from concrete examples.

Boundary Examples serve to highlight the necessity of the hypotheses of the claim and may occur before or after the statement of the claim.

Examples that Instantiate the Claim occur after the statement of the claim, and may serve to help students understand the claim, or to prepare them for the presentation of the proof.

Generic Examples occur during the proof, and are written side-by-side with the proof so that students can take aspects of the particular example and apply it to the general proof.

Instantiations of a Sub-Claim generally occur during a proof, and support the students' understanding of a sub-claim, which may not be proved if elementary enough.

Instantiations of Concepts, Definitions, or Notation serve to reinforce the mathematics content underlying the claim or proof, and may occur at any time.

Metaphorical Examples can also be used at any time. These occur when an instructor compares some aspect of a mathematical structure to a different, more familiar, mathematical structure via metaphor.

3.8.3 c. Creation of the Framework for Example Usage. I constructed pedagogical uses of examples from the interview data and inferences from the observation data, and ascribed these to the types of examples that emerged in the observation data. Then, I began to look for relationships among the types and uses of examples (Glaser & Strauss, 1967), and also noted when the different types of examples were used in relation to the presentation of the theorem or proof. I created a timeline for the presentation, and placed the different types of examples on the timeline to represent where they occurred in the data. Some of the types of examples seemed to be independent of this timeline, because they dealt with content that could be discussed at different points throughout the presentation. Arrows were drawn on the timeline to represent whether the examples were intended to motivate or support students' understanding of the claim or of the proof.

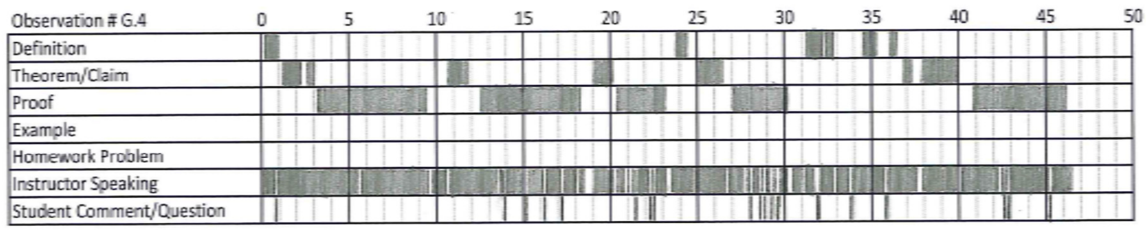
The framework that I have developed is an organizational structure that illustrates the relationship between different types of examples and the presentation of theorems and proofs in mathematics lectures. My framework serves to describe the uses and types of examples that emerged directly from my observation data, and provides insight into the pedagogical intentions of the instructors via interviews and inferences from the observation data. The framework is not intended to predict the behavior of instructors, but rather to describe the types and uses of examples in instructors' proof presentations, and to organize them chronologically with respect to the presentation of the theorem and proof.

3.8.4 Creation of Observation Timelines. To investigate the ways that instructors allocate their class time, timelines were constructed of each observation. These timelines were inspired by the timelines presented in Atman et al.'s work investigating the problem solving of novice and expert engineers (Atman, Adams, Cardella, Turns, Mosborg, & Saleem, 2007). The timelines provided a visual way to organize my observation data, allowing me to find sections of the data quickly and efficiently, and showing trends that occur within the different types of content presented and the interactions between the instructor and students.

The timelines have categories representing the different types of mathematical content that the instructor may present: definition, theorem, proof, example, or homework problem, and categories representing the source of verbalization, whether the instructor or students. The minutes in each observation are listed across the top of the timeline, and the timelines are shaded to represent when the codes were present in the observation. The timeline template was created in Microsoft Excel, and each template was shaded by hand as I watched the observation videos. A sample timeline is shown in Figure 4, and the timelines for all observations are in Appendix E. Since Dr. C's class periods were 75 minutes long and the timeline template was for 50 minutes, only the first 50 minutes of these observations were coded in this particular analysis.

To answer the question of how instructors allocate their class time, the number of minutes spent on each content code for each instructor was totaled and the percentages were calculated. To investigate the frequency of interactions between the instructor and student, the number of one-minute intervals in which there was back-and-forth interaction between the instructor and students was totaled, and the percentages were calculated for each instructor.

Figure 4: Timeline from an observation of Dr. G's lecture



It should be noted that in the timelines the code "proof" was only shaded when the instructor was actually doing a proof in class, whereas earlier the time counts for "proof presentations" included dialogue leading up to the proof, the presentation of the claim, the presentation of the proof, and any comments about the proof that followed the presentation. Thus, the percentage of time that instructors spend on *proof presentations* is more than the percentage of time that they spend on just the proofs themselves.

3.8.5 Identifying and Classifying Instructor Questions. When beginning my analysis of interaction patterns, I used the Teaching Dimensions Observation Protocol (CCHER, 2009), which is an observation instrument developed as part of the Culture, Cognition, and Evaluation of STEM Higher Education Reform, an NSF funded project at the University of Wisconsin-Madison. This instrument was designed to analyze a class observation in two-minute slices, and each two-minute chunk was coded along five different categories. One of the dimensions was *Instructor/Student Interactions*, and another dimension was *Cognitive Engagement of the Students*. The Instructor/Student Interactions had the sub-codes: instructor rhetorical question, instructor comprehension question, instructor display question, student response, student novel question, and student comprehension question.

While investigating my data with this instrument, I quickly discovered that the categories that were listed for instructor/student interactions were insufficient for my purposes. First of all, the data that I collected was from the instructor and not the students, so I was not able to do an analysis of the questions posed by the students. Secondly, all of the questions asked by the instructor that required a student response were lumped into the "instructor display questions" category. Thus, I decided to parse the "instructor display question" category using Mehan's (1979) four types of questions: Choice, Product, Process, Meta-Process.

I also decided that the categories that were used for cognitive engagement of the students were not very applicable to the advanced mathematics courses that I was observing, and it was difficult to really gauge the cognitive engagement of the students because I only collected data from the instructor. Thus, I decided to use Tallman & Carlson's (2012) adaptation of Anderson's Revised Bloom's Taxonomy to measure the cognitive engagement of the instructor questions.

While still using my adapted version of the TDOP, I found that measuring the class in 2-minute intervals was still too large of a grain size to really capture the interactions between the instructor and students. Thus, I changed my unit of analysis to be the individual questions posed by the instructor. At this point, I abandoned the TDOP entirely, and focused on analyzing questions along two dimensions: Expected Response Type and Cognitive Engagement.

Because the literature mentioned that instructors often link questions using a *funneling* pattern (Wood, 1994), I also decided to keep track of which questions were linked, and whether or not there was a student response. This way, I can track not only the frequency of different types of questions that occur, but I can also do a separate analysis on the questions that were actually answered by the students.

All of the questions that were posed by the instructor in the observation data were transcribed. Like Van Zee and Minstrell (1997), this included all utterances that had the grammatical form of a question and some alternative forms. For example, when presenting a proof, one particular instructor would often begin a statement and then pause and look at the students. In these situations the instructor appeared to be expecting the students to complete the statement, and so this situation was coded as a question. I also included questions that did not involve the explicit seeking of information, such as rhetorical questions or questions to check student understanding.

For each question, I noted whether or not there was an audible response from the students. Since the video was directed at the instructor and the board and not at the students, I could not identify or code any nonverbal responses to the questions.

Then each instructor question was coded along two dimensions, the Expected Response Dimension (adapted from CCHER, 2009; Mehan, 1979) and the Cognitive Process Dimension (Anderson & Krathwohl, 2001). First, I coded two videos from each instructor according to the descriptions of the categories in the works of CCHER (2009), Mehan (1979) and Anderson & Krathwohl (2001). Then, I looked at each category and noted the types of questions that occurred in each category. Because all of the examples of questions in the literature on Bloom's taxonomy were of lower-level mathematics content, it was necessary for me to solidify the types of questions that occur in each category in the context of proof-based mathematics. I then created a document that summarized the types of questions that occurred in each category of Anderson's Revised Bloom's Taxonomy, which can be found in Appendix F. I used this document to code the questions in the remaining videos. A description of each category is given below:

3.8.5 a. The Expected Response Dimension. The Expected Response dimension records the type of response that the instructor seems to expect from the students. The initial categories were found in the Teaching Dimensions Observation Protocol (CCHER, 2009), which used *rhetorical, comprehension,* and *display.* Rhetorical questions are questions that the instructor did not intend for the students to answer. Comprehension questions could often be answered by the students with non-verbal responses such as making eye contact, facial expressions, or nodding their heads.

So, display questions were used by the instructor when a response was expected. This category contained most of the questions, and so I saw the need to parse this category so that the instrument would describe the questioning of the instructors in more detail. Thus, the *display* category was parsed by using Mehan's (1979) four types of questions: *choice, product, process, and meta-process*. The types of questions that were used in the Expected Response Type dimension are described in more detail in the following sections.

Rhetorical Questions. The question was coded as rhetorical if the instructor either answered it himself immediately after posing it, or if he did not wait for the students to respond to the question (and it was not linked to another question).

Comprehension Questions. Questions such as "Does that make sense?" "Ok?" and "Any questions?" were coded as comprehension questions. These often had no verbal response from the students, but the instructor would look at the students and gauge their understanding based on non-verbal cues.

Choice Questions. Yes/No questions or questions where the instructor asked students to choose between two or more options were coded as choice questions.

Product Questions. Questions that require a factual response or short answer were coded as product questions.

Process Questions. When the question required more than just a short answer, or required the student to make interpretations, describe computations, or explain the mathematics content, they were coded as process questions.

Meta-Process Questions. Questions requiring the student to reflect on their thinking or formulate the grounds for their reasoning were coded as meta-process questions.

Questions requesting a student to expound upon their response, such as Minstrell's (1997) *reflective tosses,* are included as meta-process questions.

It became apparent that the most important parsing of this dimension was into rhetorical, factual (choice or product) and

3.8.5 b. The Cognitive Process Dimension. Each question was also analyzed to determine its cognitive level based on Anderson's Revised Bloom's Taxonomy (Anderson & Krathwohl, 2001). Because the Bloom's Taxonomy codes are not specifically tailored to mathematics content, one slight revision to the categories was made. The "remember" category was parsed into "remember" and "apply a procedure" as in Tallman & Carlson's framework (2012). This was done because often in mathematics students can apply a procedure without understanding the mathematics, and therefore the cognitive action of applying that procedure is more at the level of "remembering" than it is "applying." The following are brief descriptions of each level of the taxonomy as found in Anderson & Krathwohl (2001) and Tallman & Carlson (2012). For this paper, the cognitive level will be parsed into lower order (remember, apply procedure, understand) and higher order (apply understanding, analyze, evaluate, create).

Remember. Students are prompted to retrieve knowledge from long term memory.

Apply a Procedure. Students must recognize and apply a procedure.

Understand. Students are prompted to make interpretations, provide explanations, make comparisons, or make inferences that require understanding of a mathematics concept.

Apply Understanding. Students must recognize when to use a concept when responding to a question or when working a problem.

Analyze. Students are prompted to break material into its constituent parts and determine how the parts relate to one another and to an overall structure or purpose.

Evaluate. Students are prompted to make judgments based on criteria and standards.

Create. Students are prompted to reorganize elements into a new pattern or structure.

Figure 5: *Sample Coding Spreadsheet for Question Analysis*

Obs C.9	Question	Linked to Another Question	Student Response	Interaction Required	Bloom's Level	Content
0:00	So, questions on assignment 4. Does anybody have anything that they would like me to talk about this morning?	N	Y	CQ		HW
1:13	Whenever you have sets S union T subset of U, how is your proof going to go? Just give me a generic structure for the proof. Not even referring yet to what's specifically in problem 8.22.	N	Y	PRQ	A	PF
2:18	Now, in this case, when you're doing proofs about the power set, the one thing that looks a little odd is that you're not using a little letter *x* for the elements, the elements of this.. this set S is the power set of what?	N	Y	PDQ	U	PF

The transcribed questions were entered into Microsoft Excel spreadsheets. The first column contained the time in the observation when the question occurred, the second column contained the text of the transcribed question, the third column recorded whether or not the question was linked to another question, the fourth column recorded whether or not the question was answered by a student, the fifth column contained the code for Expected Response Type, the sixth column contained the code for the Cognitive Process, and the seventh column recorded the type of mathematics content that was being presented when the question was posed. For example, a section of a spreadsheet is presented in Figure 5.

Since the instructors are teaching in different content areas and the content areas or the makeup of the individual classes may have an effect on the instructors' use of questions, the results will be presented as separate but interrelated case studies. For each instructor, tables will be presented that show the types of questions asked along the two dimensions. Also, the question rate, response rate, percentage of higher-order questions, and the percentage of student-answered questions that were higher order will be computed for each instructor.

The rate at which the instructors asked questions was computed by dividing the total number of questions asked by the total amount of time observed. The response rate was computed by dividing the number of questions that received student responses by the total number of questions. The percentage of higher order questions was computed by summing up the total number of questions asked that fell into the top four Bloom's levels and dividing that number by the total number of questions. The percentage of student-answered questions that were higher order was computed by summing up the student-answered questions that fell into the top four Bloom's levels and dividing by the total number of student-answered questions. A similar computation was made to determine the percentages of instructor questions that required higher-order response types: Process and Meta-Process.

3.9 Ethical Considerations

In qualitative research, as in any research with human subjects, the protection of the subjects is a primary concern. This study was conducted under the supervision of the institution's IRB. The IRB status was expedited, because it was anticipated that no serious ethical threats were posed to any of the participants. The safeguards that were employed to ensure the protection of the rights of the participants included informed consent, confidentiality, and security of the data.

All subjects were volunteers, and written informed consent documents were obtained from all subjects. Consent documents can be found in Appendix G-H. These documents informed the participants of how the data would be used and stored.

Confidentiality means that "the researcher can identify a person's given responses, but promises not to do so publicly" (Babbie, 2007). Although background information such as the participants' specific area of research, gender, and ethnicity may give insight into their teaching philosophy, these specifics have been intentionally excluded to protect the identities of the participants. The participants have also been given pseudonyms in all reports of this research.

Cautionary measures were used to ensure the safety of all research related records and data. The data have been kept in my office and in a secure Dropbox file, and only my advisor and I have access to the data.

3.10 Issues of Trustworthiness

In quantitative research, the issues of validity (meaning that your measure accurately reflects the concept it was intended to measure) and reliability (a quality of a measurement method that suggests that the same data would be collected each time in repeated observations of the same phenomenon) are central to the quality of research (Babbie, 2007). Qualitative inquiries have similar issues of trustworthiness, although different terminology and assessment measures are used. Guba & Lincoln (1998) use the terms credibility and dependability for the standards of good and convincing research. Credibility refers to whether the participants' perceptions are accurately represented by the researcher's portrayal of them. Dependability refers to whether or not one can track the processes and procedures used to collect and interpret the data (Bloomberg & Volpe, 2012).

The criterion of credibility involves both methodological and interpretive credibility (Mason, 1996). To enhance the methodological credibility of the study, triangulation of data collection methods was used. The data gathered from the initial interviews, observations, and follow-up interviews yielded a fuller and richer understanding of the participants' teaching practices. Interpretive credibility was addressed in various ways. First, my perspectives and assumptions are clarified upfront (see Section 1.3). Secondly, to ensure that my views did not influence the portrayal of the participants' perspectives, I made use of a member-check. This allowed the participants to speak for themselves after reviewing my data analysis. Thirdly, I used "peer debriefing" by discussing my findings and data with colleagues; both fellow researchers and mathematics instructors. Data analysis has taken place over the course of three years, and during that time my methods and results have been presented at professional conferences, exposed to blind reviewers, and discussed with peers in a weekly professional seminar on proof. These constant and ongoing discussions have provided accountability and given me opportunities to discuss problematic issues in coding and analysis with colleagues who are familiar with my work.

The dependability of a qualitative study refers to whether the findings are consistent and dependable with the data collected (Lincoln & Guba, 1985). The dependability of this study was addressed by providing detailed and thorough explanations of how the data were collected and analyzed. In addition, I have kept all records of previous reports on the data and data analysis that show a record of the evolution of my thinking and document the rationale for the choices made during the research process, therefore leaving an "audit trail" (Linclon & Guba, 1985).

3.11 Limitations and Delimitations

Delimitations are boundaries that were set in place to narrow the scope of the study. Delimitations of this study include that this study focuses on the teaching practices of university instructors who are teaching proof-based courses using traditional methods. This study took place during one particular school year at a Midwestern research institution. Therefore the findings of this study cannot be directly generalizable, but aspects of the phenomenon under study may be transferrable to a similar context (Linclon & Guba, 1985).

This study contains certain limitations, some of which are related to the common critiques of qualitative research, and others that are particular to this study. Qualitative research in general is limited by researcher subjectivity, because the data analysis ultimately is determined by the thinking and choices made by the researcher. Recognizing this limitation, I attempted to be as transparent as possible throughout the research process, and discussed decisions with colleagues along the way.

A limitation of this particular study is that I only observed four instructors, and they were each teaching one course. It is therefore difficult to distinguish whether specific teaching practices are employed because of the content that is being presented, or if the practices observed are a reflection of the instructor's personal teaching style.

3.12 Chapter Summary

In summary, this chapter provided a detailed description of this study's research methodology. Qualitative case study methodology was employed to investigate the teaching practices of instructors who are teaching undergraduate proof-based mathematics courses using lecture methods. The participants in the study were four mathematics faculty members at a large Midwestern research university who were teaching proof-based courses. The participants self-identified as traditional instructors using either lecture or "modified" lecture methods. The data collected were interviews with all four participants, 6-7 video-taped observations of their teaching spread throughout the semester, and a follow-up interview to review and comment on the data analysis.

A review of the literature was used to situate the study and provide a framework for the data analysis. Four embedded units of analysis were used to address particular research questions. An analysis of each individual proof was used to address the first research question. An analysis of the time allotted for each entire class period was used to address the second research question. An analysis of each example used in proof presentations was used to address the third and fourth research questions. Additionally, an analysis of each instructor question was used to address the fifth and sixth research question. Through a comparison with the literature, interpretations and conclusions were drawn, and recommendations were made for both educational practice and further research. This study is intended to provide a basis to inform further research on teacher practice and to be useful in professional development of collegiate teachers.

CHAPTER IV

FINDINGS

The purpose of this multi-case study is to examine the teaching practices of four mathematics instructors who were teaching different proof-based mathematics courses using lecture methods. This chapter will present the findings of the data analysis used to examine the six research questions.

The first finding, presented in Section 4.1, is the identification of four proof presentation strategies from the interview data: Outline, Examples, Logical Structure, and Context. I coded each proof presentation to determine which, if any, of the strategies were used by the instructors. I found that the instructors used an identified pedagogical strategy in almost all of their proofs. Three of the four instructors used examples in half of their proof presentations, while the fourth rarely used examples in his presentations. I also determined the level of expected engagement for each proof presentation, finding that three of the four instructors expected students to contribute to 95% of their proof presentations.

The second finding, presented in Section 4.2, is that the instructors spent between 35% and 70% of their class time on proof presentations. The largest proportions of class time across all cases were spent on proof and examples. The instructors engaged students in interactive lecture to varying degrees, from 26% to 62% of their class time.

The third finding of this study, presented in Section 4.3, is the development of a descriptive framework for the instructors' example usage during their in-class proof presentations. The different types of examples are described and illustrated using excerpts from the observation and interview data.

The fourth finding of this study, presented in Section 4.4, is that instructors who are teaching proof-based mathematics courses using lecture interact with their students frequently, asking approximately one question per minute on average. Students are responding to between 35% and 52% of their instructor's questions. The instructors asked between 30% and 42% higher-order questions, and that between 29% and 57% of questions answered by students were higher-order. Thus, these instructors were asking higher-order questions, and students were responding to them at the same rate.

4.1 Pedagogical Moves in Proof Presentations

The results in this section address the first research question, "What pedagogical moves do instructors plan to use to help students understand their proof presentations, and how often do they use these moves?"

The analysis of the interview data led to the identification of four *proof presentation strategies,* and the construction of levels of *expected engagement*. The methods used to extract these categories from the interview data was described in Section 3.6.1.Then, each proof presentation in the observation data was coded according to the *proof presentation strategies* that occurred in that presentation and the level of *expected engagement* that the instructor used.

4.1.1. Proof Presentation Strategies. The four proof presentation strategies that were identified in the interview data are presented in Table 6, and the discussion that follows goes into a more detailed description of the strategies.

Table 5: *Proof Presentation Strategies*

	Description
Outline	Discussed the ideas of the proof before writing out the proof
Examples	Used examples to motivate the theorem/proof or to support students' understanding of the theorem/proof
Logical Structure	Discussed the logical structure of the statement or the proof, or made specific references to logic during the proof presentation
Context	Instructor pointed out the historical significance of the proof, or placed the proof in context of the mathematical content area.

When asked what they do to help students understand a proof presentation, several of the participants described a process of outlining or talking informally about the key ideas of the proof *before* they began to write the proof on the board. For example, Dr. N said that before he begins to present a proof in class, he tries to "*read* the… theorem, and I say, 'What's important here? What do you think of when you read this?' And I ask for ideas, what relevant theorems might be true…" This strategy was labeled *outline.* It sometimes took a form similar to Leron's (1985) top-down approach in which he begins a proof by giving an overview, but other times it was just an informal discussion about how to begin the proof. When coding, I looked for instances where the faculty member discussed with students about the ideas of the proof before starting the proof, and I also looked for when the professor was very clear about what was just informal talk and what was a formal proof.

The mathematicians also mentioned drawing pictures or using specific *examples* to help students understand the meanings of terms or statements, to motivate the proof, or to support students' understanding of the theorem or proof. In the observation data, I was looking for proofs in which the professor presented algebraic, numerical, or pictorial examples to explain the mathematical statements or the proof strategy.

Though the mathematicians who participated had differing views about whether the logic of proof writing should be taught along with content or separately, they all mentioned that they spend time talking about *logical structure*. When Dr. A begins a proof presentation, he says, "I make the statement on the board, and I point out if it's an if-then statement, or if it's a, both, if-and-only-if statement, and I talk about that, what we have to prove then." When analyzing the data, proofs in which the professor used the *outline* strategy often also emphasized *logical structure*, but not always. Indicators of *logical structure* included pointing out when hypotheses are used, explicitly discussing the structure of the statement to be proved, or summarizing the logic of the proof after the proof was completed.

Table 6: Percentages of proofs using each presentation strategy

	Dr. A	Dr. C	Dr. G	Dr. N
Total Number of Proofs in Observation Data	12	22	22	8
Outline	42%	68.2%	54%	63%
Examples	67%	50%	4%	50%
Logical Structure	25%	54.5%	36%	50%
Context	17%	13.6%	27%	38%
None of these	8.3%	0%	13.6%	0%

The final strategy that emerged from the interview data was *context.* This code was used in instances when the professor placed the ideas of the proof in historical context, pointed out standard arguments as they appeared in presentations, or highlighted the key ideas and how they fit within a larger context of the mathematical content area. For example, when describing how he presented the proof that trisecting an angle is impossible, Dr. G said that "…the whole idea that you can prove that it's impossible, no matter what you do… that's kind of a big thing to understand right there." When presenting the proof, he emphasized the standard arguments required to show that a construction is impossible.

Each proof was analyzed and coded with the presentation strategies that occurred in the proof. There were only four of the 64 proofs that used none of the identified strategies and there were some that used all four.

4.1.2. Expected engagement. By *expected engagement*, I mean the extent to which the professor seemed to expect students to contribute to the proof presentation. The individual proofs were the unit of analysis, and a code (1-5) was assigned to each proof representing different levels of expected engagement. Proofs coded 1 or 2 represented instances where the instructor did not seem to expect the students to be actively contributing to the proof presentation. The code 3 was assigned to proofs in which the instructor seemed to expect students to contribute factual information, and 4 or 5 was assigned to proofs in which the professor expected students to contribute both factual information and key ideas for the proof presentation. To determine the level of expected engagement, I looked for both the types of questions that the instructor posed as well as the amount of time that he waited for a response. For example, if the instructor posed a question and immediately answered his own question, I assumed that he did not expect the students to respond. The coding scheme for expected engagement is summarized in Table 8.

Table 7: Levels of expected engagement

	Description
1	Professor does not request active contribution during the proof presentation, and does not appear to engage students.
2	Professor does not request active contribution during the proof presentation, but seems to adapt the presentation based on monitoring, non-verbal communication with students, or feedback from closed-ended questions.
3	Professor expects students to contribute some factual information during the proof presentation.
4	Professor expects students to contribute some factual information and some key ideas of the proof during proof presentation
5	Professor expects students to generate the majority of the ideas for the proof during the proof presentation.

Of the 42 proofs that were analyzed in this phase of analysis, 16.6% of them were coded 1 or 2, 50% of them were coded with a 3, and 33.3% were coded with a 4 or 5. So, this means that in well over half of the proof presentations, the faculty members expected the students to actively contribute to the presentation, whether by providing some factual information or actually helping to contribute key ideas for the proof. The data for each professor is shown in Table 8. Since this was a study of faculty members and not students, there were no data collected about the actual number of students who participated in class, or the other ways in which students were engaging.

Table 8: Percentages of proofs for each level of expected engagement

	Dr. A	Dr. C	Dr. G	Dr. N
# Proofs in Obs. Data	12	22	22	8
Coded 1	8.3%	4.5%	0%	0%
Coded 2	41.7%	0%	4.5%	0%
Coded 3	41.7%	13.6%	54.5%	50%
Coded 4	8.3%	63.6%	36.4%	37.5%
Coded 5	0%	13.6%	4.5%	12.5%

4.1.2 a Illustration of Level 1-2 from the Data. The next few sections will give examples of different instances of proof presentations from the observation data and how they were coded for *expected engagement*. Because the video camera was directed at the instructor, any comments from the students are simply labeled 'Student.' It is not necessarily the case that the same student spoke each time. Also, single quotes are used to identify times when the instructor was writing on the chalk board while talking aloud.

The expected engagement code 1 or 2 represents proof presentations where the faculty member did not appear to expect the students to actively contribute to the proof presentation. This does not imply that there was no interaction at all, or that the students were not engaged in other ways such as note-taking or non-verbal communication. In most of these instances, the instructor used monitoring to adapt his presentation. For example, he would ask if the students understand, watch their reactions, and modify the presentation if he deemed necessary. One such example is taken from Dr. A's class, where he presented a proof that the composition of homomorphisms is a homomorphism.

Dr. A: What does that mean? '*G*, *H*, um, *P* groups. $\theta : G \to H$,' and uh, let's see… theta... let's say psi, '$\psi : H \to P$, homomorphisms, then $\psi \circ \theta$ is a homomorphism.' So, we know that composition is an operation on mappings. So, when I say theta and psi are homomorphisms, first of all, they are mappings of the underlying point-sets. So these are mappings. Hence, their composition makes sense. So, their composition is a mapping. So, the question here is… we know this is a mapping *(circles $\psi \circ \theta$).* Is it a homomorphism?

Dr. A: So, 'Proof:' What do you need to do to prove this? Um, well, we have to show that multiplication in *G*, if you take multiplication and take it under the composition, it is the multiplication of the factors in *P*, uh, of the images in *P*. So, we 'Let $g_1, g_2 \in G$,' and we 'Consider $\psi \circ \theta(g_1 g_2)$' Ok? What is that equal to? Well, first off, it is equal to '$\psi(\theta(g_1 g_2))$' that is… that's what the definition is of a composition. Ok? Theta, though, is a homomorphism. So this is '$\psi(\theta(g_1)\theta(g_2))$' Now, all these guys have big symbols here… lots of symbols. This is a single element in *H*, *(circles $\theta(g_1)$ with his finger)* this is a single element in *H (circles $\theta(g_2)$ with his finger)*. Now, would it be better if I put those stars and diamonds and dots in there for everybody, or are you ok with this? Ok? And so we have this. But this now, and we totally forget *G* for a minute, this is an h_1 *(circles $\theta(g_1)$ with his finger)*, this is an h_2 *(circles $\theta(g_2)$ with his finger)*, these are just elements in H. The image of the product is the product of the image, because psi is a homomorphism.

Dr. A: So, this is, this uh, theta a homomorphism, and now we do this as '$(\psi\theta(g_1))(\psi\theta(g_2))$' and this is because psi is a homomorphism. But now, again, we go back to the definition, this is '$((\psi \circ \theta)(g_1))((\psi \circ \theta)(g_2))$' and so, if you take this guy on the product, you end up with here, the product of the images. So, here, the image of the product is the product of the images. So, it is *(writes '$\psi \circ \theta$ is a homomorphism')*. Ok?

So, Dr. A did ask some questions, however, he did not wait for a reply from the students. The questions posed were used to model his mathematical thinking, without the expectation that the students would actively contribute to the proof presentation.

4.1.2 b. Illustration of Level 3 from the Data. Proofs coded 3 represent an instance where the professor did expect the students to actively engage in the presentation, but only to contribute factual information. There did not appear to be an expectation that the students would create any of the big ideas for the structure of the proof. An example of this level of expected engagement comes from Dr. G's presentation about the summit angles of a half-rectangle.

Dr. G: A half rectangle is a convex quadrilateral with two right angles at the base. And, what I want to prove is that in a half-rectangle, I want to look at these two angles… this is going to look like a test problem before we're through… If I look at those two angles, I claim that the greater side is across from the greater angle. So, 'In a half-rectangle, the greater side is across from the greater summit angle.' Those are the summit angles. So, what I want to do, what's the theorem? The theorem is angle 1, well, let's do it this way. 'Suppose BC is bigger than AD,' and I want to prove that angle 1 is, uh, bigger than angle 2. Suppose this is true.

Ok, Well, let's see, let's 'choose a point *E* so that *B*E*C* and *BE* congruent to *AD*.' So, pick a point so that those two things are congruent. What kind of an animal is this? *DABE*? It has a name.

Student: Saccheri rectangle?

Dr. G: It's not a Saccheri.. don't say rectangle.

Student: Saccheri Quadrilateral?

Dr. G: Saccheri quadrilateral. Ok? 'So, *ABED* is a Saccheri quadrilateral.' What do I know about summit angles of Saccheri quadrilaterals?

Student: They're congruent?

Dr. G: Yeah. We proved that on a test, I think, or on a previous test, or a homework. 'Angle 3 congruent to angle 4.' Oh, now it's gonna look like the test we just did, because angle 1, degree measure, is bigger than angle 3, which is the same in degree measure as angle 4, and this angle is… how does that compare with angle 2? How does angle 4 compare with angle 2?

Student: It's exterior.

Dr. G: It's exterior, isn't it? It's exterior to this triangle. *(re-draws part of the picture to emphasize that angle 4 is exterior)* Angle 4 is exterior to the triangle, so it's, um, bigger than angle 2. Ok.

In this proof, Dr. G expected the students to respond to most of his questions. When he is outlining the structure of the proof, he seems like he is going to ask the students how to begin the proof, but he doesn't wait for their response. He then lays out the beginning of the proof, and begins to ask the students factual questions along the way. Because he asks the same question multiple times and waits for a student response, Dr. G is communicating his expectation that students will respond. When Dr. G hears the desired response, he takes that information and immediately proceeds to the next step in the proof. The students do dialogue with Dr. G, and they contribute to the proof, however, Dr. G is clearly in control of the construction of the proof.

4.1.2 c. Illustrations of Level 4-5 from the Data. Since the expected engagement level of 4 or 5 may differ slightly from the way lecture style proof presentations have been conceptualized in the past, I included three examples, one from each faculty member. This excerpt from Dr. N's class was coded 4, because the students give the key ideas for how to construct a proof of uniqueness.

Dr. N: Um, all right, well, unfortunately after all that success, you still have uniqueness to prove. And, um, you'd like to think that maybe uniqueness was part of what you were doing, but it's really hard to justify that. So, the best way to prove uniqueness is to… I mean, what's the standard way that you prove that there's only one formula? There's sort of a standard strategy that you do. When you've done this already… we did this for the division algorithm. You remember what I said? At the end there's a uniqueness part of the division algorithm, and… how did we prove that uniqueness was actually true? Yeah, so let me write up here, 'proof of uniqueness' And there's always sort of a standard way. So, you've proved that there is a formula, and you want to show that that formula is unique, so what's the strategy for doing it?

Student: Assume there's two, and show it isn't possible.

Dr. N: Exactly! All right, that's all I want you to say is just assume that there's two. Assume there's two and then show that they are actually the same. So, 'Suppose there are two such formulas for the same n.'

Student: So are we just proving contradiction?

Dr. N: Yeah, this would be an example of proof by contradiction. What's the opposite of uniqueness? The opposite of uniqueness is that there's more than one. So, I'm going to assume that there are… this is kind of a specialized version of proof by contradiction. It always works for uniqueness. You always assume there are two, and then you try to prove that they are actually the same. So, suppose there are two such formulas, um, so the first one we got was that over there, let me just summarize it, '$n = a_k b^k + ... + a_1 b + a_0$' and for the second one, we have to use a different letter for the coefficients, b is the same, but I'm going to change the a to a c, and the k to an l, because it's possible that if I did it in some other crazy way, maybe I got more, uh, more digits or something. '$n = c_l b^l + ... + c_1 b + c_0$' And now the strategy is to prove that something is wrong with this. Suppose there are two different ones. Suppose they are actually different. We have two different formulas. Now, how would I know that these are really different? What specifically would.. what would happen that would tell me that these are really different? Now, you sort of have to hone in on what's the difference between the two formulas. You know, what's the specific difference. Now, if they are really different, something has to happen.

Student: *(mumbles)*

Dr. N: Now, you're saying *a*, but there are a lot of *a*'s, you have to be specific.

Student: a_j is not equal to c_i, or c_j.

Dr. N: No, that's not correct, or, what did you say at the end?

Student: c_j

Dr. N: c_j, ok, right, ok? So, it doesn't matter that this digit is not equal to that one, it's the corresponding ones. So, what Nathan is saying here is that '$a_j \neq c_j$ for some *j*.' Not for all of them, but for one of them, they have to be different. Ok, now this is really hard to motivate until you've seen this kind of proof, but, one of them, I know that in one of them they are different, but I want to pick, I want to sort of be real specific about *j*. I want to say, 'take *j* to be' Now, *j* could be zero, it could be 1, it could be anything in those ranges there, but I want to work specifically with *j* having a real specific property here. You know, until you get into this, it's really hard to motivate. Do you know what I want *j* to be? I want the two digits to be different, but I want to know something more about *j*. *(mumbles)* You have to learn this…

Student: The smallest one?

Dr. N: The smallest one, there we go. You got it. You know, when you're trying to hone in, you always want to go to an extreme case…

Dr. N continues to ask the students questions about how to structure the proof, and asks the same question in different ways until the students give an answer. Although the students are contributing significant ideas to the proof, this presentation is still teacher-centered. When the students give the expected answer, Dr. N spends a little bit of time explaining why the answer that the student gave is correct, and then carries on with the proof.

One common approach in Dr. G's class was to have the students identify the contradiction at the end of an indirect proof. The students didn't structure the proof, but they did contribute a key idea to the proof by identifying the contradiction. Here is an example of this type of engagement.

Dr. G: So, I was going to say, we did this proof before, but I was just going to remind you how it goes. How do I show that if Euclid 5 is true, then the converse to alternate interior angles is true? This is what I want to prove *(underlines 'converse to AIA' writes 'prove this' underneath)* But, since I'm assuming Euclid 5 is true, I get to use this. *(underlines 'Euclid 5' and writes 'Use this in my proof' underneath.)* So, let's see, I want to show the converse to alternate interior angles is true, so I say, 'suppose that k is parallel to *l*,' and I need to 'prove that angle 1 is congruent to angle 2.' So how do I do that? Well, suppose it's not. 'Suppose angle 1 is not congruent to angle 2.' Well, then what are we going to do, is use my protractor axiom to reproduce angle 1 right here. I'm running out of letters. *n*. So, let's choose *n* through *Q* so that angle 3 is congruent to angle 1.

Student: So, you're saying that angle 1 is bigger than angle 2?

Dr. G: This picture makes it look greater, It would, it's gonna be the… it's gonna be the same argument if it were smaller. So, I'm just assuming they're different. Ok, I claim that we're going to get a contradiction here already. I claim I'm done. Where is my contradiction? Well, what do I… let me… what do I know about *n* and *k*?

Student: They're lines.

Dr. G: They're lines, I do know that. That's correct, but I would hope to know more.

Student: Parallel.

Dr. G: Why?

Student: Alternate interior.

Dr. G: Alternate interior angles? Ok, so '*k* is parallel to *n* by alternate interior angles.' It's not the converse, but the theorem itself.

Student: And then Euclid 5, there's a contradiction because it says there's a unique parallel.

Dr. G: Right. I get to use Euclid 5 in my proof, and now I've just built two lines parallel to *k* through the point *Q*. Now, 'both *l* and *n* are parallel to *k* and through *Q*.' That's a contradiction. So, that didn't happen. Those angles were the same.

Dr. C presents a proof that the intersection of any collection of open sets is open. This proof was coded as a 4 because Dr. C asks a combination of factual questions and leading questions about the main ideas in the proof.

Dr. C: The intersection of any collection of open sets is open. So, the very first thing I need to do is set up notation. I take a vaguely worded statement like that with words only, and I need notation to start proving anything. So, tell me how the first couple of sentences here are. "Let.." What notation am I introducing?

Student: Let *S* be an open set?

Dr. C: Well, the phrasing here is that I have a collection of open sets. So, do I need just an *S*? What else do I need?

Student: A *T*?

Dr. C: *(smiles)*

Student: Do you just need, like, a subscript?

Dr. C: I need a subscript. Let's do a subscript. Because we need to model the kind of situation where we had an infinite collection of sets. So, let's do a subscript. […] Now, we're going to let O be the union of the sets O-alpha, where alpha runs through my index *A*. *(writes* $O = \bigcup_{\alpha \in A} O_\alpha$ *)* So, what we have to do here to use this fact is show that every point in *O* is an interior point, then I need to comment on that. So, let's "let *x* be in *O*." Now, how am I going to prove that *x* is an interior point? *(pause)* Well, an interior point is a point where there's a neighborhood around it that stays inside your point. But, if I have a collection like O_α sets, well, there's various sets involved, but if I pick a specific number x, if it's in the union of all of the sets, what can you tell me? What's the first fact you can say for sure about x? If it's..

Student: It's in some O_l.

Dr. C: Correct. So, if it's in the union, it's in one of the sets. *(writes, "Then $x \in O_\alpha$ for some α ")* So, if it's in the collection, it's in one of the sets. Now, what do I know about that set being open? What does the set O_α being in the collection… oh, the set O_α being open, what does that tell me? How am I going to prove whether x is an interior point of big O or not? Is this set known to be open or closed?

Student: Open.

Dr. C: Since O_α is open, what can I say?

Student: x is in the interior of O_α.

Dr. C: Mmm-hmm. "Since x is in $\text{int}\, O_\alpha$," so what does that tell me about an epsilon? There's an epsilon we can find, now.

Student: There is an epsilon greater than zero such that the epsilon neighborhood around x is contained…

Dr. C: That is correct. "So, there is some positive epsilon so that that neighborhood N(x, ε) is a subset of O_α" not contained in as an element, but contained in as a subset. Subset sign there, these are both sets. And, so, what does that tell me? If this neighborhood is contained in O_α, what does that say about my big conglomeration *O* that is the union of everything? What's the relationship of this neighborhood with this *O* that is the union of all the sets? Is the neighborhood contained in the union of all the sets?

Student: Yes.

Dr. C: If it's contained in one of them, it's contained in the union of everything, sure! "So, N(x, ε) is contained in *O*" And, is that enough to say that *x* is an interior point of *O*? Am I done? Yes, I'm done. "So, *x* is in the interior of *O*, so *O* is open." So, that's a good representative proof.

In all of these proofs, the instructor is still leading the discussion, but is giving a proof presentation with dialogue that leads the students through the mathematical thinking needed to construct the proof (Fuakwa-Connelly, 2012a).

4.1.3 Summary of Pedagogical Moves in Proof Presentations

Four proof presentation strategies were identified in the interview data: Outline, Examples, Logical Structure, and Context. Each of these strategies was used to varying degrees by the four instructors. It should be noted, however, that very few of the proofs observed (between 0% and 13%) used none of the identified strategies. Thus, the instructors were using the pedagogical strategies that they mentioned to help their students understand the proofs that they presented in class.

Another finding is that three of the instructors used examples in over half of their proofs, while the fourth rarely used examples at all. The ways in which the instructors used examples in their proof presentations will be discussed further in Section 4.3.

The instructors engaged their students in varying ways. Only two of the observed proofs required no engagement of the students. Three of the four participants expected active engagement (levels 3-5) in 95% of their proof presentation, while the fourth expected active engagement in 50% of his proof presentations. Thus, although the instructors all taught using lecture methods and they facilitated all of the proof presentations, they did engage their students. The questions asked by the instructors will be investigated further in Section 4.4.

4.2 Allocation of Class Time

The results presented in this section are in response to the second research question, "How do instructors allocate their class time in traditionally taught proof-based undergraduate courses?" This section presents the amount of class time that instructors spent on *proof presentations*, which were the unit of analysis in Section 4.1. Timelines were created to assess the timing and the amount of time spent on the presentation of different types of content: Definition, Theorem, Proof, Example, and Homework Problems. The timelines show instances when examples and proofs were presented in conjunction, which was mentioned as a proof presentation strategy in Section 4.1, and will be discussed further in Section 4.3. The timelines also show the source of verbalization, whether the instructor or students. An analysis of the timelines was conducted to compute the percentage of time spent in interactive lecture versus straight lecture.

When identifying a *proof presentation* in the data, I included the statement of the theorem or claim, the proof itself, and any examples or comments related to the proof that occurred either before or after the proof. Also, discussions of homework problems that required justification were also considered proof presentations. Thus, the amount of time spent on proof presentations is more than the amount of time spent just on the individual proofs. The percentage of class time spent on proof presentations is summarized in Table 9.

The first result for how instructors allocate their class time was that they spend approximately half of their class time, on average, presenting proofs.

Table 9: *Time Spent on Proof Presentations*

	Algebra	Adv Calc	Geometry	Num Thry
Instructor	Dr. A	Dr. C	Dr. G	Dr. N
# Instruction Days	44	29	44	44
# Observations	7	7	6	6
# Proofs Observed in Data	19	22	22	9
% Time on Proof Presentations	40%	49%	70%	35%

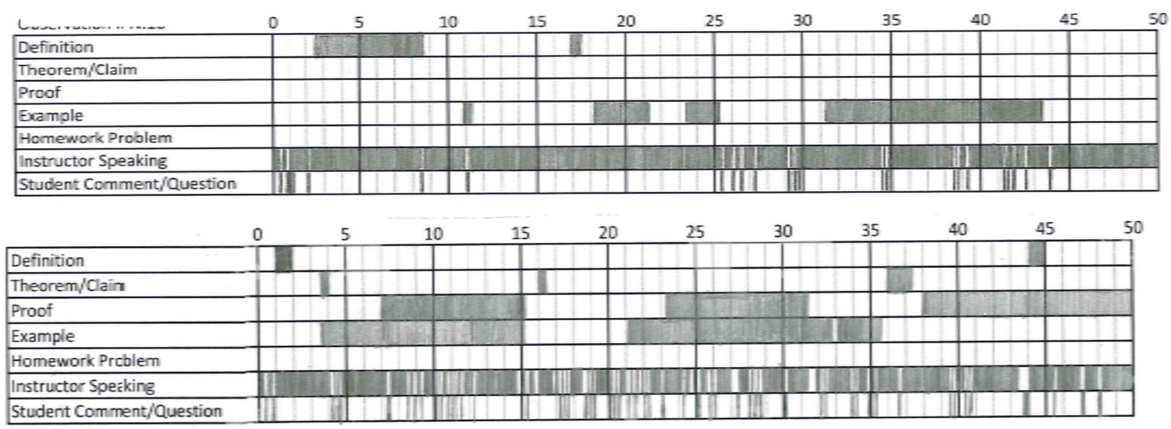
A second result presents the content to which class time was allotted. The four content codes were Definition, Theorem/Claim, Proof, Example, and Homework Problem. The actual timelines for each instructor can be found in Appendix E, and percentage of time spent on each content code is summarized in Table 10.

Table 10: *Allocation of Class Time*

	Dr. A	Dr. C	Dr. G	Dr. N
Definition	7.2%	7.2%	2.8%	3.8%
Theorem/Claim	7.2%	5.0%	13.2%	6.9%
Proof	29.2%	41.0%	54.4%	29.1%
Example	38.9%	21.8%	2.4%	26.8%
Homework Problem	2.5%	30.7%	2.4%	20.3%

Table10 shows that across all participants, the categories of *proof* and *examples* tend to be the highest percentages, with a few exceptions. It should be noted that Dr. G almost never used examples in his proof presentations. Upon further analysis of the data, it was found that he used *generic diagrams* to organize 19 of the 22 proofs that he presented in class. He would draw a picture next to the proof that served as a guide to structure the proof. He would then fill in the picture as the proof progressed. Although the picture may have been concrete to the students, it was not a specific representative of a class, because it was at the same level of generality as the proof. Therefore, the *generic diagrams* were not examples. When asked about this in the follow-up interview, Dr. G said "If I'm teaching geometry, I don't have many specific geometries that [the students] are familiar with to start with. If I'm teaching number theory, well, students have a lot of experience with integers long before we begin talking about congruences." Thus, he is suggesting that the content of Geometry is not as conducive to exemplification as Number Theory.

Figure 6: *Varying degrees of teacher-student interactions in lectures*



Homework problems were discussed in class to varying degrees. It can be noted that Dr. A and Dr. G did not frequently discuss homework problems in the observation data. When asked about this in the follow-up interview, Dr. G was surprised that the data did not capture him working homework problems. He said that he generally assigns homework problems that are due every two weeks, and that the students may not have homework questions each day, but the entire class period before the homework is due is generally spent answering questions about homework problems. Dr. A said that his philosophy was that the students should do homework on their own, and so he did not ask about the homework in class. Dr. N and Dr. C usually began each class period answering questions from the homework, and then proceeded into the content for the day.

The timelines showed that the instructors were interacting with students to varying degrees, as demonstrated by Figure 4. Each one-minute slice of the timelines was analyzed to determine if there was any back-and-forth between the instructor and students. One-minute slices in which only the instructor was speaking were coded “Straight Lecture,” while one-minute slices in which both the instructor and students were speaking were coded “Interactive Lecture.”

Table 11: *Proportion of 1-Minute Slices Spent on Interactive vs. Straight Lecture*

	Dr. A	Dr. C	Dr. G	Dr. N
Interactive Lecture	26.1%	61.5%	36.4%	45.2%
Straight Lecture	73.9%	38.5%	63.6%	54.8%

The number of one-minute segments in which there was back-and-forth interaction between the instructor and students were totaled for each instructor and the percentage of one-minute segments was calculated. This is presented in Table 11. The percentages of class time spent on interactive lecture range from 26% to 61.5%. This result shows that lecture in these classes was not always monolog, but frequently contained dialogue between the instructor and students.

4.3 Example Usage in Proof Presentations

The results presented in this section are in response to the third and fourth research questions, “What types of examples do instructors use in presentations of theorems and proofs in an upper-division proof-based mathematics course, and when do these examples occur chronologically in relation to the presentation of theorems or proofs?" and "What are the instructors' pedagogical uses for the different types of examples when presenting the statement of a theorem or a proof?" The results that are presented are an analysis of the interview data and observation data, using results from the literature to guide the analysis, as described in Sections 3.6.3 c and 3.6.3 d.

This section will present a descriptive framework that organizes the uses of examples in relation to the proof presentations. Although all of the participants taught in a lecture style, there was a rich collection of examples in the observation data. This suggests that proof presentations in these lectures do contain elements of empirical reasoning and are not strictly rigorous and formal. The framework will be presented first, followed by descriptions of the different elements of the framework. Included in these descriptions will be evidence from both of the interviews and the observation data illustrating the different types of examples that occurred and their pedagogical purposes.

4.3.1 Framework for example usage in proof presentations. Upon analysis of the interview data and the proofs that were captured in the observations, four primary purposes for using examples surfaced: to motivate the claim or proof, to persuade students of the plausibility of the claim, to generate proof insight, or to support students' understanding of the claim, proof, or the underlying mathematics content. In this paper, examples will be said to *motivate* a particular mathematical concept if they occur before the concept is presented, and *support* the concept if the example is given during or after the concept is presented. The four uses of examples in proof presentations are presented in Table 12, along with illustrative excerpts from the interview data.

The timing of the examples in relation to the statement of the claim or the presentation of the

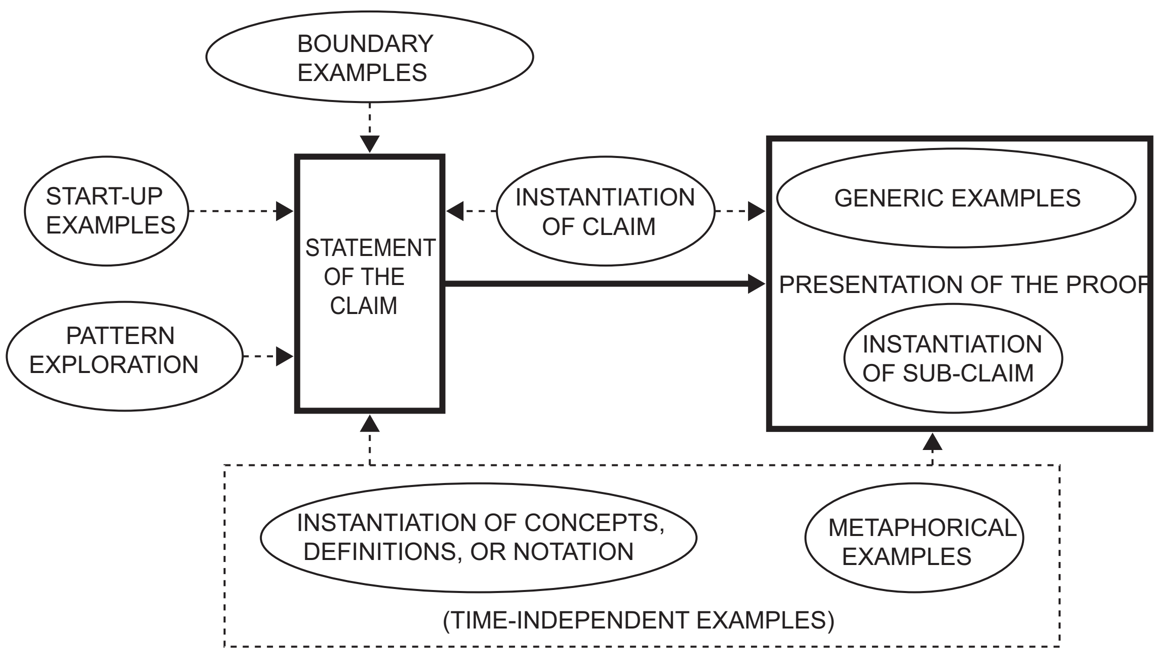
Table 12:
Pedagogical Uses of Examples in Proof Presentations

Use	Illustrative Excerpts from Interview Data
Motivate basic intuitions or claims	"I try to prepare their minds for proof, and usually that happens by example and computation."
Provide evidence for the plausibility of the claim or of a sub-claim	"It [the example] served as a plausibility argument."
Support students' understanding of the claim, proof, or underlying mathematical content	"And, when I read a complicated theoretical statement, it sounds like Chinese the first time through. And, I have to play with that statement in my brain to make sense of it. Well, how do I make sense of it? I start looking at examples. So, that's why I do that a lot [in class]."
Generate proof insight	"If you're trying to prove something, it can be very hard, and a way to start is to simplify the statement. In other words, uh, take out some of the variables, or specialize some of the variables and see if you can prove just one case. And, so I do talk about *this* to try to get an idea about how the general proof might go."

proof often sheds light on the pedagogical use that the example serves, along with the mathematical content and the context in the teacher-to-student dialogue. Although three of the four of the participants did not clearly articulate in the interviews that they use examples to persuade students of the plausibility of a mathematical claim, the fact that they often used examples in lieu of a formal mathematical proof showed that they were using examples as a form of justification.

The diagram in Figure 7 illustrates the types of examples were used and how they interact with the statement of the claim and the presentation of the proof. Flow from left to right is chronological. Start-Up Examples and Pattern Exploration Examples occur before the statement of the claim and serve to motivate the statement of the claim. Therefore the arrows from the ovals containing *Start-up Examples* and *Pattern Exploration* are pointing to the rectangle labeled *Statement of the Claim.* Boundary Examples may occur before or after the statement of the claim and serve to support students' understanding of the claim by highlighting the necessity of the hypotheses in the claim. Thus, the oval containing *Boundary Examples* is elongated on both sides of the rectangle labeled *Statement of the Claim*, and the arrow is pointing towards the Claim.

Figure 7: Framework for Example Usage in Proof Presentations



Instantiation of Claim occurs after the claim is presented and may be used by the instructor to either support the students' understanding of the claim or to motivate the method of the proof. Thus, the arrows point to both the Claim and the Proof. *Generic Examples* and *Instantiation of a Sub-Claim* are examples that are embedded within the proof itself and serve to support students' understanding of the proof. Thus, those ovals are contained within the rectangle labeled *Proof*. The examples in the dotted box are time-independent examples. Since *Instantiation of Definitions, Concepts, or Notation* or *Boundary Examples* could occur at any time throughout the presentation of the claim or proof and could be used to motivate or support either the claim or proof, they are not sequentially listed with the rest of the examples.

4.3.2 Illustrations of Example Types from the Data

4.3.2 a. Start-up Examples. These examples are used to motivate basic intuitions or claims (Michner, 1978), or, in the words of Dr. C, to "prepare the students' minds" for the statement of the claim. This type of example appeared twice in my data, both in Dr. C's class. In his interview, Dr. C said, "I think that there are people who are very abstract oriented in how they think, but I'm not, really. I think I get to abstraction in my brain through looking at patterns in concrete examples. So, that's one way I kind of grease the wheels and get them [the students] to think about something a little more abstractly. You know, as soon as they are kind of thinking about the concrete thing, then they can think about the abstract thing." Here is one instance when he used that strategy:

Dr. C began with two open intervals in **R**, (0,2) and (1,3), noting that the union of these sets is still open. Then he asked, "What about an infinite collection of open sets?" He created the collection of sets $O_j = (0, 2 - \frac{1}{j})$. Then, he prompted a discussion about whether or not 2 is included in $\bigcup_{j=1}^{\infty} O_j$, and the students decided that $\bigcup_{j=1}^{\infty} O_j = (0,2)$ was open. Next, Dr. C created a new infinite collection $O_j = (0, 2 + \frac{1}{j})$, and after some discussion the student decided that $\bigcap_{j=1}^{\infty} O_j = (0,2]$ was not open. Dr. C also did similar examples with closed sets. After the start-up examples, Dr. C presented and proved the theorem: "(a) The union of any collection of open sets is open. (b) The intersection of any finite collection of open sets is open." After the proof of that theorem, he presented the corresponding theorem regarding closed sets with no general proof.

4.3.2 b. Pattern Exploration. Examples showing a pattern that are used before the statement of the claim to lead students either to the discovery of the claim, or to provide evidence of the plausibility of the claim will be called *pattern exploration examples.*

It should be noted that the purpose of these examples is slightly different from Harel's (2001) two types of pattern generalization that are used by students to ascertain themselves or to persuade others to believe that a given claim is true. In both *result pattern generalization* and *process pattern generalization,* the claim is already given to the students, and they are attempting to use examples to construct a proof of the claim. This is different from using examples that show a pattern to motivate the statement of the claim.

Example generation by students has been shown to lead to conjectures in the context of elementary number theory (Morselli, 2006). Morselli concluded that the level of mathematical sophistication of the students has an impact on whether or not the students have a method for their example exploration or whether they explore examples "at random." Thus, when an instructor engages students in example exploration in an in-class presentation, the instructor is modeling a mature mathematical approach to selecting and exploring examples in a way that can lead to the development of a conjecture.

In his interview, Dr. N described how exploring patterns can lead to the construction of theorems: "Well, if it's a proof of a pattern, I certainly emphasize computation. First, you have to compute a lot to try to figure out what the pattern is… You're going from examples to theorems… you go through a lot of examples, you try to find something that's always true, and then you conjecture a theorem." Dr. N said that having students discover the theorem on their own by looking for patterns in examples is "a lot more fun" than just giving them the statement of the theorem.

This type of example appeared once in my data. It was used by Dr. N when he was lecturing about estimating the number of steps in Euclid's Algorithm, an algorithm used to find the greatest common divisor of any two integers *a* and *b*.

He began by stating that the "worst cases" for Euclid's Algorithm (meaning pairs of numbers that will have the maximum number of steps) happen when the two numbers are consecutive Fibonacci numbers. He illustrated Euclid's Algorithm for a=13 and b=8 to show that there are 5 steps. He then used a=144 and b=89 to show that there are 10 steps. It was clear that these examples using Fibonacci numbers would have the most steps because the quotient was 1 for each step. Next, he stated Lamé's Theorem: "The number of steps in Euclid's algorithm for integers *a* and *b* is less than or equal to five times the number of digits of the smallest of *a* or *b*." Because the pattern in the computational examples preceded and foreshadowed the statement of the theorem, this was coded as *pattern exploration.*

When asked about these examples in the follow-up interview, Dr. N agreed that they could be called pattern exploration, and that although the students may not have been able to articulate the pattern after only two examples, they served as a "plausibility argument." Thus, to the students, the two examples may not have been enough to convince them that the pattern was a *definite pattern* that was uniquely determined by its mathematical properties, but may have served to convince the students that it was a *plausible pattern* (Stylandes & Silver, 2009).

4.3.2 c. Instantiation of the Claim. When an example is used after the statement of the claim to give an instance when the claim holds, this is called *instantiation of the claim*. Instantiation of a claim may be used to persuade students of the plausibility of the claim, support students' understanding of the claim, or to generate proof insight.

Dr. N described instantiation of a claim as one of two techniques to help students understand a claim. He said, "…the other way around is looking at theorem and trying to understand how to produce examples out of it. So, I spend a lot of time doing that. So, here's a theorem, can you give me like a special case of it or something."

This type of example was used by all of the instructors, and occurred most frequently in the observation data. Dr. N and Dr. C both used instantiation of a claim twice, and Dr. G used it once. Dr. A used instantiation of a claim 6 times, and one of those times the instantiation was in lieu of a general proof. This happened when Dr. A was proving that in the symmetric group, a conjugate of a *k*-cycle is a *k*-cycle. In the follow-up interview, Dr. A said that he presented an example instead of a proof because the general proof has both heavy notation and induction, and that an example was enough to convince the students that the claim was correct.

Dr. N used instantiation when he presented the claim that "The GCD of two Fibonacci numbers f_n and f_m is $f_{\gcd(n,m)}$." He chose f_8 and f_{12}, and stated that *n* and *m* do not have to be "next to each other." He then said, "f_8 turns out to be, uh, 21? Is that right? And f_{12} is 144, and the greatest common divisor of that is 3, which is equal to f_4. So, that's an example of that. So, um, that's an amazing fact about the Fibonacci numbers." He then proceeded to give a start to a proof. He did not present a complete proof, but left the remaining details as a homework problem.

Dr. G used a pictorial representation of an example to instantiate a claim. On this occasion, Dr. G had just written this statement on the board: "In a projective plane where each line meets *n* points, (a) there are a total of $n^2 - n + 1$ points; (b) there are a total of $n^2 - n + 1$ lines." He then said, "Before we prove this theorem, let's just talk about these projective planes a little bit. Uh, what's the smallest projective plane I can possibly have? By smallest, I mean smallest number of points and smallest number of lines." With a little bit of input from the students, he concluded that the case when $n = 3$ is the smallest projective plane, with seven points and seven lines. He drew a picture of that particular projective plane, and then proceeded into a proof of the theorem.

The picture served to support the students' understanding of the statement of the claim by inserting a particular value for *n*, therefore it is a pictorial representation of an example that instantiates the claim. In his follow-up interview, Dr. G said that he used this example to "get the student involved in thinking about: What is a projective plane? What does one look like?" He said that he thought the example was used both to help students understand the claim and to lead students to an understanding of the proof.

4.3.2 d. Boundary Examples. Boundary examples serve to highlight the necessity of the hypotheses of the claim. The instructor can provide examples where some of the hypotheses do not hold, and then show that the conclusion doesn't hold. Or, they can be examples that show how a change in the hypotheses also changes the conclusion to create a different but related theorem. These examples are used to persuade students of the plausibility of the claim or to support their understanding of the claim.

Dr. A talked about using examples after the statement of a claim to "show why the hypotheses are necessary." Dr. C also referred to this type of example when he said, "through computing examples and looking for cases where the theorem does and does not hold, I think you can prepare them for understanding the parts of the hypotheses, the steps in the proof, [and] the statement [of the theorem]."

In my observation data, Dr. A used a boundary example once. He was proving the division algorithm, and used a numerical example to show why the restriction on the remainder is necessary for uniqueness. He showed that for the numbers 75 and 8, we have that $75 = 8 \cdot 8 + 11$, and $75 = 8 \cdot 10 - 5$, and then he summarizes, saying "but, the secret of this thing over here is *[circles the condition* '$0 \leq r < b$' *in the theorem]* you want that remainder right there… So, these two things *[points to the 11 and the -5]* do not fit into this condition right here. And, I would imagine most of you are happy to say that I can play the game… I can get r so that it is in this range." Thus, the example showed that there could be many ways to write the number 75, but the restriction on the remainder is what is necessary for the uniqueness result in the theorem.

Dr. C used two *boundary examples* to highlight the necessity of the hypotheses for the Heine-Borel Theorem. Dr. C proved that the set {1,3,5,7…} is not compact in **R** using the definition. Then, he presented the Heine-Borel Theorem, which states that a closed, bounded set of real numbers is compact. He asked the students why the previous set was not compact, and the students said "It is not bounded above." Dr. C then said, "Let's have another example [of a non-compact set]. What property of the Heine-Borel theorem should we contradict now?" The students said that they need an open set (note that they really only needed a set that was not closed). Dr. C then asked for an open set, and with some prodding, students chose (0,1). Dr. C then proved, using the definition, that (0,1) is not compact. These two examples showed the necessity of the closed and bounded conditions in the Heine-Borel theorem.

When asked about this classroom episode, Dr. C confirmed that he was using these examples to help students understand the hypotheses of the theorem, and commented about why he didn't give a general proof of the Heine-Borel Theorem. He said, "So, I have presented that entire proof in class, but I don't typically now. Why not? Well, it's really long, and I think that, as we get older, I think we have more of an ability to listen for a long stretch to something, and then somehow process it in one go and make sense of it. But, I think when you're younger that is very difficult. A proof that takes almost the entire class period to present is almost certainly too long." This, he explained, was why he presented some examples to help students understand and be able to apply the statement of the theorem. Note that Dr. C used the word "argument," in reference to the boundary examples, as though Dr. C felt that presenting the examples in lieu of the proof still served to convince the students of the truth of the Heine-Borel theorem.

4.3.2 e. Generic Examples. Instructors may use *generic examples* in conjunction with a general proof presentation. Generic examples are specific, often numerical, examples that mirror the general proof. They are presented side-by-side with the proof, and the instructor makes references back and forth as the presentation progresses. These examples can be used to generate proof insight or support students' understanding of the proof.

Dr. N described using a carefully chosen example to get an idea of how the proof will go in his interview: "and, so I do talk about this to try to get an idea about how the general proof might go. So, that is, that is a common approach as well. Well, commonly a theorem will say to prove that something is true for all integers *n*. And, so the most basic specialization is just to pick a particular integer and work with that one. You know, and see if you can do it for that. You know, you have to be careful about the choice of example that you're going to use this technique on."

This type of example appeared once in Dr. N's class. Dr. N was presenting a general proof of the claim that there is a unique representation of any integer in a particular base. The theorem is stated, "Let *b* be an integer greater than one. Every integer *n* greater than or equal to one has a unique representation as $n = a_k b^k + a_{k-1} b^{k-1} + ... + a_1 b + a_0$, where $0 \le a_j < b$ for all *j*, and $a_k \ne 0$." In the middle of the proof, Dr. N stopped to give an example. The computational example mirrored the general proof, which served to help the students understand the process that they were using in general. The following excerpt occurred after Dr. N had stated the theorem and started the general proof.

> **Dr. N:** Ok, so I just used the same thing again, I repeat again… I'm going to continually keep going like that… And, so we repeat this process, and so 'Repeating at the *j*th step, we have $q_j = q_{j+1} b + a_{j+1}$, where $0 \le a_{j+1} < b$.' The remainder always satisfies that it's between zero and *b*. Ok, so I'm just going to keep

dividing over and over again. So, if you do it in practice… it might actually be worth throwing some numbers up so that you can compare that against the theory, the theoretical formula. So, uh, '73 base 5'. Last class we said that $73 = 14 \cdot 5 + 3$, and this was a_0 *(labels the 3)* and this was q_0 *(labels the 14)*. So, then $14 = 2 \cdot 5 + 4$, and this is a_1 *(labels the 4)* and this is q_1 *(labels the 2)*. And then, um, the next quotient is $2 = 0 \cdot 5 + 2$, so this would be a_2, and this would be q_2. So... well, you can kind of imagine what the three steps are in there… and so now, do you notice anything about this process here that is allowing me to stop? What is it about this process that is allowing me to stop and say, 'I can finally stop dividing'?

Student 1: You get zero for…

Dr. N: I get zero for the quotient, Ok, that's right. So, that's the tip off, I get zero for the quotient. But, why are you forced to get a quotient that is zero? You have to detect a pattern in order to…

Student 2: The quotients are always deceasing.

Dr. N: The quotients are always decreasing. That's right, exactly right. If you look at the pattern, here, the quotients go from 14 to 2 to 0. And, whenever you do that the quotients are always decreasing. Ok? *(continues with the general proof)*

Dr. N used the numerical example to show the students that the quotients were decreasing, which was a crucial step in the general proof. He also tied the general notation of the proof to the numbers in the example, so that the students could have a more concrete understanding of the general proof. When asked about his use of this example, Dr. N said, "I usually… that level of the audience, or at least part of the audience is not comfortable with a large amount of algebra up on the board, and so I feel like, uh, they have grown comfortable, to some extent, with numbers. So, when you put numbers on there, that, uh, makes it easier to reach them. And, uh, so that's the reason for doing it. I feel like, uh, you learn better when you go back and forth between the two instances."

Dr. A had one instance of a generic example, although he did this example in lieu of the general proof instead of beside the general proof. He was addressing the claim that a particular map from Z_n to Z_n was well-defined. So, he chose [4] and [4+3n], which were two different representations of the same equivalence class, and showed that they mapped to the same equivalence class. In the example he presented, the 4 and 3 are merely acting as placeholders, in other words, if they were replaced by variable names the proof would have been generalized.

4.3.2 f. Instantiation of a Sub-Claim. These are examples that are used during the proof of a claim to help students understand a sub-claim or a particularly difficult logical point. These are classified as different from the examples that instantiate a claim because their purpose is not to help students understand the main claim that is to be proved, but to support students' understanding of the proof. They are often used in lieu of a proof of a sub-claim, and so they also serve to persuade students of the plausibility of a sub-claim. These types of examples were used once by Dr. N and twice by Dr. C.

In the middle of the proof of Lamé's theorem, Dr. N claimed that he can use the function $f(x) = \log_{10}(x)$ to determine the number of digits of a number. The class seemed puzzled by this notion, so Dr. N presented an example to the side of the board to *instantiate the sub-claim*. He gave the example of 5643, where $10^3 < 5643 < 10^4$, so therefore $3 < \log_{10} 5643 < 4$, and then stated that the number of digits of the number 5643 is the ceiling of $\log_{10} 5643$. Dr. N commented on this example in his follow-up interview. He said that "it's kinda funny that you have to explain that to math majors. I'm always shocked when I get asked about that stuff... I'm sure I didn't plan it." He did not spend time presenting a general proof of the sub-claim.

Dr. C was proving the statement "If x^2 is even, then x is even." He was using the fact that primes cannot be "broken up," and so he used a numerical example "If 6 divides a product *xy*, you could have 2 going in to *x* and 3 going in to *y*. Six can be broken up, you know? But that can't happen with 2." In the middle of another proof, he wanted to use the fact that the Real Numbers have no zero divisors. He claimed that her students used that fact all the time, citing the familiar exercise of solving quadratic equations by factoring. He said, "You know that $(x-2)(x-3)=0$ means either $x=2$ or $x=3$."

4.3.2 g. Instantiation of Definitions, Notation, or Concepts. These examples serve to reinforce the mathematics content underlying the claim or the proof. They may occur at any time, and often appear to be spontaneously generated by the instructor. If presented before the claim, they can be used to motivate basic intuitions or claims. They can also be used to support students' understanding of the claim or proof.

In his initial interview, Dr. A talked about how he would stop in the presentation of a proof to ask students questions about the definitions, concepts, and notation. "There is a lot, as we go down through [the proof] of recalling definitions and that kind of thing. What does it *mean* to do this? What does it *mean* to do that? Do you understand what this *symbo*l means?"

Dr. G said that "from time to time [I] try to do examples that are just illustrations of the definition: 'So, here's a new thing, a projective plane. So, let's draw a picture of a projective plane.' And I think I do that reasonably often." Although Dr. G mentioned this in his interview, the observation data did not capture him using this technique. Instantiation of definitions, concepts, and notation were used nine times by Dr. A and seven times by Dr. C.

When proving a statement about bounded sequences, Dr. C asked the class to give some examples of a bounded sequence (they had previously defined a bounded sequence). These examples served to *instantiate the definition* of a bounded sequence. One student suggested “1, 2, 3, 3, 3, 3, 3…” Dr. C said that this is a bounded convergent sequence, and asked if the students if they could think of a bounded sequence that does not converge. Another student suggested “ $(-1)^n$.” Then Dr. C asked the class what it means to say that a general sequence is bounded, and the students began to reconstruct the definition. He then proceeded to prove the general claim using this definition.

One class day, Dr. A presented a set and a relation on that set, and proved that it was an equivalence relation. In the proof, he asked the students to tell him some of the elements in an equivalence class of a particular element. This example was an instantiation of the definition of an equivalence relation, and also gave students a deeper understanding of the relation that was given.

Dr. C used an example to talk about the “universe of discourse” in a proof about divisibility. This example was prompted by a student misunderstanding about the values that *x* could take in the theorem. Dr. C said, “And so the integers are understood in that context because usually what we mean is that 6 is divisible by 3 if you can take 6 divided by 3 and the answer is still an integer. What does it mean to say pi is divisible by the square root of two? Uhh, well, we don’t even discuss it, because pi is a real number, square root of two is a real number, and so you divide them, you get a real number. Every number is divisible by every other number if you think about it that way. So, so divisibility when you talk about it this way has to be the integers.” So, the example was not used to instantiate the theorem itself, but to instantiate the concept of a universe of discourse.

Instantiation of notation was used by Dr. C when he was proving that $s_n = \frac{n+1}{n+2}$ converged. He first asked students, "If *n*=100, what is s_n?" Then he asked them to compute s_1, s_2, and s_3. These examples were used to help the students make sense of the notation and begin to see the pattern of the sequence.

4.3.2 h. Metaphorical Examples. Whenever an example was used to compare one mathematical structure to a different (more familiar) mathematical structure, I refer to this as a *metaphorical example.* This is unlike instantiation because the instructor is comparing two different structures rather than giving an instance of a structure. Metaphorical examples may occur at any time throughout the presentation of the claim or proof, and can be used to either motivate basic intuitions or claims, or to support students' understanding of the mathematical content involved in the proof. One metaphorical example was captured in my data. It was used by Dr. A.

When enumerating the different types of elements in S_4, the students tried to count four representations of the identity as different elements. Dr. A wanted to discourage this behavior and help the students to understand that though the elements are written differently, they represent the same element of S_4. Dr. A said, "Well, they're all the same, so I just do that *(writes '1-cycles: (1)')*. That's all the 1-cycles we have. Yeah, you can come up and write (4), but that's equal to that, so, I just, you wrote it different. It's not different. That's the only one cycle." Then a student asks, "So, there's only one 1-cycle?" to which Dr. A replied, "It's the identity. And, your eyes are telling me that that confuses you. Ok, like we do a lot, you can write ½ equal to $^{30}/_{60}$, that's not two fractions, that's one fraction."

Since there are mathematical differences in the two structures, this is not really instantiation. Dr. A is comparing the unfamiliar structure to a more familiar structure using a metaphor, because in both instances an element can be written in different ways. In reference to this example, he said "well, I think students get lost on, uh, equivalence relations. They don't really understand, uh, that this is one thing but it's got all kinds of different representations, and a little set of all of these different representations, but are mathematically considered the same thing. So, I think fractions help them."

4.3.3 Summary of Example Usage in Proof Presentations. A theoretical contribution of this paper is precise definition of "example" that combines the mathematical requirements found in previous definitions (Alcock & Weber, 2010; Zazkis & Leikin, 2008) with a learner-dependent requirement, as found in Mason & Watson (2005). In this paper, an "example" is a specific, concrete representative of a class of mathematical objects, where the class is defined by a set of criteria. The mathematical requirement is that the object must be a specific representative of a class of mathematical objects, but in order for an object to be an example it must be concrete to the learner. In other words, the learner must be able to either compute with it or investigate its properties.

The framework highlights four pedagogical uses for examples: to motivate basic intuitions or claims, to persuade students of the plausibility of the claim or of a sub-claim, to support students' understanding of the claim, proof, or the underlying mathematical content, or to generate proof insight.

The summary presented here categorizes the examples that were used during proof presentations on the class days that were observed in the data. Thus, it is a subset of the instructors' example usage. It is possible that these individuals may use examples differently when teaching a different course, or that different instructors may use examples in different ways when teaching a particular course. Despite the small size of the study, I have obtained a rich collection of examples that leads nicely to the construction of a framework for example usage in proof presentations. Follow-up interviews served as a member check to allow the participants to comment on their pedagogical intentions when presenting specific examples in proof presentations. A summary of each participant's example usage in proof presentations is given in Table 13.

Table 13: *Summary of Example Usage in the Data*

	Dr. C	Dr. A	Dr. G	Dr. N
Start-Up	2	0	0	0
Pattern Exploration	0	0	0	1
Boundary Examples	2	1	0	0
Instantiation of Claim	2	6	1	2
Generic Examples	0	1	0	1
Instantiation of Sub-Claim	3	0	0	1
Instantiation of Definition	3	6	0	0
Instantiation of Concepts or Notation	4	3	0	0
Metaphorical Example	0	1	0	0

The most common type of example used in proof presentations was instantiation. Pattern exploration appeared in the Number Theory course, likely because the mathematics content lends itself well to this type of exemplification (Morselli, 2006). Dr. C used start-up examples and boundary examples, which were both mentioned by Dr. C in his interview.

Dr. G rarely used examples in his proof presentations; however, he did use generic diagrams in 19 of the 22 proofs observed. In his follow-up interview, Dr. G said that he did not use examples because of the mathematics content that he was presenting. Thus, the mathematics content may have a direct influence on the types or frequency of examples used in proof presentations.

4.4 A Multi-Dimensional Analysis of Instructor Questions

Results in this section will address the fifth and sixth research questions, “How often do instructors who are teaching advanced mathematics using lecture methods interact with their students by asking questions?” and “What types of questions are asked by instructors who are teaching advanced mathematics using lecture methods, and what types of responses are expected of students?” These questions were investigated by an analysis on each question posed by the instructor.

Previous research on teaching proof-based mathematics courses, such as Weber's (2004) study showed that the students rarely participated in class discussions, while in Fukawa-Connelly (2012) the teacher asked a lot of questions, including high-level questions, but the questions that the students actually answered were lower-level, primarily factual. The analysis that follows will investigate the questioning of the four participants in their lectures. Results will be presented as separate but interrelated case studies.

The main findings are that the instructors interacted frequently with their students by asking questions, from .69 to 1.81 questions per minute, on average. Also, the students responded to between 35% and 52% of the questions that were posed.

When the questions were analyzed using Anderson’s Revised Bloom’s Taxonomy, it was found that between 30% and 54% of the instructors’ questions are higher-order (meaning Apply Understanding and above), and of the questions that were answered by students, between 31% and 57% of those were higher-order. So this study has found that there was no noticeable difference between the cognitive level of the questions posed by the instructor and the questions that were actually answered by students.

The percentage of questions that were asked that required more than just a factual response ranged from 13% to 35%. But again, the percentage of questions that students answered that required more than just a factual response ranged from 23% to 48%. Thus, this study has found that students are answering higher-level questions to a greater degree than has been indicated in past studies. The results also show that meta-process questions are infrequent.

4.4.1 Frequency of Instructor Questions. The fifth research question addresses the frequency with which instructors ask questions in advanced mathematics lectures. The average number of questions per minute for the four instructors ranged from 0.69 questions per minute to 1.81 questions per minute. So, it appears that instructors at this level do frequently interact with their students by asking questions. This shows that lectures are not necessarily monologues by the instructor, but do involve interaction with the students.

Table 14: *Frequency of Instructor Questions*

	Algebra	Adv Calc	Geometry	Num Thry
Instructor	Dr. A	Dr. C	Dr. G	Dr. N
# Students	24	9	9	14
# Instruction Days	44	29*	44	44
# Observations	7	7	6	6
Total time observed (min)	323	536	250	264
Total # Questions	224	969	262	235
Question Rate (per min)	0.69	1.81	1.05	0.88
Response Rate	35%	47%	46%	52%

* Dr. C's class meetings were 75 minutes long, while the other three courses met for 50 minutes per class.

The rate of student responses ranged from 35% to 52%, but three of the four instructors had response rates close to 50%. Note that this analysis considers rhetorical questions and comprehension questions in the question count, and those types of questions typically do not require an audible response on the part of the student.

Instructors also linked some questions together, either by re-phrasing the question or asking a related question that reduced the cognitive load, as in Wood's (1994) description of *funneling* or *focusing* questioning patterns. It was not unusual for the instructor to ask two or three linked questions before a student would give an answer. For example, Dr. C asked the following questions about the limit of a given sequence: "So, what is the limit in this case? *(pause)* Do you guys remember how to find that limit? *(pause)* When n is very large, what is happening to s_n? *(pause)* Like, when n is 100, what is s_n?" A student answered the final question, which led Dr. C into another line of questioning focused around plotting the terms of the sequence to see if they could determine the limit.

It should also be noted that although the average response rate is computed in Table 14, the questions that were asked were frequently clumped together into sequences of interactive lecture. Refer to Table 11 to see the percentages of 1-minute slices of class time that were coded interactive vs. straight lecture.

4.4.2 Illustrations of Question Types from the Data.

The types of questions were coded while watching the video observations, and thus, the codes were assigned based on the actual phrasing of the question as well as the timing and mathematics context surrounding the question. Often, the instructor's body language came into play, for example, the instructor may pose a question without looking at the students, and then answer the question himself. The body language of the instructor communicated to the students that the question was rhetorical.

4.4.2.a. Rhetorical Questions. Many of the rhetorical questions that occurred within my data were instances where the instructor was asking a question to either motivate content or to model the thinking that the instructor expects the students to do independently. Some examples include:

- Can you see that the answer is "no?" It does not require the function to be constant.
- So, this implies that *cb*=*da*, and what's this implication from? It's symmetry of equality.
- Now, what's your goal? Your goal is to take the set *S*, which is a subset of the set *T*, and show that it is finite.

The use of rhetorical questions may be a characteristic of the individual instructor, because the percentage of rhetorical questions varied wildly, from Dr. A using 40.6% rhetorical to Dr. C using 7.1% rhetorical.

4.4.2.b. Comprehension Questions. This type of question was used by instructors to check for student understanding, and often did not require an audible response on the part of the student. Typical examples of this type of question include:

- Does that make sense?
- Anything seem tricky, or was everything clear?
- Does this make sense, or do you guys have a question?
- Ok?

All of the instructors used comprehension questions in varying degrees, from Dr. G with 3.1% comprehension questions to Dr. C with 16.6% comprehension questions.

4.4.2.c. Choice Questions. Choice questions are either yes/no questions or questions that require the student to choose between two or more options. Here are some examples of choice questions that were lower order and examples of choice questions that were higher order.

- Lower order. "Now, you remember that we had a definition of even and a definition of odd that we had last time, right?" "So, therefore, is this set open or closed by our definition?"

- Higher order. “Now, if I take the intersection of open sets, is it open?” “Now, does that seem like an easier proof to you?” (asking students to look at two different proofs and evaluate which one makes more sense to them)

Choice questions were used by all of the instructors, and hit all of the Bloom’s levels except for *create*, because the nature of a choice question does not allow for the student to create a new structure.

4.4.2.d. Product Questions. Product questions require a student to give a factual response, or to give the name of a particular mathematical object. Some examples of product questions at different Bloom’s levels are:

- Lower order. “What property of a graph does (*d*) describe? There's a word for this…” “So, what do I get when I square that out?” “But, just look at the other end of the spectrum. The identity is the littlest subgroup of *G*, what’s the biggest?”
- Higher order. “Can you make a true version of the statement?” “So, now you need to help me figure out the collection of sets.” (Asking students to create a collection of sets that has certain properties)

All of the instructors used product questions frequently.

4.4.2.e. Process Questions. Process questions require the student to give their interpretations or opinions, or to describe a mathematical computation, or to explain mathematics content. Some examples of process questions include:

- Lower order. “So, the best way to prove uniqueness is to… I mean, what’s the standard way that you prove that there’s only one formula?” “We’re not ready to substitute yet until we get it in an external form that models 11.9d that we’re trying to use. So, what algebraic manipulation needs to be done?”

- Higher order. “I have to know something about this, so what would you do? Something you know about Fibonacci numbers.” “You stop when the quotient is finally zero. But, why are you forced to get a quotient that is zero? You have to detect a pattern in order to…”

All of the instructors used process questions frequently.

4.4.2.f. Meta-Process Questions. Questions that require a student to reflect upon their thinking or to formulate the grounds for their reasoning are meta-process questions. Some examples are:

- Lower order. “So, do, are any of these pictures helpful to you for any particular reason?”
- Higher order. “Absolute value of k. Keep it inside absolute values, don’t get rid of them. Correct. Why did you do that, [student]?” “So, any guesses about what I'm about to tell you I want a limit point to be, if that's the notion of limit that I've got in my head right now?”

Most of the instructors did not use meta-process questions frequently. Dr. C used them the most, with meta-process questions making up 3.7% of his total questions.

4.4.3 Case Studies of Instructor Questions. Since each instructor was teaching in a different content area with a different group of students, we must consider each classroom as a separate case study. The next sections will give specific information about each instructor’s use of questions, including some excerpts from their interviews to shed light on their perspective in regard to interacting with their students using questions.

4.4.3.a Dr. A’s Questions: In the observations of Dr. A, it was found that 40.6% of the questions that he asked were rhetorical. When asked about this, Dr. A said, “I didn’t want to take… you see, that’s the thing. I’m impatient. Very impatient. And so, you’re absolutely right. I do question, but I’m not particularly wanting them to answer.” He also said “I feel uncomfortable… I mean, if I'm asking you a question and you're sitting there and struggling with it, I don't want to just keep putting you on the spot. I want to move on. I'm very uncomfortable embarrassing anybody or making them feel uncomfortable."

Another thing that may have affected Dr. A's questioning was his view of the students in that course. In his follow-up interview, Dr. A said "Well, you know, I thought they [this class] were particularly weak." He also expressed that he didn't think the students were trying very hard, and mentioned that "they didn't do the extra credit problems… which I found very strange." It may be that the perception that Dr. A had of this particular class influenced the types of questions that he asked of the students.

Table 15: *Dr. A's Questions*

	Lower order	Higher Order	Total
Rhetorical	33.1%	11.1%	44.2%
Choice	7.6%	7.1%	14.7%
Product	16.5%	3.9%	20.4%
Process	4.9%	8.0%	12.9%
Meta-Process	0%	0%	0%
Total	62.1%	30.1%	

Dr. A asked 44.2% rhetorical and 7.5% comprehension questions. Thus, only 48.4% of his questions were questions that elicited a response from the students. His overall response rate was 35%. The table shows that 30.1% of Dr. A's questions were higher order (meaning Apply Understanding or above) and 12.9% require more than a factual response (either a process or meta-process response). When restricting to only questions that were answered by students, 31% of questions answered by students were higher order, and 13% required more than a factual response (either a process or meta-process).

4.4.3.b. Dr. C's Questions: Dr. C commented that when he lectures he tries to talk to his students and get them to talk back. He said, "I ask them to help me. I think I always ask them to help me, and I think that I have, well, I don't know. Hopefully I always ask them to help me, it depends on how much time we have. But, I want them, one of the things that I want them to do is to do some meta thinking as well as some detail thinking."

Table 16: *Dr. C's Questions*

	Lower Order	Higher Order	Total
Rhetorical	5.2%	1.8%	7.0%
Choice	8.5%	5.8%	14.3%
Product	21.5%	14.5%	36.0%
Process	6.4 %	18.3%	24.7%
Meta-Process	1.3%	2.3%	3.6%
Total	42.9%	42.7%	

When students answered incorrectly, and Dr. C would either re-direct by asking another question, or just remain silent and continue to look at the students. When asked about this strategy in the follow-up interview, he said, "I just try to let them have a chance to think about it. And, I think, if you say to somebody, 'No, you're wrong!' Then their brain doesn't have a chance to process on it. But, if you say, 'Hmmmm…' Then, all the sudden they are still thinking. Well, what's better? Is it better for me to talk or for them to think? It's better for them to think!"

It is also interesting to note that Dr. C asked some meta-process questions. He tended to have a one-on-one dialogue with individual students, and in these situations he would sometimes ask them to explain their thinking.

Dr. C asked 7% rhetorical and 14.1% comprehension questions, with an overall response rate of 47%. The table shows that 42.7% of Dr. C's questions were higher order (meaning Apply

Understanding or above) and 28.3% require more than a factual response (either a process or meta-process response). When restricting to only questions that were answered by students, 43% of questions answered by students were higher order, and 29% required more than a factual response.

4.4.3.c. Dr. G's Questions: In the initial interview, Dr. G described the types of questions that he uses when teaching this course. He said "I ask little questions like: 'why is this angle congruent to that angle?' 'What's the next step? What do I do here?' Mostly I ask questions of students and insist on an answer. The worst thing, I think, from a teacher's point of view, is if you try to ask a question, and the class just stares at you, and you can't get any response. That's when you want to throw up your hands and quit."

Table 17: *Dr. G's Questions*

	Lower Order	Higher Order	Total
Rhetorical	21.4%	11.9%	33.3%
Choice	4.5%	1.6%	6.1%
Product	10.6%	11.0%	21.6%
Process	8.4%	27.0%	35.4%
Meta-Process	0.4%	0%	0.4%
Total	45.3%	51.5%	

Most of the proofs that were presented in the Geometry class used proof by contradiction. Dr. G would often go through the proof, asking questions along the way, and then stop and ask the students to spot the contradiction. This type of questions was coded as a Process/Analysis question, and was the most frequent type of question that Dr. G asked.

Dr. G asked 33.3% rhetorical and 3.1% comprehension questions. He had a response rate of 46% overall. The table shows that 51.5% of Dr. G's questions were higher order (meaning Apply Understanding or above) and 35.8% require more than a factual response. When restricting to only questions that were answered by students, 55.4% of questions answered by students were higher order, and 35% required more than a factual response.

4.4.3.d. Dr. N's Questions: In his interview, Dr. N praised the work of Polya (1945), and talked about how he thought that using questions and interacting with students was the best way to teach. He also mentioned that because of class sizes and time constraints, he was not always able to teach in the way that he thought was best. He said, "well, my uh, usual pattern, well, it slows the class down, so I'm not always afforded the liberty of doing it the way that I like, my usual pattern is to ask questions of different pieces. Polya gives an example in his book of leading a student through a, uh, solving a problem by asking questions. So, that was his belief; that you have to ask questions and have to learn how to ask questions…"

Table 18: *Dr. N's Questions*

	Lower Order	Higher Order	Total
Rhetorical	6.3 %	4.3%	10.6%
Choice	4.3%	1.3%	5.6%
Product	13.2%	20.0%	33.2%
Process	7.3%	25.1%	32.4%
Meta-Process	0%	1.7%	1.7%
Total	31.1%	52.4%	

He asked a lot of linked questions, and when students answered incorrectly, he would take some fragment of the students' comment that was correct and re-work it into another question. When asked about that technique, he said, "when you read "How to solve it" by Polya, he has a sample dialogue with a student where he does exactly that. You're not supposed to, you're not supposed to, like, give up, you're supposed to re-word the question, or ask the students to think about a different way of asking about it, or something like that. So, that's intentional."

Dr. N asked 10.6% rhetorical and 16.6% comprehension questions in the observation data. His overall response rate was 52%. The table shows that 52.4% of Dr. N's questions were higher order (meaning Apply Understanding or above), and 34.1% require either a process or meta-process response. When restricting to only questions that were answered by students, 57% of questions answered by students were higher order, and 34% required more than just a factual response.

4.4.4 Summary of Instructor Questions. This study has shown that these instructors interacted with their students frequently during their lectures. Their question rate ranges between 0.69 and 1.81 questions per minute. Although it is the case that the occurrences of questions are often clumped together, the average longest period of straight lecture ranges from 5.7 to 11.7 minutes. Thus, although all of the instructors were using lecture methods, the lectures were not monolog. They were all engaging their students by using questioning.

The percentage of questions asked that were higher-order (apply understanding and above) ranged from 30.1% to 54.2%, and the percentage of questions requiring more than just a factual response (process or meta-process questions) ranged from 12.9% to 35.8%. This result stands in contrast to Tallman & Carlson's (2012) analysis of Calculus 1 final exams. They found that 85% of the items on the exams required no more than rote memorization or applying a procedure. The differences could be attributed to the level of the mathematics content or to the distinction between in-class questions and assessment items.

The students answered the questions at rates ranging from 35% to 52%, but three of the four participants had response rates close to 50%. Dr. A's response rate was lower because he used more rhetorical questions than the other instructors. When restricting to only the questions that were answered by the students, Table 18 shows that the percentage of answered questions that are higher order is similar to the percentage of higher-order questions asked by each instructor. Similarly, the percentage of answered questions that require more than just a factual response is not noticeably different from the percentage of questions asked that require more than just a factual response. Thus, this result shows that students were answering higher-order questions in the same proportions that instructors were asking them.

Table 19: *Summary of Instructor Questions*

	Dr. A	Dr. C	Dr. G	Dr. N
Total Number of Questions Coded	224	969	262	235
% Higher Order	54.2%	42.7%	51.5%	30.1%
% Process/Meta-Process	34.1%	28.3%	35.8%	12.9%
Response Rate	35%	47%	46%	52%
% of Answered Questions that are Higher Order	57%	43%	55.4%	31%
% of Answered Questions that are Process/Meta-Process	34%	29%	35%	13%

This result is different from other studies investigating instructor practices in proof-based lectures. Weber (2004) found that in Dr. T's analysis course "students asked questions only infrequently and rarely participated in class discussions" (p. 118). In Fukawa-Connelly's (2012a) case study of an abstract algebra instructor, he found that the instructor asked a variety of questions, including some higher-order questions, but they were generally contained in a sequence of questions so that students actually answered mostly factual questions.

A limitation of this analysis of instructor questions is the lack of inter-rater reliability. All of the questions were coded independently. To strengthen this result, inter-rater reliability should be established.

4.5 Chapter Summary

This chapter presented the findings of this multi-case study on instructional practices in proof-based undergraduate mathematics lectures. Initial analysis of the observation data showed that the instructors spent, between 35% and 70% of their class time presenting proofs. Thus, instructors at this level spend large proportions of their class time presenting proofs in class. This finding has already been cited to provide further justification for research on teacher practices when presenting proofs (Lai, et al., 2012).

The interview data were used to identify four proof presentation strategies that the instructors said that they used to help their students understand their proof presentations: Outline, Examples, Logical Structure, and Context. It was found that at least one of these pedagogical strategies was used in all but 6.3% of the total proof presentations observed. Thus, the instructors were not just presenting straight proofs, but were adapting the proof presentations based on their perceptions of their students' pedagogical needs.

The strategy of examples was used to varying degrees across the four cases. In particular, three of the four instructors used examples in half of their proof presentations, while Dr. G only used examples in one out of the 22 proof presentations. Dr. G did use *generic diagrams* to structure 19 of the proof presentations. The pictures he drew were serving as an alternative representation of the general proof, and were therefore not classified as examples. This finding set the background for a deeper analysis of the ways in which instructors use examples in their proof presentations.

The interview data were used to construct levels of expected engagement. Each proof presentation was coded according to the level of expected engagement. In three of the cases, the instructors expected the students to be actively engaged in 95% of their proof presentations by giving factual information as well as contributing big ideas for the proof. The fourth instructor expected active engagement in 50% of his proofs. Although the proofs were presented in a lecture style, the proof presentations were more dialogue than monologue, similar to Fukawa-Connelly's (2012a) finding. This result inspired an analysis of the instructors' questions.

Timelines were constructed of each observation. The timelines noted both the content that was being presented and the source of vocalization. Across all four cases, the instructors spent the most proportion of class time on examples and proofs. The timelines gave a visual representation of the interaction between the instructor and students. Each one-minute segment was coded as "straight lecture" or "interactive lecture." The proportion of 1-minute intervals that were coded "interactive lecture" ranged from 26% to 62%.

The rich collection of examples that were used in proof presentations in the observation data were used to construct a descriptive framework of instructors' example usage in proof presentations, which is illustrated in Figure 5. This framework is chronological from left to right, and demonstrates when the different types of examples are likely to occur. Arrows indicate whether the examples serve to motivate or support the students' understanding of the theorem or the proof.

Analysis of the examples used in proof presentation showed that instantiation was the most frequent type of examples used. Dr. C mentioned the use of start-up examples and boundary examples, and both were observed in his proof presentations. Instructors' example usage appeared to be content specific to some degree. Pattern exploration was only observed in Dr. N's number theory class, and examples were used very infrequently in Dr. G's geometry class.

The analysis of the instructor questions revealed that in all four cases instructors were engaging their students frequently by asking questions. The question rates for the instructors ranged from 0.69 to 1.81 questions per minute. The percentage of higher-order questions asked by the instructors ranged from 30.1% to 54.2%, and the percentage of questions requiring more than just a factual response ranged from 12.9% to 35.8%. Thus, the instructors were asking a variety of questions, including a high percentage of higher-order questions.

The percentage of questions to which students responded ranged from 35% to 52%. When restricting to only questions that were answered by students, the percentages of higher-order questions matched the overall percentage of higher order questions for all four case studies. Also, the percentages of answered questions requiring more than just a factual response matched the overall percentages as well. Thus, the students were responding to higher-order questions and to questions that required more than just a factual response.

CHAPTER V

CONCLUSION

The purpose of this multi-case study is to examine the teaching practices of four mathematics instructors who were teaching different proof-based mathematics courses using lecture methods. The conclusions from this study follow the research questions and findings and therefore address four areas: (a) pedagogical moves made by the instructors when presenting proofs in class; (b) allocation of class time among various content-specific activities and types of lecture; (c) examples used during in-class proof presentations; (d) questions posed by the instructors in class. This chapter first gives a summary of the results for each case study, and then a discussion and interpretation of the major cross-case findings in light of the literature as well as implications for future research.

5.1 Summary of Each Case Study

5.1.1 Dr. A's Abstract Algebra Class. Dr. A's class was comprised of 24 students, and he mentioned in his follow-up interview that he thought this class was particularly weak. He indicated that his perception of their ability had an effect on his teaching. He established that he was using lecture methods. He emphasized in the interviews that he believed that students should only get ideas from a lecture and should spend time thinking about the concepts at home. He also said that he likes to weave in discussions about proof techniques into the content rather than talking about them at the beginning of the semester.

When asked what strategies that he would use to help students understand his proof presentations, he said that he would talk about logical structure and have discussions with students about what the mathematics content means and ask them how they would prove a certain statement. Of the 12 presentations of theorems and proofs in his data, he used outline in 42% of those and logical structure in 25%. In his interview he only mentioned drawing pictures, but not any particular types of examples, however, he used examples in 67% of his presentations. He also did not mention context as a strategy, but he used it in 17% of his presentations. Dr. A frequently used examples to instantiate definitions, claims, and notation in his proof presentations. He was observed using a generic example once and a boundary example once. He was also the only instructor observed who used a metaphorical example. Dr. A spent 40% of his class time on proof presentations, and 29.2% of that time was on the actual proofs themselves. He only worked homework problems 2.5% of the time, which is in line with his philosophy that students should be independent. He used examples in 38.9% of his class time.

Dr. A said that he would often ask students the meaning of certain definitions or notation, and he also said that during a proof presentation he may ask students to give the next step. Half of his proof presentations were coded 3 or above, meaning that students were expected to participate in the proof presentation. Also, 73.9% of his class time was spent on straight lecture with no student interaction. He frequently asked rhetorical questions, using them to motivate the mathematics content that he was presenting. He verified this technique in the follow-up interview.

5.1.2 Dr. C's Advanced Calculus Class. Dr. C's class was comprised of 9 students, all of whom were upper classmen, and mostly mathematics or mathematics education majors. Dr. C preferred to describe his teaching style as "modified lecture." In the interview, Dr. C expressed that abstract mathematical concepts must be motivated with examples and that proof methods and techniques must be explicitly taught.

He mentioned the proof strategies of examples, logical structure, and outline in the interview. Dr. C used all of these strategies frequently, and all of the 22 proofs captured in the video data had an identified presentation strategy. Dr. C had a wide range of example types, including start-up examples, boundary examples, and different types of instantiation. Dr. C spent 49% of his class time on proof presentations, and expected active engagement in 95% of his presentations. Dr. C would involve students the most at the beginning of his proof presentations, asking students to help him set up the structure of the proof. It was also noteworthy that Dr. C spent a large portion of class time on homework problems, around 30%. Students often asked homework questions at the beginning of class.

Dr. C used interactive lecture most frequently, using 61.5% of his class time interacting with the students. He said "I try to talk to my students and get them to talk to me. I try to have some sense that they are processing what I'm saying and with me before I move on." All but one of his proof presentations required active engagement from the students. A little over 42% of Dr. C's questions were higher order, and 28% of his questions required more than just a factual response. Dr. C infrequently asked rhetorical questions.

5.1.3 Dr. G's Geometry Class. Dr. G's geometry class had 9 students enrolled, eight of whom were math or math education majors. Dr. G also preferred to call his style "modified lecture" but he contrasted his methods with inquiry-based methods, because he said he didn't "leave them with open-ended problems." Dr. G was emphatic that he expected his students to interact with him in class, and said that when his students don't respond to his questions he just wants to "throw up his hands and quit."

Dr. G talked about the proof presentation strategy of outline the most. He said that he likes to give students a "plan of attack" before presenting a complicated proof. He said that he likes to motivate the content with examples, but says that for non-Euclidian geometry "real examples… are hard to come by, because you're talking about something like the Poincare disc or the Klein bottle…" He said that it is easier to find examples in courses like number theory. He also described the importance of the historical context of many of the proofs in Geometry. In his 22 proof presentations, Dr. G only used examples in one presentation, but used outline in about half of them. Although he did not use examples, he did organize most of his proofs with a generic diagram, which was a picture that served as an alternative representation of the general proof.

Dr. G did engage his students in his proof presentations, expecting active engagement in 95% of his presentations. The most frequent way that he involved students in his presentations was to set up the proof until he got to the point where there was a contradiction, and then he would have the students explain why there was a contradiction. This may be because proof by contradiction is common in that particular mathematics content area.

Dr. G used interactive lecture in 36.4% of his class time. He asked 33% rhetorical questions, but still had a response rate of 46% overall. It was found that 51.5% of his questions were higher order, and that 55% of the questions answered by students were higher order.

5.1.4 Dr. N's Number Theory Class. The number theory course had an enrollment of 14 students, and all but one were mathematics or mathematics education majors. Dr. N confirmed that he used lecture methods because of time constraints, but said that he tries to involve students using questioning as much as possible, citing Polya's use of questioning to lead students through problem solving. He said that he likes to include group projects as well.

The proof presentation strategies that he mentioned were outline, examples, and logical structure. In his proof presentations, he used these three strategies frequently. Dr. N used examples to instantiate, and also used pattern exploration and generic examples. He also mentioned that he likes to ask questions when presenting proofs so that the students can be led through the process. Observations showed that he expected active engagement in all of the 8 proofs he presented.

Dr. N spent 35% of his class time presenting proofs, and 45% of his class time was spent using interactive lecture. He often had casual conversations with the students about the mathematics, especially about the history around the mathematics that was being presented.

Of the questions posed by Dr. N, 52.4% were higher order. The students responded to 52% of Dr. N's questions, and of the questions answered by students, 57% were higher order.

5.2 Pedagogical Moves During Proof Presentations

Literature addressing students' difficulties with mathematical proof showed that students struggled with the mathematical notation (Selden & Selden, 1995), the mathematics content (Moore, 1994), held beliefs about proof that were inconsistent with the mathematical community (Schoenfeld, 1989; Solomon, 2006), or lacked strategic knowledge (Weber, 2001).

Previous interview studies of mathematics instructors showed that they seemed to lack an arsenal of strategies for helping students understand their proof presentations (Weber, 2011; Alcock, 2010; Harel & Sowder, 2009). One explanation is that the instructors are experienced, and have well established patterns of knowledge and behavior when teaching. Schoenfeld (2011) writes that "decision making and resource access are largely automatic when people are engaged in well-practiced behavior" (p. 16). Thus, it is possible that the instructors, in an interview setting, were unable to recall the routine strategies that they use in class. Another possible explanation is that these instructors are not accustomed to discussing their teaching strategies, and may be unable to clearly articulate the methods that they use and why they use them. Therefore, it is apparent that observation of instructors as they present proofs is necessary to have a more complete understanding of the strategies that they use.

This study found that the four instructors used various proof presentation strategies and levels of expected engagement during their proof presentations. In their interviews, the instructors discussed the strategies that they use to help their students understand the proofs that they present in class. Four proof presentation strategies were identified in the interview data: *outline, examples, logical structure,* and *context.* It was found that across the four cases, the instructors were using these strategies in their proof presentations. In fact, an identified proof presentation strategy was used in all but four of the 64 total proofs observed in the data. This result shows that the instructors were not just presenting proofs to demonstrate the truth of theorems in the mathematics content, but they were adapting their proof presentations based on their perceptions of their students' pedagogical needs.

A study examining mathematicians' views of a good pedagogical proof showed that they "valued an introductory sentence that makes transparent the proof framework that will be employed" (Lai, et al., 2012). A written or verbal statement of this type was coded as *outline* in this study. This method addresses students' lack of strategic knowledge by making the proof's strategy explicit. The instructors used *outline* most frequently in between 42% and 68% of their proof presentations. Thus, the instructors in this study were using a strategy that is widely accepted by mathematicians as good pedagogical practice.

Examples were used by the instructors to varying degrees. Three of the four instructors used examples in half of their proof presentations, while the fourth rarely used examples. Comments about the logical structure of the proof were made by the instructors in 25-54% of their proof presentations. The context of the proof in history or within the mathematical content was mentioned in between 13.6% and 38% of the proofs presented.

This study found that three of the four instructors expected students to contribute to 95% of their proof presentations. The fourth instructor expected students to contribute to 50% of his proof presentations. Thus, the instructors' proof presentations were not monologue, but were similar to the *proof presentation with dialogue* strategy identified by Fukawa-Connelly (2012a).

5.3 Allocation of Class Time

In Mejia-Ramos & Inglis's (2009) literature search, they determined that there was a dearth of studies about proof presentations. Several recent studies have focused on proof presentations (Alcock, 2010; Weber, 2011; Fukawa-Connelly, 2012a; Hemmi, 2010; Yopp, 2011), and their rationale for doing so is that proof presentations are an important way in which instructors model the mathematical behavior of proof writing. This study has found that instructors spend large portions of their class time (between 35% and 70%) on proof presentations. This finding further justifies that instructor proof presentations are an important topic for further research.

The timelines constructed of each lecture were analyzed to determine the percentage of time that was spent on interactive lecture. The range of class time spent on interactive lecture was from 26% to 62%. Thus, the instructors were discussing with their students to varying degrees, but there was a significant portion of class time spent on interactions between the instructor and students.

5.4 Example Usage During Proof Presentations

The framework presented in this study describes the example usage of four instructors in their proof presentations. The framework highlights four pedagogical uses for examples: to motivate basic intuitions or claims, to persuade students of the plausibility of the claim or of a sub-claim, to support students' understanding of the claim, proof, or the underlying mathematical content, or to generate proof insight. In this framework, the term *motivate* is used when an example precedes the presentation of the mathematics content, and *support* is used when the example follows the presentation of the mathematics content. The types of examples used by the four instructors were: pattern exploration, boundary examples, instantiation of a claim, generic examples, instantiation of a sub-claim, metaphorical examples, and instantiation of definitions, notation, or concepts. These example types were organized on a timeline that visually displays when the example types were used chronologically in the presentation of the theorem or proof, as presented in Figure 7.

Many of the elements of my framework reflect the use of examples by students and experts when grappling with a mathematical concept as found in previous literature, but the organization of examples based on their pedagogical uses and timing within actual observed lectures is original. The identification of a metaphorical example that was used in a proof presentation is also original.

Two of the instructors in my study linked their own reasoning with examples to what they do in their in-class proof presentations. Thus, the literature on how mathematicians reason with examples is linked to how they use examples in their teaching. Many of the recent studies investigating experts' use of examples claim that a deeper understanding of how successful mathematicians generate and use examples can be useful for designing instructional practices (Lockwood, et al, 2012; Mejia-Ramos & Inglis, 2007). If this is the case, it is also necessary to understand how mathematics instructors are already using examples in their in-class proof presentations, which is the question addressed in this study.

This framework can be compared and contrasted with Lockwood et al.'s (2012) framework for mathematicians' example usage when conjecturing and proving. Many of the uses for examples that they identified were similar to those found in my study, such as checking or verifying, making sense of the situation, proof insight, and understand the statement of the conjecture. Since their framework dealt with conjectures and not with presenting known theorems, they included using examples to "break the conjecture" which was not found in the proof presentations in my data. Using examples to "motivate basic claims or intuitions" was also described by Michner (1978).

Mathematicians use reasoning with examples to convince themselves of the validity of a published theorem or proof (Weber & Mejia-Ramos, 2011). Three of the four instructors in my study used examples relating to the theorem in lieu of a general proof on occasion, which suggests that they were using the examples as a form of justification.

Studies have shown that students tend to be more comfortable reasoning with examples than constructing deductive proofs (Harel & Sowder, 1998). The use of examples in proof presentations has been identified by mathematics instructors as a strategy that can help students comprehend the proofs presented in class (Alcock, 2010; Weber, 2011). Although the participants in this study were unfamiliar with the literature on examples, many of the types of examples that occurred were similar to types of examples that have been previously identified.

There were also examples in the data that could not be linked to existing types of examples. Metaphorical examples were identified in this study. A metaphorical example is an example comparing properties of a mathematical structure to a different but more familiar mathematical structure.

Three of the four instructors used examples frequently in their proof presentations. The three instructors who used examples did so in half of their proof presentations, and the fourth (Dr. G) only used examples in one of the 22 proofs observed. Dr. G commented that he did not use examples because he felt that examples were not readily available in the mathematics content that he was teaching. A pattern exploration example was used by Dr. N in his number theory course, and was not observed in any of the other cases. Exploring number theoretic conjectures with examples is frequently used by mathematics education researchers because of its accessibility to students and experts alike (Harel, 2001; Lockwood, et al; Inglis, et al, 2007). The varied use of examples in proof presentations suggests that the mathematics content that they are presenting is one factor influencing the instructors' example usage.

Although Dr. G did not use examples in his proof presentations, he did use generic diagrams to organize and guide his proof presentations. These pictures were not representative of a specific member of a class of mathematical objects, but were at the same level of generality as the proof; thus they did not fit the definition of example used in this paper (see Section 3.4.5).

5.5 A Multi-Dimensional Analysis of Instructor Questions

Questions posed by instructors at the K-12 level are understood to be primarily factual (Anderson & Krathwohl, 2001; Sahin & Kulm, 2008), with the exception of exchanges in which mathematical proofs are offered (Thompson, et al, 1994). In previous studies of undergraduate level proof based mathematics courses, some instructors do not interact with their students using questions (Weber, 2004), while others ask a variety of questions, including some higher-order questions (Fukawa-Connelly, 2012a). Higher order questions were then followed by a sequence of narrowing questions so that the students eventually responded to a factual question (Fuakwa-Connelly, 2012a).

This study found that across all four case studies, instructors were frequently interacting with their students by asking a variety of questions, including between 30% to 54% higher-order questions. Between 35% to 52% of questions answered by students were higher-order. Thus, although there may be instances of funneling patterns in the instructors' questioning, the students were responding to a variety of questions, including higher-order questions.

5.6 Implications for Future Research

This study did not collect data from students and can therefore make no claims about how the various teaching practices observed in this study impacted student learning. Future research could investigate this manner. Further research into teaching practices could include studies in which I collaborate with the instructor to investigate their use of examples or questioning patterns. This collaboration could be used throughout the data collection and analysis presenting a more detailed description of the instructor's pedagogical intentions regarding their teaching practices.

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APPENDICES

Appendix A: Interview Questions

1. What courses have you taught recently (in the last 3-5 years) in which you presented proofs in class?

2. Describe the balance of proof presentation vs. application/calculation in these courses.

3. Why should we present proofs in class?

4. Why should we ask students to read and write proofs?

5. How do you decide whether or not to present a certain proof in class?

We are interested in what you do in class when you present proofs. To help you focus, think of a specific proof that you will present in class this semester, or that you've presented before. The next few questions will be focused on how you will present that proof.

6. Describe ways in which you help students understand the statement you plan to prove. (i.e. illustrate with examples, recall preliminary facts, relate to similar results, place result in larger context, etc.)

7. Describe how you present the proof. (i.e. board use, dialogue with students, etc.)

8. Describe ways in which you help the students understand the proof. (i.e. structural comments, emphasize difficult steps, do examples, draw a diagram, involve students, etc.)

9. In general, when you present proofs in class, what additional things do you do to help the students understand?

10. How do you assess students' understanding of the mathematical proofs you present in class?

BegAppendix B: Interview Analysis Codes

Theme	Codes
Value of proof in mathematics	Historical Significance
	Proof is foundational to mathematics
	Proof is good mental training
	Proof is a means for verification
	Proof is a means for communication
Emphasis in Class	Student Understanding
	Problem Solving
	Rigor
	Teaching methods of proof
	Student Discovery
	Logic
	Application
	Axiomatic Structure of Mathematics
	Independence (Think about it at home)
	Group Projects
Factors influencing proof presentations	Level of audience
	Covers the main topics
	Time constraints
	Demonstrates a method of proof
	Level of difficulty of proof
Proof presentation tools	Warm-up
	Outline
	Examples
	Draw Pictures
	Motivate steps
	Ask for student input
	Emphasize structure of proof
	Structure of statement
	Write out details
	Non-linear presentation
	Skip details
	Skip difficult parts of some proofs
Interaction	Asks questions to students
	Eye contact
	Students ask questions
	Wait for responses
	Adapting to student feedback

Appendix C: Follow-Up Interview Questions

General:

What is your overall philosophy when teaching proof-based courses?
How would you describe your teaching style in this particular course?
(Read the section about his interview data)
Do you have any comments or anything you would like to add?

Example Usage:

In what ways did you use examples when presenting proofs in class?
(Show them the model, comment on the ways that they used examples from observation data)
What effect does example usage in proofs have on student learning?
Do you have any additional comments?

Student Involvement:

In what ways and how frequently did you involve your students in class?
(Show them their graphs, and discuss them)
Comments?
How did you involve your students in class when presenting proofs?
(Here is what I observed)
How do you think these methods affected student learning?
Do you have any additional comments?

Textbook Usage:

Why did you choose the textbook that you used for this course?
What are the big idea (organizational) things that you like about this text, and things that you don't? Did you plan to deviate from the text in the way you structure the course? Why or why not?
Did you state definitions, theorems, and proofs straight from the book, or did you re-word them? Do you think that had an effect on student learning?

What about the way the text presents material do you try to emulate in your lectures? What do you try to do differently?

Strategic Thinking:

Strategic thinking is defined as a skill separate from content knowledge or the knowledge of logic and what constitutes a proof. It is defined as heuristic guidelines that students can use to recall actions that are likely to be useful or to choose which action to apply among several alternatives. This includes knowledge of the domain's proof techniques, knowledge of which theorems are important and when they are likely to be useful, and knowledge of when and when not to use symbol manipulation (or brute force tactics).
In what ways did you foster strategic thinking when presenting a proof?
How do these methods affect student learning?

Appendix D: Sample detailed log from each instructor

Obs A.8	Proof #	
0:00-9:00	Proof A.8.1	Homework Question: For a, b natural numbers, a~b iff a=b10^k some integer k, show ~ is an equivalence relation. Give a complete set of equivalence class representatives.
9:00-10:15	Organization	Looking for other examples
10:15-16:55	Example	Equivalence Class: Let C be the collection of subsets of {1,2,3,4} and X~Y iff there is a 1-1 correspondence. What are the equivalence classes? Student asks whether or not to include the empty set as an equivalence class.
16:55-29:54	Proof A.8.2	Prove that a/b~c/d iff ad=bc is an equivalence relation on the rationals.
30:00-32:24	Lecture	Where are we going? Gives an outline of the next three topics: Least Integer Principle, Division Alg, Euclidian Alg.
32:24-37:40	Lecture	Least Integer Principle: Equivalent to Principle of Induction. Can we prove this? No. It is an axiom on N
37:40-41:00	Lecture	The Division Algorithm: States the beginning, with a=bq+r. Asks what restrictions should be placed on r. Gives a numerical example: 26=4*4+10, but that's no good. We want 26=4*6+2. Why? Student says, "Because 2<4." Instructor writes $0\leq r < b$. Can we prove this? Writes "Proof," but then decides not to prove it.
41:00-43:30	Lecture	Uses of the division algorithm: modular arithmetic. a~r(modn) iff a-r=nq. This is a relation, why? Student starts to answer, but then instructor says, "any subset of Z×Z is a relation."
43:30-49:50	Proof A.8.3	Claim: equivalence modulo n is an equivalence relation.
49:50-53:00	Lecture	States Prop: Let n be a positive integer, the collection equivalence classes of integers modulo n is denoted by Z_n. Furthermore, a complete set of representatives are {0, 1, 2, …, n-1}. There are n equivalence classes. Comments on how to prove the prop using the division algorithm and uniqueness.
53:00-53:47	Example	Say n is three. Then who else is in the equivalence class of 1? Two students guess numbers, then one says "four" Students are starting to pack up and leave.

Obs C.6	Proof #	
0:00-4:00	Comments	Hands back homework and solutions to homework, discusses grades, comments about student performance on the homework. Mentions that this is a review for their exam. Students take time to look at homework and formulate questions.
4:00-14:00	Lecture HW Problem Solution	Given several different relations, decide if they are symmetric, transitive, or reflexive. A,B sets. A~B iff A contained in B. S={1,2,3} R={(1,1),(1,2),(2,2),(1,3),(3,3)} Draws a diagram with three points and arrows representing relations. x, y real numbers. x~y iff x-y is irrational. (Uses a numerical counterexample to show not transitive) x~y iff (x-y)^2<0. (Discusses the logic of an if-then statement when the antecedent is false) x~y iff |x-y|<2. (Interprets absolute value as distance on a number line. Draws a diagram of a number line)
14:00-15:37	Lecture HW Problem Solution	Show that $(A \cup B) \times C = (A \times C) \cup (B \times C)$. Projects the proof onto the board. Discusses her choice of letters of the elements in the set. Asks for questions. There are none.
15:37-17:30	Lecture HW Problem Solution	Student asks about a homework problem: Prove or give a counterexample. "For every positive integer n, n^2+4n+8 is even." She describes her solution. Instructor says "There was no flaw in your logic, you mis-read the instructions. You have to say that it's false and give a counterexample."
17:30-19:30	HW Comments	More random comments on HW problems, office hours, asks for more questions on this assignment, there are none.
19:30-	HW Comments	Projects homework solutions on the board: Choosing intervals on which given functions are one to one. Given a function on the naturals, show it is not surjective. Prove that a given function on the integers is bijective. (Sketches a graph of the function, a sequence of dots) More comments about homework solutions
25:03	Lecture	Defines the ceiling and floor functions. With numerical examples, and sketches a graph of both step functions.
27:53-	HW Comments	Goes over HW solutions: Examples and non-examples of functions from N to N that are injective, surjective, or both, or neither. Sketches graphs of some of the functions.
32:00-34:30	Comments	Informal comments about use of letters in mathematics.
34:30- 35:30	Student question	Student asks for an example of a surjective proof.
35:30-43:40	C.6.1	Proves Theorem: Suppose $f : A \to B$ and $g : B \to C$ if $g \circ f$ is surjective, then g is surjective.
43:40- 55:30	C.6.2	Instructor works problem from exam review: Prove that $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$

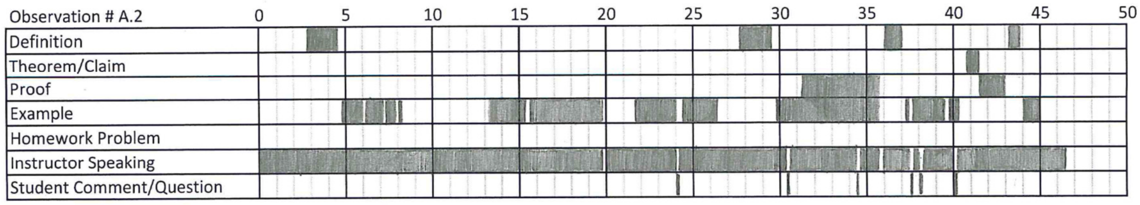
55:30-1:09:30	C.6.3	Proves $1^2+2^2+...+n^2=\frac{n(n+1)(2n+1)}{6}$ by induction.
1:09:30-1:19:30	C.6.4	Proves that for all real x>3, there is a real y<0 such that $x=\frac{3y}{2+y}$.

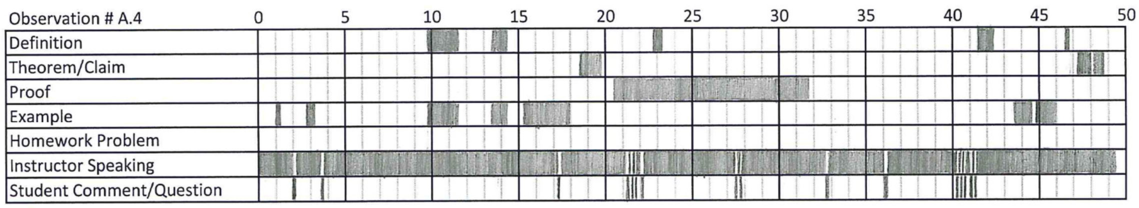
Obs G.4		
0-1:09	Lecture Definition	Draws a picture to remind students of the definition of Same side of a line and opposite sides of the line, plane separation axiom
1:09-2:40	Lecture (Prep for proof)	Plane separation axiom: Let l be a line, P, Q, R points not on l. 1. If P and Q ssl and Q, R ssl, then P, R ssl. 2. If P, Q ssl and Q, R, osl, then P, R, osl. 3. If P, Q, osl and Q, R, osl, then P, R ssl.
2:40-9:32	Proof G.4.1	If P $\neq$ Q are points, then there is a line l so that P, Q are on the opposite sides of l
10:30-18:36	Proof G.4.2	Stick Lemma
19:00-23:10	Proof G.4.3	Z Theorem
23:10-24:50	Definition	If P, Q, R are non-collinear points, then triangle PQR is the three segments $\overline{PQ}\cup\overline{QR}\cup\overline{PR}$. (draws a sketch)
24:50-30:20	Proof G.4.4	Pasch's Little Thm.
30:20-31:00	Pf Comments	Comments about how Pasch's Little Theorem relates to a future theorem that they will prove.
31:00-32:20	Def	AB and AC are opposite rays if (draws a sketch, asks student to finish the def) Student says "A*B*C"
32:20-33:00	Def	If AB and AC are not opposite rays, then angle BAC is the union of the two rays.
33:00-34:16	Lecture with student input	So, we don't allow opposite rays to be an angle. Let's think about how we would define the interior of an angle. (shades the interior on the picture) Student suggests the intersection of the two half planes.
34:16-35:40	Def	(Sketches an angle with an interior point, and refers to it as he writes the def) D is interior to angle BAC if 1. B, D are on same side of AC, 2. C,D ss of AB.
35:40-36:12	Lecture with student input	What do you think it might mean to be interior to a triangle? (sketches picture) Student suggests the intersection of the three half planes. Instructor says this will work, but we can also express it as interiors of angles.
36:12-36:30	Def	D is interior to triangle BAC if D is interior to all three angles.
36:30-38:00	Lecture	(shades the picture to illustrate that the interior of a triangle is the intersection of the interior of two of the angles) So, this is a theorem, which is exercise 4.
38:01-46:13	Proof G.4.5	Angle Chord Lemma

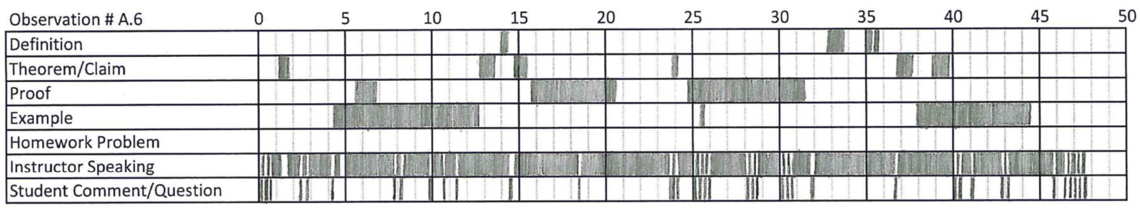
Obs N.7	Proof #	
0-2:30		Small talk with students
2:00-3:07	Lecture	Worst cases of Euclid's Algorithm, when the greatest number of steps occur.
3:07-6:21	Examples	A=13, b=8 Worst cases are Fibonacci numbers. Goes through the algorithm to show that there are 5 steps. A=144, b=89 Goes through the algorithm to show that there are 10 steps.
6:21-7:45	Theorem	Lame's Theorem: The number of steps in Euclid's algorithm applied to a, b is less than or equal to five times the number of digits of the smallest of a or b.
7:45-22:50	Proof	Follows the book on page 106. Uses an example to show that the log base 10 tells how many decimal digits a number contains.
22:50-26:47	Proof N.7.1	Homework Problem: Calculate a formula for the sum, j=1 to n of $q_j r_j$
26:47-36:44	Proof N.7.2	Homework Problem: The GCD of two Fibonacci numbers f_n and f_m is f_(n,m)
38:00-45:00		Statement of the Fundamental Theorem of Arithmetic

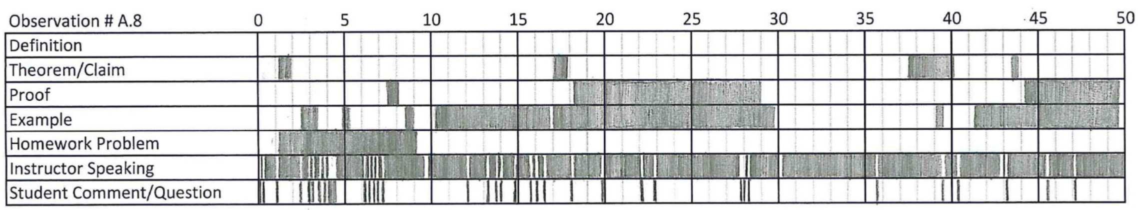
Appendix E: Timelines

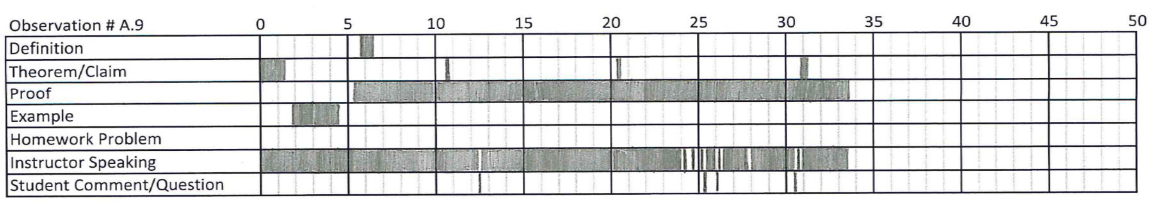
Observations for Dr. A

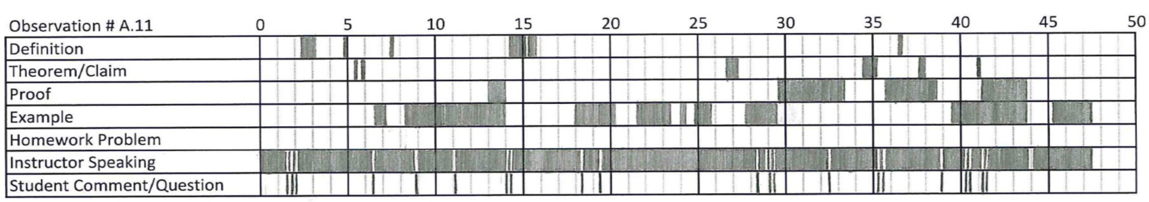


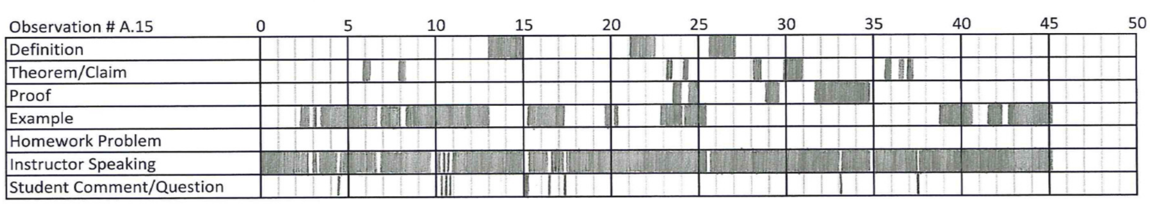




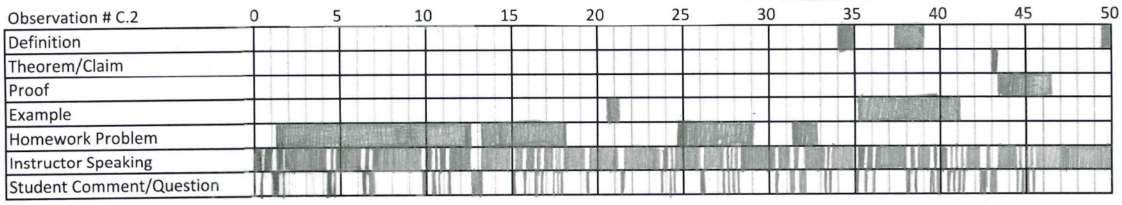


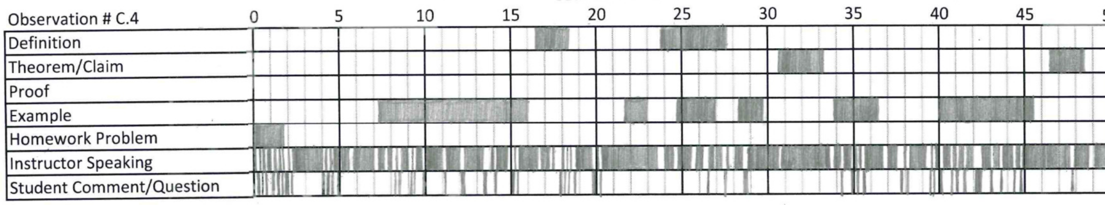


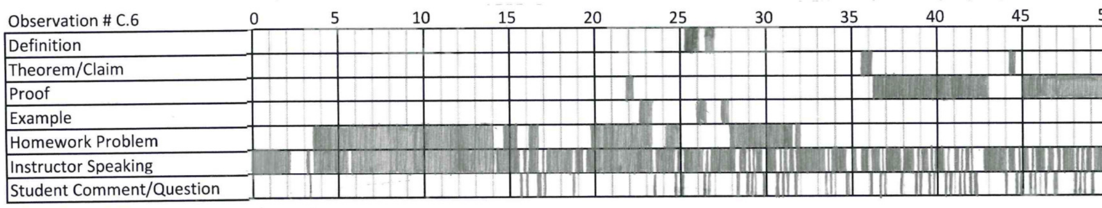


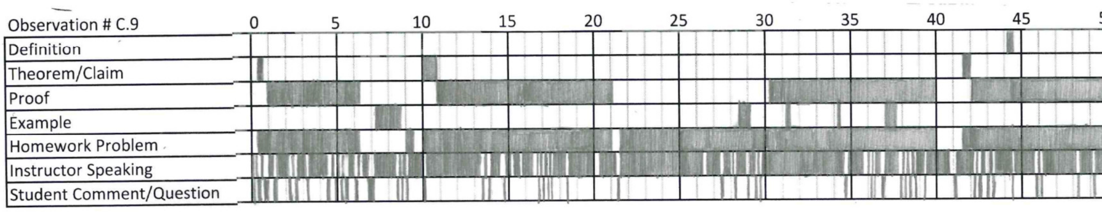


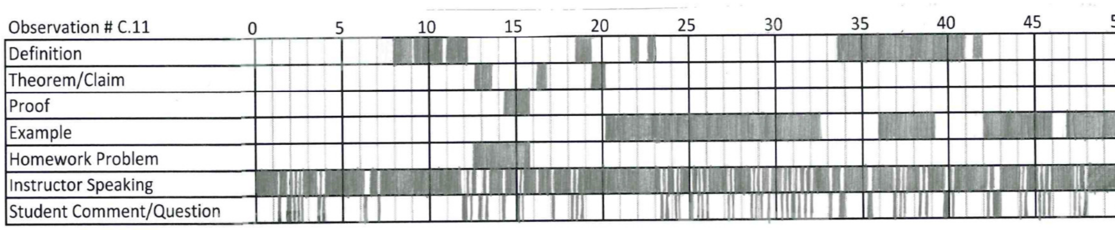
Observations for Dr. C

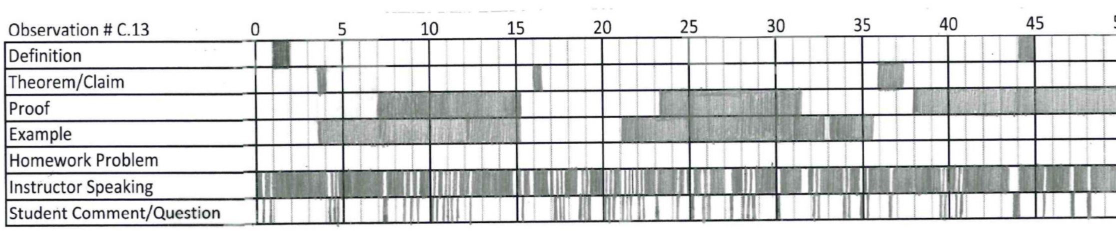


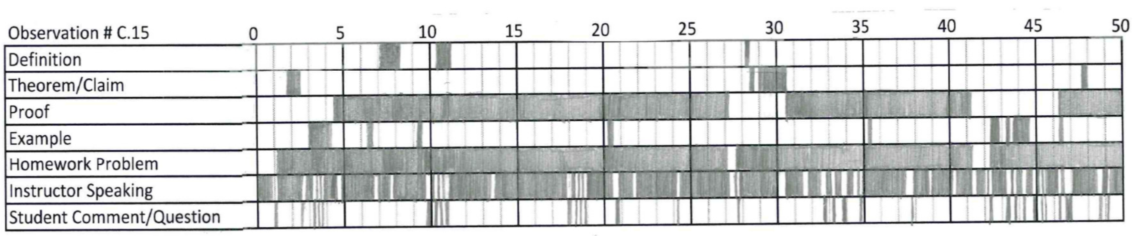


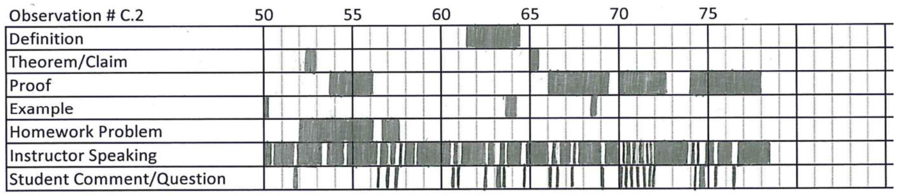








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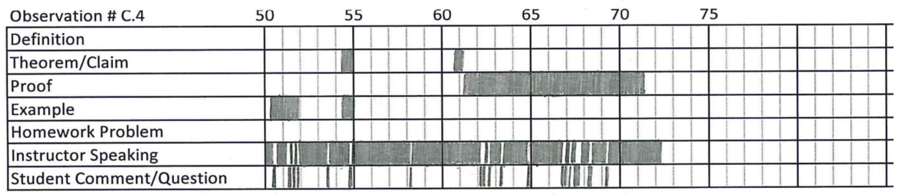
Definition
Theorem/Claim
Proof
Example
Homework Problem
Instructor Speaking
Student Comment/Question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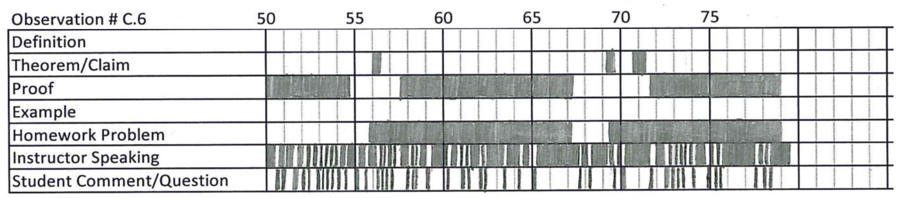
Definition
Theorem/Claim
Proof
Example
Homework Problem
Instructor Speaking
Student Comment/Question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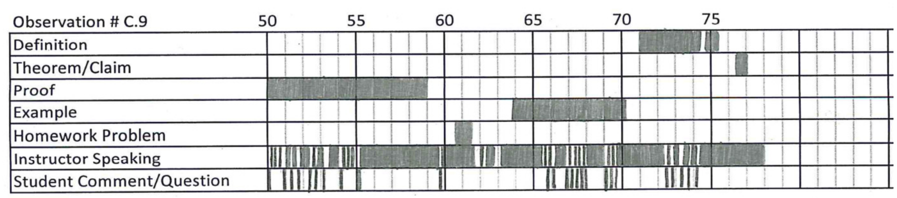
Definition
Theorem/Claim
Proof
Example
Homework Problem
Instructor Speaking
Student Comment/Question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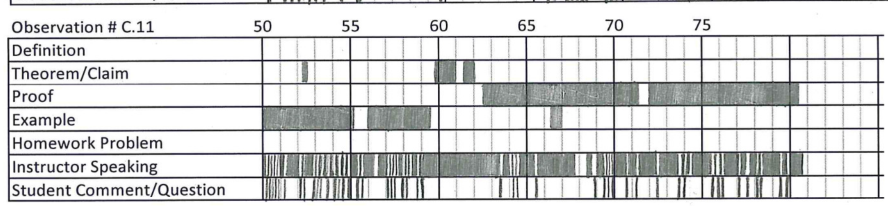
Definition
Theorem/Claim
Proof
Example
Homework Problem
Instructor Speaking
Student Comment/Question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Definition
Theorem/Claim
Proof
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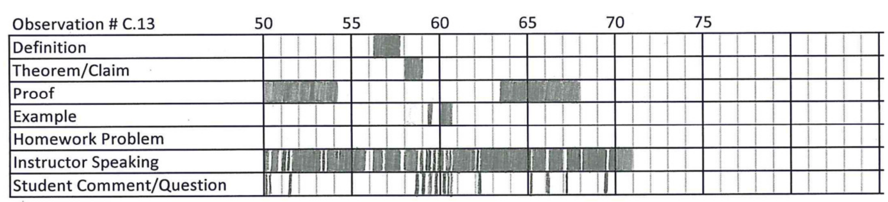
Homework Problem
Instructor Speaking
Student Comment/Question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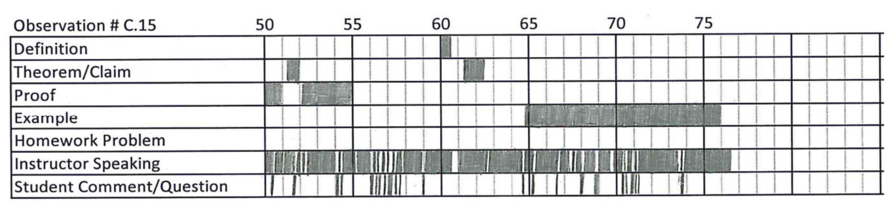
Definition
Theorem/Claim
Proof
Example
Homework Problem
Instructor Speaking
Student Comment/Question


Observation # C.15
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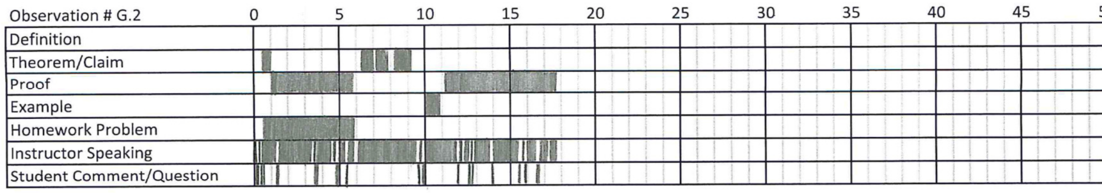
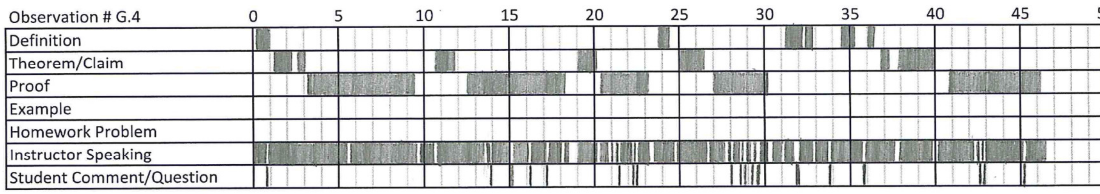
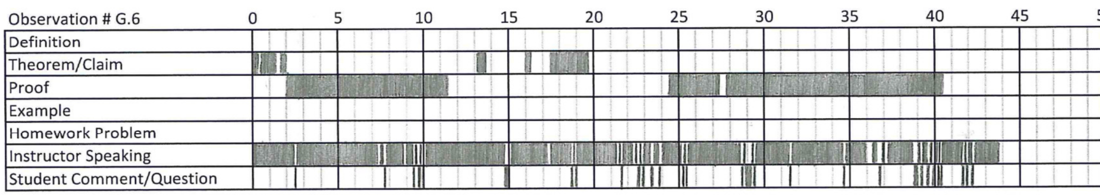
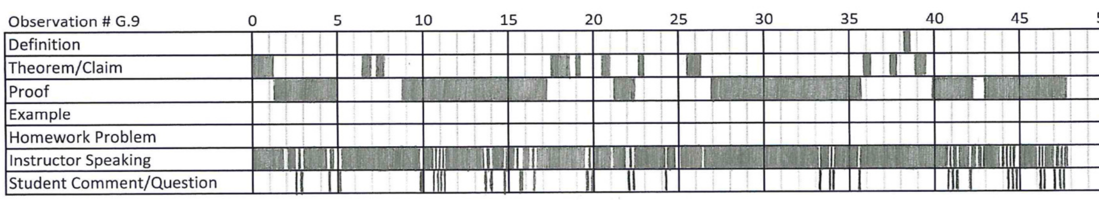
Definition
Theorem/Claim
Proof
Example
Homework Problem
Instructor Speaking
Student Comment/Question

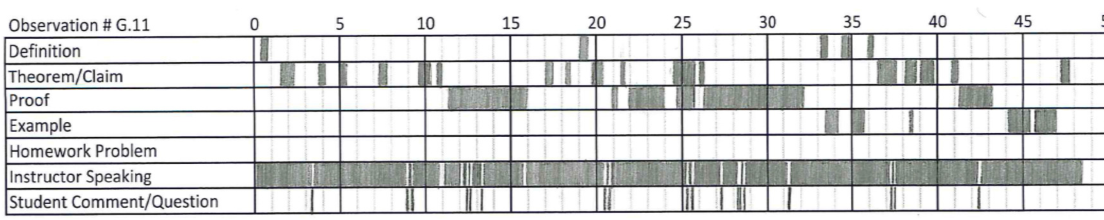
Observations for Dr. G

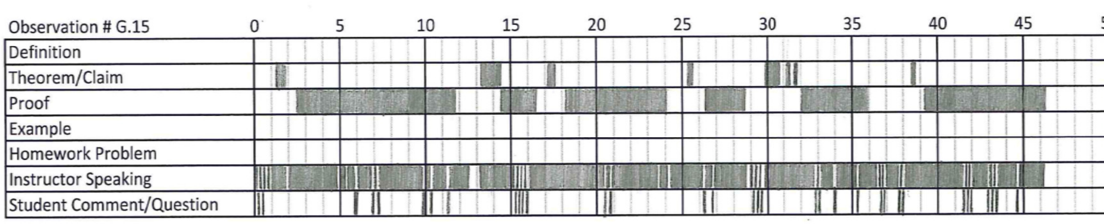




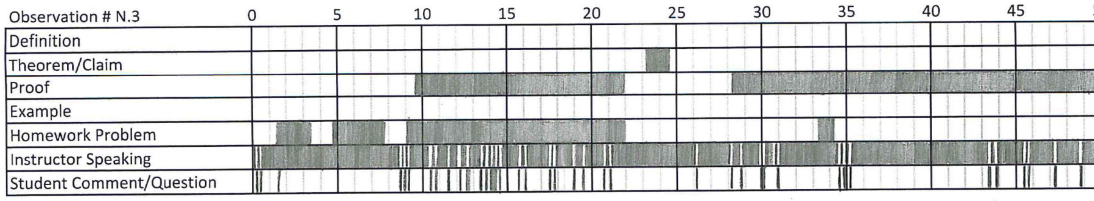


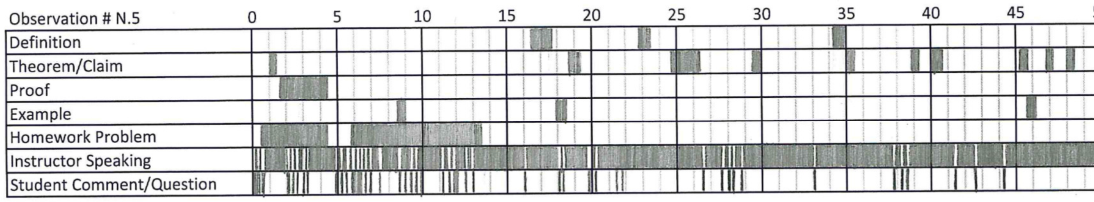


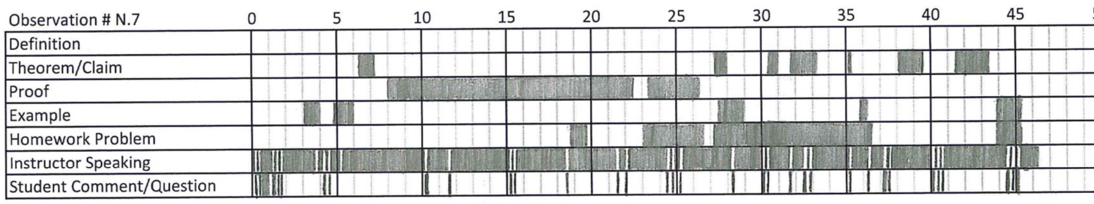


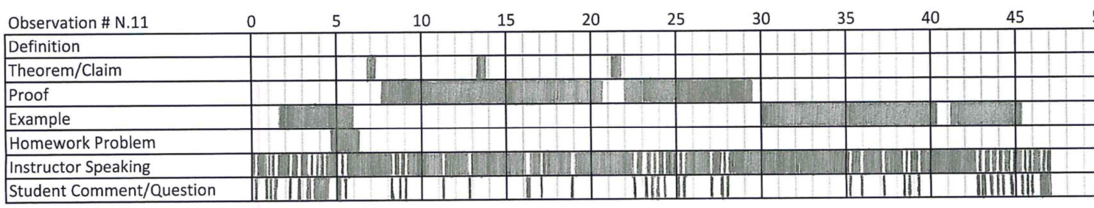


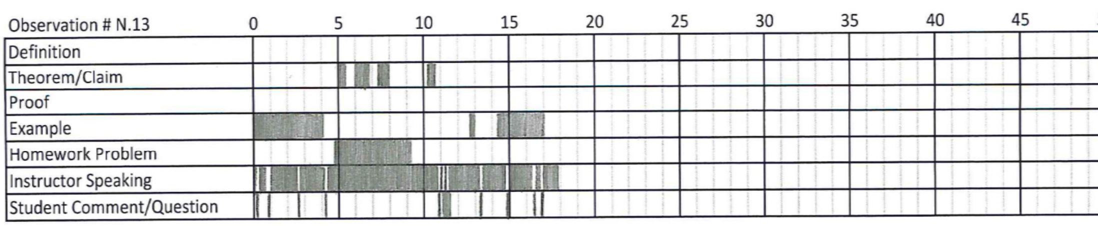
Observations for Dr. N

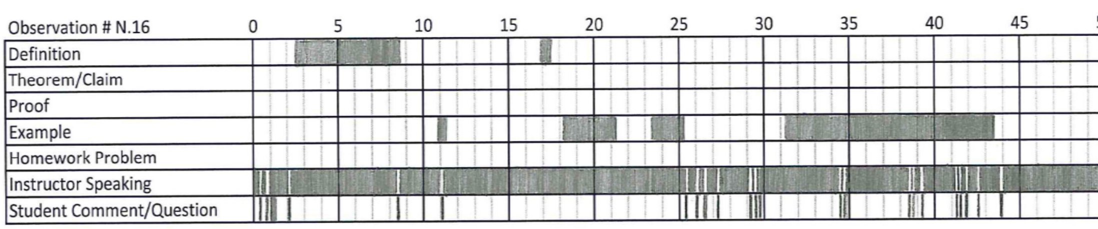












Appendix F: Bloom's Taxonomy Coding Descriptions

Remembering	Applying a Procedure	Understanding	Applying Understanding	Analyzing	Evaluating	Creating
Retrieve relevant knowledge from long-term memory Recall a familiar definition, formula, result, or proof technique Cite Identify Recall List	Students must recognize what knowledge or procedures to recall when directly prompted to do so. Perform computations in a familiar context Using a procedure that does not require deep understanding Plug-in numbers to a familiar function Compute the base case for an induction proof	Make interpretations, provide explanations, make comparisons, or make inferences that require understanding of a mathematics concept Explain in your own words Interpreting definitions Explain the implications of a condition or definition Understand and explain the steps in a given proof Explain, interpret, or modify notation Describe the elements of a given set Describe the graph of a given function Classifying	Recognize when to use or apply a concept Applying a procedure in an unfamiliar context Deciding which theorem or definition to apply and using it. Applying a theorem, result, or proof technique to an unfamiliar context. Plug-in numbers to an unfamiliar function (prime counting function) Computing in an unfamiliar context (different bases) Identifying properties of a certain function	Break material into parts and determine how they relate. Examine, organize, generalize, differentiate Detecting a contradiction Determining the next step in a proof (when it is more than just interpreting a definition) Deconstruct a statement into a plan for a proof Focusing Clarifying the statement to be proved Breaking down the proof into parts Determining how the elements fit or function within a structure	Make judgments based on criteria and standards. Conclude, testing, justify, proving, validate, defend, assess Give a more efficient computation Which proof method is best? Justify your thinking Justify a step in a proof Explain why a certain method was used.	Put elements together to form a coherent or functional whole Reorganize elements into a new pattern or structure. Hypothesizing Designing Constructing Construct a set or function that has certain properties Conjecture Formulate a new definition

Appendix G

INFORMED CONSENT DOCUMENT - INTERVIEW

Project Title: Exploring how mathematicians' instructional values influence proof presentations in the university classroom: A mixed methods investigation.

Investigators:

Melissa Mills, PhD Student, Oklahoma State University Mathematics Department

Dr. Lisa Mantini, Professor of Mathematics, Oklahoma State University Mathematics Department

Purpose:

This is an exploratory research study investigating what mathematics faculty members think about the presentation of mathematical proof, and how these thoughts manifest themselves in the faculty members' actual practices in class. This is a mixed methods study of mathematics faculty members who are teaching proof-based upper division mathematics courses.

Procedures:

This portion of the study will be an interview about your values in teaching mathematical proof, your expectations of student performance, and your ideas about presenting mathematical proof in the classroom. The interview will be audio taped and transcribed by the primary investigators. The investigators will analyze the data to look for themes, and will use these to modify the observation instrument if necessary. The interview will last approximately 1 hour.

Risks of Participation:

There are no known risks associated with this project which are greater than those ordinarily encountered in daily life.

Benefits:

The expected benefits are a chance to reflect upon teaching values, which may positively affect the instructors' confidence or ability to present mathematical proof.

Compensation:

There will be no formal compensation for participation in this study.

Confidentiality:

The records of this study will be kept private. Any written results will discuss group findings and will not include information that will identify you. Research records will be stored securely and only researchers and individuals responsible for research oversight will have access to the records. The data will be stored separately from the consent documents to ensure confidentiality. It is possible that the consent process and data collection will be observed by research oversight staff responsible for safeguarding the rights and wellbeing of people who participate in research. The data will be kept until the final draft of the report is completed.

Contacts:

If you have further questions about both the research and the subject's rights, you may contact:

Melissa Mills	Lisa Mantini
PhD Student	Faculty/Advisor
Mathematics Department	Mathematics Department
431 MSCS	410 MSCS
405-744-8412	744-5777
memills@math.okstate.edu	mantini@math.okstate.edu

If you have questions about your rights as a research volunteer, you may contact Dr. Shelia Kennison, IRB Chair, 219 Cordell North, Stillwater, OK 74078, 405-744-3377 or irb@okstate.edu.

Participant Rights:

Participation in this study is voluntary, and you may discontinue the research activity at any time without reprisal or penalty. There are no foreseeable risks should you choose to withdraw. There are no reasons that will justify terminating your participation in the study.

Signatures:

I have read and fully understand the consent form. I sign it freely and voluntarily. A copy of this form has been given to me.

________________________ _______________

Signature of Participant Date

I certify that I have personally explained this document before requesting that the participant sign it.

________________________ _______________

Signature of Researcher Date

Appendix H

INFORMED CONSENT DOCUMENT – OBSERVATION

Project Title: Exploring how mathematicians' instructional values influence proof presentations in the university classroom: A mixed methods investigation.

Investigators:

Melissa Mills, PhD Student, Oklahoma State University Mathematics Department

Dr. Lisa Mantini, Professor of Mathematics, Oklahoma State University Mathematics Department

Purpose:

This is an exploratory research study investigating what mathematics faculty members think about the presentation of mathematical proof, and how these thoughts manifest themselves in the faculty members' actual practices in class. This is a mixed methods study of mathematics faculty members who are teaching proof-based upper division mathematics courses.

Procedures:

Faculty members will be observed in the regular classroom setting. Notes will be taken using an observation instrument developed by the investigators. The faculty member and chalk board will be video taped to capture gestures and the material written on the board. The video recordings will not be transcribed, but will be used by the investigators as a reference when the observation instrument is insufficient, and short clips may be used when presenting the results at the Research in Undergraduate Mathematics Education Conference. The instrument will address how the instructor presents proofs in class, types of questions asked, usage of examples or illustrations, style of written presentation on the board, and overall proof presentation methods.

Risks of Participation:

There are no known risks associated with this project which are greater than those ordinarily encountered in daily life.

Benefits:

The expected benefits are a chance to reflect upon teaching values, which may positively affect the instructors' confidence or ability to present mathematical proof. This study will fill a gap in the current research literature by investigating the current practices of faculty members when presenting proofs, which could lead to further development of teaching practices to improve the quality of education.

Compensation:

There will be no formal compensation for participation in this study.

Confidentiality:

The records of this study will be kept private. Any written results will discuss group findings and will not include information that will identify you. Research records will be stored securely and only researchers and individuals responsible for research oversight will have access to the records. The data will be stored separately from the consent documents to ensure confidentiality. It is possible that the consent process and data collection will be observed by research oversight staff responsible for safeguarding the rights and wellbeing of people who participate in research. The data will be kept until the final draft of the report is completed.

Contacts:

If you have further questions about both the research and the subject's rights, you may contact:

Melissa Mills	Lisa Mantini
PhD Student	Faculty/Advisor
Mathematics Department	Mathematics Department
431 MSCS	410 MSCS
405-744-8412	744-5777
memills@math.okstate.edu	mantini@math.okstate.edu

If you have questions about your rights as a research volunteer, you may contact Dr. Shelia Kennison, IRB Chair, 219 Cordell North, Stillwater, OK 74078, 405-744-3377 or irb@okstate.edu.

Participant Rights:

Participation in this study is voluntary, and you may discontinue the research activity at any time without reprisal or penalty. There are no foreseeable risks should you choose to withdraw. There are no reasons that will justify terminating your participation in the study.

Signatures:

I have read and fully understand the consent form. I sign it freely and voluntarily. A copy of this form has been given to me.

__________________________ ________________

Signature of Participant Date

I will allow portions of my video-taped lectures to be used in conference presentations.

__________________________ ________________

Initials of Participant Date

I certify that I have personally explained this document before requesting that the participant sign it.

__________________________ ________________

Signature of Researcher Date

VITA

Melissa Ann Mills

Candidate for the Degree of

Doctor of Philosophy

Thesis: CASE STUDIES OF INSTRUCTIONAL PRACTICES IN ADVANCED MATHEMATICS LECTURES

Major Field: Mathematics

Biographical:

Education:

Completed the requirements for the Doctor of Philosophy in mathematics at Oklahoma State University, Stillwater, Oklahoma in July, 2013.

Completed the requirements for the Master of Science in your major at Oklahoma State University, Stillwater, Oklahoma in May 2006.

Completed the requirements for the Bachelor of Science/Arts in your major at Oklahoma State University, Stillwater, Oklahoma in December, 2003.

Experience:

Graduate TA, Oklahoma State University, January 2004-May 2006
Mathematics Instructor, Stillwater Junior High, August 2006-May 2007
Graduate TA, Oklahoma State University, August 2007-May 2013

Professional Memberships:

Mathematical Association of America
Research in Undergraduate Mathematics Education Special Interest Group of the Mathematical Association of America